Technical University of Denmark

The mortgage choice problem

Rasmussen, Kourosh Marjani; Clausen, Jens

Publication date:
2008

Document Version
Publisher's PDF, also known as Version of record

The mortgage choice problem

Kourosh Marjani Rasmussen

Kongens Lyngby 2008

IMM-PHD-2008-189

Technical University of Denmark

Informatics and Mathematical Modelling

Building 321, DK-2800 Kongens Lyngby, Denmark

Phone +45 45253351, Fax +45 45882673

reception@imm.dtu.dk

www.imm.dtu.dk

IMM-PHD: ISSN 0909-3192

Summary

In the light of the recent years' steep rise in the universe of products offered by the Danish mortgage banks an advisory model for individual homebuyers is introduced in this thesis. Taking the existing mortgage products, homebuyers risk preferences, tax rules and transaction costs into consideration, the model helps mortgage advisors find the optimal choice of mortgage loan for an individual homebuyer. The model provides the homebuyer with basis for a decision which is by far more tailored to the individual's needs as compared to current practice.

The number of mortgage products available in the Danish market has steeply increased during recent years. From a handful of products just 10 years ago, it was added up to no less than 60 according to Skovgaard (2005). With the introduction of the new covered bond legislation (SDO lovgivning) in July 2007, this number is expected to increase even further in the future. It is therefore an ever more challenging task to advise individual homebuyers on their choice of mortgage strategies. Mortgage advisors should therefore have access to tools and analysis which in an easily accessible way convey pros and cons of the decision of potential homebuyers.

Today mortgage banks provide homebuyers with information on first year payments only. With the introduction of the new covered bond legislation the banks should instead provide the annual costs in percents. The problem with both of these key figures is that they say nothing about future risk and as such they are grossly misleading. Svend Jakobsen (2007) argues that politicians have not been sufficiently ambitious on homebuyers behalf. He suggests a consequence analysis over a set of scenarios where both increasing and decreasing interest rates are considered. In this thesis we go a substantial step further towards finding the best possible decision under future uncertainty for a given homebuyer.

The thesis describes a model which solves the homebuyers optimal mortgage choice problem based on a number of optimality criteria. The model involves modeling interest rate uncertainty, mortgage pricing, homebuyers preferences for risk and return, limiting loss using the refinancing

optionality as well as transaction costs and tax rules.

Resumé

Med udgangspunkt i de senere års kraftige stigning i realkredittens produkt–palette i Danmark introduceres i denne afhandling en rådgivningsmodel, der på baggrund af bl.a. de eksisterende realkreditprodukter, låntagers præferencer, beskatning og transaktionsomkostninger skal hjælpe rådgiveren til at optimere låntagers valg af realkreditlån. Modellen giver låntageren et beslutningsgrundlag, som i langt højere grad end hidtil tager højde for den enkelte låntagers behov.

Realkreditinstitutternes produktpalette er de seneste år vokset kraftigt. For bare 10 år siden havde låntagerne kun en håndfuld forskellige produkter at vælge i mellem. I mellemtiden er antallet af låneprodukter mangedoblet. I en artikel af Skovgaard (2005) blev antallet af forskellige realkreditprodukter i Danmark således opgjort til ikke færre end 60. Den nye SDO lovgivning, der trådte i kraft juli 2007, vil formentlig betyde en yderligere udvidelse af produktpaletten. I en rådgivningssituation kan det derfor både nu, og måske specielt fremover, være svært at finde det helt rigtige produkt til kunden. I det lys er det vigtigt, at rådgiverne har kendskab og adgang til værktøjer og analyser, der på en nem og overskuelig måde kan anskueliggøre fordele og ulemper ved låntagerens valg.

Første års ydelse er det eneste nøgletal, som de fleste realkreditinstitutter oplyser i rådgivningssammenhænge i dag. I forbindelse med SDO lovgivningen er der indført skærpede krav om lånerådgivning i form af en revision af bekendtgørelsen om god skik for finansielle virksomheder. Det pålægger realkreditinstitutter at oplyse de årlige omkostninger i procent (ÅOP). Problemet med begge disse nøgletal er, at der ikke bliver taget højde for fremtidig risiko. Svend Jakobsen (2007) argumenterer for, at lovgiverne ikke har været tilstrækkeligt ambitiøse på låntagernes vegne. I artiklen foreslår Svend Jakobsen, at der skal tages udgangspunkt i en konsekvensberegning. Vi går her et stort skridt videre i retning af at stille det bedst mulige beslutningsgrundlag, under fremtidig usikkerhed, for låntageren.

Denne afhandling beskriver en model, der ud fra en række kriterier løser låntagerens problem omkring valget af det rigtige realkreditlån. Modellen

inddrager alle relevante realkreditprodukter og disses markedspriser, låntagers præferencer for risiko og gevinster, begrænsning af tab ved omlægninger samt omkostninger ved optagelse og omlægninger og beskatningsregler.

Preface

This thesis was prepared at IMM, DTU in partial fulfillment of the requirements for acquiring the Ph.D. degree in engineering.

The thesis deals with different aspects of mathematical modeling for finding the optimal choice of mortgage for an individual homebuyer. The main focus is on developing and testing a modeling framework to capture the real–life complexity of the mortgage choice problem, but also specialized interest rate modeling, appropriate choice of risk measure and the interpretation of certain mortgage products as Giffen goods are considered.

The thesis consists of a summary report and a collection of five research papers written during the period 2004–2007. The first three of these papers are at this point already published in international journals within the areas of finance and operations research.

Lyngby, November 2007

Kourosh Marjani Rasmussen

Papers included in the thesis

[A] Kourosh Marjani Rasmussen and Jens Clausen (2007), Mortgage Loan Portfolio Optimization Using Multi–Stage Stochastic Programming. *Journal of economic dynamics and control*, 31, pp 742–766.

[B] Rolf Poulsen and Kourosh Marjani Rasmussen (2007), Financial Giffen Goods: Examples and Counterexamples. *European Journal of Operational Research*, in press.

[C] Kourosh Marjani Rasmussen and Stavros A. Zenios (2007), Well ARMed and FiRM: Diversification of mortgage loans for homeowners. *The Journal of Risk*, 10, pp 67–84.

[D] Kourosh Marjani Rasmussen and Stavros A. Zenios (2007), Optimal Mortgage Loan Diversification. *Working paper available at:*

http://www2.imm.dtu.dk/ kmr/

[E] Kourosh Marjani Rasmussen and Rolf Poulsen (2007), Yield curve event tree construction for multi stage stochastic programming models. *Working paper available at: http://www2.imm.dtu.dk/ kmr/*

Acknowledgments

I thank my supervisor Professor Jens Clausen who co–authored the first paper in this thesis and helped shape the direction for further work. His approval and support for my close cooperation with the industry has resulted in a work which has already been read by many and that has become the theoretical foundation for an advisory system which has recently been put in use in Nykredit realkredit A/S.

The main innovation in this thesis is the combination of theoretical results from mathematical finance with the mathematical programming framework within optimization in finance all put into a real–life application area. My other two co-authours Professor Rolf Poulsen from University of Copenhagen and University of Gotheburg and Professor Stavros A. Zenios from University of Cyprus and the Wharton Financial Institutions Centre, University of Pennsylvania, Philadelphia have motivated this work and contributed greatly to the quality of the results. Rolf Poulsen's work in mathematical finance is widely published and acknowledged within the mathematical finance community. Stavros A. Zenios is an international front figure in the field of optimization in finance. I thank them both for their collaboration, for excellent advice and many fruitful discussions we have had during the work on the papers. An extra thanks goes to Stavros for his hospitality during my several visits in Cyprus in 2005 and 2006. I believe close collaboration in between the mathematical finance and the optimization in finance communities will result in solving several interesting realistic problems yet unsolved by each of the groups alone. The work presented here is the result of such a collaboration.

Special thanks to the Nykredit team around the project Optimus – the title for the business case and the software developed in cooperation with Nykredit – without whom this work would not have been nearly as applied and realistic as it has become. We have discussed the modeling aspects such as appropriate choices of interest rate models, pricing algorithms and optimization criteria on a daily basis. Likewise the test setup and the analysis of results have been debated intensively within the group. Kenneth Styrbæk, Søren Lolle, Thomas Kyhl, Steen H. Bertelsen, Theis Ingerslev, Michael A. Carlsen, Svend Bondorf and Henrik Hjortshøj–Nielsen have all been involved in different parts of this project. In particular Kenneth Styrbæk and Søren Lolle have developed a graphical user interface and specialized code to interact with a mortgage pricing module (ScanRate's RIO). This has eased the testing process immensely.

Also thanks to the people from ScanRate A/S in particular Professor Svend Jakobsen and Johnni Andersen for providing advice on the use of their mortgage pricing system (RIO) and its integration into our model framework.

The current work has resulted in some spin off master thesis projects which I have partially supervised alongside my work on this thesis. I would like to thank my students for showing interest in this work and I hope they will take the research work up where this thesis leaves it off.

Finally a huge thanks goes to my wife Anne Mette Rasmussen. Without her love and support I would have never finalized this work. I dedicate this thesis to my wife and to my two loving children Anna Clara and Carl Gustav.

Contents

Part I

Summary report

CHAPTER 1

Introduction

This thesis consists of a summary report, chapters 1 – 8, and a collection of five research papers in the appendices. The purpose of the summary report can be summarized as follows:

1. Chapter 1 motivates the problem and gives an overall problem description.
2. Chapter 2 describes the Danish mortgage bond market.
3. Chapter 3 compares the traditional approach on mortgage advising with our approach as suggested in this thesis.

4. Chapter 4 introduces the methods used throughout this thesis.
5. Chapter 5 summarizes the papers and clearly states their interrelation.
6. Chapter 6 points out the novel contributions achieved in this thesis.
7. Chapter 7 documents additional tests and results on model robustness which have not been fully addressed in the papers.
8. Chapter 8 draws overall conclusion and shows directions for future work.

1.1 Background and motivation

Homebuyers in most countries take up mortgages for their house financing needs. In Denmark they may loan up to 80% of the value of the house. This thesis deals with which loan or which combination of loans is optimal for an individual homebuyer.

