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ERRATUM

To the last sentence of the text on page 2 should be added:

However, this is only true when the magnetic field lines are straight. In the general case our expressions (A.6) and (A.7) differ from the formulas of Simon and Thompson ((27)), and they reduce exactly to Macmahon's result as shown in the appendix.

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On the Viscosity and Heat Conductivity of a Collisionless Plasma in a Magnetic Field

by

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Abstract

The viscous stress tensor and the heat flux tensor of a collisionless plasma immersed in a strong magnetic field are calculated by Grad's moments method. No contradiction is found between the expressions of the viscous stress tensor obtained earlier by A. Macmahon and by A. Simon and W. Thompson. It is shown that in the approximation considered the "longitudinal" heat flux is absent.

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A rarefied plasma consisting of electrons and ions is usually described by the kinetic equation in the self-consistent field approximation (Vlasov plasma). However, many problems of plasma physics can be solved by means of the more simple and descriptive hydrodynamic equations. As is known, the hydrodynamic description of ordinary neutral gases is applicable because of the smallness of the molecular free path.

In a collisionless plasma immersed in a strong magnetic field the Larmor radius (more accurately, the cyclotron radius) is an analogue of the molecular free path.

The hydrodynamic equations for a collisionless plasma immersed in a strong magnetic field were first obtained by Chew, Goldberger and Low[1]. They are equations of the zero-order approximation with the parameter $\epsilon = \frac{a}{L}$ and seem to be equivalent to the equations of ideal magnetohydrodynamics. Here, $\underline{a}$ is a Larmor radius, and L is a characteristic macroscopic length[2]. However, for solving many problems of plasma physics the Chew-Goldberger-Low zero approximation is inadequate. In such cases the effects of the finite Larmor radius (FLR) have to be taken into account.* These effects are expressed in terms of "magnetic viscosity" and "magnetic heat conductivity" in the hydrodynamic equations[3-7].

The expression for the viscous stress tensor of collisionless plasma with anisotropic pressure was first found by W. B. Thompson[3]. It has, however, been noticed in the papers refs. 8 and 9 that Thompson's results were incorrect.

A. Simon and W. B. Thompson have later corrected some errors of the paper ref. 3 and obtained an expression for the stress tensor which, in their opinion, differs from Macmahon's expression in ref. 8. Therefore, according to Simon and Thompson, the results of the paper ref. 8 are incorrect.

To make these distinctions clear, we have performed all the calculations once again in order to find the stress tensor and the heat fluxes in a collisionless plasma immersed in a strong magnetic field. Our results coincide exactly with Macmahon's. It is also shown that the expressions for the stress tensor in the works refs. 15 and 8 are exactly the same.

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* It should be noted that there are some differences between the equations of the Chew-Goldberger-Low expansion and those of the FLR theory[14].

We use Grad's method of moments[10]. This method has been used in 13 moments approximation in ref. 11 for investigating transport processes in a plasma with collisions. We consider a collisionless plasma with anisotropic pressure. For such a plasma the 13 moments approximation in general seems to be inapplicable if there are heat fluxes along the magnetic field lines.

Therefore, if there are longitudinal heat fluxes, the 20 moments approximation ought to be used.

1. A rarefied plasma is described by the Vlasov equation and Maxwell's equations (in standard notation) as

$$\frac{\partial f_a(\vec{r},\vec{v},t)}{\partial t} + \vec{v}\cdot\nabla f_a + \vec{G}_a \cdot \frac{\partial f_a}{\partial \vec{v}} + \frac{e_a}{m_a c}[\vec{v}\cdot\vec{B}]\cdot\frac{\partial f_a}{\partial \vec{v}} = 0,$$

$$\text{rot}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \qquad \text{rot}\,\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \tag{1}$$

$$\text{div}\,\vec{E} = 4\pi\rho, \qquad \text{div}\,\vec{B} = 0,$$

$$\rho = \sum_a e_a \int f_a\, d\vec{v}, \qquad \vec{j} = \sum_a e_a \int \vec{v} f_a\, d\vec{v}.$$

Here, "a" refers to particle species and is to be summed over values i and e for ions and electrons. $\vec{G}_a$ represents the acceleration of all the forces except the $\frac{e}{c}[\vec{v}\cdot\vec{B}]$ force (for instance, $\vec{G}_a = \frac{e_a}{m_a}\vec{E} + \vec{g}$, where g is an artificial gravity which approximates the magnetic field curvature effects). From now on we will omit the subscript "a".

According to the moments method of Grad, the distribution function is expanded in Hermite polynomials as follows:

$$f(\vec{r},\vec{v},t) = f^0\left\{a^{(o)} + a^{(1)}_{i_1}(\vec{r},t)\, H^{(1)}_{i_1}(\vec{\xi}) + \ldots + \frac{1}{N!}a^{(N)}_{i_1\ldots i_N} H^{(N)}_{i_1\ldots i_N} + \ldots\right\} \tag{2}$$

Here and in the following, summation over the repeated indexes is assumed.

The Hermite polynomial tensors

$$H^{(N)}_{i_1 \ldots i_N}(\xi) = (-1)^N e^{\xi^2/2} \frac{\partial^N}{\partial \xi_{i_1} \ldots \partial \xi_{i_N}} e^{-\xi^2/2}$$

satisfy the ortho-normalization conditions

$$\frac{1}{(2\pi)^{3/2}} \int e^{-\xi^2/2} H^{(N)}_{i_1 \ldots i_N}(\xi) H^{(M)}_{j_1 \ldots j_M}(\xi) d\xi = \delta_{MN} \delta^{(N)}_{ij} . \qquad (3)$$

where

$\delta^{(1)}_{ij} \equiv \delta_{ij}$ is the Kronecker symbol; $\delta^{(2)}_{ij} \equiv \delta_{i_1 i_2} \delta_{j_1 j_2} + \delta_{i_2 j_2} \delta_{i_2 j_2} +$

$+ \delta_{i_1 j_2} \delta_{i_2 j_1}$, ... ,

ξ is the dimensionless peculiar velocity relative to the fluid velocity. The first few Hermite polynomial tensors are

$$\begin{aligned} H^{(0)} &= 1, \\ H^{(1)}_i &= \xi_i, \\ H^{(2)}_{ij} &= \xi_i \xi_j - \delta_{ij}, \\ H^{(3)}_{ijk} &= \xi_i \xi_j \xi_k - (\xi_i \delta_{jk} + \xi_j \delta_{ik} + \xi_k \delta_{ij}) . \end{aligned} \qquad (4)$$

In the following we also need some general formulae[10]:

$$\frac{\partial H^{(N)}_{i_1 \ldots i_N}(\xi)}{\partial \xi_{i_r}} = H^{(N-1)}_{i_1 \ldots \breve{i_r} \ldots i_N}(\xi) \qquad (4a)$$

$$\xi_j H^{(N)}_{i_1 \ldots i_N}(\xi) = H^{(N+1)}_{j i_1 \ldots i_N}(\xi) + \delta_{j i_r} H^{(N-1)}_{i_1 \ldots \breve{i_r} \ldots i_N}(\xi) . \qquad (4b)$$

Here the notation ... $\underbrace{i_r}$ .... shows that the index i_r must be omitted.

