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Analytical theory of dark nonlocal solitons

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We investigate properties of dark solitons in nonlocal materials with an arbitrary degree of nonlocality. We employ the variational technique and describe dark solitons, for the first time to our knowledge, in the whole range of degree of nonlocality.   © 2010 Optical Society of America

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Spatial dark optical solitons are localized light intensity dips in the infinite constant background [1,2]. Like their bright counterparts, they propagate while preserving their spatial profile. This is a result of a balance between diffraction, which tends to broaden their size, and non-linearity of the medium, which counteracts this process. The existence of dark spatial solitons requires the non-linearity of the medium to be negative, or self-defocusing; i.e., the refractive index of the medium must decrease with light intensity [1]. In the simplest case of the Kerr medium, the change of the refractive index is just proportional to the light intensity. In this special situation and in the one-dimensional case, the soliton amplitude $u(x,z)$ is described analytically by the following relation: $u(x,z) = B \tanh(x - vz) + iA$, where x and z denote the transverse and propagation coordinates, respectively. B and A are constants and v is the soliton transverse velocity, $v = A/B$. In cases of a more complex relation between light intensity and nonlinearity, the soliton profiles, except for some special models, can be only found numerically (see [2] and references therein). All the typical nonlinear models are *local*, i.e., the nonlinear response in a particular point is determined solely by the light intensity in the same point. However, there has recently been strong interest in the so-called nonlocal nonlinearities [3]. In those models the relation between nonlinear response and the intensity is spatially nonlocal. Physically it means that the nonlinearity in a particular spatial location is determined by the light intensity in a certain neighborhood of this location. Typical nonlocal systems involve either transport processes, such as heat [4], or ballistic atomic transport [5] and diffusion [6], charge separation [7], or long-range interaction, as in dipolar Bose Einstein condensates [8] or nematic liquid crystals [9]. Nonlocal nonlinearity has been shown to have a stabilizing effect on nonlinear structures [3,10–13]. It also affects soliton interaction inducing long-range attractive forces between solitons [14–17]. Unlike the Kerr nonlinear models, where the solitons can be described analytically, fully nonlocal models with an arbitrary degree of nonlocality can only be treated numerically [18,19]. The only exceptions are the special cases of the so-called weak and strong nonlocality, when the steady state bright and dark soliton could be found in analytical form [20,21].

In this Letter, we describe analytically, for the first time to our knowledge, properties of dark nonlocal solitons.

To this end we employ a variational approach and show that it enables one to retrieve the major features of dark solitons in a general nonlocal regime.

The evolution of a one-dimensional optical beam with an amplitude $u(x,z)$ in a nonlocal defocusing medium is governed by the following nonlocal nonlinear Schrödinger equation (NLS) [2]:

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u \int_{-\infty}^{+\infty} R(x - \xi)|u(\xi,z)|^2 d\xi = 0. \quad (1)$$

Here we used the phenomenological model of the nonlocal nonlinearity, with $R(x)$ being the nonlocal response function. Its width determines the degree of nonlocality [20]. In particular, for $R(x) = \delta(x)$, the above equation describes a standard Kerr local medium. While this is only a phenomenological model, it nevertheless describes very well general properties of the nonlocal media. Moreover, for a certain form of the nonlocal response function, this model represents the actual physical system. For instance, $R(x) \propto \exp(-|x|/\sigma)$, with σ being the degree of nonlocality, describes the long-range interaction-mediated response of nematic liquid crystals [15] and nonlinear interaction in quadratic media [21].

To analyze the nonlocal NLS equation, we will employ the Lagrangian approach, which in case of dark solitons has been formulated in [22]. Following this work, we find that the renormalized Lagrangian density corresponding to the NLS. Equation (1) is given in the following form:

$$\mathcal{L} = \frac{i}{2}\left(u^*\frac{\partial u}{\partial z} - u\frac{\partial u^*}{\partial z}\right)\left(1 - \frac{1}{|u|^2}\right) - \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 - \frac{1}{2}(|u|^2 - 1)$$

$$\times \int_{-\infty}^{\infty} R(x - \xi)(|u(\xi,z)|^2 - 1)d\xi. \quad (2)$$

To proceed further we need to specify the nonlocality. For the sake of simplicity and analytical tractability and without loss of generality, we consider here the rectangular profile for the nonlocal response function [20] $R(x)$:

