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Analysis of Queues with Rational Arrival Process Components - A General Approach

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Abstract

In a previous paper we demonstrated that the well known matrix-geometric solution of Quasi-Birth-and-Death (QBD) processes is valid also if we introduce Rational Arrival Process (RAP) components. Here we extend those results and we offer an alternative proof by using results obtained by Tweedie.

We prove the matrix-geometric form for certain kind of operators on the stationary measure for discrete time Markov chains of GI/M/1 type. We apply this result to an embedded Markov chain modelling a queue with RAP components. We also discuss the straightforward modification of the standard algorithms for calculating the matrix R in the traditional QBD framework to this extended environment.

Finally we present examples demonstrating great reductions in dimensionality from the traditional QBD framework to the QBD – RAP framework.

Key Words : Rational Arrival Processes, Quasi-Birth-and-Death Processes, Matrix-Analytic Methods, Algorithmic Probability.

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1 Introduction

Neuts [18] introduced the matrix geometric solutions to queues with components of phase-type (PH) distributions and the versatile arrival process, later streamlined as the Markovian Arrival Process (MAP) [15]. Lipsky and coauthors [12, 13, 14] and later Asmussen and Bladt [2] showed how the PH formalism generalises into a Matrix–Exponential (ME) form with similar analytic expressions albeit without the probabilistic interpretation of sojourn times in finite state Markov chains. PH distributions belong to a strict subclass of distributions with rational Laplace transform [8], which were later shown to be equivalent to the class of ME distributions [12, 13, 14]. Bladt and Neuts [6] discussed an interpretation of ME distributions as flows leading to a possible way of extending the field of Matrix-Analytic Methods (MAM) to models involving ME-distributions.

The extension of the MAP to Rational Arrival Processes (RAP) was given by Asmussen and Bladt [3]. They also showed that the class of RAPs was the class of all point processes where a time-shifted version varied in a finite-dimensional space. See also Mitchell [17] for related work.

In [4] we showed how to generalise the method of Ramaswami [24] to the setting of a certain bivariate Markov chain, the most important sub-model probably being the RAP/ME/1 queue. Thus in that paper we showed how to extend the most modern MAM arguments to queues with RAP components. In this paper we provide an alternative proof that can be applied in a more general setting. Tweedie [27] showed how the MAM formulas in discrete time extend to cases with general phase state space turning the non-linear matrix–polynomial equation into an operator–polynomial equation. To our knowledge this work has only had limited applications so far, see Sengupta [25], Nielsen and Ramaswami [20] and Breuer [7] for examples.

In [20] Nielsen and Ramaswami demonstrated how the operator equation turns into a Neuts–type matrix equation, see e.g. [18, 19], when the underlying operators can be represented by a countable set of basis functions. The examples of Nielsen and Ramaswami might be considered somewhat contrived. Nevertheless, that paper demonstrated how a linear structure in the operator might transform into a matrix–polynomial equation. The equivalence of the RAP to the class of point processes on a finite-dimensional space and the linear nature of the RAP construction, gives hope that the operator–polynomial form will translate into some form of matrix–polynomial equation too.

In this paper we show that this is indeed the case. In contrast to Nielsen and Ramaswami [20], where the measure itself had a linear expansion, we use a kind of operator

linearity. We consider an operator (Γ), that maps measures to some vector space, such that Γ can be considered a descriptor of the measure. We then consider operators $\boldsymbol{\Pi}$ mapping measures to measures in such a way that the effect on the descriptor Γ is a linear function, identifying a matrix P, characterising the operator $\boldsymbol{\Pi}$. We then show how to obtain the value of the operator evaluated at the stationary measure without explicitly calculating the stationary measure. This is done by solving a matrix equation similar to the well-known matrix equation of the standard MAM setting. In doing so we establish a general framework in which the work by Nielsen and Ramaswami (linear expansion of the measure) and our work on queues with RAP components (linear nature of the RAP construction) both turn out to be special cases.

In Section 2 we set up the terminology and present the concept of an operator being Γ-linear, that is linear with respect to a descriptor Γ, that maps measures to a vector space. In Section 3 we apply this concept to the results of Tweedie for Markov chains of the GI/M/1-type structure, where the phase variable takes values in some general space. Now the work of Nielsen and Ramaswami arises as a special case of these more general results. We then move on to show how to make an appropriate formulation for queues with RAP-components in Section 4, where the operator Γ is the expectation operator. Specifically, we apply the theory to the GI/RAP/1 queue and the QBD with RAP-components. This latter queue was the topic of our previous paper [4]. In Section 5 we present our examples. These have been carefully chosen to demonstrate two main points. First, that such analysis is correct and the algorithms appear to be numerically stable. For this reason we have chosen examples where the representations of the underlying distributions are genuinely ME of a given order (and not PH of that order), but for which there exist PH representations using higher orders. This enables us to analyse the models directly using the theory contained in this paper and to cross-check these results with those attained using the traditional analysis involving the (larger) PH representations. Second, we consider a sequence of such examples, where the equivalent PH representation requires increasing order. This demonstrates the potential computational savings of being allowed to use the more compact ME representations and allows us to give an initial demonstration of the numerical stability of our approach, even as the underlying distributions leave the class of PH distributions. In Appendix A we develop these example processes and distributions from first principles and include detailed explanations of how they were created with the desired properties. We hope that these may become standard examples for future work in this area.

We delay the consideration of M/G/1-type structures with RAP components to future

work.

2 Operator Linearity

In this section we introduce the general framework, which is a Markov chain on the state space $\mathbb{N}_0 \times \mathbb{J}$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{N}$ is the natural numbers, and $\mathbb{J}$ is a general measurable space. In turn we define a certain property (Γ-linearity) of operators that map measures on $\mathbb{J}$ to other measures on $\mathbb{J}$.

2.1 Simple properties of operator linearity

We consider a set $\mathbb{J}$ equipped with a σ-algebra $\mathcal{J}$ and denote the set of signed finite measures on $(\mathbb{J}, \mathcal{J})$ by $\mathbb{M}$. Of particular importance is the subset $\mathbb{M}_p \subset \mathbb{M}$ consisting of the measures with total variation at most 1. Next we define the set of operators (kernels) that take an element of $\mathbb{M}_p$ to $\mathbb{M}_p$ and denote that set by $\mathbb{P}$, such that $\hat{\Pi} \in \mathbb{P} : \mathbb{M}_p \to \mathbb{M}_p$. The operator $\hat{\Pi}$ is defined through its kernel $\tilde{\Pi}(x, J)$, as $\hat{\Pi}(\varphi)(J) = \int_{\mathbb{J}} \varphi(\mathrm{d}x)\tilde{\Pi}(x, J)$, where $x \in \mathbb{J}$ and $J \in \mathcal{J}$, $\tilde{\Pi}(x, J)$ is such that $\tilde{\Pi}(x, \cdot)$ is a measure for each $x \in \mathbb{J}$, and $\tilde{\Pi}(\cdot, J)$ is measurable in $\mathbb{J}$ for fixed J. We then define the set $\mathbb{G}$ of linear continuous operators on $\mathbb{M}^\star \subseteq \mathbb{M}$ taking values in some real or complex, normed vector space $\mathbb{V}$ with a countable basis, such that $\Gamma \in \mathbb{G} : \mathbb{M}^\star \to \mathbb{V}$. Of course Γ might be defined and linear for all $\varphi \in \mathbb{M}$, in which case we can take $\mathbb{M}^\star = \mathbb{M}$. Thus an operator Γ is a descriptor that extracts some characteristic from a signed finite measure $\mu \in \mathbb{M}^\star$. We will take special interest in the restriction of Γ to $\mathbb{M}_p^\star = \mathbb{M}_p \cap \mathbb{M}^\star$ of measures of total variation at most one.

Definition 1 *An element $\hat{\Pi} \in \mathbb{P}$ is said to be Γ-linear with respect to $\mathbb{M}_p^\star \subseteq \mathbb{M}_p$ if $\hat{\Pi} : \mathbb{M}_p^\star \to \mathbb{M}_p^\star$ and $\Gamma(\hat{\Pi}(\varphi)) = \Gamma(\varphi)P$, for all $\varphi \in \mathbb{M}_p^\star$, for a unique matrix P. Whenever $\mathbb{M}_p^\star = \mathbb{M}_p$ we simply say that $\hat{\Pi}$ is Γ-linear.*

Without loss of generality, we can choose $\mathbb{V}_p^\star$, the image of $\mathbb{M}_p^\star$ in such a way that the matrix P is unique. To see this, suppose that P_1 and P_2 are two matrices such that $\Gamma(\hat{\Pi}(\varphi)) = \Gamma(\varphi)P_1$, and $\Gamma(\hat{\Pi}(\varphi)) = \Gamma(\varphi)P_2$, for all $\varphi \in \mathbb{M}_p^\star$. Therefore, $\Gamma(\varphi)P_1 = \Gamma(\varphi)P_2$ for all $\varphi \in \mathbb{M}_p^\star$ and so $P_1 - P_2$ maps all vectors in $\Gamma(\mathbb{M}_p^\star)$ to the zero vector. Consequently, we can redefine the vector space $\mathbb{V}_p^\star$ in order that P is unique. Some care is needed in doing this when more than one Γ-linear operator isconsidered simultaneously.

Lemma 1 *If $\hat{\Pi}_1 \in \mathbb{P}$ and $\hat{\Pi}_2 \in \mathbb{P}$ are both Γ-linear with respect to $\mathbb{M}_p^\star$ with matrices P_1 and P_2 respectively, then $\hat{\Pi} = \hat{\Pi}_2(\hat{\Pi}_1)$ given by the kernel $\tilde{\Pi}$: $\hat{\Pi}(\varphi)(\cdot) = \int_J \varphi(\mathrm{d}x)\tilde{\Pi}(x, \cdot) = \int_{\mathbb{J}} \int_{\mathbb{J}} \varphi(\mathrm{d}x)\tilde{\Pi}_1(x, \mathrm{d}y)\tilde{\Pi}_2(y, \cdot))$ is Γ-linear with $P = P_1 P_2$.*

Proof: For any $\varphi \in \mathbb{M}_p^\star$

$$\Gamma(\hat{\Pi}(\varphi)(\cdot)) = \Gamma\left(\int_{\mathbb{J}} \int_{\mathbb{J}} \varphi(\mathrm{d}x)\tilde{\Pi}_1(x, \mathrm{d}y)\tilde{\Pi}_2(y, \cdot)\right) = \Gamma\left(\int_{\mathbb{J}} \nu(\mathrm{d}y)\tilde{\Pi}_2(y, \cdot)\right),$$

where $\nu(\cdot) = \int_{\mathbb{J}} \varphi(\mathrm{d}x)\tilde{\Pi}_1(x, \cdot)$. Thus by the Γ-linearity of $\hat{\Pi}_2$, we have

$$\Gamma(\hat{\Pi}_2(\hat{\Pi}_1(\varphi))(\cdot)) = \Gamma(\nu(\cdot))P_2 = \Gamma\left(\int_{\mathbb{J}} \varphi(\mathrm{d}x)\tilde{\Pi}_1(x, \cdot)\right)P_2 = \Gamma(\varphi(\cdot))P_1 P_2,$$

by the Γ-linearity of $\hat{\Pi}_1$. ∎

2.2 Level-partitioned discrete-time Markov chain with a general phase space $\mathbb{J}$

We now consider a Markov chain $X_n = (L_n, \boldsymbol{J}_n)$ in discrete time on the bivariate state space $(\mathbb{N}_0 \times \mathbb{J})$, where $\mathbb{J}$ is some general measurable space. We have chosen to use the symbol $\boldsymbol{J}$ to describe the second component of the state, which we will call the phase throughout. The block-structured matrix

$$\tilde{P}(x, J) = \begin{bmatrix} \tilde{P}_{00}(x, J) & \tilde{P}_{01}(x, J) & \tilde{P}_{02}(x, J) & \dots \\ \tilde{P}_{10}(x, J) & \tilde{P}_{11}(x, J) & \tilde{P}_{12}(x, J) & \dots \\ \tilde{P}_{20}(x, J) & \tilde{P}_{21}(x, J) & \tilde{P}_{22}(x, J) & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}, \tag{1}$$

is the transition kernel of X_n, that is, $P(L_{n+1} = j, \boldsymbol{J}_{n+1} \in J | L_n = i, \boldsymbol{J}_n = x) = \tilde{P}_{i,j}(x, J)$. Here $x \in \mathbb{J}$ and $J \in \mathcal{J}$ is any measurable set in the σ-algebra $\mathcal{J}$ imposed on $\mathbb{J}$. For a discussion of Markov chains on a general state space, see the book by Meyn and Tweedie [16]. Obviously the $\tilde{P}_{i,j}$ define $\hat{\Pi}_{i,j} \in \mathbb{P}$ by

$$\hat{\Pi}_{i,j}(\varphi_i)(J) = \int_{\mathbb{J}} \varphi_i(\mathrm{d}x)\tilde{P}_{i,j}(x, J).$$