By using equations (3) it can easily be shown that the coefficients $a^{(N)}_{i_1 \dots i_N}$ are given by

$$a^{(N)}_{i_1 \dots i_N}(\vec{r}, t) = \frac{1}{n(\vec{r}, t)} \int f(\vec{r}, \vec{v}, t)\, H^{(N)}_{i_1 \dots i_N}(\vec{\xi})\, d\vec{v} \equiv \overline{H^{(N)}_{i_1 \dots i_N}} \quad . \qquad (5)$$

We observe that the coefficients $a^{(N)}_{i_1 \dots i_N}$ are proportional to combinations of moments of the distribution function $f(\vec{r}, \vec{v}, t)$. Therefore the calculation of $a^{(N)}_{i_1 \dots i_N}$ is equivalent to the calculation of the moments of the distribution function.

Further we choose the function f^o as a local Maxwellian distribution with anisotropic temperature:

$$f^o = \frac{n}{\theta_\perp \sqrt{\theta_\parallel}} \left(\frac{m}{2\pi} \right)^{3/2} \exp \left\{ - \left(\frac{m c_\perp^2}{2\theta_\perp} + \frac{m c_\parallel^2}{2\theta_\parallel} \right) \right\} , \qquad (6)$$

where $\vec{c}(\vec{r}, t) = \vec{v} - \vec{u}(\vec{r}, t)$ is the peculiar velocity of plasma particles; $\vec{u}(\vec{r}, t) = \frac{1}{n} \int \vec{v} f d\vec{v}$ is the flow velocity;

$$\vec{c}_\parallel \equiv \vec{b}(\vec{b} \cdot \vec{c}) \ ; \quad \vec{c}_\perp \equiv \vec{c} - \vec{c}_\parallel \ ; \quad \vec{b} \equiv \frac{\vec{B}}{B} \ ; \qquad (6a)$$

$\theta_\parallel$, $\theta_\perp$ are "longitudinal" and "transverse" kinetic temperatures respectively. In the isotropic case $\theta_\parallel = \theta_\perp$. As is known, the anisotropic Maxwellian distribution (6) is inconsistent with the Boltzmann equation[12]. But such a distribution is probably a very good approximation in the collisionless plasma. (About the speed of the equalization of the temperatures $\theta_\parallel$, $\theta_\perp$ see, for example ref. 13.)

In the isotropic case the pressure p is defined by the formula $p = \frac{1}{3}$ trace $\hat{P}$, where $\hat{P}$ is the pressure (stress) tensor:

$$(\hat{P})_{ij} \equiv P_{ij} = m \int c_i c_j f d\vec{v} \ . \qquad (7)$$

So it is possible to define a viscous stress tensor $\hat{\sigma}$ by the formula

$$P_{ij} = p\delta_{ij} + \sigma_{ij} \ ,$$

provided that trace $\hat{\sigma} = 0$.

In the anisotropic case

$$P_{ij} = m \int c_i c_j f d\vec{v} = p_\parallel b_i b_j + p_\perp(\delta_{ij} - b_i b_j) + \sigma_{ij} . \tag{8}$$

Then, provided that

$$\begin{aligned} &\text{trace } \hat{\sigma} = 0 , \\ &b_i b_j \sigma_i \sigma_j = 0 , \end{aligned} \tag{8a}$$

$$\begin{aligned} p_\parallel &= b_i b_j P_{ij} , \\ p_\perp &= \frac{1}{2}(\delta_{ij} - b_i b_j) P_{ij} . \end{aligned} \tag{8b}$$

So $p_\parallel + 2 p_\perp$ = trace $\hat{P}$. "Longitudinal" and "transverse" temperatures are defined by the two equations

$$\theta_\parallel = \frac{p_\parallel}{n} \quad \text{and} \quad \theta_\perp = \frac{p_\perp}{n} . \tag{8c}$$

Let us now introduce the vector of dimensionless peculiar velocity

$$\xi_i = \left\{ \sqrt{\frac{m}{\theta_\perp}} \, (\delta_{ij} - b_i b_j) + \sqrt{\frac{m}{\theta_\parallel}} \, b_i b_j \right\} c_j \equiv V_{ij}^{-1} c_j \tag{9}$$

and vice versa

$$c_i = \left\{ \sqrt{\frac{\theta_\perp}{m}} \, (\delta_{ij} - b_i b_j) + \sqrt{\frac{\theta_\parallel}{m}} \, b_i b_j \right\} \xi_j \equiv V_{ij} \xi_j , \tag{9a}$$

so that

$$V_{ik} V_{kj}^{-1} = \delta_{ij} . \tag{9b}$$

By using formulae (5), (3), (4), and (9) one can easily obtain

$$a^{(o)} = 1 \; ; \quad a_i^{(1)} = 0 \, ,$$

$$a_{ij}^{(2)} = \frac{1}{p_\perp} \left\{ \sigma_{ij} + \left(\sqrt{\frac{p_\perp}{p_\parallel}} - 1 \right) (b_i \sigma_{jk} + b_j \sigma_{ik}) \, b_k \right\} \, ,$$

$$a_{ijk}^{(3)} = \frac{1}{p_\perp} \sqrt{\frac{m}{\theta_\perp}} \left\{ M_{ijk} + \left(\sqrt{\frac{p_\perp}{p_\parallel}} - 1 \right) (b_i M_{jkn} + b_j M_{ikn} + b_k M_{ijn}) \, b_n \; + \right. \tag{10}$$

$$+ \left(\sqrt{\frac{p_\perp}{p_\parallel}} - 1 \right)^2 (b_i b_j M_{kmn} + b_i b_k M_{jmn} + b_j b_k M_{imn}) b_m b_n \; +$$

$$\left. + \left(\sqrt{\frac{p_\perp}{p_\parallel}} - 1 \right)^3 b_i b_j b_k b_l b_m b_n M_{lmn} \right\} \, .$$

Here σ_{ij} is the viscous stress tensor; M_{ijk} is the heat flux tensor:

$$M_{ijk} = m \int c_i c_j c_k \, f \, d\vec{v} \, , \tag{11}$$

so that

$$\frac{1}{2} M_{ikk} = \int c_i \, \frac{mc^2}{2} \, f \, d\vec{v} \equiv q_i \tag{11a}$$

is the heat flux vector.

