Technical University of Denmark

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Publication date:
1973

Document Version
Publisher's PDF, also known as Version of record

[redacted]

Risø Report No. 299

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October 1973

Sales distributors: Jul. Gjellerup, 87, Sølvgade, DK-1307 Copenhagen K, Denmark

Available on exchange from: Library, Danish Atomic Energy Commission, Risø, DK-4000 Roskilde, Denmark

UDC 533.9

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Abstract

Exact analytical solutions were obtained to the linearized, two-species Maxwell-Vlasov equations for the evolution of an initial density perturbation in an initially stable, neutral plasma. Two initial perturbations are considered: a step-like and a pulse-like discontinuity.

ISBN 87 550 0238 2

CONTENTS

1. INTRODUCTION

We assume that the plasma dynamics are governed by the linearized Vlasov equations for both electrons and singly charged ions coupled through Poisson's equation. The equations are solved with two different initial conditions:

I. $f_{i,e}(x,v,t=0) = n_o f_{oi,e}(v) + \Delta n(1-\epsilon(x))g_{i,e}(v)$

$\epsilon(x)$ is Heaviside's step function

II. $f_{i,e}(x,v,t=0) = n_o f_{oi,e}(v) + \Delta n g_{i,e}(v)\delta(x)$.

$f_{oi,e}(v)$ and $g_{i,e}(v)$ are assumed to be normalized Maxwellians. In our calculations we allow different drift velocities, but we neglect the B-field due to the resulting current. This approximation is necessary since we consider the problem in one dimension.

The equations are solved analytically for initial condition I using Fourier transformation in space and Laplace transformation in time. The calculations are very similar to those of Mason in ref. 1. Analytical expressions for the densities, $n_{1\,i,e}$, the distributions, $f_{1\,i,e}$, the fluxes, $F_{1\,i,e}$, the electric field, E_1, and the potential, φ_1, are obtained. The solutions to the equations with initial condition II can be found simply by differentiating the expressions above with respect to x, since initial condition II is found by differentiation of initial condition I. The solutions of the equations with an arbitrary initial perturbation

$$f_{i,e}(x,v,t=0) = n_o f_{oi,e} + \Delta n g_{i,e}(v)F(x)$$

can be found by superposition of these solutions by using

$$f_F(x,v,t) = \int_{-\infty}^{\infty} F(\gamma)\, f_\delta(x-\gamma,v,t)d\gamma \ .$$

Assuming Boltzmann distributed electrons at all times and quasi-neutrality ($n_i \simeq n_e$) we can simplify our solutions considerably. These simplified calculations are found in chapter 4.

2. ANALYTIC SOLUTIONS WITH A STEP-LIKE INITIAL CONDITION

We use the linearized Vlasov equations for both electrons and ions coupled through Poisson's equation:

$$\frac{\partial f_{1i,e}}{\partial t} + v\frac{\partial f_{1i,e}}{\partial x} = \pm \frac{e n_o}{m_{i,e}} \cdot \frac{\partial \varphi_1}{\partial x} \cdot \frac{\partial f_{oi,e}}{\partial v} \tag{1}$$

$$\frac{\partial^2 \varphi_1}{\partial x^2} = -\frac{e}{\varepsilon_o}\left(\int_{-\infty}^{\infty} f_{1i}dv - \int_{-\infty}^{\infty} f_{1e}dv\right) \equiv -\frac{e}{\varepsilon_o}(n_{1i} - n_{1e}) \tag{2}$$

where

$$\int_{-\infty}^{\infty} f_{oi,e}dv = 1.$$

These equations are to be solved with the initial condition

$$f_{1i,e}(x, v, t=0) = \Delta n(1 - \epsilon(x))g_{i,e}(v)$$

where

$\int_{-\infty}^{\infty} g_{i,e}dv = 1$ and $\Delta n \ll n_o$. $\epsilon(x)$ is Heaviside's step function.

2.1. The density, $n_{i,e}(x,t)$

We use Fourier transformation in space and Laplace transformation in time omitting for convenience the indes "1" on the perturbed quantities and get

$$s\tilde{\tilde{f}}_{i,e} - \tilde{f}_{i,e}(k, v, t=0) + ikv\tilde{\tilde{f}}_{i,e} = \pm \frac{ien_ok}{m_{i,e}} \tilde{\tilde{\varphi}} f'_{oi,e}(v) \tag{3}$$

$$k^2 \tilde{\tilde{\varphi}} = \frac{e}{\varepsilon_o}(\tilde{\tilde{n}}_i - \tilde{\tilde{n}}_e). \tag{4}$$

$\tilde{f}_{i,e}(k, v, t=0)$ must be taken as an Abel's limit, i.e.

$$\tilde{f}_{i,e}(k, v, t=0) = \lim_{\alpha\to 0} \int_{-\infty}^{0} \Delta n\, g_{i,e}(v)\, e^{-ikx} e^{-\alpha|x|}dx = \frac{i\Delta n}{k} g_{i,e}(v). \tag{5}$$

Inserting (4) in (3) we get

$$\tilde{\tilde{f}}_{i,e} = \frac{\tilde{f}_{i,e}(k, v, t=0)}{s + ikv} \pm \frac{i\omega^2_{pi,e}}{k} \frac{\tilde{\tilde{n}}_i - \tilde{\tilde{n}}_e}{s + ikv} f'_{oi,e} \tag{6}$$

where

$$\omega^2_{pi,e} = \frac{n_oe^2}{\varepsilon_o m_{i,e}}.$$

Integrating with respect to v we find

$$\tilde{\tilde{n}}_{i,e}(k, s) = S_{i,e}(k, s) \pm (\tilde{\tilde{n}}_i - \tilde{\tilde{n}}_e)\varepsilon_{i,e}(k, s) \tag{7}$$

where

$$S_{i,e}(k, s) = -\frac{i}{k}\int_{-\infty}^{\infty} \frac{\tilde{f}_{i,e}(k, v, t=0)}{v - i\frac{s}{k}} = \frac{\Delta n}{k^2}\int_{-\infty}^{\infty} \frac{g_{i,e}(v)}{v - i\frac{s}{k}}dv. \tag{8a}$$

$$\varepsilon_{i,e}(k, s) = \frac{\omega^2_{pi,e}}{k^2} \int_{-\infty}^{\infty} \frac{f'_{oi,e}}{v - i\frac{s}{k}} dv. \tag{8b}$$

This notation follows closely that of Mason[1].
Equation 7 is solved with respect to $\tilde{\tilde{n}}_i$ and $\tilde{\tilde{n}}_e$.

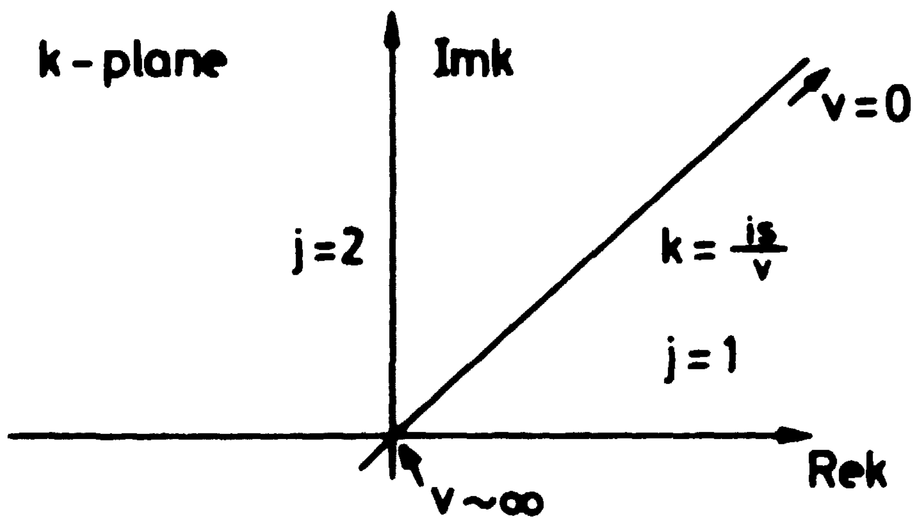
$$\tilde{\tilde{n}}_{i,e}(k, s) = S_{i,e}(k, s) \pm \frac{M_{i,e}(k, s)}{D(k, s)} \tag{9}$$

where

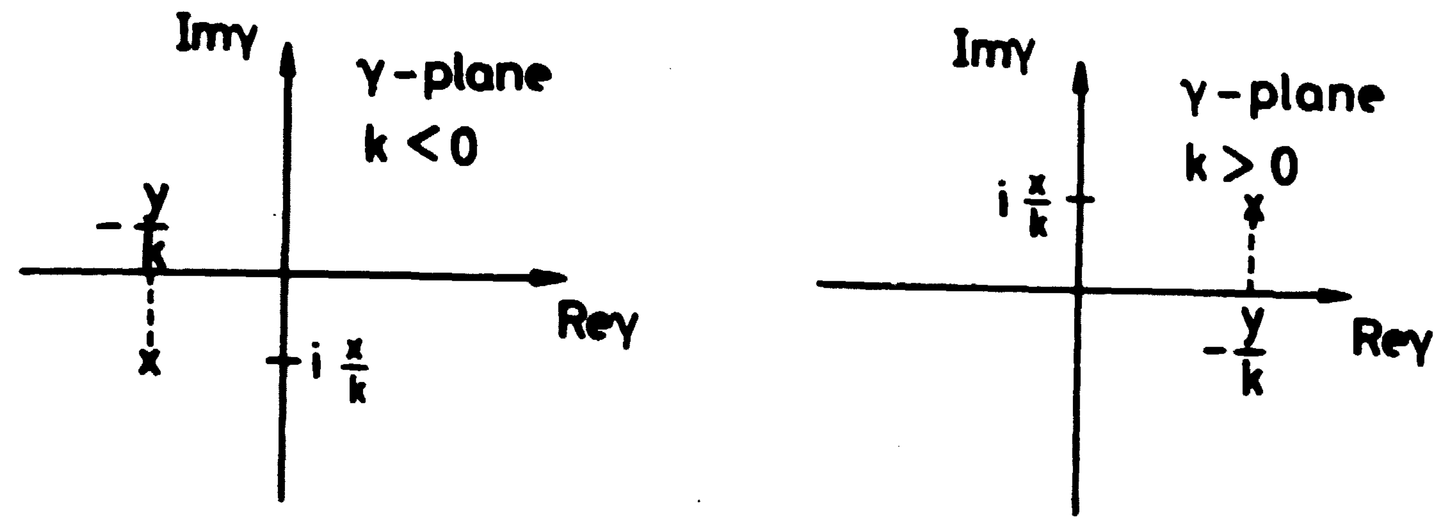
$$M_{i,e}(k, s) = \varepsilon_{i,e}(k, s)\,(S_i(k, s) - S_e(k, s))$$

$$D(k, s) = 1 - \varepsilon_i(k, s) - \varepsilon_e(k, s).$$

D(k, s) is the dielectric function for the plasma, and $S_{i,e}(k, s)$ are source functions derived from the initial conditions. If the particles are assumed to be chargeless or $f_i(x, v, t=0) = f_e(x, v, t=0)$, the density depends only on the source functions corresponding to freely streaming particles. In order to perform the inverse transformations of (9) we follow the procedure of Mason[1]. We temporarily consider s fixed with Re(s) > 0. The functions S(k, s), ε (k, s), M(k, s), and D(k, s) are analytic in the full complex k-plane except for a branch cut along the line $k = i\frac{s}{v}$, v real. The functional branches on each side of this line are denoted S_j, ε_j, M_j, and D_j with j = 1 (or 2) for k in the half-plane corresponding to Im (v) > 0 (or < 0). For Re(s) > 0 this is equivalent to j=1 (2) for k right (left) of the cut. (See fig. 1).



Then $\tilde{\tilde{n}}_{i,e}(k, s)$ are analytic in the full k-plane except at the branch cut and $D_j(k, s)$ zeroes in the respective half-planes. The denominator in the integrands at $S_{i,e}$, $M_{i,e}$, and D have zeroes for $\gamma = i\frac{s}{k} \equiv -\frac{y}{k} + i\frac{x}{k}$ ($s \equiv x + iy$, $x > 0$) as shown in fig. 2 for the case where y < 0.



Performing the Fourier inversion of (9) we get

$$\tilde{n}_{i,e}(x, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\tilde{n}}_{i,e}\, e^{ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{0} \tilde{\tilde{n}}_{i,e,2}\, e^{ikx} dk + \frac{1}{2\pi}\int_{0}^{\infty} \tilde{\tilde{n}}_{i,e,1}\, e^{ikx} dk \tag{10}$$

where the index "1" ("2") indicates an integral contour below (above) the pole. See fig. 2.

