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*Publication date:*
1980

*Document Version*
Publisher's PDF, also known as Version of record

Link back to DTU Orbit

*Citation (APA):*
Janssen, P. A. E. M., & Juul Rasmussen, J. (1980). Limit cycle behaviour of the bump-on-tail and ion-acoustic instability. (Risø-M; No. 2235).

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RISØ-M-2235

# LIMIT CYCLE BEHAVIOUR OF THE BUMP-ON-TAIL AND ION-ACOUSTIC INSTABILITY

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Abstract. The nonlinear dynamics of the bump-on-tail and current-driven ion-acoustic instability is considered. The eigenmodes have discrete k because of finite periodic boundary conditions. Increasing a critical parameter (the number density and the electron drift velocity respectively) above its neutral stable value by a small fractional amount $\Delta^2$, one mode becomes unstable. The nonlinear dynamics of the unstable mode is determined by means of the multiple time scale method. Usually, limit cycle behaviour is found. A short comparison with quasi-linear theory is given, and the results are compared with experiment.

UDC 533.951

December 1980
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ISBN 87-550-0677-9
ISSN 0418-6435

Risø Repro 1981

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## 1. INTRODUCTION

Recently, Simon and Rosenbluth[1] have considered the problem of single mode saturation of the one-dimensional bump-on-tail instability (mobile and immobile ions). The final state of the plasma, which consists of a modified equilibrium plus a steady-state oscillation and its higher harmonics, was determined by means of time-asymptotic analysis[2] which is a generalization of the methods of Bogoliuboff and co-workers[3]. The authors of Ref. 1 did however not prove that the system can indeed get from an initially linearly unstable state to this unique final steady state (limit cycle behavior). And indeed, there are examples for which there is no path in time between initial and final steady state, e.g. a) the $m = 1$ kink modes in a sharp boundary plasma pinch[4], and b) the $\vec{g} \times \vec{B}$ instability in a collisionless Finite Larmor Radius Plasma[5]. In those examples the nonlinear dynamics of the system gives rise to a modulation in the amplitude $\Gamma$ of the linearly unstable mode, instead of limit cycle behavior. In this paper we show that the dynamical equation for the bump-on-tail instability is given by the well-known nonlinear Landau equation

$$\frac{\partial}{\partial t} \Gamma = \gamma\Gamma - \beta\Gamma|\Gamma|^2, \tag{1}$$

where $\Gamma$ is the complex amplitude of the linearly unstable mode, and $\gamma$ and $\beta$ are complex coefficients. Clearly, Eq. (1) exhibits limit cycle behavior. We note that the amplitude of the Van der Pol oscillator is determined by Eq. (1), and that the suppression of plasma oscillations in a beam-plasma system by an external oscillation is well described by the Van der Pol equation with a driving term[6-9].

Another reason for the interest of the dynamical behavior of instabilities is that accurate measurements of the growth rate can be performed by making explicit use of the particular properties of the dynamics. For instance, Michelsen et al.[8] determined the growth rate of the current-driven, ion-acoustic instability in a single-ended Q-machine by means of suppression of the instability by an external oscillation. The fluctuating density was assumed to be determined by a Van der Pol equation. For this reason the ion-acoustic instability is treated in this paper also.

The plan of the paper is as follows: In Section II the linear theory of the bump-on-tail instability is presented. The eigenmodes have discrete k because of periodic boundary conditions. The critical parameter in this problem is the (electron) number density. Increasing this number density above its neutral stable value by a small fractional amount $\Delta^2$, one mode moves up to the positive slope region. This mode has a small growth rate of the order $\Delta^2$. Thus, two time scales can be distinguished in the problem of the dynamical behavior of the bump-on-tail instability, and therefore this is an appropriate opportunity to solve this problem by means of the multiple time scale method. Only the quasi-linear approximation is considered, i.e. the effect of higher harmonics is neglected. The solution is shown to conserve particles, energy and momentum. The theoretical results are compared with experiments (Sect. III). In Section IV we outline the nonlinear theory of the current-driven, ion-acoustic instability. A more natural critical parameter in this case is the electron drift velocity, at least in the limit of small wavenumbers (i.e. $k\lambda_D \ll 1$, where $\lambda_D$ is the electron Debye length).

Once again we find limit cycle behavior for the unstable mode, and the theoretical fluctuation level is compared with experiment.

In Section V we discuss our results, especially regarding the bump-on-tail instability. Conservation of the "microscopic" entropy is proved, and a short comparison with the quasi-linear theory of Drummond and Pines[10] is made (note: quasi-linear theory is not to be confused with the quasi-linear approximation). In addition, we compare two instabilities in a collisionless plasma: the bump-on-tail versus the $\vec{g} \times \vec{B}$ instability. The behavior of the dispersion relation near the threshold for instability is especially investigated. We finally discuss the relation between the dynamics of the Van der Pol oscillator and the bump-on-tail as well as the ion-acoustic instability. In Section VI our conclusions are summarized. It should be noted that we shall not go into mathematical detail, since a large portion of the calculations is already presented in Ref. 1. For instance, Ref. 1 must be consulted for a procedure to deal with integrals in which products of generalized functions occur.

## II. LINEAR THEORY OF THE BUMP-ON-TAIL INSTABILITY

Consider a one-dimensional collisionless plasma of electrons and immobile ions with a uniform density. The equations are the Vlasov-Poisson set:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{e}{m_e}\frac{\partial}{\partial x}\phi\frac{\partial}{\partial v}\right) f = 0,$$
$$-\frac{\partial^2}{\partial x^2}\phi = \frac{eN_o}{\varepsilon_o}\left(1 - \int_{-\infty}^{\infty} dv f\right), \tag{2}$$

where $f$ is the electron distribution function, $\phi$ the electrostatic potential, $-e$ the electron charge, $m_e$ its mass, and $N_o$ the ion number density. The plasma has a finite length L and periodic boundary conditions are imposed. It is clear that $f = f_o(v)$, $\phi$ = constant is a static solution if $\int f_o(v)dv = 1$, and let us assume this equilibrium to be of the bump-on-tail type (see Fig. 1).