To obtain evolution equations for the coefficients $a_{i_1 \ldots i_N}^{(N)}$ it is convenient to exchange the variables $t, \vec{r}, \vec{v}$ for the variables $t, \vec{r}, \vec{c} = \vec{v} - \vec{u}$ in the Vlasov equation (1):

$$\frac{df}{dt} + c_l \frac{\partial f}{\partial x_l} - \frac{\partial f}{\partial c_k} \left\{ \frac{du_k}{dt} + c_l \frac{\partial u_k}{\partial x_l} - G_k - \Omega \varepsilon_{klm} (c_l + u_l) b_m \right\} = 0, \tag{12}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$; $\quad \Omega = \frac{eB}{mc}$; $\quad \varepsilon_{klm}$ - Levy-Civita symbol.

Multiplying equation (12) by $H_{i_1 \ldots i_N}^{(N)}(\vec{\xi})$, integrating over $\vec{v}$ and using formulae (5), (4a) and (4b), we obtain after some calculations

$$\frac{d a^{(N)}_{i_1..i_N}}{dt} + \left(a^{(N+1)}_{li_1..i_N} + \delta_{li_s} a^{(N-1)}_{i_1..\breve{i_s}..i_N}\right)\left(\frac{\partial V_{lj}}{\partial x_j} + V_{lj} \frac{\partial \ln n}{\partial x_j}\right) +$$

$$+ V_{lj} \frac{\partial}{\partial x_j} \left(a^{(N+1)}_{li_1..i_N} + \delta_{li_s} a^{(N-1)}_{i_1..\breve{i_s}..i_N}\right) -$$

$$- \left(V_{lj} \frac{d V^{-1}_{ji_s}}{dt} - \frac{\partial u_j}{\partial x_k} V_{kl} V^{-1}_{ji_s}\right)\left(a^{(N)}_{li_1..\breve{i_s}..i_N} + \delta_{li_r} a^{(N-2)}_{i_1..\breve{i_r}..\breve{i_s}..i_N}\right) +$$

$$+ \left(\frac{du_k}{dt} - G_k\right) V^{-1}_{ki_s} a^{(N-1)}_{i_1..\breve{i_s}..i_N} -$$

$$- V_{lj} V_{mk} \frac{\partial V^{-1}_{ki_r}}{\partial x_j} \left(a^{(N+1)}_{lmi_1..\breve{i_r}..i_N} + \delta_{mi_s} a^{(N-1)}_{li_1..\breve{i_r}..\breve{i_s}..i_N} +\right.$$

$$\left. + \delta_{li_s} a^{(N-1)}_{mi_1..\breve{i_r}..\breve{i_s}..i_N} + \delta_{li_s} \delta_{mi_n} a^{(N-3)}_{i_1..\breve{i_r}..\breve{i_s}..\breve{i_n}..i_N}\right) =$$

$$= \Omega\, \epsilon_{jkl}\, b_l\, V_{km} V^{-1}_{ji_r} \left(a^{(N)}_{mi_1..\breve{i_r}..i_N} + \delta_{mi_s} a^{(N-2)}_{i_1..\breve{i_r}..\breve{i_s}..i_N}\right) . \tag{13}$$

Here, the notation $\ldots\breve{i_s}\ldots\breve{i_r}\ldots$ shows that the indices i_s, i_r must be omitted.

In the isotropic case the left-hand side of the system (13) of course coincides with the known expression[10]. The system (13) is equivalent to a Vlasov equation and represents an infinite chain of moment equations. For practical purposes it is necessary to select a method of cutting off this chain. By the Grad method it is possible to describe the properties of the system by the quantities $a^{(M)}_{i_1...i_M}$, provided that all the other coefficients $a^{(M+1)}_{i_1..i_{M+1}}$, $a^{(M+2)}_{i_1...i_{M+2}}$, .. are equal to zero. But it does not mean, of course, that all the moments of the order M+1, M+2, ... are equal to zero. In the present case, moments of the order M+1, M+2, ... are expressed in terms of moments of the 0^{th}, 1^{st}, ... M^{th} order. In practice one usually restricts oneself by setting $M = 3$ so that all coefficients $a^{(4)} = a^{(5)} = \dots = 0$. In this approximation a system is assumed to be described completely by 20 macroscopic quantities (moments) n, u_i, θ, σ_{ij}, M_{ijk}. In many cases a more simple description is possible for $M = 3$ when the 3^{rd}-order moments M_{ijk} can be expressed in terms of a heat flux vector:

$$M_{ijk} = \frac{2}{5}\left(q_i \delta_{jk} + q_j \delta_{ik} + q_k \delta_{ij}\right) . \tag{14}$$

In this approximation a system is described by 13 moments n, $\vec{u}$, θ, σ, $\vec{q}$. (For a full description of the plasma, the variables of the electromagnetic field must of course be added.) In general, however, in our anisotropic case the 13-moments approximation is invalid. In this approximation the calculated tensor M_{ijk} has the form

$$M_{ijk} = \frac{2}{5}\left\{q_i \delta_{jk} + q_j \delta_{ik} + q_k \delta_{ij} + q_\parallel\left[\left(\frac{p_\perp}{p_\parallel}\right)^{3/2} - 1\right](b_i \delta_{jk} + b_j \delta_{ik} + b_k \delta_{ij}) \right. . \tag{14a}$$

