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Extreme non-linear elasticity and transformation optics

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Abstract: Transformation optics is a powerful concept for designing novel optical components such as high transmission waveguides and cloaking devices. The selection of specific transformations is a non-unique problem. Here we reveal that transformations which allow for all dielectric and broadband optical realizations correspond to minimizers of elastic energy potentials for extreme values of the mechanical Poisson's ratio ν. For TE (H_z) polarized light an incompressible transformation $\nu = \frac{1}{2}$ is ideal and for TM (E_z) polarized light one should use a compressible transformation with negative Poissons's ratio $\nu = -1$. For the TM polarization the mechanical analogy corresponds to a modified Liao functional known from the transformation optics literature. Finally, the analogy between ideal transformations and solid mechanical material models automates and broadens the concept of transformation optics.

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1. Introduction

Transformation optics is a hot scientific topic because it provides a way to determine the material properties which ensure that novel optical devices may function with a high performance. It builds upon studies of coordinate transformations for the Maxwell's equations [1] which in the seminal references [2–6] were tailored to devices with the fascinating cloaking response (see the review by Pendry [7] for the latest cloaking developments).

Other intriguing applications of transformation optics are available in the literature: High performance waveguides and beam splitters have been studied by Rahm et al. [8–11], an optical carpet cloaking device was presented in Li and Pendry [12] and experimentally verified by Valentine et al. [13] and Liu et al. [14], and field rotators were constructed by Chen and Chan [15] and Rahm et al. [16], just to mention a few from the long list of applications.

The underlying transformations are non-unique and have mainly been determined using analytical means in a fashion similar to [8] and to a smaller extend using computational techniques such as quasi-conformal mappings known from the field of mesh generation or by solving Laplace's equation [12, 17–19].

The present contribution shows that transformations, which allow for all dielectric and broadband optical realizations, are minimizers of elastic energy potentials for extreme values of Poisson's ratio. Naturally occurring materials have a positive Poisson's ratio which quantify that

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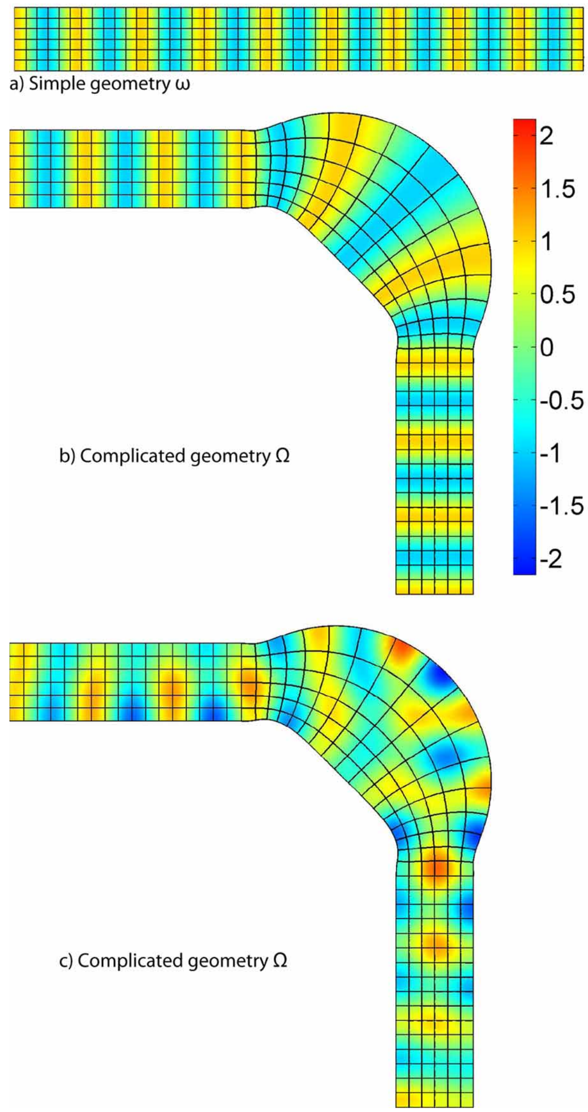

Fig. 1. Field plots of the magnetic field H_z for simple and complicated geometries with and without transformation optics. Perfect transmission in the simple geometry ω with frame of reference x_i (a). Response in the complicated geometry Ω with frame of reference $X_i = x_i + u_i$ with transformation optics and excellent transmission (b). Response in the complicated geometry Ω without transformation optics and a resulting poor transmission (c). The colorbar defines the amplitude of the wave.

they get thinner as they are stretched. A material with a negative Poisson's ratio gets thicker as it is stretched and it can be constructed by proper microstructural design see e.g. [20–24]. The present study is limited to 2D models of TE (H_z) and TM (E_z) polarized light propagation using numerical examples with waveguides cf. [9] and carpet cloaking cf. [12].