Let $\mathbb{K}$ denote the set of signed finite measures on $\mathbb{N}_0 \times \mathcal{J}$ and $\mathbb{K}_p$ denote the set of measures of total variation most 1 on $\mathbb{N}_0 \times \mathcal{J}$. It follows that $\tilde{P}$ correspondingly defines $\hat{\Pi} : \mathbb{K}_p \to \mathbb{K}_p$,

with $\boldsymbol{\varphi} = (\varphi_0, \varphi_1, \varphi_2 \dots) \in \mathbb{M}_{\mathbb{N}_0 \times \mathcal{J}}$, by

$$\hat{\Pi}(\boldsymbol{\varphi}) = \left(\sum_{i=0}^{\infty} \hat{\Pi}_{i,0}(\varphi_i), \sum_{i=0}^{\infty} \hat{\Pi}_{i,1}(\varphi_i), \dots \right)$$

$$= \left(\sum_{i=0}^{\infty} \int_{\mathbb{J}} \varphi_i(\mathrm{d}x) \tilde{P}_{i,0}(x, \cdot), \sum_{i=0}^{\infty} \int_{\mathbb{J}_1} \varphi_i(\mathrm{d}x) \tilde{P}_{i,1}(x, \cdot), \dots \right).$$

Lemma 2 *Let $\boldsymbol{\varphi} \in \mathbb{K}_p^{\star} \subseteq \mathbb{K}_p$ be a measure of total variation at most one on $(\mathbb{N}_0 \times \mathbb{J})$ with components $\varphi_i \in \mathbb{M}_p^{\star} \subseteq \mathbb{M}_p$ that are themselves measures of total variation at most one on $\mathbb{J}$. Define $\boldsymbol{\Gamma}(\boldsymbol{\varphi})$ as the infinite vector of Γ operating on the components of $\boldsymbol{\varphi}$, such that $\boldsymbol{\Gamma}(\boldsymbol{\varphi}) = (\Gamma(\varphi_0), \Gamma(\varphi_1), \dots)$. Now if $\hat{\Pi}_{i,j}$ is Γ-linear with respect to $\mathbb{M}_p^{\star}$ with matrix P_{ij} for all (i,j) then $\hat{\Pi}$ is $\boldsymbol{\Gamma}$-linear with respect to $\mathbb{K}_p^{\star}$ with matrix*

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

Proof: The result follows directly from the linearity of Γ and the Γ-linearity of $\hat{\Pi}_{i,j}$ with respect to $\mathbb{M}_p^{\star}$. Consider the jth element of $\boldsymbol{\Gamma}(\hat{\Pi}(\boldsymbol{\varphi}))$ given by $\Gamma\left(\sum_{i=0}^{\infty} \hat{\Pi}_{i,j}(\varphi_i)\right)$. Then

$$\Gamma\left(\sum_{i=0}^{\infty} \hat{\Pi}_{i,j}(\varphi_i)\right) = \Gamma\left(\sum_{i=0}^{N} \hat{\Pi}_{i,j}(\varphi_i)\right) + \Gamma\left(\sum_{i=N+1}^{\infty} \hat{\Pi}_{i,j}(\varphi_i)\right)$$

$$= \sum_{i=0}^{N} \Gamma\left(\hat{\Pi}_{i,j}(\varphi_i)\right) + \Gamma\left(\sum_{i=N+1}^{\infty} \hat{\Pi}_{i,j}(\varphi_i)\right),$$

for any positive integer N, due to the linearity of Γ. As Γ is linear and continuous, it is also bounded [9, Theorem 3.1, Page 44]. Thus $\Gamma\left(\sum_{i=N+1}^{\infty} \hat{\Pi}_{i,j}(\varphi_i)\right)$ vanishes for $N \to \infty$. By taking the limit of the right-hand side, we then have

$$\Gamma\left(\sum_{i=0}^{\infty} \hat{\Pi}_{i,j}(\varphi_i)\right) = \sum_{i=0}^{\infty} \Gamma\left(\hat{\Pi}_{i,j}(\varphi_i)\right).$$

Now we can use the Γ-linearity of the $\hat{\Pi}_{i,j}$ with respect to $\mathbb{M}_p^\star$ to show that

$$
\begin{aligned}
\mathbf{\Gamma}(\hat{\Pi}(\boldsymbol{\varphi})) &= \left(\Gamma\left(\sum_{i=0}^\infty \hat{\Pi}_{i,0}(\varphi_i)\right), \Gamma\left(\sum_{i=0}^\infty \hat{\Pi}_{i,1}(\varphi_i)\right), \dots\right) \\
&= \left(\sum_{i=0}^\infty \Gamma(\hat{\Pi}_{i,0}(\varphi_i)), \sum_{i=0}^\infty \Gamma(\hat{\Pi}_{i,1}(\varphi_i)), \dots\right) \\
&= \left(\sum_{i=0}^\infty \Gamma(\varphi_i)P_{i,0}, \sum_{i=0}^\infty \Gamma(\varphi_i)P_{i,1}, \dots\right) = \mathbf{\Gamma}(\boldsymbol{\varphi})P,
\end{aligned}
$$

where convergence of the sums are ensured by the continuity of Γ and the boundedness of $\boldsymbol{\varphi}$. ∎

2.3 Generalisations needed to cope with complex boundary behaviour

In many applications it is convenient to operate with a different state space for the phase at the boundary. This poses no real change to the essential arguments, however, it complicates notation a little. We introduce the general measure space $(\mathbb{J}_0, \mathcal{J}_0)$ and the corresponding operator Γ_0 mapping from $\mathbb{M}_{p_0}^*$ to some real or complex, normed vector space $\mathbb{V}_0$ with a countable basis, to describe the behaviour at level 0. We shall also need the mappings $\hat{\Pi}_{0i} : \mathbb{M}_{p_0}^\star \to \mathbb{M}_p^\star$ and $\hat{\Pi}_{i0} : \mathbb{M}_p^\star \to \mathbb{M}_{p_0}^\star$, for all $i \geq 1$. Now Γ–linearity of $\hat{\Pi}_{0i}$ and $\hat{\Pi}_{i0}$ means the existence of unique matrices P_{0i} and P_{i0} such that $\Gamma(\hat{\Pi}_{0i}(\varphi_0)) = \Gamma_0(\varphi_0)P_{0i}$ and $\Gamma_0(\hat{\Pi}_{i0}(\varphi)) = \Gamma(\varphi)P_{i0}$, for all $i \geq 0$. Usually, for this to be meaningful in a practical context, the spaces $\mathbb{J}_0$ and $\mathbb{J}$ would have somewhat similar structures as for instance being subsets of different Euclidean spaces, and similarly, Γ_0 and Γ would be somewhat related operators such that the definition of Γ on $\mathbb{M}_p^\star$ would lead naturally to the definition of Γ_0 on $\mathbb{M}_{p_0}^\star$. We shall see that this will indeed be the case when we deal with our main example in Section 4.

3 Tweedie's operator geometric results

Tweedie [27] considered kernels of the type in (1) with a special structure, that has been termed $GI/M/1$-type by Neuts [18]. Here $\tilde{P}_{i,j} = \tilde{A}_{i-j+1}$ for $0 < j \leq i + 1$, $\tilde{P}_{i,j} = 0$ for

$i + 1 < j$ and $\tilde{P}_{i,0} = \tilde{B}_{i+1}$. Thus

$$
\tilde{P}(x, J) = \begin{bmatrix}
\tilde{B}_1(x, J) & \tilde{A}_0(x, J) & 0 & 0 & \cdots \\
\tilde{B}_2(x, J) & \tilde{A}_1(x, J) & \tilde{A}_0(x, J) & 0 & \cdots \\
\tilde{B}_3(x, J) & \tilde{A}_2(x, J) & \tilde{A}_1(x, J) & \tilde{A}_0(x, J) \cdots \\
\vdots & \vdots & \vdots &
\end{bmatrix}.
$$

Tweedie showed [27, Theorem 2], that an invariant measure $\boldsymbol{\mu}(\cdot) = (\mu_0(\cdot), \mu_1(\cdot), \ldots)$ of the Markov chain $X_n = (L_n, \boldsymbol{J}_n)$ is of the form

$$
\mu_{i+1}(J) = \int_{\mathbb{J}} \mu_i(\mathrm{d}x) \tilde{S}(x, J), \tag{2}
$$

where the kernel $\tilde{S}(x, J)$ is the minimal non-negative solution to

$$
\tilde{S}(x, J) = \sum_{j=0}^{\infty} \int_{\mathbb{J}} \tilde{S}^j(x, \mathrm{d}y) \tilde{A}_j(y, J), \tag{3}
$$

and the kernel $\tilde{S}^j(x, J)$ is the j-th iterate

$$
\tilde{S}^j(x, J) = \int_{\mathbb{J}} \tilde{S}^{j-1}(x, \mathrm{d}y) \tilde{S}(y, J).
$$

The measure $\mu_0(\cdot)$ at level zero, subject to normalisation, can be found from

$$
\mu_0(J) = \sum_{j=1}^{\infty} \int_{\mathbb{J}} \int_{\mathbb{J}} \mu_0(\mathrm{d}x) \tilde{S}^{j-1}(x, \mathrm{d}y) \tilde{B}_j(y, J), \tag{4}
$$

where, from Proposition 1 of Tweedie [27], $\sum_{j=1}^{\infty} \int_{\mathbb{J}} \tilde{S}^{j-1}(x, \mathrm{d}y) \tilde{B}_j(y, \mathbb{J}) = 1$ for all $x \in \mathbb{J}$. Tweedie considers the $\boldsymbol{\Phi}$–irreducibility condition, see for example [26]. The stability criterion is primarily discussed when one has $\tilde{B}_k = \sum_{i=k}^{\infty} \tilde{A}_i$, in which case [27, Theorem 5] positive recurrence of the $\boldsymbol{\Phi}$-irreducible Markov chain is guaranteed when

$$
\int_{\mathbb{J}} \nu(\mathrm{d}x) \sum_{k=0}^{\infty} k \tilde{A}_k(x, \mathbb{J}) > 1,
$$

where $\nu(\cdot)$ is the unique $\boldsymbol{\Phi}$–irreducible measure of $\tilde{A}(x, J) = \sum_{k=0}^{\infty} \tilde{A}_k(x, J)$.

3.1 Complex boundary behaviour

The boundary behaviour at level 0 is sometimes such that the entrance from level 0 to level 1 can not be described by the kernel $\tilde{A}_0(x, J)$. In that case the usual structure is

$$
\tilde{P}(x, J) = \begin{bmatrix}
\tilde{B}_1(x, J) & \tilde{B}_0(x, J) & 0 & 0 & 0 & \dots \\
\tilde{B}_2(x, J) & \tilde{A}_1(x, J) & \tilde{A}_0(x, J) & 0 & 0 & \dots \\
\tilde{B}_3(x, J) & \tilde{A}_2(x, J) & \tilde{A}_1(x, J) & \tilde{A}_0(x, J) & 0 & \dots \\
\tilde{B}_4(x, J) & \tilde{A}_3(x, J) & \tilde{A}_2(x, J) & \tilde{A}_1(x, J) & \tilde{A}_0(x, J) & \dots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{bmatrix}.
$$

It is straightforward that (2) is still valid for $i \geq 1$ while (4) needs minor adjustments and an additional equation is needed.

$$
\mu_0(J) = \int_{\mathbb{J}} \mu_0(\mathrm{d}x)\tilde{B}_1(x, J) + \sum_{j=1}^{\infty} \int_{\mathbb{J}} \int_{\mathbb{J}} \mu_1(\mathrm{d}x)\tilde{S}^{j-1}(x, \mathrm{d}y)\tilde{B}_{j+1}(y, J), \tag{5}
$$

$$
\mu_1(J) = \int_{\mathbb{J}} \mu_0(\mathrm{d}x)\tilde{B}_0(x, J) + \sum_{j=1}^{\infty} \int_{\mathbb{J}} \int_{\mathbb{J}} \mu_1(\mathrm{d}x)\tilde{S}^{j-1}(x, \mathrm{d}y)\tilde{A}_j(y, J). \tag{6}
$$

From here on we deal with this more general structure.