This contradicts the general relation

$$M_{ikk} = 2 q_i . \tag{14b}$$

It is obvious that (14a) and (14b) coincide only in the absence of longitudinal heat flux ($q_\parallel \equiv \vec{q} \cdot \vec{b} = 0$). Therefore, if there are longitudinal heat fluxes, a Vlasov plasma in a strong magnetic field can be described by the quantities

$$n,\ \vec{u},\ \theta_\perp,\ \theta_\parallel,\ \sigma_{ij}\ (\text{or } a^{(2)}_{ij}),\ M_{ijk}\ (\text{or } a^{(3)}_{ijk}),\ \vec{E},\ \vec{B}.$$

The first few moments of the Vlasov equation may be written

$$\frac{\partial n}{\partial t} + \operatorname{div} n\vec{u} = 0 , \tag{15}$$

$$mn \frac{d\vec{u}}{dt} = - \nabla \cdot \hat{P} + (\vec{G} + \Omega\,[\vec{u} \cdot \vec{B}\,])mn \tag{16}$$

$$\frac{d\theta_{\parallel}}{dt} = -2\theta_{\parallel}\vec{b}\cdot(\vec{b}\cdot\nabla)\vec{u} - \frac{b_s b_r}{n}\left(2\sigma_{rm}\frac{\partial u_s}{\partial x_m} + \frac{\partial M_{rsm}}{\partial x_m}\right) + \frac{1}{n}\sigma_{rs}\frac{d}{dt}b_r b_s \tag{17}$$

$$\frac{d\theta_{\perp}}{dt} = -\theta_{\perp}\,\mathrm{div}\,\vec{u} + \theta_{\perp}\vec{b}\cdot(\vec{b}\cdot\nabla)\vec{u} - \frac{\sigma_{rs}}{n}\frac{\partial u_r}{\partial x_s} - \frac{1}{n}\mathrm{div}\,\vec{q} + $$

$$+ \frac{b_r b_s}{2n}\left(2\sigma_{rm}\frac{\partial u_s}{\partial x_m} + \frac{\partial M_{rsm}}{\partial x_m}\right) - \frac{\sigma_{rs}}{2n}\frac{d}{dt}b_r b_s . \tag{18}$$

Then, from equations (13) we obtain

$$\frac{d a_{ij}^{(2)}}{dt} + \frac{1}{\theta_{\perp}}\frac{d\theta_{\perp}}{dt}a_{ij}^{(2)} + \frac{1}{2}\left(\frac{1}{\theta_{\parallel}}\frac{d\theta_{\parallel}}{dt} - \frac{1}{\theta_{\perp}}\frac{d\theta_{\perp}}{dt}\right)b_l\left(a_{lj}^{(2)}b_i + a_{li}^{(2)}b_j\right) +$$

$$+\left(1-\sqrt{\frac{\theta_{\perp}}{\theta_{\parallel}}}\right)\left[\frac{db_l}{dt}\left(b_i a_{lj}^{(2)} + b_j a_{li}^{(2)}\right) + \sqrt{\frac{\theta_{\parallel}}{\theta_{\perp}}}\,b_l\left(a_{lj}^{(2)}\frac{db_i}{dt} + a_{li}^{(2)}\frac{db_j}{dt}\right)\right] -$$

$$-\sqrt{\frac{\theta_{\parallel}}{\theta_{\perp}}}\left(1-\sqrt{\frac{\theta_{\perp}}{\theta_{\parallel}}}\right)^2 b_k b_l b_m\frac{\partial u_k}{\partial x_m}\left(a_{lj}^{(2)}b_i + a_{li}^{(2)}b_j\right) +$$

$$+\left(\sqrt{\frac{\theta_{\perp}}{\theta_{\parallel}}}-1\right)\frac{\partial u_k}{\partial x_l}\,b_k\left(b_i a_{lj}^{(2)} + b_j a_{li}^{(2)}\right) +$$

$$+\sqrt{\frac{\theta_{\parallel}}{\theta_{\perp}}}\left(1-\sqrt{\frac{\theta_{\perp}}{\theta_{\parallel}}}\right)b_m b_l\left(\frac{\partial u_i}{\partial x_m}a_{lj}^{(2)} + \frac{\partial u_j}{\partial x_m}a_{li}^{(2)}\right) + \frac{\partial u_i}{\partial x_l}a_{lj}^{(2)} + \frac{\partial u_j}{\partial x_l}a_{li}^{(2)} +$$

$$+\sqrt{\frac{\theta_{\perp}}{m}}\frac{\partial a_{lij}^{(3)}}{\partial x_l} + \sqrt{\frac{\theta_{\parallel}}{m}}\left(1-\sqrt{\frac{\theta_{\perp}}{\theta_{\parallel}}}\right)b_l b_m\frac{\partial a_{lij}^{(3)}}{\partial x_m} + \sqrt{\frac{\theta_{\perp}}{m}}\,a_{lij}^{(3)}\frac{\partial \ln n}{\partial x_l} +$$

$$+\sqrt{\frac{\theta_{\parallel}}{m}}\left(1-\sqrt{\frac{\theta_{\perp}}{\theta_{\parallel}}}\right)b_l b_m\frac{\partial \ln n}{\partial x_m}a_{lij}^{(3)} + \frac{1}{2\sqrt{m\theta_{\perp}}}\frac{\partial\theta_{\perp}}{\partial x_k}a_{kij}^{(3)} +$$