Linearizing Eq. (2) around this static equilibrium we obtain for normal modes the linear problem

$$(\omega + kv)f_1 + \frac{ek}{m_e}\phi_1 \frac{\partial}{\partial v}f_o = 0,$$
$$k^2\phi_1 = -\frac{eN_o}{\varepsilon_o}\int_{-\infty}^{\infty} dv' f_1(v') \ . \tag{3}$$

Here, the normal mode is of the form $\vec{\psi}_1 = \begin{pmatrix}\hat{f}_1\\ \hat{\phi}_1\end{pmatrix} \exp i(\omega t+kx)$ and we have dropped the hats in Eq. (3). Elimination of $\phi_1$ yields an integral equation for $f_1$

$$(v-\nu)f_1 = -\eta(v)\int_{-\infty}^{\infty} dv' f_1(v'), \tag{4}$$

where we have introduced the notations of Case:[11]

$$\nu = -\frac{\omega}{k},\ \eta(v) = -\frac{\omega_p^2}{k^2}\frac{\partial}{\partial v}f_o,\ \omega_p^2 = \frac{e^2N_o}{\varepsilon_o m_e} \ . \tag{5}$$

As is well known, the eigenvalue problem (4) for $\nu$ results in a real continuum (the Van Kampen modes) giving a potential $\phi_1$, which decays in time (Landau damping). In addition, there may be complex values of $\nu$. For such a value, Eq. (4) has the general solution

$$f_1 = -\eta(v)/(v-\nu), \qquad (6)$$

provided $\int dv f_1 = 1$ resulting in the dispersion relation

$$1 + \int_{-\infty}^{\infty} dv \frac{\eta(v)}{v-\nu} = 0 \qquad (7)$$

If a complex solution exists (this may be the case for a velocity distribution of the bump-on-tail type) its complex conjugate also exists thus insuring instability.

In this note we are interested only in the dynamics of the growing mode(s) since the real continuum is Landau damped. A slightly unstable plasma is considered, i.e. a plasma in which only one mode is growing at a small growth rate. To this end we choose the number density

$$N_o = N_c(1 + \Delta^2), \; \Delta^2 << 1, \qquad (8)$$

where $N_c$ is the critical value for which the last pair of unstable modes has just reached the real axis. This critical value of $N_o$ exists by virtue of the assumption of discrete k so that there is only a finite number of unstable modes.

On application of the Plemelj formula to dispersion relation (7), it can easily be shown that for the critical number density

$N_c$ the phase velocity $\bar{v}$ of the marginally stable mode equals the velocity corresponding to the bottom of the well in the distribution function (see Fig. 1), thus

$$\eta(\bar{v}) = 0. \tag{9a}$$

In addition, $\bar{v}$ has to satisfy

$$1 + P\int_{-\infty}^{\infty} dv \, \frac{\eta(v)}{v-\bar{v}} = 0. \tag{9b}$$

The mode corresponding to this phase velocity is the class 1c mode defined by Case[11].

On increasing the number density above its critical value by a small fractional amount $\Delta^2$ [see Eq. (8)], one mode with the smallest possible k moves up to the positive slope region, and this mode has a small growth rate of order $\Delta^2$; namely, from the dispersion relation we obtain

$$\omega = \omega_o + \Delta^2\omega_2,$$

where

$$\omega_c = k\bar{v}, \tag{10}$$

$$\omega_2 = k \frac{\left[- P\int dv \, \frac{\eta^I}{v-\bar{v}} + \pi i \eta^I(\bar{v})\right]}{\pi^2[\eta^I(\bar{v})]^2 + \left(P\int dv \frac{\eta^I}{v-\bar{v}}\right)^2} .$$

Here, the Roman superscript denotes differentiation with respect to v. Hence, two time scales can be distinguished, namely the inverse of the oscillation frequency $\omega_0$ and the inverse of $\Delta^2\omega_2$ (containing both linear growth and linear frequency shift). Therefore, it is tempting to solve the problem of the evolution in time of the slightly unstable mode by means of the multiple time scale method. This is done in the next section.

## III. NONLINEAR THEORY OF THE BUMP-ON-TAIL INSTABILITY

We wish to obtain the nonlinear evolution in time of the slightly unstable mode of the bump-on-tail instability. According to the previous section there is only one growing mode with small growth rate if the number density $N_0$ satisfies Eq. (8). Thus, the Vlasov-Poisson set then becomes

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{e}{m_e}\frac{\partial}{\partial x}\phi\frac{\partial}{\partial v}\right)f = 0,$$
$$-\frac{\partial^2}{\partial x^2}\phi = \frac{eN_c}{\epsilon_0}(1+\Delta^2)\left[1 - \int_{-\infty}^{\infty} dv' f(v')\right]. \tag{11}$$

We solve these equations by means of the multiple time scale technique. To this end we replace the time derivative by a series in $\Delta^2$,

$$\frac{\partial}{\partial t} \Rightarrow \sum_{\ell=0} \Delta^{2\ell}\frac{\partial}{\partial \tau_{2\ell}}, \tag{12a}$$

since according to the analysis in the previous section $\tau_0 = t$, $\tau_2 = O(\Delta^2 t)$. We expand $f$ and $\phi$ in powers of $\Delta$,

$$f = \sum_{\ell=0} \Delta^{\ell} f_{\ell}, \quad \phi = \sum_{\ell=1} \Delta^{\ell} \phi_{\ell} , \tag{12b}$$

a usual assumption in asymptotic theories. The coefficients of expansion $f_\ell$ and $\phi_\ell$ are functions of all $\tau_{2\ell}$, except the equilibrium distribution function $f_o$, which is assumed to be independent of time. Of course, $f_\ell$ ($\ell \geq 1$) is also a function of x and v, and $\phi_\ell$ is also a function of x. Substitution of the expansions (12a)-(12b) in Eqs (11) results in the hierarchy

$$\Delta^{\ell}: \mathscr{L}\vec{\psi}_{\ell} = \vec{S}_{\ell}, \quad \ell = 1,2,3,\ldots, \tag{13}$$

where $\vec{\psi}_\ell = (f_\ell, \phi_\ell)$, and the linear operator $\mathscr{L}$ is of the form

$$\mathscr{L} = \begin{pmatrix} \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} & \frac{e}{m_e}\left(\frac{\partial}{\partial v} f_o\right)\frac{\partial}{\partial x} \\ \\ \frac{eN_c}{\varepsilon_o}\int_{-\infty}^{\infty} dv' & -\frac{\partial^2}{\partial x^2} \end{pmatrix} . \tag{14}$$

The source term $\vec{S}_\ell$ contains only lower order $\vec{\psi}_p$ with $p \leq \ell-1$; $\vec{S}_\ell$ will generate higher harmonics, and may also contain terms which give rise to a secular behavior of $\vec{\psi}_\ell$ within the time scale $\tau_o$. Since many time scales are assumed to be present, there is sufficient freedom to prevent these secular terms from occurring.

In order to obtain a unique solution of the hierarchy (13) initial and boundary conditions have to be specified. At t = 0 we assume that only the growing mode and its higher harmonics are excited. In addition we require periodic boundary conditions in x-space, and that $f_\ell$ vanishes sufficiently rapidly for $v \to \pm \infty$ so that all the velocity integrals exist.

In the next few subsections the hierarchy of Eqs. (13) is solved order by order, subject to the initial and boundary conditions. At every stage secularities, if present, will be avoided giving an equation for the modification of the equilibrium, and an equation for the slow time-dependence of the amplitude of the slightly unstable mode. Only the quasilinear approximation is given in detail.

