Conceptual Knowledge Representation and Reasoning

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Abstract

One of the main areas in knowledge representation and logic-based artificial intelligence concerns logical formalisms that can be used for representing and reasoning with concepts. For almost 30 years, since research in this area began, the issue of intensionality has had a special status in that it has been considered to play an important role, yet it has not been precisely established what it means for a logical formalism to be intensional. This thesis attempts to set matters straight. Based on studies of the main contributions to the issue of intensionality from philosophy of language, in particular the works of Gottlob Frege and Rudolf Carnap, we start by defining when a logical formalism is intensional. We then examine whether the current formalizations of concepts are intensional. The result is negative in the sense that none of the prevalent formalizations are intensional. This motivates the development of intensional logics for concepts. Our main contribution is the presentation of such an *intensional concept logic*.

The intensional concept logic is a development of the well-known description logic $\mathcal{ALC}$. More precisely, the logic is based, not only on a single, but on two equivalence relations. This allows us to express that concepts are co-extensional as well as to express that concepts are co-intensional. The intensional semantics of the logic is a novel algebraic semantics which is defined through abstraction of the extensional semantics of $\mathcal{ALC}$. It is shown that this approach generalizes to other logics than description logics.

Resumé

Et af hovedområderne inden for vidensrepræsentation og logikbaseret kunstig intelligens omhandler logiske formalismer, der er velegnede til at repræsentere begreber og til at foretage logiske slutninger, som involverer begreber. I næsten 30 år, siden forskning i dette emne begyndte, har problemstillingen intensionalitet haft en særstatus, idet den er blevet betragtet som værende vigtig, alligevel er det ikke blevet præcist fastlagt, hvad det vil sige, at en logisk formalisme er intensionel. Denne afhandling forsøger at råde bod på dette. Med udgangspunkt i hovedbidragene til intensionalitet, der stammer fra Gottlob Frege og Rudolf Carnap, starter vi med at definere, hvornår en logisk formalisme er intensionel. Derefter undersøger vi, hvorvidt de nuværende formaliseringer af begrebsviden er intensionelle. Resultatet er negativt, idet ingen af de fremherskende formaliseringer er intensionelle. Dette motiverer udviklingen af intensionelle logikker, der kan håndtere begrebsviden. Denne afhandlings hovedbidrag er en præsentation af en sådan *intensionel begrebslogik*.

Den intensionelle begrebslogik er en videreudvikling af den velkendte beskrivelseslogik $\mathcal{ALC}$. Den intensionelle logik er baseret på ikke alene én, men to ækvivalensrelationer, hvorved vi både kan udtrykke, at begreber har samme ekstension, samt at begreber har samme intension. Den intensionelle semantik er en ny algebraisk semantik, der er defineret ved generalisering af den ekstensionelle semantik af $\mathcal{ALC}$. Det vises, at denne fremgangsmåde kan generaliseres til andre logikker.

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Contents

1 Introduction **11**
1.1 Overview of the Thesis 16

2 Contributions to Intensionality **19**
2.1 Frege on Sense and Denotation 19
2.2 Carnap on Extension and Intension 26
2.3 Church's Logic of Sense and Denotation 30
2.4 Other Contributions 33

3 Defining Intensionality **35**
3.1 The Intuitive Notion of Intensionality 35
3.2 The Definition of Intensionality 38
3.3 Alternative Definitions 39
3.4 Intensionality Results 42
3.5 Intensional Properties of Logical Theories 50
3.6 Reduction to Extensionality 50

4 Current Concept Logic **55**
4.1 Background 55
4.2 Description Logics 56
4.3 Extensionality Results 61
4.4 Concept Theories 63

5 Intensional Concept Logic **67**
5.1 Motivation 67

5.2 Defining the Intensional Concept Logic 69
5.2.1 An Appetizer 69
5.2.2 Syntax 71
5.2.3 Extensional Semantics 72
5.2.4 Intensional Semantics 73
5.2.5 Verification of Intensionality 77
5.2.6 Relations between the Extensional and Intensional Semantics 79
5.2.7 Proof Theory 82
5.2.8 Towards Completeness 85
5.2.9 Completeness 86
5.2.10 Existence of Mediations between the Extensional and Intensional Models 90
5.3 Capturing Other Conceptions of Intensionality 91
5.3.1 Intensional Boolean Concept Logic 92
5.3.2 Other Kinds of Intensional Algebras 93
5.4 Generalizing the Logic 94
5.5 Remarks about the Intensional Concept Logics 97
5.6 Applications 98
5.6.1 Multi-Knowledge-Based Systems 100
5.6.2 Intensional Subsumption 104
5.6.3 Intensional ABoxes and Prototype Theory 105
5.6.4 Content-Based Information Retrieval 107

6 Conclusion **109**

A A Logic of Sense and Denotation **111**
A.1 Relation to Modal Logic **T** 113

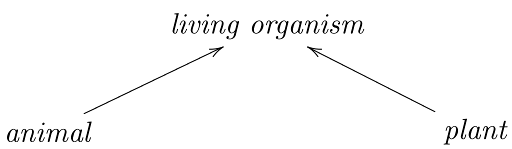
Chapter 1

Introduction

We can systematically organize the entities of a given domain into so-called *categories* or *classes*. When examining living organisms, for example, we commonly categorize those with a capacity for moving around as being *animal* and those without as being *plant*. Living organisms may therefore be divided into two categories, moreover, the categories *animal* and *plant* can be considered subcategories of *living organism*.

At least since the time of Aristotle, it has been acknowledged that categories play a fundamental role in the organization and formulation of knowledge. Categorization in general is a broad area indeed. We are working in the field of knowledge representation and logic-based artificial intelligence. Categories and classes will accordingly be referred to as *concepts*. The subject of this thesis is concept representation and reasoning, or more precisely, logics suitable for representing and reasoning with concepts.

Today, research on this subject enjoys renewed popularity. In particular, specifications of concepts—the so-called *ontologies*—are studied intensively. As an example of a graphical ontology in which *animal* and *plant* are subconcepts of (subsumed by) *living organism*, we have



Ontologies are often much more complex but an ordering of concepts, like the one above, is common to all of them. Such an ordering is called a *taxonomy*.

The reason for the interest in concept representation is, among other things, caused by the need for organizing the vast amount of information on the Internet. However, due to the fundamental role concepts play with respect to organization of knowledge, it should be clear that the range of applications of theories for concept representation and reasoning is wider.

Compared to other contributions in knowledge representation, this thesis is distinguished by *intensional formalization* of concepts. As it is not yet established what it precisely means for a formalization to be intensional, the first major aim of the thesis is to present a formal definition of when a logic is intensional. In order to accomplish this, we will go back to the origin and study contributions from philosophy of language, notably the works of Gottlob Frege and Rudolf Carnap.

Now we present an introduction to the subject of intensionality. Intensionality is basically about understanding languages, that is, about establishing linguistic meaning. And one of the first things we observe is that in order to understand what a sentence like

Don Quixote is mad

means, one must know what its expressions denote.[1] *Don Quixote* denotes the main character of the novel of the same name authored by Miguel de Cervantes Saavedra. Thus the sentence bluntly asserts that the main character of the novel Don Quixote suffers from a disordered mind.

Knowledge about denotation is *not* sufficient for understanding a language in general. To see this, let us assume the converse, i.e. that it suffices to know the denotations of the expressions of a sentence in order to determine its meaning, and let us compare the two names *Knight of the Rueful Countenance* and *Knight of the Lions*. Since both are names of Don Quixote, they both denote the character Don Quixote. But then the two sentences

(*i*) *after having confronted a pair of lions, Don Quixote calls himself the Knight of the Lions*,

(*ii*) *after having confronted a pair of lions, Don Quixote calls himself the Knight of the Rueful Countenance*

must have the same meaning, since the subexpressions of the two sentences are pairwise co-denotational. But this cannot be the case, because according to

[1] Instead of using 'denote' we could also use 'refer to', however, we adopt the terminology used by many authors in philosophy of language, like Bertrand Russell and Alonzo Church.

the novel the former is true and the latter is false, and no two sentences which mean the same can have different truth-values. Hence (*i*) and (*ii*) cannot have the same meaning. In other words, although we have not established how to determine linguistic meaning (which is difficult indeed), we have argued that linguistic meaning in general cannot be reduced to knowledge about denotations.

According to Gottlob Frege it is also necessary to know the so-called *sense* of an expression in order to determine linguistic meaning. This will be explained in details in the following chapter, but for now we say that the sense of an expression is *the way in which the expression denotes*, or simply its *mode of presentation*. Consider once again *Knight of the Rueful Countenance* and *Knight of the Lions*. They denote (i.e. refer to) Don Quixote in different ways. The first refers to his appearance, and the latter to his unrivalled courage, hence the names have different senses.

This should illustrate why intensionality is important for semantics, but it does not explain why we as computer scientists working with conceptual knowledge representation are interested in intensionality.

We have basically argued that the meaning of an expression is more than its denotation. A similar argument can now be presented for concepts, but first it should be noted that a different terminology is used for talking about what concepts mean. Instead of talking about the *denotation* of a concept, we talk about the *extension* of a concept and instead of the *sense* of a concept, we talk about the *intension* of a concept.[2] The extension of a concept is accordingly the set of individuals falling under the concept, that is, the members of the concept. The intension of a concept is closely related to the sense of a name, which suggests that the intension is something like the *way in which the concept refers to its members* or simply its *mode of presentation*. More precisely, we will later say that a concept is defined, not by its extension, but by its intension, and this merely means that we use a more restrictive condition for identifying concepts than simply assuming that concepts with the same extension are identical.

Just as there exist co-denotational names with different senses, there exist co-extensional concepts with different intensions. As a classic example we have that *creature with a heart* and *creature with a kidney* (which obviously have different intensions) have the same extension since every living creature

[2]The difference between the two ways of talking follows from two closely related traditions. The first is based on Frege's work and the latter on Rudolf Carnap's work, as we shall see in the following chapter.

with a heart has a kidney and every living creature with a kidney has a heart.[3]

In an *extensional formalization*, concepts are identified with their extension such that whenever two concepts have the same extension then they are identified (and hence substitutable). In an *intensional formalization*, concepts are identified with their intension such that co-intensional concepts are identified (substitutable). Because of the existence of different co-extensional concepts, intensional formalizations of concepts provide a more adequate representation of concepts than extensional formalizations.

This motivates why intensionality is important for conceptual knowledge representation. It is important to mention that researchers in artificial intelligence already in the 70s considered intensional formalizations of concepts to be important [McCarthy, 1977; 1979; Woods, 1975; 1991; Brachman, 1979].

In this thesis, we will, after having defined intensionality formally, investigate the current formalizations of concepts. We will show that these are extensional. There is, in other words, a need for intensional formalizations of concepts and accordingly a need for intensional logics for representing and reasoning with concepts.

The second major aim of the present work is therefore to present an intensional concept logic. The aim will be fulfilled by the introduction of an intensional concept logic in Chapter 5. The logic will be based on the assumption that conceptual knowledge can be divided in two parts, an extensional part and an intensional part, such that we have a part expressing relations between extensions and a part expressing relations between intensions. For the extensional part we simply adopt the description logic $\mathcal{ALC}$ [Baader and Nutt, 2003; Schmidt-Schauß and Smolka, 1991]. $\mathcal{ALC}$ stands out as the prominent example of a logic for formalization of concepts. The syntax of the intensional part is similar to the extensional part except that an intensional equivalence relation $=$ is used instead of the extensional equivalence relation $\equiv$ of $\mathcal{ALC}$. Thus the logic comprises two kinds of identities, meaning we can distinguish the role (modality) of a concept definition like

$$\textit{bachelor} = \textit{unmarried man}, \tag{1.1}$$

[3]It has been debated whether one should consider the actual extension or all possible individuals. If one does the latter, *creature with a heart* and *creature with a kidney* differ, since there could be a creature with a heart and no kidney. But although this is acceptable from a philosophical point of view (actually, many feel that it is not), it does not mean that it is always useful in knowledge representation, for in knowledge representation one often considers restricted domains of entities where it is useful (and maybe even imperative) to express that concepts are co-extensional, although the possibility that they are not co-extensional exists.

which defines bachelor to be unmarried man, from contingent statements like

$$\textit{bachelor} \equiv \textit{lonely hearted}, \tag{1.2}$$

which asserts that bachelors are the lonely hearted, and

$$\textit{creature with a heart} \equiv \textit{creature with a kidney},$$

which asserts that every creature with a heart also has a kidney and conversely. Notice how (1.1) uses the intensional relation $=$, whereas the others use the extensional.[4]

After defining the logic, we verify that it really is intensional. Moreover, we will show some applications and argue that the advantages of using an intensional concept logic are greater than one may expect at first. To show the versatility of the intensional logic, we will now briefly describe an application. Imagine a computer based system in which users are allowed to input facts and rules to an already existing and acknowledged knowledge base and database. One would then like to distinguish the user input like (1.2) from the acknowledged data and knowledge like (1.1), while at the same time having all the facilities for representing and reasoning with concepts as one usually has. Since we are able to ascribe different roles to the two kinds of identities ($\equiv$ and $=$), we can accomplish this, but we can actually accomplish much more.

Based on the principle from philosophy of language that intensional knowledge implies extensional, the acknowledged data and knowledge will "propagate" to the users (so a user will be able to learn (1.1)), but not the other way around.[5] This means that the original data and knowledge base is protected from inconsistencies and malicious users, although one is able to make fully use of the user contributions. Suppose a user adds

$$\textit{bachelor} \equiv \textit{not lonely hearted},$$

asserting that bachelors are *not* the lonely hearted. Then, this (together with (1.2)) causes the extensional part (the user defined part) to be inconsistent, since statements cannot both be true and false at the same time. But this will not affect the acknowledged data and knowledge. In other words, the

[4] To keep things simple, we have not used the correct syntax of the intensional concept logic.

[5] The relation between the extensional and intensional identities means that the system works as a unified knowledge base and not merely as two separate knowledge bases.

intensional concept logic can be used for uniting different types of knowledge, and yet keep them discernible.[6]

The intensional semantics of the logic will be algebraically defined; it is inspired by the property theory of George Bealer [Bealer, 1982; Bealer and Mönnich, 1989], however, despite the technical details of the semantics, our intensional concept logic should be (almost) as easy to use as $\mathcal{ALC}$, in that it in some sense basically is an intensional description logic.

1.1 Overview of the Thesis

Our work is a cross disciplinary exercise in knowledge representation (the main field of the thesis) and philosophy of language. However, since we put an effort in keeping the disciplines apart, the thesis consists of two parts. The first part, which is constituted by the chapters 2 and 3, investigates the issue of intensionality. Addressed to everyone interested in logic in general and the more technical aspects of philosophy of language in particular, this part may be read independently of the rest. The second part, which is constituted by the chapters 4 and 5, presents the intensional concept logic. A more detailed overview:

Chapter 2 describes and comments on contributions to the understanding of intensionality. The works of Gottlob Frege, Rudolf Carnap, and Alonzo Church are presented and discussed. Although these authors are famous, we put forward important aspects which we believe have not received the recognition they deserve.

Chapter 3 investigates more precisely the issue of intensionality. We present a formal condition for determining whether a logic is intensional or extensional, and apply this to well-known logics and a well-known formalization of concepts.

Chapter 4 concerns current formalizations of concepts. We describe the description logic $\mathcal{ALC}$, argue that the prevalent formalizations of concepts are not intensional and thereby reveal the need for intensional concept logic.

Chapter 5 defines an intensional logic for formalization of conceptual knowledge. It starts by clarifying our notion of a concept. It is stressed that concepts are intensional. The logic is based on the description logic

[6] The example is more detailed described in Chapter 5.

$\mathcal{ALC}$, however, it consists of an intensional part which enables intensional formalization of conceptual knowledge. The intensional semantics will be algebraically defined. We show different versions of the logic, and at the end, examples of applications are shown.

Appendix A The underlying approach of the intensional semantics may be generalized to other kinds of logics. The appendix shows a propositional logic based on the intensional semantics.

With the exception of Section 2.3 about Church's contribution (where some acquaintance with the λ-calculus is needed), the prerequisites for Chapter 2 and Chapter 3 should be covered by acquaintance with classical and modal logic. In Chapter 5 some acquaintance with universal algebra is needed.

Chapter 2

Contributions to Intensionality

This chapter describes and comments on important contributions to the subject of intensionality. The works of Gottlob Frege, Rudolf Carnap, and Alonzo Church will be presented in the following sections. We put forward important aspects of these contributions which we believe have not received the recognition they deserve.

2.1 Frege on Sense and Denotation

The subject of this section is Frege's 1892 article "Über Sinn und Bedeutung" [1984]. The article, which we consider to be the most important contribution to intensionality, establishes the foundation for research in intensionality. The article shows that in order to understand a language it is not enough to know the denotations of its expressions—one must also know the so-called senses.[1]

Frege starts by investigating equalities, that is, expressions of the form $a = b$, where a and b are names.[2] After contemplating, he rightfully recognizes

[1]Frege, who wrote in German, used 'Sinn' for 'sense' and 'Bedeutung' for 'denotation'. There are different translations of this paper, some use 'reference' or 'meaning' instead of 'denotation', however, we choose to follow the terminology used by Russell and Church.

[2]Note, equalities do not only occur in logical languages, in natural language, in which the below arguments fit more naturally, an equality could be formulated as: *a is identical to b*, or simply: *a is b*. Note also that 'name' is used generally as an expression which denotes some object.

that an equality like $a = b$ is true if a and b denote (refer to) the same. Therefore,

the morning star is the evening star

is true because both *morning star* and *evening star* denote the planet Venus.

However, this is not all what Frege has to say about equalities. Consider the two equalities $a = a$ and $a = b$ and assume both are true, i.e. that a and b are names of the same. Frege recognized that the sentences differ, for the former is trivially (analytically) true, whereas the latter is not. Frege says that they differ with respect to their cognitive value. In a criminal investigation, for example, a discovery like *the suspect is the burglar* is important, whereas *the suspect is the suspect* simply is useless as it contains no cognitive value (information).

As another example, consider the true sentence

the ancients believed that the morning star is the morning star, (2.1)

and the false sentence

the ancients believed that the morning star is the evening star. (2.2)

Although all subexpressions of both sentences have the same denotations, the sentences are obviously different since one is true and the other is false.

Now, we may ask, why do the sentences differ? Frege answers: "A difference can arise only if the difference between the signs [expressions] corresponds to a difference in the mode of presentation of the things designated [denoted]." In other words, *morning star* and *evening star* differ because the ways in which they present their denotation differ. The way *morning star* denotes Venus may be formulated as: *the brightest star or planet in the morning sky.* Similarly, the way *evening star* denotes Venus may be formulated as: *the brightest star or planet in the evening sky.* Hence, *morning star* and *evening star* have different modes of presentation, and this explains the difference between the utterances above.

Frege then concludes p. 26–27 (we use the page numbering of the original 1892 paper):

> It is natural, now, to think of there being connected with a sign (name, combination of words, written mark), besides that which the sign designates [denotes], which may be called the meaning [denotation] of the sign, also what I should like to call the *sense* of the sign, wherein the mode of presentation is contained.

In other words, the *sense* of an expression contains its mode of presentation. In terms of the above examples, it should be clear that *morning star* and *evening star* have different senses, and similarly for *suspect* and *burglar*. Moreover, it is the difference between the senses which is the cause of the difference between the cognitive values (information contents).

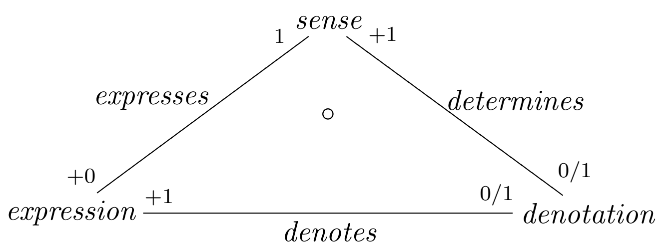
As another example, assume we have three lines a, b, c intersecting each other in the point p. Then the phrases *the intersection of a and b* and *the intersection of a and c* have the same denotation (namely p), but clearly different senses.

Frege does not describe more precisely how senses are defined, but he states some additional facts about senses. Senses are grasped "by everybody who is sufficiently familiar with the language or totality of designations to which it belongs". Moreover, on p. 27 he says:

> The regular connection between a sign, its sense, and what it means [denotes] is of such a kind that to the sign there corresponds a definite sense and to that in turn a definite thing meant [denoted], while to a given thing meant [denoted] (an object) there does not belong only a single sign. The same sense has different expressions in different languages or even in the same language.

There may be expressions which have no denotation. Frege mentions the examples *the celestial body most distant from the earth* and *the least rapidly convergent series*. Moreover, we say that an expression *expresses* its sense and *denotes* its denotation (the translation [Frege, 1984] uses 'designate' instead of 'denote'). Note that Frege accepts that exceptions to the regularities occur, especially in connection with natural language.

In order to make these characterizations more clear, we (not Frege) have made the following figure which illustrates the relations between an expression, its sense and what it denotes:



The lines indicate the relations between the expression, its sense and what it denotes. The numbers indicate cardinality constraints. Starting with the *denotes* relation, the leftmost number +1 states that for each denotation (denoted object) there are one or more expressions which are names for it. The 0/1 states that each expression either denotes a single object or does not denote. Notice that these constraints imply that there may be different names for a given object, and that every expression is a name of at most one object. The 1 at the *expresses* relation indicates that each expression has one (its definite) sense. The +0 states that for every sense there exist zero or more expressions that express it. We are not sure whether this constraint should be +1 instead.

There is an additional constraint which says that the figure commutes. This is indicated by the $\circ$. Commutativity says that the object denoted by *expression* is the same as that which its *sense* determines. Frege does not say this explicitly, however, it *has to be* an implicit assumption. Otherwise, the sense would not be the mode of presentation, since what the sense determines is then not the same as what the expression denotes, nor can there be a regular connection between an expression, its sense, and what it denotes. Commutativity will later play a significant role in our formalization of intensionality. Note, Frege did not introduce a name for the *determines* relation. Moreover, note that the constraints imply that sense uniquely determines denotation, but not conversely (for a denotation may have several senses that determine it).

It is important to note that Frege distinguishes senses from ideas (subjective thoughts). Ideas are more finely individuated than senses which are more finely individuated than denotations, such that one may have different ideas of the same sense. Senses lie therefore in between ideas and objects (denotations). (Note that Frege wasn't really interested in the role of the subjective, instead he focused on senses which are objective.)

Frege made other important contributions in the 1892 paper, like putting forward that the denotation of a sentence is its truth-value (this was later revised by Richard Montague) and the sense is the thought it expresses. However, it falls out of scope to go into further details with these issues.

In addition to proposing senses, Frege's paper is seminal because it reveals a more technical aspect of senses which, together with Carnap's contributions, leads us to propose a formal definition of intensionality in Chapter 3. Based on Leibniz's famous "identity of indiscernibles" principle, in terms of which that which may be substituted (replaced with each other) under preservation of truth are identified, Frege investigates when expressions may be substituted for each other. He assumes that the meaning of a sentence remains unchanged when a part of the sentence is replaced by an expression with the same meaning (this is his famous compositionality principle). But he discovers that the denotation of a sentence need *not* remain unchanged when subexpressions with the same denotation are substituted for each other. (2.1) and (2.2) show an example: declarative sentences with different truth-values cannot have the same denotation, and since the truth-value of (2.1) changes under the substitution of *morning star* with the co-denotational *evening star*, the denotations of (2.1) and (2.2) cannot be the same.

Now we turn to an important aspect of the issue of sense and denotation. Frege says (p. 28): "If words are used in the ordinary way, what one intends to speak of is what they mean [denote]. It can also happen, however, that one wishes to talk about the words themselves or their sense." This can be explained by an example. In the sentence

the morning star is Venus (2.3)

we speak of the denotation of *morning star*, but in (2.2) we do not speak of the denotation of *morning star*, because otherwise (2.2) would be no different from (2.1). Instead we speak about the sense of *morning star*, such that (2.2) asserts that the ancients believed that the brightest star or planet they could see in the morning is the same as the brightest star or planet they could see in the evening.

Frege then distinguishes the *customary denotation* of an expression, which is its denotation, from the *indirect denotation* which is its sense. A context, like (2.2) in contrast to (2.3), in which a name does not have its customary denotation is called an *indirect context* or an *oblique context*. Quotations and *propositional attitudes* (which involve assertions about beliefs, desires, intentions, etc.) create oblique contexts. Oblique contexts have been studied intensively, and Willard V. Quine has put forward a related contribution [1943], which appears to be independent of Frege's work as noted by Church in a review of this paper.

Now, an interesting question arises, if the denotation of an expression in an indirect context is its ordinary sense, then what is its sense in an indirect context? In other words, what is its *indirect sense*? Does it have an *indirect sense* at all? About this Frege says p. 37:

> The case of an abstract noun clause, introduced by 'that,' includes the case of indirect quotation, in which we have seen the words to have their indirect meaning [denotation], coincident with what is customarily their sense. So here, the subordinate clause has for its meaning [denotation] a thought, not a truth-value and for its

> sense not a thought, but the sense of the words 'the thought that (etc.)', which is only a part of the thought in the entire complex sentence.

What Frege actually means has been debated, cf. [Carnap, 1956; Lewy, 1949]. The translation of Max Black [Frege, 1984] has a comment after 'abstract': "Frege probably means clauses grammatically replaceable by an abstract noun-phrase: e.g. 'Smith denies *that dragons exist*'='Smith denies *the existence of dragons*'." We think, similar to [Lewy, 1949], that Frege means that the sense of an expression e in oblique contexts (which includes 'that' clauses) simply is the sense of 'the sense of 'e''. *The sense of 'e'* denotes that which is the sense of e, let us call it s, and therefore an expression *the sense of 'the sense of 'e''* denotes the sense of s, that is, the sense of the sense of e. In other words, we conclude that the oblique sense of an expression e is the sense of the sense of e.

We will make this more clear by the following example. Consider two non-identical expressions a and b with the same sense.[3] Once we admit existence of senses, we can construct the following true equalities

$$\textit{the sense of 'a'} = \textit{the sense of 'a'}$$

and

$$\textit{the sense of 'a'} = \textit{the sense of 'b'}.$$

Similar to the difference between $a = a$ and $a = b$, which motivated the introduction of senses, there is a difference between the two equalities in that the former is vacuously true, whereas the latter, which may contain useful information, is not.

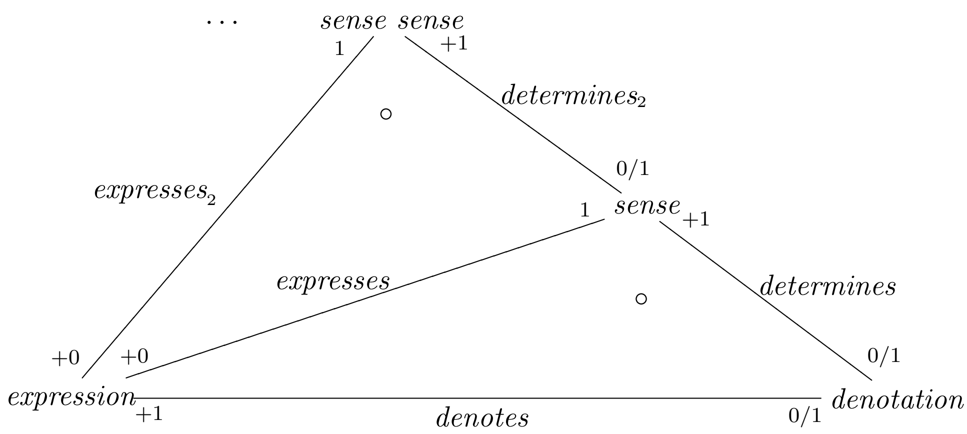
Now, the question arises, how can they be different? The difference cannot be due to senses, because the senses of a and b by assumption are equal. And we have already explained that whenever a and b have the same sense, then they also have the same denotation. Hence the difference cannot be due to a difference between the denotations of a and b. The only alternative left is simply to repeat Frege's argument. The difference must be due to a difference between the mode of presentation (i.e. the sense) of *the sense of 'a'* and *the sense of 'b'*. And since *the sense of 'a'* denotes the sense of a, the difference

[3]Frege does not give any examples of names with the same sense, nor does he, as Carnap and Church acknowledge, presents a more precise identity condition for senses. It appears as if conditions of different strengths can be formulated, and accordingly we will later talk about different *conceptions of intensionality*. As an example of two expressions which we can say have the same sense, consider *P or Q* and *Q or P*.

must be due to a difference between the sense of the sense of a and the sense of the sense of b. Instead of saying "sense of the sense", we will simply say "sense sense".

In other words, once we accept existence of senses we must also accept existence of sense senses. And once we acknowledge sense senses, it should be clear that we can repeat the construction above, and thereby show that we must also acknowledge senses of sense senses, and so on. There is in other words an infinite hierarchy of senses. It should also be clear that each level is regularly connected to its lower level, just as senses are regularly connected denotations.

Hence the relation between an expression, its denotation, its sense and sense sense, etc. can be illustrated as follows



where we have subscripted the names of relations to and from *sense sense*; the . . . indicate that the figure continues infinitely.

