Technical University of Denmark

Polynomial Vector Fields in One Complex Variable

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POLYNOMIAL VECTOR FIELDS of ONE COMPLEX VARIABLE

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Technical University of Denmark

The talk is based on
Adrien's joint work with Pierrette Sentenac,
work of Xavier Buff and Tan Lei, and
my own joint work with Kealey Dias.

IN THE MEMORY of ADRIEN DOUADY

CONTENT

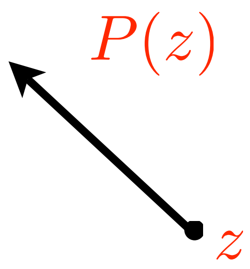
NOTATION

Consider the set $\mathcal{P}_d$ of monic centered polynomials of degree $d \geq 2$

$$P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0 \qquad \text{parametrized by } \mathbf{a} = (a_0, \ldots, a_{d-2}) \in \mathbb{C}^{d-1}$$

The associated vector field in $\mathbb{C}$ is

$$\xi_P(z) = P(z)\frac{\mathrm{d}}{\mathrm{d}z}$$



We study the maximal trajectories $t \mapsto \gamma(t, z_0)$ of ξ_P with $t \in \mathbb{R}$ i.e. maximal solutions to

$$\gamma'(t, z_0) = P(\gamma(t, z_0)) \quad \text{and} \quad \gamma(0, z_0) = z_0$$

The space of polynomial vector fields of fixed degree d is

$$\Xi_d = \{\xi_P\}_{P \in \mathcal{P}_d} \simeq \mathbb{C}^{d-1}$$

EQUILIBRIUM POINTS of ξ_P – ROOTS of P

PROPOSITION If ζ is a simple root of P with mulitiplier $\lambda = P'(\zeta)$ then ξ_P is holomorphically conjugate in a neighborhood of ζ to the linear vector field

$$\lambda z \frac{\mathrm{d}}{\mathrm{d}z} .$$

THREE CASES:

$\Re\lambda < 0$ $\qquad$ $\Re\lambda = 0$ $\qquad$ $\Re\lambda > 0$

$\Im\lambda > 0$

$\Im\lambda < 0$

sink

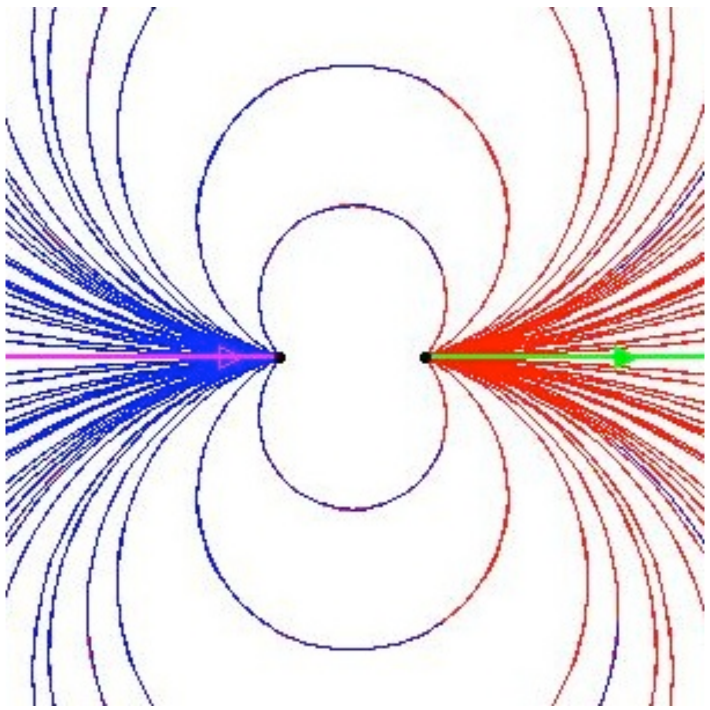
center

source

PROPOSITION If ζ is a multiple root of P of multiplicity $m > 1$ then ξ_P has $m-1$ attracting directions and $m-1$ repelling directions.

THREE QUADRATIC EXAMPLES

$P(z) = z^2 - 1$

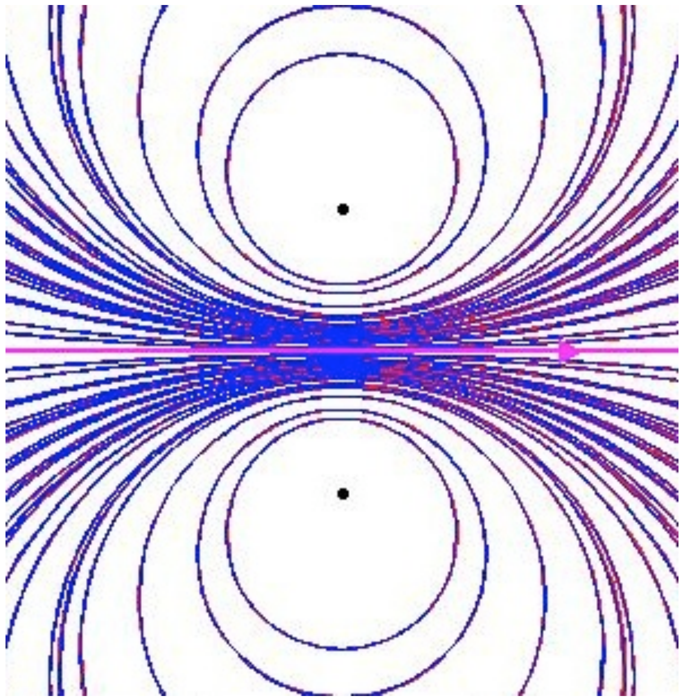


Two simple equilibrium points

$\zeta = -1$ $\zeta = 1$

a sink a source

$P(z) = z^2 + 1$

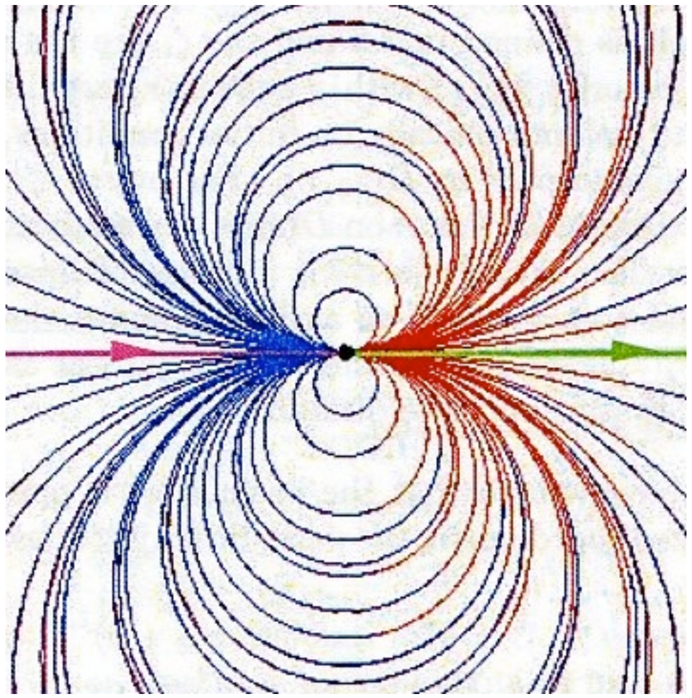


Two simple equilibrium points

$\zeta = \pm\, \mathrm{i}$

centers

$P(z) = z^2$



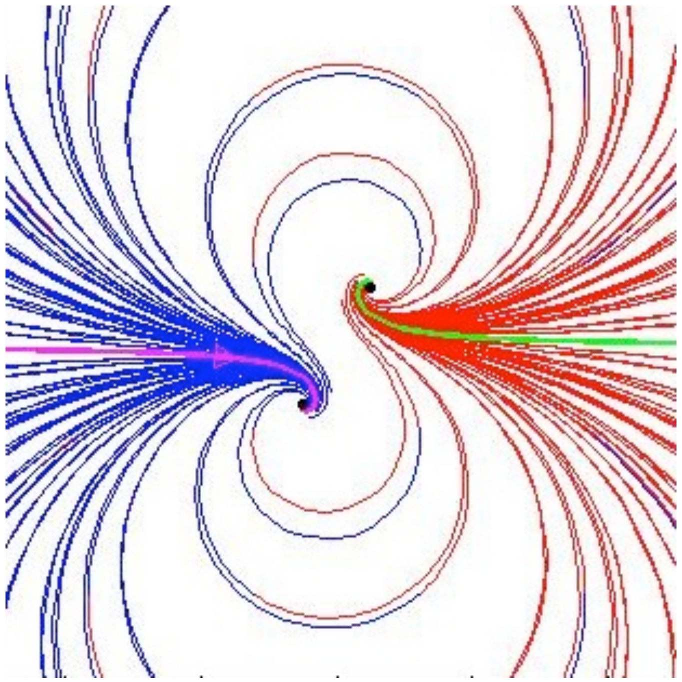
One double equilibrium point

$\zeta = 0$

THREE QUADRATIC EXAMPLES

typical:

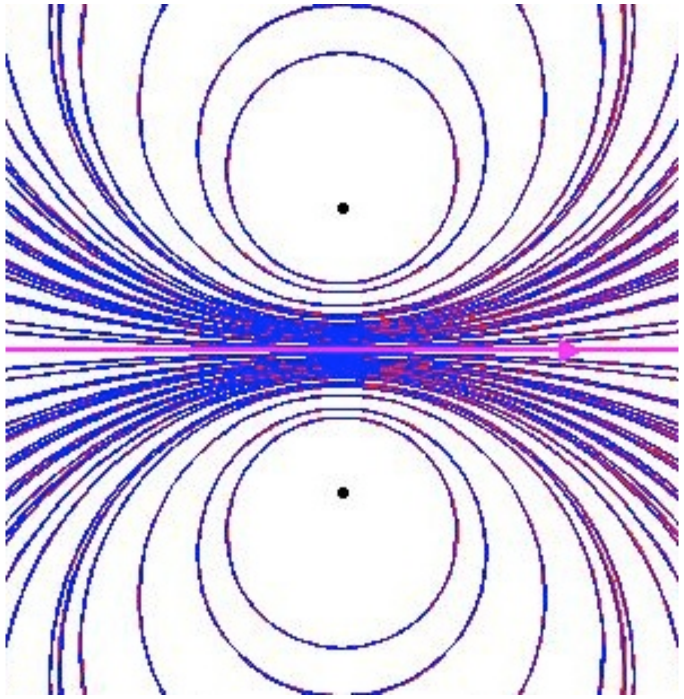
sink and source
with non-real multipliers



