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NUMERICAL ANALYSIS OF FORTH-ORDER BOUNDARY VALUE PROBLEMS IN FLUID MECHANICS AND MATHEMATICS

by

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In this paper He's variational iteration method is used to solve some examples of linear and non-linear forth-order boundary value problems. The first problem compared with homotopy analysis method solution and the other ones with the exact solution. The results show the high accuracy and speed of convergence of this method. It is found that the variational iteration method is a powerful method for solving of the non-linear equations.

Key words: *magnetohydrodynamics, variational iteration method, fourth-order differential equations, boundary value problems*

Introduction

The theoretical study of magnetohydrodynamics (MHD) channel flow has been a subject of great interest due to its widespread applications in designing cooling systems with liquid metals, MHD generators, accelerators, pumps, and flow meters.

Besides the common applications of generation and motive power from electricity, which exist in everything from an electric toothbrush to a portable recreational vehicle electrical generator, there are some more unusual applications of the motor principle, or Lenz's law. By examining some of the most complex uses of electromagnetism for motors and pumps, some very unusual and counter-intuitive applications of electromagnetic fields for propulsion will become apparent.

The MHD is a field of magnetic pumping which uses the Lenz's law to pump liquids using only an electromagnetic field. This unique concept allows MHD to pump conducting liquids with absolutely "no moving parts". There is simply a changing electromagnetic field developed through the medium being pumped which will cause it to move. The most common application of these electromagnetodynamic pumps is in the metal industry where molten metal can be pumped and stirred without contact, right through the walls of the vessel containing the molten metal. Pumps of this type are utilized in nuclear reactors where liquid sodium is pumped

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through the reactor core for cooling. The high temperature and reactivity of this molten salt would destroy any normal impeller. By pumping right through the walls of the piping the dangerously radioactive salt is isolated from the pumps themselves which is beneficial for safety and maintenance reasons [1].

Raptis *et al*. [2] considered the unsteady MHD flow of a viscous and electrically conducting fluid past to a plate by the presence of radiation. Moreover, they derived analytical solutions for the mean temperature, velocity, magnetic field, and the effect of the radiation on the temperature.

The non-Newtonian fluids are considered as more appropriate models of fluids in industrial and technological applications than Newtonian fluids. Such fluids exhibit the non-linear relationship between stress and the rate of strain at every point of flow. Due to non-linear dependence of stresses on the rate of strain for non-Newtonian fluids, the flow analysis is much more complicated in comparison with Newtonian fluids. The constitutive equations are very complex involving a number of parameters and the solutions of the resulting equations in general are more difficult to obtain. This is not only true of exact analytical solutions but even of numerical solutions. Several investigators are now engaged in finding the analytical or numerical solutions for flow problems that arise using different non-Newtonian fluids. One of the important classes of non-Newtonian fluids is viscoelastic fluid. However, even the most commonly used simplest subclass of viscoelastic fluids is that of the so-called second order fluid that can give rise to problems which are far from trivial [3].

The problem which governs the MHD boundary layer flow is [4]:

$$\frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} \quad v\frac{\partial v}{\partial y} \quad \nu\frac{\partial^2 u}{\partial y^2} \quad \frac{\alpha_1}{\rho} \quad \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} \quad u\frac{\partial^3 u}{\partial x \partial y^2} \quad \frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial y^2} \quad v\frac{\partial^3 u}{\partial y^3} \quad \frac{\sigma B_0^2}{\rho}u \tag{2}$$

$$\frac{\partial u}{\partial y} \quad v \quad 0 \text{ at } y \quad 0 \tag{3}$$

$$u \quad 0,\ v \quad \frac{V}{2} \text{ at } y \quad \frac{H}{2} \tag{4}$$

Here, ρ is the density, ν – the kinematic viscosity, σ – the electrical conductivity, H – the width of the channel, α_1 – the material parameter of second grade fluid, u and v – the velocity components in the x and y directions and the fluid injection or extraction takes place through the porous walls with velocity $V/2$. Note that $V > 0$ corresponds to the suction case and $V < 0$ for injection.