Until 1996 callable fixed rate mortgages (FRMs) were the only type of mortgages available in the Danish market. So mortgage advisors were only concerned with the timing for rebalancing an already existing mortgage. Since then the number of mortgage products has steeply increased. The main innovations have included introduction of adjustable rate mortgages (ARMs) in 1996, then interest–only (IO) versions of both FRMs and ARMs were introduced in 2003. Finally in 2005 the capped rate mortgages (CRMs) entered the market.[1] The number of mortgage products was added up to no less than 60 according to Skovgaard (2005). With the introduction of the new covered bond legislation (SDO lovgivning) in July 2007, this number is expected to increase even further in the future. It is therefore an ever more challenging task to advise individual homebuyers on their choice of mortgage strategies. Mortgage advisors should therefore have access to tools and analysis which in an easily accessible way convey pros and cons of the homebuyers decision.

The total amount of outstanding mortgage loans in Denmark in 2006 was 250 billion EURO, corresponding to 120% of the GDP. The great volume of the outstanding debt means that appropriate choices of mortgages are not only of interest for the individual household but they also have great macro economical importance. Risky choices of mortgages, combined with a house price fall and increased unemployment would result in mass de-

[1] One of the Danish mortgage banks (Totalkredit) launched the first CRMs in Denmark (BoligX lån) already in 2000. The CRMs did not gain much popularity, however, until another mortgage bank (Realkredit Danmark) introduced their first generation of CRMs in 2004 followed by other variants of the CRMs introduced by all mortgage banks in 2005.

faults on the individual homeowner side which in turn may result in a further devaluation of the housing market and may at the worst case bring major financial institutions to bankruptcy, which again may result in economical depression. The recent sub prime loans crisis is an example of how irresponsible and speculative choice of mortgages for even a partial segment of the US market has threatened financial and economical stability in several parts of the world.

The liberalization of the mortgage markets should therefore be accompanied by sufficient individual advice for homebuyers in order to suit the individual's needs and preferences while at the same time reducing default risk. The advice given today is by far not sufficient and it is certainly not tailored to the needs of the individuals.

1.2 Problem statement

The central question to be answered in this thesis can be formulated as follows:

Find the optimal choice of mortgage loan(s) and the consequent rebalancings for an individual homebuyer.

The problem statement above needs more clarification. What is the optimality criteria for a given homebuyer? What is an appropriate horizon for optimization? How are future interest rate and mortgage price uncertainties captured?

We need to answer these questions before any attempts for justifying why we consider a mortgage strategy optimal. We believe that these questions do not have a completely objective answer. There is no standard framework for modeling interest rate and mortgage price uncertainty. Most homebuyers have no clear idea of for how long they are going to keep the property and the notion of optimality for a mortgage cashflow given its price is understood differently by different groups of homebuyers. Nevertheless mortgage bank advisors should provide homebuyers with advice on their mortgage choice.

In this thesis we define what we understand by appropriate assumptions on these essentially subjective questions. When the assumptions are set, we will move on to introducing a modeling framework in which several optimality criteria, several horizons, as well several models for interest rate and mortgage price uncertainty may easily be implemented and their results tested.

Chapter 2

The Danish mortgage bond market

The Danish mortgage bond market is Europe's second largest covered bond market after the German market. Real property financing in Denmark is mainly based on mortgage loans raised through mortgage banks whose lending is funded exclusively through the issuance of mortgage bonds – covered bonds.

The purpose of this chapter is to introduce the reader to the rules of the Danish mortgage bond market as well as the products offered. The complex nature and the risks involved in these products should convince the reader that the research done in this thesis on advising homebuyers on a proper choice of mortgage loan is well justified.

2.1 The Danish mortgage finance legislation

The main principles behind the Danish mortgage finance system are:

- All loans are granted against mortgages on real property.
- The balance principle which implies that all lending is funded through the issuance of bonds and that the repayments on the loans and the payments to the bondholders must always be balanced. This balance between funding and lending eliminates the interest rate, liquidity and currency risks relating to the mortgage bank balance sheets.
- Mortgage banks have no influence on lending rates which are completely market–dependent.

The balance principle, the backbone of Danish mortgage finance, has basically not been changed since 1850. It eliminates the mortgage bank's liquidity, interest rate and currency risk. The only risk remaining for the mortgage banks is the default risk on the borrower side. Should the borrower default, however, the value of the property typically covers most of the charges. Even though borrowers default from time to time, the mortgage bondholders have not faced a single case of insolvency on the mortgage bank side during the 200–year long history of Danish mortgage bonds.

The Danish market is characterized by a high degree of concentration – at present, four major issuers account for 95% of the bond debt outstanding. The liquidity of the Danish mortgage bonds is further supported by the fact that all mortgage banks issue bonds with almost identical characteristics resulting in a unity–like market. In practice, bonds from different issuers are therefore traded on equal terms.

The liquidity of the Danish mortgage bonds, the balance principle and the long history of the Danish mortgage banks (with no insolvency cases) has resulted in an extremely efficient market – it would not be an exaggeration to consider it world's most efficient market. This means that the investors enjoy a high degree of security on their investments on Danish mortgage bonds and that the borrowers experience extremely attractive rates on their home financing. Danish homebuyers, due to the balance principle, effectively issue mortgage bonds via the mortgage banks. The mortgage banks receive a margin of approximately 0.5% for assuming the default risk on the borrower side as well as the administration costs. This margin is the lowest on any mortgage market in the world.

It is noteworthy that on the first of July 2007 a new amendment, known as the Danish covered bond legislation, was added to the Danish mortgage finance legislation. Among other things the new legislation allows separation of lending and funding within certain limits. The new law opens up for designing new mortgage products which are not simply pass–throughs. This means that mortgage banks should make a decision as to whether or not they are willing to assume some degree of the interest rate, liquidity and currency risks when issuing bonds to investors and lending money to homebuyers. So far none of the Danish mortgage banks have utilized this feature of the new law. But should they consider to make use of the new possibilities, the work done in this thesis is of even more importance not only for advising homebuyers on their mortgage choice but also for optimal product design and risk management.

The mortgage products introduced in this chapter and analyzed throughout the thesis all abide by the balance principle.

2.2 Mortgage products

Fixed rate mortgage loans (FRMs) which are funded in long–term fixed rate callable annuity bonds have traditionally dominated the Danish mortgage bond market. However, the introduction of fixed rate non–callable bullet bonds and related adjustable rate mortgages (ARMs) in the second half of the 1990's and, most recently in 2004, the successful introduction of capped long–term floating rate Cibor[1]–linked bonds and related floating rate mortgage loans with interest rate caps (CRMs) have diversified the Danish mortgage bond market, providing investors as well as borrowers with far more investment opportunities. In the following we give a short outline of the main features of these types of mortgages.

Fixed rate mortgages

Fixed rate mortgages (FRMs) are funded by fixed rate callable annuity bonds with a strike price at par. That means that the borrower should never pay more than the face value of the outstanding debt in case of prepayment of the mortgage. FRMs come both with and without interest–only options. Interest–only periods have a maximum period of 10 years. Maturities available for FRMs are 10, 20 or 30 years.

[1] Copenhagen Interbank Offered Rate.

Fixed rate callable bond series have an opening period of typically three years. That means that when a bond serie is created by the mortgage bank it has a maturity which is 3 years longer than the maturities available for FRMs. The serie remains open for lending to borrowers up to three years unless the bond price goes above par due to interest rate decreases or if the price falls way below par due to interest rate increases. This process ensures a high volume of the outstanding debt in the individual bond series and thereby reduce liquidity risk.

Adjustable rate mortgages

An adjustable rate mortgage (ARM) is funded by issuing one or more underlying bullet bonds. A bullet bond is a non–callable coupon paying bond with a single repayment of principal on the maturity date. The Danish bullet bonds have maturities of 1 to 11 years, and the Danish borrower may choose between ARMs with coupon fixing periods of 1 to 10 years (ARM1 to ARM10).

Since bullet bonds are per construction interest–only the Danish ARMs can be offered with an interest–only option without incurring any extra costs to the borrower (unlike the interest–only FRMs). ARM's are also offered as annuities by synthetic constructions of the same underlying bullet bonds. The annuity ARM does not incur any extra costs to the borrower either.

Capped rate mortgages

Capped rate mortgages (CRMs) are funded by floating rate annuity bonds (floaters). The coupons are typically based on six–month Cibor plus a fixed spread and they are subject to semi–annual coupon fixing. CRMs are offered with or without interest–only options. The interest–only periods have a maximum period of 10 years and they are slightly more expensive compared to their annuity counterparts.

CRMs have maturities of 5, 10, 20 or 30 years, and the underlying bond series have opening periods of typically three years, like the FRMs.

2.3 The delivery option

A distinct and very important feature of the Danish mortgage finance system is the delivery option also called the buyback option. It means that the Danish borrowers may terminate their loans by buying back the mortgage bonds in the bond market and delivering them to the mortgage bank. The buyback option applies to all mortgage bonds whether callable or non–callable. The buyback option constitutes a significant difference between the US and the Danish mortgage finance system. The US system only allows mortgage loan prepayment at par (100). The buyback option is an advantage to borrowers in situations with rising interest rates. As bond prices fall, the market value of borrowers' debt is reduced along with borrowers' exposure to increasing rates. This is particularly useful in case of decreasing property prices or moving to another property being forced to refinance at the higher interest rate level. For borrowers with 30–year fixed rate loans, such effect may be significant.

2.4 Refinancing and prepayment

Refinancing refers to the process of changing one or more underlying bonds behind a mortgage loan with some other bonds. For ARMs refinancing usually means adjusting of the mortgage rate to the market rate of the underlying bond. An ARM with yearly adjustments (ARM1) is refinanced once a year. In practice it means that the outstanding debt of the maturing bullet bond is paid by issuing a new one–year bullet bond. This type of refinancing is done free of charge for the borrower. The borrower can choose to change the fixing period at the refinancing point.

This incurs some refinancing fees.