Two simple equilibrium points

a sink a source

$P(z) = z^2 + 1$

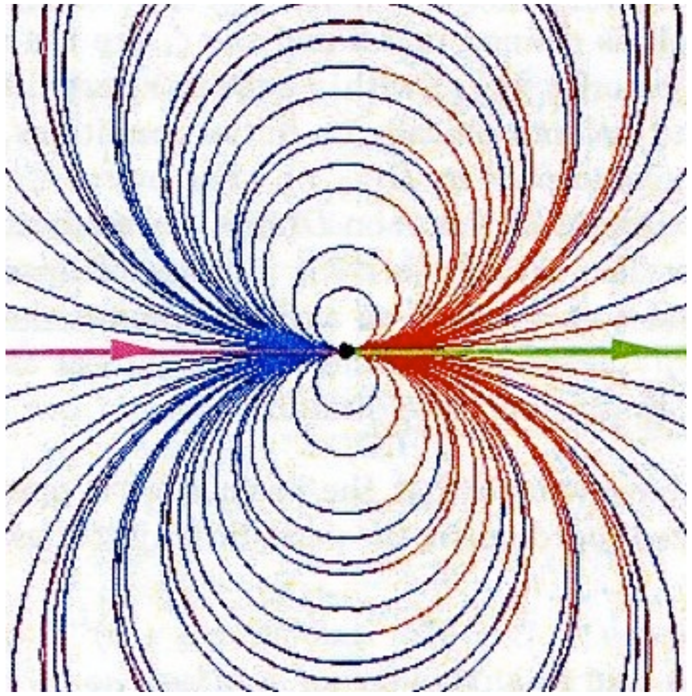


Two simple equilibrium points

$\zeta = \pm\, \mathrm{i}$

centers

$P(z) = z^2$

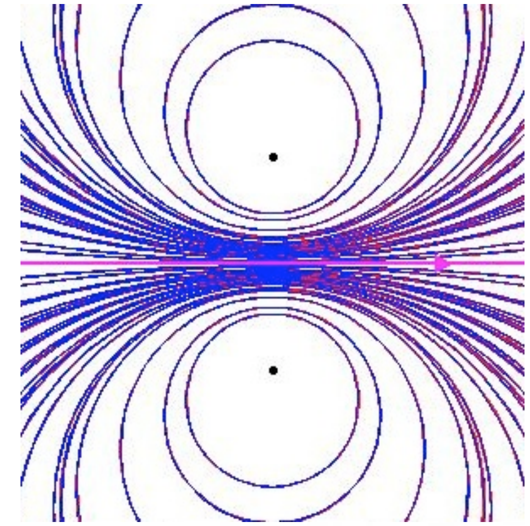


One double equilibrium point

$\zeta = 0$

THE SPACE of QUADRATIC VECTOR FIELDS $\Xi_2 \simeq \mathbb{C}$

decomposed into disjoint classes: $P(z) = z^2 + a$

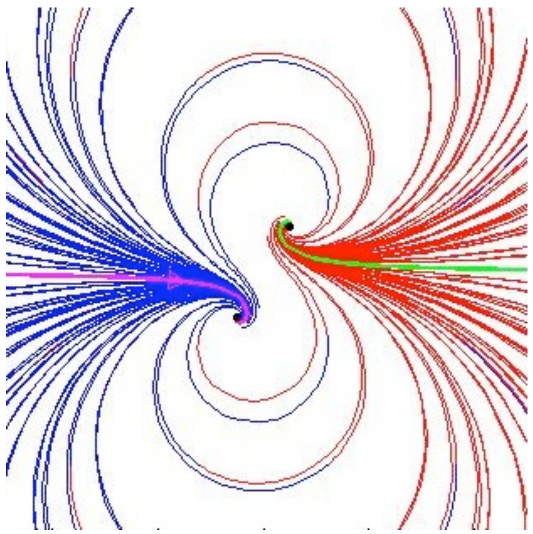


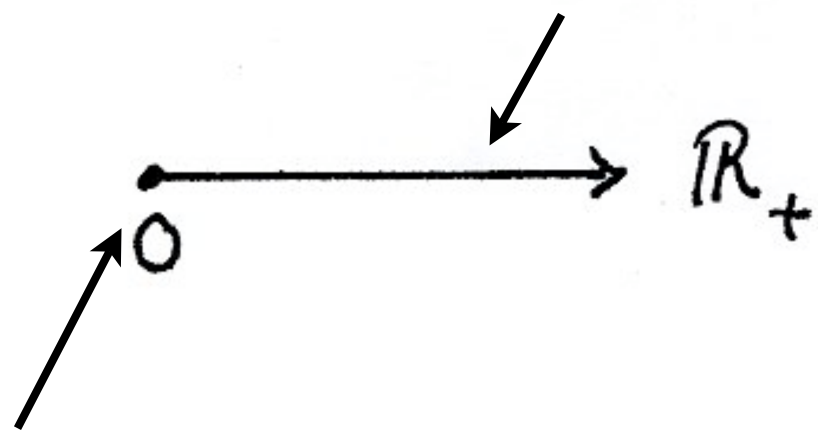
real-analytic
one-dim
submanifold

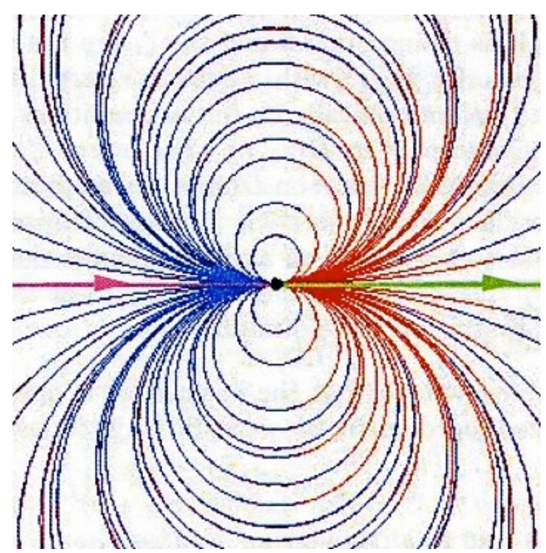
$a > 0$

complex
submanifold

$a \in \mathbb{C} \setminus \ [0, +\infty[$







a singleton

$a = 0$

GOAL

The goal is to decompose Ξ_d into disjoint classes $\mathcal{C}$ such that

- each $\mathcal{C}$ is connected
- all $\xi_P \in \mathcal{C}$ have the same qualitative dynamics
- $\mathcal{C}$ is maximal

A class is either structurally stable or part of the bifurcation locus

A class $\mathcal{C}$ is characterized by a combinatorial invariant $\mathcal{I}(\mathcal{C})$.

To each class $\mathcal{C}$ is associated two integers

$$s = s(\mathcal{C}) \geq 0 \quad \text{and} \quad h = h(\mathcal{C}) \geq 0 \quad \text{satisfying} \quad s + \tfrac{1}{2}h \leq d - 1$$

Within $\mathcal{C}$ a vector field ξ_P is uniquely determined by an analytic invariant

$$\mathcal{A}_P = (A_P^1, \ldots, A_P^s, T_P^1, \ldots, T_P^h) \in \mathbb{H}^s \times \mathbb{R}_+^h.$$

THE STRUCTURE THEOREM

Given $d \geq 2$, a combinatorial data set $\mathcal{I}$ with associated integers $s = s(\mathcal{I})$ and $h = h(\mathcal{I})$, and a tuple

$$\mathcal{A} = (A^1, \ldots, A^s, T^1, \ldots, T^h) \in \mathbb{H}^s \times \mathbb{R}_+^h .$$

Then there exists a unique $P \in \mathcal{P}_d$ such that the vector field ξ_P has combinatorial invariant $\mathcal{I}(\mathcal{C}) = \mathcal{I}$ and analytic invariant $\mathcal{A}_P = \mathcal{A}$.

MOREOVER

Each $\mathcal{C}$ is a real-analytic submanifold of $\mathbb{C}^{d-1}$ isomorphic to $\mathbb{H}^s \times \mathbb{R}_+^h$,

hence of real-dimension $2s + h$.

Any pair of vector fields ξ_1, ξ_2 in $\mathcal{C}$ are dynamically equivalent:

there exists a quasi-conformal mapping $\Psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ mapping trajectories of ξ_1 onto trajectories of ξ_2, preserving orientation but not necessarily the parametrization by time.

DUAL DESCRIPTION

Meromorphic vector fields ⟷ Meromorphic abelian differentials

$$\xi_f(z) = f(z)\frac{\mathrm{d}}{\mathrm{d}z} \qquad\qquad \omega_f = \frac{1}{f(z)}\mathrm{d}z$$

obey similar transformation laws:

If $\varphi : U \to V$ is a holomorphic coordinate change and $w = \varphi(z)$ then

$$\varphi_*(\xi_f) = \xi_g \quad \text{and} \quad \varphi^*(\omega_g) = \omega_f \quad \text{where}$$

$$\xi_g(w) = g(w)\frac{\mathrm{d}}{\mathrm{d}w} \qquad g(\varphi(z)) = \varphi'(z)f(z) \qquad \omega_g = \frac{1}{g(w)}\mathrm{d}w$$

The singularities of ξ_f and ω_f are the zeros and the poles of f.

The two descriptions complement each other. The advantage of the differentials are that they can be integrated.

RECTIFYING COORDINATES

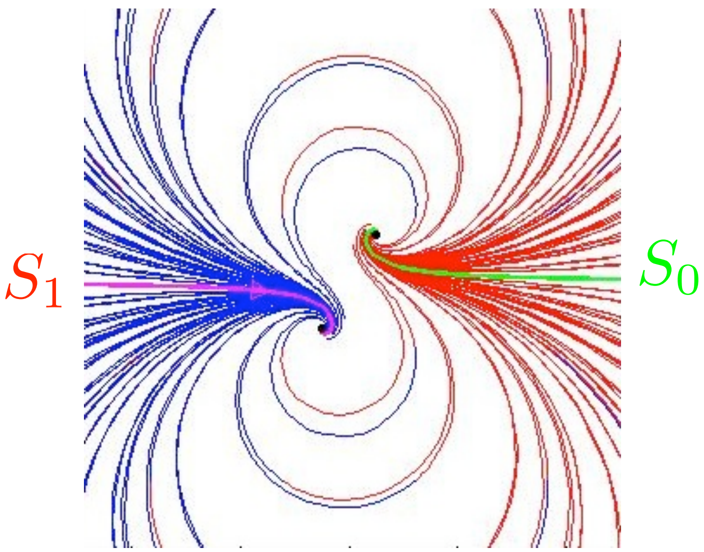
In any simply connected domain avoiding zeros of f the differential ω_f has an antiderivative, unique up to addition by a constant, $\phi(z) = \int \frac{1}{f(z)}\mathrm{d}z$

Note that $\phi_*(\xi_f) = \xi_g$ where $g(\phi(z)) = \phi'(z)f(z) = \frac{1}{f(z)}f(z) = 1$.