Defining:

$$x^* \quad \frac{x}{H}, \quad y^* \quad \frac{y}{H}, \quad u \quad Vx^* f\ (y^*), \quad v \quad Vf\,(y^*) \tag{5}$$

Equation (1) is identically satisfied and eqs. (2)-(4) reduce to eq. (8) which models MHD flow of a second grade fluid in a porous channel and was analyzed using variational iteration method. In addition to first example, we investigate two linear and non-linear fourth order boundary value problems in the following.

For ordinary and partial differential equations, some of the analytical/approximate techniques that have been developed include perturbation [5-7], variational iteration [8-23], decomposition [24-26], homotopy perturbation [27-34], *etc*. At first, almost all perturbation meth-

ods are based on an assumption that a small parameter must exist in the equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques. As is well known, an overwhelming majority of non-linear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. Furthermore, the approximate solutions solved by the perturbation methods are valid, in most cases, only for the small values of the parameters. It is obvious that all these limitations come from the small parameter assumption.

Variational iteration method (VIM) [8-23] was introduced by He [8-14] based on the use of restricted variations and correction functionals which has found a wide application for the solution of non-linear ordinary and partial differential equations. This method does not require the presence of small parameters in the differential equation, and provides the solution (or an approximation to it) as a sequence of iterates. The method does not require that the non-linearities be differentiable with respect to the dependent variable and its derivatives.

Basic idea of He's variational iteration method

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t) \tag{6}$$

where L is a linear operator, N a non-linear operator, and $g(t)$ an inhomogeneous term. According to VIM, we can write down a correction functional as follows:

$$u_{n\ 1}(t) \quad u_n(t) \quad \int_0^t \lambda[Lu_n(\tau) \quad N\tilde{u}_n(\tau) \quad g(\tau)]\mathrm{d}\tau \tag{7}$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the n^{th} approximation and $\tilde{u}_n$ is considered as a restricted variation $\delta\tilde{u}_n \quad 0$.

Numerical examples

Example 1

Consider the following non-linear fourth-order differential equation:

$$y \quad M^2 y(x) \quad \mathrm{Re}[y(x)^2 \quad y(x)y\ (x)] \quad \alpha[2y(x)y\ (x) \quad y\ (x)^2 \quad y(x)y^{\mathrm{iv}}(x)] \quad 0 \tag{8}$$

with the boundary conditions

$$y(0) \quad 0, \quad y^n(0) \quad 0; \quad y(0.5) \quad 0.5; \quad y\ (0.5) \quad 0 \tag{9}$$

where $M^2 \quad \sigma B_0^2 H^2/\mu$, Re = HV/ν, and $\alpha = \alpha_1/H_\mu$ [3]. B_0 is a constant magnetic field, applied perpendicular to the channel walls and the electric field is considered to be zero. The induced magnetic field is neglected for small magnetic Reynolds number. Re > 0 indicates the suction case and Re < 0 stands for injection. It should be noted that for $\alpha = 0$, eq. (8) is related to Newtonian fluid.

Its correction variational functional can be expressed as follows:

$$y_{n\ 1}(x) \quad y_n(x) \quad \int_0^x \lambda[y\ (\tau) \quad M^2 y(\tau) \quad \mathrm{Re}\, y(\tau)^2 \quad y(\tau)y\ (\tau)] \\ \alpha[2y(\tau)y\ (\tau) \quad y\ (\tau)^2 \quad y(\tau)y^{\mathrm{iv}}(\tau)]\ \mathrm{d}\tau \quad 0 \tag{10}$$

After some computations, we obtain the following stationary conditions:

$$\lambda^{(\mathrm{iv})} \quad \lambda \quad 0 \tag{11}$$

$$1 \quad \lambda \Big|_{\tau \; x} \quad 0, \; \lambda \Big|_{\tau \; x} \quad 0 \tag{12}$$

$$\lambda \Big|_{\tau \; x} \quad 0, \quad \lambda \Big|_{\tau \; x} \quad 0 \tag{13}$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda(t) \quad \frac{(\tau \quad x)^3}{6} \tag{14}$$

and the variational iteration formula is obtained in the form:

$$y_{n\;1}(x) \quad y_n(x) \quad \int_0^x \frac{(\tau \quad x)^3}{6} \{ y_n(\tau) \quad M^2 y_n(\tau) \quad \mathrm{Re}[y_n(\tau)^2 \quad y_n(\tau) y_n(\tau)] \quad \alpha[2 y_n(\tau) y_n(\tau) \quad y_n(\tau)^2 \quad y_n(\tau) y_n^{\mathrm{iv}}(\tau)] \} \, \mathrm{d}\tau \tag{15}$$