3.2 Operator linearity of Tweedie's $\tilde{A}_i$ and $\tilde{B}_i$ kernels

As part of the proof [27, Theorem 2] Tweedie introduced the sequence of kernels $\tilde{S}_i(x, J)$, with

$$
\tilde{S}_0(x, J) = 0, \qquad \tilde{S}_{i+1}(x, J) = \sum_{k=0}^{\infty} \int_{\mathbb{J}} \tilde{S}_i^k(x, \mathrm{d}y)\tilde{A}_k(y, J), i \geq 0,
$$

and showed that $\lim_{i \to \infty} \tilde{S}_i(x, J) = \tilde{S}(x, J)$ with $\tilde{S}_i(x, J) \leq \tilde{S}_{i+1}(x, J)$. We define $\hat{S}$, $\hat{S}_k$, and $\hat{A}_k$ to be the operators associated with the kernels $\tilde{S}$, $\tilde{S}_i$, and $\tilde{A}_k$ respectively. We then have the following important result.

Lemma 3 *If for all $k \geq 0$ the $\hat{A}_k$ are Γ-linear with respect to $\mathbb{M}_p^\star$ with matrix A_k, then all elements of the sequence $\hat{S}_i$ are Γ-linear with respect to $\mathbb{M}_p^\star$. The matrices S_i corresponding to $\tilde{S}_i$ are given by the (equivalent) matrix sequence*

$$
S_0 = 0, \qquad S_{i+1} = \sum_{k=0}^{\infty} S_i^k A_k, \quad i \geq 0.
$$

Proof: We have $\Gamma(\hat{S}_0) = 0$ due to the continuity of Γ. Now $\hat{S}_1$ is Γ-linear with respect to $\mathbb{M}_p^\star$ with matrix A_0 as $\hat{S}_1$ is identical to $\hat{A}_0$. We continue similarly to the argument used in the proof of Lemma 2.

$$
\begin{aligned}
\Gamma(\hat{S}_{i+1}(\phi)) &= \sum_{k=0}^{\infty} \Gamma\left(\hat{A}_k\left(\hat{S}_i^k(\phi)\right)\right) \\
&= \Gamma(\phi) \sum_{k=0}^{N} S_i^k A_k + \sum_{k=N+1}^{\infty} \Gamma\left(\hat{A}_k\left(\hat{S}_i^k(\phi)\right)\right).
\end{aligned}
$$

The last term can be bounded by

$$
\begin{aligned}
\left| \sum_{k=N+1}^{\infty} \Gamma\left(\hat{A}_k\left(\hat{S}_i^k(\phi)\right)\right) \right| &\leq \sum_{k=N+1}^{\infty} \left| \Gamma\left(\hat{A}_k\left(\hat{S}_i^k(\phi)\right)\right) \right| \\
&\leq \sum_{k=N+1}^{\infty} |\Gamma| \left| \hat{A}_k\left(\hat{S}_i^k(\phi)\right) \right| \leq |\Gamma| \sum_{k=N+1}^{\infty} \left| \hat{A}_k\left(\hat{S}_i^k(\phi)\right) \right|,
\end{aligned}
$$

and we see that $\sum_{k=N+1}^{\infty} \hat{A}_k\left(\hat{S}_i^k(\phi)\right)$ vanishes as $N \to \infty$. The induction hypothesis completes the proof. ∎

We are now ready to state our main result.

Theorem 4 *If for all $k \geq 0$ the $\hat{A}_k$ are Γ-linear with respect to $\mathbb{M}_p^\star$ with matrix A_k, then the operator $\hat{S}$ is Γ-linear with respect to $\mathbb{M}_p^\star$ with matrix S which is a solution to*

$$
S = \sum_{k=0}^{\infty} S^k A_k. \tag{7}
$$

Proof: Since we know that $\tilde{S}_i(x, J) \nearrow \tilde{S}(x, J)$ as $i \to \infty$, and Γ is a continuous operator, then by continuity we have that $\lim_{i\to\infty} \Gamma(\hat{S}_i(\varphi)) = \Gamma(\lim_{i\to\infty} \hat{S}_i(\varphi)) = \Gamma(\hat{S}(\varphi))$.

Further, we have just shown that $\Gamma(\hat{S}_i(\varphi)) = \Gamma(\varphi)S_i$ and so $\Gamma(\hat{S}(\varphi)) = \Gamma(\varphi)S$ with $S = \lim_{i\to\infty} S_i$, which must exist as $\Gamma(\hat{S}(\varphi))$ is well-defined and bounded. Thus we can conclude that the operator $\hat{S}$ is Γ-linear with respect to $\mathbb{M}_p^\star$ with matrix S and that S obeys

$$
S = \sum_{k=0}^{\infty} S^k A_k. \tag{8}
$$

∎

Lemma 3 and Theorem 4 provide the natural analogue of the Neuts' algorithm for determining the matrix R in a traditional QBD. They are nearly a restatement of the result

of Neuts [18] with the important distinction that there is no direct probabilistic interpretation involved. Rather, it shows that the same matrix equation governs the calculation of some derived quantities in the more generalised setting of Tweedie [27]. We shall formally present this as an algorithm for future reference.

Algorithm 1 *Given* $\epsilon > 0$,

    1. let $S_0 = 0$, *set* $i = 0$, *and*

    2. iteratively calculate, $S_{i+1} = \sum_{k=0}^{\infty} S_i^k A_k$, *while setting* $i := i+1$, *until* $||S_{i+1} - S_i|| < \epsilon$.

Finally we state the main result converting the operator-geometric result of Tweedie [27] into a matrix-geometric expression under the operation of Γ.

Corollary 5 *Assume that* $\hat{B}_k$ *and* $\hat{A}_k$ *are* Γ-*linear with respect to* $\mathbb{M}_p^\star$ *with matrices* B_k *and* A_k *for all* $k \geq 0$. *Let* $\boldsymbol{\mu} = (\mu_0, \mu_1, \mu_2, \dots)$ *be the stationary measure determined by equations (2), (5), and (6). Then* $\mu_0 \in \mathbb{M}_{p_0}^\star$, $\mu_k \in \mathbb{M}_p^\star$ *for* $k \geq 1$, *and we have* $\Gamma(\mu_{k+1}) = \Gamma(\mu_k)S$, *for* $k \geq 1$, *with* $\Gamma_0(\mu_0)$ *and* $\Gamma(\mu_1)$ *given by* $\Gamma_0(\mu_0) = \Gamma_0(\mu_0)B_1 + \Gamma(\mu_1)\sum_{k=1}^{\infty} S^{k-1}B_{k+1}$, *and* $\Gamma(\mu_1) = \Gamma_0(\mu_0)B_0 + \Gamma(\mu_1)\sum_{k=1}^{\infty} S^{k-1}A_k$.

Proof: By the Γ-linearity of $\hat{A}_k$ and $\hat{B}_k$ with respect to $\mathbb{M}_p^\star$ we have that $\hat{P}$ is Γ-linear with respect to $\mathbb{K}_p^\star$. A sequence of measures $\boldsymbol{\phi}^{(n)}$ iterated with $\hat{P}$ starting with an element $\boldsymbol{\phi}^{(0)} = (\phi_0^{(0)}, \phi_1^{(0)}, \dots) \in \mathbb{K}_p^\star$ with $\phi_0^{(0)} \in \mathbb{M}_{p_0}^\star$ and $\phi_i^{(0)} \in \mathbb{M}_p^\star$ will stay in the set $\mathbb{K}_p^\star$. The limit of the sequence $\boldsymbol{\phi}^{(n)}$ will also belong to $\mathbb{K}_p^\star$, as we show below. We have $\boldsymbol{\phi}^{(n+1)} = \hat{P}\left(\boldsymbol{\phi}^{(n)}\right)$, and suppose the limit of $\boldsymbol{\phi}^{(n)}$ is $\boldsymbol{\phi}^*$. Thus

$$
\begin{aligned}
\Gamma\left(\hat{P}(\boldsymbol{\phi}^*)\right) &= \Gamma\left(\hat{P}\left(\boldsymbol{\phi}^{(n)} + \boldsymbol{\phi}^* - \boldsymbol{\phi}^{(n)}\right)\right) = \Gamma\left(\hat{P}\left(\boldsymbol{\phi}^{(n)}\right) + \hat{P}\left(\boldsymbol{\phi}^* - \boldsymbol{\phi}^{(n)}\right)\right) \\
&= \Gamma\left(\hat{P}\left(\boldsymbol{\phi}^{(n)}\right)\right) + \Gamma\left(\hat{P}\left(\boldsymbol{\phi}^* - \boldsymbol{\phi}^{(n)}\right)\right) = \Gamma\left(\boldsymbol{\phi}^{(n)}\right)P + \Gamma\left(\hat{P}\left(\boldsymbol{\phi}^* - \boldsymbol{\phi}^{(n)}\right)\right).
\end{aligned}
$$

Now as $n \to \infty$ the right hand side tends to $\Gamma(\boldsymbol{\phi}^*)P$ and so $\boldsymbol{\phi}^* \in \mathbb{K}_p^\star$ as claimed. From the uniqueness of the stationary measure we conclude that it has the property stated in the theorem.

The expression for $\Gamma(\mu_k)$ follows immediately from equation (2) upon the application of Γ. From (5) and (6) we obtain the equations for $\Gamma(\mu_0)$ and $\Gamma(\mu_1)$ using Lemma 1.   ■

Corollary 6 *Suppose the conditions of Corollary 5 are met such that* $\mu_1 \in \mathbb{M}_p^\star$ *then*

$$
\sum_{k=0}^{\infty} \Gamma(\mu_1)S^k < \infty, \ \text{elementwise}
$$

Proof: To prove the finiteness of $\Gamma(\mu_1)\sum_{k=0}^{\infty} S^k$ consider

$$\sum_{i=0}^{\infty} \hat{S}^i(\mu_1),$$

which is a finite measure as

$$\zeta = \int \mu_1(\mathrm{d}x) \sum_{i=0}^{\infty} \tilde{S}^i(x, E) < \infty$$

since $\sum_{i=0}^{\infty} \tilde{S}^i(x, E) < \infty$ for x μ_1 a.e. ([27, Theorem 2 (iv)]). Now $\zeta^{-1} \sum_{i=0}^{\infty} \hat{S}^i(\mu_1)$ is a measure of total variation at most one, that belongs to $\mathbb{M}_p^\star$, therefore

$$\infty > \Gamma\left(\zeta^{-1}\sum_{i=0}^{\infty}\hat{S}^i(\mu_1)\right) = \sum_{i=0}^{\infty}\Gamma\left(\zeta^{-1}\hat{S}^i(\mu_1)\right) = \zeta^{-1}\sum_{i=0}^{\infty}\Gamma(\mu_1)S^i.$$

$\blacksquare$

The minimal non-negative characterisation of Neuts [18] is re-cast in the positive-recurrent case as a matrix S that is a solution to Equation (7) while obeying Corollary 6.

Note that, although we haven't analytically characterised which solution of (7) we require, we have shown that the solution delivered by Algorithm 1 is the required solution. We shall use this idea later to justify that a suite of other algorithms also deliver the required solution.