We assume $f_{oi,e}(v)$ and $g_{i,e}(v)$ to be drifting Maxwellians:

$$f_{oi,e}(v) = \frac{1}{\sqrt{2\pi}\, A_{oi,e}} \exp\left(-\left(\frac{v - v_{oi,e}}{\sqrt{2}\, A_{oi,e}}\right)^2\right)$$

$$g_{i,e}(v) = \frac{1}{\sqrt{2\pi}\, A_{i,e}} \exp\left(-\left(\frac{v - v_{i,e}}{\sqrt{2}\, A_{i,e}}\right)^2\right) \tag{11}$$

where

$$A^2_{oi,e} = \frac{\kappa T_{oi,e}}{m_{i,e}} \quad \text{and} \quad A^2_{i,e} = \frac{\kappa T_{i,e}}{m_{i,e}} .$$

Introducing the plasma dispersion function[2)] we rewrite (8):

$$S_{i,e,1}(k,s) = \frac{\Delta n}{\sqrt{2}\,A_{i,e}k^2}\; Z\left(\frac{i\frac{s}{k} - v_{i,e}}{\sqrt{2}\,A_{i,e}}\right) \tag{12}$$

$$\varepsilon_{i,e,1}(k,s) = \frac{1}{\sqrt{2}(k\,d_{i,e})^2}\; Z'\left(\frac{i\frac{s}{k} - v_{oi,e}}{\sqrt{2}\,A_{oi,e}}\right) \tag{13}$$

where

$$d_{i,e} = \left(\frac{\varepsilon_o \varkappa T_{oi,e}}{n_o e^2}\right)^{1/2} = \frac{A_{oi,e}}{\omega_{pi,e}} \quad .$$

The function $M_{i,e}(k,s)$ introduced in (9) is then given by:

$$M_{i,e,1}(k,s) = \frac{\Delta n}{2d_i^2 k^4}\; P_{i,e,1}(k,s) \tag{14}$$

where

$$P_{i,e,1} = Z'\left(\frac{i\frac{s}{k} - v_{oi,e}}{\sqrt{2}\,A_{oi,e}}\right)\left[\frac{1}{\sqrt{2}\,A_i}\; Z\left(\frac{i\frac{s}{k} - v_i}{\sqrt{2}\,A_i}\right) - \frac{1}{\sqrt{2}\,A_e}\; Z\left(\frac{i\frac{s}{k} - v_e}{\sqrt{2}\,A_e}\right)\right]$$

and the function D(k, s) is given by:

$$D_1(k,s) = 1 - \frac{1}{2(kd_i)^2}\left[Z'\left(\frac{i\frac{s}{k} - v_{oi}}{\sqrt{2}\,A_{oi}}\right) + \frac{1}{\theta}\, Z'\left(\frac{i\frac{s}{k} - v_{oe}}{\sqrt{2}\,A_{oe}}\right)\right] \tag{15}$$

where

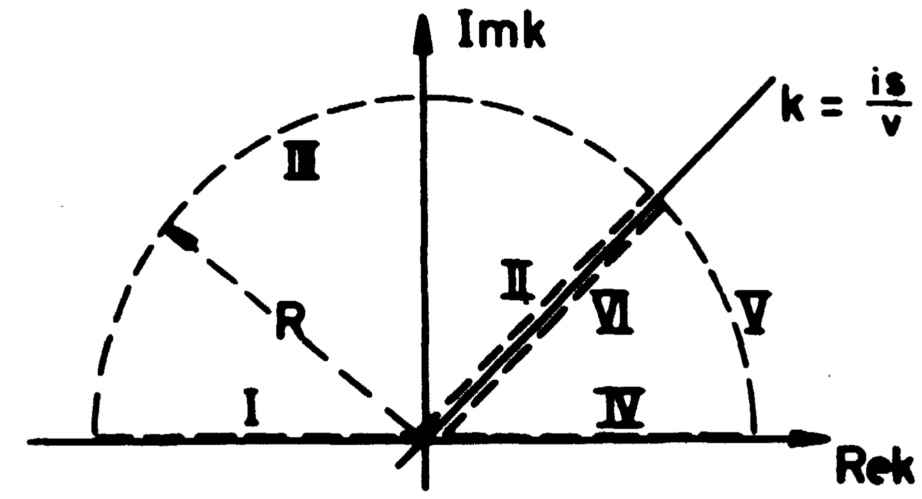
$$\theta = \left(\frac{d_e}{d_i}\right)^2 = \frac{T_{oe}}{T_{oi}} \,.$$

We shall now find the zeroes for the function $D_1(k,s)$. Substituting $k = i\frac{s}{\gamma}$ and assuming $j = 1$ (ɔ: $I_m(\gamma) \geq 0$), we find that $D_1(k = i\frac{s}{\gamma}, s) = 0$ for

$$s^2 = (s_{1,3}(\gamma))^2 = -\frac{\gamma^2}{2d_i^2}\left[Z'\left(\frac{\gamma - v_{oi}}{\sqrt{2}\,A_{oi}}\right) + \frac{1}{\theta}\, Z'\left(\frac{\gamma - v_{oe}}{\sqrt{2}\,A_{oe}}\right)\right]$$

when γ is real. For k running along the line $k = i\frac{s}{v}$, γ is running along the positive real axis. $s_1(\gamma)$ and $s_3(\gamma)$ are shown in fig. 3. See also ref. 1.

Similar results are obtained when $j = 2$. The roots of $D_2(\gamma, s)$ are denoted $s_{2,4}(\gamma)$ and $s_{2,4} = s_{1,3}^*$. When $Re(s) = s_o$ (see fig. 3) and $j = 1$, $Im(\gamma) > 0$ maps inside the closed γ figures. Similarly when $j = 2$ ($Im(\gamma) < 0$). This point is essential. We assume that s lies fully to the right of the γ figures; non-analyticity is confined to the branch cut. The Fourier inversion contour is wrapped around the cut as indicated in fig. 4.



The integral along paths III and V (see fig. 4) tends to zero as $R \to \infty$, in accordance with (12), (14), and (15). We make the substitution $k = i\frac{s}{\gamma}$ and find

$$\bar{n}_{i,e}(x,s) = \frac{1}{2\pi i}\left(\int_0^\infty \tilde{\tilde{n}}_{i,e,2}(\gamma,s)\,\frac{s}{\gamma^2}\,e^{-\frac{sx}{\gamma}}\,d\gamma - \int_0^\infty \tilde{\tilde{n}}_{i,e,1}(\gamma,s)\,\frac{s}{\gamma^2}\,e^{-\frac{sx}{\gamma}}\,d\gamma\right).$$

Inserting (9) we get

$$\bar{n}_{i,e}(x,s) = \frac{1}{2\pi i}\left(\int_0^\infty (S_{i,e,2} - S_{i,e,1})\,\frac{s}{\gamma^2}\,e^{-\frac{sx}{\gamma}}\,d\gamma \right.$$

$$\left. + \int_0^\infty \left(\frac{M_{i,e,2}}{D_2} - \frac{M_{i,e,1}}{D_1}\right)\frac{s}{\gamma^2}\,e^{-\frac{sx}{\gamma}}\,d\gamma\right) \tag{16}$$

where

$$S_{i,e,2}(\gamma,s) - S_{i,e,1}(\gamma,s) = \frac{\Delta n \gamma^2}{s^2} 2\pi i \, g_{i,e}(\gamma)$$

and

$$D_{1,2}(\gamma,s) = \frac{(s - s_{1,2}(\gamma))\,(s - s_{3,4}(\gamma))}{s^2} \,. \tag{17}$$

Thus

$$\bar{n}_{i,e}(x,s) = \Delta n \int_0^\infty \frac{g_{i,e}(\gamma)}{s} e^{-s\frac{x}{\gamma}} d\gamma$$

$$\pm \frac{\Delta n}{4\pi i d_{i,e}^2} \int_0^\infty \left(\frac{P_{i,e,2}(\gamma)}{(s-s_2)(s-s_4)} - \frac{P_{i,e,1}(\gamma)}{(s-s_1)(s-s_3)} \right) \frac{\gamma^2}{s} e^{-s\frac{x}{\gamma}} d\gamma \tag{18}$$

where

$$M_{i,e,1,2}(\gamma,s) = \frac{\Delta n \gamma^4}{2 d_i^2 s^4} P_{i,e,1,2}(\gamma) \,.$$

Performing the inverse Laplace transformation we interchange the order of the γ and s integrations. We continue (18) analytically to the full s-plane and consider the two integral terms separately.

$$1\text{st part} = \Delta n \int_0^\infty g_{i,e}(\gamma)\, \epsilon(t - \tfrac{x}{\gamma}) d\gamma$$

$$= \Delta n \int_{\frac{x}{t}}^\infty g_{i,e}(\gamma) d\gamma$$

$$= \frac{\Delta n}{2} \operatorname{erfc}\left(\frac{\frac{x}{t} - v_{i,e}}{\sqrt{2}\, A_{i,e}} \right) . \tag{19a}$$

We used (11) to obtain the last expression.

Using $s_{3,4} = - s_{1,2}$, $s_{2,4} = s_{1,3}^*$ and $P_2(\gamma) = P_1^*(\gamma)$

we find

$$2\text{nd part} = \pm \frac{1}{2\pi i} \frac{\Delta n}{4\pi i d_{i,e}^2} \int_0^\infty \int_{s_0 - i\infty}^{s_0 + i\infty} \gamma^2 \left(\frac{P_{i,e,1}^*}{s(s^2 - s_1^{*2})} - \frac{P_{i,e,1}}{s(s^2 - s_1^2)} \right) e^{s(t - \frac{x}{\gamma})} ds d\gamma$$

$$= \mp \frac{\Delta n}{2\pi d_{i,e}^2} \operatorname{Im} \int_{\frac{x}{t}}^\infty \gamma^2 \frac{P_{i,e,1}}{s_1^2} (\cosh [s_1 (t - \tfrac{x}{\gamma})] - 1) d\gamma. \tag{19b}$$

Equation 19a corresponds to freely streaming particles, while eauation 19b accounts for the collective interaction.

This result is not suitable for numerical evaluation. Following Mason[1] we therefore rewrite (19) as

$$\frac{n_{i,e}(x,t)}{\Delta n} = \int_{\frac{x}{t}}^\infty g_{i,e}(\gamma) d\gamma \pm \frac{1}{4\pi d_{i,e}^2} \operatorname{Im} \left\{ \int_{\frac{x}{t}}^\infty \gamma^2 \frac{P_{i,e}(\gamma)}{s_1^2(\gamma)} (2 - e^{s_1(\gamma)(t - \frac{x}{\gamma})}) d\gamma \right.$$

$$\left. - \int_{\frac{x}{t}}^\infty \frac{\gamma^2 P_{i,e}(\gamma)}{s_1^2(\gamma)} e^{-s_1(\gamma)(t - \frac{x}{\gamma})} d\gamma \right\} \equiv$$

$$\int_{\frac{x}{t}}^\infty g_{i,e}(\gamma) d\gamma \pm \frac{1}{4\pi d_{i,e}^2} \operatorname{Im} \{ I - II \}.$$

We consider first the last integral, II:

$$II = \int_0^{-\infty} \frac{\gamma^2 P_{i,e}(\gamma)}{s_1^2(\gamma)} e^{-s_1(\gamma)(t - \frac{x}{\gamma})} d\gamma - \int_0^{\frac{x}{t}} \frac{\gamma^2 P_{i,e}(\gamma)}{s_1^2(\gamma)} e^{-s_1(\gamma)(t - \frac{x}{\gamma})} d\gamma \,.$$

Substituting $\gamma \to -\gamma$ in the first integral we find

$$I - II = \int_{\frac{x}{t}}^{\infty} \gamma^2 \left(\frac{P_{i,e}(\gamma)}{s_1^2(\gamma)} \left(2 - e^{s_1(\gamma)(t - \frac{x}{\gamma})} \right) + \frac{P_{i,e}(-\gamma)}{s_1^2(-\gamma)} e^{-s_1(-\gamma)(t + \frac{x}{\gamma})} \right) d\gamma$$

$$+ \int_{0}^{\frac{x}{t}} \gamma^2 \left(\frac{P_{i,e}(\gamma)}{s_1^2(\gamma)} e^{-s_1(\gamma)(t - \frac{x}{\gamma})} + \frac{P_{i,e}(-\gamma)}{s_1^2(-\gamma)} e^{-s_1(-\gamma)(t + \frac{x}{\gamma})} \right) d\gamma .$$

Introducing the dimensionless variables

$$\zeta = \frac{\gamma}{\sqrt{2}A_i}, \quad \eta = \frac{x}{A_i t}, \quad \tau = \omega_{pi} t \quad \text{and} \quad \frac{s_1(\zeta)}{\omega_{pi}} \rightarrow s_1(\zeta)$$

we get

$$\frac{n_{i,e}(x,t)}{\Delta n} = \sqrt{2}\, A_i \int_{\frac{\eta}{\sqrt{2}}}^{\infty} g_{i,e}(\zeta) d\zeta \pm \frac{\sqrt{2}}{2\pi} A_{oi}\, C \,\mathrm{Im} \Bigg\{$$

$$\int_0^{\infty} \zeta^2 \left[\frac{P_{i,e}(\zeta)}{s_1^2(\zeta)} A + \frac{P_{i,e}(-\zeta)}{s_1^2(-\zeta)} e^{-s_1(-\zeta)(t + \frac{\eta}{\sqrt{2}\zeta})\tau} \right] d\zeta \Bigg\}$$

where

$$C = \begin{cases} 1 & \text{for the ions} \\ \theta^{-1} & \text{for the electrons} \end{cases}$$

and

$$A = \begin{cases} e^{-s_1(\zeta)(1 - \frac{\eta}{\sqrt{2}\zeta})\tau} & 0 \le \zeta \le \frac{\eta}{\sqrt{2}} \\ 2 - e^{s_1(\zeta)(1 - \frac{\eta}{\sqrt{2}\zeta})\tau} & \frac{\eta}{\sqrt{2}} \le \zeta . \end{cases}$$

Numerical results are shown in fig. 5.