The coordinates $w = \phi(z)$ are called *rectifying coordinates* of ξ_f.

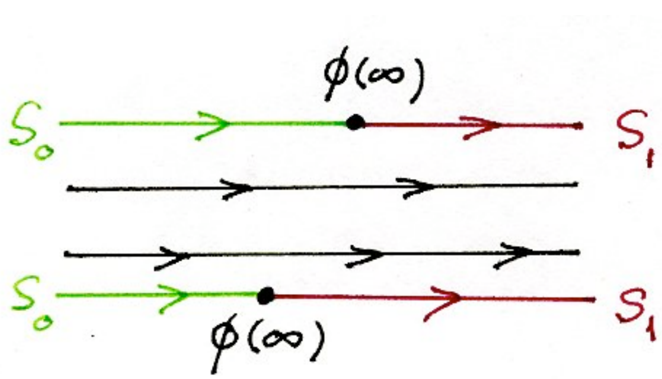
RECTIFYING COORDINATES for THE QUADRATIC EXAMPLES

One sink and one source



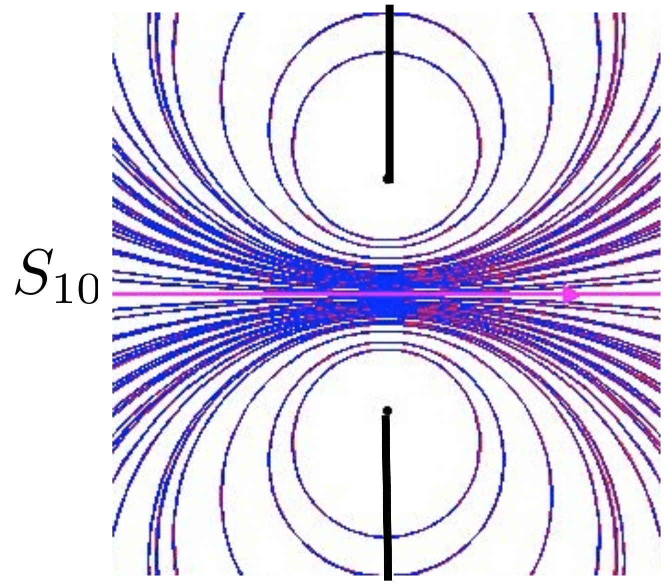
$\mathbb{C} \setminus \{S_0 \cup S_1\}$

$\downarrow \phi$



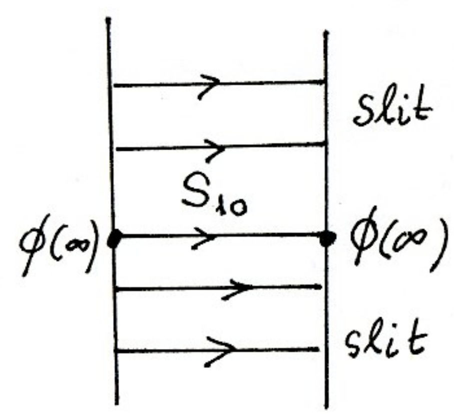
a horizontal strip

Two centers



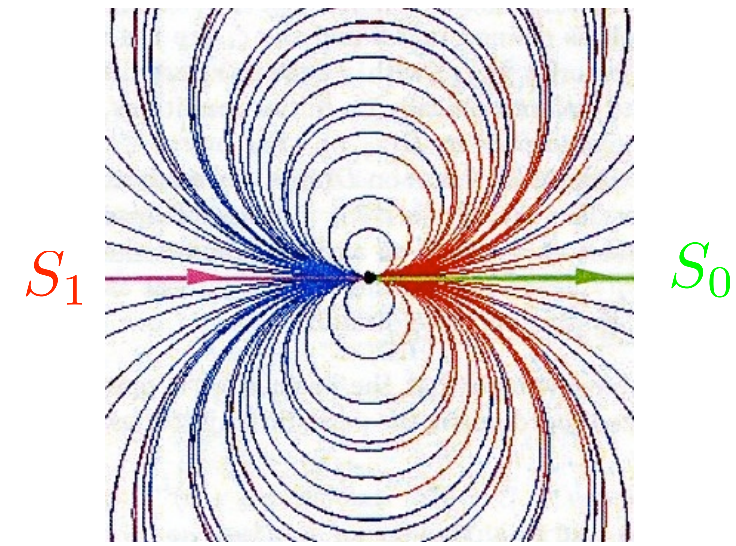
$\mathbb{C} \setminus \{\ \mathrm{i}\ ]-\infty, -1] \cup\ \mathrm{i}[+1, +\infty[\ \}$

$\downarrow \phi$



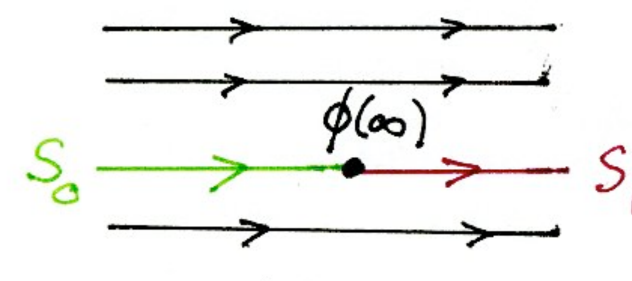
a vertical strip

One double equilibrium point



$\widehat{\mathbb{C}} \setminus \{\ 0\}$

$\downarrow \phi$



a plane

THE SINGULARITY at ∞

$(\mathbb{C}^*, \xi_P)$ is holomorphically conjugate to $\left(\mathbb{C}^*, f(z)\frac{\mathrm{d}}{\mathrm{d}z}\right)$ by $z \mapsto \frac{1}{z}$

where

$$f(z) = -\frac{1}{z^{d-2}}\left(1 + a_{d-2}z^2 + \cdots + a_0 z^d\right).$$

Hence, ξ_P has a pole of order $d-2$ at ∞.

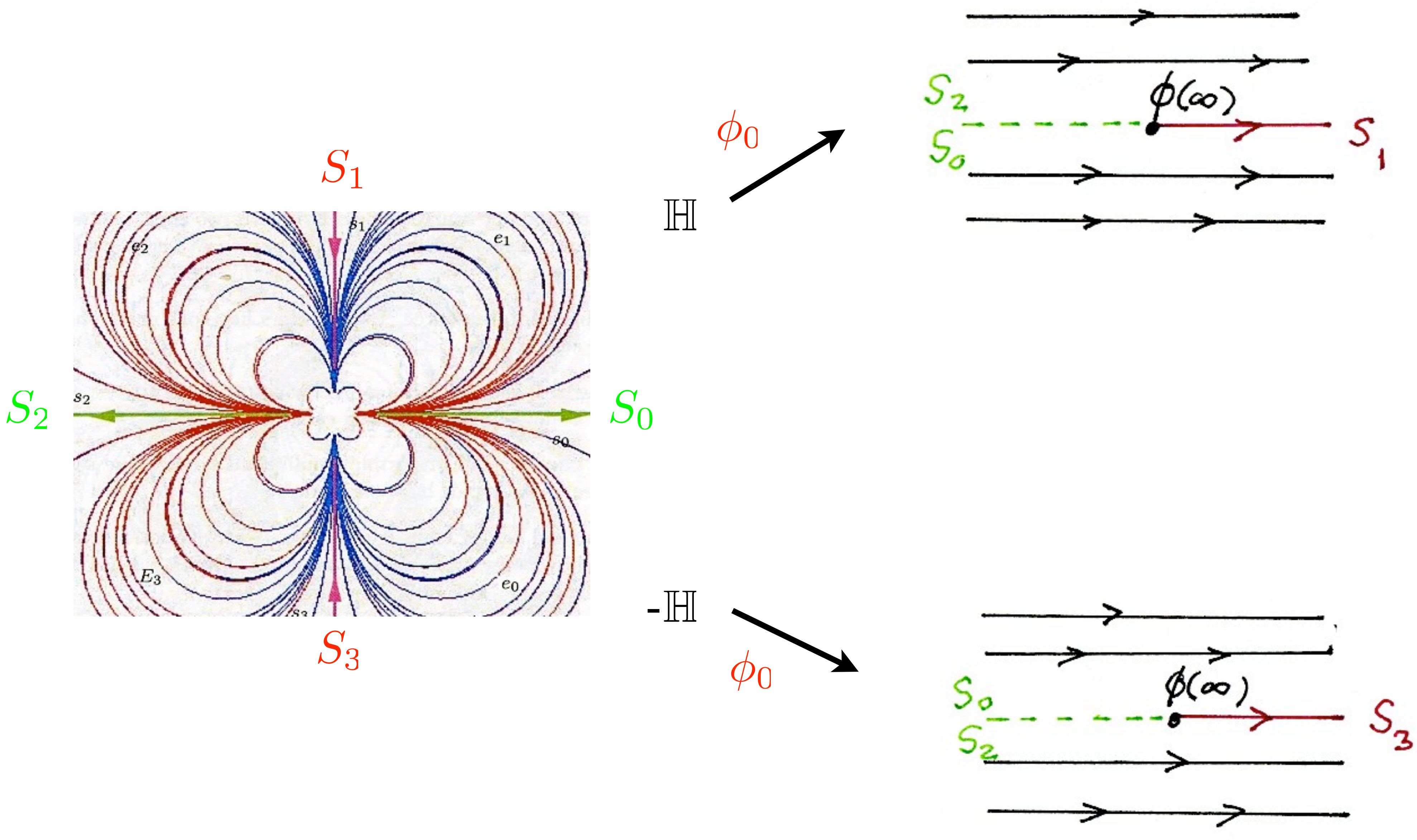
IN PARTICULAR

$\xi_0(z) = z^d \frac{\mathrm{d}}{\mathrm{d}z}$ is holomorphically conjugate to $f_0(z)\frac{\mathrm{d}}{\mathrm{d}z} = -\frac{1}{z^{d-2}}\frac{\mathrm{d}}{\mathrm{d}z}$

$\phi_0 : \widehat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$, $\quad \phi_0(z) = -\frac{1}{d-1}\frac{1}{z^{d-1}}$ is a branched covering, mapping ∞ to 0.