3.3   Birth and death structures and level censoring

In this section we consider the case when the Markov chain has a birth and death like structure, that is the transition kernel is of the form

$$\tilde{P}(x, J) = \begin{bmatrix} \tilde{B}_1(x,J) & \tilde{B}_0(x,J) & 0 & 0 & 0 & \dots \\ \tilde{B}_2(x,J) & \tilde{A}_1(x,J) & \tilde{A}_0(x,J) & 0 & 0 & \dots \\ 0 & \tilde{A}_2(x,J) & \tilde{A}_1(x,J) & \tilde{A}_0(x,J) & 0 & \dots \\ 0 & 0 & \tilde{A}_2(x,J) & \tilde{A}_1(x,J) & \tilde{A}_0(x,J) & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

If we consider this Markov chain only at successive visits to even numbered levels we obtain a new level–censored chain with kernel

$$\tilde{P}^{(1)}(x, J) = \begin{bmatrix} \tilde{B}_1^{(1)}(x,J) & \tilde{B}_0^{(1)}(x,J) & 0 & 0 & 0\dots \\ \tilde{B}_2^{(1)}(x,J) & \tilde{A}_1^{(1)}(x,J) & \tilde{A}_0^{(1)}(x,J) & 0 & 0 & \dots \\ 0 & \tilde{A}_2^{(1)}(x,J) & \tilde{A}_1^{(1)}(x,J) & \tilde{A}_0^{(1)}(x,J) & 0 & \dots \\ 0 & 0 & \tilde{A}_2^{(1)}(x,J) & \tilde{A}_1^{(1)}(x,J) & \tilde{A}_0^{(1)}(x,J) & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

where we have

$$\tilde{B}_1^{(1)}(x, J) = \int_{\mathbb{J}} \tilde{H}_0(x, \mathrm{d}y)\tilde{L}_0(y, J), \quad \tilde{B}_0^{(1)}(x, J) = \int_{\mathbb{J}} \tilde{H}_0(x, \mathrm{d}y)\tilde{H}(y, J),$$

$$\tilde{B}_2^{(1)}(x, J) = \int_{\mathbb{J}} \tilde{L}(x, \mathrm{d}y)\tilde{L}_0(y, J), \quad \tilde{A}_1^{(1)}(x, J) = \int_{\mathbb{J}} \left(\tilde{H}(x, \mathrm{d}y)\tilde{L}(y, J) + \tilde{L}(x, \mathrm{d}y)\tilde{H}(y, J) \right),$$

$$\tilde{A}_0^{(1)}(x, J) = \tilde{H}^2(x, J), \quad \text{and } \tilde{A}_2^{(1)}(x, J) = \tilde{L}^2(x, J),$$

with

$$\tilde{L}(x, J) = \sum_{i=0}^{\infty} \int_{\mathbb{J}} \tilde{A}_1^i(x, \mathrm{d}y)\tilde{A}_2(y, J), \qquad \tilde{H}(x, J) = \sum_{i=0}^{\infty} \int_{\mathbb{J}} \tilde{A}_1^i(x, \mathrm{d}y)\tilde{A}_0(y, J),$$

$$\tilde{L}_0(x, J) = \sum_{i=0}^{\infty} \int_{\mathbb{J}} \tilde{A}_1^i(x, \mathrm{d}y)\tilde{B}_2(y, J), \qquad \tilde{H}_0(x, J) = \sum_{i=0}^{\infty} \int_{\mathbb{J}} \tilde{B}_1^i(x, \mathrm{d}y)\tilde{B}_0(y, J).$$

Thus the new structure is still that of a birth and death process.

Corollary 7 *When $\tilde{A}_k = 0$ for $k > 2$ and $\tilde{B}_k = 0$ for $k > 1$ then the logarithmic reduction algorithm of Latouche and Ramaswami [11] applies verbatim to the matrices A_0, A_1, A_2, B_0 and B_1, associated with Γ-linearity.*

Proof: By following the argument in Latouche and Ramaswami [11] with the current machinery, the result follows immediately. We also need to show that we have the required solution of (7). Although the arguments presented in [11] are probabilistic, they can be expressed (albeit much more tediously) in algebraic form. This can be used to show that the solution delivered by the logarithmic reduction algorithm is the same as the solution delivered by Algorithm 1 and so the logarithmic reduction algorithm must deliver the required solution. ∎

Similarly, all the known algorithms in the QBD literature that rely on level–censored arguments, can be justified in this extended environment.

3.4 Kernels expressed in an orthonormal basis

Here we describe the model of Nielsen and Ramaswami [20] in the current framework and then show how their result can be interpreted as a special case of the more general results of the current paper.

In [20] Nielsen and Ramaswami took $\mathbb{J}$ to be the unit interval and considered kernels $\tilde{P}_{i,j}(x, J)$ mapping $x \in [0, 1]$ to some Borel set $J \in \mathbb{J}$. Further they assumed that the kernels $\tilde{P}_{i,j}(x, J)$ had a density $\tilde{p}_{i,j}(x, y)$ which could be expressed by orthonormal basis functions, $\phi_k(x) \in L^2$, in the following way

$$\tilde{p}_{i,j}(x, y) = \sum_{k,\ell} P_{i,j;k,\ell}\phi_k(x)\phi_\ell(y), \text{ here } \int_0^1 \phi_k(x)\phi_\ell(x)\mathrm{d}x = I_k(\ell),$$

where $I_k(\ell)$ is an indicator function such that $I_k(\ell) = 1$ when $\ell = k$ and 0 otherwise. They defined the set $\mathbb{M}_p^\star$ as the set of measures on the unit interval with a density that could be expressed by the same basis functions. Now define Γ as the vector of coordinates of the density of the measure when expressed in terms of the orthonormal basis functions $\phi_k(x)$. Thus, if we denote a measure in $\mathbb{M}_p^\star$ by $\varphi(\cdot)$ and its density by $\varphi'(x)$ and express the density as $\varphi'(x) = \boldsymbol{\gamma}\boldsymbol{\phi}(x) = \sum_k \gamma_k\phi_k(x)$, then $\Gamma(\varphi) = \boldsymbol{\gamma}$.

The work of Nielsen and Ramaswami can be reinterpreted in the language of this paper as showing that the $\tilde{P}_{i,j}$ are Γ-linear with respect to $\mathbb{M}_p^\star$ with matrix $P_{i,j} = (P_{i,j;k,\ell})_{k,\ell}$. To see this, consider

$$
\begin{aligned}
\tilde{\Pi}_{i,j}(\varphi')(y) &= \int_0^1 \varphi'(x)\tilde{p}_{i,j}(x, y)\mathrm{d}x = \int_0^1 \sum_m \gamma_m\phi_m(x) \sum_{k,\ell} P_{i,j;k,\ell}\phi_k(x)\phi_\ell(y)\mathrm{d}x \\
&= \sum_m \gamma_m \sum_{k,\ell} P_{i,j;k,\ell} \int_0^1 \phi_m(x)\phi_k(x)\mathrm{d}x\phi_\ell(y) = \sum_{k,\ell} \gamma_k P_{i,j;k,\ell}\phi_\ell(y) = \boldsymbol{\gamma}P_{i,j}\boldsymbol{\phi}(y),
\end{aligned}
$$

and so, $\Gamma(\tilde{\Pi}_{i,j}(\varphi)) = \boldsymbol{\gamma}P_{i,j} = \Gamma(\varphi)P_{i,j}$.

Nielsen and Ramaswami [20] considered the case of a Quasi-Birth-and-Death structure, as in (9), where $\tilde{A}_k(x, J) = 0$ for $k > 2$. Let $\mu'_k(x)$ be the density of the phase variable being x at level k under the stationary measure of the Markov chain. This can now be found by Theorem 4 and Corollary 5 as $\mu'_k(x) = \boldsymbol{\gamma}^{(k)}\boldsymbol{\phi}(x)$, and so $\Gamma(\mu_k(x)) = \boldsymbol{\gamma}^{(k)}$, where $\boldsymbol{\gamma}^{(k)} = \boldsymbol{\gamma}^{(0)}S^k$, with S being the solution to $S = A_0 + SA_1 + S^2A_2$, arising from Algorithm 1, and $\boldsymbol{\gamma}^{(0)} = \boldsymbol{\gamma}^{(0)}(B_0 + SB_1)$. Nielsen and Ramaswami finally obtained numerical results effectively, by applying Corollary 7.

4 Queues with RAP and ME components

PH distributions and MAPs are by now standard models in the applied queueing and performance literature. The MAP is a point process with finite dimensional distribution

of the first n points given by

$$f(t_1, t_2, \ldots, t_n) = \boldsymbol{\alpha} e^{Ct_1} D e^{C(t_2-t_1)} D \ldots e^{C(t_n-t_{n-1})} D \boldsymbol{e}. \tag{9}$$

That is a sequence of possibly dependent PH variables having the same sub–generator. The algebraic extension of PH distributions is termed ME distributions and goes back to [8], while the formulation as a matrix–exponential is newer. See e.g. [2] for a quite exhaustive treatment. Asmussen and Bladt [3] introduced the corresponding generalisation of a MAP as a point process of possibly dependent ME distributed random variables having the same governing matrix. Their starting point was to define a RAP as a point process, where there exists a version of the prediction process that varies in a finite dimensional space under the time shift operator. They proved that a process is a RAP if and only if it has a finite dimensional distribution given by (9). They also introduced the piecewise deterministic process $\boldsymbol{J}(t)$ (known as $\boldsymbol{A}(t)$ in [3]) as an interpretation of the behaviour of the prediction process. We use the same parameterisation as that of [3] such that $\boldsymbol{\alpha e} = 1$, $(C+D)\boldsymbol{e} = 0$, with all entries of $\boldsymbol{\alpha}, C$, and D being real. In [4] we introduced the concept of a Batch Rational Arrival Process (BRAP) as a marked RAP with countable mark space, derived directly from [3]. In that paper we also introduced the concept of a QBD with RAP components as a random walk governed by a BRAP with two different marks (-1,+1) and reflected at zero. The sojourn time at level 0 is given by an ME distribution, where the initial vector of the ME distribution might depend on the phase vector from which level 1 is left.

4.1 The GI/RAP/1 queue

We now consider the case of a queue with general renewal input governed by the distribution $F(\cdot)$ and a RAP(C, D) service process. The bivariate random variables $X_n = (L_n, \boldsymbol{J}_n)$, where L_n is the number of customers in the queue at the nth arrival and $\boldsymbol{J}_n$ is the phase vector of the RAP at the nth arrival, form a discrete time Markov chain on $\mathbb{N}_0 \times \mathbb{J}$. We adhere to the convention that the phase of the RAP is kept constant during idle periods. A change of behaviour during idle periods would manifest itself in the $\tilde{B}_k$ kernels only. The transition probability law of that Markov chain is given by

$$P(\boldsymbol{j}, J) = \begin{pmatrix} \tilde{B}_0(\boldsymbol{j}, J) & \tilde{A}_0(\boldsymbol{j}, J) & 0 & 0 & \ldots \\ \tilde{B}_1(\boldsymbol{j}, J) & \tilde{A}_1(\boldsymbol{j}, J) & \tilde{A}_0(\boldsymbol{j}, J) & 0 & \ldots \\ \tilde{B}_2(\boldsymbol{j}, J) & \tilde{A}_2(\boldsymbol{j}, J) & \tilde{A}_1(\boldsymbol{j}, J) & \tilde{A}_0(\boldsymbol{j}, J) & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

with

$$\tilde{A}_k(\boldsymbol{j}, J) = \int_0^\infty \tilde{P}_k(\boldsymbol{j}, t; J) \mathrm{d}F(t), k \geq 0 \tag{10}$$

$$\tilde{B}_k(\boldsymbol{j}, J) = \int_0^\infty \tilde{Q}_{k+1}(\boldsymbol{j}, t; J) \mathrm{d}F(t), k \geq 0. \tag{11}$$

Here $\tilde{P}_k(\boldsymbol{j}, t; J)$ denotes the probability that the RAP(C, D) has had exactly k events at time t, a customer is being served, and the phase vector $\boldsymbol{J}$ is in the set $J \in \mathcal{J}$, given it had the value $\boldsymbol{j} \in \mathbb{J}$ immediately after the last arrival; while $\tilde{Q}_k(\boldsymbol{j}, t; J)$ denotes the probability that the RAP(C, D) has had exactly k events at time t, and at the expiry of the kth event the phase vector is in the set $J \in \mathcal{J}$ and then remains there as the queue is empty, given it had the value $\boldsymbol{j} \in \mathbb{J}$ immediately after the last arrival. The role of Γ will here be taken as the expectation operator of the measure of the phase vector of the RAP, which clearly exists and is in $\mathbb{J}$ for any measure on $\mathbb{J}$, as $\mathbb{J}$ is compact and convex.