$$
\begin{aligned}
&+ \frac{1}{2}\left(\frac{1}{\sqrt{m\theta_\parallel}} \frac{\partial\theta_\parallel}{\partial x_k} - \frac{1}{\sqrt{m\theta_\perp}} \frac{\partial\theta_\perp}{\partial x_k}\right) b_k b_l a^{(3)}_{lij} \; + \\
&+ \frac{1}{2\sqrt{m\theta_\parallel}}\left(1 - \sqrt{\frac{\theta_\perp}{\theta_\parallel}}\right)\frac{\partial\theta_\parallel}{\partial x_m} b_m b_n b_l \left(b_i a^{(3)}_{lnj} + b_j a^{(3)}_{lni}\right) + \\
&+ \frac{1}{2\theta_\parallel}\sqrt{\frac{\theta_\perp}{m}}\frac{\partial\theta_\parallel}{\partial x_m} b_l\left(b_i a^{(3)}_{lnj} + b_j a^{(3)}_{lni}\right) + \\
&+ \frac{1}{2\theta_\perp}\sqrt{\frac{\theta_\parallel}{m}}\left(1 - \sqrt{\frac{\theta_\perp}{\theta_\parallel}}\right)\frac{\partial\theta_\perp}{\partial x_m} b_m b_n \left(2 a^{(3)}_{inj} - b_l b_i a^{(3)}_{lnj} - b_l b_j a^{(3)}_{lni}\right) + \\
&+ \frac{1}{2\sqrt{m\theta_\perp}}\frac{\partial\theta_\perp}{\partial x_n}\left(2 a^{(3)}_{ijn} - b_i b_l a^{(3)}_{lnj} - b_j b_l a^{(3)}_{lni}\right) - \\
&- 2\sqrt{\frac{\theta_\parallel}{\theta_\perp}}\left(1 - \sqrt{\frac{\theta_\perp}{\theta_\parallel}}\right)^2 b_i b_j b_k b_m \frac{\partial u_k}{\partial x_m} + \left(\sqrt{\frac{\theta_\perp}{\theta_\parallel}} - 1\right) b_k \left(b_i \frac{\partial u_k}{\partial x_j} + b_j \frac{\partial u_k}{\partial x_i}\right) + \\
&+ \sqrt{\frac{\theta_\parallel}{\theta_\perp}}\left(1 - \sqrt{\frac{\theta_\perp}{\theta_\parallel}}\right) b_m \left(\frac{\partial u_i}{\partial x_m} b_j + \frac{\partial u_j}{\partial x_m} b_i\right) + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \; + \\
&+ \frac{1}{\theta_\perp}\frac{d\theta_\perp}{dt}\left(\delta_{ij} - b_i b_j\right) + \frac{1}{\theta_\parallel}\frac{d\theta_\parallel}{dt} b_i b_j - \sqrt{\frac{\theta_\parallel}{\theta_\perp}}\left(\frac{\theta_\perp}{\theta_\parallel} - 1\right)\frac{d}{dt} b_i b_j = \\
&= \Omega\left(\epsilon_{ikl} a^{(2)}_{kj} + \epsilon_{jkl} a^{(2)}_{ki}\right) b_l \; . \qquad (19)
\end{aligned}
$$

The equation for $a^{(3)}_{ijk}$ is much more complicated, and we do not write it explicitly. To have a full set of equations we need the equation for $\vec{b}$. To obtain that it is necessary to use the Maxwell induction equation

$$
\mathrm{rot}\,(\vec{E}_\perp + \vec{E}_\parallel) + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \; . \qquad (20)
$$

If $\frac{\omega}{\Omega} \sim \frac{a}{L} \sim \epsilon \ll 1$, then $E_\parallel \sim \epsilon E_\perp$ [8].

So the first and the third term in (20) are zero-order quantities, and the second term is proportional to ϵ. Therefore we obtain in the lowest-order approximation

$$- \frac{B}{c} \frac{\partial \vec{b}}{\partial t} = (\hat{I} - \vec{b}\vec{b}) \cdot \text{rot}\, \vec{E_\perp} \quad , \tag{21}$$

where $\hat{I}$ is the unit tensor; $\vec{b}\vec{b}$ is the dyadic (tensor); $\vec{b} = \vec{B}/B$.

If we introduce the electric drift velocity

$$\vec{V} = c \frac{[\vec{E}_\perp \vec{b}]}{B} \quad , \tag{22}$$

then

$$\text{rot}\ \vec{E}_\perp = \frac{1}{c} \left\{ \vec{B}\, \text{div}\, \vec{V} + \vec{V} \cdot \nabla \vec{B} - \vec{B} \cdot \nabla \vec{V} \right\} . \tag{23}$$

Thus we obtain

$$\frac{\partial b_i}{\partial t} = - V_k \frac{\partial b_i}{\partial x_k} + b_k \frac{\partial V_i}{\partial x_k} - b_i b_e b_k \frac{\partial V_e}{\partial x_k} \quad , \tag{24}$$

where it is taken into account that $b_e \frac{\partial b_e}{\partial x_k} = 0$ because $\vec{b}^2 = 1$.

2. Now it is possible to find explicit expressions for the viscous stress tensor $\hat{\sigma}$ and the heat flux tensor $\hat{M}$. Let us consider σ_{ij}, M_{ijk} as being proportional to ϵ. Then in the left-hand side of equations for $\hat{\sigma}$ and $\hat{M}$ it is sufficient to use only the zero-order terms (see eq. (19)). Therefore, by using equations (17), (18) and (24) one can easily obtain

$$\Big[1 - 2\Big(\sqrt{\frac{p_\parallel}{p_\perp}} + \sqrt{\frac{p_\perp}{p_\parallel}}\Big) \Big] b_i b_j b_k b_l \frac{\partial u_k}{\partial x_l} + \Big(\sqrt{\frac{p_\perp}{p_\parallel}} - 1\Big) b_k \Big(b_i \frac{\partial u_k}{\partial x_j} + b_j \frac{\partial u_k}{\partial x_i}\Big) +$$

$$+ \sqrt{\frac{p_\parallel}{p_\perp}} \Big(1 - \sqrt{\frac{p_\perp}{p_\parallel}}\Big) b_k \Big(\frac{\partial u_i}{\partial x_k} b_j + \frac{\partial u_j}{\partial x_k} b_i \Big) + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \delta_{ij} b_k b_l \frac{\partial u_k}{\partial x_l} -$$

$$- (\delta_{ij} - b_i b_j)\text{div}\, \vec{u} + \sqrt{\frac{p_\parallel}{p_\perp}} \Big(\frac{p_\perp}{p_\parallel} - 1\Big) \Big[V_k \frac{\partial b_i b_j}{\partial x_k} - b_k \Big(b_j \frac{\partial V_i}{\partial x_k} + b_i \frac{\partial V_j}{\partial x_k}\Big) + 2 b_i b_j b_k b_l \frac{\partial V_l}{\partial x_k} \Big] =$$

$$= \Omega\, (\epsilon_{ikl}\, a^{(2)}_{kj} + \epsilon_{jkl}\, a^{(2)}_{ki}) b_l . \tag{25}$$

This equation defines the components of the tensor σ because of (10):

$$a^{(2)}_{ij} = \frac{1}{p_\perp}\left\{\sigma_{ij} + \left(\sqrt{\frac{p_\perp}{p_\parallel}} - 1\right)\left(b_i\,\sigma_{jk} + b_j\,\sigma_{ik}\right) b_k\right\} .$$

Further, in the approximation considered it is possible to assume $\vec{u} \sim \vec{V}$.