RECTIFYING COORDINATES for $\xi_0(z) = z^d \dfrac{\mathrm{d}}{\mathrm{d}z}$

EXAMPLE $d = 3$



THE SPECIAL ROLE of ∞ – SEPARATRICES

PROPOSITION Every ξ_P is holomorphically conjugate to ξ_0 in neighborhoods of infinity, by a conjugating map tangent to the identity at ∞.

CONSEQUENCES

There are $d-1$ incoming trajectories to ∞ and $d-1$ outgoing trajectories from ∞.

Their asymptotes are the half lines in directions $\delta_\ell = \exp\left(2\pi \mathrm{i}\, \frac{\ell}{2(d-1)}\right)$ where $\ell \in L = \{\ 0, 1, \ldots, 2d-3\ \}$ or $\ell \in \mathbb{Z}/2(d-1)$.

∞ is a *saddle point* for ξ_P .

Note, that for any point z_0 on such a trajectory it takes a finite amount of time to get to ∞ if incoming and to come from ∞ if outgoing.

SEPARTRICES

The separatrices are the maximal trajectories of ξ_P incoming to and outgoing from ∞ .
A separatrix is *homoclinic* if both outgoing from and incoming to ∞ .

LIMITING BEHAVIOUR of trajectories $\gamma(t, z_0), \quad t \in]\, t_-, t_+ \,[$

$$\lim_{t \to t_+} \gamma(t, z_0) = \begin{cases} \text{sink or multiple equilibrium point if } t_+ = +\infty \\ \infty \qquad t_+ \in \mathbb{R}_+ \end{cases}$$

$$\lim_{t \to t_-} \gamma(t, z_0) = \begin{cases} \text{source or multiple equilibrium point if } t_- = -\infty \\ \infty \qquad t_- \in \mathbb{R}_- \end{cases}$$

PROPOSITION Each sink or source is the landing point of at least one separatrix. Each multiple equilibrium point is the landing point of at least one separatrix tangent to any of the attracting or repelling directions.

LABELING the SEPARATRICES and THE SEPARATRIX GRAPH

A separatrix S_ℓ is labeled according to its asymptote, $\ell \in \{0.1, \dots, 2d-3\}$.

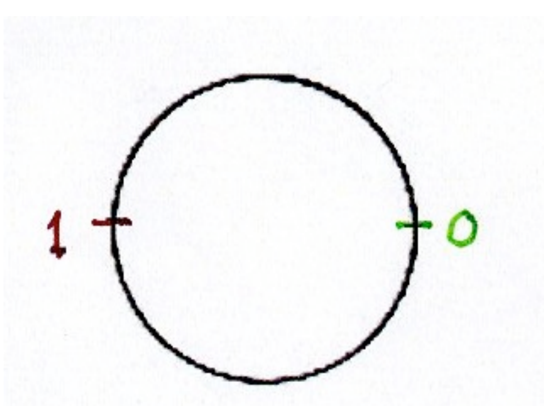
ℓ even corresponds to a separatrix incoming to ∞ .

ℓ odd corresponds to a separatrix outgoing from ∞ .

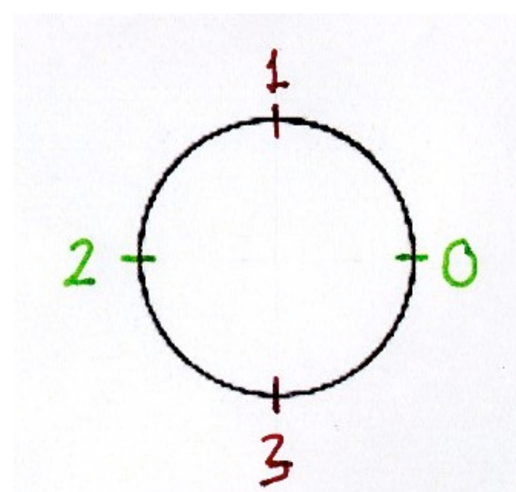
THE SEPARATRIX GRAPH Γ_P is the closure in $\widehat{\mathbb{C}}$ of the separatrices. Hence

$$\Gamma_P = \bigcup_{\ell=1,\dots,2d-3} S_\ell \quad \cup \bigcup_{\text{sink, source, mult.}} \zeta \quad \cup \{\infty\}$$

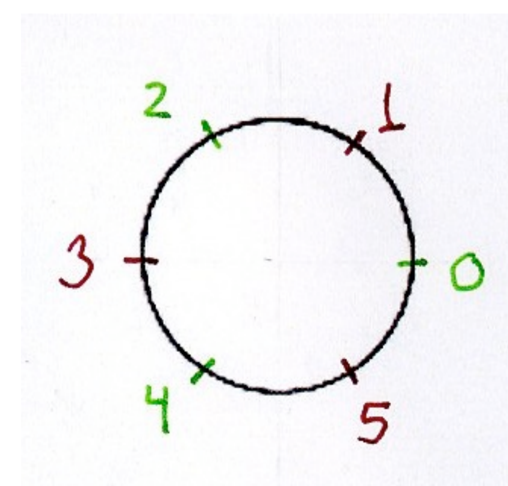
SYMBOLICALLY Mark $\delta_\ell \in \mathbb{S}^1$.



$d = 2$



$d = 3$



$d = 4$

THE COMBINATORIAL INVARIANT of $\xi_P \in \Xi_d$

DEFINITION of the COMBINATORIAL INVARIANT $\mathcal{I}_P$ of ξ_P:

$\mathcal{I}_P$ is an equivalence relation $\sim_P$ on $L = \{0, 1, \ldots, 2d-3\}$ with a specified subset $H_P \subset L$ satisfying:

- H_P consists of the labels ℓ for which S_ℓ is a homoclinic separatrix.

  For $\ell_1, \ell_2 \in H_P$:

$$\ell_1 \sim_P \ell_2 \quad \Longleftrightarrow \quad S_{\ell_1} = S_{\ell_2} \, .$$

- H_P is saturated by $\sim_P$.

- $L \setminus H_P$ consists of the labels ℓ for which S_ℓ lands at ζ, a sink, a source, or a multiple equilibrium point. For $\ell_1, \ell_2 \in L \setminus H_P$:

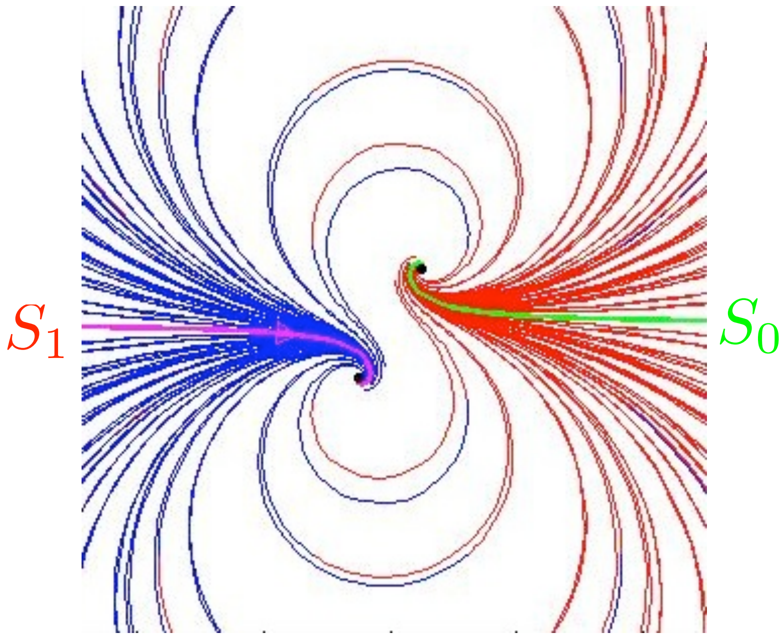
$$\ell_1 \sim_P \ell_2 \quad \Longleftrightarrow \quad S_{\ell_1} \text{ and } S_{\ell_2} \text{ lands in } \mathbb{C} \text{ at the same } \zeta \, .$$

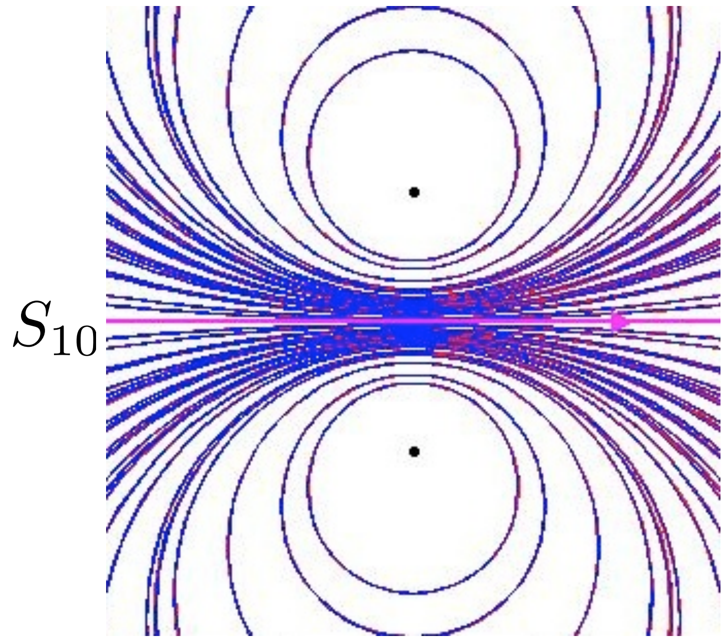
NOTE that there are three possible kinds of equivalence classes in $L \setminus H_P$:

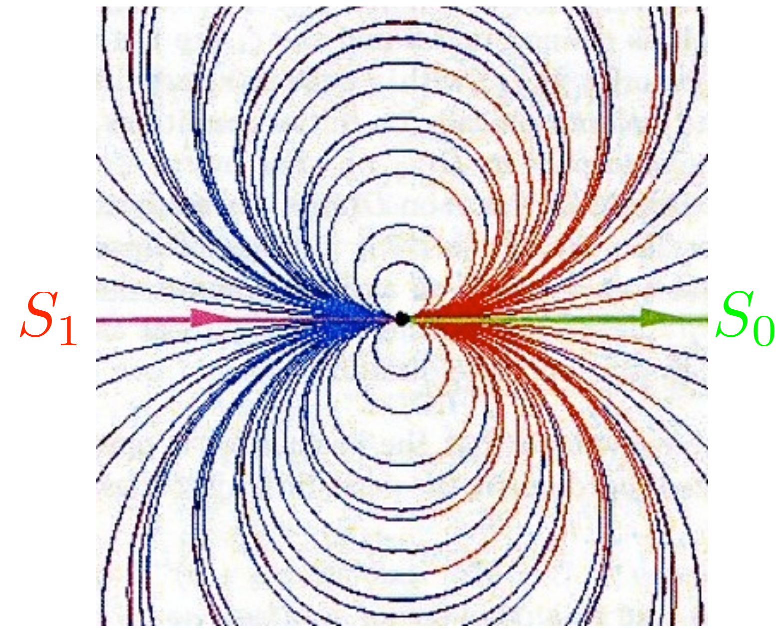
$[\ell]$ — odd / even / mixed — corresponding to ζ being a — sink / source / multiple equilibrium point

For mixed $[\ell]$ the multiplicity of ζ is m iff the ordered cyclic sequence of labels in $[\ell]$ changes parity $2(m-1)$ times.