The above presentation of sense senses is not Frege's (it is our own). It is not clear to what extend Frege realized this (and although it appears as if he realized it, he did not state it). As far as we have been able to establish, Alonzo Church was the first to clearly explain the inevitability of a hierarchy of senses, see [1951] footnote 13. In the following when we refer to this notion of senses (i.e. that we have an infinite hierarchy of senses), we will accordingly call it the *Frege-Church conception*. Among the many authors who have written about senses, few mention a hierarchy of senses. Nonetheless, it should be clear that a formalization of senses is not fully adequate unless there is an infinite hierarchy of senses.

2.2 Carnap on Extension and Intension

This section presents and discusses Rudolf Carnap's Meaning and Necessity [1956]. We will not address the issues about modal logic for which this book is widely known for and which have been discussed elsewhere, instead we focus on the issue of intensionality.

In most of his definitions Carnap uses a (non-modal) first-order logical language S_1 similar to first-order predicate logic (S_1 is, amongst other things, distinguished from first-order predicate logic by having only a finite number of predicate letters). Carnap does not restrict his investigations to S_1, however, for language systems in general he does not state explicit definitions but rather informal conventions.

The definition of a true S_1 sentence follows the traditional model-theoretic definition with the important exception that the truth-value of an atomic sentence is based on informal 'rules of designation' (1-3, 1-4, 1-5, 1-6).[4] The *rules of designation* formulate the truth-value by means of descriptions in the metalanguage such that a predicate letter followed by an individual constant is true if the individual to which the constant actually refers possesses the property to which the predicate actually refers. This generalizes to n-ary relations. This is probably more easy to understand by means of some examples. The following are of rules of designations (1-2):

'$H(x)$' is a symbolic translation of 'x is human (a human being)',
'$RA(x)$' is a symbolic translation of 'x is a rational animal',
'$F(x)$' is a symbolic translation of 'x is (naturally) featherless',
'$B(x)$' is a symbolic translation of 'x is a biped'.

Two sentences ϕ and ψ are *equivalent* if $\phi \leftrightarrow \psi$ is true (definition 1-8). As an example, Carnap says that $(\forall x)H(x) \leftrightarrow F(x) \wedge B(x)$ is true, because we can empirically verify that every human being is a (naturally) featherless biped and conversely.

A class of sentences in S_1 which contains for every atomic sentence either this sentence or its negation and no other sentences is called a *state-description* (p. 9). Carnap then introduces the notion of L-truth. A sentence is *L-true* if it is true in every state description (definition 2-2).[5] Moreover, two sentences

[4]The numbers represent the numbering used in Meaning and Necessity.

[5]L-truth is related to validity. Let ϕ be a sentence of first-order predicate logic. If ϕ is logically valid (by the traditional model-theoretic definition) then ϕ is L-true. We do not have the converse, as we shall see shortly.

ϕ, ψ are *L-equivalent* if $\phi \leftrightarrow \psi$ is L-true (definition 3-5b). Since "The English words [of the rules of designation] here used are supposed to be understood in such a way that 'human being' and 'rational animal' mean the same" (p. 4), we get that $(\forall x)H(x) \leftrightarrow RA(x)$ is L-true. In a general semantical system a sentence is *L-true* if its truth can be established on the basis of the semantical rules alone (convention 2-1).

Designators are "those expressions to which a semantical analysis of meaning is applied" (p. 6); for S_1 these are thus sentences, predicate letters, function letters, and individual expressions. Two designators *have the same extension* if they are equivalent (definition 2-1). Note, Carnap generalizes equivalence to all designators, for example, two constants a and b are equivalent if $a = b$ is true. Two designators *have the same intension* if they are L-equivalent (definition 5-2). So all L-true sentences have the same intension. For example, H and $F \wedge B$ have the same extension, while H and RA have the same intension. These definitions are generalized to any language by the conventions 4-12 and 4-13. As L-truth obviously implies truth we can say that intension determines extension.

Carnap says that the extension of a predicate letter is the corresponding class (4-14), and the intension of a predicate letter is the corresponding property (4-15). So the intension of H is by means of the rules of designation seen to be the property of being human. It is important to note that he presupposes that a property is distinct from its corresponding class, and moreover, that 'property' is to be understood in a very wide sense, including whatever can be said meaningfully about any individual. *Relation* is used in a similar way to *property*, except that relations are n-ary. *Concept* is used as a common term for properties and relations, and includes *individual concepts* (concepts with only a single member). Later Carnap shows that identity of properties may be formalized as necessary equivalence (something he has often been cited for). It is argued that the extension of a sentence is its truth-value (6-1) and its intension is the proposition expressed by it (6-2).

Now we turn to Carnap's important definitions of extensionality and intensionality. Since these, for some reason, are unnecessarily complicated, we will present them more clearly. Let χ be a sentence, and let ϕ and ψ be designators, moreover, let $\chi[\psi/\phi]$ be the result of replacing an occurrence of ϕ with ψ in χ (if ϕ does not occur in χ, it simply denotes χ). Then ϕ is *interchangeable* with ψ if for every χ, χ is equivalent to $\chi[\psi/\phi]$. And ϕ is *L-interchangeable* with ψ if for every χ, χ is L-equivalent to $\chi[\psi/\phi]$ (11-1). Now, a semantical system is *extensional* if for every ϕ, ϕ is interchangeable with any expression equivalent to ϕ (11-2). And a system is *intensional* if it is not extensional and for every ϕ, ϕ is L-interchangeable with any expression L-equivalent to ϕ (11-3). Interchangeablity has also been called *substitutability* or *substitutivity*.

Carnap also introduces the notion of intensional isomorphism which comprises "ultra-intensional" entities (using Quine's terminology). Two atomic sentences are *intensional isomorphic* if they are L-equivalent, and two non-atomic sentences are *intensional isomorphic* if their syntax trees have the same structure (are isomorphic) and all their leaves are pair-wise intensional isomorphic.[6] So $H \wedge \neg H$ and $H \wedge \neg RA$ are intensional isomorphic, whereas H and $H \vee H$ are not. Carnap suggests that intensional isomorphisms are used for analysis of belief sentences. These have proven to be difficult to analyse because one may have different beliefs about expressions that are co-intensional. For example, *N believes that P* may be true while *N believes that Q* is false although P and Q are co-intensional. Carnap then believes this can be explained by means of intensional isomorphisms, such that the above situation occurs only when P and Q are not intensionally isomorphic.[7]

Now we will comment upon Meaning and Necessity. We will discuss the relation between Frege's and Carnap's contributions. First of all, semantics has undergone significant changes since Meaning and Necessity was written. For instance, empirical investigations or extra-linguistic knowledge are not part of formal semantics today.[8] In the light of this, some parts of Meaning and Necessity are more of historical interest. Nevertheless, Meaning and Necessity makes important contributions to (amongst other things) Frege's notion of sense by making it more precise.

Carnap's semantical analysis of language systems is called the *method of extension and intension*. The method is based on the distinction between understanding the meaning of an expression (this is explicated by means of intension) and investigating whether it holds in some context (this is explicated by means of extension). The method is intended to be a "suitable method for the semantical analysis of meaning" (p. 2).

The method of extension and intension is closely related to Frege's notions of denotation and sense, however, Carnap's method is distinguished by the

[6] As before, we have presented a more simple definition instead of Carnap's more comprehensive definition. Note, this definition is not fully precise for we have not defined when variables are intensional isomorphic (because Carnap does not define this either).

[7] We do not find this fully satisfying, for among other things intensional isomorphisms do not allow us to discern between co-intensional atomic formulas.

[8] Today's prominent formal semantical theories are *truth-conditional*, meaning that the semantics of an utterance is the conditions under which it is true. This means amongst other things that the dubious rules of designations can be dispensed.

fact that extension and intension remain the same in all contexts, in contrast to Frege's suggestion where they change in oblique contexts. This is actually clearly stated in [Carnap, 1956], for example on page 125: "For any expression, its ordinary nominatum [denotation] (in Frege's method) is the same as its extension (in our method)." Moreover, on page 126: "For any expression, its ordinary sense (in Frege's method) is the same as its intension (in our method)." In other words, extension is the same as denotation in ordinary contexts and intension is the same as sense in ordinary contexts. Moreover, sense sense (oblique sense) and sense sense sense (oblique sense sense) etc. seem to be the same as intension, because the oblique intension simply is the intension.

It is interesting to elaborate on this. First of all, Carnap is often said to adopt a possible-world semantics of concepts (such that the intension of a concept is formalized as a mapping from possible worlds to extensions). However, the close similarity between intensions and Frege's more general senses, suggests that Carnap's notion of a concept is more general than that of the possible-world semantics.

Second, and more importantly, we have argued that Frege's notion of sense leads to an infinite hierarchy of senses. Now, Carnap claims that his two notions of extension and intension are suitable, that is, that the infinity can be avoided. How can that be?

First of all, we can repeat the arguments which show the need for an infinite hierarchy of senses in Carnap's method. Consider two expressions A and B which are assumed to be co-intensional, and the sentences:

$$A \textit{ has same intension as } A,$$

and

$$A \textit{ has same intension as } B.$$

Since the former, in contrast to the latter, is vacuously true, the two statements differ. But why do they differ? It cannot be due to difference of intensions because A and B are assumed to have the same intension, nor can it be due to a difference of extensions—sameness of intension implies sameness of extension. Since we accept the need for introducing intensions, we must therefore also accept the need for introducing intensions of intensions.

One may argue that Carnap is saved by the fact that his notions of intension and extension belong to the metalanguage, and only expressions of the object-language are subjects to the semantical analysis. However, assuming the method of extension and intension is generally applicable, this argument does not hold—the method ought to work even in cases where the metalanguage is studied, and Carnap actually *does* consider the metalanguage by means of a meta-metalanguage. Moreover, we could construct the example in the object language simply by comparing the sentences $A\ is\ A$ with $A\ is\ B$.

As we see the issue, it appears that Carnap fails to understand the full consequences of Frege's contribution. When discussing why Frege distinguishes the oblique sense (sense sense) from the sense, Carnap says on p. 129: "It is not easy to say what his [Frege's] reasons were for regarding them as different [...] It does not appear, at least not to me, that it would be unnatural or implausible to ascribe its ordinary sense to a name in an oblique context." It should additionally be noted that a reviewer of Meaning and Necessity, C. Lewy [1949], says that Carnap has failed to understand Frege's sense senses.

Basically, this shows that the method of extension and intension fails as a suitable method for semantical analysis of meaning in general. Carnap may to some extend actually have agreed about this, because—as already noted—he introduces the notion of intensional isomorphism in addition to extension and intension for analyzing the intricate belief sentences. Unfortunately Carnap does not compare intensional isomorphisms with higher level senses, but there could be a rather close relationship, which suggests that the differences between Frege's work and Carnap's work may be small indeed.

This does nevertheless *not* mean that the method of extension and intension is useless, it merely shows that the method has some shortcomings. Later we will adopt the method, and it will prove to be useful for concept logics, since it is customary to divide the semantical analysis of concepts in two parts.

2.3 Church's Logic of Sense and Denotation

Alonzo Church made diverse contributions to the issue of intensionality. As noted by Carnap [1956], he was responsible for the renewed interest in Frege's work about sense and denotation in the symbolic logic community. Moreover, he clearly stated (as mentioned earlier) the inevitability of an infinite hierarchy of senses. Additionally, he proposed his own formalization of Frege's notion of sense. In this section we present only an overview of this contribution, which Church called *A Formulation of the Logic of Sense and Denotation*.

It is not trivial to make an unified presentation of this contribution, for Church made five papers [Church, 1946; 1951; 1973; 1974; 1993] over a period of almost 50 years, and in the process he presented different alternatives, and made major revisions as previous formulations were unsound and faulty. We will simply refer to the unified presentation as *the Formulation* below.[9] It should be noted that, as far as we are aware, these particular contributions of Church have not received much attention.

The Formulation is based on the typed λ-calculus, or more precisely on Church's own *A Formulation of the Simple Theory of Types* [1940], which we assume the reader is familiar with. Besides a few exceptions, which should be obvious, we follow Church's notation.

The most important notion in the Formulation is that of being 'a concept of'. It is introduced as follows (p. 11 [Church, 1951]):

> In order to describe what the members of each type are to be, it will be convenient to introduce the term *concept* in a sense which is entirely different from that of Frege's *Begriff*, but which corresponds approximately to the use of the word by Russell and others in the phrase "class concept" and rather closely to the recent use of the word by Carnap, in *Meaning and Necessity.* Namely anything which is capable of being the sense of a name of x is called a *concept of* x.

In terms of Frege's work, we can say that if there exists a name which has the sense y and denotes x, then Church says that y is a concept of x. However, this does not mean that if y is a concept of x then there necessarily exists a name which has y as sense and denotes x, because there may be more concepts (namely uncountably many) than names, Church says. Note that the Formulation is not aimed at presenting a Fregean semantics which describes how to determine the meaning of sentences by means of Frege's notions of sense and denotation. The Formulation is a logic (or a foundation, we can say) for intensional entities.

Church presented three alternatives for identifying senses, called *Alternative* (0), *Alternative* (1), and *Alternative* (2). Alternative (0) corresponds to Carnap's notion of intensional structure, such that senses roughly speaking are identified if they are intensionally isomorphic. He explains the other alternative as (the 1993 paper p. 141): "Under Alternative (1) we identify *propositions* with Frege's *Gedanken*, i.e., concepts of truth-values, and the proposal is that propositions in this sense shall be taken as objects of assertion of belief." Under Alternative (2) the sense of the names A and B are identified if and only if the equation $A = B$ is logically valid. The last alternative turns out to be very similar to the Montague-Gallin logic (see [Gallin, 1975]) as noted by C. A. Anderson [1984].

[9] Note, when Church refers to the Formulation he speaks of his 1951 paper.

As noted in [Anderson, 1984] Alternative (0) is suitable for constructing a general intensional logic. In the following we therefore concentrate on this alternative only. Our aim is not to present the entire Alternative (0), we merely want to show the underlying ideas behind it as well as its relation to the rest of this thesis.

We have the following simple types: $o_0, o_1, o_2, \ldots$ and $\iota_0, \iota_1, \iota_2, \ldots$ (o_0 and ι_0 are written as o and ι). The type o is to consist of truth-values (true and false), and ι is to consist of individuals.[10] Greek letters α, β, γ are used as variables whose values are type symbols. The type α_{n+1} is to consist of concepts of the members of type α_n, thus o_1 consists of concepts of truth-values.

Among the primitive constants, $\Delta^m_{o_n \alpha_{n+1} \alpha_n}$ plays an important role in the Formulation.[11] Below we will not use subscripts which can be derived from the fact that the formulas are well-formed.

Now, $\Delta^0_{o\alpha_1\alpha}$ denotes a binary function whose value (for a pair of arguments) is truth in case the second argument is a concept of the first, and otherwise false. The essential axioms of the Formulation allow proofs of the form $\Delta M_\alpha M_{\alpha_1}$, which expresses that M_{α_1} is a concept of M_α. We now present the most relevant of these (they may be found in [Church, 1974]). First, we have the axiom schema called ($15^{m\alpha\beta}$):

$$(\forall f_{\alpha\beta} \forall f_{\alpha_1\beta_1} \forall x_\beta \forall x_{\beta_1})\, \Delta^m f_{\alpha\beta} f_{\alpha_1\beta_1} \rightarrow (\Delta^m x_\beta x_{\beta_1} \rightarrow \Delta^m (f_{\alpha\beta} x_\beta)(f_{\alpha_1\beta_1} x_{\beta_1}))$$

which we interpret as: functional application preserves the *concept-of* relation.

Second, we have the axiom schema called ($16^{m\alpha\beta}$):

$$(\forall f_{\alpha\beta} \forall f_{\alpha_1\beta_1} \forall x_\beta \forall x_{\beta_1})\, (\Delta^m x_\beta x_{\beta_1} \rightarrow \Delta^m (f_{\alpha\beta} x_\beta)(f_{\alpha_1\beta_1} x_{\beta_1})) \rightarrow \Delta^m f_{\alpha\beta} f^1_{\alpha_1\beta_1}$$

which basically is the converse of ($15^{m\alpha\beta}$). Note that we, by increasing the subscripts and superscript by 1 in the rightmost subformula, get the sense $f^1_{\alpha_1\beta_1}$ of $f_{\alpha\beta}$.

Third, we have the axiom schema called ($17^{m\alpha}$):

$$(\forall x_\alpha \forall y_\alpha \forall x_{\alpha_1})\, \Delta^m x_\alpha x_{\alpha_1} \rightarrow (\Delta^m y_\alpha x_{\alpha_1} \rightarrow x_\alpha = y_\alpha)$$

which asserts that a concept can at most be a concept of one thing.

[10] Note that we say 'is to', Church presented namely no models of the Formulation in general.

[11] The superscript m was added in the 1974 paper in order to avoid antinomies. Note, λ-abstraction is also subscripted, i.e. $\lambda_n x_{\beta_n} M_{\alpha_n}$ is a well-formed formula of type $\alpha_n\beta_n$ (using the typing convention of Simple Type Theory).

If we compare the axioms with Frege's contribution, we see that the last axiom formalizes the relation between senses and what they determine (the figure on page 25 shows this relation). Moreover, the two first axioms appear not to be inconsistent with his work. It is not clear to us whether the remaining axioms of the Formulation (which we have not presented here) are in accordance with Frege's contribution, but Church admits in the 1951 paper (page 4) that "we do make certain changes to which he [Frege] would probably not agree."

Later we present a completely different algebraic approach for formalizing concepts. But it is interesting to note that there are similarities to Church's Formulation. First of all, we admit (in the logic of Section 5.4) an infinite hierarchy of senses, just as Church does (this should be clear since there is no limit on the subscripts $o_0, o_1, o_2, \dots$). Moreover, our functions on intensions (senses) will preserve functional application, as we shall see later.

2.4 Other Contributions

Stacked on top of each other, the volume of the contributions of Frege, Carnap, and Church take up only a fraction of the entire volume of the collection of writings which in some way or another are related to the issue of intensionality. We can divide these writings into two categories.

First of all there are the pre-symbolic-logic authors, like Antoine Arnauld (Port-Royal logic), Immanuel Kant, Gottfried W. Leibnitz, and John S. Mill. As it falls out of scope to address historical and general philosophical issues, these will not be considered.

Secondly there are the recent (and formal contributions) like those of Peter Aczel [1980], Jon Barwise and John Perry [1983], Paul Gilmore [2001], Michael Jubien [1989], Yiannis Moschovakis [1994], and Edward Zalta [1988]. These contributions present very different theories, and it falls out of scope to describe them in details, however, it should be noted that the issue of an infinite hierarchy of senses does not seem to have been addressed in these contributions.

There are also the contributions of Richard Montague [1974c; 1974b; 1974a]. As these, after some years, became widely renowned and have been presented and discussed intensively elsewhere (see for example [Gallin, 1975; Anderson, 1984; Gamut, 1991b]), we find no need for yet another presentation of this work. Moreover, the intensional logic we present later (Chapter 5) is, unlike Montague's work, not based on possible-world semantics.

Finally there are the contributions of George Bealer [Bealer, 1982; Bealer and Mönnich, 1989] and the related [Menzel, 1986; Swoyer, 1998]. They present an approach which is closely related to our approach. These contributions are described in Chapter 5, however, it should be noted that they do not address the issue of the infinite hierarchy of senses either.

Chapter 3

Defining Intensionality

The aim of this chapter is to present a formal definition of intensionality based on the contributions described in the previous chapter. The first section discusses the intuitive notion of intensionality as a step towards the formal definition which is presented in the second section and discussed in the third. Thereafter we examine whether some well-known logics are extensional or intensional. The fifth section describes how to determine whether a logical theory is intensional. The last section presents a theory which at first may appear to be intensional; then we show that the theory is extensional. This motivates a formal definition of intensionality.

3.1 The Intuitive Notion of Intensionality

Before we present the formal definition of intensionality, it is important to note that there is what we call an intuitive notion of intensionality in terms of which a language is intensional if denotation is distinguished from sense, that is, if both a denotation and a sense is ascribed to some of its expressions. This notion is simply adopted from Frege's contribution described in the previous chapter.

The reason why we do not adopt the intuitive notion is that it is not well established what the sense of an expression precisely is in general. Of course, following Frege, we know that sense contains mode of presentation, that sense determines denotation, and senses are grasped (see section 2.1).

However, these conditions are not sufficient, it seems, for determining whether something really is the sense of an expression.

Several suggestions on how to formalize senses (or intensions) have been presented. A contribution by Moschovakis [1994] suggests that sense is algorithm and denotation is the value of the algorithm. This seems to be coherent with Frege's work. However, there is a problem of formulating senses of senses, but maybe they are some sort of higher-order algorithms that computes algorithms.

The possible world semantics of modal logic provides another suggestion in which intension arises through functional abstraction.[1] For example, a proposition in propositional logic is interpreted as a truth-value. Under the possible world semantics (of propositional modal logic) it is interpreted as a mapping from possible worlds (contexts) to truth-values. The extension of a proposition can accordingly be seen as its truth-value in the actual world and the intension as the mapping from possible worlds to truth-values. This greatly enhances the expressivity, for example, distinct propositions may have the same truth-values in some of the possible worlds, meaning we can discern between co-extensional propositions. However, the formalization of propositional attitudes is inadequate, because one may have different propositional attitudes towards propositions which have the same truth-values in every possible world, i.e. co-intensional propositions. This has been extensively discussed in the literature, see for example [Anderson, 1984; Bäuerle and Cresswell, 1989; Carnap, 1956]. Reinhard Muskens [1991] presents a possible solution to this problem which is based on the possible world semantics, however, it assumes that propositions are abstracted even further (as mappings of mappings and so on).

It should be noted that the intensional semantics we present later proposes another suggestion for formalizing senses.

The intuitive notion suggests a more technical condition for defining intensionality. As noted by Frege and Carnap, expressions with the same denotation (extension) need not have the same sense (intension). This means, as we saw in the previous chapter, that the truth-value of a sentence may be altered if co-denotational expressions are substituted for each other. Hence co-denotational expressions are not *substitutable* in general. It seems as if

[1] Often 'intension' has been used exclusively in connection with possible world semantics, however, we use (as many others) 'intension' in a more wide sense as described in Chapter 2. 'Intensional logic' has actually been used as a general term for modal logic, temporal logic, and Montague's IL [van Benthem, 1988; Gamut, 1991b], but we will later show that other intensional logics exist.

this only provides a necessary condition for being intensional in terms of the intuitive notion, however, since it is not established what precisely senses are, we will identify intensional languages with languages that do *not* allow substitution of co-denotational formulas.

We therefore have the following *characterization of intensionality*:

> *a language is extensional if all co-denotational expressions can be substituted for each other, and a language is intensional if it is not extensional.*

This is similar to (and adopted from) Carnap's definitions of extensionality and intensionality (see page 27), except that we identify intensional with non-extensional (Carnap had additionally that co-intensional expressions must be substitutable). This identification is similar to the general linguistic characterization of intensionality used today, see e.g. [Audi, 1999]. Moreover, we use 'co-denotational' instead of 'co-extensional' as explained earlier. The characterization is an important step towards the definition of intensionality. Since the definition should be formal, we have restricted it to languages with a formal notion of consequence, viz. logics.

There exists a vast amount of writings about intensionality and issues related to intensionality, however, very few precisely define what it means to be intensional. Carnap represents an exception, however, his definition is not formal. In many cases, intensionality is associated with natural language understanding, in particular, the feature of natural language that co-denotational phrases cannot necessarily be substituted for each other. Note that this use of intensionality is related to the intuitive notion of intensionality (our introduction of the intuitive notion was actually motivated by this use of intensionality).

This vagueness is probably caused by the fact that intensionality is an intricate notion. This has for example been acknowledged by R. H. Thomason. He writes [1985] p. 2:

> The point [. . .] is that there's really no such thing as a naive theory or even a naive account of intensionality. This is a subject for sophisticates, who have bought the program of giving a certain type of semantic theory for a sufficiently rich language, and who are concerned about the interpretations of phrases—in particular, of *sentences*.

3.2 The Definition of Intensionality

In his seminal paper where he defines a complete semantics of modal logic **S5**, Kripke says [1959] p. 3:

> It is noteworthy that the theorems of this paper can be formalized in a metalanguage (such as Zermelo set theory) which is "extensional," both in the sense of possessing set-theoretic axioms of extensionality *and* in the sense of postulating no sentential connectives other than the truth-functions. Thus it is seen that at least a certain non-trivial portion of the semantics of modality is available to an extensionalist logician.

The quotation has inspired us to propose that intensionality should be studied in well-known and well accepted mathematical settings, i.e. in extensional settings (if nothing else, this should at least show to what extend being intensional can be made rigorous). The present work attempts to follow this suggestion.

It is not trivial to define extensionality and intensionality formally (which also suggests why it, to our knowledge, has not been done before). One problem is that we want the definition to work in general and not merely for distinguishing between modal logic and classical logic. Another problem is that the notions, like co-denotation, that are presupposed in the characterization are not precise in general. For example, what does it mean to be co-denotational in FDE (first degree entailment)? And what does it mean to be co-denotational in a description logic?[2]

The characterization of intensionality implicitly says: *if co-denotational then substitutable.* Hence it comprises a conditional, but there are many ways to formalize conditionals. In order to be general, we formalize it on the meta-level by means of logical consequence (entailment). Moreover, there may be several notions of logical consequence in a given logic.

It also turns out to be a problem that we want our definition to be in accordance with the intensionality results which are part of the logical folklore where modal logic is said to be intensional and first-order predicate logic extensional.

Definition 1 *Let ϕ, ψ and χ be formulas and Γ a set of formulas of the logic in question. Let $\models$ be the consequence relation, $\leftrightarrow$ the biimplication and let*

[2]This will be answered later.

$\chi[\psi/\phi]$ be the result of substituting (replacing) an occurrence of ϕ with ψ in χ. Then the logic is extensional *if*

$$\textit{whenever } \Gamma \models \phi \leftrightarrow \psi, \textit{ then } \Gamma \models \chi \textit{ implies } \Gamma \models \chi[\psi/\phi]$$

for all ϕ, ψ, χ and Γ. The logic is intensional *if it is not extensional.*

It is important to note that the notions on which the definition is founded need not be uniquely defined in general; we may for instance have several definitions of logical consequence. In such cases we may have several intensionality results.

The definition is in accordance with Carnap's definition of extensionality and intensionality, assuming that ϕ and ψ are co-extensional if $\phi \leftrightarrow \psi$ is satisfied, which commonly is the case. Actually, Carnap reaches a result (thus it is not a definition) which is quite similar to our definition of intensionality, see [1956] 12-1. b. Moreover, the definition does also seem to be in accordance with the intuitive notion of intensionality, as we will argue below. It should also be noted that it is similar to the use of extensionality and intensionality in [Gamut, 1991a; 1991b], although they do not define extensionality nor intensionality.

3.3 Alternative Definitions

The above definition of extensionality and intensionality is only one of several alternatives. Now we describe some of these.

In Definition 1 intensionality was defined by means of the meta-logical notion of entailment. Alternatively it could be defined internally such that the conditional used in the characterization of intensionality is formalized by means of implication. Then the logic could be defined as extensional if

$$(\phi \leftrightarrow \psi) \rightarrow (\chi \rightarrow \chi[\psi/\phi])$$

is valid for all formulas ϕ, ψ, χ.[3] Such a definition has not been used because description logics do not comprise a general implication within the logic, moreover, this definition does not, informally speaking, allow us to discern between different definitions of entailments (and as we shall see, different entailment relations give rise to different intensionality results).

[3] Which implies $(\Gamma \rightarrow (\phi \leftrightarrow \psi)) \rightarrow ((\Gamma \rightarrow \chi) \rightarrow (\Gamma \rightarrow \chi[\psi/\phi]))$, an "internal" version of Definition 1, assuming $\rightarrow$ is material implication; Γ is now a formula.

Intensionality has been defined in terms of a biimplication connective. However, we could also define it directly in terms of co-denotation. We can accomplish this if the notions of *satisfaction* and *model* are well defined for the particular logic in question. (This is inspired by Tarski's contemplations about the concept of logical consequence [1956].) Intensionality could then be defined as follows.

> Let ϕ, ψ, and χ be formulas and Γ a set of formulas (of the logic in question). If in every model which satisfies Γ, the interpretation of ϕ is equal to the interpretation of ψ, then we say that ϕ and ψ are *co-denotational in* Γ. The logic is said to be *extensional* if it satisfies the following: whenever ϕ and ψ are co-denotational in Γ, then if χ is a logical consequence of Γ then $\chi[\psi/\phi]$ is a logical consequence of Γ, for all ϕ, ψ, χ and Γ. If the logic is not extensional then we say that it is *intensional*.

This is in accordance with Carnap's definition. However, we have not said anything about when to identify models, and this may not be clear in general. So far our investigations show that this definition gives the same intensionality results as Definition 1.

As yet another alternative, one may found the definition of extensionality on another (more meta-logical) understanding of co-denotation in which formulas are co-denotational if they are consequences of each other. Then the logic could be defined as extensional if

$$\phi \models \psi \text{ and } \psi \models \phi \text{ implies } \chi \models \chi[\psi/\phi] \tag{3.1}$$

for all ϕ, ψ, χ. However, under this definition modal logic becomes extensional, so this definition is not coherent with the intuitive notion of intensionality.