The authors realize that the contribution of the paper may be easily misunderstood since it builds on theory and ideas from the two normally disconnected areas - optics and non-linear elasticity. Hence, in the following we summarize what the paper is *not* about. The paper does not consider transformation acoustics or elasticity nor does it consider acoustic or elastic wave cloaking [25–27]. The static non-linear elasticity theory used in the paper is only used as a convenient (and well-proven) mathematical way of introducing *transformations* that ensure realizability of transformation optics problems and hence the realizability of the used extremal elastic materials is not an issue either.

The paper is organized as follows: Section 1 is the introduction, section 2 describes the connection between transformation optics and the theory of finite deformation elasticity, section 3 considers the ideal transformation for TE polarized light, section 4 treats the case of TM polarized light, section 5 elaborates on manufacturing aspects and section 6 concludes on the study.

2. Transformation optics formulated using finite deformation elasticity

The goal of transformation optics is to determine the optical material parameters which provide complex wave propagation (like cloaking) in a complicated geometry by a geometrical transformation from a simpler geometry. To fix ideas consider Helmholtz's equation governing a plane optical wave with TE (H_z) polarization posed on the simple geometry $\omega \in \mathbb{R}^2$

$$\begin{aligned}&\text{Find } H_z \in \mathscr{V} \text{ such that:}\\ &\int_\omega \big(-(\nabla v \cdot \eta \nabla H_z) + k_0^2 \mu_{33} v H_z\big)\,\mathrm{d}\mathbf{x} = 0 \quad \forall v \in \mathscr{W}\end{aligned} \tag{1}$$

where H_z is the out-of-plane component of the magnetic field, $\mathbf{x} = [x_1\ x_2]^{\mathrm{T}}$ denotes the spatial coordinates with gradient $\nabla = [\frac{\partial}{\partial x_1}\ \frac{\partial}{\partial x_2}]^{\mathrm{T}}$, k_0 is the wave number, $\eta = \varepsilon^{-1}$ is the relative impermittivity tensor being the inverse of the relative electric permittivity tensor ε associated with the 2D model, and μ_{33} is the out-of-plane component of the relative magnetic permeability tensor. Moreover, $\mathscr{V}$ and $\mathscr{W}$ are the trial and test function spaces for H_z and v, respectively. Thus a harmonic problem is at hand and the optical material parameters (η, μ_{33}) are taken to be isotropic and spatially uniform. Figure 1(a) shows the response of a simple geometry, being a straight wave guide with perfect electrically conducting boundary conditions, and consequently with a perfect transmission.

Mapping or deforming the simple geometry ω to a more complicated geometry $\Omega \in \mathbb{R}^2$ that uses the frame of reference X_i, transforms the governing equation to the form

$$\begin{aligned}&\text{Find } H_z \in \mathscr{V} \text{ such that:}\\ &\int_\Omega \big(-\big(\nabla_{\mathbf{X}} v \cdot \eta' \nabla_{\mathbf{X}} H_z\big) + k_0^2 \mu'_{33} v H_z\big)\,\mathrm{d}\mathbf{X} = 0 \quad \forall v \in \mathscr{W}\end{aligned} \tag{2}$$

where $\nabla_{\mathbf{X}} = [\frac{\partial}{\partial X_1}\ \frac{\partial}{\partial X_2}]^{\mathrm{T}}$ is the gradient wrt. to the coordinates of the complicated geometry. The material properties in Ω are denoted (η', μ'_{33}) which are to be determined such that a perfect transmission is obtained. It is done by applying a push-forward operation [28] to Eq. (1), i.e. a

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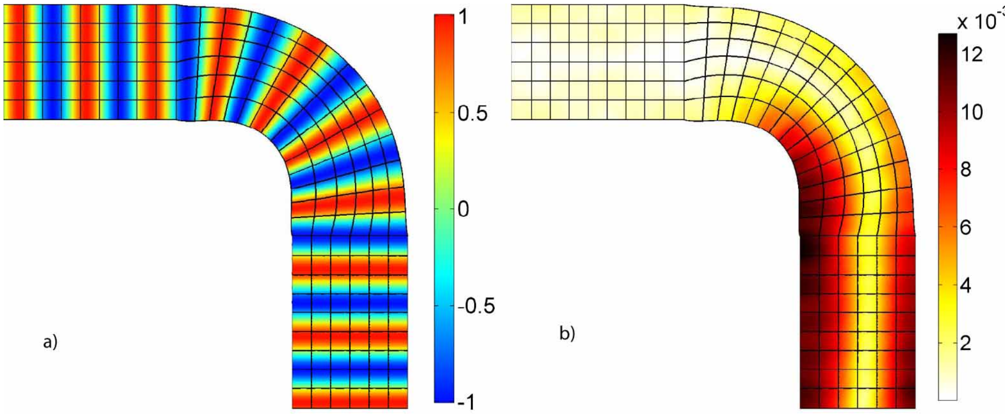