2. 2. The Distribution Function, $f_{i,e}(x, v, t)$

Inserting (9) in (6) we find

$$\tilde{f}_{i,e}(k, v, s) = \frac{i\Delta n}{k} \frac{g_{i,e}(v)}{s + ikv} \pm \frac{i\omega_{pi,e}^2}{k} \frac{S_i - S_e}{D} \frac{f'_{oi,e}(v)}{s + ikv} . \qquad (20)$$

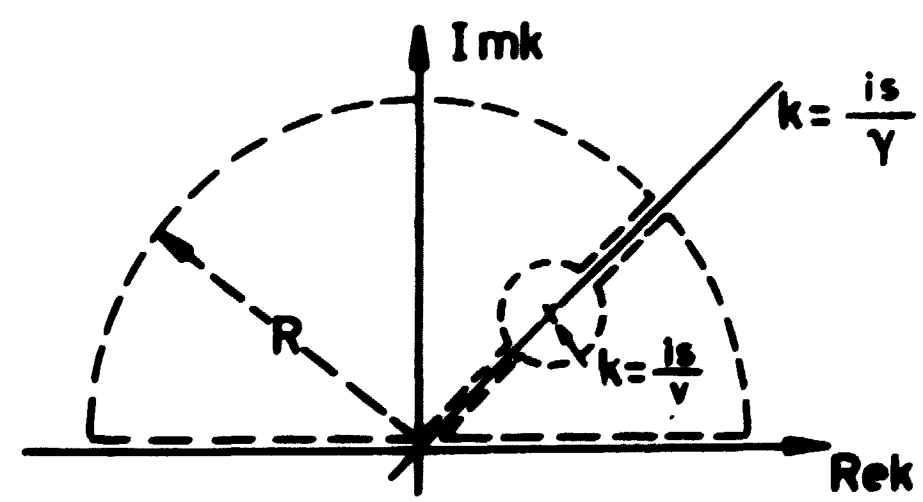
Performing the inverse transformations of (20) we find $f_{i,e}(x, v, t)$. The calculations are similar to those in section 2.1 except for contribution due to the pole for $s + ikv = 0$. We therefore only consider this contribution in detail.

$$\bar{f}_{i,e}(x, v, s) = \frac{i\Delta n\, g_{i,e}(v)}{2\pi} \left(\int_{-\infty}^{0} \frac{e^{ikx}}{k(s + ikv)} dk + \int_{0}^{\infty} \frac{e^{ikx}}{k(s + ikv)} dk \right)$$

$$\pm \frac{i\omega_{pi,e}^2 f'_{oi,e}(v)}{2\pi} \left(\int_{-\infty}^{0} \frac{1}{k(s + ikv)} \frac{S_{i2} - S_{e2}}{D_2} e^{ikx} dk \right.$$

$$\left. + \int_{0}^{\infty} \frac{1}{k(s + ikv)} \frac{S_{i1} - S_{e2}}{D_2} e^{ikx} dk \right) . \qquad (21)$$

We change the path of integration in the k-plane as indicated in fig. 6.



We make the substitution $k = i\frac{s}{\Upsilon}$ and find

$$\bar{f}_{i,e} = -\frac{i\Delta n g_{i,e}(v)}{2\pi}\left(\underset{\smile}{\int}_0^{\infty} \frac{e^{-s\frac{x}{\Upsilon}}}{s(\Upsilon - v)}\, d\Upsilon + \overset{\frown}{\int}_{\infty}^{0} \frac{e^{-s\frac{x}{\Upsilon}}}{s(\Upsilon - v)}\, d\Upsilon\right)$$

$$\mp \frac{i\omega^2_{pi,e} f'_{oi,e}(v)}{2\pi}\left(\underset{\smile}{\int}_0^{\infty} \frac{1}{s(\Upsilon - v)} \frac{S_{i,2} - S_{e,2}}{D_2}\, e^{-s\frac{x}{\Upsilon}}\, d\Upsilon\right.$$

$$\left. + \overset{\frown}{\int}_{\infty}^{0} \frac{1}{s(\Upsilon - v)} \frac{S_{i,1} - S_{e,1}}{D_1}\, e^{-s\frac{x}{\Upsilon}}\, d\Upsilon\right)$$

where $\underset{\smile}{\int}$ and $\overset{\frown}{\int}$ imply that the pole is above and below the integration path respectively. We consider the two terms separately:

$$\text{1st term} = -\frac{i\Delta n\, g_{i,e}(v)}{2\pi}\, 2\pi i \,\mathrm{Res}(\Upsilon = v)$$

$$= \Delta n\, g_{i,e}(v)\, \frac{e^{-s\frac{x}{v}}}{s}\ .$$

$$\text{2nd term} = \mp \frac{i\Delta n\omega^2_{pi,e}}{2\pi} f'_{oi,e}(v)\left\{P\int_0^{\infty} \frac{1}{s(\Upsilon - v)}\left(\frac{S_{i,2} - S_{e,2}}{D_2} - \frac{S_{i,1} - S_{e,1}}{D_1}\right) e^{-s\frac{x}{\Upsilon}} d\Upsilon\right.$$

$$\left. + i\pi\left(\frac{S_{i,2} - S_{e,2}}{D_2} + \frac{S_{i,1} - S_{e,1}}{D_1}\right)\frac{e^{-s\frac{x}{v}}}{s}\right\}$$

where P denotes principal value.

Using (12) we get

$$S_{i,1,2}(\Upsilon, s) - S_{e,1,2}(\Upsilon, s) = -\Delta n \frac{\Upsilon^2}{s^2}\, Q_{1,2}(\Upsilon)$$

where

$$Q_1(\Upsilon) = \frac{1}{\sqrt{2}\, A_i}\, Z\left(\frac{\Upsilon - v_i}{\sqrt{2}\, A_i}\right) - \frac{1}{\sqrt{2}\, A_e}\, Z\left(\frac{\Upsilon - v_e}{\sqrt{2}\, A_e}\right)$$

$$Q_2(\Upsilon) = Q_1^*(\Upsilon)\ .$$

Using these equations and (17) we find

$$\text{2nd term} = \pm \frac{i\Delta n\omega^2_{pi,e}}{2\pi} f'_{oi,e}(v)$$

$$\left\{P\int_0^{\infty} \frac{\Upsilon^2}{s(\Upsilon - v)}\left(\frac{Q_2(\Upsilon)}{(s - s_2(\Upsilon))(s - s_4(\Upsilon))} - \frac{Q_1(\Upsilon)}{(s - s_1(\Upsilon))(s - s_3(\Upsilon))}\right) e^{-s\frac{x}{\Upsilon}} d\Upsilon\right.$$

$$\left. + i\pi v^2\left(\frac{Q_2(v)}{(s - s_2(v))(s - s_4(v))} + \frac{Q_1(v)}{(s - s_1(v))(s - s_3(v))}\right) e^{-s\frac{x}{v}}\right\}.$$

Performing the inverse Laplace transformation in analogy with section 2.1 we find

$$f_{i,e}(x, v, t) = \Delta n\, g_{i,e}(v)\, \epsilon\left(t - \frac{x}{v}\right) \pm \frac{\Delta n\omega^2_{pi,e}}{\pi} f'_{oi,e}(v)$$

$$\left\{\mathrm{Im}\; P\int_{\frac{x}{t}}^{\infty} \frac{\Upsilon^2}{\Upsilon - v} \frac{Q_1(\Upsilon)}{s_1^2(\Upsilon)} \left(\cosh\left[s_1(\Upsilon)\left(t - \frac{x}{\Upsilon}\right)\right] - 1\right) d\Upsilon\right.$$

$$\left. - \mathrm{Re}\left(\pi v^2 \frac{Q_1(v)}{s_1^2(v)}\left(\cosh\left[s_1(v)\left(t - \frac{x}{v}\right)\right] - 1\right)\right) \epsilon\left(t - \frac{x}{v}\right)\right\} \quad (22)$$

where $\epsilon(\zeta)$ is Heaviside's step function.

For $v < \frac{x}{t}$ $f_{i,e}(x, v, t)$ reduces to

$$f_{i,e}(x, v, t) = \pm \frac{\Delta n\omega^2_{pi,e}}{\pi} f'_{oi,e}(v)\; \mathrm{Im} \int_{\frac{x}{t}}^{\infty} \frac{\Upsilon^2}{\Upsilon - v} \frac{Q_1(\Upsilon)}{s_1^2(\Upsilon)} \left(\cosh\left[s_1(\Upsilon)\left(t - \frac{x}{\Upsilon}\right)\right] - 1\right) d\Upsilon.$$

This contribution is solely due to collective interaction since $vt < x$, i.e. the freely streaming particles have not yet arrived at the observation point x.

For $v > \frac{x}{t}$ we get

$$f_{i,e}(x,v,t) = \Delta n\, g_{i,e}(v) \mp \frac{\Delta n \omega^2_{pi,e}}{\pi} f'_{oi,e}(v)$$

$$\mathrm{Im} \int_{-\infty}^{\frac{x}{t}} \frac{\gamma^2}{\gamma} \frac{Q_1(\gamma)}{s_1^2(\gamma)} (\cosh[s_1(\gamma)(t-\frac{x}{\gamma})] - 1)\, d\gamma.$$

The first term corresponds to freely streaming particles, and the second term is due to collective interaction. To obtain this expression we used the identity

$$\mathrm{Re}\,[\pi v^2 \frac{Q_1(v)}{s_1^2(v)} (\cosh[s_1(v)(t-\frac{x}{v})]-1)] =$$

$$\mathrm{Im}\, P \int_{-\infty}^{\infty} \frac{\gamma^2}{\gamma - v} \frac{Q_1(\gamma)}{s_1^2(\gamma)} (\cosh[s_1(\gamma)(t-\frac{x}{\gamma})] - 1)\, d\gamma$$

(Hilbert transformation).

$f_{i,e}(x,v,t)$ is not defined for $v = \frac{x}{t}$. This does not mean that our result is unphysical. Since $f_{i,e}(x,v,t)$ is a distribution function, it is sufficient to know how to use it for calculating averages, i.e. how to integrate it after multiplication by other functions. This amounts to saying that $f_{i,e}(x,v,t)$ must be a "distribution" in the sense of L. Schwartz. For more realistic initial perturbations than the present one $f_{i,e}(x,v,t)$ may very well be defined for all x, v, t. We shall revert to this question later on.

2.3. The Flux, $F_{i,e}(x,t)$

The flux is defined as

$$F_{i,e}(x,t) \equiv \int_{-\infty}^{\infty} v f_{i,e}(x,v,t)\, dv.$$

Since the density, $n_{i,e}(x,t)$, is known explicitly (19), the easiest way of calculating $F_{i,e}(x,t)$ is to use the equation of continuity

$$\frac{\partial n_{i,e}}{\partial t} + \frac{\partial}{\partial x} F_{i,e}(x,t) = 0.$$

Inserting the expression for the density we find

$$\frac{\partial}{\partial x} F_{i,e}(x,t) = -\Delta n \frac{x}{t^2} g_{i,e}(\frac{x}{t})$$

$$\pm \frac{\Delta n}{2\pi d^2_{i,e}} \mathrm{Im} \int_{\frac{x}{t}}^{\infty} \gamma^2 \frac{P_{i,e,1}(\gamma)}{s_1^2(\gamma)} \sinh[s_1(\gamma)(t-\frac{x}{\gamma})]\, d\gamma.$$

Integrating from x to infinity we get

$$F_{i,e}(x,t) = \Delta n \int_x^{\infty} \frac{\zeta}{t^2} g_{i,e}(\frac{\zeta}{t}) d\zeta$$

$$\mp \frac{\Delta n}{2\pi d^2_{i,e}} \mathrm{Im} \int_x^{\infty} \int_{\frac{\zeta}{t}}^{\infty} \gamma^2 \frac{P_{i,e,1}(\gamma)}{s_1(\gamma)} \sinh[s_1(\gamma)(t-\frac{\zeta}{\gamma})]\, d\gamma\, d\zeta.$$

The integration constant is zero since $F_{i,e}(x\to\infty, t) = 0$ for all t. We consider the two terms separately. Using the tranformation

$\gamma = \frac{\zeta}{t}$, we get the 1st part $= \Delta n \int_{\frac{x}{t}}^{\infty} \gamma\, g_{i,e}(\gamma) d\gamma$.

Calculating the expression for the 2nd part we first consider the double integral

$$\int_x^a \int_{\frac{\zeta}{t}}^b F(\zeta,\gamma) d\gamma\, d\zeta = \int_{\frac{x}{t}}^b \int_x^{t\gamma} F(\zeta,\gamma) d\zeta\, d\gamma, \quad b = \frac{a}{t}$$

where we changed the order of integration. Letting $a, b \to \infty$ we get

$$\int_x^{\infty} \int_{\frac{\zeta}{t}}^{\infty} F(\zeta,\gamma) d\gamma\, d\zeta = \int_{\frac{x}{t}}^{\infty} \int_x^{t\gamma} F(\zeta,\gamma) d\zeta\, d\gamma. \tag{23}$$

Using this identity we find

$$F_{i,e}(x,t) = \Delta n \Bigg(\int_{\frac{x}{t}}^{\infty} \gamma\, g_{i,e}(\gamma) d\gamma$$

$$\mp \frac{1}{2\pi d_{i,e}^{2}} \, \mathrm{Im} \int_{\frac{x}{t}}^{\infty} \gamma^{3} \, \frac{P_{i,e}(\gamma)}{s_1^2(\gamma)} \left(\cosh\left[s_1(\gamma)\left(t - \frac{x}{\gamma}\right)\right] - 1\right) d\gamma \Bigg). \quad (24)$$

Using (11) we can simplify the first term

$$\int_{\frac{x}{t}}^{\infty} \gamma\, g_{i,e}(\gamma) d\gamma = \frac{A_{i,e}}{\sqrt{2\pi}} e^{-\left(\frac{\frac{x}{t} - v_{i,e}}{\sqrt{2} A_{i,e}}\right)^2} + \frac{v_{i,e}}{2} \, \mathrm{erfc}\left(\frac{\frac{x}{t} - v_{i,e}}{\sqrt{2} A_{i,e}}\right) .$$

The first term in (24) is due to freely streaming particles, the second term is due to collective interaction.