$$R(x) = \begin{cases} \frac{1}{2\sigma} & -\sigma \le x \le \sigma, \\ 0 & \text{otherwise}, \end{cases} \quad (3)$$

with σ defining the degree of nonlocality. The accuracy of the Lagrangian approach depends on the functional choice of the variational solution. Here we use the obvious

ansatz

$$u(x,z) = B \tanh[D(x - x_0)] + iA, \qquad A^2 + B^2 = 1, \quad (4)$$

which represents an exact soliton solution in the local regime. All parameters A, B, D, and x_0 are assumed to be functions of the propagation variable z. Substituting Eqs. (4) and (3) into the Lagrangian density Eq. (2) and integrating over x, we get the "averaged" Lagrangian:

$$L = \int_{-\infty}^{\infty} \mathcal{L}(u) dx = 2 \frac{dx_0}{dz} \left[-AB + \tan^{-1}\left(\frac{B}{A}\right) \right] - \frac{2}{3} B^2 D$$
$$+ \frac{B^4}{D} \left[\mathrm{csch}^2(D\sigma) - \frac{\coth(D\sigma)}{D\sigma} \right]. \quad (5)$$

Then we find from the corresponding Euler–Lagrange equations that $B = \mathrm{const}$, and

$$\frac{\coth(D\sigma)}{D\sigma} \left[\frac{1}{D^2} - \sigma^2 \mathrm{csch}^2(D\sigma) \right] = \frac{1}{3B^2},$$
$$\frac{dx_0}{dz} = A \left[\frac{D}{3B} - \frac{B}{D} \left(\mathrm{csch}^2(D\sigma) - \frac{\coth(D\sigma)}{D\sigma} \right) \right]. \quad (6)$$

Notice that for $\sigma = 0$, the above formulas give $B = D = \mathrm{const}$ and $dx_0/dz = A$, which are the exact soliton parameters for local Kerr solitons [22]. The formulas of Eqs. (6) constitute the main result of this Letter. They give, for the first time to our knowledge, the analytical relation among parameters of the dark nonlocal soliton, for an arbitrary degree of nonlocality. In particular, one can use it to show the dependence of the soliton width ($\propto D^{-1}$) as a function of the degree of nonlocality (σ). This relation is depicted in Fig. 1 by the solid curve. Here the soliton width is normalized to its value in the local regime. The graph clearly shows that the width of the dark soliton is a nonmonotonic function of the nonlocality. It decreases first for small σ, reaches its minimum, and then monotonically increases. This nontrivial analytical result agrees with that found earlier in numerical simulations (see Fig. 3 in [21]). On physical grounds, the initial decrease of the soliton width can be explained by the fact that weak nonlocality causes the nonlinear index change to advance towards regions of lower light intensity. As a result, the waveguide induced by the soliton
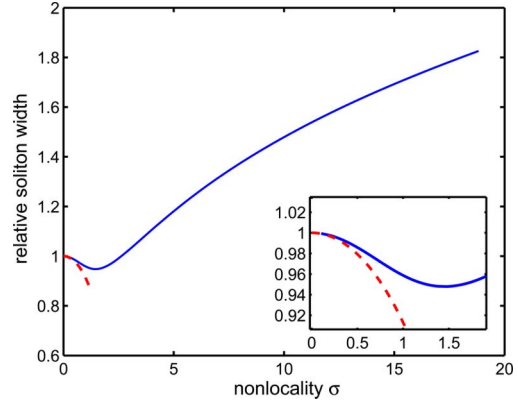

Fig. 1. (Color online) Width of the dark nonlocal soliton (normalized to its value in the local limit) as a function of the degree of nonlocality σ: solid blue curve, variational results with rectangular response function; dashed red curve, exact solution in a weakly nonlocal limit [20]. The inset depicts the magnified region of weak nonlocality. Here, $B \approx 1$ and $A \approx 0$.

becomes slightly narrower, and so does the soliton. The situation becomes different for a high degree of nonlocality. For large σ, the refractive index modulation expressed by the convolution integral in Eq. (1) becomes weaker and broader, acquiring a wide rectangular profile and resulting in increased width of the soliton.