Fig. 2. Response for TE (H_z) polarized light in the complicated geometry (a) where only the impermittiviy η is transformed and the transformation is the minimizer of the elastic energy potential for a nearly incompressible material with $K/G = 1000$. The error measure $|H_z - H_z^{\text{ref}}|/|H_z^{\text{ref}}|$ where H_z^{ref} is the response where both η and μ_{33} are transformed (b). The colorbars define the amplitude of the wave and the error measure, respectively.

change of variables from x_i to X_i

$$\text{Find } H_z \in \mathscr{V} \text{ such that:}$$
$$\int_\Omega \left(-\left(\nabla_{\mathbf{X}} v \cdot \left(\Lambda \eta \Lambda^{\mathrm{T}}\right) \nabla_{\mathbf{X}} H_z\right) + k_0^2 \mu_{33} v H_z\right) \frac{\mathrm{d}\mathbf{X}}{\det(\Lambda)} = 0 \quad \forall v \in \mathscr{W} \tag{3}$$

where $X_i = x_i + u_i$ with u_i denoting the displacement field which deforms ω into Ω. $\Lambda_{ij} = \frac{\partial X_i}{\partial x_j} = \delta_{ij} + u_{i,j}$ is the Jacobian where δ_{ij} denotes Kronecker's delta and $i, j = \{1,2\}$. Thus by inspection of Eq. (2) and (3) one identifies the non-uniform parameters which deliver perfect transmission as

$$\eta' = \frac{\Lambda \eta \Lambda^{\mathrm{T}}}{\det(\Lambda)}, \qquad \mu'_{33} = \frac{\mu_{33}}{\det(\Lambda)} \tag{4}$$

where $(\cdot)'$ denotes the transformed optical parameter. Figure 1(b-c) show the response for the transformed parameters as well as for isotropic and spatially uniform parameters (i.e. $\eta' = \eta$ and $\mu' = \mu$) in the complicated geometry. It is clear that in the latter case the transmission is very poor.

Similarly, TM (E_z) polarized light is governed by

$$\text{Find } E_z \in \mathscr{V} \text{ such that:}$$
$$\int_\Omega \left(-\left(\nabla_{\mathbf{X}} v \cdot \left(\Lambda \mathbf{B} \Lambda^{\mathrm{T}}\right) \nabla_{\mathbf{X}} E_z\right) + k_0^2 \varepsilon_{33} v E_z\right) \frac{\mathrm{d}\mathbf{X}}{\det(\Lambda)} = 0 \quad \forall v \in \mathscr{W} \tag{5}$$

where $\mathbf{B} = \mu^{-1}$ is the relative impermeability tensor being the inverse of the relative magnetic permeability tensor μ associated with the 2D model and ε_{33} is the out-of-plane component of the relative electric permittivity tensor. Thus one finds the transformed parameters

$$\mathbf{B}' = \frac{\Lambda \mathbf{B} \Lambda^{\mathrm{T}}}{\det(\Lambda)}, \qquad \varepsilon'_{33} = \frac{\varepsilon_{33}}{\det(\Lambda)}. \tag{6}$$

On this basis several conclusions can be drawn. First, one achieves perfect transmission in the complicated geometry if one can control the spatial material variation according to Eq. (4)

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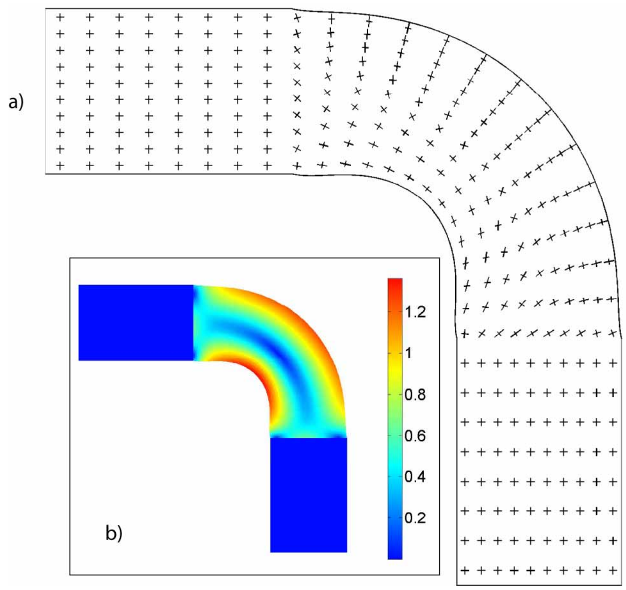