Following the procedure shown in section 2.1 we put (24) on a form suitable for numerical evaluation:

$$F_{i,e}(\eta,\tau) = \Delta n \Big\{ 2 A_i^2 \int_{\frac{\eta}{\sqrt{2}}}^{\infty} \gamma\, g_{i,e}(\gamma) d\gamma \mp C \frac{A_{oi}^2}{\pi} \, \mathrm{Im} \Big\{$$

$$\int_{\frac{\eta}{\sqrt{2}}}^{\infty} \zeta^3 \left[\frac{P_{i,e}(\zeta)}{s_1^2(\zeta)} \left[e^{s_1(\zeta)\left(1 - \frac{\eta}{\sqrt{2}\zeta}\right)\tau} - 2 \right] + \frac{P_{i,e}(-\zeta)}{s_1^2(-\zeta)} e^{-s_1(-\zeta)\left(1 + \frac{\eta}{\sqrt{2}\zeta}\right)\tau} \right] d\zeta$$

$$- \int_{0}^{\frac{\eta}{\sqrt{2}}} \zeta^3 \left[\frac{P_{i,e}(\zeta)}{s_1^2(\zeta)} e^{-s_1(\zeta)\left(1 - \frac{\eta}{\sqrt{2}\zeta}\right)\tau} - \frac{P_{i,e}(-\zeta)}{s_1^2(-\zeta)} e^{-s_1(-\zeta)\left(1 + \frac{\eta}{\sqrt{2}\zeta}\right)\tau} \right] d\zeta$$

where

$$C = \begin{cases} 1 & \text{for the ions} \\ \theta^{-1} & \text{for the electrons} \end{cases}$$

Numerical results are shown in fig. 7.

2.4. The Electric Field, $E(x, t)$

We integrate Poisson's equation from x to infinity:

$$E(x,t) = -\frac{e}{\varepsilon_0} \int_{x}^{\infty} [\, n_i(\xi,t) - n_e(\xi,t) \,] \, d\xi .$$

The integration constant is zero since $E(x \to \infty, t) = 0$. Inserting (19) and using formula 23 we find

$$E(x,t) = \frac{-\Delta n_e}{\varepsilon_0} \Big\{ \int_{\frac{x}{t}}^{\infty} \gamma \left(t - \frac{x}{\gamma}\right)(g_i(\gamma) - g_e(\gamma)) d$$

$$- \frac{1}{2\pi d_i^2} \, \mathrm{Im} \int_{\frac{x}{t}}^{\infty} \gamma^3 \frac{R(\gamma)}{s_1^3(\gamma)} \left(\sinh\left[s_1(\gamma)\left(t - \frac{x}{\gamma}\right)\right] - s_1(\gamma)\left(t - \frac{x}{\gamma}\right)\right) d\gamma \quad (25).$$

where

$$R(\gamma) = P_{i,1}(\gamma) + \frac{1}{\theta} P_{e,1}(\gamma) .$$

Using (11) we find

$$\int_{\frac{x}{t}}^{\infty} \gamma \left(t - \frac{x}{\gamma}\right)(g_i(\gamma) - g_e(\gamma)) \, d\gamma =$$

$$\frac{t}{\sqrt{2\pi}} \left[A_i \exp\left(-\left(\frac{\frac{x}{t} - v_i}{\sqrt{2} A_i}\right)^2\right) - A_e \exp\left(-\left(\frac{\frac{x}{t} - v_e}{\sqrt{2} A_e}\right)^2\right) \right]$$

$$+ \frac{1}{2} \left((t v_i - x) \, \mathrm{erfc}\left(\frac{\frac{x}{t} - v_i}{\sqrt{2} A_i}\right) - (t v_e - x) \, \mathrm{erfc}\left(\frac{\frac{x}{t} - v_e}{\sqrt{2} A_e}\right) \right) .$$

2.5. The Potential, $\varphi(x, t)$

We integrate $E(x,t) = -\frac{\partial}{\partial x} \varphi(x,t)$ from x to infinity, inserting (25) and using formula 23, and find

$$\varphi(x,t) = -\frac{\Delta n}{\varepsilon_o} e\left\{\frac{1}{2}\int_{\frac{x}{t}}^{\infty}\gamma^2(t-\frac{x}{\gamma})^2(g_i(\gamma)-g_e(\gamma))d\gamma\right.$$

$$\left. -\frac{1}{2\pi d_i^2}\ \mathrm{Im}\int_{\frac{x}{t}}^{\infty}\gamma^4\frac{R(\gamma)}{s_1^4(\gamma)}(\cosh[s_1(\gamma)(t-\frac{x}{\gamma})]-\frac{1}{2}s_1^2(\gamma)(t-\frac{x}{\gamma})^2-1)d\gamma\right\}.$$

Using (11) we get

$$\int_{\frac{x}{t}}^{\infty}\gamma^2(t-\frac{x}{\gamma})^2(g_i(\gamma)-g_e(\gamma))\,d\gamma =$$

$$\frac{1}{2}t^2\left\{\sqrt{\frac{2}{\pi}}A_i(v_i-\frac{x}{t})\exp\left(-\left(\frac{\frac{x}{t}-v_i}{\sqrt{2}A_i}\right)^2\right)-\sqrt{\frac{2}{\pi}}A_e(v_e-\frac{x}{t})\exp\left(-\left(\frac{\frac{x}{t}-v_e}{2A_e}\right)^2\right)\right.$$

$$\left.+(A_i^2+(v_i-\frac{x}{t})^2)\,\mathrm{erfc}\left(\frac{\frac{x}{t}-v_i}{\sqrt{2}A_i}\right)-(A_e^2+(v_e-\frac{x}{t})^2)\mathrm{erfc}\left(\frac{\frac{x}{t}-v_e}{\sqrt{2}A_e}\right)\right\}.$$

3. ANALYTIC SOLUTIONS WITH A PULSE-LIKE INITIAL CONDITION

Equations 1 and 2 are solved for the case where the initial condition is given by

$$f_{i,e}(x,v,t=0) = \Delta n\, g_{i,e}(v)\,\delta(x)$$

and

$$n_{i,e}(x,t=0) = \Delta n\,\delta(x).$$

As mentioned earlier these solutions can be found simply by differentiating those obtained in chapter 2, but the following procedure may give a better physical insight.

3.1. The Density, $n_{i,e}(x,t)$

We first consider the following two initial value problems:

a) $n_{i,e}(x,t=0) = \Delta n\, h[1-\epsilon(x)]$

b) $n_{i,e}(x,t=0) = \Delta n\, h[1-\epsilon(x-\frac{1}{h})]$.

Notice that $[h] = L^{-1}$ and $[\Delta n] = \text{particles}\cdot L^{-2}$. Referring to section 2.1 we find (omitting the indices "1" on $P_{i,e}$)

$$n_{ai,e}(x,t) = \Delta n h\left\{\int_{\frac{x}{t}}^{\infty}g_{i,e}(\gamma)d\gamma \mp \frac{1}{2\pi d_{i,e}^2}\ \mathrm{Im}\int_{\frac{x}{t}}^{\infty}\gamma^2\frac{P_{i,e}(\gamma)}{s_1^2(\gamma)}(\cosh[s_1(\gamma)(t-\frac{x}{\gamma})]-1)d\gamma\right\}$$

and

$$n_{bi,e}(x,t) = \Delta n h\left\{\int_{\frac{x+1/h}{t}}^{\infty}g_{i,e}(\gamma)d\gamma\right.$$

$$\left.\mp\frac{1}{2\pi d_{i,e}^2}\ \mathrm{Im}\int_{\frac{x+1/h}{t}}^{\infty}\gamma^2\frac{P_{i,e}(\gamma)}{s_1^2(\gamma)}(\cosh[s_1(\gamma)(t-\frac{x+1/h}{\gamma})]-1)d\gamma\right\}.$$

The solution to (1) and (2) for the case of a pulse-like initial condition is now found by $\lim\limits_{h\to\infty}[n_a(x,t)-n_b(x,t)]$. We find

$$n_{ai,e}-n_{bi,e} = \Delta n h\left\{\int_{\frac{x}{t}}^{\frac{x+1/h}{t}}g_{i,e}(\gamma)d\gamma \mp \frac{1}{2\pi d_{i,e}^2}\ \mathrm{Im}\right.$$

$$\int_{\frac{x}{t}}^{\frac{x+1/h}{t}}\gamma^2\frac{P_{i,e}(\gamma)}{s_1^2(\gamma)}(\cosh[s_1(\gamma)(t-\frac{x}{\gamma})]-1]\ d\gamma$$

$$\left.+\int_{\frac{x+1/h}{t}}^{\infty}\gamma^2\frac{P_{i,e}(\gamma)}{s_1^2(\gamma)}(\cosh[s_1(\gamma)(t-\frac{x}{\gamma})]-\cosh[s_1(\gamma)(t-\frac{x+1/h}{\gamma})])d\gamma\right\}.$$

The three integral terms are denoted I_1, I_2, and I_3 respectively. We consider each term separately.

$$I_1 = \frac{1}{ht}\, g_{i,e}(\eta_1)$$

$$I_2 = \frac{1}{ht}\, \eta_2^2\, \frac{P_{i,e}(\eta_2)}{s_1^2(\eta_2)} \left(\cosh(s_1(\eta_2)(t - \frac{x}{\eta_2}))-1\right)$$

where

$$\frac{x}{t} \leq \begin{matrix}\eta_1 \\ \eta_2\end{matrix} \leq \frac{x + 1/h}{t} .$$

We used the mean-value theorem. Evidently $h\, I_2 \simeq 0$ when $h \gg 1$ since $\cosh(s_1(\eta_2)(t - \frac{x}{\eta_2})) \simeq 1$ when $\eta_2 \simeq \frac{x}{t}$ so

$$\lim_{h\to\infty} (h\, I_2) = 0 \quad \text{and} \quad \lim_{h\to\infty} (h\, I_1) = \frac{1}{t}\, g_{i,e}(\frac{x}{t}) .$$

In order to evaluate the last term we consider

$$\lim_{h\to\infty} \left\{ h\left(\cosh\left[s_1(\gamma)(t - \frac{x}{\gamma})\right] - \cosh\left[s_1(\gamma)(t - \frac{x + 1/h}{\gamma})\right]\right) \right\} =$$

$$\lim_{h\to\infty} \left\{ \frac{h}{2}\left[(1 - e^{-\frac{s_1(\gamma)}{h\gamma}})\, e^{s_1(\gamma)(t - \frac{x}{\gamma})} + (1 - e^{\frac{s_1(\gamma)}{h\gamma}})\, e^{-s_1(\gamma)(t - \frac{x}{\gamma})} \right] \right\} =$$

$$\frac{s_1(\gamma)}{\gamma} \sinh\left[s_1(\gamma)(t - \frac{x}{\gamma})\right] . \tag{26}$$

The final result is then

$$n_{i,e}(x,t) = \Delta n \left(\frac{1}{t}\, g_{i,e}(\frac{x}{t}) \mp \frac{1}{2\pi d_{i,e}^2}\, \mathrm{Im} \int_{\frac{x}{t}}^{\infty} \gamma\, \frac{P_{i,e}(\gamma)}{s_1(\gamma)} \sinh\left[s_1(\gamma)(t - \frac{x}{\gamma})\right] d\gamma \right)$$

We want to show that this result satisfies the initial condition:

$$\lim_{t\to 0} n_{i,e}(x,t) = \Delta n \lim_{t\to 0} \frac{1}{t}\, g_{i,e}(\frac{x}{t})$$

$$= \Delta n \lim_{t\to 0} \frac{1}{t} \int_{-\infty}^{\infty} g_{i,e}(v)\, \delta(v - \frac{x}{t})\, dv$$

$$= \Delta n \lim_{t\to 0} \int_{-\infty}^{\infty} g_{i,e}(v)\, \delta(vt - x)\, dv$$

$$= \Delta n \int_{-\infty}^{\infty} g_{i,e}(v)\, \delta(-x)\, dv$$

$$= \Delta n\, \delta(x)$$

since $\int_{-\infty}^{\infty} g_{i,e}(v)\, dv = 1$.

3.2. The Distribution Function, $f_{i,e}(x,v,t)$

Using

$$f_{i,e}(x,v,t) = \lim_{h\to\infty} \left(f_{ai,e}(x,v,t) - f_{bi,e}(x,v,t)\right)$$

together with (22) we find (omitting the indices "1" on $Q(\gamma)$)

$$f_{ai,e} - f_{bi,e} = \Delta n\, h\, g_{i,e}(v) \left(\epsilon(t - \frac{x}{v}) - \epsilon(t - \frac{x + 1/h}{v})\right)$$

$$\pm \frac{\Delta n\, h\, \omega_{pi,e}^2}{\pi}\, f'_{oi,e}(v)$$

$$\left\{ \mathrm{Im}\, P \int_{\frac{x}{t}}^{\frac{x+1/h}{t}} \frac{\gamma^2}{\gamma - v}\, \frac{Q(\gamma)}{s_1^2(\gamma)} \left(\cosh\left[s_1(\gamma)(t - \frac{x}{\gamma})\right] - 1\right) d\gamma \right.$$

$$+ \mathrm{Im}\, P \int_{\frac{x+1/h}{t}}^{\infty} \frac{\gamma^2}{\gamma - v}\, \frac{Q(\gamma)}{s_1^2(\gamma)} \left(\cosh\left[s_1(\gamma)(t - \frac{x}{\gamma})\right] - \cosh\left[s_1(\gamma)(t - \frac{x+1/h}{\gamma})\right]\right) d\gamma$$

$$- \pi v^2 \,\mathrm{Re}\Big[\frac{Q(v)}{s_1^2(v)} \big((\cosh[s_1(v)(t-\frac{x}{v})]\, \epsilon(t-\frac{x}{v})$$

$$- (\cosh[s_1(v)(t - \frac{x+1/h}{v}) - 1)\,\epsilon(t-\frac{x+1/h}{v})\big)\Big]\Big\}.$$

Using

$$\lim_{h\to\infty} h\big(\epsilon(t-\frac{x}{v}) - \epsilon(t - \frac{x+1/h}{v})\big) = \delta(vt-x) \quad \text{(see App.)}$$

we find for the first term: $\Delta n\, g_{i,e}(v)\,\delta(vt-x)$.