After having defined extensionality and intensionality, we discovered that D. M. Gabbay [1994] gives a general and formal definition of extensionality. Let $\mid\!\sim$ be the consequence relation of the logic in question (we use the notation of [Gabbay, 1994]). Then a connective $\sharp(A_1, \dots, A_n)$ is *extensional* if

$$\frac{x_i \mid\!\sim y_i, y_i \mid\!\sim x_i, i = 1 \dots n}{\sharp(x_1, \dots, x_n) \mid\!\sim \sharp(y_1, \dots, y_n)}$$

Moreover, a logic is extensional if all its connectives are extensional in all their variables. We get that Gabbay's definition of extensionality is equivalent to our definition of extensionality formulated in (3.1), since n applications of

(3.1) give

$$\frac{x_1 \models y_1 \quad y_1 \models x_1}{\natural(x_1, x_2, \ldots, x_n) \models \natural(y_1, x_2, \ldots, x_n)}$$

$$\vdots$$

$$\frac{x_n \models y_n \quad y_n \models x_n}{\natural(x_1, \ldots, x_{n-1}, x_n) \models \natural(x_1, \ldots, x_{n-1}, y_n)}$$

which give (assuming uniform substitution of multiple variables is sound)

$$\frac{x_i \models y_i, y_i \models x_i, i = 1 \ldots n}{\natural(x_1, \ldots, x_n) \models \natural(y_1, \ldots, y_n)}$$

This shows that our definition of extensionality subsumes Gabbay's. The converse is easy to see (by structural induction on χ), meaning that the two definitions are equivalent.

Gabbay [1994] does not discuss other definitions of extensionality, nor does he mention intensionality. Notwithstanding, it shows that different definitions of extensionality have been described in the literature.

Notice how Gabbay's definition of intensionality is similar to the compatibility property of a congruence relation.[4] This suggests that there is a relation between whether a logic is extensional and whether the consequence relation gives rise to a congruence.

There is one problem, though, the consequence relation is not an equivalence relation (in common cases, at least). Let us therefore define a binary relation $\mathrel{|\!\sim\!|}$ on the formulas by $x \mathrel{|\!\sim\!|} y$ if and only if $x \mathrel{|\!\sim} y$ and $y \mathrel{|\!\sim} x$. Clearly, $\mathrel{|\!\sim\!|}$ is symmetric, it is also reflexive, and it is transitive (assuming $\mathrel{|\!\sim}$ is transitive), because if $x \mathrel{|\!\sim\!|} y$ and $y \mathrel{|\!\sim\!|} z$, then $x \mathrel{|\!\sim} y$, $y \mathrel{|\!\sim} x$, $y \mathrel{|\!\sim} z$, and $z \mathrel{|\!\sim} y$, which give $x \mathrel{|\!\sim} z$ and $z \mathrel{|\!\sim} x$, thus $x \mathrel{|\!\sim\!|} z$.

Then it should be clear from the above that the logic in question is extensional by the definition formulated in (3.1) if and only if $\mathrel{|\!\sim\!|}$ is a congruence, that is, if and only if

$$\frac{x_1 \mathrel{|\!\sim\!|} y_1 \quad \cdots \quad x_n \mathrel{|\!\sim\!|} y_n}{\natural(x_1, \ldots, x_n) \mathrel{|\!\sim\!|} \natural(y_1, \ldots, y_n)}$$

It falls out of scope to pursue this relation further.

[4] Let **A** be an algebra and let θ be an equivalence relation on A. The *compatibility property* holds for θ if $x_1 \theta y_1, \ldots, x_n \theta y_n$ implies $f(x_1, \ldots, x_n) \theta f(y_1, \ldots, y_n)$ for all n-ary function symbols f and for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$.

3.4 Intensionality Results

The following lemma is used for simplifying the proof of extensionality of propositional logic.

Lemma 2 *For every set of formulas* Γ *and formulas* ϕ, ψ *and* χ *in propositional logic we have that*

$$\textit{whenever } \Gamma \models \chi \leftrightarrow \chi[\psi/\phi], \textit{ then } \Gamma \models \chi \textit{ implies } \Gamma \models \chi[\psi/\phi]. \tag{3.2}$$

Proof. In order to keep track of the different quantifications and implications, we show this by encoding the result in first-order predicate logic. $\Gamma \models \phi$ is defined as follows: if for every assignment α of truth values to the propositional letters, we have that if every member of Γ is true under α (in which case we simply write $t(\alpha, \Gamma)$), then ϕ is true under α (in which case we write $t(\alpha, \phi)$). Thus, $\Gamma \models \phi$ if and only if $\forall\alpha(t(\alpha, \Gamma) \to t(\alpha, \phi))$ (we have not defined $t(\cdot, \cdot)$, but this is straightforward). In fact this is a misuse of notation, strictly speaking the arguments of t are not formulas (or sets of formulas) of propositional logic, but terms denoting formulas. An encoding of the formulas can be accomplished by means of Gödel numbers. However, as this is not important, we skip these technicalities.

Then we have the following encoding of (3.2)

$$\forall\alpha(t(\alpha, \Gamma) \to t(\alpha, \chi \leftrightarrow \chi[\psi/\phi])) \to \\ (\forall\alpha(t(\alpha, \Gamma) \to t(\alpha, \chi)) \to \forall\alpha(t(\alpha, \Gamma) \to t(\alpha, \chi[\psi/\phi])))$$

which gives, by the semantics of biimplication in propositional logic

$$\forall\alpha(t(\alpha, \Gamma) \to (t(\alpha, \chi) \leftrightarrow t(\alpha, \chi[\psi/\phi]))) \to \\ (\forall\alpha(t(\alpha, \Gamma) \to t(\alpha, \chi)) \to \forall\alpha(t(\alpha, \Gamma) \to t(\alpha, \chi[\psi/\phi]))).$$

This formula is valid, because it is implied by the universal generalization of an instance of a tautology, $(p \to (q \leftrightarrow r)) \to ((p \to q) \to (p \to r))$. Hence the encoding of (3.2) is valid, meaning that (3.2) holds. □

Theorem 3 *Propositional logic is extensional.*

Proof. Because of Lemma 2 it suffices to show

$$\Gamma \models \phi \leftrightarrow \psi \text{ implies } \Gamma \models \chi \leftrightarrow \chi[\psi/\phi].$$

We proceed by induction on the structure of χ. Note, if ϕ does not occur in χ then $\chi[\psi/\phi]$ simply is χ, meaning we get $\Gamma \models \chi \leftrightarrow \chi$, which obviously holds. Moreover, if χ is identical to ϕ, then $\Gamma \models \chi \leftrightarrow \chi[\psi/\phi]$ becomes $\Gamma \models \phi \leftrightarrow \psi$ which is the assumption. These two obvious cases will not be shown in the following.

The base case when χ is a propositional letter p. If ϕ does not occur in p then we get the abovementioned obvious case. Otherwise ϕ occurs in p, which means that ϕ and p are identical; this leads as mentioned above also to an obvious case.

For the case $\neg\chi'$ the induction hypothesis, $\Gamma \models \phi \leftrightarrow \psi$ implies $\Gamma \models \chi' \leftrightarrow \chi'[\psi/\phi]$, gives that χ' and $\chi'[\psi/\phi]$ have the same truth value under every assignment which satisfies every member of Γ. But this means, by the truth table for $\neg$, that $\neg\chi'$ and $\neg\chi'[\psi/\phi]$ have the same truth value under every assignment which satisfies every member of Γ, hence $\Gamma \models \neg\chi' \leftrightarrow \neg\chi'[\psi/\phi]$.

For the case $\chi_1 \wedge \chi_2$, let us assume $\Gamma \models \phi \leftrightarrow \psi$. The induction hypotheses $\Gamma \models \phi \leftrightarrow \psi$ implies $\Gamma \models \chi_i \leftrightarrow \chi_i[\psi/\phi]$, $i = 1, 2$, give that χ_i and $\chi_i[\psi/\phi]$ have the same truth value under every assignment which satisfies every member of Γ. But this means, by the truth table for $\wedge$, that $\chi_1 \wedge \chi_2$ and $\chi_1[\psi/\phi] \wedge \chi_2[\psi/\phi]$ i.e. that $\chi_1 \wedge \chi_2$ and $(\chi_1 \wedge \chi_2)[\psi/\phi]$ have the same truth value under every assignment which satisfies every member of Γ, hence $\Gamma \models (\chi_1 \wedge \chi_2) \leftrightarrow (\chi_1 \wedge \chi_2)[\psi/\phi]$.

By the adequacy of the negation and conjunction connectives, the remaining cases need not be shown. □

The result is as we would expect because all the connectives of propositional logic are truth-functions, so the truth-value of a connective depends only of the truth-values of its arguments which means that formulas with the same truth-value are substitutable.

One may think that our definition of extensionality is unnecessary complicated and suggest that we simply define extensionality by $\Gamma \models \phi \leftrightarrow \psi$ implies $\Gamma \models \chi \leftrightarrow \chi[\psi/\phi]$. This would then mean that we can skip Lemma 2. Unfortunately, this definition of extensionality is not sufficiently general: substitutivity is assumed to hold if $\chi \leftrightarrow \chi[\psi/\phi]$, but in general (as in Chapter 5) we may have several biimplications. And in such cases it is not enough that substitutivity holds only for one of the biimplications. This explains our more general definition of extensionality where the implication is formulated in the metalanguage.

Now we turn to modal logic. There are two definitions of logical consequence in modal logic, the local consequence relation and the global consequence relation [Blackburn *et al.*, 2001]. For a formula χ to be a *local* *consequence* of a set of formulas Γ, notation $\Gamma \models \chi$, we must have that whenever all formulas of Γ are true in a possible world in a model then χ is true in the same possible world in that model. For χ to be a *global consequence* of Γ, notation $\Gamma \models^g \chi$, we must have that whenever Γ is true in every possible world of a model then χ is true in every possible world of the model too.[5]

Lemma 4 *For every set of formulas Γ and formulas ϕ, ψ and χ in the modal logics* **K**–**S5** *we have that*

$$\textit{whenever } \Gamma \models^g \chi \leftrightarrow \chi[\psi/\phi],\ \textit{then } \Gamma \models^g \chi \textit{ implies } \Gamma \models^g \chi[\psi/\phi]. \tag{3.3}$$

Proof. This is proven similarly to Lemma 2, but instead of quantifying over assignments, we quantify over Kripke models and possible worlds. This makes the proof more technical, but since it follows the same approach as the proof of Lemma 2, it is not presented it here. □

Theorem 5 *The modal logics* **K**–**S5** *are*

1. *intensional with respect to the local consequence relation,*
2. *extensional with respect to the global consequence relation.*

Proof. 1. We simply show a counter example for **S5**. Let p and q be propositional letters then

$$p \leftrightarrow q \models p \leftrightarrow q \text{ and } p \leftrightarrow q \models \Box p \leftrightarrow \Box p,$$

but

$$p \leftrightarrow q \not\models \Box p \leftrightarrow \Box p[q/p].$$

To see this, let (W, R, V) be a Kripke model with $W = \{w_1, w_2\}$ and let $R = W \times W$ (which is an equivalence relation), and let $V(p) = \{w_1, w_2\}$ and $V(q) = \{w_1\}$. We verify that $(W, R, V), w_1 \Vdash p \leftrightarrow q$ and $(W, R, V), w_1 \nVdash \Box p \leftrightarrow \Box q$.

[5] More precisely, a *Kripke model* is a tuple (W, R, V) consisting of a set of possible worlds W, a binary accessibility relation R on W and a valuation V mapping the propositional letters to 2^W. The local consequence relation $\models$ is defined as follows: $\Gamma \models \phi$ if for every Kripke model $M = (W, R, V)$ (of the appropriate class) and possible world $w \in W$, if $M, w \Vdash \Gamma$ then $M, w \Vdash \phi$. Where $\Vdash$ is the usual satisfaction relation in modal logic (following the notation of [Blackburn *et al.*, 2001]). The global consequence relation $\models^g$ is defined as: $\Gamma \models^g \phi$ if for all Kripke models $M = (W, R, V)$ (of the appropriate class), if for all $w \in W$, $M, w \Vdash \Gamma$ then for all $w \in W$, $M, w \Vdash \phi$.

For obtaining the results for the weaker modal logics note that $\phi \models \psi$ if and only if $\phi \to \psi$ is valid under the usual possible world definition of validity. Then, since every formula which is valid in one of the weaker logics is valid in **S5**, and $(p \leftrightarrow q) \to (\Box p \leftrightarrow \Box q)$ is not valid in **S5**, it cannot be valid in the weaker modal logics, which therefore must be intensional too.

2. Because of Lemma 4, we merely have to show

$$\Gamma \models^g \phi \leftrightarrow \psi \text{ implies } \Gamma \models^g \chi \leftrightarrow \chi[\psi/\phi]$$

for all ϕ, ψ, χ, Γ. We proceed by induction on the structure of χ. All cases but necessitation are similar to the proof of Theorem 3.

For the case $\Box\chi'$, let us assume $\Gamma \models^g \phi \leftrightarrow \psi$. The induction hypothesis $\Gamma \models^g \phi \leftrightarrow \psi$ implies $\Gamma \models^g \chi' \leftrightarrow \chi'[\psi/\phi]$ gives that for every Kripke model (W, R, V), which for every possible world satisfies every member of Γ, that $(W, R, V), w \Vdash \chi' \leftrightarrow \chi'[\psi/\phi]$ for all $w \in W$. Hence χ' and $\chi'[\psi/\phi]$ have the same truth value in every possible world w of the model. But this means that χ' and $\chi'[\psi/\phi]$ have the same truth-value in all possible worlds accessible from w for all $w \in W$, hence, by the semantics of $\Box$, $(W, R, V), w \Vdash \Box\chi' \leftrightarrow \Box\chi'[\psi/\phi]$ for all $w \in W$, which shows the wanted $\Gamma \models^g \Box\chi' \leftrightarrow \Box\chi'[\psi/\phi]$. $\Box$

It should be noted that the local intensionality result generalizes to other modal logics than the ones above, but not all. In principle it is possible for a modal logic to be locally extensional, for instance, a normal modal logic with the axiom $p \leftrightarrow \Box p$ is extensional.[6] Such logics, however, seem to be of little relevance—as modal logics at least. This, along with the fact that the local consequence relation is the most common definition of consequence, means that a general characterization of modal logic as being intensional is acceptable.

The extensionality result for the global semantics of modal logic can fairly easy be shown by means of the algebraic semantics of modal logic.[7] One of the advantages of the algebraic semantics is that once we have established

[6] A *normal modal logic* is a set of formulas containing the propositional tautologies, the axiom K, $\Box(p \to q) \to (\Box p \to \Box q)$, and which is closed under modus ponens, uniform substitutions and generalization (if ϕ is a member then so is $\Box\phi$) [Blackburn *et al.*, 2001].

[7] Now we briefly present the algebraic semantics of modal logic. This is described in details in [Blackburn *et al.*, 2001]. An interpretation is a homomorphism $\tilde{v}$ from the formulas to an algebra with a greatest element 1 (a complex algebra). Now, $\Gamma \models \phi$ if for all $\omega \in \Gamma$, $\tilde{v}(\omega) = 1$ implies $\tilde{v}(\phi) = 1$. Then we have the extensionality result because if $\tilde{v}(\phi \leftrightarrow \psi) = 1$, that is, if $\tilde{v}(\phi) = \tilde{v}(\psi)$, then $\tilde{v}(\chi) = \tilde{v}(\chi[\psi/\phi])$. This can be proven by induction on the structure of χ. For the case $\Diamond\chi'$, we have for instance $\tilde{v}(\Diamond\chi') = f_\Diamond(\tilde{v}(\chi')) = f_\Diamond(\tilde{v}(\chi'[\psi/\phi])) = \tilde{v}(\Diamond\chi'[\psi/\phi])$, where $f_\Diamond$ is the operator for $\Diamond$. The remaining cases are similar.

one extensionality proof, it is easy to see that logics which may be given a similar semantics must be extensional too (we will use this later).

Now we turn to first-order predicate logic. Similar to modal logic, we may present two notions of consequence. A *local consequence* defined as: $\Gamma \models \chi$ if for every interpretation and valuation, whenever all formulas of Γ are satisfied by the valuation in the interpretation then χ is satisfied by the valuation in the interpretation. And a *global consequence* defined as: $\Gamma \models^g \chi$ if for every interpretation whenever every formula of Γ is satisfied by every valuation in the interpretation then χ is satisfied by every valuation in the interpretation (this is also known as *semantical entailment*). Moreover, we may say that there are two versions of predicate logic: one in which open formulas are allowed (meaning for example that $p(x) \leftrightarrow p(x)$ is valid), and one in which only sentences (closed formulas) are considered.

Lemma 6 *For every set of formulas Γ and formulas ϕ, ψ and χ in first-order predicate logic we have that*

whenever $\Gamma \models^g \chi \leftrightarrow \chi[\psi/\phi]$, *then* $\Gamma \models^g \chi$ *implies* $\Gamma \models^g \chi[\psi/\phi]$, *and*
whenever $\Gamma \models \chi \leftrightarrow \chi[\psi/\phi]$, *then* $\Gamma \models \chi$ *implies* $\Gamma \models \chi[\psi/\phi]$.

Proof. The proofs are similar to the proofs of the previous lemmas. □

Theorem 7 *First-order predicate logic is*

1. *extensional with respect to the global consequence relation,*
2. *extensional with respect to the local consequence relation if only sentences are considered,*
3. *intensional with respect to the local consequence relation.*

Proof. 1. Because of Lemma 6 we only have to show (for all ϕ, ψ, χ, Γ)

$$\Gamma \models^g \phi \leftrightarrow \psi \text{ implies } \Gamma \models^g \chi \leftrightarrow \chi[\psi/\phi].$$

The proof proceed by induction on the structure of χ, and all cases but quantification are similar to the proof of Theorem 3.

For the case $\forall x \chi'$, let us assume $\Gamma \models \phi \leftrightarrow \psi$. Then the induction hypothesis gives $\Gamma \models \chi' \leftrightarrow \chi'[\psi/\phi]$. Thus every interpretation, which for every valuation satisfies Γ, satisfies $\chi' \leftrightarrow \chi'[\psi/\phi]$ for every valuation, but then the interpretation also satisfies $\forall x(\chi' \leftrightarrow \chi'[\psi/\phi])$ for every valuation, by the semantics for quantification. This implies that the interpretation satisfies $\forall x\chi' \leftrightarrow \forall x\chi'[\psi/\phi]$, hence $\Gamma \models \forall x\chi' \leftrightarrow \forall x\chi'[\psi/\phi]$.

2. By Lemma 6 it suffices to show

$$\Gamma \models \phi \leftrightarrow \psi \text{ implies } \Gamma \models \chi \leftrightarrow \chi[\psi/\phi]$$

for all sentences ϕ, ψ and χ and every set of sentences Γ. Notice that we now consider closed formulas. Instead of the above we will show

$$\Gamma \models \phi \leftrightarrow \psi \text{ implies } \Gamma \models \forall x_1 \cdots \forall x_n(\chi \leftrightarrow \chi[\psi/\phi]),$$

where we generalize χ to a formula, and $\{x_1, \ldots, x_n\}$ contains all the variables occurring in χ. This is easily seen to imply the wanted by the axiom $(\forall x\phi) \to \phi$ and the fact that every sentence is a formula.

We proceed by induction on χ. For the case when χ is an atomic formula χ'; if ϕ is identical to χ' then we must show $\Gamma \models \forall x_1 \cdots \forall x_n(\chi' \leftrightarrow \psi)$. This follows from the assumption $\Gamma \models \chi' \leftrightarrow \psi$ which may be generalized by $x_1, \ldots, x_n$ since neither χ' nor ψ contain any free variable—they are by assumption sentences. Otherwise, we get an obvious case (no substitution).

The case $\forall y\chi'$. The induction hypothesis gives $\Gamma \models \forall x_1 \cdots \forall x_n(\chi' \leftrightarrow \chi'[\psi/\phi])$. If y is a free variable of χ' then there exists an integer $i \leq n$ such that x_i is identical to y since $\{x_1, \ldots, x_n\}$ contains the free variables of χ'. Hence we have $\Gamma \models \forall x_1 \cdots \forall x_n \forall y(\chi' \leftrightarrow \chi'[\psi/\phi])$ which implies the wanted $\Gamma \models \forall x_1 \cdots \forall x_n(\forall y(\chi') \leftrightarrow \forall y(\chi'[\psi/\phi]))$. If y is not a free variable of χ' then the result is easily seen to hold.

The two remaining cases are similar to the proof of Theorem 3.

3. We show a counter example. We have

$$p(x) \leftrightarrow q(x) \models p(x) \leftrightarrow q(x) \text{ and } p(x) \leftrightarrow q(x) \models \forall x(p(x) \leftrightarrow p(x))$$

but

$$p(x) \leftrightarrow q(x) \models \forall x(p(x) \leftrightarrow q(x))$$

does *not* hold. □

Although we, when formulating the semantics of predicate logic, allow valid formulas to be open, it is more natural to consider formulas to be universally quantified at outermost level (corresponding to the first case above) or to consider only closed formulas (corresponding to the second case above). Thus a general characterization of predicate logic as being extensional is acceptable.

Now we consider the three-valued logic of Łukasiewicz and FDE (first degree entailment), see e.g. [Priest, 2001; Urquhart, 2001].[8]

Proposition 8 *1. The three-valued logic of Łukasiewicz is extensional;*

2. First degree entailment (FDE) is intensional.

Proof. 1. Similar to the earlier extensionality proofs, we establish that if

$$\Gamma \models \phi \leftrightarrow \psi \text{ implies } \Gamma \models \chi \leftrightarrow \chi[\psi/\phi]$$

holds for every ϕ, ψ, χ, Γ then the logic is extensional. As this is proven similar to the proof of Lemma 2, we will not show it.

Now we can prove that the three-valued logic of Łukasiewicz is extensional by induction on χ. Although we have three truth values instead of two, the proof is similar to the proof of Theorem 3. It will therefore not be shown here. □

2. We present a counter example of extensionality. We have

$$p \leftrightarrow q \models p \leftrightarrow q \text{ and } p \leftrightarrow q \models (r \wedge p) \leftrightarrow (r \wedge p)$$

but

$$p \leftrightarrow q \not\models (r \wedge p) \leftrightarrow (r \wedge q).$$

To see that the latter is the case, consider the interpretation $i(p) = 1, i(q) = b$, and $i(r) = n$. We get $i(p \leftrightarrow q) = b$ which is a designated truth-value, and $i((r \wedge p) \leftrightarrow (r \wedge q)) = n$ which is not a designated truth-value. □

[8]Now we briefly describe the semantics of these two logics. The three-valued logic of Łukasiewicz has three truth-values: $1, \frac{1}{2}$ and 0. The truth-tables for the connectives are

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

$\leftrightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

FDE has four truth-values $1, b, n, 0$. The truth-tables are

$\neg$	
1	0
b	b
n	n
0	1

$\vee$	1	b	n	0
1	1	1	1	1
b	1	b	1	b
n	1	1	n	n
0	1	b	n	0

$\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	0	0
n	n	0	n	0
0	0	0	0	0

$\leftrightarrow$	1	b	n	0
1	1	b	n	0
b	b	b	1	b
n	n	1	n	n
0	0	b	n	1

Under valid inferences all so-called designated truth-values are preserved. So let D be a set of *designated truth-values* which is a subset of the truth-values, then $\phi \models \psi$ if for every interpretation i, $i(\phi) \in D$ implies $i(\psi) \in D$. For the three-valued logic of Łukasiewicz $D = \{1\}$ and for FDE $D = \{1, b\}$.

Both the three-valued logic of Łukasiewicz and FDE are non-classical logics. The proposition shows therefore that extensional logic is not the same as classical logic and that intensional logic is not the same as non-classical logic.

Now we comment on the results. The extensionality and intensionality results appear to be in accordance with the intuitive notion of intensionality. If we, for example, consider the local semantics of modal logic (where the consequence relation is $\models$), it appears to be intensional in terms of the intuitive intensionality notion because of the distinction between denotation (extension) in the actual world and the denotation (intension) in all possible worlds. However, under the global semantics of modal logic (where the consequence relation is $\models^g$), this distinction vanishes on the level of entailment, since every formula of the antecedent must be satisfied in every possible world and not merely in the actual. Thus, in terms of the intuitive notion, the global semantics is not intensional.

The results are also in correspondence with what seems to be common knowledge in the logic community, with the exceptions that we say that modal logic is extensional under the global consequence relation and that predicate logic is intensional under the local consequence relation, in contrast to the general agreement that modal logic is intensional and predicate logic is extensional. One may therefore complain that this means that there must be something wrong with our definition of extensionality and intensionality. However, we find the results satisfying, because once we accept that modal logic is intensional, then first-order predicate logic must at least in some sense be intensional too, because of the well-known result of modal logic which says that modal formulas are equivalent to first-order formulas in one free variable [Blackburn *et al.*, 2001]. This also explains the similarities between the proofs of Theorem 5 and 7. The proofs of Theorem 5 and 7 show actually that the intensionality property of modal logic can be seen as stemming from lack of quantification of open formulas.

In the proof of intensionality of FDE (Proposition 8), the denotation of p and q are actually not the same. One may therefore argue that the example does not show non-extensionality because co-denotation is not presupposed. However, p and q are both interpreted as designated truth-values, hence in terms of the consequence relation of FDE (where we have two designated truth-values) they are equivalent, and accordingly we can consider them—at least in some sense—to denote the same. (Note that modal logic has a similar "problem" with respect to the local semantics where $p \leftrightarrow q$ means same denotation in the actual possible world.)

We find that this discussion justifies our definition of intensionality. Moreover, the discussion shows that intensionality is a comprehensive notion of which there does not seem to be a universal definition, but a plethora of candidates of which we have surveyed only a few.

3.5 Intensional Properties of Logical Theories

Our investigations of intensionality have been restricted to the level of formulas (to the level of the logical connectives), however, for logics which syntactically admit individuals, like first-order predicate logic, we additionally have intensionality properties for this level. Note that we have an intensionality property for the logical axioms of the logic in question, and intensionality properties for the particular theories formulated in the logic. Now we consider whether theories formulated in the logic are intensional.

Consider, for example, axiomatic set-theory. It is said to be extensional—not because it is formulated in an extensional logic—but because of its extensionality axiom (or extensionality principle, depending on the particular theory). This suggests that a set-theory without the extensionality axiom is non-extensional (intensional).

We will not go into details with this aspect, but it should be clear that the definition of intensionality (Definition 1) can be altered to handle this kind of intensionality. This can be illustrated by an example. Consider a theory of names which is formulated in first-order predicate logic. We have a binary predicate *co-den* expressing that two names are co-denotational. Now, assume we have the axiom

$$\forall x \forall y (x = y \rightarrow \textit{co-den}(x, y))$$

which says that identical names have the same denotation. If the converse does not hold, it should be clear that the theory is intensional because co-denotational names are not necessarily identical, meaning that co-denotational names cannot be substituted.

3.6 Reduction to Extensionality

A common principle of intensionality states that the intension of a concept is the set of properties common to all the individuals falling under the concept. In this section we show that if we follow this principle, concepts with the same extension are identified. In other words, this principle reduces to an extensional formalization of concepts.

We start by formalizing the principle. We have a set of individuals, a set of properties, and a set of concepts (properties are distinguished from concepts). Individuals are said to posses properties, meaning there is a binary relation A between individuals and properties. We write $A(x,p)$ when individual x is ascribed property p (such a relation is often called an *instance relation*). Individuals fall under (are members of) concepts, meaning there is a binary relation E between individuals and concepts. We write $E(x,c)$ when individual x falls under concept c. Concepts have intensions which are sets of properties, so we have a binary relation between properties and concepts. We write $I(p,c)$ when p belongs to the intension of c, in which case we say p is a *mark of* c. The idea of this principle is that the extension of a concept is formalized as the set of all individuals falling under the concept and the intension is formalized as the set of all its marks.

Concepts are identified with their intension. This means, among other things, that we are able to define when concepts subsume each other. We say that c is a *subconcept* of d if the intension of d is a subset of the intension of c (note the ordering).

An individual x falls under a concept c if and only if x is ascribed all the marks of c, thus

$$\forall x \forall c (\forall p (I(p,c) \to A(x,p)) \leftrightarrow E(x,c)). \tag{3.4}$$

Moreover, p is a mark of c if and only if all the individuals falling under c are ascribed p, thus

$$\forall p \forall c (\forall x (E(x,c) \to A(x,p)) \leftrightarrow I(p,c)). \tag{3.5}$$

Notice the close relationship between (3.4) and (3.5). (It is actually this duality which is responsible for the extensionality property of the theory.)

Now, assume we have two co-extensional concepts d and d', that is,

$$\{x \mid E(x,d)\} = \{x \mid E(x,d')\}$$

then (3.5) yield

$$\forall p (\forall x (E(x,d) \to A(x,p)) \leftrightarrow I(p,d))$$

and

$$\forall p (\forall x (E(x,d') \to A(x,p)) \leftrightarrow I(p,d')),$$

which by the assumption $\forall x(E(x,d) \leftrightarrow E(x,d'))$ give

$$\forall p(I(p,d) \leftrightarrow I(p,d')).$$

In other words, $\{p \mid I(p,d)\} = \{p \mid I(p,d')\}$, meaning that d and d' are co-intensional. Thus, co-extensionality implies co-intensionality. As concepts are identified with their intension, this shows that co-extensional concepts are identified, meaning that the principle reduces to extensionality. This is not the first principle which turns out to reduce to extensionality. Church admits in [1973] that parts of his earlier formulation [1951] reduces to extensionality.