Fig. 3. Anisotropy of $\varepsilon = (\eta')^{-1}$ illustrated with crosses oriented in the principal directions (a). The relative difference between the principal directions (b) defined as $(q_1 - q_2) / \left(\frac{1}{2}(q_1 + q_2)\right)$ where q_i are the eigenvalues. In the straight entrance and exit of the geometry the permittivity is isotropic with a value of $\varepsilon_{\mathrm{inout}} = 3$ such that the eigenvalues in Ω satisfy $1.31 \leq q_2 \leq q_1 \leq 6.89$. The transformation is nearly incompressible and governed by Eq. (7).

or Eq. (6). If the values of the transformed parameters can not be realized by known materials one may succeed in scaling the transformed parameters at the price of changing the wavenumber. Moreover, the transformed parameters are tied to the underlying polarization model since it is (η, μ_{33}) (not (ε, μ_{33})) which are transformed for the TE case, and similarly, $(\mathbf{B}, \varepsilon_{33})$ (not (μ, ε_{33})) which are transformed for the TM case. Finally, from a manufacturing and practical point of view it is desirable to have only one spatially varying parameter, preferably the permittivity, i.e. either η' or ε'_{33}. Spatial variation of epsilon on a scale smaller than the optical wavelength is practically realizable (see e.g. [13]) and ensures lossless and broadband response whereas spatial modulation of the permeability μ usually is associated with high losses and difficulties in manufacturing.

Prescribing a pure dielectric realization limits the relevant transformations to those which yield a spatially uniform $\det(\Lambda)$ for the TE case (ensuring constant $\mu'_{33} = \mu_{33}$ in (4)) and spatially constant and isotropic $\mathbf{B}' = \mathbf{B}$ for the TM case in (6). In the following two sections we show that extreme non-linear elasticity provides transformations that satisfy these requirements for purely dielectric and broadband realizations of transformation optics results. Based on numerical experiments two different material models have been chosen from the mechanical engineering literature. They have their range of applicability for modeling incompressible and compressible material response, corresponding to the applications for E and H polarization, respectively.

3. Dielectrically realizable transformation for the TE case

There is a simple analogy between the ideal deformation associated with the TE case and a deformation of an incompressible material where $\det(\Lambda) = 1$ corresponding to the maximum

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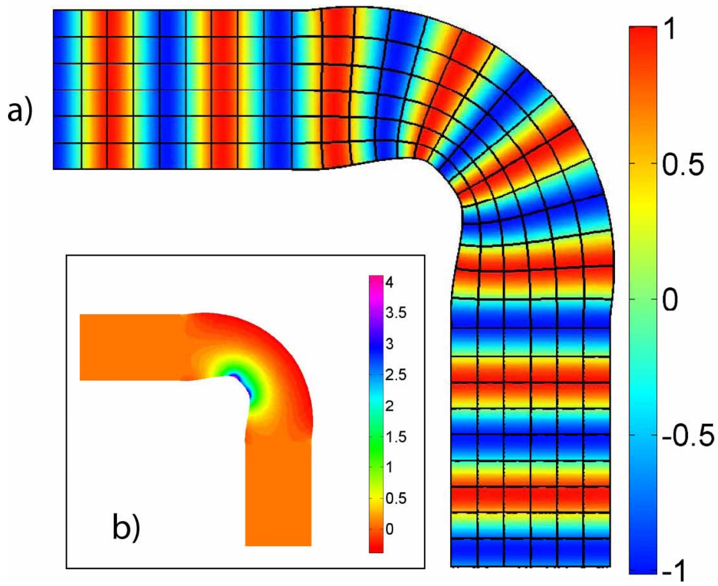

Fig. 4. Response for TM (E_z) polarized light in the complicated geometry (a) where only ε_{33} is transformed and the transformation is the minimizer of the elastic energy potential for a negative Poisson's ratio material with $G/K = 1000$ cf. Eq. (12). The relative difference between the $\left(\varepsilon_{33}' - \varepsilon_{\text{inout}}\right)/\varepsilon_{\text{inout}}$ where $\varepsilon_{\text{inout}} = 2$ is the background permittivity of the straight entrance and exit of the geometry (b). The transformed permittivity belongs to the interval $1.25 \le \varepsilon_{33}' \le 10.2$.