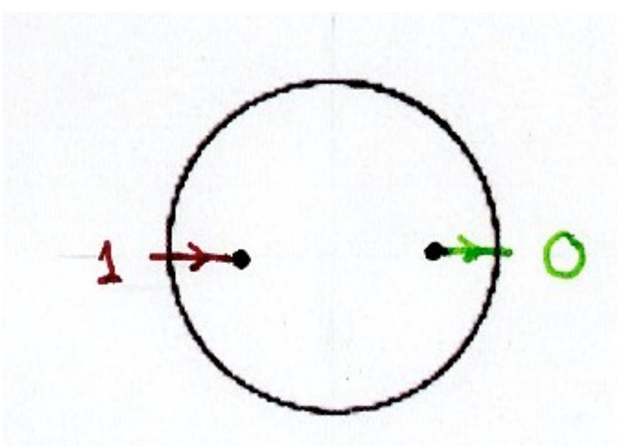
COMBINATORIAL INVARIANTS for the QUADRATIC EXAMPLES





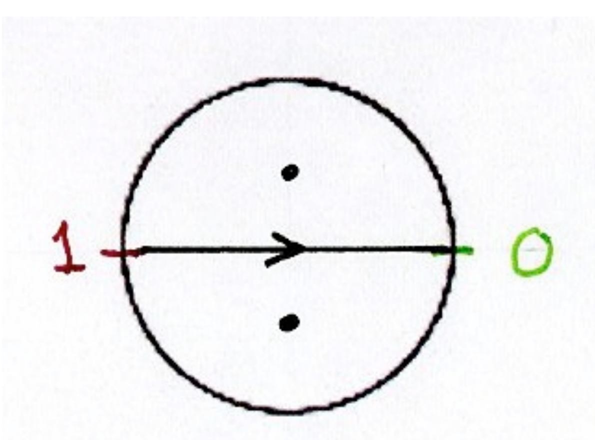


EQUIVALENCE CLASSES of $\sim_P$



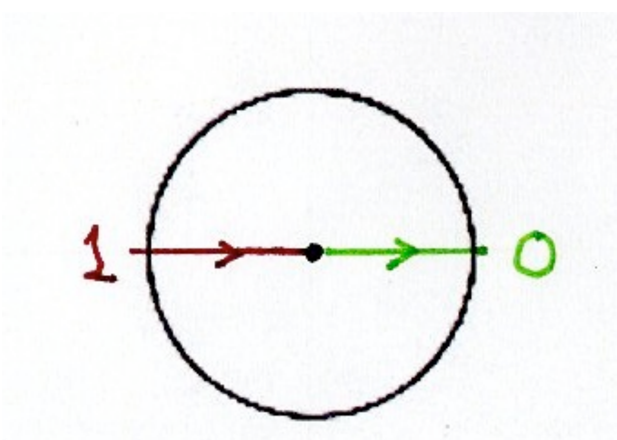
[0] even

[1] odd



$H_P = \{0, 1\}$

$[0, 1]$



$[0, 1]$ mixed

A CUBIC EXAMPLE with $H_P = \emptyset$ and its COMBINATORIAL INVARIANT

$P(z) = (z+1)^2(z-2)$

$\zeta = -1$ a double equilibrium point

$\zeta = 2$ a source

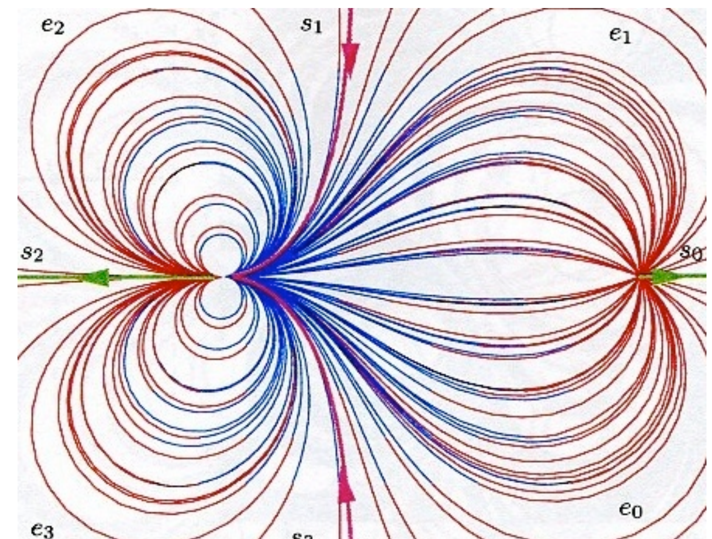
Attracting petal of $\zeta = -1 :=$
$\mathbb{C} \setminus \{\overline{S_0} \cup \overline{S_2}\}$

Repelling petal of $\zeta = -1 :=$
Half plane left of $\overline{S_1} \cup \overline{S_3}$

Two sepals := the intersection of
the attracting and the repelling petal.