## A. First Order Theory

In first order the linear problem that has been investigated in Sec. II results, and because of the particular choice of the number density [cf. Eq. (8)], only the mode with the smallest possible k is unstable. In view of the initial conditions the solution reads

$$f_1 = - \Gamma \frac{\eta(v)}{v-\bar{v}} \exp i\theta \ + \text{c.c.} \quad ,$$
$$\phi_1 = - \Gamma \frac{eN_c}{\varepsilon_0 k^2} \exp i\theta \ + \text{c.c.} \quad , \qquad (15)$$

where $\theta = \omega\tau_0 + kx$, $\Gamma$ is a complex amplitude, which is still an unknown function of the time scales $\tau_2$, $\tau_4$, ... The phase velocity $\bar{v}$ is the velocity corresponding to the bottom of the well in the equilibrium velocity distribution and satisfies Eq. (9b).

B. Second Order Theory

In second order we obtain $\mathcal{L}\vec{\psi}_2 = \vec{S}_2 = \begin{pmatrix} S_{21} \\ S_{22} \end{pmatrix}$ , where

$$S_{21} = -\frac{e}{m_e}\frac{\partial}{\partial x}\phi_1 \frac{\partial}{\partial v} f_1, \qquad (16)$$

$$S_{22} = 0.$$

The solution to the second order problem is given by

$$f_2 = f_{20}(v,\tau_2) + f_{22}(v,\tau_2)\exp 2i\theta + \text{c.c.}, \qquad (17)$$

$$\phi_2 = \phi_{20}(\tau_2) + \phi_{22}(\tau_2)\exp 2i\theta + \text{c.c.},$$

where the dependence on the time scales $\tau_4$, ... etc. has been suppressed. In the quasi-linear approximation the effect of higher harmonics is neglected.

In second order the term giving modification of the equilibrium, $f_{20}$, is still undetermined but will be in fourth order. It should be noted that Simon and Rosenbluth[1] determined $f_{20}$ in second order by making the phase velocity slightly complex; in the multiple time scale method this procedure is, however, not necessary as will be seen in Subsec. III D.

## C. Third Order Theory

The third order problem reads $\mathcal{L}\vec{\psi}_3 = \vec{S}_3 = \begin{pmatrix} S_{31} \\ S_{32} \end{pmatrix}$, where

$$S_{31} = -\frac{e}{m_e}\left(\frac{\partial}{\partial x}\phi_1 \frac{\partial}{\partial v} f_2 + \frac{\partial}{\partial x}\phi_2 \frac{\partial}{\partial v} f_1\right) - \frac{\partial}{\partial \tau_2} f_1 \,,$$
$$S_{32} = -\frac{eN_c}{\varepsilon_o}\int dv' f_1 \,. \tag{18}$$

The source term $\vec{S}_3$ gives rise to secularities in $\vec{\psi}_3$. The requirement that secularity be absent is already formulated by Simon and Rosenbluth[1]. In our case we obtain

$$\int_{\smile} dv\, \chi_1^*\left(-\frac{\partial}{\partial \tau_2} f_1 - i\frac{ek}{m_e}\phi_1 \frac{\partial}{\partial v} f_{20}\right) - \int_{\smile} dv \chi_2^* \int_{\smile} dv \frac{eN_c}{\varepsilon_o} f_1 = 0, \tag{19}$$

where $\vec{\chi} = (\chi_1, \chi_2)$ exp $i\theta$ is the solution of the adjoint problem $\mathcal{L}^{+}\vec{\chi}=0$: $\chi_1 = P(v-\bar{v})^{-1} + \tilde{\lambda}\delta(v-\bar{v}), \chi_2 = -ik\varepsilon_o\eta(v)/eN_c(v-\bar{v})$. The integrals in (19) are taken along the Landau contour[1], since we are interested in the growing mode. At this stage it is not possible to evaluate Eq. (19) as $f_{20}$ is still unknown. To this end the third order problem is solved and the secularity condition in fourth order is considered. The reason for going to fourth order in $\Delta$ is that we need an equation involving $\frac{\partial}{\partial \tau_2} f_{20}$ and this quantity is fourth order.

The assumptions

$$f_3 = f_{30} + f_{31}\, e^{i\theta} + f_{33}\, e^{3i\theta} + \text{c.c.},$$
$$\phi_3 = \phi_{30} + \phi_{31}\, e^{i\theta} + \phi_{33}\, e^{3i\theta} + \text{c.c.}, \tag{20}$$

solve the third order problem, and we obtain, e.g. for $f_{31}$

$$f_{31} = \left(\frac{i}{k}\frac{\partial}{\partial \tau_2} f_1 - \frac{e}{m_e}\phi_1 \frac{\partial}{\partial v} f_{20}\right)\left[\frac{P}{v-\bar{v}} + \lambda\delta(v-\bar{v})\right]. \tag{21}$$

To determine $f_{20}$ we only need $f_{31}$.

D. Fourth Order Theory

In fourth order the source term $\vec{S}_4 = \begin{pmatrix} S_{41} \\ S_{42} \end{pmatrix}$ is given by

$$S_{41} = -\frac{\partial}{\partial \tau_2} f_2 - \frac{e}{m_e}\left(\frac{\partial}{\partial x}\phi_3 \frac{\partial}{\partial v} f_1 + \frac{\partial}{\partial x}\phi_2 \frac{\partial}{\partial v} f_2 + \frac{\partial}{\partial x}\phi_1 \frac{\partial}{\partial v} f_3\right),$$

$$S_{42} = -\frac{eN_c}{\varepsilon_o}\int dv\, f_2 \tag{22}$$

The $\tau_o$- and x-independent part of $\vec{S}_4$ is non-vanishing because there is a phase shift between $\phi_1$ and $f_{31}$ [namely, through the term proportional to $i\frac{\partial}{\partial \tau_2} f_1$, cf. Eq. (21)]. Hence, writing

$$\vec{S}_4 = \vec{S}_4^{(0)} + \vec{S}_4^{(1)} e^{i\theta} + \text{etc.}$$

for the source term $\vec{S}_4$, we obtain

$$S_{41}^{(0)} = -\frac{\partial}{\partial \tau_2} f_{20} - \frac{e}{m_e}\left(\frac{\partial}{\partial x}\phi_1^* \frac{\partial}{\partial v} f_{31} + \frac{\partial}{\partial x}\phi_1 \frac{\partial}{\partial v} f_{31}^*\right),$$

$$S_{42}^{(0)} = -\frac{eN_c}{\varepsilon_o}\int dv\, f_{20} \tag{23}$$

The source term $\vec{S}^{(0)}$ leads to secularity; avoiding this we obtain an equation for the $\tau_2$-dependence of $f_{20}$,

$$\frac{\partial}{\partial \tau_2} f_{20} + \frac{e}{m_e}\left(\frac{\partial}{\partial x}\phi_1^* \frac{\partial}{\partial v} f_{31} + \frac{\partial}{\partial x}\phi_1 \frac{\partial}{\partial v} f_{31}^*\right) = 0, \tag{24}$$

showing that the electron distribution function is modified because of interactions between particles and the unstable wave.