value of Poisson's ratio $\nu = \frac{1}{2}$. Therefore one may use the minimizer of the elastic energy potential of an incompressible material as the transformation. As an example a Neo-Hookean material model [29] suited for finite deformations is employed. It is derived from the potential

$$\psi = \int_{\omega} \left(\frac{1}{2} G \left(I_1 - 2 - 2 \ln J \right) + \frac{1}{2} K (J-1)^2 \right) \mathrm{d}\mathbf{x} \tag{7}$$

where $I_1 = \operatorname{trace}\left(\Lambda^{\mathrm{T}}\Lambda\right)$, i.e. the trace of the metric tensor (aka the right Cauchy-Green deformation tensor), $J = \det(\Lambda)$, G is the shear modulus and K is the bulk modulus. It is here clear that a large value of K compared to G will ensure $J = \det(\Lambda) = 1$ and hence a purely dielectric realization according to Eq. (4). Remark here that $K \gg G$ corresponds to a material with Poisson's ratio $\nu \approx 1/2$, i.e. an almost incompressible material.

Figure 2 shows the optical response where only η is transformed and the transformation is defined as the minimizer of Eq. (7) with $K/G = 1000$. The straight parts of Ω are prescribed by mechanical boundary conditions while the shape of the remaining curved part is determined by the minimization of ψ. Since the material is only nearly incompressible the transformation introduces a small relative error in the amplitude of the magnetic field, i.e. $|\max|H_z| - |H_z^{\text{ref}}||/|H_z^{\text{ref}}| = 1.5\%$, where H_z^{ref} is the ideally transformed magnetic field. Moreover, Fig. 3 shows the anisotropy of the corresponding permittivity tensor, i.e. $(\eta')^{-1}$, measured by the relative difference

$$r_q = \frac{q_1 - q_2}{\frac{1}{2}(q_1 + q_2)} \tag{8}$$

where $q_1 \ge q_2$ are the principal values of $(\eta')^{-1}$.

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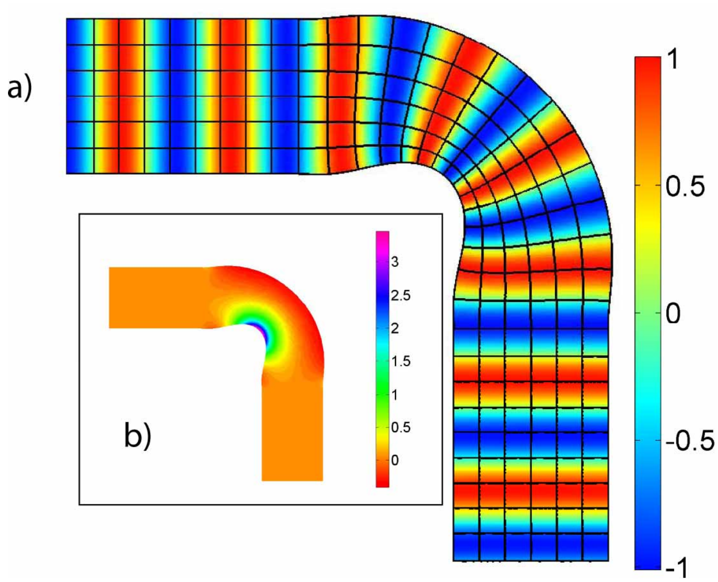

Fig. 5. Response for TM (E_z) polarized light in the complicated geometry (a) where only ε_{33} is transformed and the transformation is the minimizer of the modified Liao functional cf. Eq. (13). The relative difference between the $\left(\varepsilon'_{33}-\varepsilon_{\text{inout}}\right)/\varepsilon_{\text{inout}}$ where $\varepsilon_{\text{inout}}=2$ is the background permittivity of the straight entrance and exit of the geometry (b). The transformed permittivity belongs to the interval $1.20 \leq \varepsilon'_{33} \leq 8.92$.

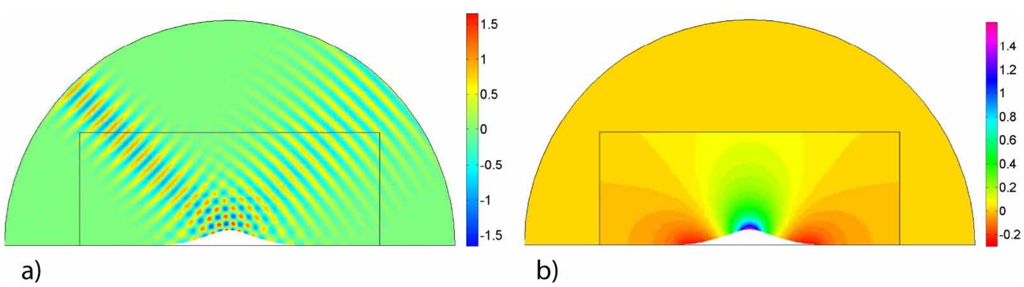

Fig. 6. A carpet cloaking example [12] where the transformation is generated by the modified Liao functional (a) with the amplitude of the wave given in the colorbar. The spatial variation of the permittivity (b), illustrated by the measure $\left(\varepsilon'_{33}-\varepsilon_{\text{inout}}\right)/\varepsilon_{\text{inout}}$ where $\varepsilon_{\text{inout}}=2$.