By weakening the assumption that the intension is the set of properties common to all the individuals of the extension of the concept, that is, by replacing (3.5) with

$$\forall p \forall c\,(\forall x\, E(x,c) \rightarrow A(x,p)) \leftarrow I(p,c), \tag{3.6}$$

the principle does not reduce to extensionality. Note that this means that the intension of a concept is a subset of the set of properties common to all individuals of the extension of the concept, because (3.6) yields

$$\{p \mid I(p,c)\} \subseteq \{p \mid \forall x(E(x,c) \rightarrow A(x,p))\}.$$

The problem with this solution is that there is no sufficient condition for identifying concepts, that is, we have no sufficient condition for $I(p,c)$.

The above theory is closely related to formal concept analysis [Ganter and Wille, 1999], although they present their theory differently. The revised theory comprising (3.6) instead of (3.5) is closely related to the notion of *semiconcept* in formal concept analysis.

Now we turn to a different but related issue, namely that of Galois connections.[9] Above we saw an example of an instance relation A between individuals and the properties ascribed to them. It is noteworthy that such a relation gives rise to a Galois connection. This fact is well-known (see e.g. [Ganter and Wille, 1999]), but we will show it anyway because it proves the existence of an *inverse relation between extension and intension*, which have been acknowledged for a long time in the philosophical literature, see e.g. [Ar-

[9] A *Galois connection* is a pair of mappings $(f : P \rightarrow Q, g : Q \rightarrow P)$ between the partial orders $(\leq, P)$ and $(\leq, Q)$ which are dually adjoint, that is, such that $p \leq g(q)$ if and only if $q \leq f(p)$ holds for all $p \in P$ and $q \in Q$. In a Galois connection we have $p \leq p'$ implies $f(p') \leq f(p)$, and $q \leq q'$ implies $g(q') \leq g(q)$.

nauld and Nicole, 1996] (first edition published in 1662) and [Weingartner, 1974].[10] Moreover, we do not show the same as [Ganter and Wille, 1999].

Given an instance relation $A \subseteq I \times P$, define the mapping $i : 2^I \to 2^P$ by and $e : 2^P \to 2^I$ by

$$i(X) = \{y \in P \mid \forall x \in X\ A(x, y)\},$$

and define $e : 2^P \to 2^I$ by

$$e(Y) = \{x \in I \mid \forall y \in Y\ A(x, y)\}.$$

We can think of i as some kind of intension mapping in the sense that it gives the properties common to all the individuals, and e as some kind of extension mapping in the sense that it gives the individuals to which all the properties are ascribed.

We have, then, for a subset of individuals $X \subseteq I$ and a subset of properties $Y \subseteq P$ that

$$X \subseteq e(Y) \text{ if and only if } Y \subseteq i(X),$$

because

$$\begin{aligned}
&\forall x(x \in X \to x \in \{x \in I \mid \forall y \in Y\ A(x, y)\}) \iff \\
&\forall x(x \in X \to \forall y \in Y\ A(x, y)) \iff \\
&\forall x \forall y(x \in X \to (y \in Y \to A(x, y))) \iff \\
&\forall x \forall y(y \in Y \to (x \in X \to A(x, y))) \iff \\
&\forall y(y \in Y \to \forall x \in X\ A(x, y)) \iff \\
&\forall y(y \in Y \to y \in \{y \in P \mid \forall x \in X\ A(x, y))\}.
\end{aligned}$$

Hence (i, e) is a Galois connection, meaning that i and e are order reversing, so if $X \subseteq X'$ then $i(X') \subseteq i(X)$ and if $Y \subseteq Y'$ then $e(Y') \subseteq e(Y)$. This shows (among other things) the inverse relation between extension and intension.

Note, the terminology and principles introduced in this section, Section 3.6, will not be used in the remainder. Note also that the terminology 'intension' is actually a bit misleading since the principle reduces to extensionality.

[10] The "inverse relation between extension and intension" means merely that if one generalizes a concept (i.e. decreases the intension) then one increases the extension, and conversely. For example the extension of *dog* is less than the extension of *animal*, but the intension of *dog* is larger (comprises more marks like *domesticated*) than the intension of *animal*.

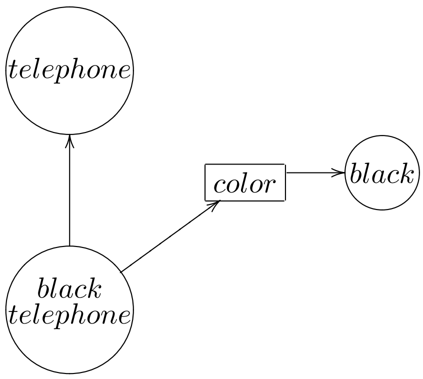
Chapter 4

Current Concept Logic

This chapter describes the approaches currently used for formalization of conceptual knowledge. First we describe the background of these approaches, then description logics will be presented, after which we examine whether description logics are intensional. Finally we examine other concept theories.

4.1 Background

Description logics stem from semantic networks which form a large group of graphical languages and systems used mainly in the 1970s to represent and reason with conceptual knowledge. The following figure illustrates a simple semantic network



in which the concept *black telephone* is a subconcept of *telephone* and has an attribute *color* with the value *black*.

The problem with semantic networks was that they did not have a rigorously defined semantics. Commenting upon the situation before semantic networks were formalized, Brachman and Levesque state in [1985] p. 217:

> Until this time, the semantics of semantic network languages were mostly a mystery, with the meanings of various constructs relying strictly on the intuitions of the reader and based normally only on suggestive naming conventions.

The semantic network above could for example stand for a concept definition such that *black telephone* is defined as *telephone* with *color black*, however, it could also assert a relation between the concept *telephone* and the color *black* such that every (or some) telephone is assumed to have the color black, cf. [Woods, 1975]. The figure in isolation is not enough to disambiguate.

The lack of formal semantics of semantic networks was firstly criticized by William Woods [1975].[1] (Woods' paper was actually predated by a contribution of Patrick Hayes [1974], but Hayes' paper does not seem to be as widely recognized.) After this, different versions of description logics with formal semantics appeared (but it was not until recently that they were known as 'description logics'—at one time they were actually known as 'concept logics'). One of the most important papers was [Levesque and Brachman, 1987] which investigates the trade-off between computational complexity and expressiveness, something which has turned out to be one of the major research themes of description logics. With the introduction of $\mathcal{ALC}$ [Schmidt-Schauß and Smolka, 1991] description logics reached their current form. During this period the notion of a concept changed from a data-structure intended to represent human memories to the formal notion used today which is defined in the following section.

4.2 Description Logics

Stemming from semantic networks, description logics are tailored for representing and reasoning with conceptual knowledge. As many description logics

[1]The paper by Woods [1975], which is often referred to as a 'milestone' in the history of semantic networks, is particular interesting for us, because this paper, as well as his later papers, like [Woods, 1991], stresses the importance of intensionality in connection with semantic networks.

have nice computational properties (compared to other logics), they provide a means for representing and reasoning with conceptual knowledge in practice.

We focus on the prototypical description logic $\mathcal{ALC}$ [Baader and Nutt, 2003; Schmidt-Schauß and Smolka, 1991]. Since the majority of description logics are more expressive (or eventually less expressive) versions of $\mathcal{ALC}$ and have a model-theoretic semantics similar to $\mathcal{ALC}$, it suffices (for our purpose) to consider $\mathcal{ALC}$. We attempt to use the same notation as the Description Logic Handbook [Baader *et al.*, 2003].[2]

Description logics have a rather special variable-free syntax (such that is it possible for non-logicians to use them), which makes them similar to equational logics. The terms of $\mathcal{ALC}$ are called *concept descriptions*; they are defined by means of a set C of *atomic concepts* and a set R of *atomic roles*. Atomic roles are used for expressing binary relations between concepts.

For the remainder of this chapter let c and d be any concept descriptions and r any atomic role.

The set of *concept descriptions* are formed by means of the following syntax rule (a more logical definition is presented in Definition 10):

$$
\begin{array}{lll}
c, d \rightarrow & a \mid & \text{(atomic concept)} \\
& \top \mid & \text{(universal concept)} \\
& \neg c \mid & \text{(concept negation)} \\
& c \sqcap d \mid & \text{(concept conjunction)} \\
& \forall r.c & \text{(value restriction).}
\end{array}
$$

Moreover, we have the following abbreviations:

$$
\begin{array}{ll}
\bot \text{ stands for } \neg\top & \text{(bottom concept)} \\
c \sqcup d \text{ stands for } \neg(\neg c \sqcap \neg d) & \text{(concept disjunction)} \\
\exists r.c \text{ stands for } \neg(\forall r.(\neg c)) & \text{(existential quantification).}
\end{array}
$$

Note that these are similar to the classical abbreviations.

$\mathcal{ALC}$ is given a model-theoretic semantics. An *interpretation* $(\cdot)^{\mathcal{I}}$ consists of a mapping $C \rightarrow 2^U$ where U is a non-empty set called the *domain of the interpretation*, and a mapping $R \rightarrow 2^{U \times U}$. It extends to a mapping on

[2] Among other things, we do *not* write concept descriptions in capital letters, and we give entailments a unified presentation, linking together TBoxes and ABoxes.

concept descriptions by the following definition:

$$\begin{aligned}\top^{\mathcal{I}} &= U\\ (c \sqcap d)^{\mathcal{I}} &= c^{\mathcal{I}} \cap d^{\mathcal{I}}\\ (\neg c)^{\mathcal{I}} &= \complement(c^{\mathcal{I}}) = U \backslash (c^{\mathcal{I}})\\ (\forall r.c)^{\mathcal{I}} &= \{x \in U \mid \forall y(\langle x, y\rangle \in r^{\mathcal{I}} \to y \in c^{\mathcal{I}})\}.\end{aligned}$$

We can verify that the abbreviations have a straightforward semantics:

$$\begin{aligned}(c \sqcup d)^{\mathcal{I}} &= (\neg(\neg c \sqcap \neg d))^{\mathcal{I}} = \complement(\complement c^{\mathcal{I}} \cap \complement d^{\mathcal{I}}) = c^{\mathcal{I}} \cup d^{\mathcal{I}}\\ \bot^{\mathcal{I}} &= (\neg\top)^{\mathcal{I}} = \emptyset.\end{aligned}$$

Similarly, it can easily be shown that

$$(\exists r.c)^{\mathcal{I}} = \{x \in U \mid \exists y(\langle x, y\rangle \in r^{\mathcal{I}} \wedge y \in c^{\mathcal{I}})\}.$$

In description logics, formulas are separated into two components, TBoxes and ABoxes. TBoxes formulate the relations between concepts and ABoxes formulate membership relations between individuals and concepts on one hand and individuals and roles on the other. Thus when we want to state that the concept *bachelor* is a subconcept of *man* then the assertion belongs to a TBox, and when we want to say that *John* is a particular *bachelor*, the assertion belongs to an ABox.

This separation means that there are two kinds of formulas. The former kind are called *terminological axioms* (in accordance with the name TBox). There are two types of terminological axioms:

$$c \equiv d,$$

which expresses that c is equivalent to d, and

$$c \sqsubseteq d,$$

which expresses that c is subsumed by d, i.e. that c is a *subconcept* of d. The subsumption relation is also known as the *is-a* relation.

Now, a *TBox* is simply a set of terminological axioms. Note, sometimes it is in addition required that a TBox has a particular simple structure.

The semantics of TBoxes is as follows. Let $(\cdot)^{\mathcal{I}}$ be an interpretation and $\mathcal{T}$ a TBox. Then $(\cdot)^{\mathcal{I}}$ *satisfies* $c \equiv d$ if $c^{\mathcal{I}} = d^{\mathcal{I}}$, and $(\cdot)^{\mathcal{I}}$ *satisfies* $c \sqsubseteq d$ if $c^{\mathcal{I}} \subseteq d^{\mathcal{I}}$. Moreover, $(\cdot)^{\mathcal{I}}$ *satisfies* $\mathcal{T}$ if $(\cdot)^{\mathcal{I}}$ satisfies all terminological axioms of $\mathcal{T}$; such an interpretation is called a *model* of $\mathcal{T}$.

In order to describe ABoxes, we need individuals.[3] Let a and b be any individuals. There are two types of formulas in ABoxes:

$$c(a) \quad \text{and} \quad r(a,b),$$

called a *concept assertion* and a *role assertion*. Now, an *ABox* is simply a set of concept assertions and role assertions.

The semantics of ABoxes follows the semantics of TBoxes, however, the interpretation now also maps individuals to members of the domain of the interpretation. Let $(\cdot)^{\mathcal{I}}$ be an interpretation and $\mathcal{A}$ an ABox, then $(\cdot)^{\mathcal{I}}$ *satisfies* $c(a)$ if $a^{\mathcal{I}} \in c^{\mathcal{I}}$, and $(\cdot)^{\mathcal{I}}$ *satisfies* $r(a,b)$ if $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in r^{\mathcal{I}}$. Moreover, $(\cdot)^{\mathcal{I}}$ *satisfies* $\mathcal{A}$ if $(\cdot)^{\mathcal{I}}$ satisfies all concept and role assertions of $\mathcal{A}$; such an interpretation is called a *model* of $\mathcal{A}$.

Now we present some examples.

$$\textit{string-instrument} \equiv \textit{instrument} \sqcap \exists \textit{has-part}.\textit{string}$$

expresses that a string instrument is an instrument which has a part which is a string (and conversely that an instrument which has a part which is a string is a string instrument). And

$$\textit{guitar} \sqsubseteq \textit{string-instrument}$$

expresses that every guitar is a string instrument, i.e. that *guitar* is a subconcept of *string-instrument*.

Moreover,

$$\textit{odd-number} \equiv \textit{natural-number} \sqcap \forall \textit{has-divisor}.\textit{odd-number}$$

expresses that an odd number is a natural number such that all its divisors are odd numbers (and conversely). (We have treated natural numbers as concepts, but they could also be individuals.) Notice that this axiom is recursive.[4]

[3] Similar to [Baader and Nutt, 2003] we do not introduce a set of individuals.

[4] There is an easy way to read concept descriptions if you are a logician. An atomic concept stands for a unary predicate and a concept description stands for a formula of first-order predicate logic in one free variable. The above examples translate to the formulas:

$$\forall x(\textit{string-instrument}(x) \leftrightarrow \textit{instrument}(x) \wedge \exists y(\textit{has-part}(x,y) \wedge \textit{string}(y)))$$
$$\forall x(\textit{guitar}(x) \rightarrow \textit{string-instrument}(x))$$
$$\forall x(\textit{odd-number}(x) \leftrightarrow \textit{natural-number}(x) \wedge \forall y(\textit{has-divisor}(x,y) \rightarrow \textit{odd-number}(y))).$$

Moreover,

$$guitarist(John)$$

expresses that the individual named *John* falls under the concept *guitarist*, i.e. that a John is a guitarist.

We are now ready to present the important definition of entailment (logical consequence) which formalizes what knowledge we can infer from TBoxes and ABoxes.

Let $\mathcal{T}$ be a TBox and $\mathcal{A}$ an ABox. If every model of $\mathcal{T} \cup \mathcal{A}$ satisfies $c \equiv d$ (or $c \sqsubseteq d$), we say that $\mathcal{T} \cup \mathcal{A}$ *entails* $c \equiv d$ (or $c \sqsubseteq d$) and write

$$\mathcal{T} \cup \mathcal{A} \models^{\mathcal{ALC}} c \equiv d \text{ (or } \mathcal{T} \cup \mathcal{A} \models^{\mathcal{ALC}} c \sqsubseteq d).$$

If every model of $\mathcal{T} \cup \mathcal{A}$ satisfies $c(a)$ (or $r(a,b)$), we say that $\mathcal{T} \cup \mathcal{A}$ *entails* $c(a)$ (or $r(a,b)$) and write

$$\mathcal{T} \cup \mathcal{A} \models^{\mathcal{ALC}} c(a) \text{ (or } \mathcal{T} \cup \mathcal{A} \models^{\mathcal{ALC}} r(a,b)).$$

Notice that this means that $\mathcal{ALC}$ is a fragment of first-order predicate logic, since everything that one can express in $\mathcal{ALC}$ can be expressed in first-order predicate logic (see also footnote 4).

Now we have for example

$$\models^{\mathcal{ALC}} \top(a) \text{ and } \not\models^{\mathcal{ALC}} \bot(a)$$

for every individual a.

There are other description logics than $\mathcal{ALC}$. The expressiveness of $\mathcal{ALC}$ can for example be enhanced by adding *role constructions* to the language (such that we not only have primitive roles but also role descriptions). Role conjunction $r_1 \sqcap r_2$ is defined as

$$(r_1 \sqcap r_2)^{\mathcal{I}} = r_1^{\mathcal{I}} \cap r_2^{\mathcal{I}}.$$

Role complement $\neg r$ is defined as

$$(\neg r)^{\mathcal{I}} = U \times U \backslash (r^{\mathcal{I}}).$$

Then we have for example

$$\not\models^{\mathcal{ALC}} (r \sqcap \neg r)(a,b) \text{ and } \models^{\mathcal{ALC}} \forall (r \sqcap \neg r).c \equiv \top$$

for every (a,b) and every c.

There are many aspects of description logics which we have not considered here (because they are not important for the present work). We have for example not considered how to construct algorithms that can perform deductions, nor have we considered complexity issues (entailment deduction of terminological axioms in $\mathcal{ALC}$ is ExpTime complete [Donini, 2003]).[5]

4.3 Extensionality Results

Theorem 9 *The description logic $\mathcal{ALC}$ is extensional.*

Proof. Let c, d, e, f be concept description, $\mathcal{T}$ a set of terminological axioms, concept assertions and role assertions, and let $(e \equiv f)[d/c]$ be the result of substituting an occurrence of c with d in $e \equiv f$. Then we have to show, for all $c, d, e, f, \mathcal{T}$:[6]

$$\text{whenever } \mathcal{T} \models^{\mathcal{ALC}} c \equiv d, \text{ then } \mathcal{T} \models^{\mathcal{ALC}} e \equiv f \text{ implies } \mathcal{T} \models^{\mathcal{ALC}} (e \equiv f)[d/c].$$

We proceed by induction on the structure of e. Note, we actually have two cases, one for e and one for f, but as these are similar, we only show the former. Note, if c does not occur in e then $e[d/c]$ simply is e, meaning we get $\mathcal{T} \models^{\mathcal{ALC}} e \equiv f$, which is the assumption. Moreover, if c is identical to e, then $\mathcal{T} \models^{\mathcal{ALC}} (e \equiv f)[d/c]$ becomes $\mathcal{T} \models^{\mathcal{ALC}} d \equiv f$ which follows from the assumption $\mathcal{T} \models^{\mathcal{ALC}} c \equiv d$ and $\mathcal{T} \models^{\mathcal{ALC}} c \equiv f$. These two obvious cases will not be shown in the following.

The base case when e is a primitive concept. If c does not occur in e then we get the abovementioned obvious case. Otherwise c occurs in e, which means that c and e are identical; this leads as mentioned above also to an obvious case.

For the case $\neg e'$, assume $\mathcal{T} \models^{\mathcal{ALC}} c \equiv d$. Then the induction hypothesis gives that $\mathcal{T} \models^{\mathcal{ALC}} e' \equiv f$ implies $\mathcal{T} \models^{\mathcal{ALC}} e'[d/c] \equiv f$ for any f, that is, for every model $(\cdot)^{\mathcal{I}}$ of $\mathcal{T}$, $(e')^{\mathcal{I}} = (e'[d/c])^{\mathcal{I}}$. Thus, if we assume $\mathcal{T} \models^{\mathcal{ALC}} \neg e' \equiv f$, that is, if we for every model $(\cdot)^{\mathcal{I}}$ of $\mathcal{T}$ assume that $f^{\mathcal{I}} = (\neg e')^{\mathcal{I}} = \complement(e')^{\mathcal{I}}$, then we get $f^{\mathcal{I}} = \complement(e'[d/c])^{\mathcal{I}} = (\neg e'[d/c])^{\mathcal{I}}$. Hence $\mathcal{T} \models^{\mathcal{ALC}} \neg e'[d/c] \equiv f$.

For the case $e_1 \sqcap e_2$, assume $\mathcal{T} \models^{\mathcal{ALC}} c \equiv d$ and $\mathcal{T} \models^{\mathcal{ALC}} e_1 \sqcap e_2 \equiv f$. Then we get, for every model $(\cdot)^{\mathcal{I}}$ of $\mathcal{T}$, $(e_1 \sqcap e_2)^{\mathcal{I}} = e_1^{\mathcal{I}} \cap e_2^{\mathcal{I}} = (e_1[d/c])^{\mathcal{I}} \cap$

[5]ExpTime may sound frightening to some, however, there do exist implemented description logic systems which can deal with large ontologies (more than 100.000 concept introduction axioms) [Haarslev and Möller, 2001].

[6]We also have to show extensionality for the cases when $e \equiv f$ is a concept or role assertion, but as these are easily seen to hold, they will not be shown.

$(e_2[d/c])^{\mathcal{I}} = ((e_1 \sqcap e_2)[d/c])^{\mathcal{I}}$, where the second equality follows from the induction hypotheses. Hence $\mathcal{T} \models^{\mathcal{ALC}} (e_1 \sqcap e_2)[d/c] \equiv f$.

For the case $\forall r.e'$, assume $\mathcal{T} \models^{\mathcal{ALC}} c \equiv d$ and $\mathcal{T} \models^{\mathcal{ALC}} \forall r.e' \equiv f$. Then we get, for every model $(\cdot)^{\mathcal{I}}$ of $\mathcal{T}$, $(\forall r.e')^{\mathcal{I}} = \{x \mid \forall y \, \langle x, y\rangle \in r^{\mathcal{I}} \rightarrow y \in (e')^{\mathcal{I}}\} = \{x \mid \forall y \langle x, y\rangle \in r^{\mathcal{I}} \rightarrow y \in (e'[d/c])^{\mathcal{I}}\} = (\forall r.e'[d/c])^{\mathcal{I}}$, where the second equality follows from the induction hypothesis. Hence $\mathcal{T} \models^{\mathcal{ALC}} \forall r.e'[d/c] \equiv f$. □

Extensionality results can be shown for description logics similar to $\mathcal{ALC}$.

Some authors characterize TBoxes as 'intensional' [Nardi and Brachman, 2003; Calvanese *et al.*, 1998]. In terms of the contributions described in Chapter 2, it should be clear that this is not correct.

There is a close relation between modal logics and description logics. Klaus Schild [1991] has shown this. The relationship is so close that $\mathcal{ALC}$ is said to be "a notational variant of modal logic $\mathbf{K}_m$".[7] We will show the close relationship by showing how $\mathcal{ALC}$ formulas are translated to equivalent $\mathbf{K}_m$ formulas. The motivation for showing this is that the relation between $\mathbf{K}_m$ and $\mathcal{ALC}$ is important for our intensionality results.

Let us without loss of generality consider the atomic concepts to be propositional variables and the modalities to be indexed by the set of primitive roles R. Then we can translate a concept description c into a $\mathbf{K}_m$ formula c' by:

$$(c \sqcap d)' = c' \wedge d'$$
$$(\neg c)' = \neg c'$$
$$(\forall r.c)' = \Box_r c'$$

We can also translate $\mathcal{ALC}$ interpretations to Kripke models. This can be done as follows. Let $(\cdot)^{\mathcal{I}} : C \rightarrow 2^U$ and $(\cdot)^{\mathcal{I}} : R \rightarrow 2^{U \times U}$ be an $\mathcal{ALC}$ interpretation (recall, interpretations consist of two mappings although we often identify them with their first component). Then the associated Kripke model is $(U, (r^{\mathcal{I}})_{r \in R}, (\cdot)^{\mathcal{I}} : C \rightarrow 2^U)$. Thus an individual of U is considered to be a possible world, the accessibility relations become the interpretations of the roles, and the valuation becomes the interpretation of concepts. It is easy to see that both of these translations are bijective.

Let $c \in T_C$ be a concept description. It maps by the above translation to a $\mathbf{K}_m$ formula ϕ. And let $(\cdot)^{\mathcal{I}} : C \rightarrow 2^U$ be an $\mathcal{ALC}$ interpretation. Then we have the equivalence between satisfaction in $\mathcal{ALC}$ and in $\mathbf{K}_m$, for every i:

$$i \in c^{\mathcal{I}} \text{ if and only if } (U, (r^{\mathcal{I}})_{r \in R}, (\cdot)^{\mathcal{I}}), i \Vdash \phi.$$

[7] Modal logic $\mathbf{K}_m$ is simply a modal logic with m pairs of modalities each axiomatized as modal logic $\mathbf{K}$, that is, $\Box_i(\phi \rightarrow \psi) \rightarrow (\Box_i \phi \rightarrow \Box_i \psi)$. Thus $\mathbf{K}_1$ is simply modal logic $\mathbf{K}$.

This is proven by induction on c, consider the case $\forall r.d$ which translates to $\Box_r \psi$. We have

$$i \in (\forall r.d)^{\mathcal{I}} \Leftrightarrow \forall y \in U(\langle i, y\rangle \in r^{\mathcal{I}} \rightarrow y \in d^{\mathcal{I}})$$

by the definition of satisfaction in $\mathcal{ALC}$. This translates to the equivalent

$$(U, (r^{\mathcal{I}})_{r \in R}, (\cdot)^{\mathcal{I}}), i \Vdash \Box_r \psi \Leftrightarrow \forall y \in U(\langle i, y\rangle \in r^{\mathcal{I}} \rightarrow (U, (r^{\mathcal{I}})_{r \in R}, (\cdot)^{\mathcal{I}}), y \Vdash \psi)$$

by the Kripke semantics for necessity. This shows that $\mathcal{ALC}$ concept descriptions and $\mathbf{K}_m$ formulas are notational variants. (But it does, strictly speaking, not show that $\mathcal{ALC}$ and $\mathbf{K}_m$ are notational variants.)

Now the interesting question arises, how we can say that $\mathcal{ALC}$ is extensional (Theorem 9) when modal logic is intensional (Theorem 5), and $\mathcal{ALC}$ and modal logic are so closely related? The answer is that there is only a global definition of logical entailment in description logics (although one could define a local one, it does not appear to be useful, nor has it been done, as far as we are aware). The extensionality result of $\mathcal{ALC}$ is simply another formulation of the extensionality result for the global entailment relation for modal logic (Theorem 5 item 2). Note that this contributes to the understanding of the relation between modal logic and description logic. Usually one has only considered the relation between concept descriptions of $\mathcal{ALC}$ and $\mathbf{K}_m$ formulas, but there is also a relation with respect to entailment, in which case modal logic in some sense is richer than description logics, as it has more notions of logical consequence.

4.4 Concept Theories

Description logics do not provide the only means for representing and reasoning with concepts. Among the many alternatives, first-order predicate logic and in a few cases some first-order modal logic have been used, see e.g. [McCarthy, 1979; Welty and Guarino, 2001; Palomäki, 1994].

The straightforward way to formalize a concept c in first-order logic is to represent it as unary predicates p_c such that $p_c(t)$ if and only if the individual t falls under c. This provides an intuitive as well as expressive formalization, which for example is used in [Welty and Guarino, 2001; Hayes, 1979]. The subconcept relation is simply formalized by means of implication. For example, $\forall x(\mathit{guitar}(x) \rightarrow \mathit{string\text{-}instrument}(x))$ asserts that *guitar* is a subconcept of *string-instrument*. Moreover, $\mathit{guitarist}(\mathit{John})$ expresses that John is a guitarist. Every TBox and ABox can be represented under this approach, which therefore is more expressive than $\mathcal{ALC}$.

Because first-order predicate logic is extensional, such formalizations are unfortunately extensional (but if one uses first-order modal logic, such formalizations may be intensional). Note that many of the logics we meet in artificial intelligence, like the knowledge interchange format (KIF) [Genesereth and Fikes, 1992; Genesereth, 1991] and conceptual graphs [Sowa, 2000] have the same semantics as first-order predicate logic. Hence similar concept theories formulated in these languages are extensional too.

It should now be clear that the prevalent formalizations of concepts are extensional.

Alternatively, more general kinds of concept theories have been put forward. McCarthy suggests in [1977; 1979] that concepts should be represented as objects, i.e. as constants and variables, in first-order predicate logic (note that McCarthy only considers individual concepts, that is, concepts with a single member). Subsumption could then be formalized by means of a binary predicate *is-a*, meaning that the above example becomes

$$\textit{is-a}(\textit{guitar}, \textit{string-instrument}).$$

For distinguishing between whether a constant is a concept or an individual we could use predicates, and to express that John is a guitarist, we could use an instance relation. Such theories may be intensional. However, because of the poor computational properties of predicate logic, such theories are more useful for illustrating particular aspects of concepts, for example, showing modal aspects of concepts like [McCarthy, 1979], rather than formulating particular ontologies. Moreover, it is also a problem that these formulations often do not provide a condition for identifying concepts; they simply remove the extensional condition that concepts with the same members are identified and present no alternative.