a sepal　　S_1　　an interpetal

S_2



S_0

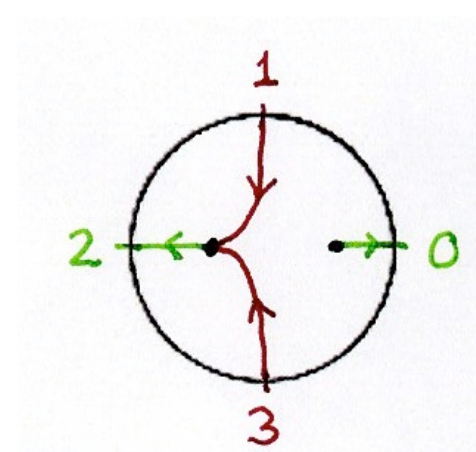
S_3

a sepal

EQUIVALENCE CLASSES of $\sim_P$

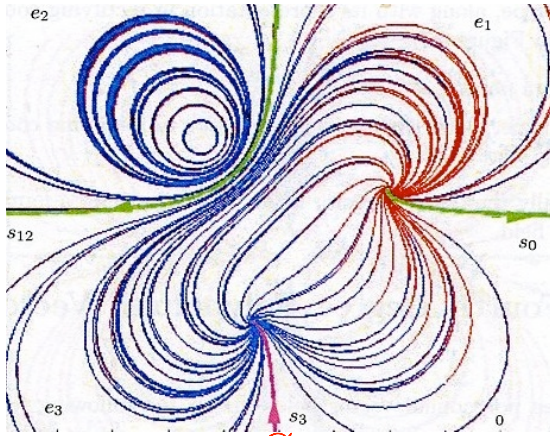
$[0]$ odd

$[1, 2, 3]$ mixed



CUBIC EXAMPLES with $H_P \neq \emptyset$ and their COMBINATORIAL INVARIANT

S_{12}

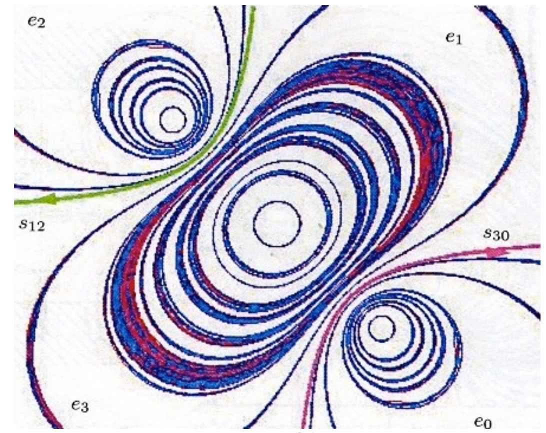


S_0

S_3

one center,
one sink, one source

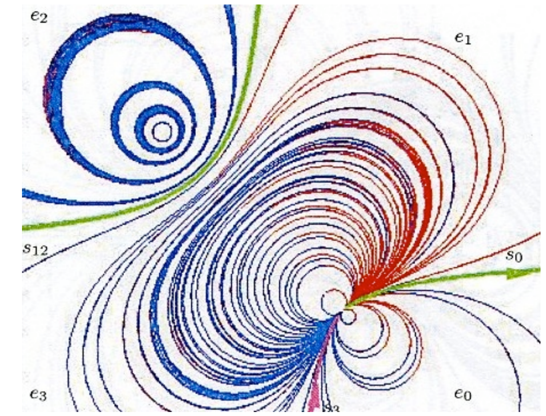
S_{12}



S_{30}

three centers

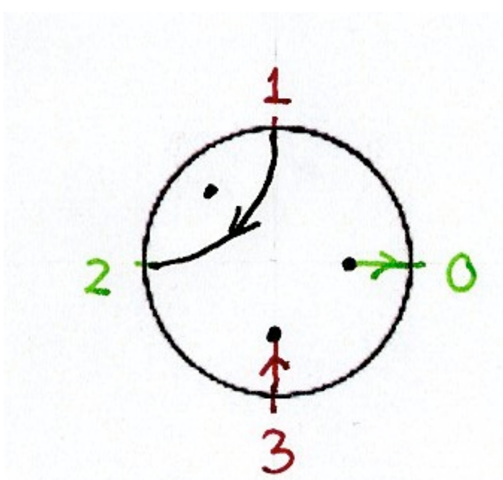
S_{12}



S_0

S_3

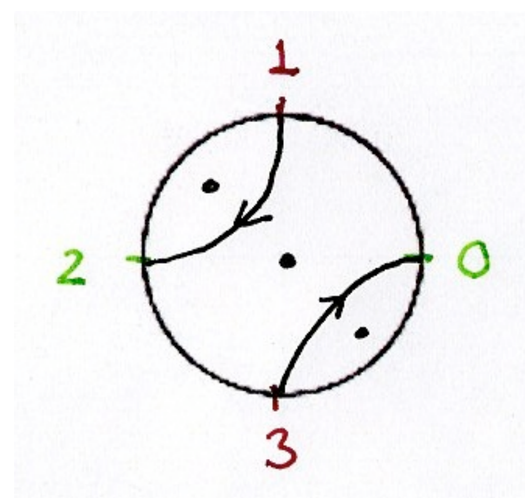
one center,
one double equilibrium



$H_P = \{1,2\} = [1,2]$

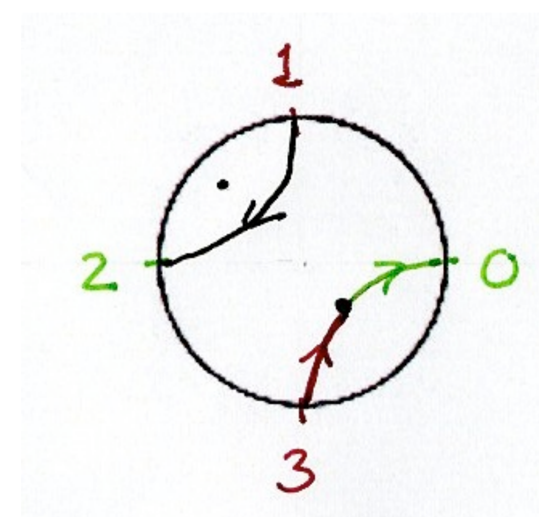
$[0]$ odd

$[3]$ even



$H_P = \{0,1,2,3\}$

$[1,2]$, $[3,0]$



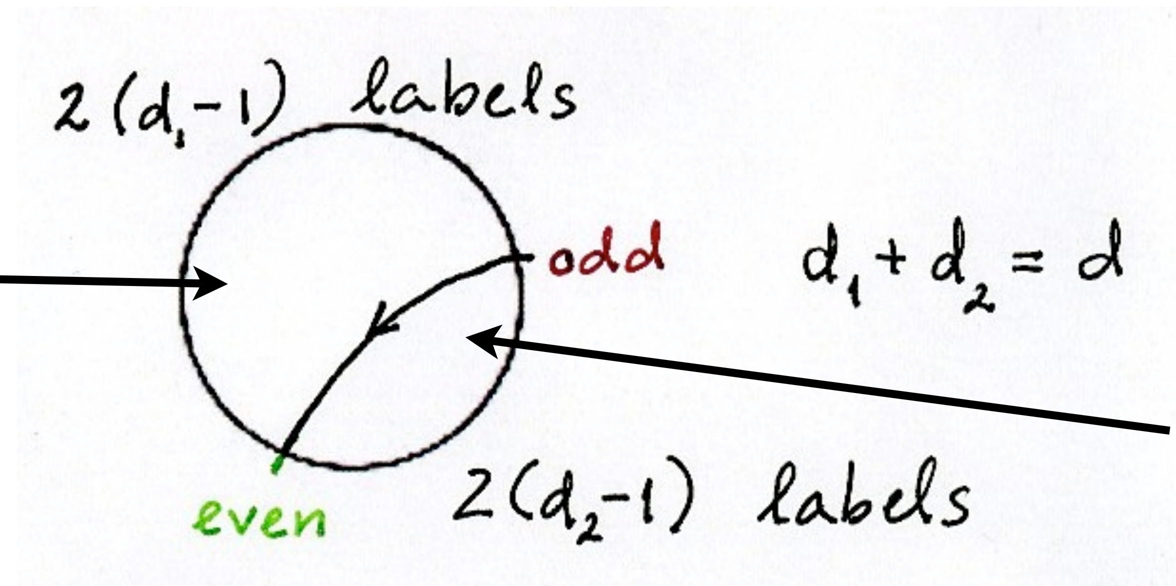
$H_P = \{1,2\} = [1,2]$

$[3,0]$ mixed

SUBDIVISION of LABELS if $H_P \neq \emptyset$:

polynomial-like of

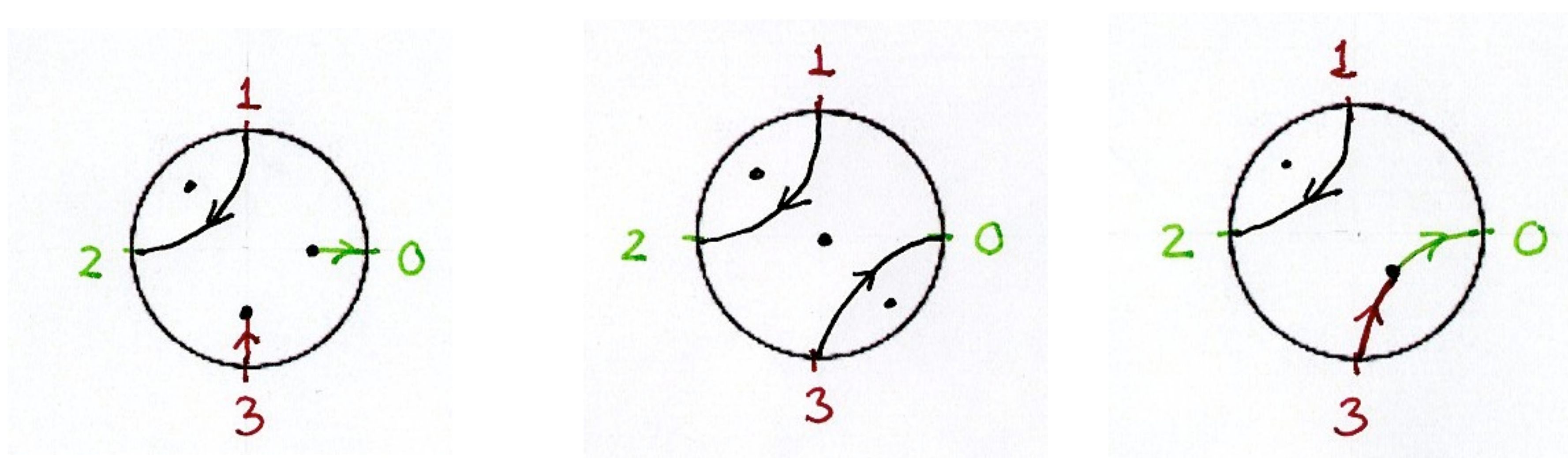
degree d_1



polynomial-like of
degree d_2

$d_1 = 1$

$d_2 = 2$



TYPES of ZONES

A zone Z is a connected component of $\widehat{\mathbb{C}} \setminus \Gamma_P$. There are three types, classified by the type of the holomorphic conjugacy

$$\phi : (Z, \xi_P) \to \left(\bullet \, , \frac{\mathrm{d}}{\mathrm{d}z} \right)$$

- An $\alpha\omega$-zone is isomorphic to a HORIZONTAL STRIP. $\exists$ two distinct equilibrium points:

  ζ_α a source or a multiple equilibrium,

  ζ_ω a sink or a multiple equilibrium, such that $\forall$ $z_0 \in Z$

  the $\begin{matrix} \alpha\text{-limit} \\ \omega\text{-limit} \end{matrix}$ of $\gamma(t, z_0)$ is $\begin{matrix} \zeta_\alpha \\ \zeta_\omega \end{matrix}$

  ∂Z consists of one or two incoming separatrices and one or two outgoing separatrices, and possibly some homoclinics.

- A sepal-zone is isomorphic to an UPPER or LOWER HALF PLANE.

  $\exists$ a multiple equilibrium ζ such that $\forall$ $z_0 \in Z$ the α-limit and the ω-limit is ζ .

  ∂Z consists of one incoming separatrices and one outgoing separatrices, and possibly some homoclinics.

- A center-zone contains one center ζ and $Z \setminus \zeta$ is isomorphic to a HALF UPPER or LOWER CYLINDER. $\forall\, z_0 \in Z \setminus \zeta$ $\gamma(t, z_0)$ is periodic of period $T = \dfrac{2\pi}{|P'(\zeta)|}$.

  ∂Z consists of one or several homoclinics.

COMBINATORIAL CLASSES

A combinatorial class $\mathcal{C}$ consists of all ξ_P with $\mathcal{I}_{\mathcal{P}} = \mathcal{I}(\mathcal{C})$. The integers

$$s = s(\mathcal{C}) \qquad \text{and} \qquad h = h(\mathcal{C})$$

are the numbers of $\alpha\omega$-zones (numbers of strips) and the number of homoclinics (half the number of labels in H_P) respectively.

A COMBINATORIAL DATA SET

DEFINITION

Given $d \geq 2$, an equivalence relation $\sim$ on $L = \{0, 1, \ldots, 2d-3\}$, and a specified subset $H \subset L$ consisting of $2h \geq 0$ labels, h odd and h even.

$(\sim, H)$ is *a combinatorial data set* if it satisfies:

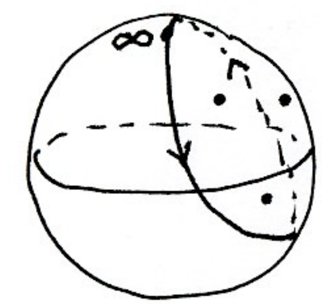
- $\sim$ is non-crossing.
- H is saturated by $\sim$, and each equivalence class in H consists of an odd and an even label.
- Zones in the disc model are of the three types: $\alpha\omega$, sepal, center.

ANALYTIC INVARIANTS for a given ξ_P

Each homoclinic separatrix for ξ_P is assigned the positive real time T_P it takes to travel along the oriented trajectory from ∞ to ∞ .

Each $\alpha\omega$-zone of ξ_P is assigned the complex "time" A_P it takes to travel along the *transversal* in the zone closest to the α-limit point ζ_α from ∞ to ∞.

We choose $A_P \in \mathbb{H}$.

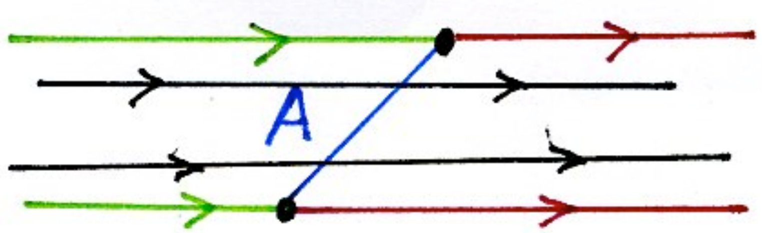


In each case the invariant can be expressed as

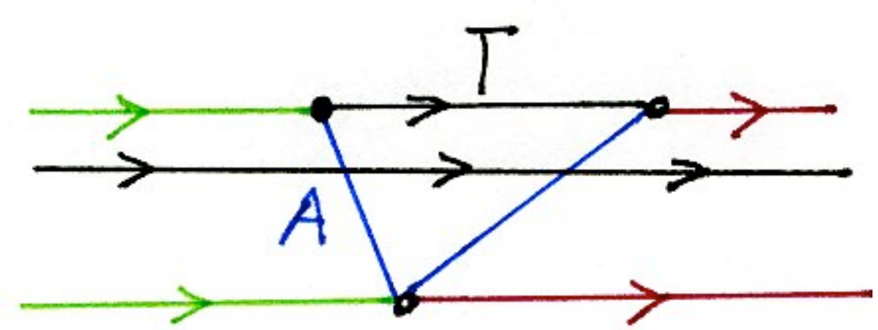
$$\int_{\mathrm{loop}} \frac{1}{P} = 2\pi \mathrm{i} \sum_{\zeta \ \text{left of loop}} \mathrm{Res}\left(\frac{1}{P}, \zeta\right) = -2\pi \mathrm{i} \sum_{\zeta \ \text{right of loop}} \mathrm{Res}\left(\frac{1}{P}, \zeta\right)$$

where the loop is either the homoclinic or the transversal and the summation is over all equilibrium points ζ left of the loop (or all right of the loop).