Elimination of $\phi_1$ and $f_{31}$ gives

$$\frac{\partial}{\partial \tau_2} f_{20} + \left(\frac{\omega_p}{k}\right)^2 \left\{\frac{\partial}{\partial v} \left[\left(\frac{P}{(v-\bar{v})} + \lambda \delta(v-\bar{v})\right) \frac{\eta(v)}{v-\bar{v}}\right] \Gamma^* \frac{\partial}{\partial \tau_2} \Gamma + c.c.\right\} = 0.$$

(25)

From conservation of momentum (see below), $\lambda$ is shown to be real. Then, Eq. (25) can be integrated at once,

$$f_{20} = g(v) - \left(\frac{\omega_p}{k}\right)^2 |\Gamma|^2 \frac{\partial}{\partial v} \left[\left(\frac{P}{(v-\bar{v})} + \lambda \delta(v-\bar{v})\right) \frac{\eta(v)}{v-\bar{v}}\right], \quad (26)$$

where the arbitrary function g(v) is determined by the initial condition $f_{20}(\tau_2=0) = 0$, hence

$$f_{20} = - \left(\frac{\omega_p}{k}\right)^2 (|\Gamma|^2 - |\Gamma(0)|^2) \frac{\partial}{\partial v} \left[\left(\frac{P}{v-\bar{v}} + \lambda \delta(v-\bar{v})\right) \frac{\eta(v)}{v-\bar{v}}\right]. \quad (27)$$

The quantity $\lambda$ in Eq. (27) can be determined from the requirement that all solutions of the Vlasov equation must conserve momentum, in an analogous fashion as was done by Simon and Rosenbluth[1].

In addition, it can be shown that particles and energy are conserved by the solution, i.e.

$$\frac{d}{dt} \int dx\, dv\, f = 0, \quad \frac{d}{dt} \int dx\, dv \left(\frac{m_e N}{2} v^2 f + \frac{1}{2} \varepsilon_o E^2\right) = 0. \quad (28)$$

E. Dynamics of the Unstable Mode

As $f_{20}$ is now known the secularity condition (19) can be evaluated; dealing with the integrals in a like fashion as in Ref. 1, we obtain the nonlinear Landau equation for the complex amplitude $\Gamma$ of the unstable mode:

$$A \frac{\partial}{\partial \tau_2} \Gamma + B\Gamma + C\Gamma(|\Gamma|^2 - |\Gamma(0)|^2) = 0, \tag{29}$$

where the complex constants are given by

$$A = A_1 + iA_2 = i \int_{\mathcal{V}} \frac{\eta^{I}}{v-\bar{v}} dv ,$$

$$B = B_1 + iB_2 = k \int_{\mathcal{V}} \frac{\eta}{v-\bar{v}} dv ,$$

$$C = C_1 + iC_2 = \frac{k}{12} \left(\frac{\omega_p}{k}\right)^4 \int_{\mathcal{V}} dv \frac{\eta^{IV}}{v-\bar{v}} .$$

Again the integrals are taken along the Landau contour. It should be noted that the equation for the damped mode is found if the integrations are performed along the anti-Landau contours. Here we may emphasize that the result (29) is quite general; no restrictions are imposed on the form and position of the bump. Eq. (29) can be solved by assuming $\Gamma = \rho \exp i\sigma$, which results in an equation for the amplitude $\rho$ and the phase $\sigma$ of the mode:

$$\begin{aligned} &a) \quad \frac{\partial}{\partial \tau_2} \rho + \frac{\vec{A}.\vec{B}}{|A|^2} \rho + \frac{\vec{A}.\vec{C}}{|A|^2} \rho(\rho^2 - \rho^2(0)) = 0, \\ &b) \quad \frac{\partial}{\partial \tau_2} \sigma = - \frac{A_2}{A_1} \frac{\partial}{\partial \tau_2} \ln\rho - \frac{B_2}{A_1} - \frac{C_2}{C_1}(\rho^2 - \rho^2(0)), \end{aligned} \tag{30}$$

where $\vec{A} = (A_1, A_2)$, $\vec{B} = (B_1, B_2)$ and $\vec{C} = (C_1, C_2)$.

The nonlinear Landau equation (30a) can easily be solved, and, as is well known, exhibits limit cycle behavior. For large $\tau_2$, $\rho^2$ saturates to the level $\rho_0^2 = - \vec{A}.\vec{B}/\vec{A}.\vec{C}$; the same expression is obtained in Ref. 1 by means of a time-asymptotic analysis (in the quasi-linear approximation). The results, as presented here, can easily be extended to the full nonlinear case (hence, the effect of second harmonics is included) but no purpose is served in reproducing calculations which for a large part can be found in Ref. 1. We note, however, that inclusion of second harmonics does not change the form of the equation for the unstable mode, it merely changes the coefficients in Eq. (29).

For the special case of a small bump far out on the tail of the main distribution, Simon and Rosenbluth[1] have given an analytic estimate of the saturation level [Eq. (58a) of Ref. 1]. If $\tilde{\phi}$ is the amplitude of the potential fluctuation their result can be written as

$$\frac{e\tilde{\phi}}{\kappa T_e} = \Delta \frac{v_e}{\bar{v}} 4\sqrt{6} \,, \tag{31}$$

where $\kappa$ is the Boltzmann constant, $T_e$ is the electron temperature, $v_e = (\kappa T_e/m_e)^{\frac{1}{2}}$, and we have used $e\tilde{\phi}/\kappa T_e = 2\Delta\rho_0/(\kappa\lambda_D)^2$ (Eq. 15). The main distribution function was assumed to be Maxwellian. Note that Eq. (31) is valid only for $\bar{v}/v_e \gtrsim 10$.