Finally, it should be noted that there (of course) are other notions of concepts and accordingly completely different concept theories. Formal concept analysis [Ganter and Wille, 1999] presents an interesting theory which as shown in Section 3.6 allows both extensional and intensional concept formalization (or *mathematization* as it is called). However, because of its nice mathematical properties, there are far more results for the extensional formalization. As this theory has arisen from a different tradition (lattice theory) than ours (symbolic logic), and is founded on a different setting (so-called *formal contexts*) than ours, it will take us too far afield to go into details about this (see also Section 3.6).

Peter Gärdenfors has proposed a concept theory [2000] based on a geometrical notion of conceptual structures. Prototype theory, see e.g. [Hampton, 1993], presents another principle where a concept is defined by its prototypical members, that is, the members which exemplify the concept particularly well. This means that an individual is a member of a concept up to a certain degree, instead of being either a member or not (as we assume). It should be noted that the two latter proposals are informal.

The above investigations show that the currently used approaches do not provide a viable means for formalization of conceptual knowledge. This will be remedied below when we present a decidable, intensional concept logic.

Chapter 5

Intensional Concept Logic

This chapter concerns the intensional concept logic. After a motivation, the second section provides a formal definition of the logic. The following three sections present different kinds of variants of the logic, and finally different applications are described.

5.1 Motivation

We follow, as noted earlier, the tradition in knowledge representation and artificial intelligence. This does unfortunately not mean that it is precisely established what concepts are. On the contrary—most authors presuppose the notion of a concept, although one has not been precisely established (maybe because it appears intuitive what concepts are). Nevertheless, we will briefly touch upon this subject, because it is important for how concepts should be represented.

Our notion of a concept is similar to Carnap's in the sense that our concepts are names for what Carnap calls concepts, with the exception that we consider concepts to be unary—roles capture the binary case.[1] This notion of a concept seems to be in accordance with the tradition in knowledge representation, in particular the more philosophically rooted tradition followed by people like John McCarthy [1977; 1979] and William Woods [1991;

[1]It would probably be more appropriate if we called concepts for *concept names* but this is not common in knowledge representation.

1975].[2]

The set of members of a concept is called its *extension*. Concepts are not necessarily defined by their extension, meaning that distinct concepts with the same extension exist. In the introduction we mentioned the concepts *creature with a heart* and *creature with a kidney* as examples of distinct co-extensional concepts. As another often used example we have *mermaid* and *unicorn* which have no members (or at least no actual members), and therefore are vacuously co-extensional. Moreover, *naturally featherless biped*, *rational animal*, and *human* constitute another example. And *equilateral triangle* and *equiangular triangle* constitute an infamous and debatable example. These examples may appear somewhat speculative. In practice, where one often considers a restricted application domain and not all (possible) states of our universe, there are many other relevant cases in which we would like to assert co-extensionality although co-extensionality may be falsified in general. When modeling hospitals, for example, we may assert that *operation* is a *surgical procedure*, i.e. that *operation* and *operation* $\sqcap$ *surgical procedure* are co-extensional, and that every patient (and only patients) has a medical record, i.e. that *patient* and $\exists$*has.medical-record* are co-extensional.[3]

In other words, concepts are non-extensional (intensional). This is in accordance with many papers in knowledge representation and artificial intelligence, cf. [McCarthy, 1977; 1979; Woods, 1975; 1991; Brachman, 1979; Nilsson and Palomäki, 1998; Boman *et al.*, 1997].[4]

Now, an important question arises, if the extension does not define a concept, then what does? The contributions described in Chapter 2, provide an answer, and following Carnap, we say that the *intension* defines a concept such that concepts with the same intension are identified. Note that we use 'intension' more generally than the possible-world tradition. This answer is not without problems, for as already noted there does not seem to be a gen-

[2]Concepts should not be confused with ideas or thoughts. Concepts are related to ideas and thoughts as described in Section 2.1, however, concepts are linguistic entities of formal languages and not mental entities. This should also explain why we talk about the intension of a concept and compare it to senses. Recall, senses are associated with names, and the difference between senses and intensions is that, whereas we have senses of senses, there are no distinct intensions of intensions.

[3]This shows a more pragmatic motivation for considering non-extensional concept formalization, for in practice we often assert that concepts, which we clearly consider to be different concepts, are co-extensional.

[4]The only work we have seen which clearly states the opposite i.e. that concepts (here called *classes*) are extensional is [Goodman and Quine, 1947]. However, since then it has been established that intensional logic (at least the ones we consider) can be defined in extensional settings, so we do not consider this to be incoherent with the present work.

erally acknowledged answer to what senses and intensions are.[5] On the other hand—as we see the situation—this does not prevent formalization of intensions, it merely means that there are different formalizations of intensions, and in the following sections we present some alternatives which are coherent with the contributions described in Chapter 2.

In terms of extensional formalizations, in contrast to intensional formalizations, we formally identify co-extensional concepts although we can and may want to discern between them. This explains why we are interested in intensional formalizations of concepts and intensional concept logics.

Concepts are related to properties as they are construed in *property theory* [Bealer and Mönnich, 1989; Jubien, 1989; Menzel, 1986; Swoyer, 1998; Turner, 1987; Weingartner, 1974], since both concepts and properties are said to be intensional. However, 'property' appears to be used more generally than 'concept', and foundational issues are often the motivation for property theories, instead of knowledge representation.

It should be noted that some papers concerning intensional representation and intensional semantics of concepts exist, see [Maida and Shapiro, 1982; Cappelli and Mazzeranghi, 1994]. We do not find these papers fully satisfying. Amongst other things, the former is almost informal, whereas the latter adopts the semantics of data types in programming languages. Nonetheless, the papers show that intensionality has been combined with knowledge representation before.

5.2 Defining the Intensional Concept Logic

5.2.1 An Appetizer

The intensional concept logic is based upon Carnap's distinction between extension and intension. We therefore recognize two kinds of conceptual knowledge: extensional and intensional. A concept logic should facilitate expressions of both, and our logic allows one to express

- extensional relations between concepts; more precisely, the logic has a connective for expressing that concepts have the same extension;
- intensional relations between concepts; more precisely, the logic has a connective for expressing that concepts have the same intension.

[5] When we say that concepts are defined by their intensions, it means, from a formal point of view, merely that we use a more restrictive condition for identifying concepts than simply assuming that co-extensional concepts are identical.

To express that two concepts c and d have the same extension, i.e. are co-extensional, we shall write

$$c \equiv d.$$

This relation will actually be similar to the $\equiv$ relation of $\mathcal{ALC}$ in that they have the same semantics.

To express that two concepts c and d are co-intensional, we shall write

$$c = d.$$

By means of these two kinds of expressions we can capture the infamous example regarding the relations between the concepts *featherless biped*, *rational animal* and *human* which are assumed to be co-extensional but not co-intensional:

$$\begin{aligned} \textit{human} &\equiv \textit{featherless biped}, \\ \textit{human} &= \textit{rational animal}. \end{aligned}$$

The former equation expresses that *human* and *featherless biped* are co-extensional. The latter equation expresses that *rational animal* and *human* are co-intensional, meaning that *human* is defined as *rational animal*. The two equivalence relations are related such that whenever we have $c = d$ then $c \equiv d$ (in correspondence with the principle that intension determines extension), thus we have that *human* and *rational animal* are co-extensional.

We also have an *extensional subsumption* relation, $c \sqsubseteq d$, which expresses that every individual which belongs to the extension of c also belongs to the extension of d. And an *intensional subsumption* relation, $c \leq d$, which expresses that c is intensionally included in d. Now,

$$\begin{aligned} \textit{human} &\sqsubseteq \textit{biped}, \\ \textit{human} &\leq \textit{animal}, \end{aligned}$$

express that every human is a biped, and that *human* is an intensional subconcept of *animal*. (We explain more about this later.)

If we compare our two notions of equivalence with Carnap's notion of equivalence and L-equivalence, we get that extensional equivalence ($\equiv$) is related to Carnap's equivalence, and that intensional equivalence ($=$) is related to L-equivalence. There is one important difference, though, our notions of equivalence are not dependent on extra-linguistic knowledge, since the semantics of the logic will be formally defined.

It should be noted that we (deliberately) have not (yet) said anything about what the intension of a concept is. The logic consists merely of a connective for expressing "sameness" and subsumption of intensions, and not for expressing what intensions *are*.

5.2.2 Syntax

The syntax of the logic is closely similar to the syntax of the description logic $\mathcal{ALC}$. Besides the $=$ relation, the two logics will have the same syntax as we can see from the following two definitions.

Definition 10 *Let C be a set of atomic concepts and let R be a set of atomic roles. The set of concept descriptions T_C is defined as the least set satisfying*

1. *Every atomic concept, including the* universal concept $\top$*, is a concept description.*
2. *If c and d are concept descriptions then the* concept conjunction $c \sqcap d$ *is a concept description.*
3. *If d is a concept description then the* concept negation $\neg d$ *is a concept description.*
4. *If d is a concept description and r is an atomic role, then the* value restriction $\forall r.d$ *is a concept description.*

We use the same abbreviations as $\mathcal{ALC}$, so the *bottom concept* $\bot$ stands for $\neg\top$, *concept disjunction* $c \sqcup d$ stands for $\neg(\neg c \sqcap \neg d)$, and *existential quantification* $\exists r.c$ stands for $\neg\forall r.(\neg c)$.

Then we can define the different kinds of concept axioms.

Definition 11 *Let $c, d \in T_C$ be concept descriptions, then*

1. $c \equiv d$ *is a* concept equivalence *between c and d;*
2. $c = d$ *is a* concept identity *between c and d;*
3. $c \sqsubseteq d$ *is an* extensional subsumption relation *between c and d;*
4. $c \leq d$ *is an* intensional subsumption relation *between c and d.*

Concept axiom *is used as common term for concept equivalence, concept identity, extensional subsumption relation, or intensional subsumption relation. In order to have a common notation, let*

$$c \cong d$$

denote either a concept equivalence or a concept identity.

5.2.3 Extensional Semantics

The following extensional semantics is an algebraic formulation of the semantics of $\mathcal{ALC}$.

Definition 12 *Let $\mathcal{T}$ be a set of concept axioms and let U be a universe of discourse. An* extensional model *of $\mathcal{T}$ is a mapping $\varepsilon : C \rightarrow 2^U$ (as well as a mapping $\varepsilon : R \rightarrow 2^{U \times U}$) which extends to a homomorphism $\tilde{\varepsilon} : \mathbf{T}_C \rightarrow \mathbf{2}^U$ by the following definition:*

$$\begin{aligned} \tilde{\varepsilon}(\top) &= U \\ \tilde{\varepsilon}(c \sqcap d) &= \tilde{\varepsilon}(c) \cap \tilde{\varepsilon}(d) \\ \tilde{\varepsilon}(\neg c) &= \complement \tilde{\varepsilon}(c) = U \backslash \tilde{\varepsilon}(c) \\ \tilde{\varepsilon}(\forall r.c) &= \{x \in U \mid (\forall y \in U)\ \langle x, y \rangle \in \varepsilon(r) \rightarrow y \in \tilde{\varepsilon}(c)\} \end{aligned}$$

for all $c, d \in T_C$ and $r \in R$, and which satisfies *the axioms of $\mathcal{T}$, that is, for all $c \equiv d \in \mathcal{T}$ we have $\tilde{\varepsilon}(c) = \tilde{\varepsilon}(d)$, and for all $c = d \in \mathcal{T}$ we have $\tilde{\varepsilon}(c) = \tilde{\varepsilon}(d)$, and for all $c \sqsubseteq d \in \mathcal{T}$ we have $\tilde{\varepsilon}(c) \subseteq \tilde{\varepsilon}(d)$, and for all $c \leq d \in \mathcal{T}$ we have $\tilde{\varepsilon}(c) \subseteq \tilde{\varepsilon}(d)$.*

We say that $\mathcal{T}$ entails *$c \equiv d$ and write*

$$\mathcal{T} \models c \equiv d$$

if for all extensional models ε of $\mathcal{T}$ we have $\tilde{\varepsilon}(c) = \tilde{\varepsilon}(d)$. Moreover, we say that $\mathcal{T}$ entails *$c \sqsubseteq d$ and write*

$$\mathcal{T} \models c \sqsubseteq d$$

if for all extensional models ε of $\mathcal{T}$ we have $\tilde{\varepsilon}(c) \subseteq \tilde{\varepsilon}(d)$.

If $\mathcal{T}$ does *not* entail $c \equiv d$, we write $\mathcal{T} \not\models c \equiv d$.

This definition should be compared to the definition of the semantics of the description logic $\mathcal{ALC}$. There is no essential difference between the two (except that we (for now) do not consider ABoxes). This means that the extensional semantics of the intensional concept logic is the same as the semantics of $\mathcal{ALC}$, hence if $\mathcal{T}$ consists only of concept equivalences and extensional subsumption relations then $\mathcal{T} \models^{\mathcal{ALC}} c \equiv d$ if and only if $\mathcal{T} \models c \equiv d$ for all $c, d \in T_C$.[6]

Under the extensional semantics there is no difference between extensional concept axioms ($\equiv, \sqsubseteq$) and intensional concept axioms ($=, \leq$), in the sense that $c \equiv d$ is satisfied by ε if and only if $c = d$ is satisfied by ε and similarly for subsumptions. Under the intensional semantics, we will (of course) discern between extensional and intensional axioms. Later it will become apparent that this (together with the remaining definitions) means that concept identity implies concept equivalence.

The codomain of an extensional interpretation is a power set algebra, which satisfies $X = X \cap Y$ if and only if $X \subseteq Y$ for all $X, Y \subseteq U$. This means that we get $c \sqsubseteq d$ is satisfied by ε if and only if $c \equiv c \sqcap d$ is satisfied by ε for all concept descriptions c and d. From a semantical point of view, the extensional subsumption relation $\sqsubseteq$ is therefore superfluous since it can be expressed by means of extensional equivalence.

The fact that description logics may be given an algebraic semantics is not new, see [Brink and Schmidt, 1992; Brink *et al.*, 1994], and [Blackburn *et al.*, 2001] which presents the algebraic semantics of the related modal logics. The idea of following an algebraic approach to knowledge representation was brought to my attention by Prof. Jørgen Fischer Nilsson.

5.2.4 Intensional Semantics

It is useful to go into more details about the algebraic formulation of the extensional semantics. The intensional semantics is namely an algebraic generalization of the extensional semantics.

The syntactic domain, that is, the set T_C of concept descriptions, may be considered an algebra $(T_C, \sqcap, \neg, (\forall r.(\cdot))_{r \in R}, \top)$. This algebra includes a whole family $(\forall r.(\cdot))_{r \in R}$ of operations, one for each atomic role. The codomain of a given extensional model $\varepsilon : C \to 2^U$ is first of all a Boolean algebra $(2^U, \cap, \complement, U)$, but it is endowed with additional structure due to the value restrictions. A family of operations $(f_r^{\cap})_{r \in R}$ exists, each corresponding to the interpretation of an atomic role (note, $\forall r.(\cdot)$ is a piece of syntax, $f_r^{\cap}$ is a corresponding operator). Each mapping $f_r^{\cap} : 2^U \to 2^U$ of this family is

[6] The reason for this difference is that this chapter follows an algebraic approach to logic where definitions like the one above are customary, in contrast to the more common approach followed by description logicians.

defined by

$$f_r^{\cap}(Z) = \{x \in U \mid (\forall y \in U)\langle x, y\rangle \in \varepsilon(r) \rightarrow y \in Z\}$$

given an extensional model ε. The codomain of an extensional model is therefore an algebra $(2^U, \cap, \complement, (f_r^{\cap})_{r\in R}, U)$, and such an algebra is a concrete example of a *Boolean algebra with operators* [Jónsson and Tarski, 1951]; it is also known as a *complex algebra*.[7] An extensional model ε is a mapping from C to a complex algebra:

$$\varepsilon : C \longrightarrow (2^U, \cap, \complement, (f_r^{\cap})_{r\in R}, U).$$

Moreover, it extends by definition to a homomorphism:

$$\tilde{\varepsilon} : (T_C, \sqcap, \neg, (\forall r.(\cdot))_{r\in R}, \top) \longrightarrow (2^U, \cap, \complement, (f_r^{\cap})_{r\in R}, U).$$

The intensional semantics works in principle the same way. An intensional model is also (more precisely, it *consists of*) a mapping from C which extends to a homomorphism, however, there is one important difference, the codomain of the mapping is not a complex algebra. Instead, we are inspired by the property theory of George Bealer, see [Bealer, 1982; Bealer and Mönnich, 1989] and the related [Menzel, 1986; Swoyer, 1998].[8] The underlying idea is that the intensional semantics arises through interpretation over weaker structures, that is, over algebras satisfying fewer identities. In our case these algebras are called *intensional algebras*, and they are simply semilattices endowed with additional structure due to value restrictions.

To contrast the algebras of the extensional semantics with the intensional algebras, we will simply call these algebras for *extensional algebras*.

Definition 13 *An* intensional algebra *is a tuple* $(I, \times, \sim, (f_r^{\times})_{r\in R}, 1_I)$ *consisting of a non-empty set* I*, a binary operation* $\times$ *on* I*, a unary operation* $\sim$ *on* I*, a constant* $1_I \in I$*, and an* R*-indexed family of mappings on* I *which*

[7] Our operators are structure preserving with respect to $\cap$, meaning they are dual of the ones in [Jónsson and Tarski, 1951].

[8] Our approach is, amongst other things, distinguished from Bealer's because we have both extensional and intensional interpretations, commutativity, and (later) an infinite hierarchy of senses. On the other hand, we consider more simple logics than Bealer who considers first-order logic.

satisfies

$$x \times x = x$$
$$x \times y = y \times x$$
$$x \times (y \times z) = (x \times y) \times z$$
$$x \times 1_I = x$$
$$f_r^{\times}(x \times y) = f_r^{\times}(x) \times f_r^{\times}(y)$$
$$f_r^{\times}(1_I) = 1_I$$

for all $x, y, z \in I$ *and* $r \in R$.

We may give the intensional algebras the following informal interpretation. Intensional algebras are formalizations of intensions of concepts, meaning the operations $\times, \sim, f_r^{\times}$, and 1_I are operations on intensions. The axioms simply state that *intensional conjunction* $\times$ is idempotent, commutative, and associative. It has been philosophically justified in [Swoyer, 1998] that conjunction has these algebraic properties.

We had several considerations regarding the axiomatization of negation. In an earlier paper about this work [Oldager, 2003], *intensional negation* was axiomatized as involution, viz. $\sim\sim x = x$, following a suggestion of [Swoyer, 1998]. Alternatively, intensional negation could be defined such that it is order reversing, that is, as $\sim(x \times y) \times (\sim x) = \sim x$. However, as there are other viable axiomatizations of negation, and as we wanted the intensional semantics to be as general as possible (under a certain assumption which is to be described later), we have simply dropped the axiom, such that there are no axioms for intensional negation. When we in Section 5.3 discuss that different axiomatizations of the intensional algebras give rise to different *conceptions of intensionality*, this choice should be motivated.

With this axiomatization the conjunction of an intension with its negation does not (necessarily) yield the least intension, although this is the case in the extensional algebras where $x \cap \complement x = \emptyset$, because we are not guaranteed existence of a least element.

As we shall see shortly, the axiom $x \times 1_I = x$ means that 1_I is the greatest intension. It may be considered a neutral intension with respect to conjunction. Moreover, since the roles preserve the structure of their arguments, they are endomorphisms on I. We find this intuitively acceptable, unfortunately, we have not found any justification for this in the philosophical literature. (This should not be taken to mean that the opposite has been justified, but merely that value restrictions, as far as we know, have not been discussed (yet) in the philosophical literature.)

Before we present the intensional semantics the following relation needs to be introduced. Let $(I, \times, \sim, (f_r^\times)_{r\in R}, 1_I)$ be an intensional algebra. We can define a partial order $\preceq$ on I by

$$x \preceq y \text{ if and only if } x = x \times y.$$

It can be shown that $\preceq$ is an order-theoretic semilattice, that is, a partial order with binary infima (greatest lower bounds). Conversely, given a partial order $\preceq'$ on I with binary infima we can define a binary operation $\times'$ on I by $x \times' y$ is equal to the infimum of x and y. It can also be shown that $(I, \times')$ is a semilattice, and that the definitions are the inverses of each other such that $\preceq = \preceq'$ if and only if $\times = \times'$. The proofs can be found in most books on lattice theory, see for instance [Davey and Priestley, 1990].

In other words, there is a one-to-one correspondence between the partial orders with binary infima and semilattices. Semilattices are the most general algebras with this property as any other axiomatization either would be less general than the suggested axiomatization (satisfy more axioms) or would not have the correspondence between conjunction and subsumption.

The partial order $\preceq$ will be used for formalizing intensional subsumption relations between concepts. In other words, $\preceq$ formalizes an intensional taxonomy. It is therefore no coincidence that we have chosen semilattices as intensional algebras, since we then have the most general algebras where there is a correspondence between concept conjunction and concept subsumption, both of which are fundamental in knowledge representation.

The definition below presents the intensional semantics of the concept logic. It should be noted that in the following the algebras will not be concrete (set-theoretic) but abstract.

Definition 14 *Let $\mathcal{T}$ be a set of concept axioms, $(I, \times, \sim, (f_r^\times)_{r\in R}, 1_I)$ an intensional algebra, and $(E, \wedge, \neg, (f_r^\wedge)_{r\in R}, 1_E)$ an extensional algebra. An* intensional model *of $\mathcal{T}$ is a pair (ι, τ) of mappings $\iota : C \to I$ and $\tau : I \to E$ such that $\tau \circ \iota$ is an extensional model of $\mathcal{T}$, and ι extends to a homomorphism $\tilde{\iota} : \mathbf{T}_C \to \mathbf{I}$ by the following definition:*

$$\begin{aligned}
\tilde{\iota}(\top) &= 1_I \\
\tilde{\iota}(c \sqcap d) &= \tilde{\iota}(c) \times \tilde{\iota}(d) \\
\tilde{\iota}(\neg c) &= \sim \tilde{\iota}(c) \\
\tilde{\iota}(\forall r.c) &= f_r^\times(\tilde{\iota}(c))
\end{aligned}$$

for all $c, d \in T_C$ *and* $r \in R$*, and which* satisfies *the intensional axioms of* $\mathcal{T}$*, that is, for all* $c = d \in \mathcal{T}$ *we have* $\tilde{\iota}(c) = \tilde{\iota}(d)$*; and for all* $c \leq d \in \mathcal{T}$ *we have* $\tilde{\iota}(c) \preceq \tilde{\iota}(d)$*.*

We say that $\mathcal{T}$ entails $c = d$ *and write*

$$\mathcal{T} \models c = d$$

if for all intensional models (ι, τ) *of* $\mathcal{T}$ *we have* $\tilde{\iota}(c) = \tilde{\iota}(d)$*. Moreover, we say that* $\mathcal{T}$ entails $c \leq d$ *and write*

$$\mathcal{T} \models c \leq d$$

if for all intensional models (ι, τ) *of* $\mathcal{T}$ *we have* $\tilde{\iota}(c) \preceq \tilde{\iota}(d)$*.*

If $\mathcal{T}$ does not entail $c = d$ we write $\mathcal{T} \not\models c = d$.

Similar to extensional subsumption, the intensional subsumption relation may be dispensed since it can be expressed by means of concept conjunction and concept identity, i.e. $c \leq d$ is satisfied if and only if $c = c \sqcap d$ is satisfied, as argued above. For the next sections (until we consider applications), we restrict our investigations to concept identities and concept equivalences.

5.2.5 Verification of Intensionality

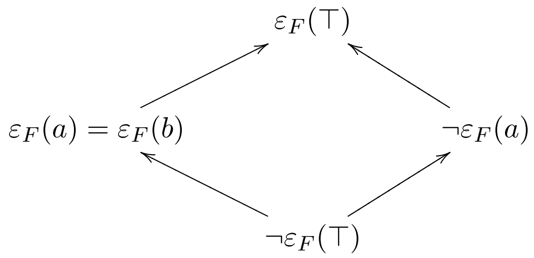
The following shows the intensionality result, which verifies that the concept logic is intensional.

Theorem 15 *The intensional concept logic is intensional.*

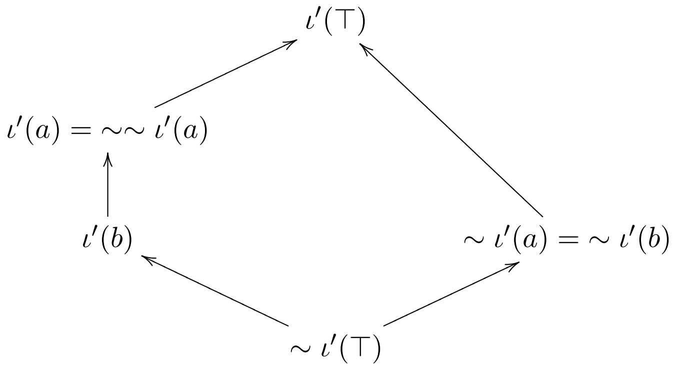
Proof. We show by means of a counter example where $a, b \in C$ that

$$a \equiv b \not\models a = b,$$

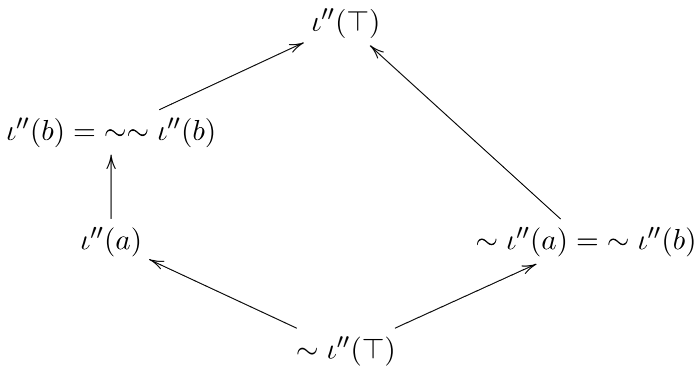
which shows the wanted as $a \equiv b \models a = a$ holds. The free extensional model for $a \equiv b$ is



Instead of considering the free intensional model generated by $\{a, b\}$, which is infinitely large, we construct two intensional models (ι', τ') and (ι'', τ'') as follows:



where $\tau'(\iota'(a)) = \tau'(\iota'(b)) = \varepsilon_F(a)$, $\tau'(\sim \iota'(a)) = \neg\varepsilon_F(a)$; and



where $\tau''(\iota''(a)) = \tau''(\iota''(b)) = \varepsilon_F(a)$ and $\tau''(\sim \iota''(a)) = \neg\varepsilon_F(a)$. We verify that (ι', τ') and (ι'', τ'') are intensional models for $a \equiv b$ in that the codomains of ι', ι'' are semilattices, and $\tau' \circ \iota'$ and $\tau'' \circ \iota''$ are extensional models. We have $\iota'(a) \prec \iota'(b)$ and $\iota''(b) \prec \iota''(a)$, thus $a = b$ is not satisfied in these models.[9] □

As another result we have that co-intensional formulas may be freely substituted for each other. This means that the intensional concept logic satisfies Carnap's definition of intensionality where co-intensional expressions are substitutable.

Proposition 16 *Let $c, d, e \in T_C$ be concept descriptions, then we have*

$$c = d \models e = e[d/c]$$

[9]The two intensional models show that neither $\mathcal{T} \models a \leq b$ nor $\mathcal{T} \models b \leq a$.

and

$$c = d \models e \equiv e[d/c]$$

for all c, d, e, *where* $e[d/c]$ *denotes the result of substituting an occurrence of* c *with* d *in* e.

Proof. The proofs proceed by structural induction on the concept description e. (We need not show the full proofs as argued in connection with the algebraic extensionality result for modal logic in Chapter 3.) □

As a corollary of this proposition, we get that concept identity implies concept equivalence, that is,

$$c = d \models c \equiv d \tag{5.1}$$

for all $c, d \in T_C$.

Many other results may be derived. We have for example

$$c \equiv d, d = e \models c \equiv e$$
$$c \equiv d, d = e \not\models c = e$$

for all $c, d, e \in T_C$.

5.2.6 Relations between the Extensional and Intensional Semantics

As shown above, the extensional and the intensional semantics are related (as we would expect). Now we examine this relation more closely.

One of the aims of this section is to show the relation between the algebraic semantics and the informal discussions about intensionality in Chapter 2, hence motivate that the algebraic semantics is in accordance with the contributions.

The proposition below shows that the intensional semantics subsumes the extensional in the sense that every extensional model gives rise to an intensional model. Note that the extensional semantics is equivalent to modal logic $\mathbf{K}_m$, hence the result shows that the intensional semantics subsumes this logic.

Proposition 17 *Let* $\varepsilon : C \to 2^U$ *be an extensional model of a set* $\mathcal{T}$ *of concept axioms and let* id_{2^U} *be the identity mapping on* 2^U. *Then* $(\varepsilon, \mathrm{id}_{2^U})$ *is an intensional model of* $\mathcal{T}$.