We now show that $\tilde{A}_k$ and $\tilde{B}_k$ are expectation-linear. In order to show this we first show that $\tilde{P}_k(\boldsymbol{j}, t; J)$ and $\tilde{Q}_k(\boldsymbol{j}, t; J)$ are expectation-linear for all k and t. We have from the definition of the RAP(C, D) that

$$\tilde{P}_0(\boldsymbol{j}, t; J) = \boldsymbol{j} e^{Ct} \boldsymbol{e} I_J \left(\frac{\boldsymbol{j} e^{Ct}}{\boldsymbol{j} e^{Ct} \boldsymbol{e}} \right),$$

where $I_J(\boldsymbol{J})$ is an indicator function for the event $\boldsymbol{J} \in J$. For $\tilde{P}_1$ and $\tilde{Q}_1$ we get

$$\tilde{P}_1(\boldsymbol{j}, t; J) = \int_0^t \boldsymbol{j} e^{Ct_1} D e^{C(t-t_1)} \boldsymbol{e} I_J \left(\frac{\boldsymbol{j} e^{Ct_1} D e^{C(t-t_1)}}{\boldsymbol{j} e^{Ct_1} D e^{C(t-t_1)} \boldsymbol{e}} \right) \mathrm{d}t_1,$$

$$\tilde{Q}_1(\boldsymbol{j}, t; J) = \int_0^t \boldsymbol{j} e^{Ct_1} D \boldsymbol{e} I_J \left(\frac{\boldsymbol{j} e^{Ct_1} D}{\boldsymbol{j} e^{Ct_1} D \boldsymbol{e}} \right) \mathrm{d}t_1,$$

and for $k \geq 2$,

$$\tilde{P}_k(\boldsymbol{j}, t; J) = \int_0^t \int_{\mathbb{J}} \tilde{P}_1(\boldsymbol{j}, t_1; \mathrm{d}\boldsymbol{y}) \tilde{P}_{k-1}(\boldsymbol{y}, t - t_1; J) \mathrm{d}t_1,$$

$$\tilde{Q}_k(\boldsymbol{j}, t; J) = \int_0^t \int_{\mathbb{J}} \tilde{P}_1(\boldsymbol{j}, t_1; \mathrm{d}\boldsymbol{y}) \tilde{Q}_{k-1}(\boldsymbol{y}, t - t_1; J) \mathrm{d}t_1.$$

Lemma 8 *The kernels $\tilde{P}_k(\boldsymbol{j}, t; J)$, for all $k \geq 0$, and $\tilde{Q}_k(\boldsymbol{j}, t; J)$, for all $k \geq 1$, are expectation-linear, that is*

$$\int_{\mathbb{J}} \boldsymbol{y} \tilde{P}_k(\boldsymbol{j}, t; \mathrm{d}\boldsymbol{y}) = \boldsymbol{j} P_k(t)$$

for the set of matrices $P_k(t)$, $k \geq 0$, given by

$$P_k(t) = \begin{cases} e^{Ct}, & k = 0, \\ \int_0^t e^{Ct_1} De^{C(t-t_1)} \mathrm{d}t_1, & k = 1, \\ \int_0^t P_1(t_1) P_{k-1}(t-t_1) \mathrm{d}t_1, & k > 1, \end{cases} \tag{12}$$

and

$$\int_{\mathbb{J}} \boldsymbol{y} \tilde{Q}_k(\boldsymbol{j}, t; \mathrm{d}\boldsymbol{y}) = \boldsymbol{j} Q_i(t)$$

for the set of matrices $Q_k(t)$, $k \geq 1$, given by

$$Q_k(t) = \begin{cases} \int_0^t e^{Ct_1} D \mathrm{d}t_1, & k = 1, \\ \int_0^t P_1(t_1) Q_{k-1}(t-t_1) \mathrm{d}t_1, & k > 1. \end{cases} \tag{13}$$

Proof: First

$$\begin{aligned} \int_{\mathbb{J}} \boldsymbol{y} \tilde{P}_0(\boldsymbol{j}, t; \mathrm{d}\boldsymbol{y}) &= \int_{\mathbb{J}} \boldsymbol{y} \boldsymbol{j} e^{Ct} \boldsymbol{e} I_{\mathrm{d}\boldsymbol{y}} \left(\frac{\boldsymbol{j} e^{Ct}}{\boldsymbol{j} e^{Ct} \boldsymbol{e}} \right) = \boldsymbol{j} e^{Ct} \boldsymbol{e} \int_{\mathbb{J}} \boldsymbol{y} I_{\mathrm{d}\boldsymbol{y}} \left(\frac{\boldsymbol{j} e^{Ct}}{\boldsymbol{j} e^{Ct} \boldsymbol{e}} \right) \\ &= \boldsymbol{j} e^{Ct} \boldsymbol{e} \left(\frac{\boldsymbol{j} e^{Ct}}{\boldsymbol{j} e^{Ct} \boldsymbol{e}} \right) = \boldsymbol{j} e^{Ct} = \boldsymbol{j} P_0(t). \end{aligned}$$

Also,

$$\begin{aligned} \int_{\mathbb{J}} \boldsymbol{y} \tilde{P}_1(\boldsymbol{j}, t; \mathrm{d}\boldsymbol{y}) &= \int_{\mathbb{J}} \boldsymbol{y} \int_0^t \boldsymbol{j} e^{Ct_1} De^{C(t-t_1)} \boldsymbol{e} I_{\mathrm{d}\boldsymbol{y}} \left(\frac{\boldsymbol{j} e^{Ct_1} De^{C(t-t_1)}}{\boldsymbol{j} e^{Ct_1} De^{C(t-t_1)} \boldsymbol{e}} \right) \mathrm{d}t_1 \\ &= \int_0^t \boldsymbol{j} e^{Ct_1} De^{C(t-t_1)} \boldsymbol{e} \int_{\mathbb{J}} \boldsymbol{y} I_{\mathrm{d}\boldsymbol{y}} \left(\frac{\boldsymbol{j} e^{Ct_1} De^{C(t-t_1)}}{\boldsymbol{j} e^{Ct_1} De^{C(t-t_1)} \boldsymbol{e}} \right) \mathrm{d}t_1 \\ &= \int_0^t \boldsymbol{j} e^{Ct_1} De^{C(t-t_1)} \boldsymbol{e} \left(\frac{\boldsymbol{j} e^{Ct_1} De^{C(t-t_1)}}{\boldsymbol{j} e^{Ct_1} De^{C(t-t_1)} \boldsymbol{e}} \right) \mathrm{d}t_1 \\ &= \boldsymbol{j} \int_0^t e^{Ct_1} De^{C(t-t_1)} \mathrm{d}t_1 = \boldsymbol{j} P_1(t). \end{aligned}$$

Then, for $k > 1$,

$$\begin{aligned} \int_{\mathbb{J}} \boldsymbol{y} \tilde{P}_k(\boldsymbol{j}, t; \mathrm{d}\boldsymbol{y}) &= \int_{\mathbb{J}} \boldsymbol{y} \int_0^t \int_{\mathbb{J}} \tilde{P}_1(\boldsymbol{j}, t_1; \mathrm{d}\boldsymbol{z}) \tilde{P}_{k-1}(\boldsymbol{z}, t-t_1; \mathrm{d}\boldsymbol{y}) \mathrm{d}t_1 \\ &= \int_0^t \int_{\mathbb{J}} \tilde{P}_1(\boldsymbol{j}, t_1; \mathrm{d}\boldsymbol{z}) \int_{\mathbb{J}} \boldsymbol{y} \tilde{P}_{k-1}(\boldsymbol{z}, t-t_1; \mathrm{d}\boldsymbol{y}) \mathrm{d}t_1 \\ &= \int_0^t \int_{\mathbb{J}} \tilde{P}_1(\boldsymbol{j}, t_1; \mathrm{d}\boldsymbol{z}) \boldsymbol{z} P_{k-1}(t-t_1) \mathrm{d}t \\ &= \int_0^t \boldsymbol{j} P_1(t_1) P_{k-1}(t-t_1) \mathrm{d}t = \boldsymbol{j} \int_0^t P_1(t_1) P_{k-1}(t-t_1) \mathrm{d}t_1, \end{aligned}$$

where the third equality is due to the induction hypothesis.

Similar arguments apply for the $\tilde{Q}_k(\boldsymbol{j}, t; J)$. ∎

Corollary 9 *The kernels $\tilde{A}_k(\boldsymbol{j}, J)$ and $\tilde{B}_k(\boldsymbol{j}, J)$, for $k \geq 0$, are expectation-linear with matrices $A_k = \int_0^\infty P_k(t)\mathrm{d}F(t)$ and $B_k = \int_0^\infty Q_{k+1}(t)\mathrm{d}F(t)$, respectively.*

Proof: Application of Lemma 8 to equations (10) and (11) yields the result immediately. ∎

Thus this corollary establishes that we can apply Theorem 4 and Corollary 5 to the $GI/RAP/1$ queue, effectively obtaining exactly the same non-linear matrix equation as in [18]. We can also use Lemma 3 to determine the required solution, say R, to that equation. We could also use the cyclic-reduction algorithm of Bini and Meini [5] in this environment, as it can be justified using level-censoring arguments (see Hunt [10]). As it delivers the same solution as does Algorithm 1, the cyclic-reduction algorithm must also deliver the required solution, R. The only - but important - differences are the more general model to which these apply and the change in the interpretation of the R matrix.

4.2 A QBD with RAP-components

The model in Section 4.1 was naturally formed in discrete time. We will now pay attention to a model which following traditional analysis would be formulated in continuous time. However, in order to apply the methodology of Tweedie we have to analyse the queue in discrete time then later consider an interpretation in continuous time, which not surprisingly happens to be straightforward and natural in our case. The most natural queue of this type is the queue with RAP input and ME service time where the space $\mathbb{J}$ would be the Cartesian product of the two spaces $\mathbb{J}_{\mathrm{RAP}}$ and $\mathbb{J}_{\mathrm{ME}}$ of the two components respectively. See [4] for a more thorough discussion of these issues and how to describe more complex boundary behaviour. To analyse this QBD with RAP-components within the framework of Tweedie we will consider the state of the process at level changes only.