Refinancing FRM's and CRM's refers to the process of paying back the outstanding debt before horizon (prepayment) by issuing new bonds. The borrower uses either the call option or the buyback option to prepay a loan. Prepayment usually occurs as a consequence of the callability of FRMs at par. In the case of decreasing interest rates the borrower prepays the mortgage with a higher coupon by issuing a new mortgage with a lower coupon. The new mortgage may be an FRM, an ARM or a CRM. Another reason for prepayment is reduction of outstanding debt. When interest rates increase the prices of FRMs and CRMs fall, so the underlying bonds may be bought back at a cheap price. This transaction is funded by either an ARM, FRM or CRM of a higher price and probably higher rate. The result is an outstanding debt reduction which approximately corresponds to the difference of the old and the new bonds. Prepayment also occurs simply due to selling the property. Prepayment incurs extra fees as compared to refinancing of ARMs to different fixing periods at fixing times.

Chapter 3

Our approach versus the traditional mortgage advice

A valid question at this point would be "what is the value added by introducing a new mortgage advising system?" This chapter answers this question by comparing the mortgage advising practice today with the one we suggest in this thesis.

3.1 Traditional mortgage advice

Today mortgage banks are only required to provide homebuyers with information on first year payments. With the introduction of the new covered bond legislation the banks should also provide the annual costs in percent. The problem with both of these key figures is that they say nothing about future risk and as such they are grossly misleading. Svend Jakobsen (2007) argues that politicians have not been sufficiently ambitious on homebuyers behalf. He suggests a consequence analysis over a set of scenarios where both increasing and decreasing interest rates are considered. Indeed some mortgage banks have taken up the idea and as an extra advisory service they provide payment calculations under a few interest rate scenarios for a given choice of mortgage loan. Even though this approach provides more information to homebuyers than first year payments and annual costs in percent, it has the following flaws:

1. The interest rate scenarios are generated on an ad hoc basis. Market information is not used to capture the overall tendencies in the dynamics of the term structure of interest rates.

2. It is not possible to calculate rebalancings before horizon. Most mortgagors rebalance their mortgage as market movements warrant

it along the way. Therefore a decision on the choice of mortgage loan here and now should consider future rebalancing possibilities under different market conditions.

3. The analysis is done for one mortgage loan at a time. Even if one allows for a combination of loans and perhaps some ad hoc rebalancings along the way, the analysis will not reveal what the best strategy is according to some criteria for example lowest average payments, least variability, least maximum payments, etc.

3.2 Our approach

In this thesis we go a large step further from the existing methods towards finding the best possible decision under future uncertainty for a given homebuyer.

Figure (3.1) gives an example of what we understand by an optimal loan strategy.

For simplicity of this illustrative example we have made the following assumptions:

1. We only consider two mortgage loans, an adjustable mortgage loan

with yearly adjustments (ARM1) and a fixed rate mortgage with 4% coupon payments (FRM 4%).

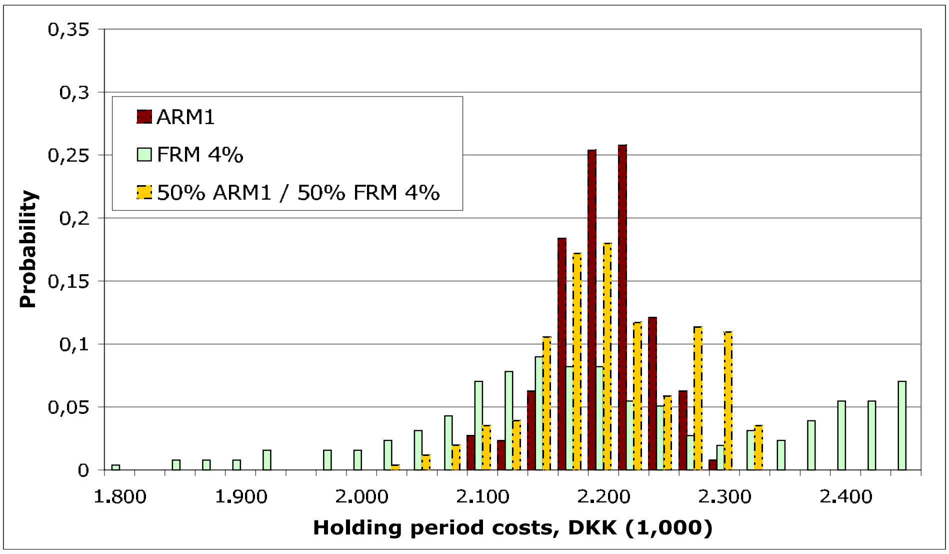
2. We wish to compare the holding period costs over a five–year period.

3. We consider only issue and hold strategies, i.e. no rebalancings are allowed.

4. We wish to find the combination of loans which results in the smallest average holding period cost for the highest 10% of the holding period cost scenarios.

Comparing the two frequency distributions for ARM1 and FRM 4% it is obvious that the ARM1 distribution has a smaller right tail. Now given that the homebuyer of our example wish to minimize the average of the 10% right tail, the question is whether a combination of the two loans will result in a smaller right tail than that of ARM1. In the existing consequence analysis systems one may simulate several combinations of these two loans and compare the right tails obtained. We have tried this once with a 50–50 combination of the two loans, which clearly does not result in a smaller right tail than that of the ARM1 alone. We could continue these calculations for several other combinations until some threshold for possible improvement is reached. The problem with this approach is that it is neither computationally efficient, nor does not render a guarantee for finding the optimal combination. Applying our optimization model we can within a few seconds find the optimal combination which is an 81–19 combination of ARM1 and FRM 4%.

The overall theme of this thesis is to make such an example as realistic as computational resources and the existing uncertainty embedded in the nature of this problem allow us. Our model framework allows for several mortgage products, future rebalancing possibilities under uncertainty as well as several different optimization criteria. In the next chapter we take a step back and introdue the methods needed to achieve the objectives of this thesis.



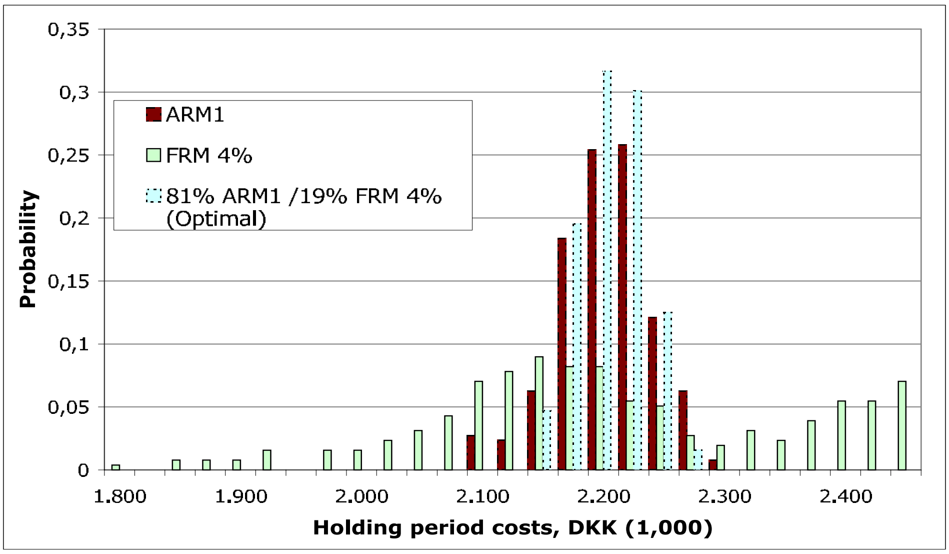


Figure 3.1: Comparison of single loan strategies with loan portfolios. Top: An arbitrary combination of the two loans is compared with the two single loan strategies. Down: The optimal combination of the two loans is compared with the two single loan strategies. Here the optimization criteria is to minimize the average of the highest 10% of the holding period costs.

Chapter 4

Fundamental elements and methods for the mortgage choice problem

This work can be characterized as an integration of different models into a system which provides mortgagors with individual advice. The integration is illustrated in Figure 4.1.

In this chapter we will not show *our* way of applying these modeling paradigms to the problem at hand – this is explained in chapters 5, 6

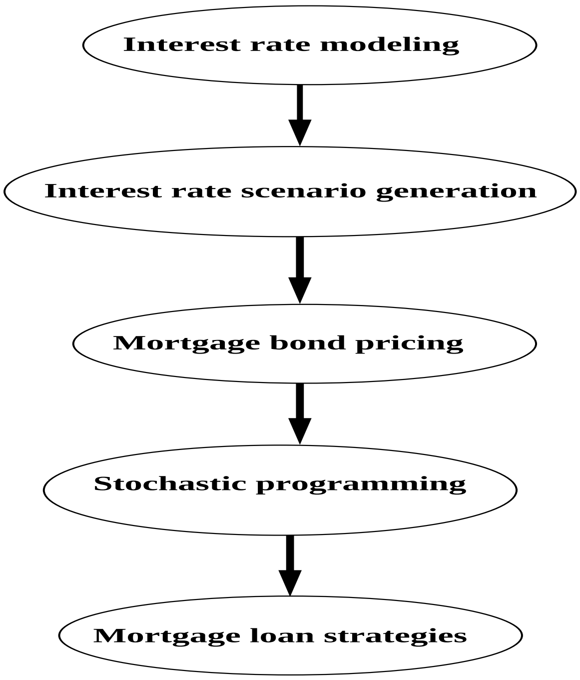


Figure 4.1: The modeling paradigms and their interactions in this thesis.

and 7 as well as in the papers in the appendices. We will, however, briefly go through the basic terminology and the intuition behind the particular methods which we build upon. A complete coverage of these methods and theories is beyond the scope of this thesis.

4.1 Interest rate modeling

The predominant risk affecting the cashflow payments of a mortgagor is the risk associated with changes in the general level of interest rates. When the interest rates increase the cashflow payments of short term financing increase as well. On the other hand the value of outstanding debt for long term fixed rate financing decreases[1]. Interest rate models are mathematical descriptions of interest rate dynamics. They describe possible movements of the entire term structure of interest rates.