Proof. We verify that $(2^U, \cap, \complement, (f_r^{\cap})_{r\in R}, U)$ is an intensional algebra. First, $\cap$ is idempotent, commutative, and associative. Second, U is maximum, and third, each operator $f_r^{\cap} : 2^U \to 2^U$ is structure preserving (an endomorphism):

$$\begin{aligned}
f_r^{\cap}(W \cap Z) &= \{x \in U \mid (\forall y \in U)\langle x, y\rangle \in \varepsilon(r) \to y \in (W \cap Z)\} \\
&= \{x \in U \mid (\forall y \in U)\langle x, y\rangle \in \varepsilon(r) \to y \in W\} \cap \\
&\quad\ \{x \in U \mid (\forall y \in U)\langle x, y\rangle \in \varepsilon(r) \to y \in Z\} \\
&= f_r^{\cap}(W) \cap f_r^{\cap}(Z)
\end{aligned}$$

for all $W, Z \subseteq U$ and $r \in R$;

$$f_r^{\cap}(U) = \{x \in U \mid (\forall y \in U)\langle x, y\rangle \in \varepsilon(r) \to y \in U\} = U.$$

Moreover, every concept identity and every intensional subsumption relation of $\mathcal{T}$ are satisfied by $(\varepsilon, \mathrm{id}_{2^U})$ because ε is an extensional model of $\mathcal{T}$. Finally, we see that $\mathrm{id}_{2^U} \circ \varepsilon = \varepsilon$ is an extensional model of $\mathcal{T}$. □

As a corollary of this proposition we get that

$$\mathcal{T} \models c = d \text{ implies } \mathcal{T} \models c \equiv d, \tag{5.2}$$

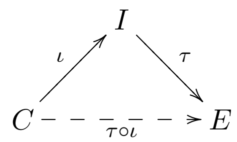
because, by contraposition, if $c \equiv d$ is not satisfied by an extensional model ε of $\mathcal{T}$, then $\tilde{\varepsilon}(c) \neq \tilde{\varepsilon}(d)$ and $(\varepsilon, \mathrm{id})$ is an intensional model of $\mathcal{T}$ meaning there is an intensional model $(\varepsilon, \mathrm{id})$ in which $\tilde{\varepsilon}(c) \neq \tilde{\varepsilon}(d)$. Hence $\mathcal{T} \not\models c = d$. Note that this result is similar to (5.1), however, it is obtained differently.

Similarly, we have the corollary

$$\mathcal{T} \models c \leq d \text{ implies } \mathcal{T} \models c \sqsubseteq d.$$

The contributions to intensionality (see Chapter 2) assert that intension (sense) determines extension (denotation). This relation can be made precise in our setting. Every intensional model (ι, τ) is defined such that $\tau \circ \iota$ is an extensional model. This extensional model is the unique mapping with this property (any other extensional model satisfying the commutativity condition will obviously be equal to $\tau \circ \iota$).

The converse is not true, however. As we saw in the proof of Theorem 15, given an extensional model ε, there may exist several intensional models $(\iota', \tau'), (\iota''\tau'')$ such that commutativity holds, that is, such that $\tau' \circ \iota' = \tau'' \circ \iota'' = \varepsilon$ although $\iota' \neq \iota''$. In other words, an intensional model gives rise to (determines) a unique extensional, however, an extensional model does not give rise to (determines) a unique intensional model. This may be presented by the following commuting diagram



Notice that this diagram is similar to the diagram describing the relations between sense and denotation in Chapter 2.

We may comment further on the relation between the extensional and the intensional semantics. An intensional model is a pair (ι, τ) such that $\tau \circ \iota$ is an extensional model. By means of ι the model formulates the intensions of the concepts, and by means of τ (together with ι) it formulates their extensions; τ is—we say—a *mediation* between the intensions and the extensions. The following proposition shows that τ forms a homomorphism.

Proposition 18 *Let $(\iota : C \to I, \tau : I \to E)$ be an intensional model, let $\iota[C]$ denote the image of ι and let $\mathbf{I}_\iota$ denote the image of $\tilde{\iota}$. Then the restriction $\tau_\iota : \iota[C] \to E$ of τ extends to a homomorphism $\tilde{\tau} : \mathbf{I}_\iota \to \mathbf{E}$ such that $\tilde{\tau} \circ \tilde{\iota} = \widetilde{\tau \circ \iota}$.*

Proof. Let the homomorphic extension $\tilde{\tau}$ of τ_ι be defined as usual. We proceed by induction on the structure of the argument. For the base case:

$$\tilde{\tau} \circ \tilde{\iota}(a) = \tilde{\tau}(\iota(a)) = \tau \circ \iota(a) = \widetilde{\tau \circ \iota}(a)$$

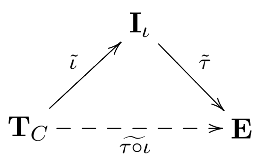
where $a \in C$. For concept conjunctions we get

$$\tilde{\tau} \circ \tilde{\iota}(c \sqcap d) = \tilde{\tau} \circ \tilde{\iota}(c) \wedge \tilde{\tau} \circ \tilde{\iota}(d) = \widetilde{\tau \circ \iota}(c) \wedge \widetilde{\tau \circ \iota}(d) = \widetilde{\tau \circ \iota}(c \sqcap d)$$

where $c, d \in T_C$. The remaining cases are shown similarly. $\square$

It should be noted that τ need not be a homomorphism.[10]

We have therefore shown that the commuting diagram above may be extended to concept descriptions, meaning commutativity also holds for models:



[10] The inconvenience that there may be a difference between τ and $\tilde{\tau}$ is not important. It follows merely from the fact that we wanted the definition of an intensional model to look nice.

The fact that $\tau : I \to E$ forms a homomorphism means that we can define $\tilde{\tau}$ set-theoretically because the extensional algebra $\mathbf{E}$ is a complex algebra $(2^U, \cap, \complement, (f_r^{\cap})_{r \in R}, U)$:

$$
\begin{aligned}
\tilde{\tau}(1_I) &= U \\
\tilde{\tau}(x \times y) &= \tilde{\tau}(x) \cap \tilde{\tau}(y) \\
\tilde{\tau}(\sim x) &= \complement(\tilde{\tau}(x)) = U \backslash \tilde{\tau}(x) \\
\tilde{\tau}(f_r^{\times}(x)) &= f_r^{\cap}(\tilde{\tau}(x)) \\
&= \{x \in U \mid (\forall y \in U)\langle x, y\rangle \in \tilde{\tau}(r) \to y \in \tilde{\tau}(x)\}
\end{aligned}
$$

5.2.7 Proof Theory

There are several approaches to follow when one wants to present a proof theory. The proof theory of the intensional concept logics follows the approach used in equational logic. This casts further light on the algebraic approach to logic which rarely has been followed in knowledge representation.

Basically, the intensional concept logic comprises two equational logics, one follows from the concept equivalences and the other from the concept identities. Moreover, these logics are united into one, in that the latter implies the former, cf. (5.1). This means that the proof theory to a large extend can follow the approach used in equational logic. Note that axiomatic proof theory is often not considered in description logics. Tableau algorithms, which are less intuitive but more appropriate for implementation, are considered instead.

The results and definitions from equational logic that are not described in details below may be found in [Burris and Sankappanavar, 1981] which is available on the Web.

Concept equivalence is axiomatized as a Boolean algebra with unary operators. We have therefore the following *logical axioms* for equivalence:

$$
\begin{aligned}
c \sqcap d &\equiv d \sqcap c & c \sqcup d &\equiv d \sqcup c \\
c \sqcap (d \sqcap e) &\equiv (c \sqcap d) \sqcap e & c \sqcup (d \sqcup e) &\equiv (c \sqcup d) \sqcup e \\
c \sqcap (d \sqcup e) &\equiv (c \sqcap d) \sqcup (c \sqcap e) & c \sqcup (d \sqcap e) &\equiv (c \sqcup d) \sqcap (c \sqcup e) \\
c \sqcap \top &\equiv c & c \sqcup \bot &\equiv c \\
c \sqcap \neg c &\equiv \bot & c \sqcup \neg c &\equiv \top \\
\forall r.(c \sqcap d) &\equiv \forall r.(c) \sqcap \forall r.(d) & \exists r.(c \sqcup d) &\equiv \exists r.(c) \sqcup \exists r.(d) \\
\forall r.\top &\equiv \top & \exists r.\bot &\equiv \bot
\end{aligned}
$$

for all $c, d, e \in T_C$ and $r \in R$. $c \sqcup d$ is an abbreviation of $\neg(\neg c \sqcap \neg d)$, $\bot$ an abbreviation of $\neg\top$, and $\exists r.c$ an abbreviation of $\neg(\forall r.(\neg c))$ as mentioned earlier.

Concept identity is axiomatized as a bounded semilattice with unary operators (similar to the intensional algebras, Definition 13). Therefore, we have the following *logical axioms* for identity:

$$
\begin{aligned}
c \sqcap c &= c \\
c \sqcap d &= d \sqcap c \\
c \sqcap (d \sqcap e) &= (c \sqcap d) \sqcap e \\
c \sqcap \top &= c \\
\forall r.(c \sqcap d) &= \forall r.(c) \sqcap \forall r.(d) \\
\forall r.\top &= \top
\end{aligned}
$$

for all $c, d, e \in T_C$ and $r \in R$. Note that since the logical axioms hold for all concept descriptions, they are actually axiom schemas.

We have the following *proof rules* for all concepts descriptions c, d, and e

Reflexivity	$\dfrac{}{c \equiv c}$	$\dfrac{}{c = c}$
Symmetry	$\dfrac{c \equiv d}{d \equiv c}$	$\dfrac{c = d}{d = c}$
Transitivity	$\dfrac{c \equiv d \quad d \equiv e}{c \equiv e}$	$\dfrac{c = d \quad d = e}{c = e}$
Replacement	$\dfrac{c \equiv d}{e(c) \equiv e(d)}$	$\dfrac{c = d}{e(c) = e(d)}$
Intensionality	$\dfrac{c = d}{c \equiv d}$	

where $e(c)$ denotes a term e in which c occurs as a subterm, and $e(d)$ denotes the result of replacing the occurrence of c with d in e. All proof rules, but the last one, are well-known in equational logic. The last proof rule is motivated by the fact that concept identity implies concept equivalence, cf. (5.1).

Definition 19 *Let $\mathcal{T}$ be a set of concept axioms. If there exists a finite sequence of concept axioms*

$$c_1 \cong d_1, \ldots, c_n \cong d_n$$

such that $c_n \cong d_n$ is $c \cong d$, and every member of the sequence is a logical axiom, or a member of $\mathcal{T}$, or the consequent of a proof rule where the *antecedent(s) are previous members of the sequence, we say that* $\mathcal{T}$ proves $c \cong d$*, and write*

$$\mathcal{T} \vdash c \cong d.$$

The sequence $c_1 \cong d_1, \ldots, c_n \cong d_n$ *is called a* formal deduction of $c \cong d$*.*

The reader familiar with equational logic may have recognized that a substitution proof rule is missing. The reason for this omission is that the concept logic does not contain any variables. A substitution rule would therefore be "idle".

We *could have* included variables, however, as this is far from being common in concept logics (recall, the special variable-free syntax of $\mathcal{ALC}$), we have not done it. Let us assume that we included variables, then we should have added proof rules for uniform variable substitutions:

$$\text{Substitution}^{11} \quad \frac{c \equiv d}{(c \equiv d)[e/x]} \qquad \frac{c = d}{(c = d)[e/x]}$$

where $(c \equiv d)[e/x]$ denotes the result of substituting all occurrences of the variable x with e in $c \equiv d$, and similarly for $(c = d)[e/x]$. Now we can see that the primitive concepts cannot be variables, because (by means of the Substitution rule) we get, for example,

$$\{a_1 \equiv a_2\} \vdash a_1 \equiv a_3, \ \{a_1 \equiv a_2\} \vdash a_1 \equiv a_4, \ldots$$

where $a_1, a_2, \ldots$ are primitive concepts.

Thus, given a concept axiom $\{a_1 \equiv a_2\}$, we get that all primitive concepts are equivalent, and this is not what is intended by the concept axiom $a_1 \equiv a_2$ (cf. the extensional semantics). Hence the primitive concepts cannot be variables.

Instead, the primitive concepts are constants. This means that the type (algebraically speaking) of the concept logic actually is

$$(\sqcap, \neg, (\forall r.(\cdot))_{r \in R}, \top, a_1, a_2, \ldots)$$

rather than $(\sqcap, \neg, (\forall r.(\cdot))_{r \in R}, \top)$. Moreover, the term algebra is denoted by $\mathbf{T}_C(\emptyset)$ to emphasize that the set of variables is empty. However, this is not important for the results, so we will abstract from these technicalities.

[11] Note that these two proof rules are not part of the proof theory of the intensional concept logic.

5.2.8 Towards Completeness

This section presents definitions necessary for proving completeness. Although closely related, our approach is distinguished from that of equational logic [Burris and Sankappanavar, 1981] by a number of issues: we have two types of identities ($\equiv$ and $=$), we have an additional proof rule (Intensionality), we lack variables and Substitution proof rules, and we show completeness with respect to every set of concept axioms.

Let $\text{Id}(C)$ denote the set of concept axioms (such that all of its concept descriptions belong to T_C).

Let us define the following mappings

$$\begin{aligned}\gamma(S) &= \{(c,d) \mid c \cong d \in S\}\\ \gamma_{\equiv}(S) &= \{(c,d) \mid c \equiv d \in S\}\end{aligned}$$

and let $\gamma_{\equiv}^{-1}$ be the inverse of $\gamma_{\equiv}$. Then $\gamma_{\equiv}^{-1} \circ \gamma$ maps a set of concept axioms to a set of concept equivalences.

Let S be a set of concept equivalences over T_C. The *deductive closure $D(S)$ of S* is the least set of concept equivalences over T_C containing S such that for all $c \in T_C$, $c \equiv c \in D(S)$, and such that

1. if $c \equiv d \in D(S)$ then $d \equiv c \in D(S)$,
2. if $c \equiv d \in D(S)$ and $d \equiv e \in D(S)$ then $c \equiv e \in D(S)$,
3. if $c \equiv d \in D(S)$ and $e(c) \in T_C$ where c occurs as a subterm of e, let $e(d)$ denote the result of replacing the occurrence of c with d in e, then $e(c) \equiv e(d) \in D(S)$.

These properties are called *closure properties*. A set closed under the third property, for example, is said to be *closed under replacement*, moreover, we say that a right hand side of a closure property is *derived from* the left hand side; for example, $d \equiv c$ is derived from $c \equiv d$ under the first closure property.

Later we do not allow use of the Intensionality proof rule. We therefore introduce an additional proof relation $\vdash_E$ which is defined by: $\mathcal{T} \vdash_E c \equiv d$ if $\mathcal{T} \vdash c \equiv d$ and the Intensional proof rule has not been used in the formal deduction of $c \equiv d$. Moreover, we will split a set of concept axioms $\mathcal{T}$ into two sets of concept equivalences by the following definitions

$$\begin{aligned}\mathcal{T}^E &= \{c \equiv d \mid c \equiv d \in \mathcal{T}\}\\ \mathcal{T}^I &= \{c \equiv d \mid c = d \in \mathcal{T}\}.\end{aligned}$$

A *fully invariant congruence* θ on an algebra $\mathbf{A}$ is a *congruence* (an equivalence relation satisfying the compatibility property) such that for all $x, y \in A$ and every endomorphism α on $\mathbf{A}$, $(x, y) \in \theta$ implies $(\alpha x, \alpha y) \in \theta$. Let X be a binary relation over A, then $\Theta_{\mathrm{FI}}(X)$ denotes the *least fully invariant congruence containing* X.

5.2.9 Completeness

Because of the special syntax of the concept logic (which does not include implications hence no general notion of entailment within the logic), we will show more general soundness and completeness results. When formulating knowledge bases, it is of little relevance to know that the logic is sound and complete with respect to empty knowledge bases only. Hence instead of proving $\models c = d$ if and only if $\vdash c = d$, we will prove soundness and completeness with respect to any set of concept axioms.

Now we are ready to show that the intensional concept logic is sound and complete.

Theorem 20 *(Soundness and Completeness.) Let $\mathcal{T}$ be a set of concept axioms and c and d concept descriptions. Then $\mathcal{T} \models c \equiv d$ if and only if $\mathcal{T} \vdash c \equiv d$, and $\mathcal{T} \models c = d$ if and only if $\mathcal{T} \vdash c = d$.*

Proof. We only show the result for concept equivalences, which is the most difficult case because of the additional proof rule (Intensionality). We have to refer to the logical axioms, so let Σ denote the set of logical axioms.

The proof consists of four steps:

1. $\mathcal{T} \vdash c \equiv d$ if and only if $\mathcal{T}^I \cup \mathcal{T}^E \vdash_E c \equiv d$,
2. $\mathcal{T}^I \cup \mathcal{T}^E \vdash_E c \equiv d$ if and only if $c \equiv d \in D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$,
3. $c \equiv d \in D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$ if and only if $(c, d) \in \Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$,
4. $(c, d) \in \Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$ if and only if $\mathcal{T} \models c \equiv d$.

(1. $\Rightarrow$) If $\mathcal{T} \vdash c \equiv d$ then there may be several formal deductions of $c \equiv d$. Among these there exists a formal deduction in which the Intensionality rule is either used only on the concept identities of $\mathcal{T}$ or not used at all. Now we show how to see this. If the Intensionality rule is used at step i in the formal deduction, that is, if we have that $c_i \equiv d_i$ is a consequent of $c_i = d_i$ then there are different cases to consider. If $c_i = d_i$ is a logical axiom or a consequent of the Reflexivity rule, then there is a corresponding extensional

axiom or rule $c_i \equiv d_i$ which we obtain without use of Intensionality. If $c_i = d_i$ is a consequent of another proof rule, say Symmetry, then we have a subsequence $d_i = c_i, c_i = d_i, c_i \equiv d_i$ of the formal deduction, but then there is an alternative subsequence that starts by using the Intensionality rule and then use Symmetry: $d_i = c_i, d_i \equiv c_i, c_i \equiv d_i$. Similar arguments can be made for the remaining proof rules. This presents a method in which the Intensionality rule is used gradually earlier and earlier in the sequence. Since a proof consists of a finite sequence, this method will eventually terminate, such that the Intensionality rule is used only on concept identities of $\mathcal{T}$—if it is used at all.

Now, if the Intensionality rule is not used for establishing $\mathcal{T} \vdash c \equiv d$ then we have $\mathcal{T}^E \vdash_E c \equiv d$, thus also $\mathcal{T}^I \cup \mathcal{T}^E \vdash_E c \equiv d$. Otherwise, if the Intensionality rule is used, then the abovementioned formal deduction exists, which means that $\mathcal{T}^I \cup \mathcal{T}^E$ proves $c \equiv d$, moreover, once the identities of $\mathcal{T}$ are transformed into equivalences, we need not use the Intensionality rule, hence $\mathcal{T}^I \cup \mathcal{T}^E \vdash_E c \equiv d$, which shows the wanted.

(1. $\Leftarrow$) Assume $\mathcal{T}^I \cup \mathcal{T}^E \vdash_E c \equiv d$. Every member of $\mathcal{T}^I$ used in the formal deduction of $c \equiv d$ can be obtained from $\mathcal{T}$ by means of the Intensionality proof rule. Since we for establishing $\mathcal{T} \vdash c \equiv d$ may use the same logical axioms and more proof rules than we use for establishing $\mathcal{T}^I \cup \mathcal{T}^E \vdash_E c \equiv d$, then we also have $\mathcal{T} \vdash c \equiv d$.

(2. $\Rightarrow$) All logical axioms that are concept equivalences and all members of $\mathcal{T}^I \cup \mathcal{T}^E$ are also members of $D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$. Moreover, we have used properties under which the deductive closure is closed in the construction of a formal ($\vdash_E$) deduction.

(2. $\Leftarrow$) By means of the Reflexivity proof rule we can prove $c \equiv c$ for all $c \in T_C$, moreover, every axiom of $\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T})$ can be proven, since we can prove $\mathcal{T}^I \cup \mathcal{T}^E$ and all logical axioms that are equivalences.

If we can prove $c \equiv d$, then there is a formal deduction $c_1 \equiv d_1, \ldots, c_n \equiv d_n$ of $c \equiv d$. By the Symmetry proof rule, we have a sequence

$$c_1 \equiv d_1, \ldots, c_n \equiv d_n, d_n \equiv c_n.$$

Thus we have proved $d \equiv c$.

If we can prove $c \equiv d$ and $d \equiv e$, we have two formal deductions

$$c_1 \equiv d_1, \ldots, c_n \equiv d_n \text{ and } d'_1 \equiv e_1, \ldots, d'_m \equiv e_m.$$

But then we have, by applying the Transitivity rule on the last members of each of the sequences (since d_n is d'_m)

$$c_1 \equiv d_1, \ldots, c_n \equiv d_n, d'_1 \equiv e_1, \ldots, d'_m \equiv e_m, c_n \equiv e_m,$$

which is a formal deduction of $c \equiv e$.

If we can prove $c \equiv d$, there is a formal deduction $c_1 \equiv d_1, \ldots, c_n \equiv d_n$. If $e \in T_C$ contains c as a subterm and $e(d)$ denotes the result of replacing the occurrence of c with d in e, then by the Replacement rule, we have a formal deduction

$$c_1 \equiv d_1, \ldots, c_n \equiv d_n, e(c) \equiv e(d)$$

of $e(c) \equiv e(d)$.

We have therefore shown that

$$c \equiv d \in D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T})) \text{ implies } \mathcal{T}^I \cup \mathcal{T}^E \vdash_E c \equiv d.$$

(3.) By the definition of the deductive closure, we have that for all $c \in T_C$, $c \equiv c \in D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$, which together with the first two closure properties ensure that $\gamma D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$ is an equivalence relation.

By the third closure property we get that $\gamma D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$ is a congruence, because if $c \equiv d \in D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$ then since $D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$ is closed under replacement and $\neg c \equiv \neg c \in D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$, we get $\neg c \equiv \neg d \in D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$. (This only shows the congruence property with respect to negation; the other cases are similar).

$D(S)$ is vacuously closed under uniform variable substitutions, since we admit no variables. And as endomorphisms on the term algebra simply are uniform variable substitutions, $D(S)$ is fully invariant.

Since $\gamma(\Sigma \cup \mathcal{T}) \subseteq \gamma D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$, $\gamma D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$ is a fully invariant congruence containing $\gamma(\Sigma \cup \mathcal{T})$, and since $\Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$ is the least fully invariant congruence containing $\gamma(\Sigma \cup \mathcal{T})$ we get

$$\Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T})) \subseteq \gamma D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T})).$$

Moreover, as $D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T}))$ is the least set satisfying the properties that make it a fully invariant congruence containing $\gamma(\Sigma \cup \mathcal{T})$, we get the wanted converse

$$\gamma D(\gamma_{\equiv}^{-1} \circ \gamma(\Sigma \cup \mathcal{T})) \subseteq \Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T})).$$

(4. $\Rightarrow$) Consider the binary relation $\{(c,d) \mid \mathcal{T} \models c \equiv d\}$. Since

$$\begin{aligned}
&\mathcal{T} \models c \equiv c \\
&\mathcal{T} \models c \equiv d \text{ implies } \mathcal{T} \models d \equiv c \\
&\mathcal{T} \models c \equiv d \text{ and } \mathcal{T} \models d \equiv e \text{ implies } \mathcal{T} \models c \equiv e
\end{aligned}$$

for all $c, d, e \in T_C$, we see that $\{(c,d) \mid \mathcal{T} \models c \equiv d\}$ is an equivalence relation.

We see that $\{(c,d) \mid \mathcal{T} \models c \equiv d\}$ is a congruence, because if

$$(c_1, d_1), (c_2, d_2) \in \{(c,d) \mid \mathcal{T} \models c \equiv d\}$$

then for every extensional model ε of $\mathcal{T}$, $\tilde{\varepsilon}(c_1) = \tilde{\varepsilon}(d_1)$ and $\tilde{\varepsilon}(c_2) = \tilde{\varepsilon}(d_2)$, hence

$$\tilde{\varepsilon}(c_1 \sqcap c_2) = \tilde{\varepsilon}(c_1) \wedge \tilde{\varepsilon}(c_2) = \tilde{\varepsilon}(d_1) \wedge \tilde{\varepsilon}(d_2) = \tilde{\varepsilon}(d_1 \sqcap d_2),$$

i.e. $(c_1 \sqcap c_2, d_1 \sqcap d_2) \in \{(c,d) \mid \mathcal{T} \models c \equiv d\}$. The cases for $\neg$ and $\forall r.(\cdot)$ are similar.

We also see that $\{(c,d) \mid \mathcal{T} \models c \equiv d\}$ is fully invariant, because if α is an endomorphism on $\mathbf{T}_C(\emptyset)$ then, since the primitive concepts are nullary operators and we have no variables, $\alpha(c)$ is vacuously equal to c for every concept description c. Thus if $(c,d) \in \{(c,d) \mid \mathcal{T} \models c \equiv d\}$ then $(\alpha(c), \alpha(d)) \in \{(c,d) \mid \mathcal{T} \models c \equiv d\}$.

Since all logical axioms and concept axioms of $\mathcal{T}$ are entailed by $\mathcal{T}$, we have $(c,d) \in \gamma(\Sigma \cup \mathcal{T})$ implies $(c,d) \in \{(c,d) \mid \mathcal{T} \models c \equiv d\}$. And since $\Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T})$ is the least fully invariant congruence containing $\gamma(\Sigma \cup \mathcal{T})$ we get

$$(c,d) \in \Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T})) \text{ implies } \mathcal{T} \models c \equiv d.$$

(4. $\Leftarrow$) Consider the quotient algebra $\mathbf{T}_C/\Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$. We can define a mapping $\varepsilon_{FI} : C \rightarrow T_C/\Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$ such that a primitive concept is mapped to its equivalence class, i.e., $\varepsilon_{\mathrm{FI}}(a) = [a]_\Theta$.[12] By the usual definition this map extends to a homomorphism $\tilde{\varepsilon}_{\mathrm{FI}} : \mathbf{T}_C \rightarrow \mathbf{T}_C/\Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$ such that for all $c \in T_C$, $\tilde{\varepsilon}_{\mathrm{FI}}(c) = [c]_\Theta$. As $\gamma(\Sigma) \subseteq \Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$, we get that $\mathbf{T}_C/\Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$ is an extensional algebra. We verify, for example, that meet is commutative, that is, for all $c, d \in T_C$:

$$[c]_\Theta \wedge_{\mathbf{T}_C/\Theta} [d]_\Theta = [c \sqcap d]_\Theta = [d \sqcap c]_\Theta = [d]_\Theta \wedge_{\mathbf{T}_C/\Theta} [c]_\Theta,$$

where the second identity follows from the fact that $c \sqcap d \equiv d \sqcap c \in \Sigma$. Moreover, as $\gamma(\mathcal{T}) \subseteq \Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$ we have that every concept axiom $c \cong d \in \mathcal{T}$ is satisfied by $\tilde{\varepsilon}_{\mathrm{FI}}$, since for every $c, d \in T_C$,

$$\tilde{\varepsilon}_{\mathrm{FI}}(c) = [c]_\Theta = [d]_\Theta = \tilde{\varepsilon}_{\mathrm{FI}}(d)$$

assuming that $c \cong d \in \mathcal{T}$. But this means that $\tilde{\varepsilon}_{\mathrm{FI}}$ is an extensional model of $\mathcal{T}$, thus by the assumption $\mathcal{T} \models c \equiv d$ we get

$$\tilde{\varepsilon}_{\mathrm{FI}}(c) = \tilde{\varepsilon}_{\mathrm{FI}}(d),$$

[12]To keep things simple, we use Θ as index instead of the more correct $\Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$.

that is, $[c]_\Theta = [d]_\Theta$. Hence $(c, d) \in \Theta_{\mathrm{FI}}(\gamma(\Sigma \cup \mathcal{T}))$ if $\mathcal{T} \models c \equiv d$. $\square$

As a corollary we get that $\mathcal{ALC}$ is sound and complete (with respect to $\vdash_E$), because if we have no concept identities then the intensional logic simply is $\mathcal{ALC}$. Notice that if soundness and completeness of $\mathcal{ALC}$ has been established, we may skip the last three steps of the proof.

5.2.10 Existence of Mediations between the Extensional and Intensional Models

Now we return to the issue about the mediations (Section 5.2.6). The intensional algebras are defined as abstract algebras and not concretely (set-theoretically). A mediation τ maps, as shown above, abstract intensions to concrete extensions. It is therefore interesting to establish when a non-trivial mediation exists, i.e. a mapping $\tau : I \to 2^U$ such that U is non-empty, because such a mapping gives a concrete (set-theoretic) and non-trivial interpretation of the abstract intensional domain.

We consider therefore the following problem. Given a set of concept axioms $\mathcal{T}$, imagine that we have a mapping $\iota : C \to I$ satisfying $\mathcal{T}$. Then we want to establish whether there exists a $\tau : I \to 2^U$ such that $\tau \circ \iota$ is an extensional model of $\mathcal{T}$ and $U \neq \emptyset$.