In continuous time we denote the process by $X(t) = (L(t), \boldsymbol{J}(t))$ and denote by T_n the time of the nth level change. Now we define $X_n = X(T_n) = (L_n, \boldsymbol{J}_n)$. Due to the piecewise deterministic nature of the RAP this process is a Markov chain on the state space $\mathbb{N}_0 \times \mathbb{J}$. The block kernels of Tweedie's framework take on the specific form

$$P(L_{n+1} = i + 1, \boldsymbol{J}_{n+1} \in J | L_n = i, \boldsymbol{J}_n = \boldsymbol{j}) = \tilde{A}_0(\boldsymbol{j}, J), \quad n \geq 1,$$

and

$$P(L_{n+1} = i-1, \boldsymbol{J}_{n+1} \in J | L_n = i, \boldsymbol{J}_n = \boldsymbol{j}) = \tilde{A}_2(\boldsymbol{j}, J), \quad n \geq 2,$$

$$P(L_{n+1} = 1, \boldsymbol{J}_{n+1} \in J | L_n = 0, \boldsymbol{J}_n = \boldsymbol{j}) = \tilde{B}_0(\boldsymbol{j}, J)$$

and

$$P(L_{n+1} = 0, \boldsymbol{J}_{n+1} \in J_0 | L_n = 1, \boldsymbol{J}_n = \boldsymbol{j}) = \tilde{B}_2(\boldsymbol{j}, J),$$

while all other $\tilde{A}_i$'s and $\tilde{B}_i$'s are 0. The kernels $\tilde{A}_i$, $i = 0, 2$ are expressed in the parameters of the QBD with RAP components, as

$$\tilde{A}_i(\boldsymbol{j}, J) = \int_0^\infty \boldsymbol{j} e^{A_1 t} A_i \boldsymbol{e} I_J \left(\frac{\boldsymbol{j} e^{A_1 t} A_i}{\boldsymbol{j} e^{A_1 t} A_i \boldsymbol{e}} \right) \mathrm{d}t.$$

Lemma 10 *The kernels $\tilde{A}_i(\boldsymbol{j}, J)$, $i = 0, 2$ are expectation linear with matrices $(-A_1)^{-1} A_i$.*

Proof: By definition, for $i = 0, 2$ we have

$$
\begin{aligned}
\int_{\mathbb{J}} \boldsymbol{y} \tilde{A}_i(\boldsymbol{j}, \mathrm{d}\boldsymbol{y}) &= \int_{\mathbb{J}} \boldsymbol{y} \int_0^\infty \boldsymbol{j} e^{A_1 t} A_i \boldsymbol{e} I_{\mathrm{d}\boldsymbol{y}} \left(\frac{\boldsymbol{j} e^{A_1 t} A_i}{\boldsymbol{j} e^{A_1 t} A_i \boldsymbol{e}} \right) \mathrm{d}t \\
&= \int_0^\infty \boldsymbol{j} e^{A_1 t} A_i \boldsymbol{e} \int_{\mathbb{J}} \boldsymbol{y} I_{\mathrm{d}\boldsymbol{y}} \left(\frac{\boldsymbol{j} e^{A_1 t} A_i}{\boldsymbol{j} e^{A_1 t} A_i \boldsymbol{e}} \right) \mathrm{d}t \\
&= \int_0^\infty \boldsymbol{j} e^{A_1 t} A_i \boldsymbol{e} \frac{\boldsymbol{j} e^{A_1 t} A_i}{\boldsymbol{j} e^{A_1 t} A_i \boldsymbol{e}} \mathrm{d}t \\
&= \int_0^\infty \boldsymbol{j} e^{A_1 t} \mathrm{d}t A_i = \boldsymbol{j} (-A_1)^{-1} A_i.
\end{aligned}
$$

$\blacksquare$

Lemma 11 *The kernels $\tilde{B}_i(\boldsymbol{j}, J)$, $i = 0, 2$ are expectation linear with matrices $(-B_1)^{-1} B_0$ $(-A_1)^{-1} B_2$ respectively.*

Proof: The proof follow exactly the same lines as the proof of Lemma 10, with obvious minor modifications. $\blacksquare$

Now we can apply the results of Section 3 to the stationary measure $\boldsymbol{\mu}(\cdot) = (\mu_0(\cdot), \mu_1(\cdot), \ldots)$. By choosing Γ to be the expectation operator, we let $\boldsymbol{\nu}_i = \Gamma(\mu_i) = \int_{\mathbb{J}} \boldsymbol{y} \mu_i(\mathrm{d}\boldsymbol{y}) = \mathbb{E}(\boldsymbol{J}_n I(L_n = i))$. Then Theorem 4 and Corollary 5 show that $\boldsymbol{\nu}_i$ is given by

$$\boldsymbol{\nu}_{i+1} = \boldsymbol{\nu}_i S,$$

where S is the solution to

$$S = (-A_1)^{-1}A_0 + S^2(-A_1)^{-1}A_2, \tag{14}$$

delivered by Algorithm 1. Here $\boldsymbol{\nu}_0$, $\boldsymbol{\nu}_1$ are given by $\boldsymbol{\nu}_0 = \boldsymbol{\nu}_1(-A_1)^{-1}B_2$ and $\boldsymbol{\nu}_1 = \boldsymbol{\nu}_0(-B_1)^{-1}B_0 + \boldsymbol{\nu}_1 S(-A_1)^{-1}A_2$.

4.2.1 Stability condition

For the irreducible case, from Tweedie [27] we have that the stability condition is equal to

$$\boldsymbol{\phi}(-A_1)^{-1}A_0\boldsymbol{e} < \boldsymbol{\phi}A_2(-A_1)^{-1}\boldsymbol{e}$$

where $\boldsymbol{\phi}(-A_1)^{-1}(A_0 + A_2) = \boldsymbol{\phi}$. Now inserting the relation $\boldsymbol{\phi} = \dfrac{\boldsymbol{\theta}(A_0 + A_2)}{\boldsymbol{\theta}(A_0 + A_2)\boldsymbol{e}}$, where $\boldsymbol{\theta}A = \mathbf{0}$, we easily obtain the natural analogue of the standard stability condition.

4.2.2 From the embedded process to the time stationary process

In the previous section, all probabilistic calculations were done from the perspective of the distribution of the phase at the time of level changes in the stationary process. In this section we exploit those results and determine the expectation of the phase at an arbitrary time-point in the stationary process.

Let the stationary measure of the QBD with RAP components be denoted by $\boldsymbol{\pi}(\cdot) = (\pi_0(\cdot), \pi_1(\cdot), \ldots)$ and recall that $\boldsymbol{\mu}(\cdot) = (\mu_0(\cdot), \mu_1(\cdot), \ldots)$ is the stationary measure of the embedded process. Again, choosing Γ to be the expectation operator, we let $\boldsymbol{\theta}_i = \Gamma(\pi_i) = \int_{\mathbb{J}} \boldsymbol{y}\pi_i(\mathrm{d}\boldsymbol{y}) = \mathbb{E}(\boldsymbol{J}_t \, I(L_t = i))$. Now, for some normalising constant K

$$\pi_i(J) = K \int_{t=0}^{\infty} \int_{\mathbb{J}} \mu_i(\mathrm{d}\boldsymbol{y})\boldsymbol{y}e^{A_1 t}\boldsymbol{e}I_J\left(\frac{\boldsymbol{y}e^{A_1 t}}{\boldsymbol{y}e^{A_1 t}\boldsymbol{e}}\right)\mathrm{d}t,$$

and applying the expectation operator Γ to this yields

$$
\begin{aligned}
\boldsymbol{\theta}_i &= \Gamma(\pi_i) = K \int_{\mathbb{J}} \boldsymbol{z} \int_{t=0}^{\infty} \int_{\mathbb{J}} \mu_i(\mathrm{d}\boldsymbol{y})\boldsymbol{y}e^{A_1 t}\boldsymbol{e}I_{\mathrm{d}\boldsymbol{z}}\left(\frac{\boldsymbol{y}e^{A_1 t}}{\boldsymbol{y}e^{A_1 t}\boldsymbol{e}}\right)\mathrm{d}t \\
&= K \int_{t=0}^{\infty} \int_{\mathbb{J}} \mu_i(\mathrm{d}\boldsymbol{y})\boldsymbol{y}e^{A_1 t}\boldsymbol{e} \int_{\mathbb{J}} \boldsymbol{z}I_{\mathrm{d}\boldsymbol{z}}\left(\frac{\boldsymbol{y}e^{A_1 t}}{\boldsymbol{y}e^{A_1 t}\boldsymbol{e}}\right)\mathrm{d}t = K \int_{t=0}^{\infty} \int_{\mathbb{J}} \mu_i(\mathrm{d}\boldsymbol{y})\boldsymbol{y}e^{A_1 t}\boldsymbol{e}\frac{\boldsymbol{y}e^{A_1 t}}{\boldsymbol{y}e^{A_1 t}\boldsymbol{e}}\mathrm{d}t \\
&= K \int_{t=0}^{\infty} \int_{\mathbb{J}} \boldsymbol{y}\mu_i(\mathrm{d}\boldsymbol{y})e^{A_1 t}\mathrm{d}t = K \int_{t=0}^{\infty} \boldsymbol{\nu}_i e^{A_1 t}\mathrm{d}t, = K\boldsymbol{\nu}_i(-A_1)^{-1}.
\end{aligned}
$$

Thus using the results of the previous section, we know that

$$\boldsymbol{\theta}_{i+1} = K\boldsymbol{\nu}_{i+1}(-A_1)^{-1} = K\boldsymbol{\nu}_i S(-A_1)^{-1} = \boldsymbol{\theta}_i(-A_1)S(-A_1)^{-1} = \boldsymbol{\theta}_i R,$$

with $R = (-A_1)S(-A_1)^{-1}$, and where S is the solution of equation (14). Now, this can be rewritten as

$$(-A_1)^{-1}R(-A_1) = (-A_1)^{-1}A_0 + (-A_1)^{-1}R^2A_2.$$

Premultiplication by A_1 gives the expression,

$$R^2A_2 + RA_1 + A_0 = 0,$$

which is well-known in the traditional QBD framework. The vector $\boldsymbol{\theta}_0$ and possibly $\boldsymbol{\theta}_1$ is determined from boundary equations.

This result was proved in Bean and Nielsen [4] for exactly this class of problems using arguments based on taboo-probabilities and the last time of entering a set of states. We can now apply Lemma 3 and Corollary 7 directly in this environment and so have shown that all algorithms for standard QBDs that rely on level–censoring arguments can be applied directly to QBDs with RAP components.

5   Example

We consider some families of examples, which we believe to be of generic interest, particularly in the interface between ME and PH distributions. We hope that these may become standard example distributions in this area.

- A family of ME distributions of order 3 governed by the parameter $a \geq 0$, which can be made to have smaller coefficient of variation than the Erlang distribution of order p. For $a > 0$ we develop a TPH representation of order 5 (which may not be of minimal order), but for $a = 0$ there is no PH representation.

- A family of ME distributions of order 3 governed by the parameter $\eta \geq 1$. For $\eta > 1$, we develop a minimal order (> 3) PH representation. As $\eta \to 1$ the minimal order of the PH representation tends to ∞.

In Appendix A we present the detailed derivations of these families of distributions, to assist others in devising different families with similar characteristics.

In our queueing example, we consider a RAP/ME/1 queue which for specific choices of parameters has an alternative formulation as a MAP/PH/1 queue, although, generally of higher (in some cases significantly higher) order.

Service time distribution

As service time distribution we choose a distribution from the family parameterised by $a \geq 0, \epsilon \geq 0$ with density

$$f(x) = \frac{\frac{\lambda}{2}((\lambda x - \epsilon)^2 + a\epsilon^2))}{1 - \epsilon + \frac{1+a}{2}\epsilon^2} e^{-\lambda x},$$

which is an ME distribution of order 3 with $\boldsymbol{\alpha}$ and S given by

$$\boldsymbol{\alpha} = \frac{1}{1 + \frac{1+a}{2}\epsilon^2 - \epsilon}(1, -\epsilon, \frac{1+a}{2}\epsilon^2), \text{ and } S = \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & -\lambda \end{bmatrix}.$$

For certain values of a and ϵ the minimal PH order is at least 4. We give an explicit PH representation for this distribution of order 5 in Appendix A.