We consider the three terms II_1, II_2, II_3 within the brackets $\{\ \}$ separately.

1st term

$$\lim_{h\to\infty} (h\, II_1) = 0\,.$$

2nd term

Using (26) we get

$$\lim_{h\to\infty}(h\, II_2) = \mathrm{Im}\, P \int_{x/t}^{\infty} \frac{\gamma}{\gamma - v}\,\frac{Q(\gamma)}{s_1(\gamma)}\, \sinh[s_1(\gamma)(t-\frac{x}{\gamma})]\,d\gamma\,.$$

3rd term

We consider

$$M \equiv \cosh[s_1(v)(t-\frac{x}{v})]\,\epsilon(t-\frac{x}{v}) - \cosh[s_1(v)(t-\frac{x+1/h}{v})]\,\epsilon(t-\frac{x+1/h}{v}).$$

Expressing Heaviside's step function as a Fourier integral (see Appendix) we get

$$M = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\big(1-\exp(-(ip+\frac{s_1(v)}{v})\frac{1}{h})\big)\exp(s_1(v)(t-\frac{x}{v}))$$

$$+ \big[1-\exp(-(ip-\frac{s_1(v)}{v})\frac{1}{h})\big]\exp(-s_1(v)(t-\frac{x}{v}))\Big]\frac{\exp(ip(vt-x))}{2ip}\,dp$$

and

$$\lim_{h\to\infty}(h M) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\cosh[s_1(v)(t-\frac{x}{v})] + \frac{s_1(v)}{ipv}\sinh[s_1(v)(t-\frac{x}{v})]\Big)\exp(ip(vt-x))\,dp$$

$$= \frac{1}{v}\cosh[s_1(v)(t-\frac{x}{v})]\,\delta(t-\frac{x}{v})$$

$$+ \frac{s_1(v)}{v}\sinh[s_1(v)(t-\frac{x}{v})]\,\epsilon(t-\frac{x}{v})$$

using the Appendix.

Thus

$$\lim_{h\to\infty}(h\, II_3) = -\pi v^2\,\mathrm{Re}\Big[\frac{Q(v)}{s_1^2(v)}\,\big(\frac{1}{v}\cosh[s_1(v)(t-\frac{x}{v})]\big)\,\delta(t-\frac{x}{v})$$

$$+ \frac{s_1(v)}{v}\sinh[s_1(v)(t-\frac{x}{v})]\,\epsilon(t-\frac{x}{v}) - \frac{1}{v}\delta(t-\frac{x}{v})\Big].$$

The last term is found using the Appendix. Using $F(x)\,\delta(x) = F(o)\,\delta(x)$, we get

$$\lim_{h\to\infty}(h\, II_3) = -\pi v\,\mathrm{Re}\Big[\frac{Q(v)}{s_1(v)}\sinh[s_1(v)(t-\frac{x}{v})]\,\epsilon(t-\frac{x}{v})\Big]\,.$$

The final result is then

$$f_{i,e}(x,v,t) = \Delta n \frac{1}{v} g_{i,e}(v)\,\delta(t-\frac{x}{v}) \pm \frac{\Delta n\,\omega^2_{pi,e}}{\pi} f'_{oi,e}(v)$$

$$\left\{ \mathrm{Im}\, P \int_{\frac{x}{t}}^{\infty} \frac{\gamma}{\gamma - v} \frac{Q(\gamma)}{s_1(\gamma)} \sinh[s_1(\gamma)(t-\frac{x}{\gamma})]\,d\gamma \right.$$

$$\left. - \pi v \,\mathrm{Re}\,(\frac{Q(v)}{s_1(v)} \sinh[s_1(v)(t-\frac{x}{v})])\,\epsilon(t-\frac{x}{v}) \right\} \qquad (27)$$

or

$$f_{i,e}(x,v,t) = \begin{cases} \Delta n \frac{1}{v} g_{i,e}(v)\,\delta(t-\frac{x}{v}) \pm \frac{\Delta n\,\omega^2_{pi,e}}{\pi} f'_{oi,e}(v) \\ \quad \left\{ \mathrm{Im} \int_{\frac{x}{t}}^{\infty} \frac{\gamma}{\gamma-v} \frac{Q(\gamma)}{s_1(\gamma)} \sinh[s_1(\gamma)(t-\frac{x}{\gamma})]\,d\gamma \right\} & v < \frac{x}{t} \\ \text{not defined} & v = \frac{x}{t} \\ \Delta n \frac{1}{v} g_{i,e}(v)\,\delta(t-\frac{x}{v}) \mp \frac{\Delta n\,\omega^2_{pi,e}}{\pi} f'_{oi,e}(v) \\ \quad \left\{ \mathrm{Im} \int_{-\infty}^{\frac{x}{t}} \frac{\gamma}{\gamma-v} \frac{Q(\gamma)}{s_1(\gamma)} \sinh[s_1(\gamma)(t-\frac{x}{\gamma})]\,d\gamma \right\} & v > \frac{x}{t} \end{cases} \qquad (28)$$

For $t \to 0$ we find, using (27)

$$\lim_{t\to 0} f_{i,e}(x,v,t) = \Delta n \frac{1}{v} g_{i,e}(v)\,\delta(-\frac{x}{v})$$

$$= \Delta n\, g_{i,e}(v)\,\delta(x).$$

The result satisfies the initial condition.

3.3. The Flux, $F_{i,e}(x,t)$

We have $F_{i,e}(x,t) = \lim_{h\to\infty} [F_{ai,e}(x,t) - F_{bi,e}(x,t)]$.

Using (24) and following the method in the previous section we find

$$F_{i,e}(x,t) = \Delta n \left\{ \frac{x}{t^2} g_{i,e}(\frac{x}{t}) \mp \frac{1}{2\pi d^2_{i,e}} \int_{\frac{x}{t}}^{\infty} \gamma^2 \frac{P_{i,e}(\gamma)}{s_1(\gamma)} \sinh[s_1(\gamma)(t-\frac{x}{\gamma})]\,d\gamma \right\}. \qquad (29)$$

3.4. The Electric Field, $E(x,t)$

We find

$$E(x,t) = -\frac{\Delta n\, e}{\epsilon_o} \left\{ \int_{\frac{x}{t}}^{\infty} (g_i(\gamma) - g_e(\gamma))\,d\gamma \right.$$

$$\left. - \frac{1}{2\pi d_i^2}\, \mathrm{Im} \int_{\frac{x}{t}}^{\infty} \gamma^2 \frac{R(\gamma)}{s_1^2(\gamma)} (\cosh[s_1(\gamma)(t-\frac{x}{\gamma})] - 1)\,d\gamma \right\}. \qquad (30)$$

Using (11)

$$\int_{\frac{x}{t}}^{\infty} (g_i(\gamma) - g_e(\gamma))\,d\gamma = \frac{1}{2}\left(\mathrm{erfc}\left(\frac{\frac{x}{t} - v_i}{\sqrt{2}\,A_i}\right) - \mathrm{erfc}\left(\frac{\frac{x}{t} - v_e}{\sqrt{2}\,A_e}\right)\right) .$$

3.5. The Potential, $\varphi(x,t)$

$$\varphi(x,t) = -\frac{\Delta n\, e}{\epsilon_o} \left\{ \int_{\frac{x}{t}}^{\infty} \gamma(t-\frac{x}{\gamma})(g_i(\gamma) - g_e(\gamma))\,d\gamma \right.$$

$$\left. - \frac{1}{2\pi d_i^2}\, \mathrm{Im} \int_{\frac{x}{t}}^{\infty} \gamma^3 \frac{R(\gamma)}{s_1^3(\gamma)} (\sinh[s_1(\gamma)(t-\frac{x}{\gamma})] - s_1(\gamma)(t-\frac{x}{\gamma}))\,d\gamma \right\} \qquad (31)$$

where

$$\int_{\frac{x}{t}}^{\infty}\gamma\left(t-\frac{x}{\gamma}\right)\left(g_i(\gamma)-g_e(\gamma)\right)d\gamma =$$

$$\frac{t}{\sqrt{2}}\left(A_i \exp\left(-\left(\frac{\frac{x}{t}-v_i}{\sqrt{2}\,A_i}\right)^2\right) - A_e \exp\left(-\left(\frac{\frac{x}{t}-v_e}{\sqrt{2}\,A_e}\right)^2\right)\right)$$

$$+\frac{1}{2}\left((tv_i-x)\,\mathrm{erfc}\left(\frac{\frac{x}{t}-v_i}{\sqrt{2}\,A_i}\right) - (tv_e-x)\,\mathrm{erfc}\left(\frac{\frac{x}{t}-v_e}{\sqrt{2}\,A_e}\right)\right).$$

4. APPROXIMATE SOLUTIONS

In this chapter we want to give a simplified description of the problem treated in the previous chapters. For this purpose we assume the electrons to be Boltzmann distributed at all times so the linearized Vlasov equation for the electrons is replaced by

$$n_e = n_o\, e^{e\varphi/\varkappa T_e} \qquad \text{or} \qquad E = -\frac{\varkappa T_e}{e n_o}\frac{\partial n_e}{\partial x}, \qquad T_e \text{ const.}$$

The linearized Poisson equation will be replaced by

$$E = -\frac{\varkappa T_e}{e n_o}\frac{\partial n_i}{\partial x} + d_e^2\,\frac{\partial^2 E}{\partial x^2}$$

where

$$d_e^2 = \frac{\epsilon_o \varkappa T_e}{e^2 n_o}.$$

If we further assume quasi-neutrality ($n_e \simeq n_i$), the original set of equations ((1), (2)) reduces to

$$\frac{\partial f_i}{\partial t} + v\frac{\partial f_i}{\partial x} = C_e^2\,\frac{\partial n_i}{\partial x}\,f'_{o,i}(v) \qquad (32)$$

where

$$C_e^2 = \frac{\varkappa T_e}{m_i}.$$

The assumption of quasi-neutrality implies $d_e^2 \simeq 0$. This assumption will be justifiable for low electron temperatures, T_e, and high background densities, n_o, if $\frac{\partial^2 E}{\partial x^2}$ is not too large, but also if the initial condition is such that the E-field varies smoothly ∂: $\frac{\partial^2 E}{\partial x^2} \ll 1$. The assumption of Boltzmann distributed electrons implies the assumption $m_e \simeq 0$. This assumption is justified if the initial condition involves a perturbation of the ions. This is verified numerically in ref. 1 for the special case considered there. We shall now consider (32) in detail. The solution to this equation with the initial conditions of interest here is given in refs. 3, 4, and 5. We only show the results:

For $\quad f_i(x, v, t=0) = \Delta n\, g(v)\,(1-\epsilon(x))$

the solution to (32) is

$$f_i(x,v,t) = C_e^2\,\Delta n\, f'_o(v)\, P\int_{\frac{x}{t}}^{\infty}\frac{h(\gamma)}{v-\gamma}\,d\gamma + \left\{\Delta n\, g(v) - C_e^2\,\Delta n\, f'_o(v)\, P\int_{-\infty}^{\infty}\frac{h(\gamma)}{v-\gamma}\,d\gamma\right\}\epsilon\left(v-\frac{x}{t}\right) \qquad (33)$$

$$n_i(x,t) = \Delta n\int_{\frac{x}{t}}^{\infty} h(\gamma)\,d\gamma \qquad (34)$$

$$F_i(x,t) = \Delta n\int_{\frac{x}{t}}^{\infty}\gamma\, h(\gamma)\,d\gamma \qquad (35)$$

$$E(x,t) = \frac{\Delta n}{t}\,\frac{\varkappa T_e}{e n_o}\,h\left(\frac{x}{t}\right) \qquad (36)$$

$$\varphi(x,t) = \Delta n\,\frac{\varkappa T_e}{e n_o}\int_{\frac{x}{t}}^{\infty} h(\gamma)\,d\gamma\,. \qquad (37)$$

By differentiation with respect to x we find the solution for the case where

$$f_i(x, v, t = 0) = \Delta n\, g(v)\, \delta(x)$$

to be

$$f_i(x, v, t) = \Delta n \frac{1}{t} g(v)\, \delta(v - \frac{x}{t}) + \Delta n\, C_e^2 f'_o(v) \frac{h(\frac{x}{t})}{vt - x} \tag{38}$$

$$n_i(x, t) = \frac{\Delta n}{t} h(\frac{x}{t}) \tag{39}$$

$$F_i(x, t) = \frac{\Delta n}{t} (\frac{x}{t}) h(\frac{x}{t}) \tag{40}$$

$$E(x, t) = \frac{\Delta n}{t} \frac{\varkappa T_e}{e n_o} \frac{\partial h(\frac{x}{t})}{\partial x} \tag{41}$$

$$\varphi(x, t) = \Delta n \frac{\varkappa T_e}{e n_o} \frac{1}{t} h(\frac{x}{t}) . \tag{42}$$