First of all we note that in general such a non-trivial mediation does not exist. To see this, consider the case where $\mathcal{T}$ is $\{a \equiv b\}$. Now, we have the following $\iota : C \to I$ which (vacuously) satisfies $\mathcal{T}$:

$$\begin{array}{c} \iota(a) = \sim\iota(a) \\ \uparrow \\ \iota(b) = \sim\iota(b) \end{array}$$

If there exists a $\tau : I \to 2^U$ such that $\tau \circ \iota$ is an extensional model of which U is non-empty, then by Proposition 18 τ forms a homomorphism $\tilde{\tau}$ such that $\tilde{\tau} \circ \tilde{\iota}$ is equal to the extension $\widetilde{\tau \circ \iota}$ of the extensional model $\tau \circ \iota$. However, then we get that $\tilde{\tau}(\sim\tilde{\iota}(a)) = \tilde{\tau}(\tilde{\iota}(a))$, that is, $\neg\widetilde{\tau \circ \iota}(a) = \widetilde{\tau \circ \iota}(a)$ contradicting that U is non-empty (only a trivial Boolean algebra satisfies $\neg x = x$). In other words, a non-trivial mediation τ cannot exist in this case.

There is another reason for investigating the existence of a mediation τ, because if $\tilde{\tau}$ always is an isomorphism (actually it merely needs to be injective), then there is no difference between the extensional and intensional entailment relations. To see this, note firstly that it suffices to show $\mathcal{T} \models c \equiv d$ implies $\mathcal{T} \models c = d$. Now, by contraposition, if (ι, τ) is an intensional model of $\mathcal{T}$ such that $\tilde{\iota}(c) \neq \tilde{\iota}(d)$ then since $\tilde{\tau}$ is injective $\tilde{\tau}(\tilde{\iota}(c)) \neq \tilde{\tau}(\tilde{\iota}(d))$, which by Proposition 18 yields $\widetilde{\tau \circ \iota}(c) \neq \widetilde{\tau \circ \iota}(d)$, hence $\mathcal{T} \not\models c \equiv d$. In other words, the mediations say something about the difference between the extensional and intensional semantics.

Consider the proof of Proposition 17 which showed that the intensional semantics subsumes the extensional. In this proof, we constructed an intensional model from an extensional. It is then interesting to note that the mediation was defined to be the identity, hence isomorphic. This means that we so far actually have not shown that "genuine" intensional models in which there is a difference between the extensional and intensional entailment relations exist in general. This will be remedied below by considering the free algebras.

Let $\mathbf{F}_C^E$ denote the free algebra over the class of the codomains of the extensional models of $\mathcal{T}$ (the extensional algebras) and let $\mathbf{F}_C^I$ denote the free algebra over the class of codomains of the first coordinate of the intensional models of $\mathcal{T}$ (the intensional algebras). Note, as we from an algebraic point of view have no variables, these algebras are freely generated by the empty set.

Since the codomain of every extensional model is the codomain of the first coordinate of an intensional model (Proposition 17), $\mathbf{F}_C^E$ is member of the class of codomains of the first coordinate of the intensional models of $\mathcal{T}$. Now, by the universal mapping property for free algebras, this means that there exists a unique homomorphism $\mathbf{F}_C^I \to \mathbf{F}_C^E$ (the mediation $\tilde{\tau}$). This shows that when we consider free algebras, there exists a mediating mapping from the intensional algebra to the extensional algebra, which is non-trivial if $\mathbf{F}_C^E$ is non-trivial. Moreover, we see that if $\mathbf{F}_C^E$ and $\mathbf{F}_C^I$ are not isomorphic then $\tilde{\tau}$ is not an isomorphism, meaning that there is a difference between the extensional and intensional entailments.

The above shows that there may be a difference between whether we consider each mapping into an intensional algebra or the models altogether (the free algebras). As we already have shown, this does not affect the completeness of the intensional concept logic.

5.3 Capturing Other Conceptions of Intensionality

When discussing Frege's contribution, we noted that there are conditions for identifying senses (hence also intensions) of different strengths (see the footnote on page 24). The identity conditions may be seen as following from different *conceptions of intensionality.* The fact that we have different conceptions of intensions (or properties) is not new. It is for instance noted in [Swoyer, 1998]. This section shows how other conceptions of intensionality can be captured by modifying the intensional concept logic.

5.3.1 Intensional Boolean Concept Logic

The intensional algebras need not be semilattices. Now we show that the intensional algebras can be defined as extensional algebras. Recall, the extensional algebras are complex algebras which are concrete examples of Boolean algebras with operators, or simply BAOs. In other words, the extensional algebras are simply Boolean algebras (with additional structure). We show that the intensional properties of the logic, which we call the *intensional Boolean concept logic*, are preserved.

The definition of entailment (which we simply superscript with BAO) follows the earlier definition under the intensional semantics (Definition 14), it should therefore not be necessary to go through all details. Let $\mathcal{T}$ be a set of concept axioms, and let

$$(E_1, \wedge_1, \neg_1, (f_r^{\wedge,1})_{r\in R}, 1_1) \text{ and } (E_2, \wedge_2, \neg_2, (f_r^{\wedge,2})_{r\in R}, 1_2)$$

be extensional algebras. An *intensional Boolean model* is a pair $(\varepsilon_2, \varepsilon_1)$ of mappings $\varepsilon_2 : C \to E_2$ and $\varepsilon_1 : E_2 \to E_1$ such that $\varepsilon_1 \circ \varepsilon_2$ is an extensional model of $\mathcal{T}$, and such that the homomorphic extension $\tilde{\varepsilon}_2$ of ε_2 satisfies all intensional concept axioms of $\mathcal{T}$. Thus now $\mathbf{E}_1$ formalizes the extensions and $\mathbf{E}_2$ formalizes the intensions which now are assumed to obey the laws of Boolean algebras (with operators).

We write

$$\mathcal{T} \models^{\text{BAO}} c = d$$

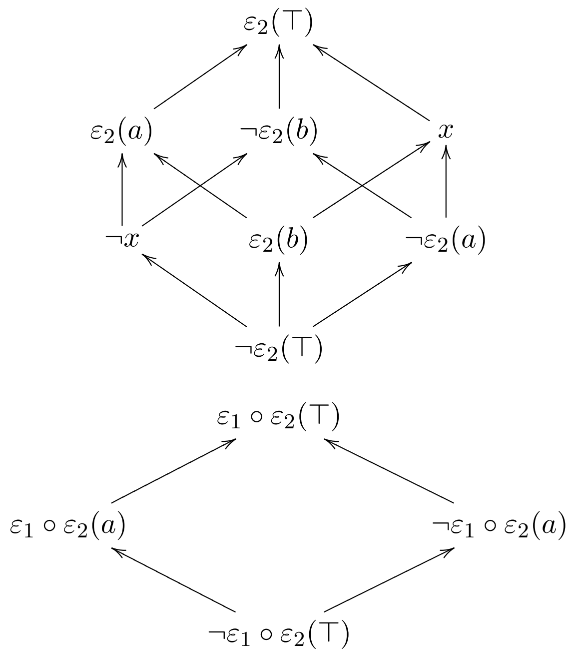
if for every intensional Boolean model $(\varepsilon_2, \varepsilon_1)$ of $\mathcal{T}$ we have $\tilde{\varepsilon}_2(c) = \tilde{\varepsilon}_2(d)$. The extensional semantics is as before, thus $\mathcal{T} \models^{\text{BAO}} c \equiv d$ if and only if $\mathcal{T} \models c \equiv d$.

The following theorem shows that the intensional properties of this altered concept logic are preserved. Although it is fairly obvious (due to the discussion in Section 5.2.10), it is noteworthy because it shows that we have an intensional logic although we make interpretations over extensional algebras only. What gives rise to intensionality is simply the inclusion of the models. This suggests that intensionality, from a formal point of view, is about considering supersets of models.

Theorem 21 *The intensional Boolean concept logic is intensional, moreover, for all* $c, d, e \in T_C$*:*

$$c = d \models^{\mathrm{BAO}} e = e[d/c].$$

Proof. The first proof is similar to the one of Theorem 15. As a counter example we have $a \equiv b \not\models a = b$ where $a, b \in C$. We verify that the appropriate model exists:



where $\varepsilon_1 \circ \varepsilon_2(b) = \varepsilon_1 \circ \varepsilon_2(a)$ and $\varepsilon_1(x) = \varepsilon_1 \circ \varepsilon_2(\top)$.

The last proof is, similar to the proof of Proposition 16, easily seen to hold. □

5.3.2 Other Kinds of Intensional Algebras

Having seen the previous sections, it should be clear that any class of algebras weaker (in the sense, satisfying fever identities) or just as weak as complex algebras may be used as intensional algebras without destroying the intensional properties of the logic.

Each class of intensional algebras (maybe one should only consider equationally defined classes) formalizes a different conception of intensionality. For example, under the intensional Boolean concept logic we have

$$\models^{BAO} c \sqcap (d \sqcup e) = (c \sqcap d) \sqcup (c \sqcap e),$$

meaning $c \sqcap (d \sqcup e)$ and $(c \sqcap d) \sqcup (c \sqcap e)$ are intensionally identified, because every complex algebra satisfies the distributive law.[13] This is the weakest possible conception of intensionality, since any stronger axiomatization of the intensional algebra means that the universal mapping property for the corresponding free intensional algebra to the free extensional algebra would not exist.

The strongest conception of intensionality arises when the intensional algebra is the term algebra. Under this conception only syntactically identical concept descriptions are identified, so not even $c \sqcap (d \sqcap e)$ and $(c \sqcap d) \sqcap e$ are intensionally identified.

It seems natural to ask, then, what conception of intensionality we should employ. Unfortunately, a general answer seems to be unattainable at the moment, and more investigations have to be carried out. Note, this is somewhat similar to the situation in modal logic where we also have different modalities for the different axiomatizations. The conception of intensionality in which the intensional algebras are semilattices (Definition 14) seems relevant for knowledge representation since it is the most general in which there is a correspondence between conjunction and subsumption.

5.4 Generalizing the Logic

In the previous section we showed that the intensional algebra need not be a semilattice. This enabled us to propose other versions of the intensional concept logic. In this section we show another way in which the logic may be altered.

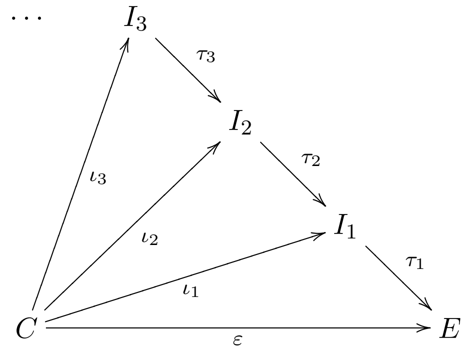
We have already mentioned that our intensional logic follows Carnap's ideas about intensionality. We have accordingly extensional and intensional algebras which give rise to two kinds of identities, $=$ and $\equiv$ (two kinds of satisfaction). However, if we follow the Frege-Church conception, there are not merely two levels of semantical entities but an infinite hierarchy instead.

We are actually also able to formalize this. Instead of letting an intensional model be a pair of mappings, we let it be an infinite tuple of mappings corresponding to the infinite hierarchy of senses. This can be defined as follows.

Let $\mathbf{E}$ be an extensional algebra and $\mathbf{I}_1, \mathbf{I}_2, \ldots$ be intensional algebras, then a *general intensional interpretation* is a tuple $(\varepsilon, \iota_1, \iota_2, \ldots, \tau_1, \tau_2, \ldots)$ where

[13] This suggests that we can identify the intension of a concept c with the equivalence class $\{d \in T_C \mid \models c = d\}$ in terms of a conception of intensionality. Carnap [1956] has suggested something similar to this.

$\varepsilon : C \to E$, $\iota_i : C \to I_i$ and $\tau_{i+1} : I_{i+1} \to I_i$ such that $\iota_i = \tau_{i+1} \circ \iota_{i+1}$ and $\varepsilon = \tau_1 \circ \iota_1$. Moreover, all mappings must extend to homomorphism. The relations between the mappings are illustrated in the commuting diagram below



Notice the similarity with the illustration of Frege's contribution (see page 25).

This means that we can define infinitely many intensional identities $=_1, =_2$, ... (instead of the single intensional identity $=$ we had earlier). The definition of *general-satisfaction* is a generalization of the earlier definition of satisfaction, such that a general intensional interpretation $(\varepsilon, \iota_1, \iota_2, \ldots, \tau_1, \tau_2, \ldots)$ satisfies $c =_j d$ if whenever $i \leq j$ then $\tilde{\iota}_i(c) = \tilde{\iota}_i(d)$ (only mappings "less" intensional than the j'th have to satisfy $c =_j d$). Entailment is defined as previously, but now we use the notation $\models^\infty$.

Now it should be clear that we have

$$c =_j d \models^\infty c =_i d$$

and

$$c =_i d \not\models^\infty c =_j d$$

for all $i \leq j$, showing the intensional properties of the logic.

The $\mathbf{I}_i$s need not be intensional algebras, nor do they need to be of the same definition. $\mathbf{I}_1$ could, for instance, be a complex algebra, and $\mathbf{I}_2$ a semilattice and so on. In order for the intensional properties to hold, $\mathbf{I}_{i+1}$ must be weaker or just as weak as $\mathbf{I}_i$.

Under this semantics there is no such thing as *an* intension of a concept—a concept has an intension only with respect to a given level of the intensional algebras (level of intensionality). This may appear strange at first (at least in terms of Carnap's semantical division into extensions and intensions), but in terms of the Frege-Church conception it seems reasonable. It is accordingly probably more appropriate to call the members of I_1 for senses instead of intensions, and the members of I_2 for sense senses and so on.

Note that the generalized formalization is coherent with Church's axiom $(15^{m\alpha\beta})$, which asserts that function application preserves the *concept of* relation (in Church's terminology). To see this, let x and y be concepts of a and b respectively, that is, let $\tilde{\tau}_{i+1}(x) = a$ and $\tilde{\tau}_{i+1}(y) = b$, then since $\tilde{\tau}_{i+1}$ is homomorphic, we have for every function on senses, like for example $\times_{i+1}$, that

$$\tilde{\tau}_{i+1}(x \times_{i+1} y) = \tilde{\tau}_{i+1}(x) \times_i \tilde{\tau}_{i+1}(y) = a \times_i b,$$

showing that $x \times_{i+1} y$ is a concept of $a \times_i b$. In other words, axiom $(15^{m\alpha\beta})$ corresponds in our framework to the fact that the mediations are homomorphic.

If we consider the proof of soundness and completeness (Section 5.2.9), it should be clear that it may be generalized to the general intensional concept logic, where we have infinitely many kinds of identities. We "only" need an Intensionality proof rule for each level of intensionality:

$$\frac{c =_{i+1} d}{c =_i d}, \ i = 1, 2, \ldots$$

It should also be clear that the soundness and completeness results hold no matter how the intensional algebras are axiomatized (as long as $\mathbf{I}_{i+1}$ is weaker than $\mathbf{I}_i$ or just as weak, of course). In other words, we have vast number (infinitely many) of sound and complete concept logics.

Each axiomatization of the intensional algebras may be associated with a conception of intensionality, as already noted. Let us consider the class of all concept identities entailed by the empty set of concept axioms under a given conception of intensionality (for example $a \sqcap b = b \sqcap a$ belongs to the conception in which intensional algebras are semilattices, Definition 14). A well-known result of universal algebra says that if we want to determine whether a concept axiom belongs to the class, it suffices to make interpretations over the free algebra over the class (because the free algebra has the universal mapping property). This means that each conception of intensionality may be associated with the corresponding free algebra.

This leads to a natural way of ordering the different notions of intensionality. Let us index the free algebras corresponding to the different notions of intensionality by a set K. Define an ordering on K by $k \leq l$ iff there is a (unique) map $\mathbf{F}_C^l \to \mathbf{F}_C^k$, where $\mathbf{F}_C^k$ and $\mathbf{F}_C^l$ are the free algebras associated with the k'th and l'th conception of intensionality. By the fact that we have identity mappings on the free algebras, by the unique mapping property for free algebras, and by the fact that maps commute, it should be clear that $\leq$ is reflexive, antisymmetric, and transitive. Hence $(K, \leq)$ is a partial order (and not merely a chain). This ordering formulates the relations between the different notions of intensionality, because if $k \leq l$, then

$$\mathcal{T} \models c =_l d \text{ implies } \mathcal{T} \models c =_k d.$$

In other words, we can order conceptions of intensionality. It should then be clear that the algebraic semantics sheds new light on intensionality.

Among all the different notions of intensionality, it should be clear that the three we have described earlier have a special status: the term algebra $\mathbf{T}_C(\emptyset)$ is maximum in this order, and the free algebra over the class of codomains of the first coordinate of the intensional Boolean models (described in Section 5.3.1) is minimum, and $\mathbf{F}^I_C$ (described in Section 5.2.10) is the greatest in which there is a one-to-one correspondence between subsumption and conjunction.

Finally it should be noted that we have shown that we have a class of logics of which some are in accordance with Carnap's method of extension and intension and some are in accordance with the Frege-Church conception.

5.5 Remarks about the Intensional Concept Logics

Although we have not shown how, it should be fairly obvious that we can define intensional semantics of other description logics than $\mathcal{ALC}$ by using an approach similar to the ones described above. Moreover, the underlying approach of the intensional semantics may, with some corrections, be used for other kinds of logics than description logics. Such extensions are not easy to formalize and presuppose that the approach is modified even further, however, in order to show that it can be done, we have defined an intensional propositional logic based on the Frege-Church conception of intensionality. The logic can be found in Appendix A. The result that it subsumes modal logic **T** is also shown.

Although computational properties play a significant role in description logic, we have not focused on this issue in the present work. However, we can establish that the complexity of the intensional Boolean concept logic is the same as that of $\mathcal{ALC}$. More precisely, the decision problem of determining if $c \cong d$ belongs to the set of concept axioms which is entailed by $\mathcal{T}$, i.e. determining if $\mathcal{T} \models^{BAO} c \cong d$, is ExpTime-complete. To see this, recall from the completeness proof that we can divide a set of concept axioms $\mathcal{T}$ in two,

$\mathcal{T}^E = \{c \equiv d \mid c \equiv d \in \mathcal{T}\}$ and $\mathcal{T}^I = \{c \equiv d \mid c = d \in \mathcal{T}\}$, such that

$$\mathcal{T} \models^{BAO} c = d \text{ if and only if } \mathcal{T}^I \models^{\mathcal{ALC}} c \equiv d$$

and

$$\mathcal{T} \models^{BAO} c \equiv d \text{ if and only if } \mathcal{T}^I \cup \mathcal{T}^E \models^{\mathcal{ALC}} c \equiv d.$$

This shows that in order to determine whether a concept axiom is entailed by $\mathcal{T}$ we can use the entailment relation of $\mathcal{ALC}$ (and conversely). The two decision problems must therefore belong to the same complexity class.

This result does not only hold for the intensional Boolean concept logic, it holds also for the general intensional concept logic (where we have an infinite hierarchy of intensional algebras) presupposing that each of the intensional algebras are complex algebras. To see this, let n be the highest level of intensionality we find in a set of concept axioms $\mathcal{T}$, that is, for all $c =_i d \in \mathcal{T}$ we have $i \leq n$. Then we can define general-satisfaction only by means of n numbers of mappings into intensional algebras. Let us use the notation $\models^n$ for the entailment relation which is defined in terms of interpretations consisting of tuples $(\varepsilon, \iota_1, \ldots, \iota_n, \tau_1, \ldots, \tau_n)$. Then we see that $\mathcal{T} \models^\infty c =_i d$, where $i \leq n$ if and only if $\mathcal{T} \models^n c =_i d$, since there are no concept axioms in $\mathcal{T}$ which has a higher level of intensionality than n. Hence it should be clear that we can make a similar construction as the one above (but now we need n sets of concept axioms instead of just $\mathcal{T}^I$ and $\mathcal{T}^E$), meaning that this decision problem also belongs to the same complexity class as $\mathcal{ALC}$. This is interesting because it shows that we have a decidable logic based on the Frege-Church conception.

From a practical point of view (if one wants to implement the logics and use them for formalizing ontologies) the fact that entailment is ExpTime-complete, which is often considered to be intractable, need not be a serious problem. Because there exist efficient implementations of description logics even more expressive (more complex) than $\mathcal{ALC}$, see e.g. [Haarslev and Möller, 2001; Horrocks and Sattler, 2002]. The proposed intensional concept logics provide therefore a viable alternative to the extensional concept logics.

5.6 Applications

The expressiveness of the intensional concept logic is higher than the expressiveness of the description logic $\mathcal{ALC}$. This should be clear because the intensional algebras may be weaker than the extensional algebras of $\mathcal{ALC}$, moreover, the expressiveness is higher because we have a logic with two notions of equivalence. This means that there is a number of novel applications of the intensional concept logic. This section describes some of these.

First of all, we have an obvious application of the intensional concept logic in that we are able to discern between co-extensional concepts. As already noted, it means that we, for example, can assert

$$\textit{bachelor} \equiv \textit{lonely hearted},$$

i.e. that bachelors are the lonely hearted, without this entails that *bachelor* and *lonely hearted* are identified in the concept logic, for we have

$$\textit{bachelor} \equiv \textit{lonely hearted} \not\models \textit{bachelor} = \textit{lonely hearted}.$$

Similarly,

$$\textit{creature} \sqcap \exists \textit{has-part.heart} \equiv \textit{creature} \sqcap \exists \textit{has-part.kidney}$$

captures the earlier mentioned co-extensionality of *creature with a heart* and *creature with a kidney* without identifying the two concepts.[14]

The two kinds of identity and the two kinds of subsumption provide us with a logic which can be used to formalize more well-structured taxonomies (i.e. also more well-structured ontologies). In a series of papers, Guarino and Welty, see e.g. [Welty and Guarino, 2001], have introduced a methodology for constructing more well-structured taxonomies. The methodology consists in removing, adding, or rearranging subsumption vertices (*is-a* links) which are found not to be acceptable in terms of philosophically motivated meta-properties. It would take us too far afield to go into details about this, so we merely illustrate the possibilities by means of an example from [Welty and Guarino, 2001].

In a given taxonomy *apple* is asserted to be a subconcept of both *fruit* and *food*. Now they suggest that in a more properly structured ontology *apple* is only a subconcept of *fruit*. Unfortunately, this means that the knowledge that an apple is a kind of food is removed from the taxonomy. However, for applications within e.g. nutrition, this knowledge is quite relevant. So it would be nice if we did not have to give it up. By means of our two kinds of subsumption, their proposal can be captured without giving up knowledge:

$$\begin{aligned} \textit{apple} &\leq \textit{fruit} \\ \textit{apple} &\sqsubseteq \textit{food}. \end{aligned}$$

[14] Here we have formalized the relation by means of value restrictions. This ensures that the part (the heart and kidney) exists, however, if one wants to say that exactly one part exists, we need to use more expressive description logics.

Unlike the first taxonomy, this does not mess up the taxonomy, because the two kinds of knowledge are clearly separated. In a graphical presentation of the taxonomy, extensional relations could for instance be removable.

This rearranging may appear arbitrary at first, but it is interesting to note that researchers in terminology, which is a different, more linguistic research area (see e.g. [Madsen, 1999]), employ a somewhat similar division when discerning between

- the *characteristic (defining) attributes*, and
- the *supplementary attributes*

of a concept. An characteristic attribute, for example that an apple contains (has part) seeds, could in the intensional logic be formalized as

$$apple = apple \sqcap \exists has\text{-}part.seed.$$

A supplementary attribute, like the fact that an apple may be eaten, could be formalized as

$$apple \equiv apple \sqcap food.$$

Here we could also have used roles. Notice the different use of intensional and extensional relations.[15]

This shows that the intensional concept logic supports at least some of the non-classical principles (which description logics do not support) that modellers working with construction of ontologies adopt in practice in order to represent and reason with concepts.

There are other applications of the intensional concept logic. In the following we show some examples.

5.6.1 Multi-Knowledge-Based Systems

Now we describe the example mentioned in Chapter 1 in more details. The application is called *multi-knowledge-based systems* because we show how to unify different knowledge bases by means of the intensional concept logic.

Assume we have a knowledge base which is considered to give a correct formalization of a given domain, but not a complete formalization in the sense that there are additional facts about the domain to be formalized. One may

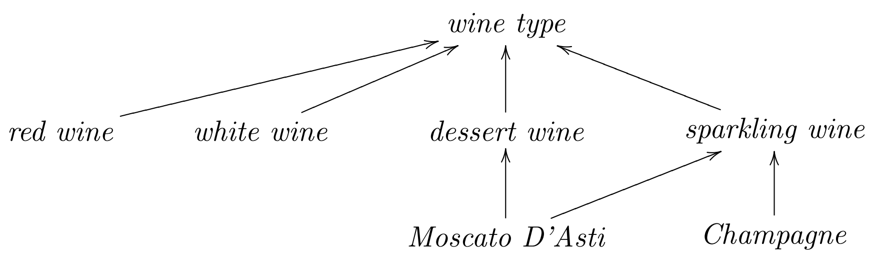
[15] As always, this could have been formalized differently. In a general ontology we would probably define *fruit* to have seeds as attribute, and this would then by inheritance entails that apple has seeds.

therefore want the users of the knowledge base to alter it (or maybe only privileged users—we abstract from such technical details).

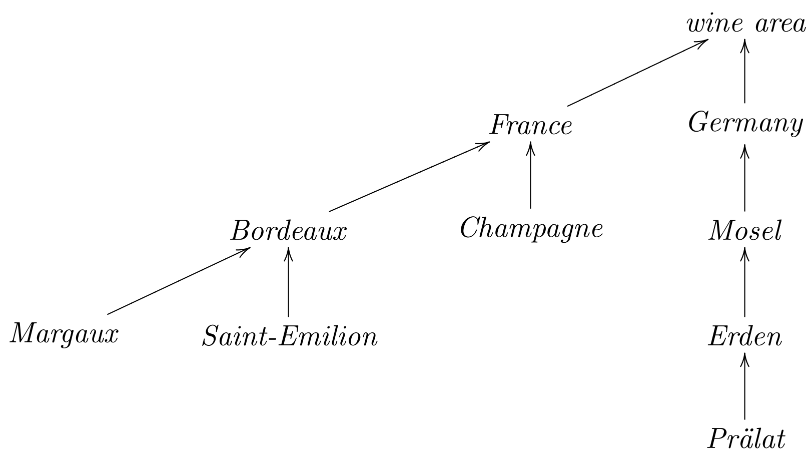
However, as it probably takes a lot of resources to construct the knowledge base in the first place (and to keep it up-to-date as well), one would like some sort of protection, especially since it can be very easy to make the knowledge base inconsistent and thereby useless.

If we, for example, have a rule saying $c \equiv d$ and someone adds $c \equiv \neg d$ then we have an inconsistent knowledge base, from which any conclusion might be drawn, hence a useless knowledge base since $c \equiv d, c \equiv \neg d \models e \equiv f$ for all $e, f \in T_c$.[16]

To make this less speculative, let us imagine that Alice, an expert on wines, owns a wine shop. Through years of studies, Alice has made a comprehensive wine ontology, describing different types of wines, like



Different wine producing areas ranging from countries to wine fields, like:[17]



[16] Note that inconsistencies may arise in less obvious ways, and that it is not only inconsistencies that may harm the knowledge base.

[17] Notice that it need not be a trivial assignment to construct ontologies, *Champagne* is for instance used both as a region and a wine type.

It should be noted that in this example we formalize knowledge in a special way such that everything is formalized as concepts (to keep things simple). For example, the wine areas are formalized by means of the subsumption relation such that

$$\textit{Bordeaux} \leq \textit{France}.$$

It is probably more appropriate to formalize these relations by means of a *part-of* relation, like *part-of*$(Bordeaux, France)$ (but this is not crucial for the example).

Different types of grapes, like *Cabernet Sauvignon, Merlot, Chardonnay, Riesling* are also represented. Now, we could have a rule stating that all wines made from the Riesling grape are either white wines or dessert wines.

$$\forall \textit{made from.Riesling} \leq \textit{white wine} \sqcup \textit{dessert wine}.$$

The wine producers follow many rules and traditions. Being a passionate wine expert (drinker) has left Alice with little time to address this aspect of the wine world, and since she moreover finds this part tedious, she has left it to the users of her knowledge base (which could include her customers) to define this part. Alice has no time to control that all the rules and data entered are correct, so in order to protect her ontology, the users enter their knowledge by means of the extensional relations $(\equiv, \sqsubseteq)$, whereas she uses $(=, \leq)$.

Now, some user adds the rule that every wine from the Erden village is made from the Riesling grape:

$$\textit{Erden} \sqsubseteq \forall \textit{made from.Riesling}.$$

Notice that the users (including Alice) do not even have to know that there are both extensional and intensional relations, because they can only use one relation.

The knowledge about wines may be combined with the customers needs. Users can accordingly add information about what kind of wines they like, such that Alice can order the wines which the customers like, inform users about special sales, wine tastings of their favorite wines etc. For expressing favorite wines the *likes* relation is used. Bob likes dessert wines and white wines, he therefore asserts

$$\textit{dessert wine} \sqcup \textit{white wine} \sqsubseteq \exists \textit{likes}.\textit{Bob}.$$

One day Alice gets a special dessert wine for sale, an Erdener Prälat Beerenauslese, which Alice simply calls *EPBA*. This wine comes from the

Prälat wine field, so Alice adds

$$EPBA \leq Prälat.$$

Let us call the entire wine knowledge base including both Alice's ontology and the users contributions for $\mathcal{W}$. Now, we have

$$\mathcal{W} \models EPBA \sqsubseteq \exists likes.Bob$$

meaning that it can be deduced that Bob will like *EPBA*. Notice how we, in order to derive this, must use both Alice's ontology and the user contributions. Notice also that we have

$$\mathcal{W} \not\models EPBA \leq \exists likes.Bob.$$

In such cases Alice (or Bob) can see that the deduction presupposes extensional knowledge (supplied by the users), which means that it is possible to ascribe less importance to the result.