4.1.1 Term structure of interest rates

The term structure of interest rates, or the yield curve, is the set of interest rates for different investment periods or maturities. Yield curves can display a wide variety of shape as seen in Figure 4.2. Mostly, a yield curve slopes upwards, with longer term rates being higher. Such curves are called *normal*. But several examples of historical inverse yield curves have been observed too. One such example is shown in Figure 4.2 for the Danish yield curve on the 30/08/2000.

[1] This is a special case in Denmark due to the buyback delivery option on the underlying fixed rate bonds as explained in chapter 2.

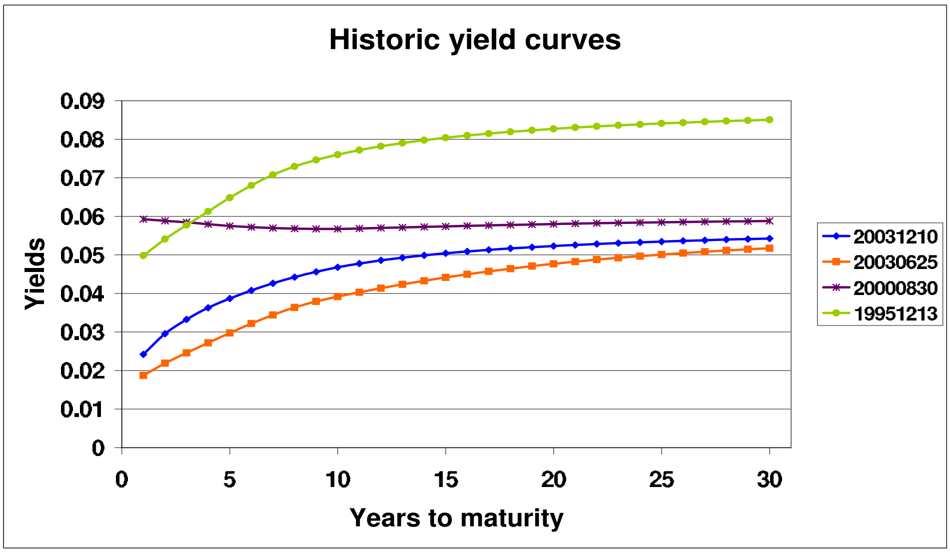


Figure 4.2: Danish yield curves from 4 different historical time points.

Principal component analysis of interest rates in several fixed income markets have shown that changes in level, slope and curvature of the yield curves can explain almost all variation. Looking at Figure 4.2 one can see that parallel shifts of the yield curves are not the only way yield curves move in the Danish market either. For more details on this subject see paper E in the appendices.

4.1.2 Examples of interest rate models

Three elementary interest rate models with the short rate r_t being the underlying state variable are defined below:

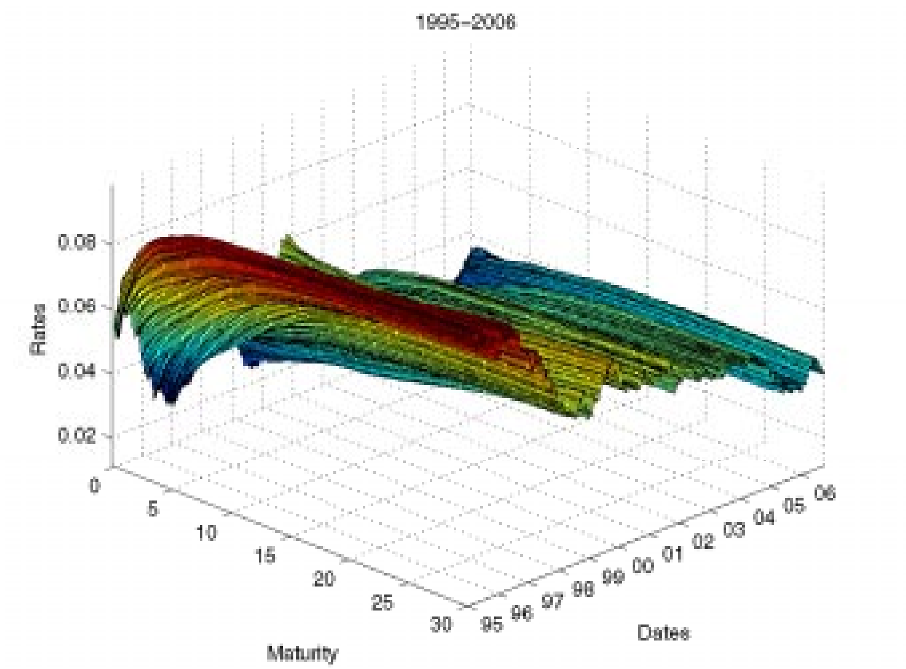


Figure 4.3: Historical data on Danish yield curves for the period 1995 to 2006.

- Extended Vasicek (time–varying mean, Hull & White (1993)),

$$dr_t = \alpha(\theta(t) - r_t)dt + \sigma dz_t;$$

- Extended CIR (time–varying mean, Cox, Ingersoll & Ross (1985) and Jamshidian (1995)),

$$dr_t = \alpha(\theta(t) - r_t)dt + \sigma r_t^{\frac{1}{2}} dz_t;$$

- CKLS (Chan, Karolyi, Longstaff & Sanders (1992)),

$$dr_t = \alpha(\mu - r_t)dt + \sigma r_t^{\gamma} dz_t.$$

All models have a mean reversion level – the time–varying $\theta(t)$ in the extended Vasicek and extended CIR models and the constant μ in CKLS. The parameter α decides the height of the interest rate jumps at each step. The models also have variance σ and a stochastic Wiener process z_t. The extended CIR model has a factor $r_t^{\frac{1}{2}}$ in its volatility which can ensure that rates do not become negative. The volatility function in the CKLS model is slightly more flexible.

A large number of scientific papers have been written on interest rate models. The models offer numerous variations of the simple models mentioned above and they add each some special features to them. As some of the most important enhancements to these models one could mention:

1. Adding the number of state variables (n–factor models) to better capture the dynamics of the whole yield curve of the underlying market.

2. Using alternative stochastic processes for different monetary regimes, for example allowing jumps in times of hyperinflation and allowing

only positive rates with high volatility in times of very low interest rates.

We will not go any further on exploring these special features. Two excellent books on interest rate modeling are Brigo & Mercurio (2006) and James & Webber (2000).

4.2 Interest rate scenario generation

The mortgage choice problem does not have closed–formed solutions in continuous time and state. This is due to the fact that we have several instruments with complex cashflows in a dynamic setting and that market frictions such as variable and fixed transaction costs and tax regulations play an important role on the optimal portfolio choice. The uncertainty space needs to be discretized both in time and state. We refer to the process of generating discrete yield curve scenarios as interest rate scenario generation.

In the following we introduce some scenario generation methods for use in stochastic programming applications. These methods are general and may be used for discretizing any underlying stochastic process. Kaut & Wallace (2003) give a review of these methods. To the best of our knowledge, no comparative studies on the suitability of these methods for use in stochastic programming have been published up to the time of this writing.

4.2.1 Bootstrapping

Bootstrapping is the simplest approach for generating scenarios. It does not involve using any underlying interest rate model. Instead it uses the available historical data directly as future scenarios. For example yield curves observed the last 120 months may be used to indicate possible yield curve scenarios in a month, a year or in five years. The strength of this approach, besides being simple, is that it preserves the observed historical correlation. However, there are serious shortcomings:

1. It can only be used for one–period models, since there is no mechanism to capture the conditional moments in between the periods.
2. The information about the current level of the stochastic variable is ignored.
3. The volatility of the historical data is only correctly captured if we use disjunct observations of the same length as the period length for the scenarios. For example the 120 monthly observations of yield

curves may only be used for generating scenarios over the next month.

4. The method never suggests a scenario not observed historically.

5. It does not necessarily generate consistent yield curve scenarios with for example no existence of arbitrage.

4.2.2 Sampling

The most common method for generating scenarios in finance is sampling from an underlying stochastic process such as an interest rate model. Sampling does not suffer from the shortcomings of the bootstrapping method, since the underlying stochastic process may be quite advanced. What is more, sampling is almost as easy as bootstrapping in that it is essentially a question of generating random numbers from the distribution of an underlying random variable.

The main problem with sampling is the curse of dimensionality. It is common that over 1000 scenarios are generated to match the statistical properties of a continuous one–factor stochastic process. The number grows exponentially as the number of periods in the scenario tree increases. A multi–factor stochastic process with non–perfectly correlated variables has a similar effect on the number of scenarios.

Due to this curse of dimensionality a number of variance reduction methods have been developed. Variance reduction is a procedure used to increase the precision of the estimates that can be obtained for a given number of iterations. Every randomly generated variable from the simulation is associated with a variance which limits the precision of the simulation results. Variance reduction methods are then used to reduce this variance. The main methods are: Common random numbers, antithetic variates, control variates, importance sampling and stratified sampling.

4.2.3 Moment matching

Høyland & Wallace (2001) suggest a simple moment matching approach to generate scenarios for stochastic programs. Unlike in sampling, moment matching uses optimization to generate scenarios which match some statistical properties of an underlying stochastic process. Such properties may include mean, covariance, skewness, kurtosis, percentiles, higher co–moments and so on.

Given a set of statistical properties s_l, their estimated values VAL_{s_l} and a weight w_{s_l} attached to every statistical property the moment matching problem is formulated as the following optimization problem:

$$\min \sum_{s_l=s_1}^{s_L} w_{s_l}(f_{s_l}(x_n,p_n) - VAL_{s_l})^2$$

wrt.

$$\sum_n p_n = 1$$

$$p_n \geq 0 \qquad\qquad \text{for all } n \in 1,\cdots,N.$$

Here, x_n is the value of the stochastic variable found by the optimization model at every scenario n, the function f_{s_l} takes all such values with their probability p_n and returns the value for the statistical property in question. Note that in this formulation both x_n and the scenario probability p_n are defined as variables. In many cases the probabilities p_n are fixed beforehand to reduce the non–linearity of the problem.