4. Dielectrically realizable transformation for the TM case

For the TM case the ideal transformation ensures that $\mathbf{B}'$ is isotropic, i.e. it has a double eigenvalue. For the case of an isotropic reference permeability one has $\mathbf{B}=\frac{1}{\mu}\mathbf{I}$ (with $\mathbf{I}$ denoting the identity matrix) and the difference between the eigenvalues of $\mathbf{B}'$ is thus controlled by $\Lambda\Lambda^{\mathrm{T}}$. By diagonalizing $\Lambda\Lambda^{\mathrm{T}}$ one finds

$$\Lambda\Lambda^{\mathrm{T}}=\mathbf{V}^{-1}\begin{bmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{bmatrix}\mathbf{V} \tag{9}$$

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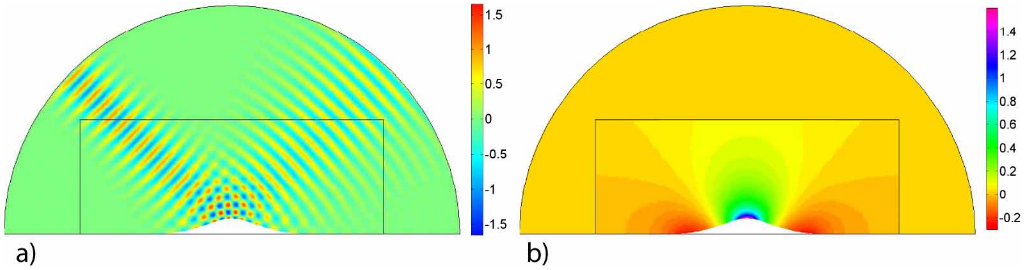

Fig. 7. A carpet cloaking example [12] where the transformation is generated by the elastic energy potential for a negative Poisson's ratio material with $G/K = 1000$ (a) with the amplitude of the wave given in the colorbar. The spatial variation of the permittivity (b), illustrated by the measure $\left(\varepsilon'_{33} - \varepsilon_{\text{inout}}\right)/\varepsilon_{\text{inout}}$ where $\varepsilon_{\text{inout}} = 2$.

where $\mathbf{V}$ is a matrix containing the eigenvectors. (γ_1, γ_2) are the eigenvalues which may be expressed by

$$\det\left(\Lambda\Lambda^{\mathrm{T}}\right) = \det\left(\Lambda^{\mathrm{T}}\Lambda\right) = \gamma_1\gamma_2 = J^2, \qquad \operatorname{trace}\left(\Lambda\Lambda^{\mathrm{T}}\right) = I_1 = \gamma_1 + \gamma_2. \tag{10}$$

Thus, one obtains an ideal transformation by minimization of $(\gamma_1 - \gamma_2)^2$, which may be written as

$$(\gamma_1 - \gamma_2)^2 = \gamma_1^2 + \gamma_2^2 - 2\gamma_1\gamma_2 = I_1^2 - 4J^2. \tag{11}$$

This again reveals a mechanical analogy, since the integrand of the elastic energy potential of a Saint Vernant-Kirrchhoff material [28]

$$\psi = \int_\omega \left(\frac{1}{8}G\left(I_1^2 - 4J^2\right) + \frac{1}{8}K\left(I_1 - 2\right)^2\right)\mathrm{d}\mathbf{x} \tag{12}$$

reduces to Eq. (11) (up to a constant scaling) in the asymptotic limit $G \gg K$. Moreover, it is noted that the limit $G/K \to +\infty$ corresponds to the minimum value of the Poisson's ratio $\nu = -1$.