ζ_α



ζ_α

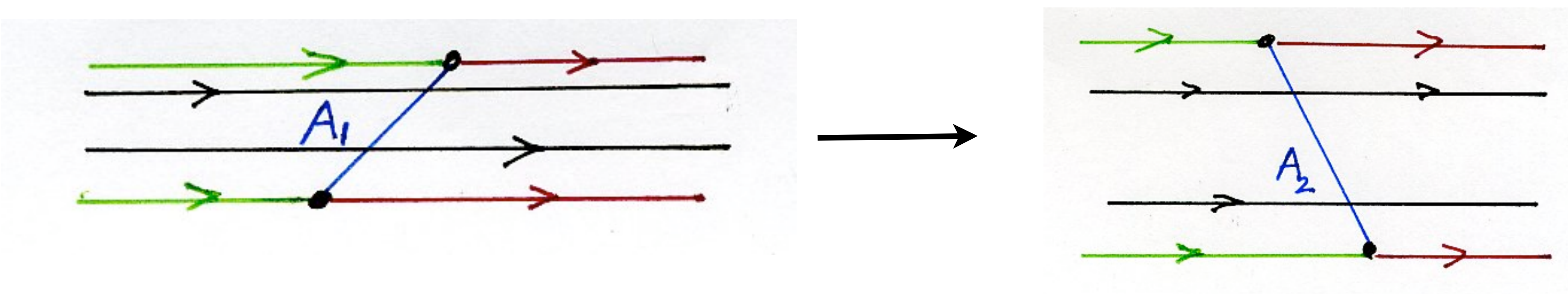


QC DYNAMICAL EQUIVALENCE

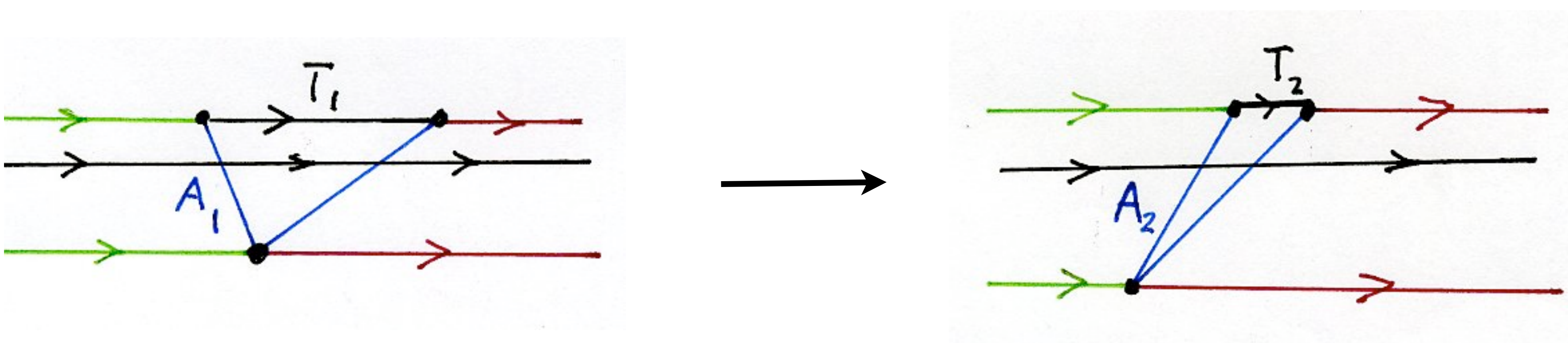
Suppose ξ_{P_1}, ξ_{P_2} belong to the same combinatorial class. Then they have the same qualitative dynamics. In rectifying coordinates the equivalence is given through piecewise affine mappings.

AMONG $\alpha\omega$-zones , represented in rectifying coordinates:

affine map, mapping the base $\{1, A_1\}$ to the base $\{1, A_2\}$.



piecewise affine map; on the triangle, mapping the base $\{T_1, A_1\}$ to the base $\{T_2, A_2\}$.

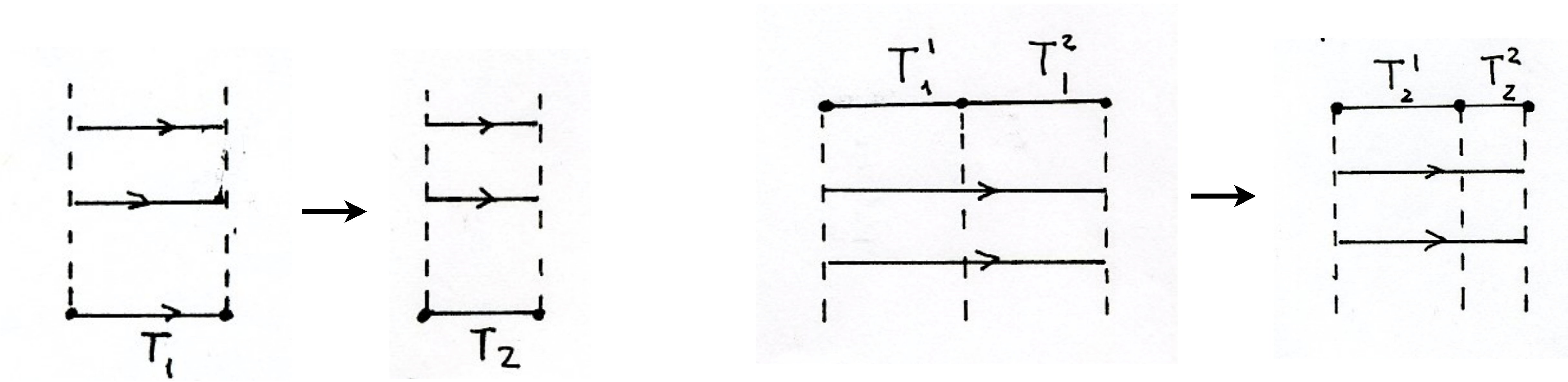


QC DYNAMICAL EQUIVALENCE

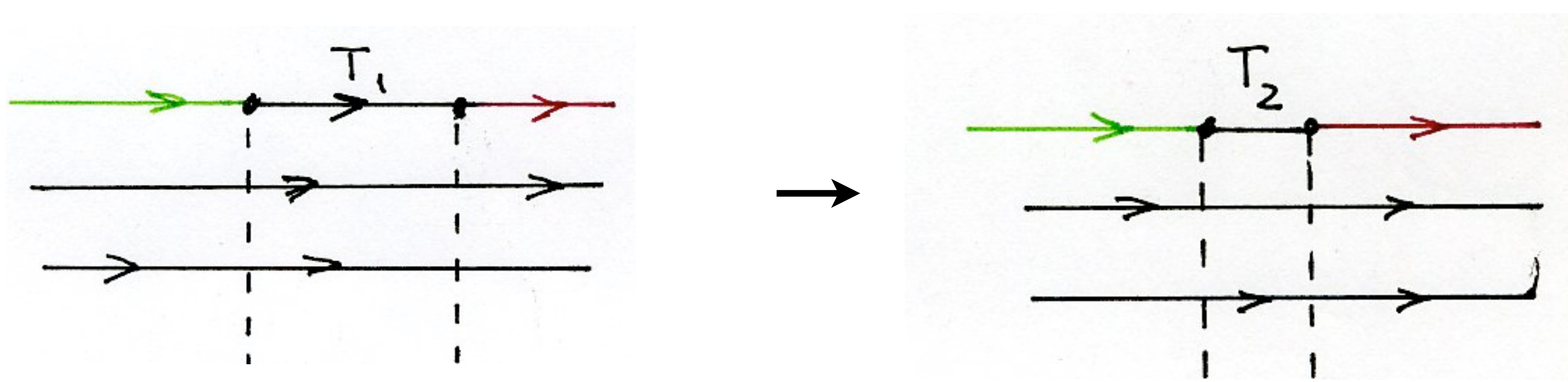
Suppose ξ_{P_1}, ξ_{P_2} belong to the same combinatorial class.

AMONG center-zones: affine or piecewise affine mappings,

mapping the base $\{T_1, \mathrm{i}\}$ to the base $\{T_2, \mathrm{i}\}$, or the base $\{T_1^j, -\mathrm{i}\}$ to the base $\{T_2^j, -\mathrm{i}\}$.



AMONG sepal-zones with homoclinics : piecewise affine mappings.



THE STRUCTURE THEOREM

Given $d \geq 2$, a combinatorial data set $(\sim, H)$ and

$$\mathcal{A} = (A^1, \ldots, A^s, T^1, \ldots, T^h) \in \mathbb{H}^s \times \mathbb{R}^h_+$$

where s is the number of $\alpha\omega$-zones for $(\sim, H)$ and h is the number of equivalence classes in H. There exists a unique $\xi_P \in \Xi_d$ realizing the above, i.e. $(\sim_P, H_P) = (\sim, H)$ and $\mathcal{A}_P = \mathcal{A}$.

PROOF by surgery. From the rectified building blocks we construct a Riemann surface $\mathcal{M}$ with a vector field $\xi_{\mathcal{M}}$ and prove that $\mathcal{M}$ is isomorphic to $\widehat{\mathbb{C}}$ and that there exists a unique $P \in \mathcal{P}_d$ such that $(\mathcal{M}, \xi_{\mathcal{M}})$ is holomorphically conjugate to $(\widehat{\mathbb{C}}, \xi_P)$.