A moment matching approach ensures statistical accuracy by definition as it matches the statistical moments. In that respect the method is much more efficient than sampling – fewer scenarios are needed to match the moments. However, the approach is too general for many applications. Extra conditions need to be added to meet the particular correctness and consistency criteria for individual applications. We give such an example in paper E.

4.2.4 Optimal discretization

Pflug (2001) and Hochreiter & Pflug (2002) introduce a number of optimization models for scenario tree generation using what they refer to as “optimal discretization”. Optimal discretization is essentially different from all the other discretization methods, in that the focus is *not* on capturing the characteristics of an underlying stochastic process as closely as possible. Instead the method generates scenario trees such that the discretization error in the objective function of the underlying stochastic programming model is minimized. The discretization error of the objective function can, however, only be determined within some lower and upper bounds, which are not necessarily tight, meaning that optimal discretization does not with guarantee overperform other methods such as moment matching. More work is needed to investigate the effectiveness of this method in practical applications.

4.3 Mortgage bond pricing

This section reviews the pricing models applied to fixed rate callable mortgage bonds (the bonds behind FRMs) as well as Cibor linked floating rate callable mortgage bonds (the bonds behind CRMs). Conceptually, the pricing of non callable bullet mortgage bonds (the bonds behind ARMs) is straightforward. The payments of a bullet bond are discounted with for example the swap curve plus a constant yield curve spread (which generally increases with the maturity of the bond). The pricing of fixed rate callable mortgage bonds and Cibor linked floating rate callable mortgage bonds is, however, more complex due to the embedded options.

4.3.1 Pricing of fixed rate callable bonds

In principle, a fixed rate callable bond constitutes a portfolio of a non callable bond and a short position in a Bermudan call option on that bond (with a strike price of 100) reflecting the embedded prepayment option. However, for pricing purposes, the prepayment option cannot be treated as a standard Bermudan call option since borrowers do not pursue rational exercise strategies. There is no prepayment risk when a mortgage bond trades below par (since the bond trades at market price), but for bonds trading above par the prepayment option is in the money and therefore there is a substantial prepayment risk.

Empirical prepayment models based on historical data are needed to price fixed rate callable mortgage bonds. Such models predict the prepayment rate for a given payment date as a function of the yield curve and other factors affecting the level of prepayments such as the size of the loans.

The most important factor affecting the prepayment rate is the gain from refinancing to a lower rate. The gain is defined as the percentage reduction in the mortgage payments on the new loan, taking taxation and prepayment costs into account. When prepaying a loan, borrowers face both fixed and variable costs. The gain calculation is based on the total payment for the next year or the present value of all remaining payments using the after tax refinancing rate on the new loan as the discount rate. On average, borrowers prepay large loans more actively than smaller loans. This fact has to be taken into account by the prepayment model as well.

4.3.2 Pricing of capped floaters

Capped floaters carry a floating rate, are callable and have an embedded option in the form of an interest rate cap. The cap has a fixed strike throughout the maturity of the bond, typically up to 30 years. The repayment profile will be of the annuity type where amortization may be deferred for the first 10 years. A characteristic of Danish capped floaters is that the annuity rate tracks the six–month Cibor. This means that the repayment profile of the bonds is stochastic as the annuity rate is fixed on the basis of the development in six–month Cibor. As the bonds have embedded options, a stochastic yield curve model is required for the pricing. This model must be calibrated to basis options (such as caps and swaptions) matching the implied options embedded in the capped floaters.

With such a model at hand the pricing of capped floaters is done in a straight forward manner, i.e. without a need for a prepayment model. The embedded call option in capped floaters is insignificant and will theoretically or practically never go above the strike price of 105.

4.4 Stochastic programming

Stochastic programming is a framework for modeling optimization problems that involve uncertainty. Whereas deterministic optimization problems are formulated with known parameters, real world problems almost invariably include some unknown parameters. When the parameters are known only within certain bounds, one approach to tackling such problems is called robust optimization. Here the goal is to find a solution which is feasible for all such data and optimal in some sense. Stochastic programming models are similar in style but take advantage of the fact that probability distributions governing the data are known or can be estimated. The goal here is to find a policy that is feasible for all (or almost all) the possible data instances and maximizes the expectation of some function of the decisions and the random variables. More generally, such models are formulated, solved analytically or numerically, and analyzed in order to provide useful information to a decision maker.[2] Two classical books on stochastic programming are Birge & Louveaux (1997) and Kall & Wallace (1994).

4.4.1 Two–stage stochastic programs

The most widely applied and studied stochastic programming models are two–stage linear programs. Here the decision maker takes some action in the first stage, after which a random event occurs affecting the outcome of the first-stage decision. A recourse decision can then be made in the second stage that compensates for any bad effects that might have been

[2] This definition of stochastic programming is taken from the Stochastic Programming Community homepage: http://www.stoprog.org/

experienced as a result of the first–stage decision. The optimal policy from such a model is a single first–stage policy and a collection of recourse decisions (a decision rule) defining which second–stage action should be taken in response to each random outcome.

Let (Ω, P) be a probability space, $\omega \in \Omega$ be the realization of the uncertain data parameters and $p(\omega)$ the corresponding probability. Let A, b, c be deterministic parameters and x the first stage deterministic decision variable. We define a two-stage stochastic program as:

$$\begin{aligned}
\min Z =& cx + E_\omega Q(x,\omega) \\
&\text{wrt.} \\
&\quad Ax = b \\
&\quad x \geq 0
\end{aligned}$$

where the recourse function $Q(x, \omega)$ is defined as follows:

$$Q(x,\omega) = \min fy(\omega)$$

wrt.

$$D(\omega)y(\omega) = d(\omega) + B(\omega)x$$

$$y(\omega) \geq 0$$

The parameters $D(\omega)$, $d(\omega)$ and $B(\omega)$ as well as the recourse variable $y(\omega)$ are stochastic and defined over Ω. The two–stage stochastic program may now be rewritten as:

$$\min Z = cx + E_{\omega}[fy(\omega)]$$

wrt.

$$Ax = b$$

$$-B(\omega)x + D(\omega)y(\omega) = d(\omega)$$

$$x, y(\omega) \geq 0;$$

4.4.2 The deterministic equivalent of the two–stage stochastic program with recourse:

Once the uncertainty space is represented as a set of discrete scenarios then the stochastic programs can be formulated as deterministic ones. For the two–stage stochastic program the deterministic equivalent is formulated as follows:

$$
\begin{aligned}
\min Z = & cx + p^1 f y^1 + p^2 f y^2 + \cdots + p^k f y^k \\
& \text{wrt.} \\
& Ax = b \\
& -B^1 x + D^1 y^1 = d^1 \\
& -B^2 x + \qquad D^2 y^2 = d^2 \\
& \quad \vdots \qquad\qquad \ddots \quad \ddots \\
& -B^k x + \qquad\qquad D^k y^k = d^k \\
& \\
& x, y^1, y^2, \cdots, y^k \geq 0; \\
& 0 \leq p^\omega \leq 1 \quad \text{and} \quad \sum_\omega p^\omega = 1.0
\end{aligned}
$$

Here, the set of scenarios ω are enumerated from $1, \cdots, k$. Note that in the second stage, we take some function f of the recourse variables $y^1, \cdots, y^k$ first and then average the return values.

4.4.3 Multistage stochastic programs

Two–stage stochastic programs can be extended to several stages in the following manner:

$$\min_{x_1} = \Big\{ c_1 x_1 + E_{\omega_2}\Big[\min_{x_2} c_2 x_2 + E_{\omega_3|\omega_2}\Big[\min_{x_3} c_3 x_3 + \cdots + E_{\omega_T|\omega_{T-1}|\cdots|\xi_2} \min_{xT} c_T x_T \Big]\Big]\Big\}$$

wrt.

$$\begin{aligned}
&A_{11}x_1 && = b_1 \\
&A_{21}x_1 + A_{22}x_2 && = b_2 \\
&A_{31}x_1 + A_{32}x_2 + A_{33}x_3 && = b_3 \\
&\vdots \qquad\qquad \ddots && \vdots \\
&A_{31}x_1 + A_{32}x_2 + A_{33}x_3 + \cdots + A_{TT}x_T && = b_T,
\end{aligned}$$

where x_1 is a deterministic first stage decision variable and x_2 – x_T are stochastic recourse variables for periods 2 – T. ω_t is the realization of the uncertain data parameters for times $t = 2, \cdots, T$. The uncertainty unfolds at a given time t conditioned on the states of the uncertainty realized at time $t-1$.

4.4.4 Two formulations of a stochastic program

Consider the scenario trees in Figure 4.4:

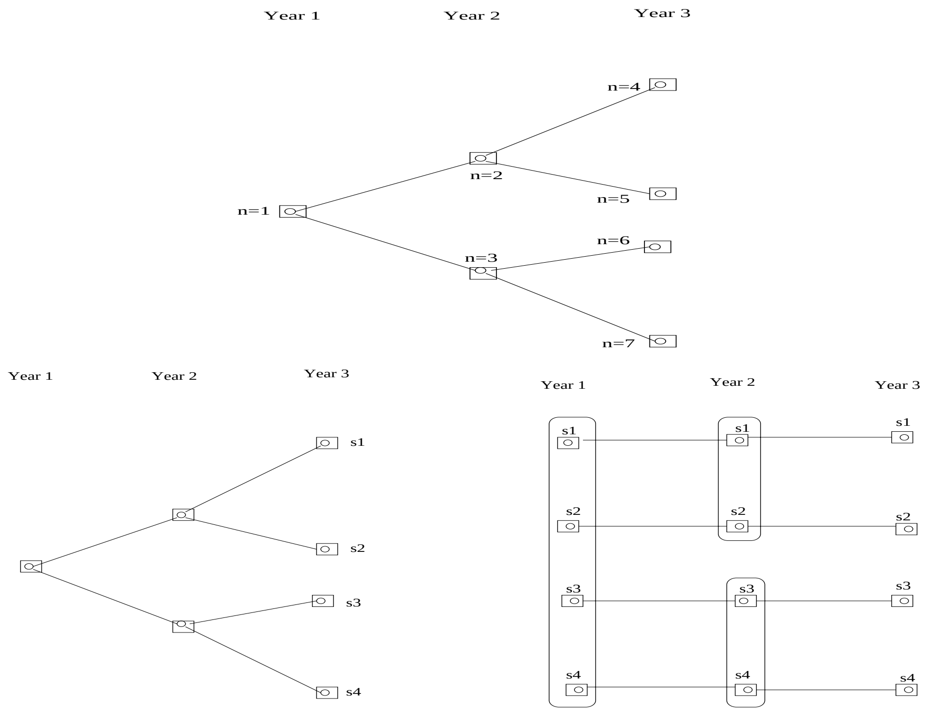


Figure 4.4: A scenario tree may be represented either by a number of nodes (top) or by a number of scenarios and time points (down).