As follows from (25), in the local co-ordinate system with the z-axis along the magnetic field lines, the components of the σ tensor are

$$\begin{aligned}
\sigma_{xx} &= -\sigma_{yy} = -\frac{p_\perp}{2\Omega}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right),\\
\sigma_{xy} &= \sigma_{yx} = \frac{p_\perp}{2\Omega}\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right),\\
\sigma_{xz} &= \sigma_{zx} = -\frac{1}{\Omega}\left\{p_\perp \frac{\partial u_z}{\partial y} + (2p_\parallel - p_\perp)\frac{\partial u_y}{\partial z}\right\},\\
\sigma_{yz} &= \sigma_{zy} = \frac{1}{\Omega}\left\{p_\perp \frac{\partial u_z}{\partial x} + (2p_\parallel - p_\perp)\frac{\partial u_x}{\partial z}\right\},\\
\sigma_{zz} &= 0 .
\end{aligned} \tag{26}$$

This result is exactly the same as Macmahon's[8), 9)]. A. Simon and W. B. Thompson state in their work[15)] that the components σ_{xx}, σ_{yy}, σ_{xy} are correct, but Simon and Thompson disagree with Macmahon's results for σ_{xz} and σ_{yz}, and they give their formulae for these components as follows:

$$\begin{aligned}
\sigma_{xz} &= -\frac{1}{\Omega}\left\{p_\parallel \frac{\partial u_y}{\partial z} + p_\perp\,\vec{b}\cdot\frac{\partial \vec{u}}{\partial y} + (p_\parallel - p_\perp)\,\vec{i}_y\cdot\frac{d\vec{b}}{dt}\right\} ,\\
\sigma_{yz} &= \frac{1}{\Omega}\left\{p_\parallel \frac{\partial u_x}{\partial z} + p_\perp\,\vec{b}\cdot\frac{\partial \vec{u}}{\partial x} + (p_\parallel - p_\perp)\,\vec{i}_x\cdot\frac{d\vec{b}}{dt}\right\} .
\end{aligned} \tag{27}$$

At first sight these formulae seem to differ from (26). But if one reduces formulae (27) by using equation (24), one can easily convince oneself that expressions (26) and (27) coincide exactly (in the given local co-ordinate system). So, actually, there is not any difference between the results of

Macmahon and those of Simon and Thompson. We note that the calculated viscous stress tensor σ automatically gives an expression for the coefficient of the "magnetic viscosity" $\nu = \frac{p}{2\rho\Omega} = \frac{1}{4}\Omega a^2$, where a is the Larmor radius of charged particles. It can easily be shown that, unlike the usual "collision" viscosity[9], the "magnetic viscosity" is not connected with a dissipation of energy.

To find the components of the tensor M_{ijk} one can obtain the first-order equation for $a^{(3)}_{ijk}$ from (13):

$$\frac{3}{\sqrt{m\theta_\parallel}}\left(1-\sqrt{\frac{\theta_\perp}{\theta_\parallel}}\right)\frac{\partial\theta_\perp}{\partial x_l}b_ib_jb_kb_l+\frac{1}{\theta_\perp}\sqrt{\frac{\theta_\parallel}{m}}\left(1-\sqrt{\frac{\theta_\perp}{\theta_\parallel}}\right)\frac{\partial\theta_\perp}{\partial x_m}b_m(b_i\delta_{jk}+b_j\delta_{ik}+b_k\delta_{ij}-3b_ib_jb_k)+$$

$$+\sqrt{\frac{\theta_\perp}{m}}\left(1-\sqrt{\frac{\theta_\perp}{\theta_\parallel}}\right)^2 b_m\left\{2b_ib_j\frac{\partial b_k}{\partial x_m}+2b_ib_k\frac{\partial b_j}{\partial x_m}+2b_jb_k\frac{\partial b_i}{\partial x_m}\right.+$$

$$\left.+\sqrt{\frac{\theta_\parallel}{\theta_\perp}}\left(b_i\frac{\partial b_jb_k}{\partial x_m}+b_j\frac{\partial b_ib_k}{\partial x_m}+b_k\frac{\partial b_ib_j}{\partial x_m}\right)\right\}+$$

$$+\sqrt{\frac{\theta_\parallel}{m}}\left(1-\frac{\theta_\perp}{\theta_\parallel}\right)\left\{b_i\left(\frac{\partial b_j}{\partial x_k}+\frac{\partial b_k}{\partial x_j}\right)+b_j\left(\frac{\partial b_i}{\partial x_k}+\frac{\partial b_k}{\partial x_i}\right)+b_k\left(\frac{\partial b_i}{\partial x_j}+\frac{\partial b_j}{\partial x_i}\right)\right\}+$$

$$+\frac{1}{\theta_\parallel}\sqrt{\frac{\theta_\perp}{m}}\left(b_ib_k\frac{\partial\theta_\parallel}{\partial x_j}+b_jb_k\frac{\partial\theta_\parallel}{\partial x_i}+b_ib_j\frac{\partial\theta_\parallel}{\partial x_k}\right)+$$

$$+\frac{1}{\sqrt{m\theta_\perp}}\left[\frac{\partial\theta_\perp}{\partial x_i}(\delta_{jk}-b_jb_k)+\frac{\partial\theta_\perp}{\partial x_j}(\delta_{ik}-b_ib_k)+\frac{\partial\theta_\perp}{\partial x_k}(\delta_{ij}-b_ib_j)\right]=$$

$$=\Omega b_l(\epsilon_{iml}a^{(3)}_{mjk}+\epsilon_{jml}a^{(3)}_{mik}+\epsilon_{kml}a^{(3)}_{mij})-$$

$$-\Omega\epsilon_{nrl}b_l\left(1-\sqrt{\frac{\theta_\perp}{\theta_\parallel}}\right)\left\{\sqrt{\frac{\theta_\parallel}{\theta_\perp}}\left(1-\sqrt{\frac{\theta_\perp}{\theta_\parallel}}\right)b_nb_rb_m(b_ia^{(3)}_{mjk}+b_ja^{(3)}_{mik}+b_ka^{(3)}_{mij})\right.+$$

$$\left.+b_n(b_ia^{(3)}_{rjk}+b_ja^{(3)}_{rik}+b_ka^{(3)}_{rij})-\sqrt{\frac{\theta_\parallel}{\theta_\perp}}b_mb_r(\delta_{in}a^{(3)}_{mjk}+\delta_{jn}a^{(3)}_{mik}+\delta_{kn}a^{(3)}_{mij})\right\}$$