F. Comparison with Experiments

As an illustration of the present theory we consider the experiments reported in Ref. 7 and 9. The authors investigated the problem of suppression of plasma oscillations by an external wave in a beam-plasma system of the feedback type (i.e. an experiment in which the beam is reflected at the end of the machine). The properties of the oscillation system were well described by the Van der Pol equation. As is well known, the amplitude of the Van der Pol oscillator is determined by the nonlinear Landau equation given in Eq. (29); thus there is qualitative agreement between these experiments and our theory (see also Sec. V). In addition, only a single unstable mode seems to be present in these experiments. To be specific, we have plotted the experimental velocity distribution of Ref. 9 in Fig. 1 and we have also drawn the possible phase velocities of the system: the oscillation frequency f was about 50 MHz and the length L of the system was about 17.5 cm. Clearly, only the mode with mode number $\ell = 5$ is on the positive slope region, hence is unstable. Furthermore, a small growth rate ($\gamma/\omega \lesssim 0.05$) was found. Therefore, the theory of this section seems applicable to this experiment. In addition, we calculate the saturation level of the instability. From Fig. 1 we obtain $\Delta^2 = \frac{v_{f,5}-\bar{v}}{\bar{v}} \approx 0.08$, and $\bar{v}/v_e \approx 4$ ($T_e \approx 5$ eV). Using the full expression for $\rho_o^2$ and assuming the distribution function to be Maxwellian, we obtain $e\tilde{\phi}/\kappa T_e \approx 0.26$ ($\tilde{\phi} \approx 1.3$ V). Amemiya and Nakamura[9] reported a fluctuation amplitude $\tilde{\phi}$ of around 1 V. In view of the idealized theory (e.g. no dissipative effects are included; a one-dimensional plasma is considered) we believe that there is satisfactory agreement between theory and experiment.

It is interesting to note that if beam trapping was the dominant saturation effect the saturation level is given by (see e.g. Appendix B of Ref. 1)

$$\frac{e\tilde{\phi}}{\kappa T_e} = (\gamma/\omega_p)^2 \frac{1}{2\sqrt{2}(k\lambda_D)^2}$$

which for the present experiment gives $e\tilde{\phi}/\kappa T_e \lesssim 0.01$. This seems to indicate that the initial trapping regime has been overcome. The same conclusion seems to hold for the computer experiment of Armstrong and Montgomery[12] where the saturated amplitude is $e\tilde{\phi}/\kappa T_e \approx 0.6$ for the dominant mode, while the amplitude calculated from the above expression is $\approx 0.01$.

Finally we emphasize that we are dealing with the saturation of a single unstable mode. This situation is entirely different from that of the usual quasilinear theory[10], where many unstable modes are assumed to be present. This may be the case when the bump is sufficiently gentle and the system sufficiently long. A very good agreement with the predictions from quasilinear theory in such a system was found in the experiment by Roberson et al.[13].

## IV. THE ION-ACOUSTIC INSTABILITY

The results presented in the previous section can easily be generalized to the case of mobile ions, as was already pointed out by Simon[14]. Inclusion of mobile ions merely amounts to the replacement of e.g. $\eta_e \equiv -(\omega_{pe}/k)^2 f_{oe}'$ by $\eta_e + \eta_i$. The form of the equation for the unstable mode is however not changed so that for mobile ions limit cycle behavior may also be found. We should mention that the results thus obtained are fairly general, since no assumptions have been made regarding the form of the ion and electron velocity distribution, except for the condition that there is a threshold for instability. Thus, the bump-on-tail, the ion-acoustic as well as the ion-ion beam instability can be treated on the same footing. Here, only the ion-acoustic instability is considered.

To this end we assume a Maxwellian velocity distribution for the ions which are at rest, and the electrons which drift with a velocity $v_o$:

$$f_{oi} = \frac{1}{v_i\sqrt{2\pi}} \exp - \frac{v^2}{2v_i^2} , \quad f_{oe} = \frac{1}{v_e\sqrt{2\pi}} \exp - \frac{(v-v_o)^2}{2v_e^2} , \tag{32}$$

where the ions and electrons have thermal velocity $v_i$ and $v_e$ respectively. The linear dispersion relation for the ion acoustic instability is given by Eq. (7) with $\eta = \eta_e + \eta_i$; by using Eq. (32) it can be written as

$$\frac{T_e}{T_i} Z' \left(\frac{\nu}{\sqrt{2}v_i}\right) + Z' \left(\frac{\nu-v_o}{\sqrt{2}v_e}\right) = 2(k\lambda_D)^2, \tag{33}$$

where Z' is the derivative of the plasma dispersion function[15], and $\nu = -\frac{\omega}{k}$. In the limit of $\frac{T_e}{T_i} >> 1$ we get the approximate expressions for frequency, $\omega$, and growth rate, $\gamma$:

$$\omega \approx \frac{-kc_s}{[1+(k\lambda_D)^2]^{\frac{1}{2}}} \quad \text{and} \tag{34a}$$

$$\frac{\gamma}{\omega} = \frac{\pi}{2}\left(\frac{\omega}{\omega_{pi}}\right)^2 \eta\left(\frac{\omega}{k}\right), \tag{34b}$$

where $c_s^2 = \kappa(T_e + 3T_i)/m_i$ is the ion-acoustic speed, $\omega_{pi}$ the ion plasma frequency, and $\eta = \eta_e + \eta_i$. From (34) we see that the condition $v_i << \nu < v_o << v_e$ can be met for $T_e/T_i >> 1$ and we can thus write the growth rate in the following form:

$$\frac{\gamma}{\omega} \simeq -\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{\nu}{c_s}\right)^2\left\{\frac{v_o}{v_e} - \frac{\nu}{v_e}\left[1 + \frac{T^{3/2}}{\delta}\exp\left(-\frac{\nu^2}{2v_i^2}\right)\right]\right\}, \tag{35}$$

where $T \equiv T_e/T_i$ and $\delta \equiv (m_e/m_i)^{\frac{1}{2}}$.

Since in experimental situations $k\lambda_D << 1$, we infer from Eq. (35) that the growth rate is only weakly dependent on the number density N. If one now uses the number density N as the critical parameter (as was done in the previous sections) very small growth rates will be obtained for N near its critical value.

However, another critical parameter for the ion-acoustic instability is the drift velocity $v_o$. Its critical value $v_c$ can be obtained from Eq. (35) by setting $\gamma = 0$ for the mode with the smallest possible $k = k_{min}$:

$$\frac{v_c}{v_e} = \frac{\nu_{min}}{v_e}\left[1 + \frac{T^{3/2}}{\delta}\exp\left(-\frac{\nu_{min}^2}{2v_i^2}\right)\right], \tag{36}$$

where $\nu_{min}$ corresponds the mode with the smallest possible wave number.

For a plasma with a drift velocity slightly above the critical value, i.e.

$$v_o = v_c(1+\Delta^2), \quad \Delta^2 << 1, \tag{37}$$

the growth rate of the unstable mode corresponding to $k = k_{min}$ becomes (Eq. 34b)

$$\frac{\gamma}{\omega} = - \frac{\pi}{2}\left(\frac{\omega}{\omega_{pi}}\right)^2 \Delta^2 v_c \, \eta_e^I(\bar{v}) . \tag{38}$$

All other modes remain stable provided the fractional increase in $v_o$ is not too large. As an example we take $k_{min}\lambda_D = 10^{-1}$, $T = 7$, $\delta = \frac{1}{40}$. Then, the mode with $k = 2\,k_{min}$ becomes unstable if the fractional increase in $v_o$ is more than 20%. Thus, once again we deal with an instability which may have a single mode with small growth rate.