In (33)-(42) $\epsilon(x)$ denotes Heaviside's step function, $\int_{-\infty}^{\infty} g(v)dv = 1$ and

$$h(\Upsilon) = \frac{1}{\pi} \operatorname{Im} \left[\frac{P \int_{-\infty}^{\infty} \frac{g(v)}{v - \Upsilon} dv + i\pi g(\Upsilon)}{1 - C_e^2 P \int_{-\infty}^{\infty} \frac{f'_o(v)}{v - \Upsilon} dv - i\pi C_e^2 f'_o(\Upsilon)} \right] .$$

We notice that the solutions are self-similar, i.e. of the form

$$\frac{1}{t^{\alpha}} H(\frac{x}{t}) .$$

It should be mentioned that the free-streaming contribution in these cases is always self-similar. In the following we shall only consider (33)-(37). Figs. 8a-e show numerical calculations of $h(\Upsilon)$, $n_i(\frac{x}{t})$, $F_i(\frac{x}{t})$, and $f(\frac{x}{t}, v)$. In the calculations we assumed drifting Maxwellians for $f_o(v)$ and $g(v)$. We chose slightly different drift velocities and temperatures for

$f_o(v)$ and $g(v)$ since otherwise the figures would show misleading symmetries. We notice that the E-field is positive for all (x, t) unless we choose distributions $g(v)$ and $f_o(v)$ which have a difference in drift velocities which is great compared to their thermal velocities. The main effect of collective interaction will therefore be an acceleration of all ions, so $f_i(\frac{x}{t}, v)$ is negative for $v < v_{Drift}$ and positive for $v > v_{Drift}$, as shown in fig. 8e. We notice again that $f_i(\frac{x}{t}, v)$ is not defined for $v = \frac{x}{t}$. Close to $\frac{x}{t}$ the dependence on v is roughly given by $\ln(|v - \frac{x}{t}|)$ so f_i is integrable with respect to v, and this is all that is needed as mentioned in section 2.2. What is worse: for $v \to \frac{x}{t}$, $f \to \infty$ so the assumption of linearization ($\Delta n f_i \ll n_o f_o$) breaks down in this region. We shall show that this is due to our unphysical initial condition which implies an infinite electric field for $t = 0$.

We consider a more realistic initial condition, namely

$$f_i(x, v, t = 0) = \frac{\Delta n\, g(v)}{\exp(\frac{x}{d}) + 1}$$

where d may represent the physical dimension of the exiter, e.g. a grid. A detailed calculation gives

$$n_i(x, t) = \int_{-\infty}^{\infty} \frac{h(\Upsilon)}{\exp(\frac{x - t\Upsilon}{d}) + 1} d\Upsilon \tag{43}$$

$$f_i(x, v, t) = \frac{\Delta n\, g(v)}{\exp(\frac{x - vt}{d}) + 1} +$$

$$C_e^2 \Delta n f'_o(v)\, P \int_{-\infty}^{\infty} \frac{h(\Upsilon)}{v - \Upsilon} \frac{d\Upsilon}{\exp(\frac{x - t\Upsilon}{d}) + 1} - \frac{C_e^2 \Delta n f'_o(v)}{\exp(\frac{x - tv}{d}) + 1} P \int_{-\infty}^{\infty} \frac{h(\Upsilon)}{v - \Upsilon} d\Upsilon . \tag{44}$$

The solution for the initial condition

$$f_i(x, v, t = 0) = \frac{1}{d} \frac{\Delta n\, g(v) \exp(x/d)}{(\exp(x/d) + 1)^2}$$

may be found by differentiation (remember the "minus"-sign). f(x, v, t) is limited and continuous for all (x, v, t) so if Δn is small enough, the linearization is justified. If we insert $d = \frac{1}{n}$ and let n take the values 1, 2, 3, ..., we get a sequence of functions which converge to the solution given in (33) since $\lim_{d\to o}(\exp(x/d) + 1)^{-1} = (1 - \epsilon(x)) = \epsilon(-x)$. Consideration of the characteristics in the v, x/t plane offers a convenient way of comparing the linear results with the non-linear numerical results shown in ref. 6. The linear calculations give a separatrix v = x/t which divides the v, x/t plane into two halves. Such a separatrix v = S(x/t) is recovered in the non-linear calculations. v = S(x/t) lies entirely above v = x/t, but has v = x/t as an asymptote for large v. S(x/t) - x/t will be small as long as $T_e \simeq T_i$. There is overall agreement between the linear and non-linear calculations of the distribution function, density, and E-field. We therefore conclude that the results obtained from a linear calculation are sufficient also in the non-linear region as long as $T_e \simeq T_i$. This is confirmed in the experiments reported in ref. 3. For high temperature ratios, T_e/T_i, the calculations break down since we expect collisionless shock formation in this region.

ACKNOWLEDGEMENT

This work was performed in part under the contract of association between the Danish Atomic Energy Commission and Euratom.

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APPENDIX

Fourier transformation of Heaviside's step function $\epsilon(t - \frac{x}{v})$

$$f(p) = \int_{-\infty}^{\infty} e^{ipx}\,\epsilon(t - \frac{x}{v})dx = \int_{-\infty}^{vt} e^{ipx}dx = \frac{e^{ipvt}}{ip}$$

and

$$\epsilon(t - \frac{x}{v}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(ip(vt - x))}{ip}\,dp \,.$$

For $x \rightarrow x + 1/h$

$$\epsilon(t - \frac{x + 1/h}{v}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(ip(vt - x - 1/h))}{ip}\,dp$$

thus

$$\epsilon(t-\frac{x}{v}) - (t - \frac{x + 1/h}{v}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1 - \exp(-ip/h))}{ip}\,\exp(ip(vt - x))dp \,.$$

For $h \rightarrow \infty$ we find

$$\lim_{h \rightarrow \infty} h(\epsilon(t - \frac{x}{v}) - \epsilon(t - \frac{x + 1/h}{v})) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ip(vt - x))dp = \delta(vt - x) \,.$$

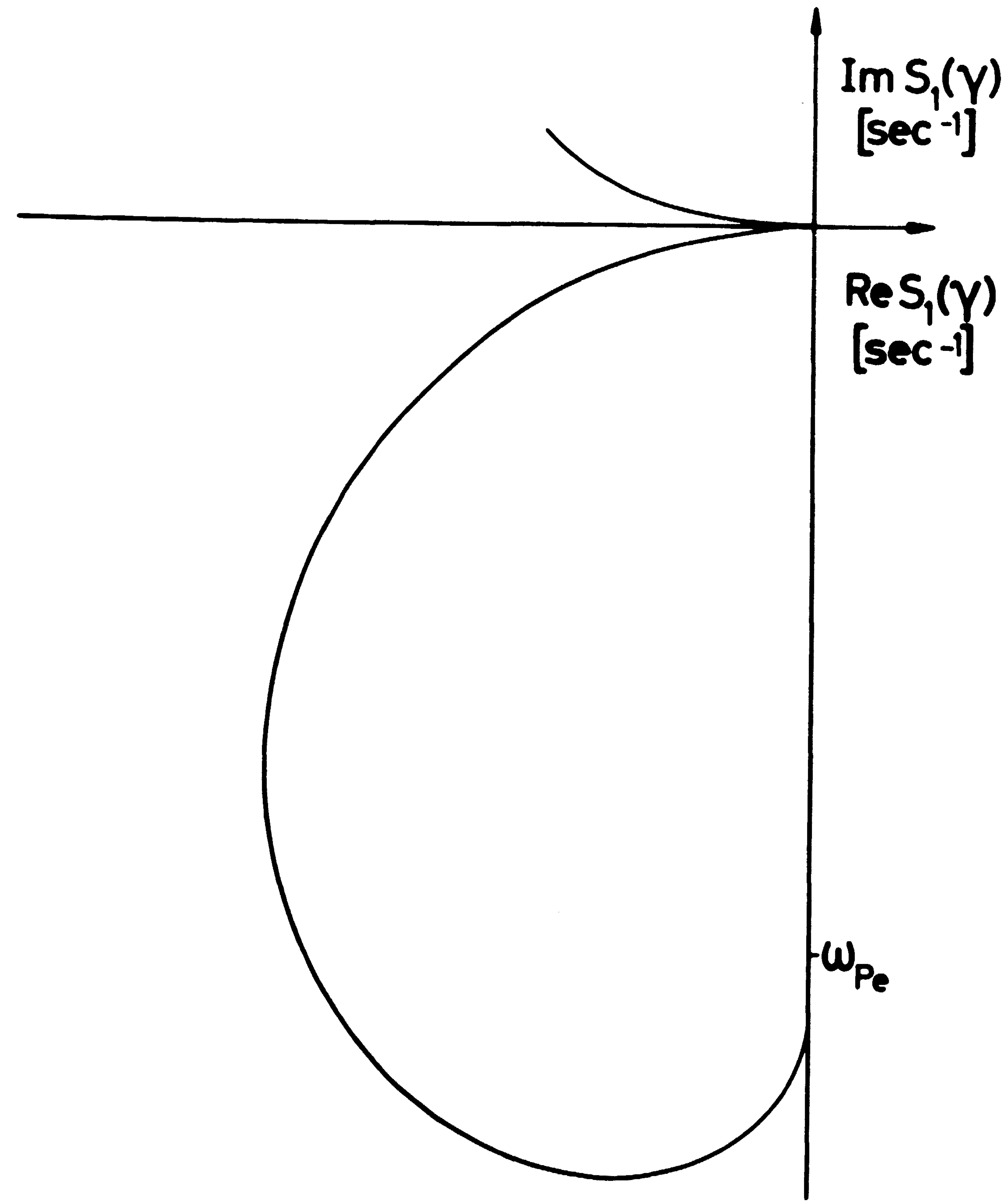

Fig. 3a. The function $s_1(\gamma)$.

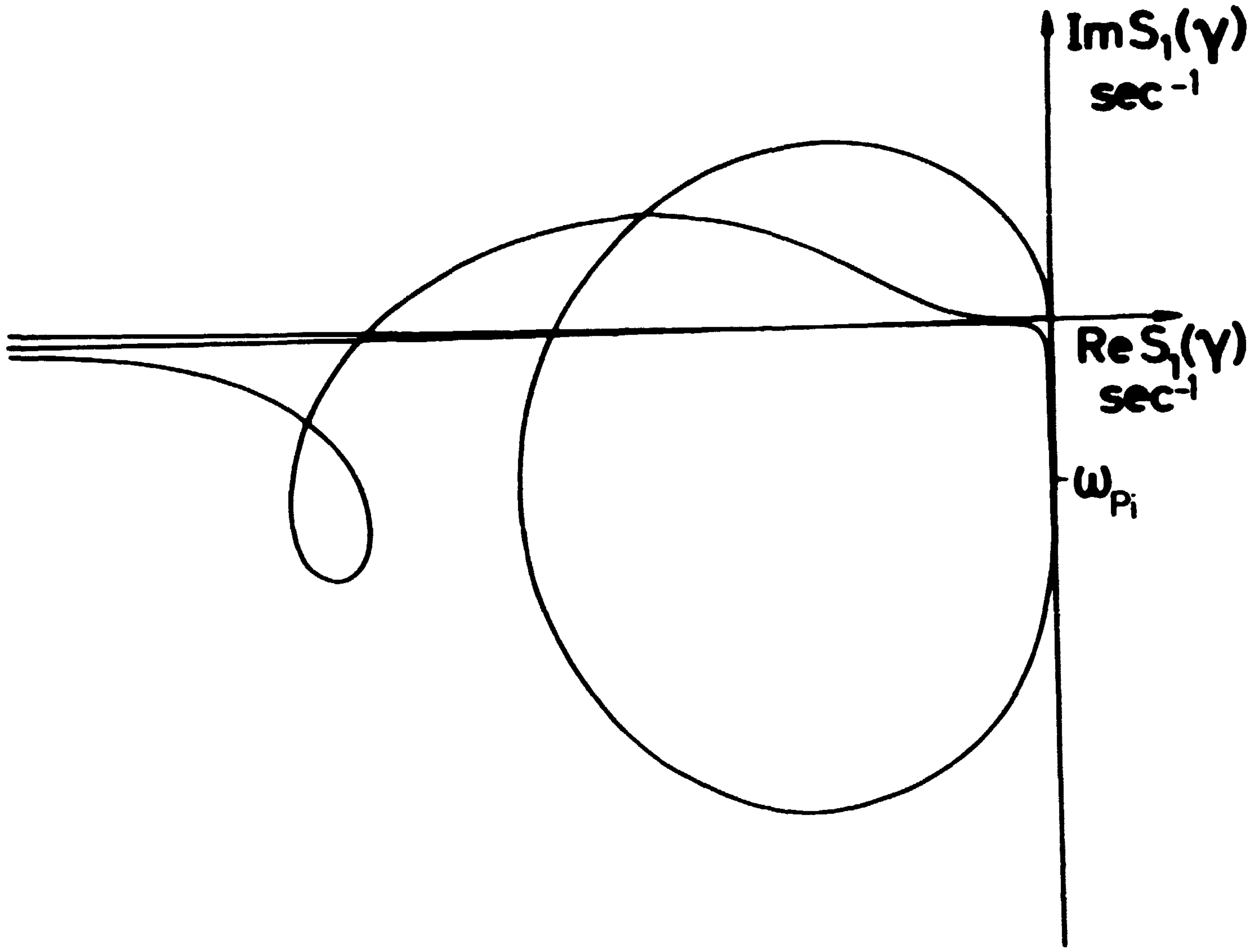

Fig. 3b. Enlarged reproduction of the region close to the origin. The scale on the axis is indicated by the plasma frequency.