Stochastic programs can be formulated either by using nodes or by using scenarios. In the node formulation the uncertainty variables are represented in harmony with the way the uncertainty is unfolded. In the scenario formulation (also called split–variable formulation) the number of uncertainty variables at each time point is multiplied by the number of scenarios. So in this formulation we need explicitly to make sure that not more than one decision is made at any given node. This is done by adding a number of constraints known as "non–anticipativity" constraints. For the example shown in Figure 4.4 (down) and given a stochastic variable $x_{t,s}$ defined over all times t and scenarios s we need to add the following constraints:

$$x_{t_1,s_1} = x_{t_1,s_2} = x_{t_1,s_3} = x_{t_1,s_4}$$

$$x_{t_2,s_1} = x_{t_2,s_2}$$

$$x_{t_2,s_3} = x_{t_2,s_4}$$

Chapter 5

Summary of the papers

The research efforts in this work are within the domain of optimization in finance and applied mathematical finance. The focus has been on realistic problem solving. That involved developing and testing several mathematical models in the one hand and financial analysis and interpretation and discussion of the findings in the other. Along the way the research has also resulted in a theoretical proof. In this chapter we review the main features of our work as presented in papers A through E and point out how the papers are interrelated around the same central theme, namely the mortgage choice problem.

5.1 Interest rate modeling

Interest rate dynamics is a very well researched area. Thousands of research papers and several books are written with the main focus on interest rate modeling. These models are, however, developed mostly in order to provide the underlying dynamics for pricing of interest rate sensitive instruments here and now rather than ensuring that future interest rate dynamics are captured in a realistic manner. Their success criteria is resulting in realistic present values for interest rate sensitive instruments.

In our setting we not only need prices of mortgages here and now but we also need approximative prices under different market conditions for rebalancing purposes in some future scenarios. Our contribution within interest rate modeling is presented in paper E. Our model is to the best of our knowledge the only one which uses the three factors level, slope and curvature directly and thereby produces a real–life–like variation over term structure predictions. The model is an specialization of a vector autoregressive model with lag 1 (VAR1). It is easy to calibrate to historic time series with some time step, say weekly observations. The length of the prediction steps does not need to be equal to the step length for the historic observations. Interest rates can be predicted over varying time steps (say annually or bi–annually) without having to simulate over steps of same length as in the calibration data. This means, besides computational efficiency, that the scenario trees generated based on this model are reproducible which is an important quality for testing.

5.2 Scenario generation

The scenario generation is a two–step process:

1. An event tree of the term structures of interest rates is built.
2. Mortgages are priced in every node of the scenario tree.

5.2.1 Interest rates

With an interest rate model at hand we need to generate a scenario tree of interest rates. Our scenario generation approach is explained fully in paper E. We define a number of quality requirements for a scenario tree of term structures. Our scenario generation approach is an extension of the moment matching approach of Høyland & Wallace (2001).

We developed the new scenario generation approach in the later part of the Ph.D. project which is why the method is not tried in the optimization models from the papers represented in this thesis. In paper A we use the one factor model of Black, Derman & Toy (1990) and in papers C and D we use a specialized version of the Vasicek model developed by Jensen & Poulsen (2002). We have, however, compared the results of the optimization models based on the new scenario generation approach with the results from the above mentioned papers. These results are presented in chapter 7 of this summary report under a discussion on model robustness.

5.2.2 Mortgage bond prices

Once a scenario tree of interest rates is built the universe of available mortgage bonds need to be priced in the nodes of the tree. While this is an straight forward calculation for bullet bonds which are the funding instruments behind ARMs, it becomes an extremely challenging task when it comes to pricing callable fixed rate mortgage bonds which are long term annuities with embedded Bermudan call options as well as buyback delivery options. Pricing such bonds asks for a proper prepayment (burn out) model which predicts the exercise of the embedded options under different interest rate scenarios. Besides the models used for pricing such bonds normally add a so called option adjusted spread (OAS) to the theoretical prices found in order to match market prices of the product. Likewise, capped floaters – the funding instruments behind CRMs – involve path dependent option pricing.

We do not develop new pricing algorithms for the bonds behind FRMs and CRMs, since we believe this would take us far from the central question in this project. Instead we apply existing "state of the art" pricing models to every path of the scenario tree. In paper A we use Nykredit's internal mortgage bond pricing model (Nyklib), whereas in papers C and D we use approximative pricing approaches similar to those suggested in Nielsen & Poulsen (2004). Finally we have tried ScanRate's RIO pricing system (see http://www.scanrate.dk) on our VAR1 interest rate trees and the optimization results based on these scenarios are reported in chapter 7 of this summary report.

5.3 Optimization framework

With a scenario tree of mortgage bond rates and prices at hand we want to find optimal mortgage strategies for homebuyers with different objectives. We develop an optimization framework which is completely separated from the scenario generation process. A given scenario tree is only one possible input to the optimization model. In this way we obtain maximal flexibility with regards to personal preferences on the choice of an uncertainty model.

But why do we need optimization? After all one might argue that if we all agree on a complete representation of the uncertainty which reproduces market prices of mortgages, then all mortgages are equally attractive in average. The answer is, that even under these unrealistic assumptions the homebuyers personal risk preferences ask for an optimization model in order to find the best mortgage choice. In section 3.2 we saw an example of a homebuyer who was interested in finding a mortgage portfolio which yields the smallest average of the highest 10% of the holding period costs over 5 years. Answering such questions is simply not possible without an optimization model. But even if we do not consider personal risk preferences, it is by far a questionable assumption that all mortgages should be equally attractive in average. We give the following reasons:

- The mortgage market is incomplete, i.e. there are more states of the world than mortgages.
- Market frictions such as transaction costs and tax affects have an impact on the mortgagors choice.
- The prepayment behavior for mortgages with embedded options is suboptimal.

Given this background, using optimization techniques for the mortgage choice problem is indeed well justified. Most of the work in the papers A, C and D is concentrated around developing and testing optimization models for the mortgage choice problem.

The work was inspired by a paper of Nielsen & Poulsen (2004). They design a trinomial scenario tree using an underlying two–factor model of interest rates for pricing existing and synthetic mortgage bonds. Furthermore they introduce a stochastic programming model to find the optimal initial loan strategy among a number of ARMs and FRMs and to advise the mortgagor on optimal readjustments along the way. Their optimization model, however, does not include a risk measure and the effects of fixed–mortgage origination costs were ignored. In paper A we extend the model to include fixed–mortgage origination costs and budget constraints. Different objective functions are tried in this paper:

1. Minimizing average holding period costs.
2. Minimizing the highest holding period cost scenario. (Minmax)
3. Minimizing the average holding period cost with budget constraints.
4. Minimizing the average holding period cost with budget and outstanding debt constraints.

The conclusion is that a minmax mortgagor or a mortgagor with budget constrains benefits from choosing an initial portfolio of an ARM and a FRM, given that there are only these two types of products to choose from. The budget constraints provide indirect means for risk control, but no explicit risk measure is considered in this paper either. We incorporate the scenario reduction algorithm of Heitsch & Römisch (2003) to reduce the size of the tree. We observe, however, that the scenario reductions introduces a high degree of arbitrage opportunities in the scenario tree and even though arbitrage is not allowed to be exercised in our problem, the optimal solutions found in the reduced trees become biased. We also introduce a simple iterative algorithm for solving the LP–relaxed version of the 0–1 stochastic program just using a few iterations.

We add an explicit risk measure for this class of problems in paper C. Here we develop a single–period stochastic programming model to trade off the present value of average holding period costs against the Conditional Value at Risk (CVaR[1]) value. We introduce the notion of a Mean/CVaR efficient frontier for a mortgagor and show that diversified mortgage loan strategies outperform single mortgage loan strategies. Figure D.1 highlights our findings which speak strongly in favor of diversification.

[1]For a review of CVaR as a coherent risk measure see Artzner et al. (1999), Rockafellar & Uryasev (2000) and Zenios (2007).

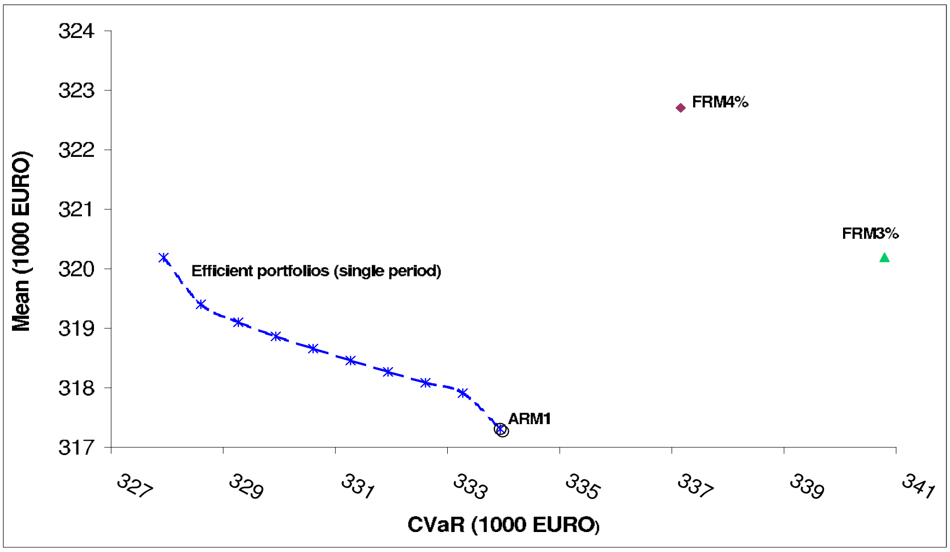


Figure 5.1: For a mortgagor with a seven year horizon a mix of variable and fixed–rate mortgages provide low payments and low risk, here measured by the 10% CVaR value.