The production size of dessert wines like *EPBA* may be microscopic although the demand for such wines is high. A character like Alice has a special principle which keeps the prices on these wines, the so-called rarities, from rising to astronomical heights. Alice insists on only selling rarities to specially selected customers. And by means of the following rule

$$\forall buy.rarity \leq selected\ customer$$

she can enforce this system.

As mentioned in Chapter 1, Alice's ontology is protected from the incorrect rules users may add. For instance, if Bob in despair over not being allowed to buy *EPBA* adds

$$EPBA \equiv \neg EPBA,$$

and causes the extensional part of the knowledge base to be inconsistent, the intensional part remain (and he is therefore still prevented from buying the wine).[18]

The system need not be divided into two parts only, it may be divided into an arbitrary number of parts. This could be necessary if the staff of the wine shop and wine producers also may add rules. In such cases we have to

[18]Nevertheless, it should be noted that inconsistencies (of course) are harmful to the system, they are just not as harmful as in a usual system.

use the logic with n-equivalence relations (see Section 5.4), but the principles are the same.[19]

This describes some of the possibilities of how the intensional logic can be used to formalize integrations of several knowledge bases. The logic is of course only really useful in the cases where some of the knowledge bases are ordered (ascribed different modalities). As another (more important) domain, we have a health care system, with doctors, nurses, patients, visitors, medicine, patient records, prescriptions, schedulings of operations, etc. (In this domain there is also an ordering of the users and their knowledge.)

This shows that the intensional concept logic is useful for these applications, because it allows different kinds of knowledge to be united, and yet keep them discernible.

As far as we know, no other decidable logic would be able do what we have described above.[20]

5.6.2 Intensional Subsumption

Now we describe different ways to understand the intensional relations, in particular, the difference between extensional subsumption and intensional subsumption. As already noted, $\leq$ is to be used as an intensional subsumption relation, and $\sqsubseteq$ as an extensional subsumption relation.

It should be clear that different roles are ascribed to the extensional and intensional axioms. They may therefore be interpreted as modalities. Then it should be clear that we may interpret $c = d$ as expressing a necessary relationship between c and d, whereas $c \equiv d$ expresses a contingent relationship. We could for example have the following

$$\mathit{bachelor} = \mathit{unmarried} \sqcap \mathit{man} \tag{5.3}$$

$$\mathit{bachelor} \equiv \mathit{lonely\ hearted.} \tag{5.4}$$

which entails that *bachelor* is an intensional subconcept of *man* (and *unmarried*), hence

$$\mathit{bachelor} \leq \mathit{man}$$

[19]Note we could also have had one equivalence relation for each user, such that every user is protected from the other users.

[20]It should be noted that the above example only employs *closed world reasoning*. The particular examples could therefore be successfully implemented in a logic programming language. However, in general, where we have ABoxes, such implementations may not give sound answers.

but it does not entail that *bachelor* is an intensional subconcept of *lonely hearted*, that is, we have

$$\textit{bachelor} \not\leq \textit{lonely hearted}.$$

But we have of course that *bachelor* is an extensional subconcept of *lonely hearted*

$$\textit{bachelor} \sqsubseteq \textit{lonely hearted}.$$

The intensional relations can alternatively be used for expressing concept definitions (this is related to the approach employed in terminology with its characteristic and supplementary attributes). For example, *vitamin-C* is defined as a *vitamin*, therefore we know by definition that *vitamin-C* is a subconcept of *vitamin*, in which case we write

$$\textit{vitamin-C} \leq \textit{vitamin}.$$

However, the knowledge that vitamin C inhibits oxidation, that is, that *vitamin-C* is an *antioxidant* follows from empirical investigations (chemical experiments). In this case we would therefore use the extensional relation and write

$$\textit{vitamin-C} \sqsubseteq \textit{antioxidant}.$$

5.6.3 Intensional ABoxes and Prototype Theory

The intensional semantics has only been defined for concept axioms, that is, for TBoxes. Now we show how to define an intensional semantic for concept assertions, that is, for ABoxes.[21]

Just as the intensional semantics for TBoxes needed two kinds of expressions for discerning between extensional and intensional knowledge, we need two kinds of membership assertions. The *extensional concept assertion* is the usual concept assertion of $\mathcal{ALC}$, where we by writing $c(a)$ express that the individual a is a member of the concept c. The *intensional concept assertion* has the following notation

$$c[a],$$

[21] The reason why this semantics is described as an application of the logic is that not all details are worked out. Moreover, the contributions described in Chapter 2, which formed the basis for the intensional semantics, do not seem to acknowledge intensionality on the levels of individuals (note, we are not talking about individual concepts but about individuals of the domain of interpretation).

which should be read as: *a is an intensional member of c* (we later describe what this is to mean).

For defining the intensional semantics of ABoxes we use the same approach as when we defined the intensional semantics of TBoxes: intensionality arises through algebraic abstraction. Recall, under the extensional semantics $c(a)$ is satisfied by an $\mathcal{ALC}$ interpretation $(\cdot)^{\mathcal{I}}$ if $a^{\mathcal{I}} \in c^{\mathcal{I}}$. Since it is not particular important, we skip membership of roles ($r(a,b)$); and once the first is defined, the other follows by means of products of algebras.

Now we can define the semantics. Let A be the set of individuals. An *extensional interpretation* is a mapping $\varepsilon : C \cup A \to 2^U$ such that for all $a \in A$, $\varepsilon(a)$ is an atom of U. Moreover, ε must extend to a homomorphism on C. Then ε *satisfies* $c[a]$ if $\varepsilon(a) \subseteq \tilde{\varepsilon}(c)$ and ε *satisfies* $c(a)$ if $\varepsilon(a) \subseteq \tilde{\varepsilon}(c)$. The interpretation is an *extensional model* if it satisfies all concept axioms and concept assertions.

An *intensional interpretation* is a pair $(\iota : C \cup A \to I, \tau : I \to 2^U)$, where $\mathbf{2}^U$ is a complex algebra and $\mathbf{I}$ is an intensional algebra (of appropriate conception) such that for all $a \in A$, $\iota(a)$ is a generator of $\mathbf{I}$ and $\tau \circ \iota(a)$ is an atom of U. Moreover, ι must extend to a homomorphism on C and $\tau \circ \iota$ must be an extensional model. Then $c[a]$ is *satisfied* by (ι, τ) if $\iota(a) = \iota(a) \times \tilde{\iota}(c)$ (recall, this means that $\iota(a)$ is less than or equal to $\tilde{\iota}(c)$). An *intensional model* is an intensional interpretation which satisfies all concept identities and intensional concept assertions.

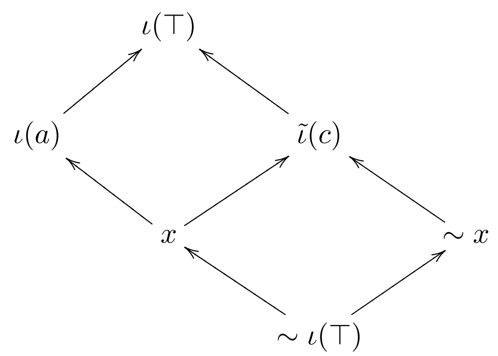
Now it is easy to see that we have for all $a \in A, c \in T_C$

$$c[a] \models c(a). \tag{5.5}$$

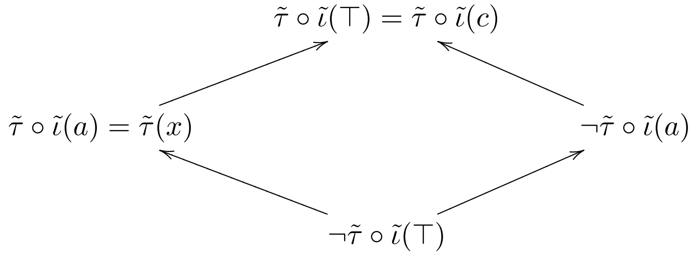
On the other hand we have

$$c(a) \not\models c[a]. \tag{5.6}$$

To see this, consider



where $\sim x = \sim \iota(a)$, $x = \sim\sim x$, $\tilde{\iota}(c) = \sim\sim \tilde{\iota}(c)$, and $\sim \tilde{\iota}(c) = \sim \iota(\top)$ and



which form an intensional model of $c(a)$. We verify, amongst other things, that $c[a]$ is not satisfied because $\iota(a) \times \tilde{\iota}(c) = x \neq \iota(a)$.

These two results, (5.5) and (5.6), can be seen as intensionality results for ABoxes in that they show that intensional membership implies extensional membership (but not conversely).

How should intensional membership be understood? If $\mathcal{T} \models c[a]$ then we may think of a as a prototypical member of c, that is, a member which is a particular good example or representative of the concept c. Within cognitive science there is a theory called *prototype theory*, where one works with this distinction, see e.g. [Hampton, 1993]. For example, a prototypical wine could be a common red wine, in contrast to an aromatic dessert wine like *EPBA*. In order to capture these ideas we actually need to have an infinity of different degrees of membership, but the presented theory can easily be generalized to capture this if we follow the approach described in Section 5.4. (It should not be necessary to show all the technicalities.)

Unfortunately, it is not entirely clear that this provides an appropriate formalization of prototype theory (if anything does), because by the definitions above, we have the following

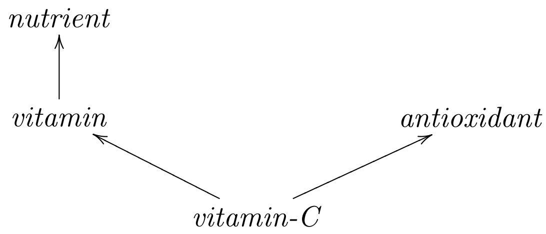
$$c \leq d, c[a] \models d[a]$$

which, for example, means that if we assert that *EPBA* is a prototypical dessert wine then this entails that it is also a prototypical wine, and this does not seem to be in fully accordance with prototype theory.

5.6.4 Content-Based Information Retrieval

Now we change subject to information retrieval. Information retrieval is about selecting data which are similar to a given query. Information retrieval is for example used by search engines on the Web. Similarity is (most often) determined by comparing two words with each other. In traditional key word based information retrieval, words are judged to be similar if they are identical. In the OntoQuery research project (see `www.ontoquery.dk`), a more elaborate strategy based upon ontologies is used. Basically, similarity is determined by measuring the distance (the number of vertices) between the corresponding nodes in the ontology.

Assume we have the following part of a nutrition ontology



Then it should be clear that *vitamin-C* is more similar to *vitamin* than *nutrient*. This means that if one requests information about nutrient, one is more likely to get information about vitamin than information about vitamin C. (Of course one is most likely to get information about nutrient, but in case there is no, vitamin is retrieved.)

By means of the intensional concept logic this can be refined even further. This is simply done by asserting that the distances between intensional subsumption relations are less than corresponding extensional relations. If we follow the previous example, where *vitamin-C* is an extensional subconcept of *antioxidant* and an intensional subconcept of *vitamin*, it means that if one requests information about vitamin C, one is more likely to get information about vitamin than antioxidant. This does seem acceptable.

Chapter 6

Conclusion

An intensional logic for representing and reasoning with concepts has been presented. The logic was based on the description logic $\mathcal{ALC}$ which we augmented with an intensional equivalence relation. It was given an algebraic semantics, but despite the technical details, the underlying principle of the intensional semantics was strikingly simple: we considered simply pairs (in the general case, tuples) of commuting models, instead of merely single models as one usually does. By letting the codomain of the first model be weaker (more general) than the codomain of the second, it was shown that we obtain an intensional logic.

It was shown that a wealth of different versions of this semantics exists, and that the underlying principle, with some modifications, can be used for other kinds of logics than description logics. The intensional semantics proposes therefore a general approach for formalizing intensionality in which intensionality arises through abstraction (generalization) of properties. This may at first appear unmotivated, but the relation between abstraction and concepts (intensional entities) has actually been known for a long time. For example, Antoine Arnauld and Pierre Nicole assert page 38 of [1996] (first edition published in 1662):[1]

> if I draw an equilateral triangle on a piece of paper, and if I concentrate on examine it on this paper alone with all the accidental

[1]Note, many things have changed since then, and what they call *idea* is related to what we call *concept*.

> circumstances determining it, I shall have an idea of only a single triangle. But if I ignore all the particular circumstances and focus on the thought that the triangle is a figure bounded by three equal lines, the idea I form will [...], be able to represent all equilateral triangles. Suppose I go further and, ignoring the equalities of lines, I consider it only as a figure bounded by three straight lines. I will then form an idea that can represent all kinds of triangles.

Now we touch upon the issue of related work once again. From a general perspective, the present work is related to many of other theories in that we have proposed a logic with two connectives, $=$ and $\equiv$, interpreted as equivalence relations such that the former is a subset of the latter. This occurs for example in some modal logics. In **T** we have that $\Box(\phi \leftrightarrow \psi) \rightarrow (\phi \leftrightarrow \psi)$ is valid, showing that strict equivalence implies equivalence. As another example, Peter Aczel and Solomon Feferman [1980] have proposed a logic with an intensional equivalence operator. Our work is also related to formal concept analysis, however, it is not yet clear to what extend. The present work appears, amongst other things, to be distinguished by its algebraic semantics and its (the general concept logic of Section 5.4) connection to the Frege-Church conception of intensionality with its infinite hierarchy of senses.

Future work include a more thorough comparison between the present approach and related approaches. There is also work to be done on investigating how the underlying intensional semantics can be applied to other kinds of logics. Part of this work has already been carried out as shown in Appendix A. There is also work to be done on implementing the intensional concept logic. It is for example not clear whether the current tableau algorithms of description logics offer the most efficient implementation.

Appendix A

A Logic of Sense and Denotation

The underlying approach of our intensional semantics may, with some modifications, be used for other kinds of logic than description logics. Now we show how it can be used for defining a propositional logic that allows a distinction between sense and denotation.

The syntax of the present logic is that of propositional logic with an additional unary connective $[\cdot]$. $[\phi]$ should be read as: *the sense of* ϕ *is the sense of* $\top$, where $\top$ is the nullary connective which is interpreted as *true*. $[\phi \leftrightarrow \psi]$ should be read as: ϕ *and* ψ *have the same sense*. Frege asserted that sense determines denotation, in the present logic this becomes formalized as:

$$[\phi] \to \phi$$

is valid.

In the following let $\mathcal{P}$ be the set of propositional letters. The formal definition of the semantics is given below.

Definition 22 *An* S-interpretation *is a tuple* $(\varepsilon_0 : \mathcal{P} \to E_0, \varepsilon_1 : \mathcal{P} \to E_1, \ldots, \tau_1 : E_1 \to E_0, \tau_2 : E_2 \to E_1, \ldots)$ *where for every* $i \in \mathbb{N}$, $(E_i, \wedge_i, \neg_i, 1_i)$ *is a Boolean algebra and* $\tau_{i+1} \circ \varepsilon_{i+1} = \varepsilon_i$ *and* ε_i *extends to a homomorphism, that is, it satisfies*

$$\begin{aligned} \varepsilon_i(\top) &= 1_i \\ \varepsilon_i(\phi \wedge \psi) &= \varepsilon_i(\phi) \wedge_i \varepsilon_i(\psi) \\ \varepsilon_i(\neg\phi) &= \neg_i \varepsilon_i(\phi), \end{aligned}$$

and

$$\varepsilon_i([\phi]) = \begin{cases} 1_i & \textit{if } \varepsilon_{i+1}(\phi) = 1_{i+1} \\ 0_i & \textit{otherwise.} \end{cases}$$

The idea is that a formula which does not contain any $[\cdot]$ is interpreted in E_0, such that the formula is true if it is true in E_0, i.e., equal to 1_0. It should be clear that we then obtain the classical propositional tautologies. Moreover, a formula enclosed in a single $[\cdot]$ is interpreted in E_1, such that the whole formula is true if it is true in E_1, i.e., equal to 1_1. E_1 formulates the domain of the senses of an interpretation. Formulas enclosed in nested $[\cdot]$ are interpreted similarly by means of the E_i's.

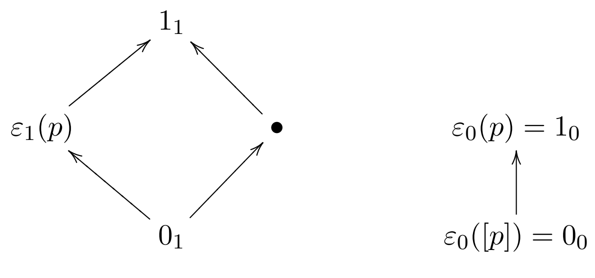
It can easily be shown that τ_i is a Boolean homomorphism (when its domain is restricted to the image of ε_i).

Definition 23 *A formula* ϕ *is* S-valid *if for every* S-*interpretation*

$$(\varepsilon_0, \varepsilon_1, \ldots, \tau_1, \tau_2, \ldots) \textit{ we have } \varepsilon_0(\phi) = 1_0.$$

As an example we get that $[\phi] \to \phi$ is S-valid. If $\varepsilon_1(\phi) = 1_1$ then by applying τ_1, we have $\tau_1(\varepsilon_1(\phi)) = \tau_1(1_1)$, which by commutativity and the fact that τ_1 preserves bounds since it is a homomorphism yields $\varepsilon_0(\phi) = 1_0$. So, $\varepsilon_0([\phi] \to \phi) = 1_0$. In the other case where $\varepsilon_1(\phi) \neq 1_1$ we get by definition that $\varepsilon_0([\phi]) = 0_0$, meaning $\varepsilon_0([\phi] \to \phi) = 1_0$. In all cases $[\phi] \to \phi$ is true in E_0, thus it is S-valid.

As another example we see that $\phi \to [\phi]$ is not S-valid. As a counter model we have



E₁ **E₀**

Verifying that we have commutativity, we have $\neg_0 \varepsilon_0(p) \vee_0 \varepsilon_0([p]) = 0_0$, meaning that the formula $p \to [p]$ is not S-valid.

In a similar way we can show that

$$(\phi \leftrightarrow \psi) \rightarrow ([\phi] \leftrightarrow [\psi])$$

is not S-valid, meaning that the logic of sense and denotation is intensional.

A.1 Relation to Modal Logic T

It is natural to compare the logic of sense and denotation with modal logic. There is a natural translation of a modal logic formula ϕ to a formula ϕ' of the logic of sense and denotation, by mapping $\Box\psi$ to $[\psi]$ and otherwise map ψ to ψ.

The following theorem shows a relation between the logics.

Theorem 24 *If ϕ is a formula valid in modal logic* **T***, then the translated formula ϕ' is S-valid.*

Proof. We must show that the axioms of **T** are S-valid, and that the inference rules of **T** preserve S-validity.

The translation of the K-axiom $[\phi \rightarrow \psi] \rightarrow ([\phi] \rightarrow [\psi])$ is S-valid, because if $\varepsilon_0([\phi \rightarrow \psi]) = 1_0$, that is, if $\varepsilon_1(\phi) \leq_1 \varepsilon_1(\psi)$, then if $\varepsilon_0([\phi]) = 1_0$, that is, if $\varepsilon_1(\phi) = 1_1$, we get $\varepsilon_1(\psi) = 1_1$, meaning $\varepsilon_0([\psi]) = 1_0$. Hence $\varepsilon_0([\phi] \rightarrow [\psi]) = 1_0$.

The translation of the T-axiom, $[\phi] \rightarrow \phi$, is S-valid as shown in the example above.

If ϕ is a tautology of propositional logic then ϕ is S-valid, since propositional logic may be interpreted over any Boolean algebra as a well-known result shows.

If $\phi \rightarrow \psi$ is S-valid then $\varepsilon_0(\phi) \leq_0 \varepsilon_0(\psi)$ and if ϕ is S-valid then $\varepsilon_0(\phi) = 1_0$. Therefore $\varepsilon_0(\psi) = 1_0$ meaning ψ is S-valid.

If ϕ is S-valid and $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \tau_1, \tau_2, \tau_3, \ldots)$ is any S-interpretation, then we get that $(\varepsilon_1, \varepsilon_2, \ldots, \tau_2, \tau_3, \ldots)$ is an S-interpretation, so we must have $\varepsilon_1(\phi) = 1_1$ meaning that $\varepsilon_0([\phi]) = 1_0$. Thus $[\phi]$ is S-valid.

We show that uniform substitutions preserve S-validity. Let x be a propositional letter (variable), and let $\phi(\psi/x)$ denote the result of substituting every occurrence of x with ψ in ϕ. Assume ϕ is S-valid. Given an S-interpretation $(\varepsilon_0, \varepsilon_1, \ldots, \tau_1, \tau_2, \ldots)$ in which $\varepsilon_i(\psi) = m_i$, we have an S-interpretation $(\varepsilon'_0, \varepsilon'_1 \ldots, \tau_1, \tau_2, \ldots)$ defined by $\varepsilon'_i(x) = m_i$, otherwise $\varepsilon'_i = \varepsilon_i$. We verify that $(\varepsilon'_0, \varepsilon'_1 \ldots, \tau_1, \tau_2, \ldots)$ is an S-interpretation in that

$$\tau_{i+1}(\varepsilon'_{i+1}(x)) = \tau_{i+1}(m_{i+1}) = \tau_{i+1}(\varepsilon_{i+1}(\psi)) = \varepsilon_i(\psi) = \varepsilon'_i(x).$$

Now we see that $\varepsilon'_i(\phi) = \varepsilon_i(\phi(\psi/x))$ for all i. We proceed on induction on the structure of ϕ. If ϕ is equal to a propositional letter y, then for all i if $y = x$, $\varepsilon'_i(y) = m_i = \varepsilon_i(\psi)$, otherwise $\varepsilon'_i(y) = \varepsilon_i(\psi)$ for all i. For conjunctions we have for all i, $\varepsilon'_i(\phi' \wedge \phi'') = \varepsilon'_i(\phi') \wedge_i \varepsilon'_i(\phi'') = \varepsilon_i(\phi'(\psi/x)) \wedge_i \varepsilon_i(\phi''(\psi/x)) = \varepsilon_i((\phi' \wedge \phi'')(\psi/x))$. For negations we have for all i, $\varepsilon'_i(\neg\phi) = \neg_i \varepsilon'_i(\phi) = \neg_i \varepsilon_i(\phi(\psi/x)) = \varepsilon_i(\neg\phi(\psi/x))$. For necessitation we have for all i

$$\begin{aligned}
&\varepsilon'_i([\phi]) = 1_i \text{ if and only if}\\
&\varepsilon'_{i+1}(\phi) = 1_{i+1} \text{ if and only if}\\
&\varepsilon_{i+1}(\phi(\psi/x)) = 1_{i+1} \text{ if and only if}\\
&\varepsilon_i([\phi](\psi/x)) = 1_i.
\end{aligned}$$

Then, since ϕ is S-valid, we have $\varepsilon_0(\phi(\psi/x)) = \varepsilon'_0(\phi) = 1_0$. Thus $\phi(\psi/x)$ is S-valid. □

The logic of sense and denotation is therefore a normal modal logic. It is not yet clear whether the logic of sense and denotation is subsumed by **T**.

The definitions 22 and 23 may be altered, which may give rise to different kinds of logics of sense and denotation. If we only have one sense domain, or more precisely, if all sense domains E_i for $i > 1$ are isomorphic then it should be clear that $[\phi] \to [[\phi]]$ becomes valid. In this case, it appears that the modal logic **S4**, which satisfies $\Box\phi \to \Box\Box\phi$, is captured. Further alternations may capture other, more stronger, modal logics.

It does not appear to be possible to capture weaker modal logics than **T**, like **K**, because in the logic of sense and denotation, it must be the case, as mentioned in Section 2.1, that we have a mapping from the sense domain to their denotations, that is, we must have a mapping τ_i. Moreover τ_i extends to a homomorphism, so $[\phi] \to \phi$, which corresponds to axiom T, will be valid.

It is also possible to change the system such that we for sure will know that the logic we obtain is not equivalent to or subsumes a normal modal logic. If the sense domains, for instance, are weaker than Boolean algebras (as we have in the intensional concept logic, where there is one sense domain which is a semilattice) then we obtain a logic which does not subsume **T**, because we fail to have what corresponds to a necessitation rule, i.e. if ϕ is a propositional tautology then we do not necessarily have $[\phi]$. This shows that the system in this sense is richer than modal logics.

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Index

$\mathcal{ALC}$, 57

ABox, 59, 105
Alternative (0), 31
atomic concept, 71
atomic concept, $\mathcal{ALC}$, 57
atomic role, 57

bottom concept, 71
bottom concept, $\mathcal{ALC}$, 57

Carnap, 26, 37, 67, 69
characteristic attribute, 100
characterization of intensionality, 37
Church, 25, 30
classical logic, 49
closed under, 85
closure property, 85
commutativity, 22
compatibility property, 41
completeness, 86, 96
complex algebra, 74
complexity, 61, 97
concept, 11, 27, 67
  assertion, 59
  axiom, 72
  conjunction, 71
  conjunction, $\mathcal{ALC}$, 57
  description, 71
  description, $\mathcal{ALC}$, 57
  disjunction, 71
  disjunction, $\mathcal{ALC}$, 57
  equivalence, 71
  identity, 71
  negation, 71
  negation, $\mathcal{ALC}$, 57
concept of, 31, 96
conception of intensionality, 24, 91, 97
conceptual graphs, 64
congruence, 86
customary denotation, 23

decidable, 98
deductive closure, 85
derived from, 85
designated truth-value, 48
domain of interpretation, 57

entail, 72, 77
entailment, $\mathcal{ALC}$, 60
EPBA, 102
existential quantification, 71
existential quantification, $\mathcal{ALC}$, 57

extension, 13, 68
extensional, 39, 40
  algebra, 74
  concept assertion, 105
  formalization, 14, 69
  model, 72
  semantics, 72
  subconcept, 105
  subsumption, 70, 71
extensionality
  of $\mathcal{ALC}$, 61
  of first-order predicate logic, 46
  of modal logic, 44
  of propositional logic, 42
  of three-valued logic of Łukasiewicz, 48

FDE, 48
formal concept analysis, 52, 64
formal context, 64
formal deduction, 84
free algebra, 91, 96
Frege, 19, 28, 31, 35, 95, 111
Frege-Church conception, 25, 97
fully invariant congruence, 86

Galois connection, 52
general intensional interpretation, 94
general-satisfaction, 95, 98
generator, 106
global consequence, 44, 46

hierarchy of senses, 25, 29, 94

indirect context, 23
indirect denotation, 23
individual concept, 27, 64
information retrieval, 107
instance relation, 51
intension, 13, 36, 68, 75
intension of intension, 29, 68
intensional, 39, 40
  algebra, 74
  Boolean concept logic, 92
  Boolean model, 92
  concept assertion, 105
  conjunction, 75
  formalization, 14, 69
  isomorphic, 28
  model, 76
  negation, 75
  semantics, 76
  subconcept, 104
  subsumption, 71, 76
intensional subsumption, 70
intensionality
  of ABoxes, 106
  of FDE, 48
  of first-order predicate logic, 46
  of general intensional concept logic, 95
  of intensional Boolean concept logic, 92
  of intensional concept logic, 77
  of modal logic, 44
  of the logic of sense and denotation, 113
Intensionality proof rule, 83, 96
interchangeable, 27

knowledge interchange format, KIF, 64
Kripke, 38
Kripke model, 44, 62

L-equivalence, 27
L-interchangeable, 27
L-truth, 26
least fully invariant congruence, 86

level of intensionality, 95
local consequence, 44, 46
logical axioms, 82, 83

mark of, 51
mediation, 81
method of extension and intension, 28, 97
modal logic, 45
  $\mathbf{K}_m$, 62
  K, 38, 44
  S5, 44
  T, 110, 113
mode of presentation, 13
model, $\mathcal{ALC}$, 58, 59
multi-knowledge-based systems, 100

non-classical logic, 49
normal modal logic, 45

oblique context, 23
ontology, 11, 100
OntoQuery, 108

possible world, 36
possible world semantics, 36
proof rules, 83
property, 27, 69
property theory, 69
propositional attitudes, 23, 36
prototype theory, 64, 107
prove, 84

Reflexivity proof rule, 83
relation, 27
Replacement proof rule, 83
role assertion, 59
role constructions, 60
rules of designation, 26

S-interpretation, 111
S-valid, 112
satisfy, 72, 77
semantic networks, 55
semantical entailment, 46
sense, 13, 21, 28
sense sense, 25
soundness, 86, 96
state description, 26
strict equivalence, 110
subconcept, 51, 58
substitutable, 36
supplementary attributes, 100
Symmetry proof rule, 83

taxonomy, 12, 99
term algebra, 84
terminological axiom, $\mathcal{ALC}$, 58
terminology, 100
the Formulation, 31
three-valued logic of Łukasiewicz, 48
Transitivity proof rule, 83

universal concept, 71
universal concept, $\mathcal{ALC}$, 57

value restriction, 71
value restriction, $\mathcal{ALC}$, 57