We start with the initial approximation of $y_0(x)$, but since no initial approximation of $y_0(x)$ is available, we make one in the form of a polynomial as:

$$y_0(x) \quad a \quad bx \quad cx^2 \quad dx^3 \tag{16}$$

which depends on the order of differentiation, and a, b, c, and d are unknown constants to be later determined.

Using the above iteration formula (15), we can directly obtain other components as:

$$\begin{aligned} y_1(x) \quad & 0.5ax^2 \quad 0.01666a^2 \,\mathrm{Re}\, x^6 \quad cx \quad a^2 dx^2 \quad 0.16666bd \,\mathrm{Re}\, x^2 \quad ax^3 \\ & bx^2 \quad 0.1a^3 x^5 \quad 0.03333ab \,\mathrm{Re}\, x^5 \quad 0.02777b^2 \,\mathrm{Re}\, x^4 \quad 0.5a^3 x^4 \\ & 0.04166aM^2 x^4 \quad 0.16666a^2 bx^4 \quad 0.05555bM^2 x^3 \quad 0.33333ab^2 x^2 \\ & 0.083333c^2 \,\mathrm{Re}\, x^2 \quad 0.08333cM^2 x^2 \quad 0.33333a^2 cx^3 \quad 0.66666a^2 bx^3 \\ & 0.05555bc \,\mathrm{Re}\, x^3 \quad 0.16666x^3 ad \,\mathrm{Re} \quad d \end{aligned} \tag{17}$$

For a special case: $M = 2$, $\mathrm{Re} = 0$, and $\alpha = 0.2$, $y_1(x)$ will be:

$$\begin{aligned} y_1(x) \quad & 0.5ax^2 \quad cx \quad 0.2a\mathrm{d}x^2 \quad ax^3 \quad bx^2 \quad 0.02a^2 x^5 \quad 0.1a^2 x^4 \\ & 0.16666ax^4 \quad 0.033333abx^4 \quad 0.22222bx^3 \quad 0.06666b^2 x^2 \\ & 0.33333cx^2 \quad 0.06666acx^3 \quad 0.13333abx^3 \quad d \end{aligned} \tag{18}$$

Incorporating the boundary conditions, eq. (9), into $y_1(x)$, we have:

$$y_1(0) = d = 0 \tag{19}$$

$$y_2(0) \quad a \quad 0.4ad \quad 2b \quad 0.13333b^2 \quad 0.66666c \quad 0 \tag{20}$$

$$\begin{aligned} & y_1(0.5) \quad 0.010416a \quad 0.58333c \quad 0.05ad \quad 0.27777b \\ & 0.00562a^2 \quad 0.01458ab \quad 0.01666b^2 \quad 0.00833ac \quad d \quad 0.5 \end{aligned} \tag{21}$$

$$y_1(0.5) \quad 0.33333a \quad 1.33333c \quad 0.2ad \quad 1.16666b$$
$$0.04375a^2 \quad 0.08333ab \quad 0.06666b^2 \quad 0.05ac \quad 0 \tag{22}$$

Solving the system of equations simultaneously, we obtain:

$$a = -1.246731807, \; b = -1.044604271, \; c = 1.481954721, \; d = 0 \tag{23}$$

Therefore, we obtain the following first-order approximate solution for special case, $M = 2$, Re = 0, $\alpha = 0.2$:

$$y_1(x) = 0.03108x^5 - 0.31981x^4 - 1.77568x^3 + 0.5\ 10^{-9}x^2 + 1.481954721x \tag{24}$$

In the same manner, the rest of the components of the iteration formula can be obtained.