Finally in paper D we develop a multi–stage version of our earlier model and show that improved results can be obtained by introducing dynamic trading into the model. It will be seen that the budget–constrained model of paper A is subsumed by the bilinear Mean/CVaR minimizing model. Furthermore, we consider Capped Rate Mortgages CRMs as part of our universe of loans and suggest a simple approach to determine whether the cap option comes at a fair price for a given mortgagor with a certain risk appetite. Figure (5.2) compares a mean/CVaR efficient frontier for a single–period model with that of a multi–stage model.

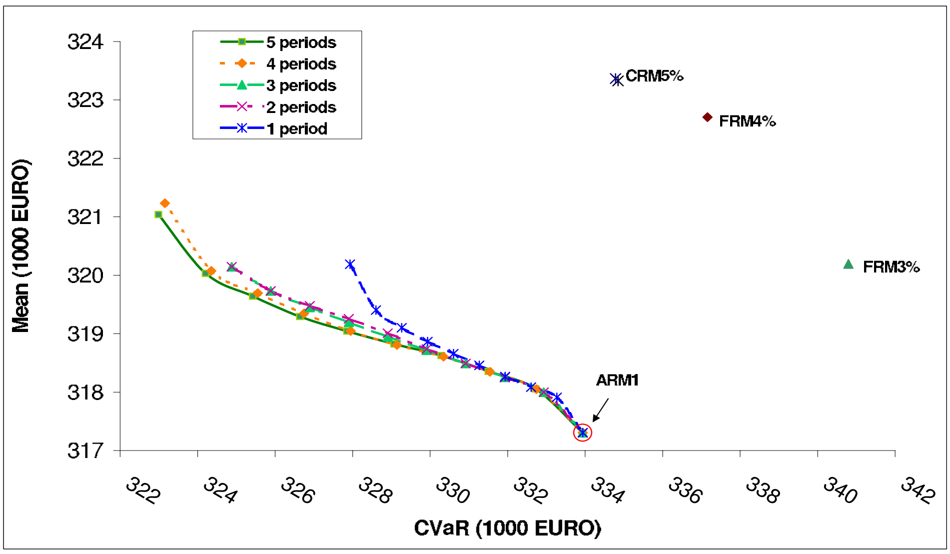


Figure 5.2: As more decision stages are added to the problem the solution quality is improved. The improvement is, however, marginal after adding three extra decision stages.

More optimization results are compared by using different scenario generation approaches, several loans and many optimization models in chapter 7. These results have yet not been published in any paper.

5.4 Financial Giffen goods

Paper B may at a first reading seem to be a deviation from the central theme of this thesis. That is not the case. We show in this paper that financial Giffen goods can not exist in a Markowitz mean variance setting. We argue that it makes good financial sense to allow their existence in optimal portfolio models and we show that such goods do exist in more realistic models such as those developed in papers A, C and D. In other words we provide additional evidence as to why we do not consider portfolio variance but rather budget constraints or more generally Conditional Value at Risk as our measure of risk.

A Giffen good is one for which demand goes down if its price goes down. At first, it is counter intuitive that such goods exist at all. But most introductory text books in economics will tell you that they do; some with stories about potatoes and famine in Ireland, some with first order conditions for constrained optimization. In paper B we study similar effects

— by which we mean a negative relation between expected return and demand — in portfolio choice models. Surprising dependence on expected rates of return is not uncommon in finance. In complete models, option prices do not depend on the stock's growth rate. And quite generally call option prices increase with the interest rate; immediately you would think that cashflows are discounted harder, but in fact the replicating strategy which entails a short position in the bank account becomes more expensive, and hence the call option does too.

We first show that in the basic Markowitz mean/variance model, there are no Giffen goods; if a stock's expected rate of return goes up, its weight in any efficient portfolio goes up. This seems a text-book comparative statics result. We have, however, only been able to find it indirectly stated, for instance one could view it as a corollary or lemma related to the Harmony Theorem from Luenberger (1998, Section 7.8). So we give a simple proof. We then look at Merton's dynamic investment framework. In its basic version demand for any asset depends positively on its expected rate of return, but if a subsistence level is included, demand for the risk free asset may fall with the interest rate.

Skeptics would say that Giffen goods exist in *and only in* economic text books. We end the paper by illustrating that it is not so. Our example uses a the multistage stochastic programming framework from papers A, C and D and shows that some – completely rational – mortgagors react to lower costs of long-term financing (reflecting a smaller market price of risk) by using more short term financing.

In the next chapter the main features and novelties of this thesis are summarized.

Chapter 6

Research contributions

The research efforts in this work are within the domain of optimization in finance and applied mathematical finance. The focus has been on realistic problem solving. That involved developing and testing several mathematical models as well as financial analysis and interpretation and discussion of the findings. Along the way the research has also resulted in a theoretical proof on lack of Giffen goods in a Markowitz mean variance setting. We show then that such goods do exist in more realistic models such as those developed in this thesis. In this chapter the main research contributions are summarized:

6.1 Optimization models

The optimization models developed in this project are novel. In particular:

- In paper A we develop a number of multistage stochastic programs to represent the homebuyers mortgage choice problem. The emphasis of the modeling work is its realism, i.e variable fixed and transaction costs, tax effects, mortgage rebalancings and early repayments are modeled. Likewise homebuyers budget constraints can be added.
- In paper C we generalize the budget constraints by introducing an explicit measure of risk (CVaR). The model is developed as a single stage model in order to study the incremental effects of moving from single loan issue and hold strategies to optimal portfolios of loans though still in an issue and hold setting.
- In paper D we introduce the multistage version of the model from paper C and show that initial diversification and future rebalancings improves the optimal payment/risk frontiers from the single stage setting.
- In paper E we develop an extended moment matching model for generating scenario trees of the term structure of interest rates. The

model is an extension of Høyland & Wallace (2001) and it results in realistic representations of interest rate uncertainty.

6.2 A theoretical result on financial Giffen goods

In paper B we show first that in the basic Markowitz mean/variance model, there are no Giffen goods; if a stock's expected rate of return goes up, its weight in any efficient portfolio goes up. We then look at Merton's dynamic investment framework. In its basic version demand for any asset depends positively on its expected rate of return, but if a subsistence level is included, demand for the risk–free asset may fall with the interest rate. We end the paper by illustrating a generalized version of the multi–stage stochastic programming framework from Rasmussen & Clausen (2007) and show that some – completely rational – mortgagors react to lower costs of long–term financing (reflecting a smaller market price of risk) by using more short–term financing.

6.3 A term structure scenario generation model

Our term structure scenario generation approach in paper E is novel. We define a number of quality requirements for a scenario tree of term structures and we extend the moment matching approach of Høyland & Wallace (2001) in order to generate multiperiod scenario trees of term structures which abide by these requirements.

Chapter 7

New results on model robustness

One of the advantages of the mortgage advising system developed in this project is its modularity. In particular the following parts of the model can be replaced by the analyst's choice of models in order to suit the particular needs or subjective expectations of the homebuyer:

1. Scenario trees of term structure of interest rate.

2. Mortgage pricing models.

3. Objective functions of the optimization problem.

The high degree of flexibility necessitates a discussion of robustness of the conclusions. In particular it is important to know how robust the conclusions of an instance of the optimization problem are given different choices of interest rate models and mortgage pricing models.

In this chapter we discuss model robustness by showing some results which have not been discussed in any of the papers presented in this thesis. The background for this extra analysis is that we in paper E introduce a new interest rate scenario generation model which we argue gives a more realistic representation of interest rate uncertainty than the Vasicek model used in papers C and D. Besides in the advisory system developed in cooperation with Nykredit A/S we use ScanRate's RIO to price mortgages instead of using the approximative approach of Nielsen & Poulsen (2004) as is the case in papers C and D. We will now present the new results and compare them with those reported in papers C and D.

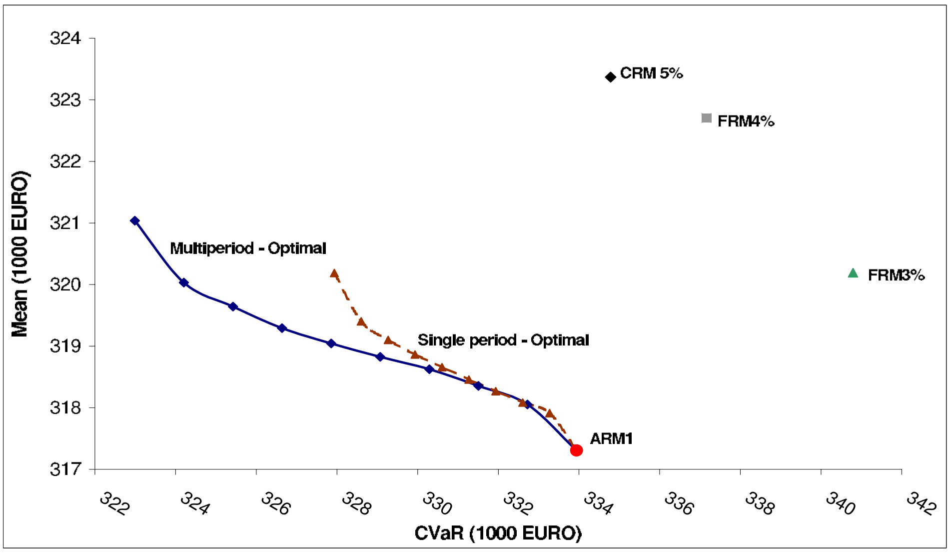
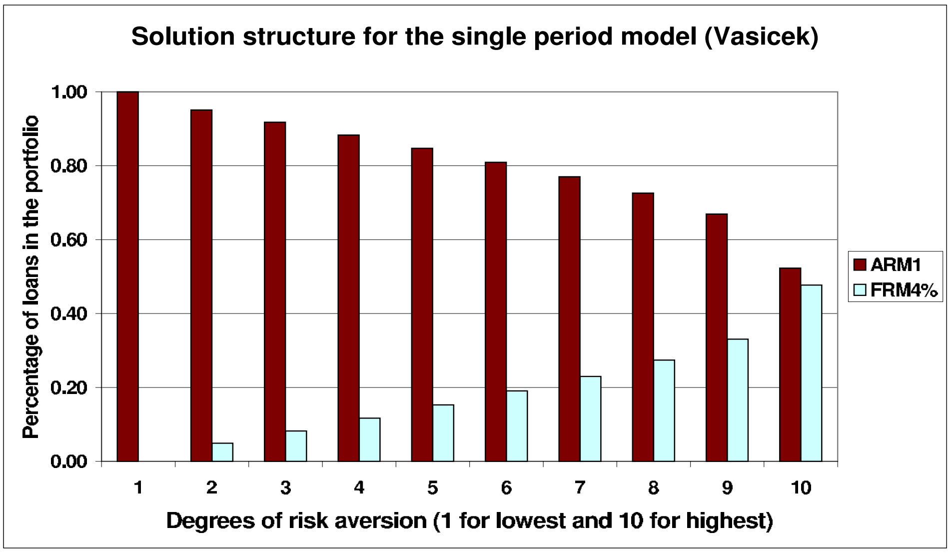