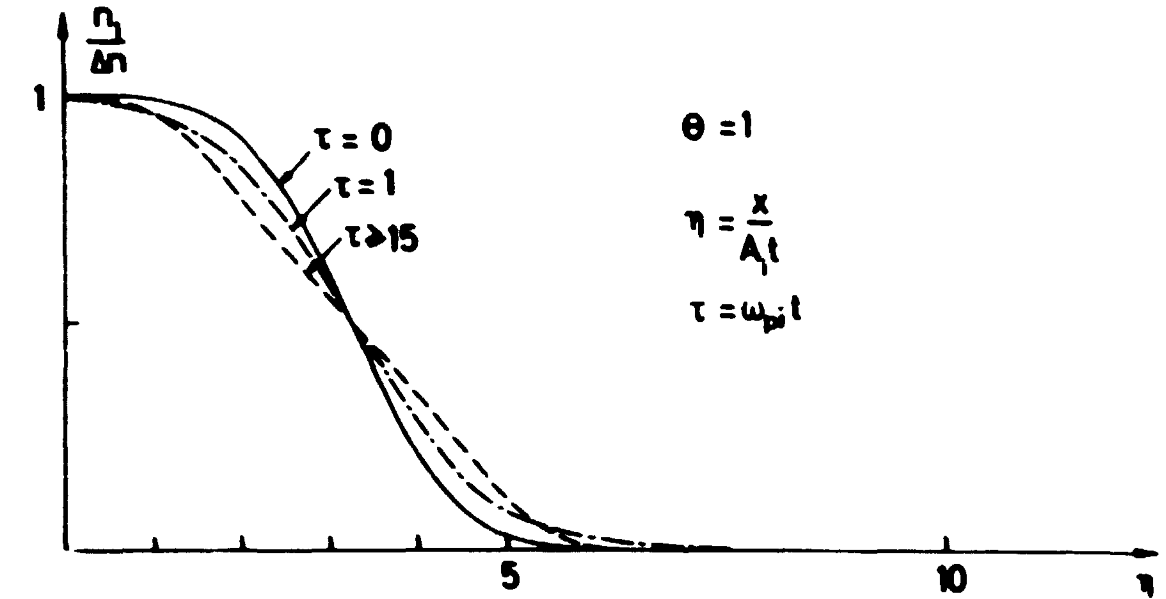

Fig. 5a. The perturbed density as a function of η and with τ as a parameter. The drift velocities in the background plasma, described by $f_{oi,e}(v)$, are $v_{oi} = 1200$ m/s, $v_{oe} = 0$ m/s and in the perturbation, $g_{i,e}(v)$; $v_i = 1200$ m/s, $v_e = 0$ m/s. The temperatures are $T_{oi} = 2200$ K and $T_i = 2000$ K. $T_{oe}/T_{oi} =$ [illegible] The background density is $n_o = 10^{14}$ m^{-3}, the ion mass $2.21 \cdot 10^{-25}$ kg (Cs). Notice the self similarity for $\tau \geq 15$.

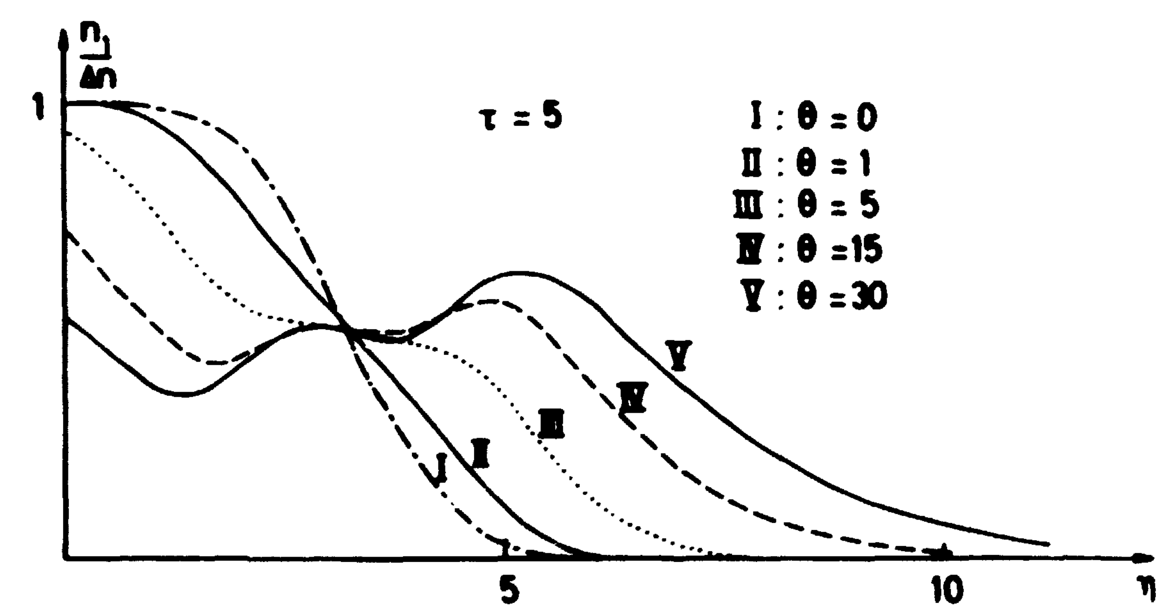

Fig. 5b. The perturbed density as a function of η for various θ with $\tau = 5$. Plasma parameters as in fig. 5a. Related (nonlinear) numerical calculations are shown in ref. 7. We notice that the linear results seem to be a fairly good approximation, way into the nonlinear regime provided that T_e/T_i is not too large.

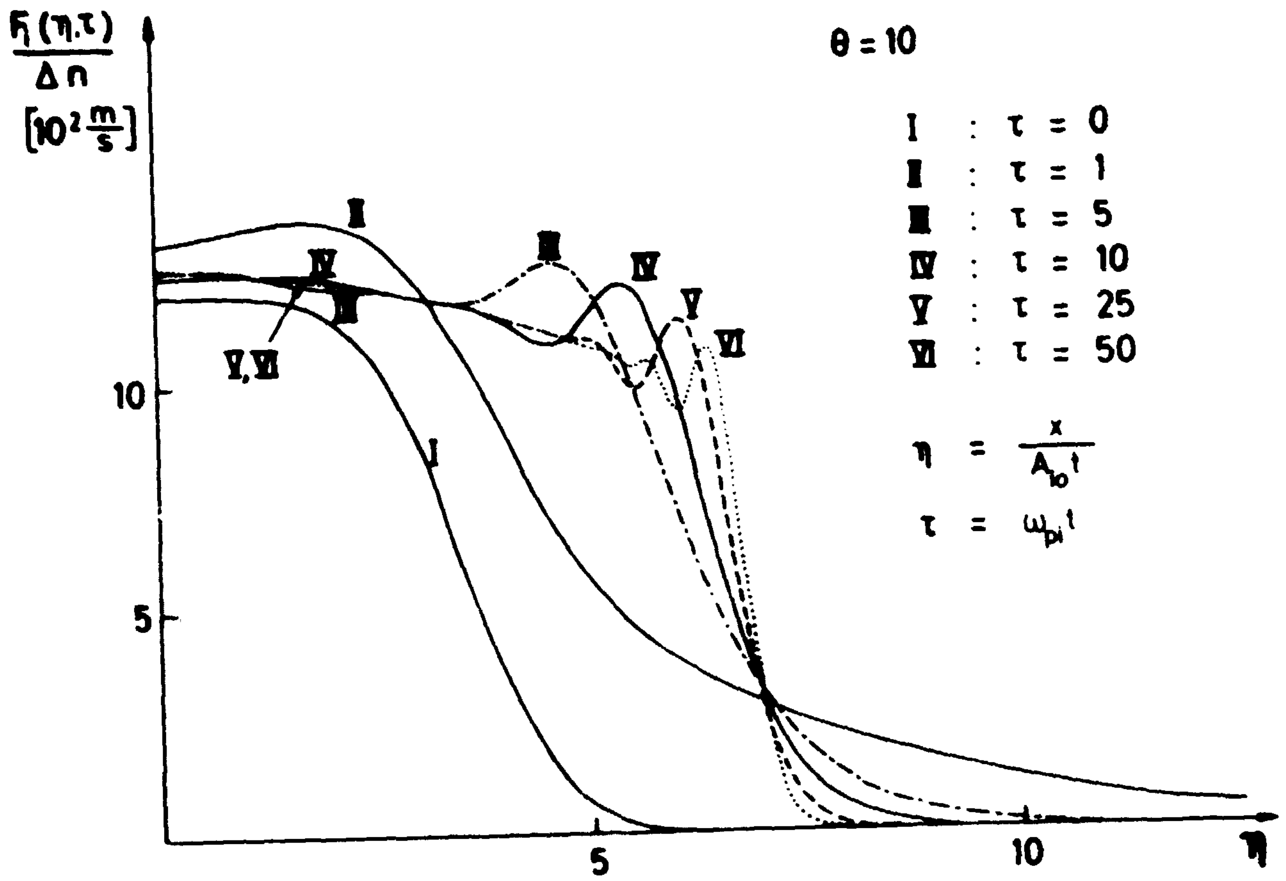

Fig. 7a. The perturbed flux as a function of η and with τ as a parameter. The plasma parameters are the same as in fig. 5.

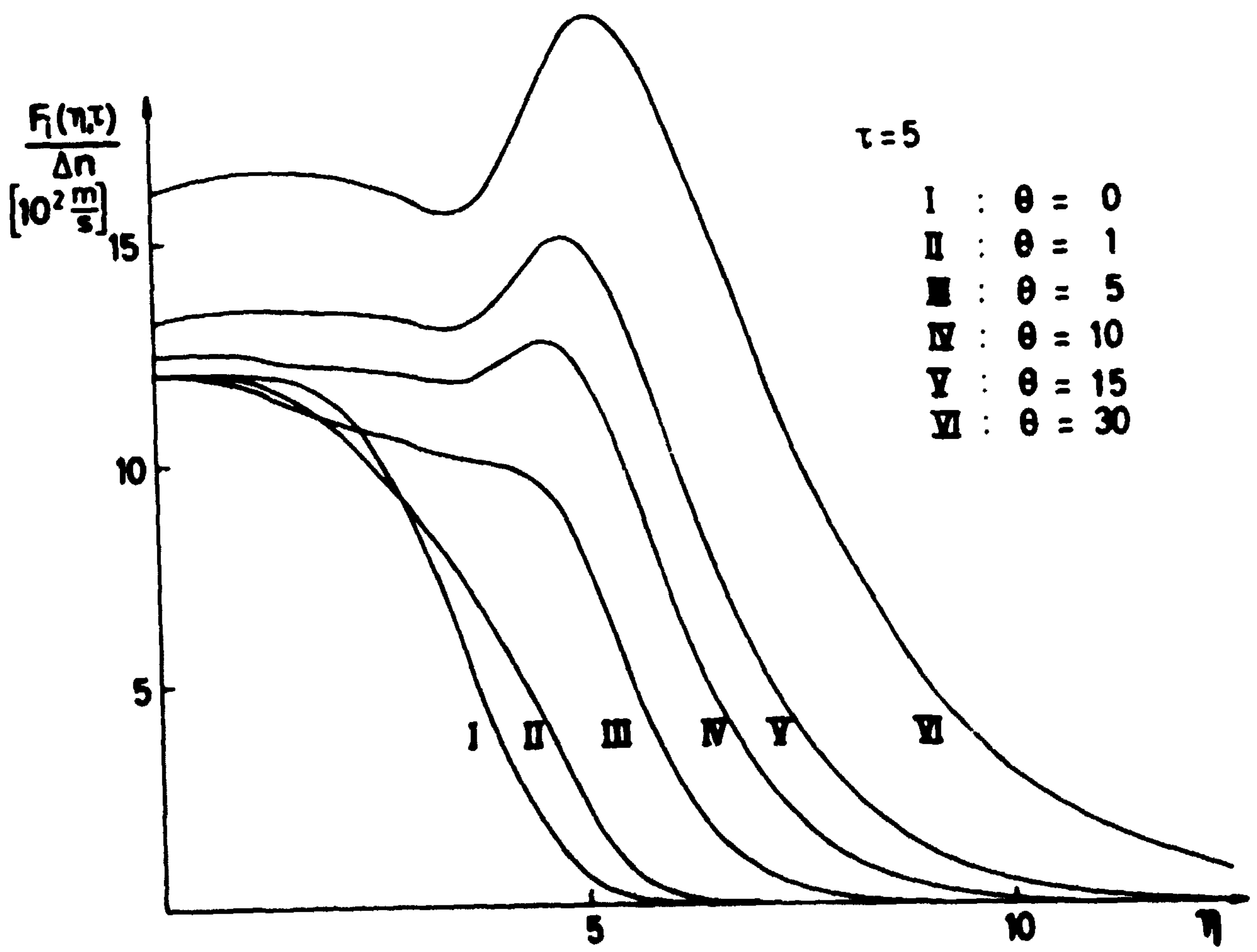

Fig. 7b. The perturbed flux as a function of η for various θ with $\tau = 5$. Measurements related to figs. 7a, 7b are reported by V. Vanek and T. C. Marshall[8].