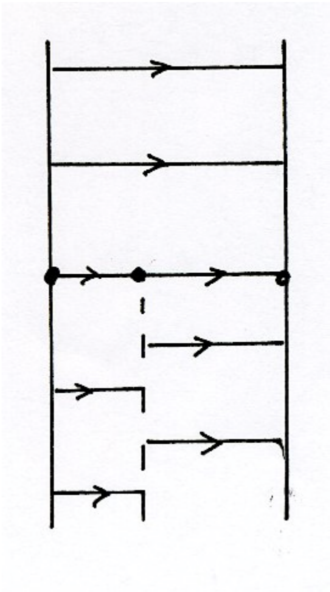
STRUCTURALLY STABLE

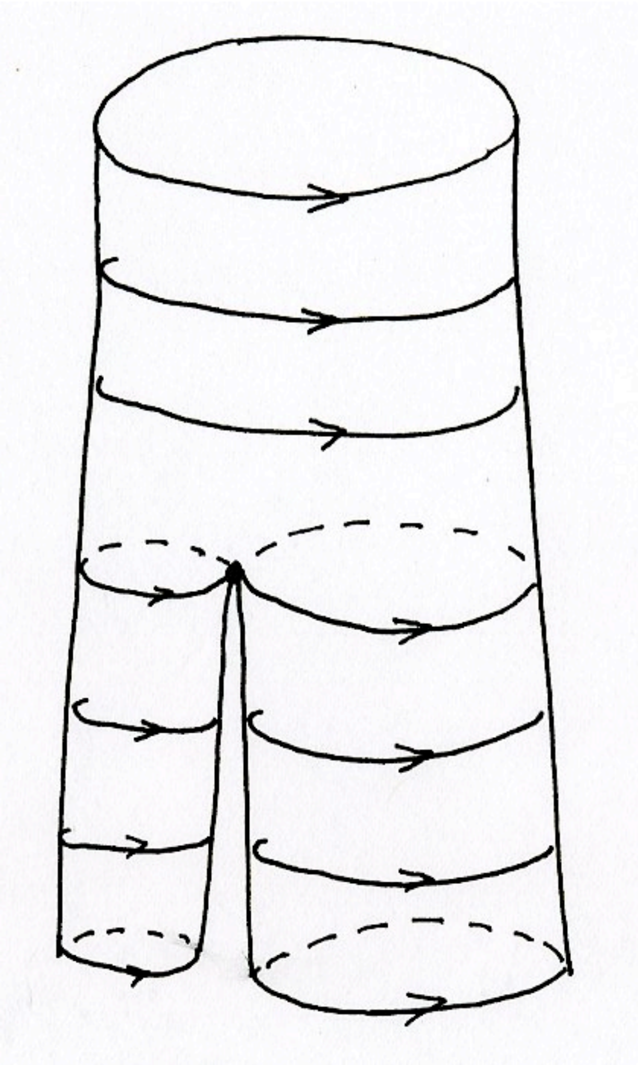
If ξ_P has only sinks and sources, no homoclinic separatrices and no multiple equilibrium points, then the number of $\alpha\omega$-zones takes its maximal value $s = d - 1$. It follows that ξ_P belongs to a class, which is isomorphic to $\mathbb{H}^{d-1}$. Hence ξ_P is structurally stable.

BIFURCATION SET

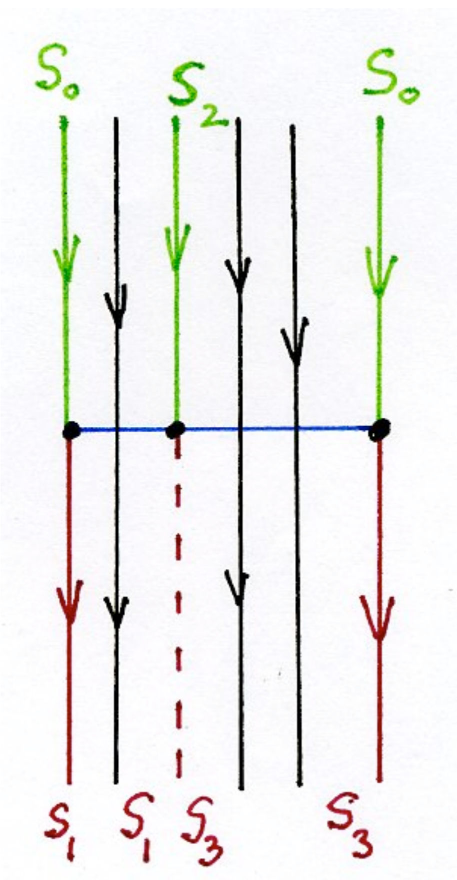
If ξ_P has a homoclinic separatrix or a multiple equilibrium point then $s < d - 1$ and $2s + h < 2(d - 1)$. Hence ξ_P is in the bifurcation set.

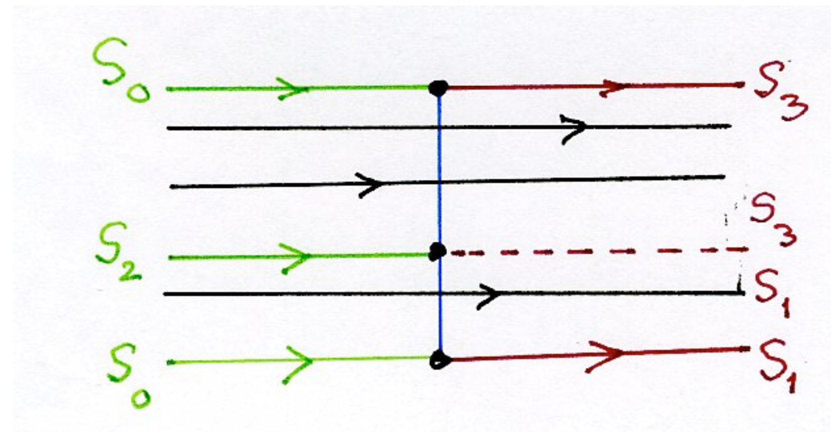
ILLUSTRATING the SURGERY in a FAMILIAR case $d = 3$

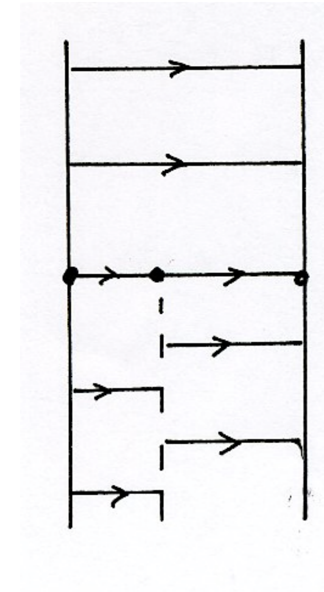


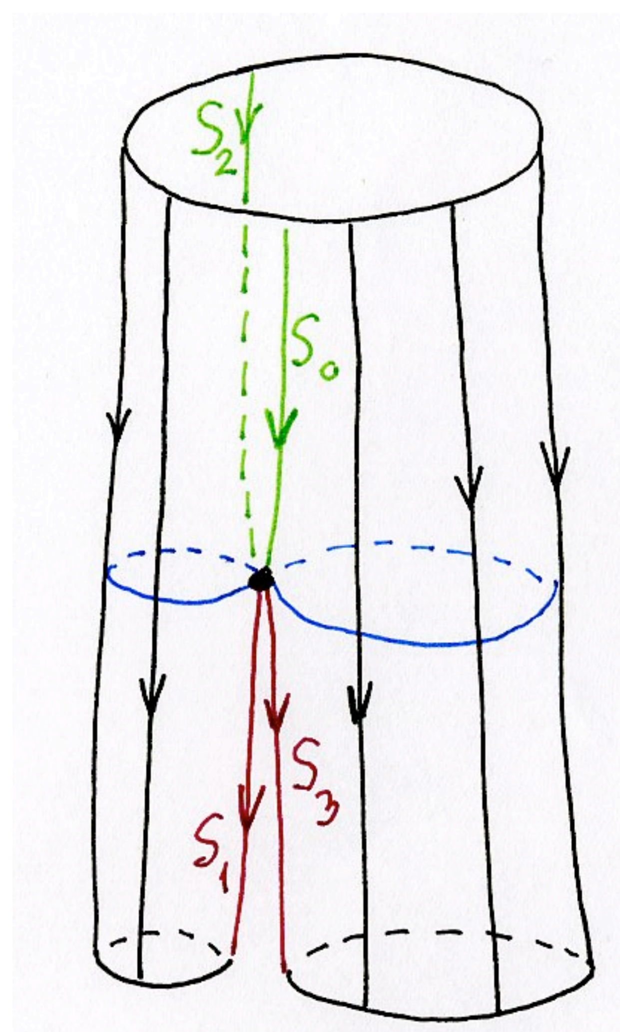


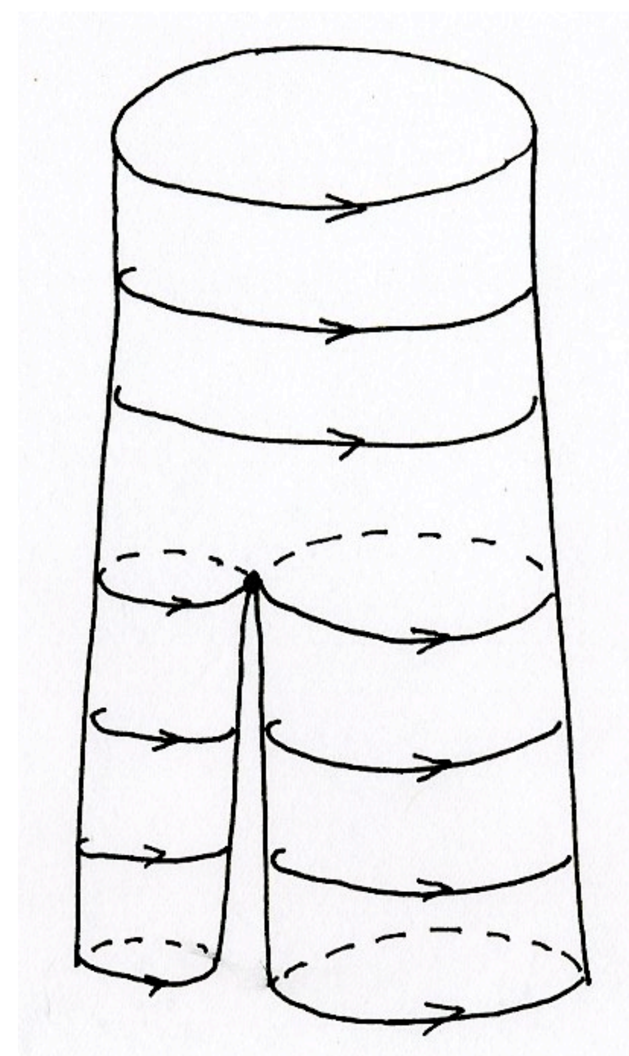
ILLUSTRATING the SURGERY in a FAMILIAR case $d = 3$











ILLUSTRATING the SURGERY in a FAMILIAR case $d = 3$