It is instructive to compare the above variational calculations with the exact analytical solutions, which can be obtained in two limiting regimes of weak and strong nonlocality. In the former case, when the response function is much narrower than the soliton width, the convolution term in Eq. (1) can be expanded in a Taylor series, leading to the following form of the nonlinear response:

$$\int_{-\infty}^{+\infty} R(x - \xi)|u(\xi,z)|^2 d\xi \approx |u(x)|^2 + \gamma \frac{\partial^2 |u(x)|^2}{\partial x^2},$$

where $\gamma = \sigma^2/6$. The NLS with such a nonlinear term can be solved analytically (see [20]). It can be shown that the weakly nonlocal limit can be recovered from the general variational solutions of Eqs. (6) by expanding them into a Taylor series around $\sigma = 0$ [23].

The exactly found width of a dark soliton in a weakly nonlocal regime solution is shown in Fig. 1 by the dashed curve. It is evident that the agreement between variational
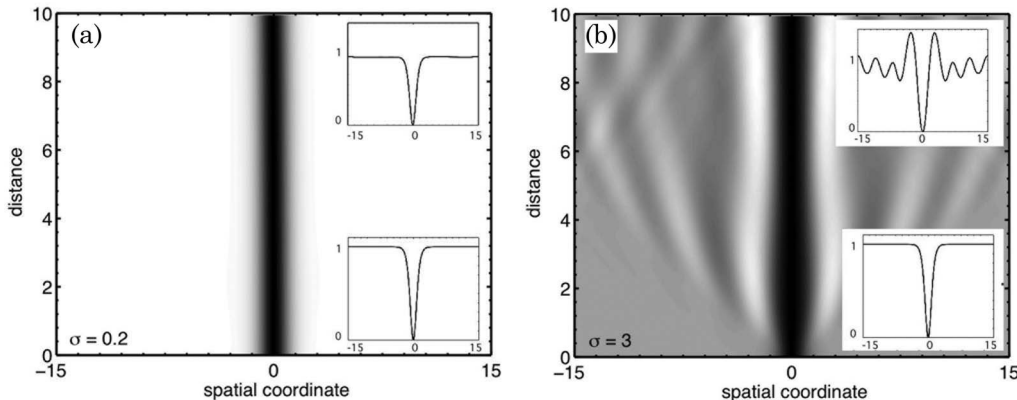

Fig. 2. Dynamics of formation of dark spatial solitons in (a) almost local, $\sigma = 0.2$, and (b) nonlocal, $\sigma = 3$ media with rectangular response function. The insets depict the soliton profile at the beginning (bottom) and end (top) of the propagation distance.

and exact solutions is indeed very good. This is an obvious consequence of the fact that the ansatz, Eq. (4), is actually pretty close to the exact solution in this regime. A different situation arises in the so-called highly nonlocal limit. It has been shown that in such a limit, the nonlocal NLS becomes linear with the convolution term being just proportional to the response function. In this regime, this equation can be solved exactly. For the rectangular response function, the solution is $u(x) = B \sin([\pi(x - x_0)]/2\sigma) + iA$, which gives a monotonic increase of the soliton width with nonlocality $D^{-1} \propto \sigma$. This, of course, differs from the variational solution of Eqs. (6), which for a large degree of nonlocality gives $D^{-1} \propto \sigma^{1/3}$. This discrepancy is most likely caused by the inadequacy of our ansatz in the highly nonlocal regime, as will be evident below.

In Fig. 2 we illustrate the effect of nonlocality on the formation and propagation of a single dark spatial soliton. We used the split-step Fourier method to integrate numerically the nonlinear Schrödinger equation. Figure 2 depicts the dynamics of a dark soliton excited by the initial conditions, represented by the variationally calculated profiles. Figures 2(a) and 2(b) correspond to the almost local ($\sigma = 0.2$) and nonlocal ($\sigma = 3$) cases, respectively. It is clear that nonlocality tends to expand the width of the soliton. The strong dispersive waves visible in Fig. 2 (b) reflect the fact that the initial conditions do not correspond to the stationary soliton profile.

In conclusion, we studied properties of single dark solitons in nonlocal nonlinear media. We used the variational approach and the rectangular model of the nonlocality to derive, for the first time to our knowledge, analytical relations for soliton parameters for an arbitrary degree of nonlocality. We showed that this approach, while approximate, faithfully represent properties of dark nonlocal solitons.

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