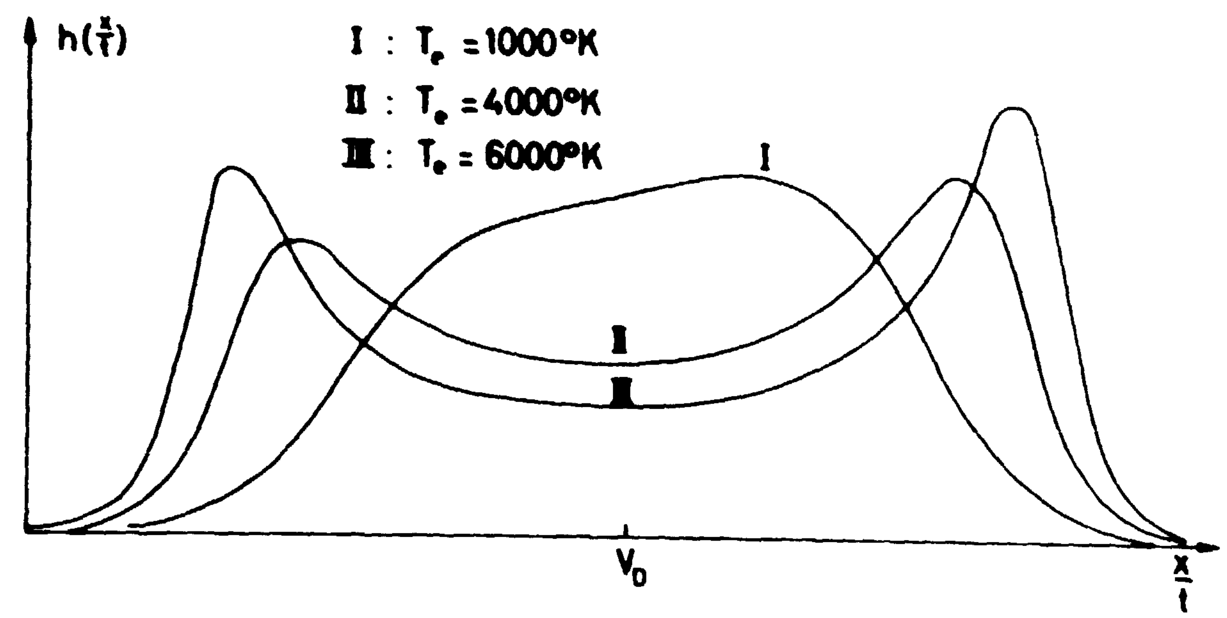

Fig. 8a. The function h(x/t) for different electron temperatures, T_e. The plasma parameters are: $v_{oi} = v_o = 1340$ m/s, $v_i = 1400$ m/s, $T_{io} = T_i = 2000$ K. The values are taken from the experiment reported in ref. 5. The background density, n_o, does not appear in this approximation.

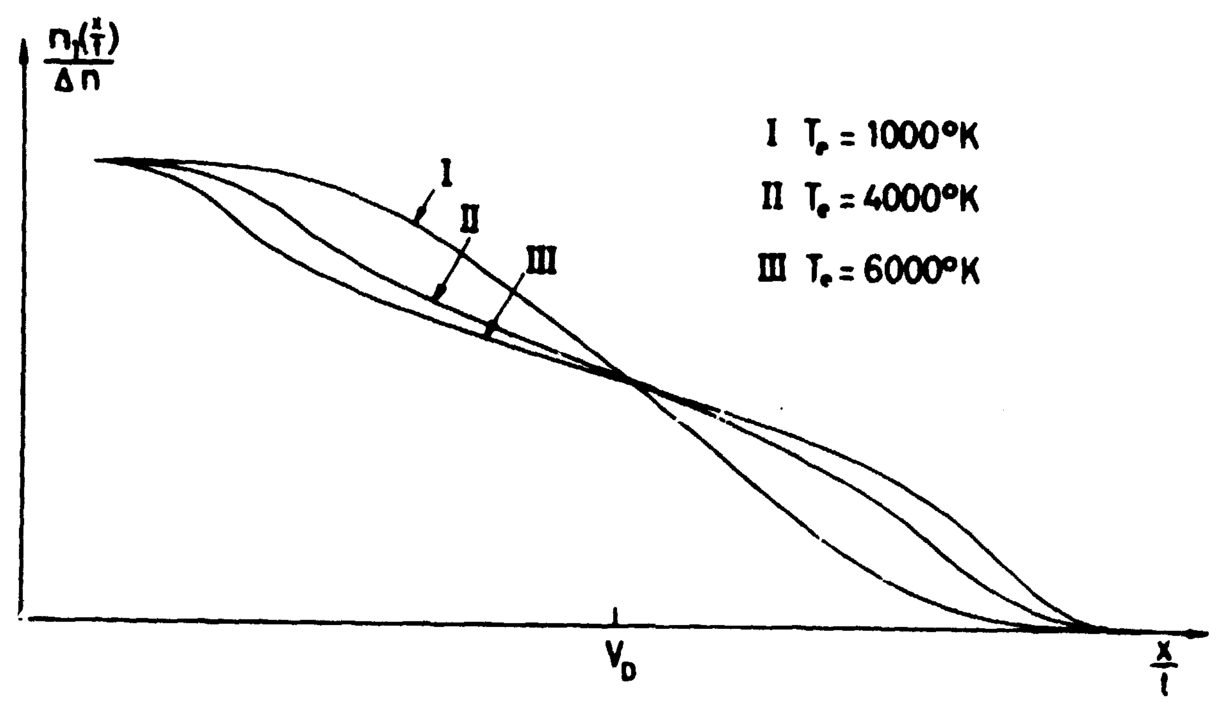

Fig. 8b. The perturbed density, n(x/t), for three values of T_e.

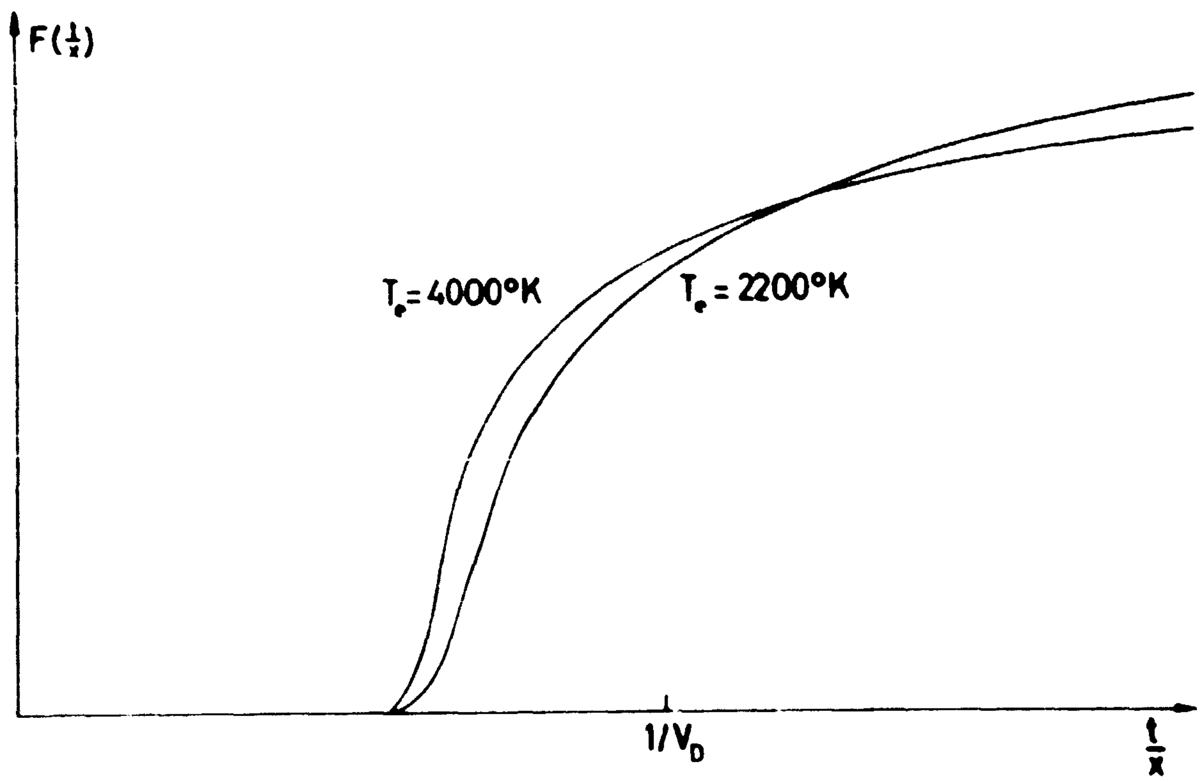

Fig. 8c. The perturbed flux, F(t/x). (The t/x dependence allows direct comparison with measurements. (See refs. 5 and 9).

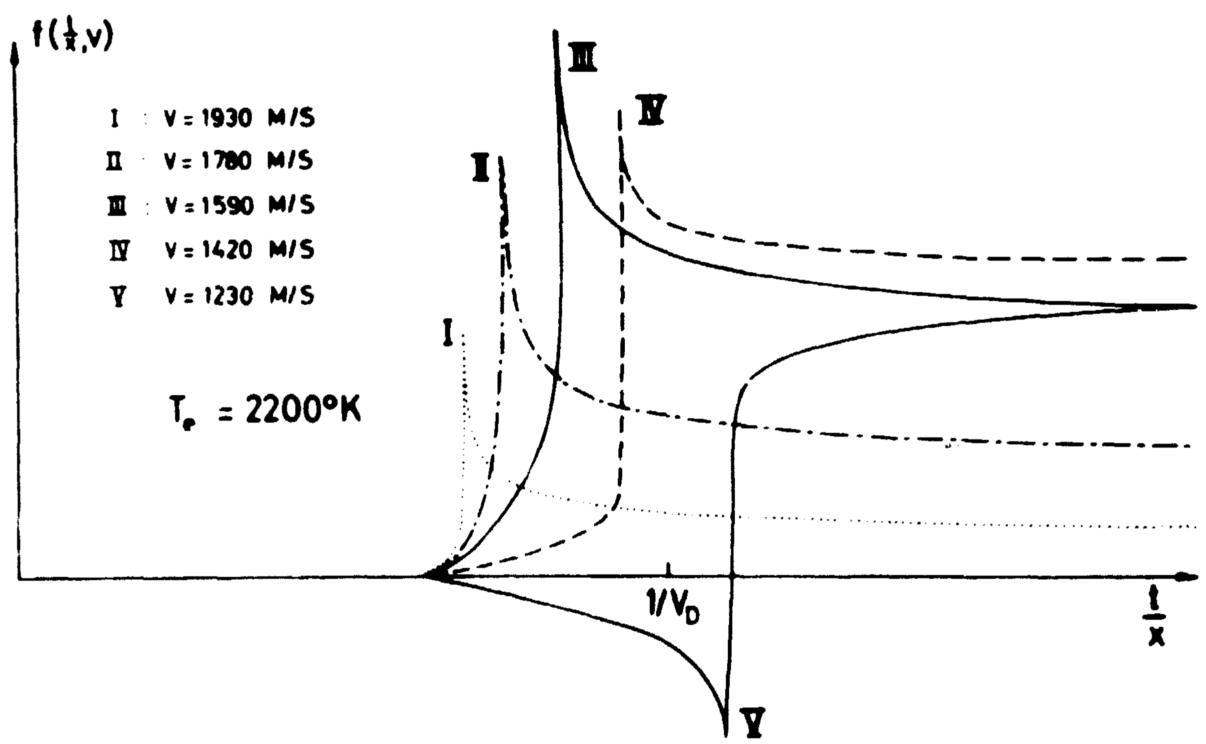

Fig. 8d. The perturbed ion distribution function, f(t/x, v), as a function of t/x and with v as a parameter.

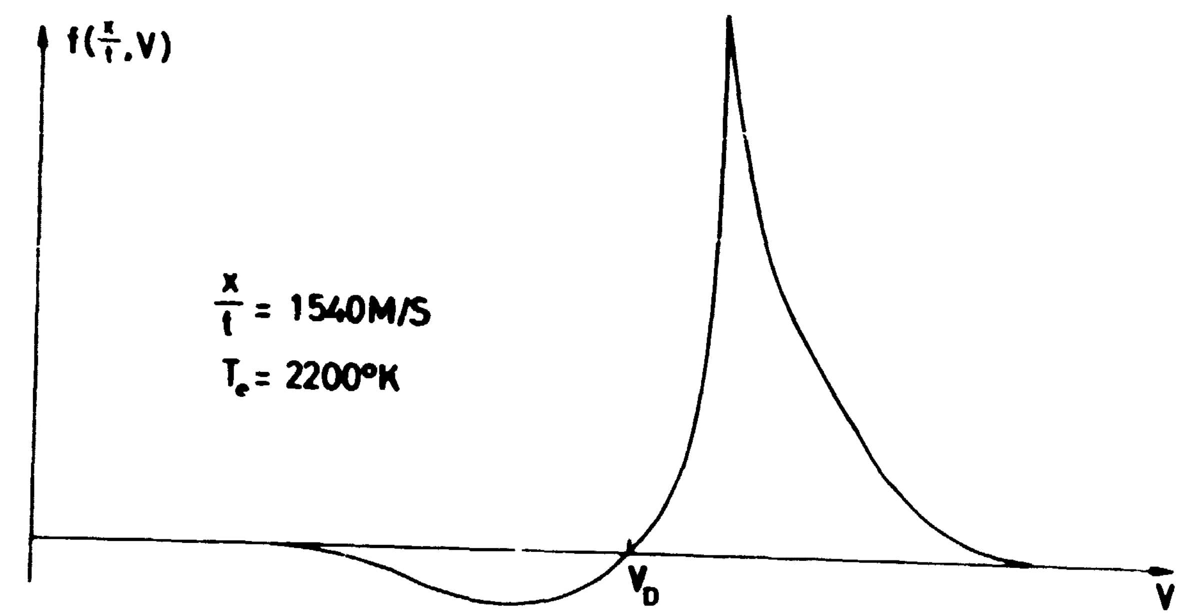

Fig. 8e. The perturbed ion distribution function, f(x/t, v), as a function of v and with x/t as a parameter.