Arrival process

As arrival process we consider a process switching between a high and a low activity regime. In each regime arrivals occur according to a Poisson process with rate γ_1 and γ_2 respectively. When $\gamma_2 = 0$ the process is an ON-OFF process. The sojourn time in the high regime is governed by an ME-distribution of the second family , with density

$$\frac{\lambda_1(\lambda_2^2 + \omega^2)}{\lambda_2^2 + \omega^2 + b\lambda_1\lambda_2} \left(e^{-\lambda_1 t} + b e^{-\lambda_2 t} \cos(\omega t) \right).$$

The relationship between λ_1 and λ_2 needs to be such that $\lambda_2 = \eta\lambda_1$ where $\eta \geq 1$. We have Poisson arrivals with rate γ_1 while visiting this regime. The sojourn time in the second regime is exponential with intensity λ_3. We have Poisson arrivals with rate γ_2 while visiting this regime. The C and D matrices for this RAP arrival process are

$$C = \begin{pmatrix} -(\lambda_1 + \gamma_1) & 0 & 0 & \lambda_1 \\ \frac{(-\lambda_1 + \lambda_2 - \omega)(\lambda_2^2 + \omega^2)}{\lambda_2^2 + \omega^2 + b\lambda_1\omega} & -(\lambda_2 + \gamma_1) & \frac{(\lambda_2^2 + \omega^2 + b\lambda_1\lambda_2)\omega}{\lambda_2^2 + \omega^2 + b\lambda_1\omega} & \frac{\lambda_1(\lambda_2^2 + \omega^2)}{\lambda_2^2 + \omega^2 + b\lambda_1\omega} \\ \frac{(-\lambda_1 + \lambda_2 + \omega)(\lambda_2^2 + \omega^2)}{\lambda_2^2 + \omega^2 + b\lambda_1\lambda_2} & -\frac{(\lambda_2^2 + \omega^2 + b\lambda_1\omega)\omega}{\lambda_2^2 + \omega^2 + b\lambda_1\lambda_2} & -(\lambda_2 + \gamma_1) & \frac{\lambda_1(1+b)(\lambda_2^2 + \omega^2)}{\lambda_2^2 + \omega^2 + b\lambda_1\lambda_2} \\ 0 & 0 & \lambda_3 & -(\lambda_3 + \gamma_2) \end{pmatrix},$$

$$D = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 \\ 0 & \gamma_1 & 0 & 0 \\ 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & \gamma_2 \end{pmatrix}.$$

η	1.01	1.1	1.5	2	3
n	4945	505	105	65	35

Table 1: Dimension of the matrices in the $MAP/PH/1$ queue as a function of the parameter η.

The alternative MAP formulation, valid when $\eta > 1$, (with n states) and $b = -1$ is

$$C = \begin{pmatrix} T - \gamma_1 I & \lambda(1-p)\boldsymbol{e}_n \\ \lambda_3 \boldsymbol{\alpha}_2 & -\lambda_3 - \gamma_2 \end{pmatrix},$$

where $\boldsymbol{e}_n$ is a column vector of zeros with a one in the last place, and where the specific forms of T and $\boldsymbol{\alpha}_2$ are given in Appendix A, as are the values of λ and p. Finally,

$$D = \begin{pmatrix} \gamma_1 I & 0 \\ 0 & \gamma_2 \end{pmatrix}.$$

5.1   Numerical findings

We present two main experiments, both chosen so that the mean of the service time distribution is 1, and the arrival process is an ON-OFF process with $\gamma_1 = 1.9$ and $\gamma_2 = 0$. For the arrival process, we fixed $\omega = \pi$. Finally λ_3 was chosen such that the mean time in the high and low regime would be the same.

In the first experiment, we let $a = 1$ and then choose $\epsilon = 0.18350$ so that the service-time distribution has minimal form-factor of 1.311, but still has a TPH(5) representation (see Appendix A). For the arrival process, we ensure $\eta > 1$ and allow it to range over a set of values $\eta \in \{3, 2, 1.5, 1.1, 1.01\}$. We then chose λ_1 such that the mean sojourn-time in the high regime was 100. We have experienced no problems with numerical stability whatsoever. In all the cases we have tried, the maximum relative error between the two queue length distributions of the two comparable queues was less than 10^{-10}. However, obviously the computing time for the $MAP/PH/1$ queue grows rapidly with increasing η. In Table 1 we show how the dimensionality of the problem grows as η gets closer to 1. The dimension of the problem in the $RAP/ME/1$ formulation is always 12.

In the second experiment, we chose the parameters so that there was no comparable model possible in the traditional QBD framework. In other words, we analysed a pure RAP/ME/1 system. Therefore, for the service-time distribution we let $a = 0$ and then choose $\epsilon = 0.41577$ so that the service-time distribution again has minimal form-factor, this time of 1.277. However, in this case, because $a = 0$ there is no TPH(5) representation

and in fact there can be no PH representation at all (see Appendix A). For the arrival process we then let $\eta = 1$ and so again there can be no PH representation for this ME distribution either. Further, we chose λ_1 so that the mean sojourn-time in the high regime was 100. Again, we experienced no problems with numerical stability whatsoever.

For curiosity we present the queue length distribution (probability mass function) for a non-empty system in Figure 1. The probability of the system being empty is 0.05. The
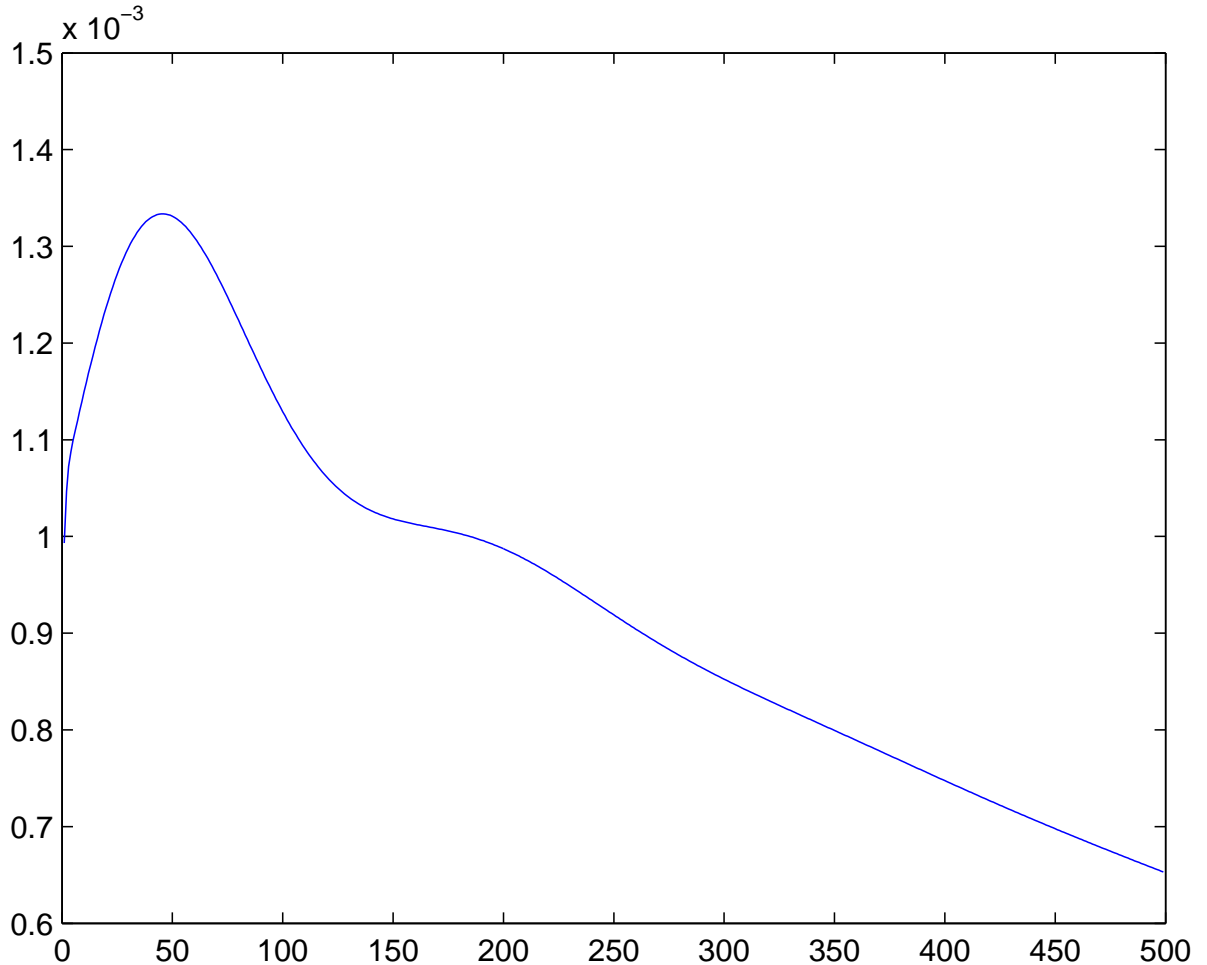

Figure 1: The queue length probability distribution (where the queue length is greater than 0) for a QBD-structured queue with RAP-components, that has no QBD parallel.

figure is constructed for the pure $RAP/ME/1$ case. However, for values of η close to one there is practically no difference. Apparently the minor difference between the two service time distributions is also quite insignificant.

For all cases we have tried with $\eta > 1$ and a PH service time distribution all the

eigenvalues of R in the $RAP/ME/1$ formulation were included in the set of eigenvalues for (the much larger) R in the $MAP/PH/1$ case. We conjecture that this is a general property without providing any proof.

6   Conclusion

In this paper we introduced the concept of Γ–linearity on order to rapidly reproduce results by Nielsen and Ramaswami [20] and Bean and Nielsen [4]. We then used the concept of Γ–linearity to produce new results for the GI/RAP/1 queue and finally justified that all algorithms based on level censoring arguments for the analysis of the standard QBD queue can be used in our more general setting.

Acknowledgements

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A   Derivation of distributions families for examples

In this appendix we discuss the derivation of the distributions of our example section, Section 5, in some more detail, as we find that our considerations leading to these examples might be of more general interest.

A.1   Service time distribution

Aldous and Shepp [1] proved that the least variable PH distribution among distributions with generators of order n is Erlang-n. O'Cinneide [23] showed that there exist ME distributions with generators of order n, that are less variable - in terms of the coefficient of variation - than the Erlang-n distribution. In this section we provide a specific family of such distributions, parameterised by $a \geq 0$ and of 3. When $a > 0$ we offer an alternative formulation as an order 5 phase type distribution. We do not claim the latter to be of minimal PH-order, however we certainly claim that the minimal PH-order is larger than 3 due to the results of Aldous and Shepp, and O'Cinneide. Consider the following mixture of the first three Erlang distributions

$$
\begin{aligned}
f(x) &= \frac{\frac{\lambda}{2}((\lambda x - \epsilon)^2 + a\epsilon^2))}{1 - \epsilon + \frac{1+a}{2}\epsilon^2} e^{-\lambda x} \\
&= \frac{\left(\frac{\lambda^3}{2}x^2 - \epsilon\lambda^2 x + \frac{1}{2}(1+a)\epsilon^2\lambda\right)}{1 - \epsilon + \frac{1+a}{2}\epsilon^2} e^{-\lambda x}.
\end{aligned}
$$

This distribution has $\boldsymbol{\alpha}$ and S given by

$$
\boldsymbol{\alpha} = \frac{1}{1 + \frac{1+a}{2}\epsilon^2 - \epsilon}(1, -\epsilon, \frac{1+a}{2}\epsilon^2), \text{ and } S = \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & -\lambda \end{bmatrix},
$$

with mean

$$
M_1 = \frac{6 - 4\epsilon + (1+a)\epsilon^2}{\lambda(2 - 2\epsilon + (1+a)\epsilon^2)},
$$

which is 1 when λ is chosen so that

$$
\lambda = \left(\frac{6 - 4\epsilon + (1+a)\epsilon^2}{2 - 2\epsilon + (1+a)\epsilon^2}\right).
$$

The ratio of the second non–central moment to the first moment squared (Palm's form-factor) is

$$
\frac{2(12 - 6\epsilon + (1+a)\epsilon^2)(2 - 2\epsilon + (1+a)\epsilon^2)}{(6 - 4\epsilon + (1+a)\epsilon^2)^2}.
$$