In complete analogy with what was done in Sec. III the dynamical equation for this unstable mode can be obtained by means of the multiple time scale formalism. As a result limit cycle behavior is found, expressed by an equation similar to (29), with the coefficients:

$$A = i\int_{\smile} dv \, \frac{\eta^I}{v-\bar{v}} ,$$

$$B = - v_c k\int_{\smile} dv \, \frac{\eta_e^I}{v-\bar{v}} ,$$

$$C = \frac{k}{12}\left(\frac{\omega_p}{k}\right)^4 \int_{\smile} dv \, \frac{\eta^{IV}}{v-\bar{v}} ,$$

giving the saturation level:

$$\rho_o^2 = \frac{12 v_c\left(\eta_e^I P\int dv \, \frac{\eta^I}{v-\bar{v}} - \eta^I P\int dv \, \frac{\eta_e^I}{v-\bar{v}}\right)}{\left(\frac{\omega_p}{k}\right)^4 \eta^{IV} P\int dv \, \frac{\eta^I}{v-\bar{v}} - \eta^I P\int dv \, \frac{\left(\frac{\omega_p}{k}\right)^4 \eta^{IV}}{v-\bar{v}}} , \tag{39}$$

where $\rho_o$ has the same meaning as in the previous section,

$\eta^I = \eta_e^I + \eta_i^I$, $\left(\frac{\omega_p}{k}\right)^4 \eta^{IV} = \left(\frac{\omega_{pe}}{k}\right)^4 \eta_e^{IV} + \left(\frac{\omega_{pi}}{k}\right)^4 \eta_i^{IV}$, and $\bar{v} = - \omega/k$.

In order to obtain Eq. (39) the effect of higher harmonics has been neglected. Stabilization of the unstable mode is achieved by local flattening of the ion as well as the electron velocity distribution function, which may be inferred from Eq. (27) with $(\omega_p/k)^2\eta = (\omega_{pi}/k)^2\eta_i + (\omega_{pe}/k)^2\eta_e$.

In the limit of large T we can expand the integrals in (39) and calculate the relative fluctuation level $e\tilde{\phi}/\kappa T_e$ (where $\tilde{\phi}$ is the amplitude of the potential fluctuation; from (15): $e\tilde{\phi}/\kappa T_e = 2\Delta\rho_o/(k\lambda_D)^2$)

$$\frac{e\tilde{\phi}}{\kappa T_e} = 0.9\ \Delta\ \sqrt{\frac{v_c}{\delta v_e}};$$

this approximate expression for the fluctuation level is valid only for $T > 50$. Since the latter condition cannot be met experimentally, we have calculated the saturation level directly from Eq. (39), using the plasma dispersion function[15] where $\bar{v}$ is determined from the dispersion relation (33). In Fig. 2 we plotted $e\tilde{\phi}/\kappa T_e\Delta$ as a function of T for $k\lambda_D = 0$ and $m_e/m_i = 1/1836$. It should be noted that for $k\lambda_D < 0.2$ there is hardly no dependence on $k\lambda_D$. The order of magnitude of the saturation level is in agreement with experimental observations[16,17]. That is, Nakamura et al.[16] reported a peak-peak saturation amplitude $\delta n/n_o = e\tilde{\phi}/\kappa T_e = 0.2$ for a temperature ratio of $T = 10$, while Schrittwieser[17] found an amplitude in the range $0.2 < \delta n/n_o < 0.3$ for a somewhat lower temperature ratio. In these experiments the single mode structure was produced by the finite geometry giving rise to a standing wave. Another situation leading to single mode saturation is realized when a testwave is excited at the same boundary as that where the electron beam is injected, so that the testwave has a large start over the unstable noise. This case is considered by e.g. Wong et al.[18]; their saturation level ($e\tilde{\phi}/\kappa T_e$ = 0.1 (T = 15)) is also of the same order of magnitude as our results. Furthermore, they observe a clear flattening of the electron distribution function around the phase velocity of the wave as it saturates, in agreement with the predictions above. We emphazise, however, that the test wave case should be described by a boundary value problem, rather than the initial value problem considered above, and only qualitative agreement is expected. Finally, we note that Albright[19] has calculated a saturation level for the Wong et al.[18] experiment considering electron trapping as the dominant mechanism, and finds $e\tilde{\phi}/\kappa T_e \approx 0.01$, thus here the conclusion of Sec. IIIF also holds.

## V. DISCUSSION OF THE RESULTS

In this section our results are discussed. Although only the bump-c -tail instability is considered in detail, our conclusions hold for the ion-acoustic instability as well.

We have already demonstrated that particles, momentum and energy are conserved by the solution. In addition, this solution can be shown to conserve the "microscopic" entropy S which, apart from a constant, is given by

$$S = -\int dx\, dv\, f \ln f\,, \qquad (41)$$

By means of Eqs. (12a) and (12b), we obtain for $\frac{d}{dt}$ S to fourth order in $\Delta$

$$\frac{d}{dt} S = - \Delta^4 \int dx\, dv \left[\frac{f_1}{f_0} \frac{\partial}{\partial\tau_2} f_1 + (1+\ln f_0) \frac{\partial}{\partial\tau_2} f_{20}\right], \qquad (42)$$

Only the terms given in Eq. (42) survive the averaging activity of the integration over x. Then, using the expressions for $f_1$ [Eq. (15)] and $\partial f_{20}/\partial\tau_2$ [Eq. (25)], conservation of S follows at once.

As is well known, the Vlasov-Poisson set conserves the entropy S, and it is therefore quite obvious that the solution found in Sec. III conserves this entropy, since this solution contains the same essential information as the Vlasov-Poisson set. This contrasts with the quasi-linear theory as given by Drummond and Pines[10], and Bernstein and Engelmann[20]. There one is interested in some average of the distribution function and the electric field. Because of averaging, information about specific details is lost and therefore the entropy, defined by means of the averaged distribution function, is not conserved[21]. This is also evident from the equations for the modification of the equilibrium. In the quasi-linear theories of Ref. 10 and 20 this equation is of the diffusion type, whereas in our case [Eq. (25)] this is certainly not true.