Li and Pendry [12] used the minimizer of the modified Liao functional

$$\Phi = \frac{1}{A}\int_\omega \left(\frac{I_1}{\sqrt{J^2}}\right)^2 \mathrm{d}\mathbf{x}, \qquad A = \int_\omega 1\mathrm{d}\mathbf{x} \tag{13}$$

for their transformation for the carpet cloaking device. Up to a constant, this integrand is found to be a scaled version of Eq. (11) (corresponding to diagonalizing $\Lambda\Lambda^{\mathrm{T}}/J$ in the above derivation), since

$$\Phi = \left(\frac{\gamma_1 - \gamma_2}{J}\right)^2 + 4. \tag{14}$$

This analysis motivates the conclusion that both the elastic energy functional $\psi|_{G\gg K}$ as well as the modified Liao functional Φ minimize the anisotropy of $\mathbf{B}'$ required for an ideal transformation, albeit in a L_2 (integral) sense. In the following a waveguide and a carpet cloaking example are considered.

Figure 4 considers the bend of the same waveguide shown in Fig. 2 but now for the TM polarization. Only ε_{33} is transformed and the transformation is defined as the minimizer of Eq. (12) with $G/K = 1000$. Since the permeability is kept constant a small relative error in the amplitude of the electric field $|\max|E_z| - |E_z^{\text{ref}}||/|E_z^{\text{ref}}| = 2.0\%$ is introduced, where E_z^{ref}

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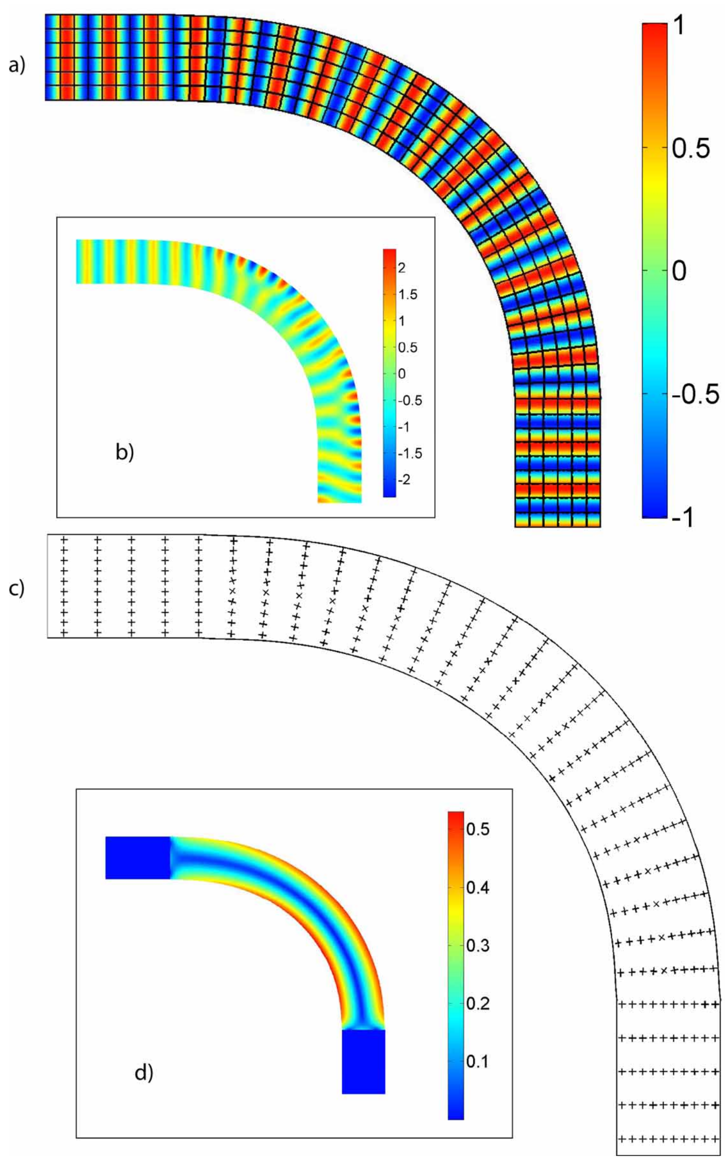

Fig. 8. Response for TE (H_z) polarized light in the complicated geometry where only η is transformed (a) and without transformation optics and $\varepsilon_{11} = \varepsilon_{22} = 4$ (b). The principal directions of $\varepsilon = (\eta')^{-1}$ are shown (c) and the relative difference between the eigenvalues $(q_1 - q_2) / \left(\frac{1}{2}(q_1 + q_2)\right)$ where q_i are the eigenvalues (d). The transformation is the minimizer of the elastic energy potential for a nearly incompressible material with $K/G = 1000$. The relative error measure of the amplitude is $|\max|H_z| - |H_z^{\mathrm{ref}}||/|H_z^{\mathrm{ref}}| = 1.2\%$. In the straight entrance and exit of the geometry the permittivity has been scaled to $\varepsilon_{\mathrm{inout}} = 4$ such that the eigenvalues of ε in Ω satisfy $3.05 \leq q_2 \leq q_1 \leq 5.25$. The colorbars define the amplitude of the waves and the relative difference between the eigenvalues, respectively.