This function has a global minimum at a point where a is negative. Minimising with respect to ϵ, for fixed values of non-negative a, is obtained by solving the cubic in ϵ,

$$(a^2 - 1)\epsilon^3 + 9(1 + a)\epsilon^2 - 18(1 + a)\epsilon + 6 = 0.$$

We did not manage to reduce the analytical solution of this equation to anything simple. For $a = 0$ the minimal form-factor is 1.277 (as compared to $\frac{4}{3}$ for the Erlang distribution), obtained when $\epsilon = 0.41577$. For $a = 1$, the minimal form-factor is 1.311, obtained when $\epsilon = 0.18350$. The distribution can not be in PH for $a = 0$ as the density becomes 0 when $x = \epsilon/\lambda$. However, the distribution is in PH for $a > 0$, see O'Cinneide [21]. The distribution is even in TPH (Phase-Type with upper triangular(bidiagonal) generator) as all the poles of the Laplace Stieljtes transform of the distribution are real [22]. However, a TPH representation might not be of minimal order as a general PH-distribution with a generator of lower order than the one needed in the class of TPH might suffice. We now claim, that we can find a distribution in TPH with a specific form of generator. We proceed by deriving an initial probability vector such that this is indeed the case. We consider distributions in TPH(5) with the following generator

$$T = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda_1 \end{bmatrix}.$$

Our choice is motivated by the following considerations. We need to have at least three poles at $-\lambda$ which, given the diagonal structure of the TPH distribution, leads to at least three phases with $-\lambda$ on the diagonal. We need at least one phase with a diagonal value other than $-\lambda$ in order to get a density with negative coefficients for some of the Erlang components. The current choice is close to the most simple one could hope for. The TPH(5) distribution includes a subset of the set of distributions with basis functions (densities) given by

$$\boldsymbol{f}(t) = \begin{bmatrix} \frac{\lambda^3}{2} t^2 e^{-\lambda t} \\ \lambda^2 t e^{-\lambda t} \\ \lambda e^{-\lambda t} \\ \lambda_1 e^{-\lambda_1 t} \end{bmatrix}.$$

Any PH density of the generator T can be expressed as $f(t) = \boldsymbol{\alpha}_1 \mathbf{H} \boldsymbol{f}(t)$ with the coefficient matrix K given by

$$
\mathbf{H} = \begin{bmatrix}
\frac{\lambda_1}{\lambda_1 - \lambda} & \frac{-\lambda_1 \lambda}{(\lambda_1 - \lambda)^2} & \frac{\lambda_1 \lambda^2}{(\lambda_1 - \lambda)^3} & \frac{-\lambda^3}{(\lambda_1 - \lambda)^3} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

We want to find $\boldsymbol{\alpha}_1$ such that $\boldsymbol{\alpha}_1 \mathbf{H} = (\boldsymbol{\alpha}, 0)$. This linear system is under-determined leaving us some room to manipulate to get a solution of a usable form, namely

$$
\boldsymbol{\alpha}_1^T = \begin{bmatrix}
\frac{(\rho-1)^3 y}{1 + 1/2\,\epsilon^2 a - \epsilon + 1/2\,\epsilon^2} \\
\frac{1 - (\rho-1)^2 \rho\, y}{1 + 1/2\,\epsilon^2 a - \epsilon + 1/2\,\epsilon^2} \\
\frac{(\rho-1)\rho\, y - \epsilon}{1 + 1/2\,\epsilon^2 a - \epsilon + 1/2\,\epsilon^2} \\
\frac{1/2\,(1+a)\epsilon^2 - y\rho}{1 + 1/2\,\epsilon^2 a - \epsilon + 1/2\,\epsilon^2} \\
\frac{y}{1 + 1/2\,\epsilon^2 a - \epsilon + 1/2\,\epsilon^2}
\end{bmatrix},
$$

where $\rho = \frac{\lambda_1}{\lambda}$ and y is the free parameter in the solution. For $a = 1$ we can choose $\rho = 1 + \frac{1}{\epsilon}$ and $y = \frac{\epsilon^3}{1+\epsilon}$ so that

$$
\boldsymbol{\alpha}_1 = \begin{bmatrix}
\frac{1}{(\epsilon^2 - \epsilon + 1)(\epsilon + 1)} & 0 & 0 & 0 & \frac{\epsilon^3}{(\epsilon^2 - \epsilon + 1)(\epsilon + 1)}
\end{bmatrix}.
$$

A.2   Generic ME - distribution for RAP/MAP arrival process

We consider the set of distributions with basis functions (densities) given by

$$
\boldsymbol{g}(t) = \begin{bmatrix}
\lambda_1 e^{-\lambda_1 t} \\
\frac{\lambda_1 (\lambda_2^2 + \omega^2)}{\lambda_2^2 + \omega^2 + b\lambda_1 \lambda_2} \left(e^{-\lambda_1 t} + b e^{-\lambda_2 t} \sin(\omega t) \right) \\
\frac{\lambda_1 (\lambda_2^2 + \omega^2)}{\lambda_2^2 + \omega^2 + b\lambda_1 \lambda_2} \left(e^{-\lambda_1 t} + b e^{-\lambda_2 t} \cos(\omega t) \right)
\end{bmatrix}
$$

for sojourn time in the high regime. In particular we pick the third component of these. This distribution has mean

$$
\frac{2\,\lambda_2{}^2 \omega^2 + \lambda_2{}^4 + \omega^4 - \omega^2 \lambda_1{}^2 b + \lambda_2{}^2 \lambda_1{}^2 b}{\left(\lambda_2{}^2 + \omega^2 + b\,\lambda_1\,\lambda_2 \right) \lambda_1 \left(\lambda_2{}^2 + \omega^2 \right)}.
$$

Following the steps of Asmussen and Bladt [3, page 135] we derive the ME-generator of
this distribution to be

$$C = \begin{pmatrix} -\lambda_1 & 0 & 0 \\ \frac{(-\lambda_1+\lambda_2-\omega)(\lambda_2^2+\omega^2)}{\lambda_2^2+\omega^2+b\lambda_1\omega} & -\lambda_2 & \frac{(\lambda_2^2+\omega^2+b\lambda_1\lambda_2)\omega}{\lambda_2^2+\omega^2+b\lambda_1\omega} \\ \frac{(-\lambda_1+\lambda_2+\omega)(\lambda_2^2+\omega^2)}{\lambda_2^2+\omega^2+b\lambda_1\lambda_2} & -\frac{(\lambda_2^2+\omega^2+b\lambda_1\omega)\omega}{\lambda_2^2+\omega^2+b\lambda_1\lambda_2} & -\lambda_2 \end{pmatrix},$$

where the initial vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ will pick up the three components of
$\boldsymbol{g}(t)$. We now turn to possible PH-distributions of this form. A distribution with rational
Laplace transform is in PH if it has a positive density for all positive arguments and if the
pole of maximum real part is unique and real, see O'Cinneide [21]. As a consequence the
distribution will be in PH whenever $|b| \leq 1$ and $\lambda_1 < \lambda_2$. We know from O'Cinneide [22]
that the minimum number of phases in a PH-representation n is given by:

$$\frac{\omega}{\lambda_2 - \lambda_1} \leq \cot\left(\frac{\pi}{n}\right) \Leftrightarrow n \geq \frac{\pi}{\arctan\left(\frac{\lambda_2-\lambda_1}{\omega}\right)}. \tag{15}$$

We now consider PH distributions of order n with generator T, where $T_{ii} = -\lambda, i = 1, \ldots n$,
$T_{i,i+1} = \lambda$, $i = 1, \ldots, n-1$, $T_{n,1} = p\lambda$ and all other entries are zero. We denote the initial
vector of this distribution by $\boldsymbol{\alpha}_2$. The densities of this PH distribution are given by $\boldsymbol{\alpha}_2\boldsymbol{h}(t)$,
where $\boldsymbol{h}(t) = e^{Tt}(-T\boldsymbol{e})$. By straightforward algebra we find the LST's of the l'th element
of $\boldsymbol{h}(t)$, $h_l(t)$, to be

$$\tilde{H}_l(s) = \frac{\lambda^{n+1-l}(1-p)(s+\lambda)^{l-1}}{(s+\lambda)^n - p\lambda^n}.$$

We denote the roots of $(s+\lambda)^n - p\lambda^n$ by z_j where $z_j = -\lambda + \lambda p^{\frac{1}{n}}u_j$, where u_j are
the n solutions to the equation $u^n = 1$, such that $z_{n-j} = \bar{z}_j$. Now by partial fraction
decomposition we have, for some $\kappa_{ij}, j = 0, \ldots, n-1$,

$$\tilde{H}_l(s) = \sum_{j=0}^{n-1} \frac{\kappa_{lj}}{s + z_j}.$$

We find

$$\kappa_{lj} = \frac{\lambda^{n+1-l}(1-p)(z_j+\lambda)^{l-1}}{\prod_{k\neq j}(z_j - z_k)} = \lambda(1-p)p^{\frac{l}{n}}\frac{u_j^{l-1}}{\prod_{k\neq j}(u_j - u_k)}.$$

Now a standard result in complex analysis tells us that $\prod_{k\neq j}(u_j - u_k) = n\bar{u}_j$. We thus get

$$\kappa_{lj} = \frac{\lambda}{n}(1-p)p^{\frac{l}{n}}\frac{u_j^{l-1}}{\bar{u}_j} = \frac{\lambda}{n}(1-p)p^{\frac{l}{n}}u_j^l = \frac{\lambda}{n}(1-p)p^{\frac{l}{n}}\left(\cos\left(\frac{2\pi jl}{n}\right) + i\sin\left(\frac{2\pi jl}{n}\right)\right).$$

With $\boldsymbol{u}_l = (u_0^{l-1}, u_1^{l-1}, u_2^{l-1}, \ldots, u_{n-1}^{l-1})$ we have

$$\boldsymbol{\Psi} = \{\kappa_{lj}\} = \frac{\lambda}{n}(1-p)\Delta(\boldsymbol{p}^\star) \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_n \end{bmatrix},$$

where $\Delta(\boldsymbol{c})$ is a diagonal matrix of the elements of the vector $\boldsymbol{c}$ and the l'th element of $\boldsymbol{p}^\star$ is $p^{\frac{l}{n}}$. It is straightforward to verify that

$$\boldsymbol{\Psi}^{-1} = \frac{1}{\lambda(1-p)} \begin{bmatrix} \boldsymbol{u}_1' & \boldsymbol{u}_n' & \boldsymbol{u}_{n-1}' & \cdots & \boldsymbol{u}_2' \end{bmatrix} \Delta(\boldsymbol{p}^\star)^{-1}.$$

If we set $b = -1$, then the distribution we are aiming for is a scalar multiple of $e^{u_0 t} - \frac{1}{2}\left(e^{u_1 t} + e^{\bar{u}_1 t}\right) = e^{u_0 t} - \frac{1}{2}\left(e^{u_1 t} + e^{u_{n-1} t}\right)$. Here the unnormalised vector $\boldsymbol{\alpha}_2$ is determined by multiplying the vector $\left(1, -\frac{1}{2}, 0, 0, \ldots, 0, -\frac{1}{2}\right)$ by $\boldsymbol{\Psi}^{-1}$. Finally we see that the following phase type distribution with $\alpha_{2,i} = K\left(1 - \cos\left(\frac{i2\pi}{n}\right)\right)p^{\frac{-i}{n}}$ and $\{T_{ii} = -\lambda, i = 1, \ldots, n, T_{i,i+1} = \lambda, i = 1, \ldots, n-1, T_{n,1} = p\lambda\}$ is identical to the ME distribution with $\lambda_1 = \lambda\left(1 - p^{\frac{1}{n}}\right)$, $\lambda_2 = \lambda\left(1 - p^{\frac{1}{n}}\cos\left(\frac{2\pi}{n}\right)\right)$, and $\omega = \lambda p^{\frac{1}{n}}\sin\left(\frac{2\pi}{n}\right)$. For a given ME distribution (i.e. given $\lambda_1, \lambda_2,$ and ω) if

$$n = \frac{2\pi}{\arcsin\left(\frac{2\omega(\lambda_2 - \lambda_1)}{(\lambda_2 - \lambda_1)^2 + \omega^2}\right)} \in \mathbb{Z}_+,$$

then there is an equivalent PH distribution with $(\boldsymbol{\alpha}, T)$ representation where

$$p = \left(\frac{(\lambda_2 - \lambda_1)^2 + \omega^2}{(\lambda_2 + \lambda_1)(\lambda_2 - \lambda_1) + \omega^2}\right)^n, \qquad \lambda = \frac{(\lambda_2 + \lambda_1)(\lambda_2 - \lambda_1) + \omega^2}{2(\lambda_2 - \lambda_1)},$$

and the representation is minimal. The minimality can be seen by inserting the values of $\lambda_1, \lambda_2,$ and ω in expression (15) and expressing $\cos\left(\frac{2\pi}{n}\right)$ and $\sin\left(\frac{2\pi}{n}\right)$ in terms of $\cos\left(\frac{\pi}{n}\right)$ and $\sin\left(\frac{\pi}{n}\right)$.