It is of interest to compare the dynamics of these microscopic instabilities with a macroscopic one, namely the $\vec{g} \times \vec{B}$ instability in a collisionless Finite Larmor Radius plasma[5]. There, the gravity constant g is increased above its neutral stable value by the fractional amount $\Delta^2$ such that only one mode is unstable with growth rate $O(\Delta)$, instead of $O(\Delta^2)$ as found in the case of bump-on-tail. As a consequence, a different type of dynamics is to be expected for the $\vec{g} \times \vec{B}$ instability. And indeed, it was found in Ref. 5 that periodic modulation in the amplitude $\Gamma$ of the unstable mode results, i.e. $\partial^2\Gamma/\partial\tau^2 = \gamma\Gamma + \beta\Gamma|\Gamma|^2$. This second order differential equation does not exhibit limit cycle behavior. In order to see the differences more clearly, let us consider the behavior of the modes near the critical point. The dispersion relation of the $\vec{g} \times \vec{B}$ instability in a collisionless Finite Larmor Radius plasma is quadratic in $\omega$: $a\omega^2 + b\omega + c = 0$, with real a, b and c. Below the neutral stable value of the gravity constant g, all the modes are stable, slightly above this neutral stable value two modes become unstable, one which is damped and the other growing. We have shown this transition in Fig. 3.

In case of the bump-on-tail instability the transition is quite different. Below the neutral stable value of the density, there is a real continuum of modes giving a potential $\phi$, which decays in time through phase mixing (Landau damping). Slightly above this neutral stable value, two modes become unstable, one which is damped and the other growing (see Fig. 1). The transition behavior in case of the bump-on-tail instability is typical for limit cycles in many respects, except that above the neutral stable value of the density there is both a damped and a growing mode. Then, if one reverses time the growing mode becomes the damped one, and vice versa. Therefore, the system is symmetrical with respect to the initial point of departure, in agreement with the invariance of the Vlasov-Poisson equations under the transformation $t \to -t$, $v \to -v$.

The properties of nonlinear oscillations in a beam plasma system have been found experimentally to be well described in terms of the Van der Pol Equation[6-9,22], as mentioned in the previous sections. In Ref. 6-7 (dealing with the ion-acoustic and electron-beam instability, respectively), the Van der Pol equation was derived from the fluid equations including ionization and recombination terms. These source terms are necessary for giving the nonlinear terms in the Van der Pol equation. However, also in collisionless plasmas without source terms the Van der Pol equation has proven to be a good phenomenological model for the ion-acoustic instability[8]. Noting that the amplitude of a Van der Pol oscillator is determined by a nonlinear Landau equation, (e.g. (30a)) we may take our results as justification of the use of the Van der Pol equation for the self-sustained, ion-acoustic instability as well as the self-sustained bump-on-tail instability. To be more specific, consider the Van der Pol equation for the potential in the form:

$$\frac{\partial^2\phi}{\partial t^2} + (\alpha + \beta\phi^2)\frac{\partial\phi}{\partial t} + \omega_o^2\phi = 0. \tag{43}$$

For $\alpha$, $\beta \ll \omega_o$ Eq. (43) has the solution[23]

$$\phi = \tilde{\phi}(t_1)\cos(\omega_o t - \sigma(t_1)),$$

where the slowly varying amplitude $\tilde{\phi}$ and phase $\sigma$ are determined by:

$$\frac{\partial\tilde{\phi}}{\partial t_1} + \frac{\alpha}{2}\tilde{\phi} + \frac{\beta}{8}\tilde{\phi}^3 = 0, \tag{44a}$$

$$\frac{\partial\sigma}{\partial t_1} = 0. \tag{44b}$$

We see that Eq. (44a) takes the same form as Eq. (30a) and using $\tilde{\phi} = -2\rho\kappa T_e/e(k\lambda_D)^2$, we can identify the coefficients α and β in the Van der Pol equation in terms of A, B and C:

$$\alpha = 2\,\frac{\vec{A}.\vec{B}}{|\vec{A}|^2}\ ,\ \text{and}\ \beta = 2(k\lambda_D)^4\,\frac{e^2}{(\kappa T_e)^2}\,\frac{\vec{A}.\vec{C}}{|\vec{A}|^2}$$

where A, B and C must be determined for the problem at hand. On the other hand for the classical Van der Pol equation there is no phase shift up to third order (see Eq. (44b)), while it is evident from Eq. (30b) that in the present system, there is a phase shift. This could be introduced in the Van der Pol model by a nonlinear restoring force in addition to the nonlinear damping term. However, the Van der Pol equation has mainly been used to describe the suppression of an instability by an external oscillation in the cited experiments[6-9,22], and for this purpose only the amplitude variation is of importance.

The cited experiments are dealing with self-sustained oscillations where the feed-back mechanism is reflections at the boundaries of the finite system. An alternative approach for describing the instability suppression in such a system has recently been suggested by Kato et al.[24], who derived a linear Mathieu-type equation with an inhomogeneous term describing density oscillations in the cold beam plasma system. But while this equation explains the resonances of the system and the quenching of the unstable oscillations it can obviously not describe the nonlinear properties e.g. the amplitude dependence of the applied frequency, which is given by the van der Pol equation in good agreement with the experimental observations. In the case where the instability is not self-sustained, the suppression by an externally excited wave is also observed. However, here the suppression is caused by a deformation of the electron beam distribution function as clearly observed by e.g. Fukumasa and Itatani[25]. In this case the unstable oscillation is still in its linear regime when the external wave is applied and this might, having a sufficient amplitude, alter the dispersion relation and suppress the instability. While in the case of a self-sustained oscillation, as we have been dealing with in the present work, the instability is already in the saturated state, when applying the external wave. It thus resembles a steady state oscillator and the suppression is brought about because the modes compete for the energy available. (For further discussion see Ref. 22 which contains references to related work).

Finally we mention that Walsh and Hagelin[26] recently derived a nonlinear Landau equation describing the low density, cold beam plasma instability, by expanding the dispersion relation around the most unstable root and the perturbed beam electron orbit, thus justifying the use of the Van der Pol equation in such a system.

## VI. CONCLUSION

We have considered the nonlinear evolution in time of the bump-on-tail and ion-acoustic instability where periodic boundary conditions allow only one mode to be unstable for a particular choice of a critical parameter. The growth rate of this mode is of the order $\Delta^2$, where $\Delta^2$ equals the fractional increase in the critical parameter over its neutral stable value. Then, two time scales can be distinguished permitting the Vlasov-Poisson set to be solved by means of the multiple time scale method. As a result, limit cycle behavior is found, in agreement with experiment. In addition, the theoretical fluctuation level seems to agree with experimental results.

## ACKNOWLEDGMENTS

One of the authors (P.J.) is pleased to acknowledge useful discussions with M.P.H. Weenink and P.P.J.M. Schram.
This work was performed as part of the research program of the association of Euratom and the foundation FOM.

## REFERENCES

1. A. Simon and M.N. Rosenbluth, Phys. Fluids 19, 1567 (1976).

2. A. Simon, Phys. Fluids 11, 1181 (1968).

3. W. Kryloff and N. Bogoliuboff, Introduction to Nonlinear Mechanics (Princeton University, Princeton, N.J. 1947).

4. Young-ping Pao, Phys. Fluids 21, 765 (1978).

5. P.A.E.M. Janssen, Phys. Fluids 23, 372 (1980).

6. B.E. Keen and W.H.W. Fletcher, J. Phys. D: Appl. Phys. 3, 1868 (1970).

7. Y. Nakamura, J. Phys. Soc. Japan 28, 1315 (1970), J. Phys. Soc. Japan 31, 273 (1971); A. Itakura, T. Tsuru, Y. Nakamura, and S. Kojima, J. Phys. Soc. Japan 34, 176 (1973).