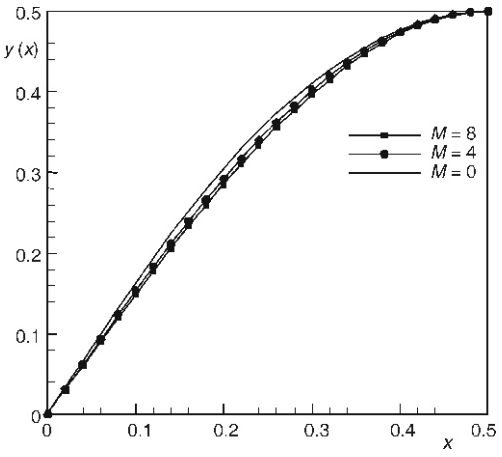

Figure 1. Variation of $y(x)$ for different values of M for $\alpha = 0.2$, Re = 2 : (HAM and VIM)

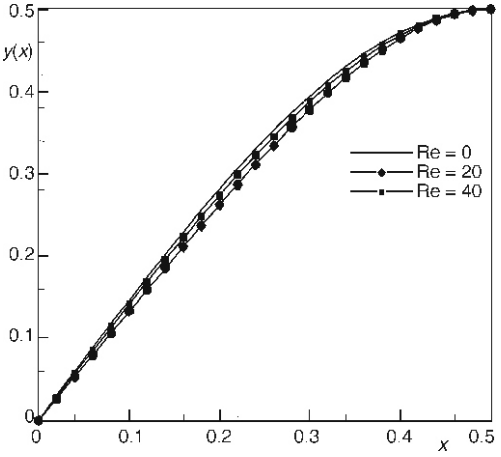

Figure 2. Variation of $y(x)$ for different values of Re for $\alpha = 0.2$, $M = 2$: (HAM and VIM)

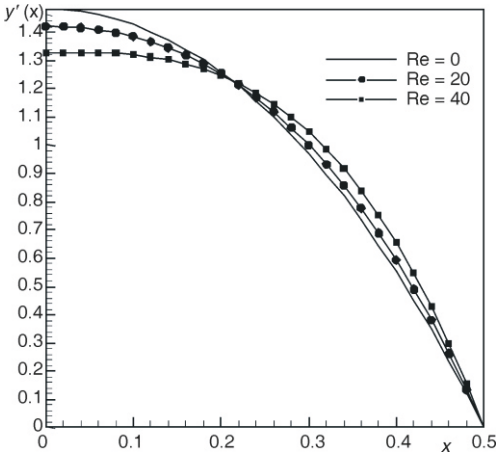

Figure 3. Variation of d$y(x)$/dx for different values of Re for $\alpha = 0.2$, $M = 2$: (HAM and VIM)

Figures 1-3 show comparison of homotopy analysis method HAM [3] and VIM results which reveals excellent agreement between the methods.

Example 2

Now consider another non-linear fourth-order BVP:

$$y^{iv}(x) \quad y(x)y'(x) \quad 4x^7 \quad 24 \quad 0 \tag{25}$$

Subject to the boundary conditions:

$$\begin{aligned} &y(0) \quad 0 \quad y'(0.25) \quad 6 \\ &y'(0.5) \quad 3 \quad y(1) \quad 1 \end{aligned} \tag{26}$$

Its correction variational functional in can be expressed as:

$$y_{n\ 1}(x) \quad y_n(x) \quad \int_0^x \lambda[y_n^{\mathrm{iv}}(\tau) \quad y_n(\tau)\, y_n(\tau) \quad 4x^7 \quad 24]\mathrm{d}\tau \tag{27}$$

After some computations, we obtain the following stationary conditions:

$$\lambda^{\mathrm{iv}} = 0 \tag{28}$$

$$1 \quad \lambda \Big|_{\tau\ x} \quad 0 \quad \lambda \Big|_{\tau\ x} \quad 0 \tag{29}$$

$$\lambda \Big|_{\tau\ x} \quad 0 \quad \lambda\Big|_{\tau\ x} \quad 0 \tag{30}$$

The Lagrangian multiplier can, therefore, be identified as:

$$\lambda(t) \quad \frac{(x \quad \tau)^3}{6} \tag{31}$$

and the variational iteration formula is obtained in the form:

$$y_{n\ 1}(x) \quad y_n(x) \quad \int_0^x \frac{(x \quad \tau)^3}{6}[y_n^{\mathrm{iv}}(\tau) \quad y_n(\tau) \quad y_n(\tau)4x^7 \quad 24] \ \mathrm{d}\tau \tag{32}$$