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is the ideally transformed electric field. Moreover, Fig. 4b shows the spatial variation of ε'_{33}. Figure 5 shows the optical response when the modified Liao functional is used to generate the transformation. In this case a smoother shape variation is observed and the error in the amplitude is slightly smaller $|\max|E_z| - |E_z^{\mathrm{ref}}||/|E_z^{\mathrm{ref}}| = 1.1\%$.

Figure 6 and Fig. 7 display a carpet cloaking example cf. [12] using a scaling of the background material of $\varepsilon_{33} = 2$. The cloaking performance is quantified by the relative far field error

$$e_{\mathrm{rel}} = \frac{\int_{\widetilde{\Gamma}} |E_z - F_z|^2 \mathrm{d}s}{\int_{\widetilde{\Gamma}} |F_z|^2 \mathrm{d}s} \tag{15}$$

where E_z is the response in Ω with the carpet bump, F_z is the response corresponding to a flat carpet ($u_i = 0$), $\widetilde{\Gamma}$ is the far-field (circular) boundary of Ω without the ground plane and s a parameter along the boundary. Using this error measure the modified Liao functional has $e_{\mathrm{rel}} = 37.7\%$ and yields $1.44 \leq \varepsilon'_{33} \leq 5.01$. Using the elastic energy potential one finds $e_{\mathrm{rel}} = 31.8\%$ at the price of a slightly larger contrast since $1.45 \leq \varepsilon'_{33} \leq 5.09$.

5. Realization

From a manufacturing point of view one may realize the anisotropic ε', relevant for the TE (H_z) polarization, by a composite material with a microstructure which blends a low and high index material. In practice one may use a homogenization (averaging) approach (c.f. [12, 30, 31]) to determine the effective properties of a given microstructure and to solve the inverse problem of finding the lamination parameters resulting in the desired effective properties.

One complication occurs, however, if the relative difference between the principal values of ε' is large. In the example shown in Fig. 3 the difference is $r_q \leq 1.36$ which makes the realization difficult. To illustrate this, consider a simple Rank-1 material, cf. e.g. [31], where the eigenvalues (q_1, q_2) of the permittivity tensor reduce to the harmonic and arithmetic averages of the low and high index materials, i.e.

$$q_1 = m\varepsilon^- + (1-m)\varepsilon^+, \qquad q_2 = \left(\frac{m}{\varepsilon^-} + \frac{(1-m)}{\varepsilon^+}\right)^{-1} \tag{16}$$

where ε^- is the low index material, ε^+ the high index material and m the volume fraction of ε^-. In that case, the anisotropy measure given in Eq. (8) has its maximum for $m = \frac{1}{2}$, moreover for $\varepsilon^+/\varepsilon^- = 12$ one finds $r_q(m = \frac{1}{2}, \varepsilon^+/\varepsilon^- = 12) = 1.12$. That is, a Rank-1 material will not do the job for the waveguide seen in Fig. 3.

The above realizability problem can be resolved by increasing the bending radius (cf. Fig. 8) and hence relieving the contrast requirement. The relative error in the amplitude for this example is $|\max|H_z| - |H_z^{\mathrm{ref}}||/|H_z^{\mathrm{ref}}| = 1.2\%$ and the relative difference between the principal values is small enough ($r_q \leq 0.53$) to ensure a simple realization with low contrast materials.

For both polarizations effective medium theory requires the wavelength to periodicity ratio λ_w/λ_p to be large in order to secure a broadband (non-resonant) response. However, it is reported in [13] that a ratio of the order of $\lambda_w/\lambda_p = 3$ is sufficient to ensure reliable estimates and hence easy manufacturing with standard optical lithography techniques.

6. Conclusions

This study shows that the elastic energy potential for materials with extreme values of Poisson's ratio ν generate transformations which allow for all dielectric and broadband realizations of complex optical devices. The ideal transformations for TE (H_z) and TM (E_z) polarized light are obtained by using the extreme Poisson's ratio values $\nu = \frac{1}{2}$ and $\nu = -1$, respectively. To the

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authors best knowledge, no previous works have presented a transformation that ensures all-dielectric realizations for the TE polarization case. Since computational mechanics is a mature field both in academia and in an industrial setting this discovery opens doors for an automated and broader application of transformation optics based on advanced material modelling tools. These tools allow a high geometric resolution, possess a high computational efficiency and are designed for the workflow of industrial research and development.

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