8. P. Michelsen, H.L. Pécseli, J. Juul Rasmussen and R. Schrittwieser, Plasma Phys. 21, 61 (1979).

9. H. Amemiya and Y. Nakamura, Proc. XIIIth ICPIG, Berlin, D.D.R. p. 875 (1977).

10. W.E. Drummond and D. Pines, Ann. Phys. (N.Y.) 28, 478 (1964).

11. K.M. Case, Ann. Phys. (N.Y.) 7, 349 (1959).

12. T.P. Armstrong and D. Montgomery, Phys. Fluids 12, 2094 (1969).

13. C. Roberson, K.W. Gentle, and P. Nielsen, Phys. Rev. Lett. 26, 226 (1971).

14. A. Simon, Phys. Fluids 20, 79 (1977).

15. B.D. Fried and S.D. Conte, The Plasma Dispersion Function (Academic, New York, 1961).

16. Y. Nakamura, Y. Nomura, and T. Itoh, Phys. Rev. Lett. 39, 1622 (1977).

17. R. Schrittwieser, Phys. Lett. 65A, 235 (1978), and private communication (1979).

18. A.Y. Wong, B.H. Quon, and B.H. Ripin, Phys. Rev. Lett. 30, 1299 (1973).

19. N.W. Albright, Phys. Fluids 17, 206 (1974).

20. I.B. Bernstein and F. Engelmann, Phys. Fluids 9, 937 (1966).

21. H. Grad, Comm. on Pure and Appl. Math. XIV, 323 (1961).

22. O. Fukumasa and R. Itatani, Proc. Int. Conf. Plasma Phys., Nagoya, Japan, p. 234 (1980); and IPPJ 488 (1980) Inst. Plasma Phys., Nagoya Univ., Japan.

23. R.C. Davidson, Methods in Nonlinear Plasma Theory, (Academic Press, New York, 1972), Chap. 1.

24. T. Kato, T. Okazaki, and T. Ohsawa, J. Phys. Soc. Japan 46, 277 (1979).

25. O. Fukumasa and R. Itatani, Phys. Lett. 68A, 59 (1978).

26. J.E. Walsh and J.S. Hagelin, Phys. Fluids 19, 339 (1976).

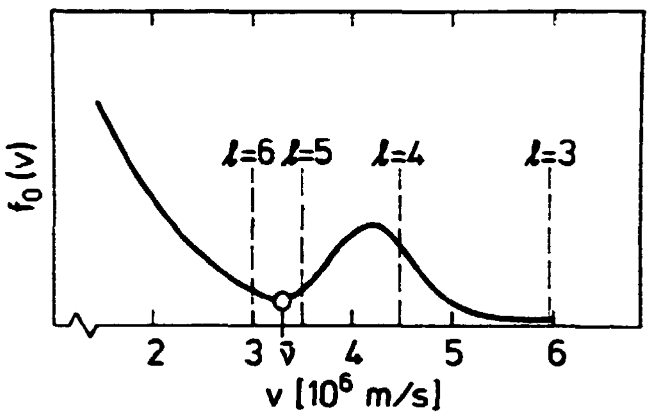


Fig. 1. Velocity distribution in the Amemiya-Nakamura experiment[9]. Vertical lines indicate the phase velocities of the various modes: $v_\ell = \frac{2fL}{\ell}$, $k = \frac{\pi\ell}{L}$, $\ell$ is the mode number.

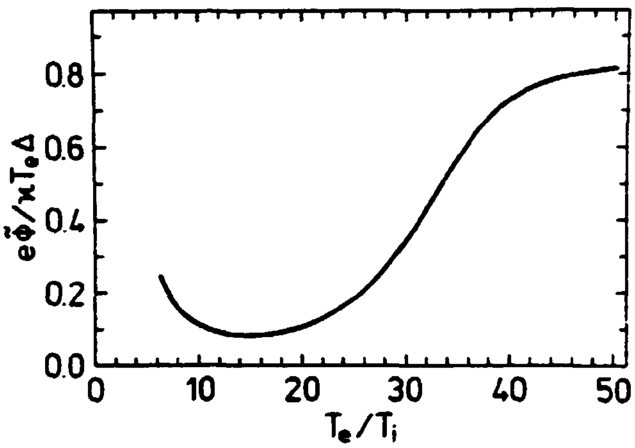


Fig. 2. The saturation level of the ion-acoustic instability as function of the temperature ratio $T_e/T_i$ for $k\lambda_D = 0$ and $m_e/m_i$ = 1/1836.

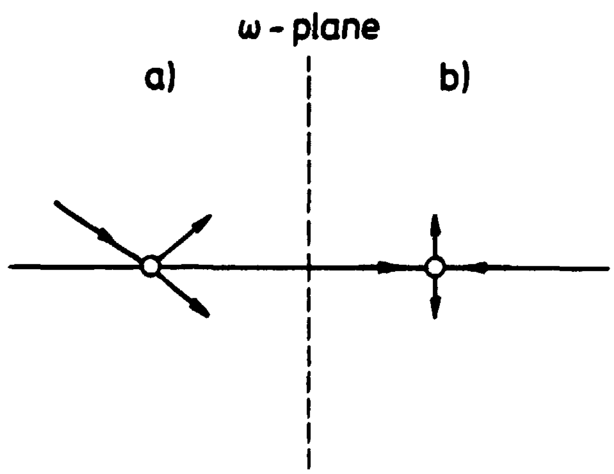


Fig. 3. Transition between stable and unstable modes for a) the bump-on-tail instabiblity and b) the $\vec{g}\times\vec{B}$ instability.

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| LIMIT CYCLE BEHAVIOR OF THE BUMP-ON-TAIL AND ION-ACOUSTIC INSTABILITY<br><br>Peter A.E.M. Janssen<br>J. Juul Rasmussen | Department or group<br><br>Physics<br><br>Group's own registration number(s) |
| 34 pages + tables + 3 illustrations | |

Abstract

The nonlinear dynamics of the bump-on-tail and current driven, ion-acoustic instability is considered. The eigenmodes have discrete k because of finite periodic boundary conditions. Increasing a critical parameter (the number density and the electron drift velocity respectively) above its neutral stable value by a small fractional amount $\Delta^2$, one mode becomes unstable. The nonlinear dynamics of the unstable mode is determined by means of the multiple time scale method. Usually, limit cycle behavior is found. A short comparison with quasi-linear theory is given, and the results are compared with experiment.

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