(28)

In the local co-ordinate system with the z-axis along the magnetic field lines the tensor M_{ijk} has the components, *as follows from* (28) and (10):

$$
\begin{aligned}
M_{111} &= -\frac{3p_\perp}{m\Omega}\frac{\partial\theta_\perp}{\partial y} \quad ; \qquad M_{122} = -\frac{p_\perp}{m\Omega}\frac{\partial\theta_\perp}{\partial y} \quad . \\
M_{211} &= \frac{p_\perp}{m\Omega}\frac{\partial\theta_\perp}{\partial x} \quad ; \qquad M_{222} = \frac{3p_\perp}{m}\frac{\partial\theta_\perp}{\partial x} \quad . \\
M_{133} &= -\frac{p_\perp}{m\Omega}\frac{\partial\theta_\parallel}{\partial y} \quad ; \qquad M_{233} = \frac{p_\perp}{m\Omega}\frac{\partial\theta_\parallel}{\partial x} \quad . \\
M_{123} &= 0 \quad ; \qquad \frac{\partial\theta_\parallel}{\partial z} = \frac{\partial\theta_\perp}{\partial z} = 0 \quad .
\end{aligned}
\tag{29}
$$

The components $M_{311} = M_{322}$ and M_{333} remain undetermined (as in Macmahon's work[8]). But these components define a longitudinal heat flux vector

$$
\vec{q}_\parallel \equiv \vec{b}(\vec{b}\cdot\vec{q}) = \frac{1}{2}\left\{0,\ 0,\ M_{311} + M_{322} + M_{333}\right\} .
$$

It is naturally assumed that the vector $\vec{q}_\parallel$ is determined by the gradients $\frac{\partial\theta_\perp}{\partial z}$, $\frac{\partial\theta_\parallel}{\partial z}$. However, in accordance with (29), $\frac{\partial\theta_\perp}{\partial z} = \frac{\partial\theta_\parallel}{\partial z} = 0$. Thus we see that in the frame considered longitudinal heat fluxes must be absent: $q_\parallel = 0$. This means that the 13 moments theory can be used in the anisotropic case also (in the approximation considered).

Now let us introduce two vectors (in dyadic notation):

$\vec{q}^{\parallel} = \frac{1}{2}\vec{M} : \vec{b}\,\vec{b}$ - "longitudinal" heat flux, and

$\vec{q}^{\perp} = \frac{1}{2}\vec{M} : (I - \vec{b}\,\vec{b})$ - "transverse" heat flux.

Their transverse parts have the components

$$
\vec{q}^{\perp}_{\perp} = \frac{1}{2}\left\{M_{111} + M_{122}\ ;\quad M_{211} + M_{222}\ ;\quad 0\right\} ,
$$

$$
\vec{q}^{\parallel}_{\perp} = \frac{1}{2}\left\{M_{133}\ ;\quad M_{233}\ ;\quad 0\right\} .
$$

By means of formulae (29) we obtain

$$\vec{q}_{\perp}^{\perp} = \frac{2p_{\perp}}{m\Omega}\left[\vec{b}\cdot\nabla\Theta_{\perp}\right],$$

$$\vec{q}_{\perp}^{\,\parallel} = \frac{p_{\perp}}{2m\Omega}\left[\vec{b}\cdot\nabla\Theta_{\parallel}\right]. \qquad (31)$$

Exactly the same results follow from the general formulae of Macmahon[8] in the case where the distribution function is a product: $f(\vec{c}) = f_1(c_\perp)\, f_2(c_\parallel)$.

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Appendix

We want here to give a more detailed and general proof that the results of Macmahon and those of Simon - Thompson are the same.

According to Macmahon[8] the "magnetic viscous" stress tensor of a collisionless plasma has the following components (in dyadic notation):

$$\sigma_{22} = -\sigma_{33} = -\frac{p_\perp}{2\Omega}(\vec{e}_2\vec{e}_3 + \vec{e}_3\vec{e}_2) : (\nabla\vec{u}) ,$$

$$\sigma_{23} = \sigma_{32} = \frac{p_\perp}{2\Omega}(\vec{e}_2\vec{e}_2 - \vec{e}_3\vec{e}_3) : (\nabla\vec{u}) ,$$

$$\sigma_{12} = \sigma_{21} = -\frac{1}{\Omega}\{ p_\perp(\vec{e}_1\vec{e}_3) : (\nabla\vec{u}) + (2p_\parallel - p_\perp)(\vec{e}_3\vec{e}_1) : (\nabla\vec{u})\} ,$$

$$\sigma_{13} = \sigma_{31} = \frac{1}{\Omega}\{ p_\perp(\vec{e}_1\vec{e}_2) : (\nabla\vec{u}) + (2p_\parallel - p_\perp)(\vec{e}_2\vec{e}_1) : (\nabla\vec{u})\} . \qquad (A.1)$$

Here $\vec{e}_1 \equiv \vec{b}$, $\vec{e}_2$, $\vec{e}_3$ are unit vectors forming a right-handed orthogonal system, $(\vec{a}\vec{b}) : (\nabla\vec{u}) \equiv a_i b_j \frac{\partial}{\partial x_j} u_i$.

In the local co-ordinate system with $\vec{b} = (0,0,1)$, $\vec{e}_2 = (1,0,0)$ and $\vec{e}_3 = (0,1,0)$ eqs. (A.1) are reduced to eq. (26).