Now we assume that the initial approximation has the form:

$$y_0(x) \quad a \quad bx \quad cx^2 \quad \mathrm{d}x^3 \tag{33}$$

where a, b, c, and d are unknown constants to be further determined. Using the iteration formula (32), we can directly obtain the other components as follows:

$$\begin{aligned} y_1(x) \quad & 0.00005x^{11} \quad 0.00298abx^8 \quad 0.00476acx^7 \quad 0.00238bx^7 \\ & 0.00833adx^6 \quad 0.00833bcx^6 \quad 0.01666bdx^5 \quad 0.00833cx^5 \\ & x^4 \quad 0.04166cdx^4 \quad ax^3 \quad 0.00009ax^2 \quad bx^2 \quad cx \quad d \end{aligned} \tag{34}$$

Incorporating the boundary conditions, eq. (26), into $y_1(x)$, we have:

$$y_1(0) \quad d \quad 0 \tag{35}$$

$$\begin{aligned} y_1(1) \quad & 0.00833ac \quad 0.00297ab \quad 0.00833bc \quad 0.00009a^2 \\ & 0.00238b^2 \quad 0.00833c^2 \quad 1.005 \quad a \quad b \quad c \quad 1 \end{aligned} \tag{36}$$

$$\begin{aligned} y_1(0.5) \quad & 0.00625ac \quad 0.0026041\,ab \quad 0.015625\,bc \quad 0.00055a^2 \\ & 0.00312b^2 \quad 0.02083c^2 \quad 3a \quad 2b \quad 3.00011 \quad 3 \end{aligned} \tag{37}$$

$$\begin{aligned} y_1(0.25) \quad & 0.00391ac \quad 0.00097ab \quad 0.015625bc \quad 0.000122a^2 \\ & 0.0019531b^2 \quad 0.03125c^2 6a \quad 6.00001 \quad 6 \end{aligned} \tag{38}$$

Solving the system of equations simultaneously, we obtain:

$$a \quad 0.0000012703, \quad b \quad 0.000052345, \quad c \quad 0.00045143, \quad d \quad 0 \tag{39}$$

Therefore, we obtain the following first-order approximate solution, in the form:

$$\begin{aligned} y_1(x) \quad & 5.0505 \ 10^{\ 4}x^{11} \quad 1.60104 \ 10^{\ 15}x^9 \quad 1.979125 \ 10^{-13}x^8 \\ & 9.25486 \ 10^{\ 12}x^7 \quad 1.96921 \ 10^{\ 10}x^6 \quad 1.69826 \ 10^{\ 9}x^5 \quad x^4 \\ & 0.127 \ 10^{\ 4}x^3 \quad 0.5234 \ 10^{\ 4}x^2 \quad 0.4514 \ 10^{\ 3} \end{aligned} \tag{40}$$

The exact solution is $y(x) = x^4$. Plotting the exact and VIM solutions, it is clear that the results are in excellent agreement (fig 4).

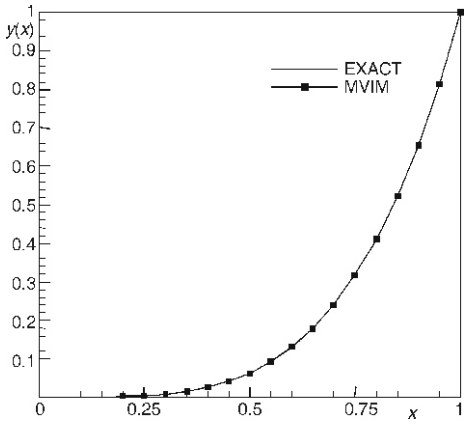

Figure 4. The comparison between the exact and VIM solutions

Example 3

Consider the linear boundary value problem as follows:

$$y^{\mathrm{iv}}(x) \quad y(x) \quad y\ (x) \quad e^x(x \quad 3) \quad 0 \tag{41}$$

Subject to the following conditions:

$$\begin{aligned} &y(0) \quad 1, \quad y\ (0) \quad 0 \\ &y(1) \quad 0, \quad y\ (1) \quad e \end{aligned} \tag{42}$$