From eq. (19) one obtains in first-order approximation

$$-2\sqrt{\frac{p_\parallel}{p_\perp}}\left(1-\sqrt{\frac{p_\perp}{p_\parallel}}\right)^2 b_i b_j(\vec{b}\vec{b}) : (\nabla\vec{u}) + \left(\sqrt{\frac{p_\perp}{p_\parallel}} - 1\right) b_k \left(b_i \frac{\partial u_k}{\partial x_j} + b_j \frac{\partial u_k}{\partial x_i}\right) +$$

$$+\sqrt{\frac{p_\parallel}{p_\perp}}\left(1-\sqrt{\frac{p_\perp}{p_\parallel}}\right)(b_j \vec{b}\cdot\nabla u_i + b_i \vec{b}\cdot\nabla u_j) + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \qquad (A.2)$$

$$+\frac{1}{\theta_\perp}\frac{d\theta_\perp}{dt}(\delta_{ij} - b_i b_j) + \frac{1}{\theta_\parallel}\frac{d\theta_\parallel}{dt} b_i b_j - \sqrt{\frac{p_\parallel}{p_\perp}}\left(\frac{p_\perp}{p_\parallel} - 1\right)\frac{d}{dt} b_i b_j =$$

$$= \Omega(\epsilon_{ikl} a^{(2)}_{kj} + \epsilon_{jkl} a^{(2)}_{ki}) b_l .$$

By using eqs. (17) and (18) in zero-order approximation and the definition (10) one gets

$$\left[1-2\left(\sqrt{\frac{p_\parallel}{p_\perp}}+\sqrt{\frac{p_\perp}{p_\parallel}}\right)\right]b_i b_j(\vec b\,\vec b):(\nabla\vec u)+\left(\sqrt{\frac{p_\perp}{p_\parallel}}-1\right)b_k\left(b_i\frac{\partial u_k}{\partial x_j}+b_j\frac{\partial u_k}{\partial x_i}\right)+$$

$$+\left(\sqrt{\frac{p_\parallel}{p_\perp}}-1\right)\left[b_j(\vec b\cdot\nabla)u_i+b_i(\vec b\cdot\nabla)u_j\right]+\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}+\delta_{ij}(\vec b\,\vec b):(\nabla\vec u)\;-$$

$$-(\delta_{ij}-b_i b_j)\nabla\cdot\vec u-\frac{p_\perp-p_\parallel}{\sqrt{p_\perp p_\parallel}}\frac{d}{dt}b_i b_j= \qquad \text{(A. 3)}$$

$$=\frac{\Omega}{p_\perp}\left\{(\epsilon_{ikl}\sigma_{kj}+\epsilon_{jkl}\sigma_{ki})b_l+\left(\sqrt{\frac{p_\perp}{p_\parallel}}-1\right)(\epsilon_{ikl}b_j+\epsilon_{jkl}b_i)b_l b_m\sigma_{mk}\right\}.$$

These are the equations from which the components of the tensor $\hat\sigma$ must be determined.

By calculating the scalar product of (A. 3) and the dyadic $\vec e_2\vec e_2$ we find

$$2(\vec e_2\vec e_2):(\nabla\vec u)+(\vec b\vec b):(\nabla\vec u)-\nabla\cdot\vec u=$$

$$=\frac{\Omega}{p_\perp}(e_{2i}e_{2j}\epsilon_{ikl}b_l\sigma_{kj}+e_{2i}e_{2j}\epsilon_{jkl}b_l\sigma_{ki})=$$

$$=\frac{\Omega}{p_\perp}(e_{3k}e_{2j}\sigma_{kj}+e_{3k}e_{2i}\sigma_{ki})\equiv\frac{2\Omega}{p}\vec e_3\vec e_2:\hat\sigma\equiv\frac{2\Omega}{p_\perp}\sigma_{32}\;,$$

where we have taken into account the relations

$$\vec e_2\cdot\vec b=0,\qquad \vec e_2^{\,2}=1,\qquad \epsilon_{ikl}e_{2i}b_l\equiv-[\vec e_2\times\vec b]_k=e_{3k}\;.$$

As $\vec e_2\vec e_2+\vec e_3\vec e_3+\vec b\vec b=\hat 1$, we finally obtain

$$\sigma_{32}=\frac{p_\perp}{2\Omega}(\vec e_2\vec e_2-\vec e_3\vec e_3):(\nabla\vec u)\;. \qquad \text{(A. 4)}$$

In the same way, by multiplication by

$\vec e_2\vec e_3$, $\quad \vec e_2\vec e_1\equiv\vec e_2\vec b \quad$ and $\quad \vec e_3\vec e_1\equiv\vec e_3\vec b$, we get

$$\sigma_{33}-\sigma_{22}=\frac{p_\perp}{\Omega}(\vec e_2\vec e_3+\vec e_3\vec e_2):(\nabla\vec u)\;, \qquad \text{(A. 5)}$$

$$\sigma_{13} = \frac{1}{\Omega} \left\{ p_\perp (\vec{b}\vec{e}_2) : (\nabla\vec{u}) + p_\parallel (\vec{e}_2\vec{b}) : (\nabla\vec{u}) + (p_\parallel - p_\perp) \vec{e}_2 \cdot \frac{d\vec{b}}{dt} \right\} , \quad (A.6)$$

$$\sigma_{12} = - \frac{1}{\Omega} \left\{ p_\perp (\vec{b}\vec{e}_3) : (\nabla\vec{u}) + p_\parallel (\vec{e}_3\vec{b}) : (\nabla\vec{u}) + (p_\parallel - p_\perp) \vec{e}_3 \cdot \frac{d\vec{b}}{dt} \right\} . \quad (A.7)$$

Here $\sigma_{\alpha\beta} = \vec{e}_\alpha \vec{e}_\beta : \hat{\sigma}$ by definition; $\alpha, \beta = 1, 2, 3$.
As $Tr\ \hat{\sigma} = 0$, and $\vec{b}\vec{b} : \hat{\sigma} \equiv \sigma_{11} = 0$, we *obtain from (A.5)*

$$\sigma_{33} = - \sigma_{22} = \frac{p_\perp}{2\Omega} (\vec{e}_2\vec{e}_3 + \vec{e}_3\vec{e}_2) : (\nabla\vec{u}) . \quad (A.8)$$

Thus the components σ_{22}, σ_{32} in (A.4), (A.8) are exactly the same as those in Macmahon's result (A.1). The components σ_{13}, σ_{12} in (A.6), (A.7) are identical with those of Simon - Thompson[15)] in the local co-ordinate system. By substituting Maxwell's induction equation (24) in (A.6) we obtain

$$\sigma_{13} = \frac{1}{\Omega} \left\{ p_\perp (\vec{b}\vec{e}_2) : (\nabla\vec{u}) + p_\parallel (\vec{e}_2\vec{b}) : (\nabla\vec{u}) + (p_\parallel - p_\perp) (\vec{e}_2\vec{b}) : (\nabla\vec{V}) \right\} . \quad (A.9)$$

As in the zero-order approximation the electric drift velocity $\vec{V} \sim \vec{u}$, we see that (A.9) does not differ from Macmahon's result (A.1).