Its correction variational functional can be expressed as:

$$y_{n\ 1}(x) \quad y_n(x) \quad \int_0^x \lambda[y_n^{\mathrm{iv}}(\tau) \quad y_n(\tau) \quad y_n(\tau) \quad e^x(x \quad 3)]\mathrm{d}\tau \tag{43}$$

After some computations, we obtain the following stationary conditions:

$$\lambda \quad 0 \tag{44}$$

$$1 \quad \lambda\,\big|_{\tau\ x} \quad 0 \quad \lambda\big|_{\tau\ x} \quad 0 \tag{45}$$

The Lagrangian multiplier can, therefore, be identified as:

$$\lambda(t) \quad \frac{(\tau \quad x)^3}{6} \tag{46}$$

and the variational iteration formula is obtained in the form:

$$y_{n\ 1}(x) \quad y_n(x) \quad \int_0^x \frac{(\tau \quad x)^3}{6}[y_n^{\mathrm{iv}}(\tau) \quad y_n(\tau) y\ n(\tau) \quad e^x(x \quad 3)]\ \mathrm{d}\tau \tag{47}$$

Now we assume that the initial approximation has the form:

$$y_0(x) = a + bx + cx^2 + dx^3 \tag{48}$$

Where a, b, c, and d are unknown constants to be further determined. Using the above iteration formula (47), we can directly obtain the other components as:

$$\begin{aligned} &y_1(x) \quad 0.00119ax^7 \quad 0.00277bx^6 \quad (0.05a \quad 0.00833c)x^5 \\ &(0.08333b \quad 0.04166d)x^4 \quad (a \quad 0.66666)x^3 \quad (b \quad 2.5)x2 \\ &(c \quad 6)x \quad e^x(x \quad 7) \quad d \quad 7 \end{aligned} \tag{49}$$

Incorporating the boundary conditions, eq. (42), into $y_1(x)$, and solving the equations, we obtain:

$$y_1(0) \quad d \quad 1 \tag{50}$$

$$y_1(1) \quad 1.05119a \quad 1.08611b \quad 1.00833c \quad 6e \quad 17.20833 \quad 0 \tag{51}$$

$$y_1(0) \quad 6 \quad c \quad 6e^0 \quad 0 \tag{52}$$

$$y_1(1) \quad 3.25833a \quad 2.35b \quad 0.42473 \quad e \tag{53}$$

Solving the system of equations simultaneously, we obtain:

$$a = -0.35486, \quad b = -0.48394, \quad c = 0, \, d = 1 \tag{54}$$

In the same manner, the rest of the components of the iteration formula can be obtained as:

$$\begin{aligned} y_1(x) \quad & 0.00042x^7 \quad 0.00134x^6 \quad 0.01774x^5 \quad 0.00134x^4 \\ & 0.31180x^3 \quad 2.01606x^2 \quad e^x(x \quad 7) \quad 6x \quad 8 \end{aligned} \tag{55}$$

The exact solution for this problem is: $(1 - x)e^x$

Figure 5 shows the comparison between VIM and exact solution (fig. 5).

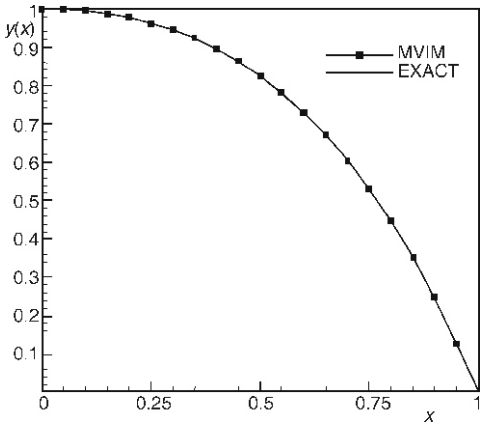

Figure 5. The comparison between the exact and VIM solutions

Conclusions

In this work, we studied the application of the VIM fourth order boundary value problems.

The figures clearly show that the results by VIM are in excellent agreement with the HAM and exact solutions. VIM provides highly accurate numerical solutions in comparison with other methods, and it is expected here that VIM as a powerful mathematical tool can solve a large class of linear and non-linear differential system and equations used in engineering and physics.

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