Figure 7.1: Comparison of single issue and hold strategies with optimal passive and active strategies. The underlying interest rate model is a 1–factor Vasicek model.

7.1 Comparison of two scenario generation approaches

Recall that one of the central conclusions in papers C and D was summarized in Figure (7.1).

The corresponding initial solutions for the single period and the multi-stage cases are shown in Figure (7.2).



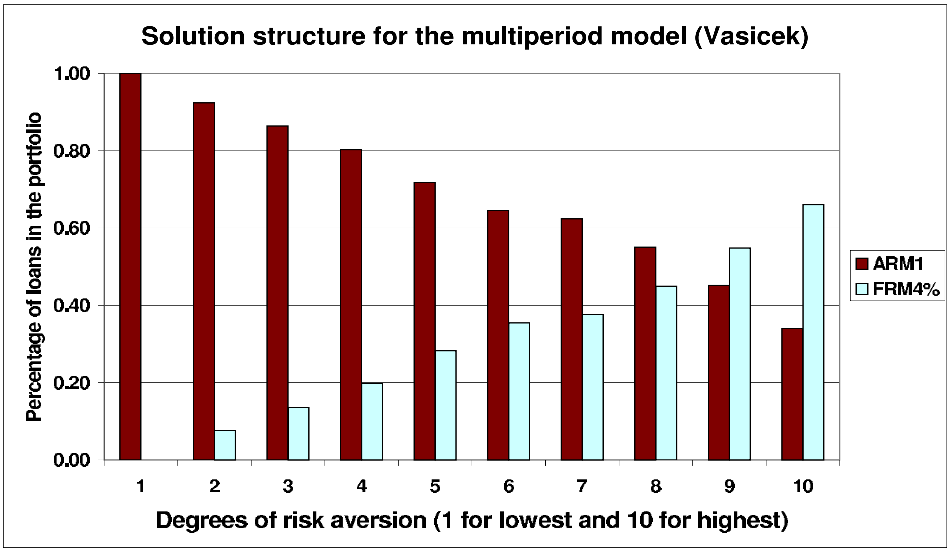


Figure 7.2: First stage solutions for different degrees of risk aversion for a passive (single period) and active (multiperiod) mortgagor.The underlying interest rate model is a 1–factor Vasicek model.

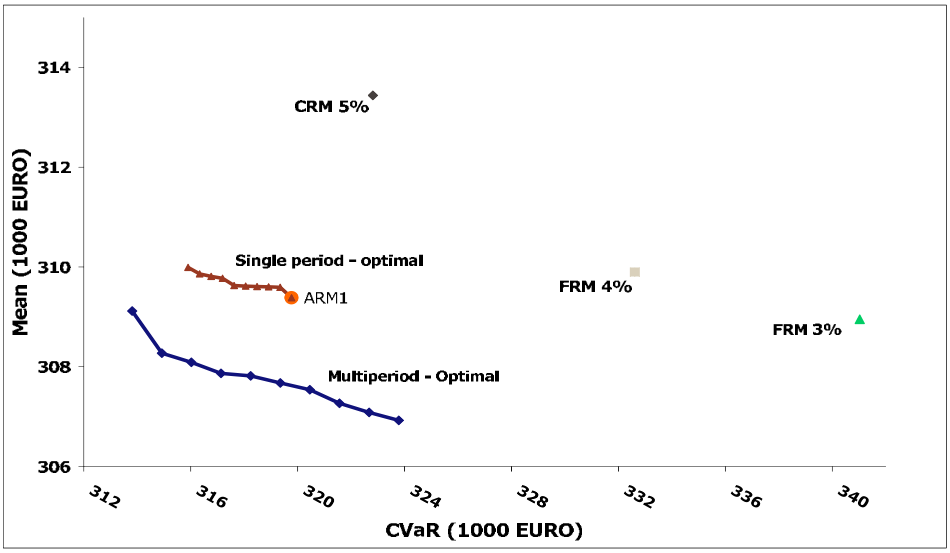


Figure 7.3: Comparison of single issue and hold strategies with optimal passive and active strategies. The underlying interest rate model is our 3–factor VAR1 model.

In comparison when we use the VAR1 interest rate model of paper E together with ScanRate's RIO mortgage pricing we get the efficient frontier given in Figure (7.3) and the corresponding initial solutions in Figure (7.4).

The experiments are based on market data from February 2005. Similar solution patterns are obtained for quarterly updates of data until August 2007.

We make the following observations on robustness:

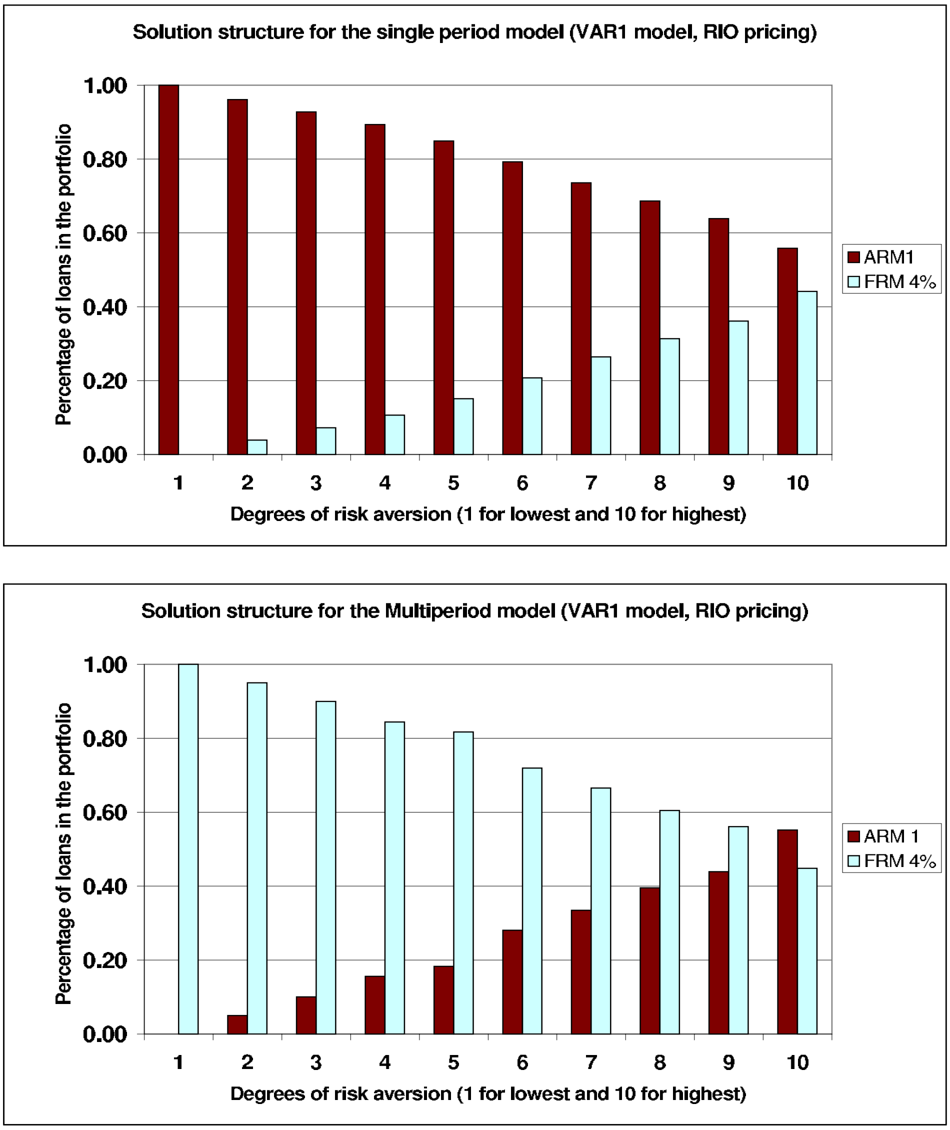


Figure 7.4: First stage solutions for different degrees of risk aversion for a passive (single period) and active (multiperiod) mortgagor.The underlying interest rate model is our 3–factor VAR1 model.

- Combinations of loans and rebalancings improve the results as compared to single loan issue and hold strategies.

- Risk averse mortgagors start with an initial portfolio of loans rather than a single loan regardless of the underlying uncertainty representation.

- Single period models are more robust than multistage models.

The following qualitative conclusions may then be made:

- It is safe (robust) to advise risk averse mortgagors to start with a loan portfolio made of two mortgages rather than one.

- Multistage models add value but they include an element of speculation on the underlying uncertainty representation.

Chapter 8

Final remarks

8.1 Conclusions and Empirical findings

We have shown that research in the area of optimization in finance answers real world financial problems not touched in continuous mathematical finance. Our models are similar to those from the well–known case studies such as the Russel–Yasuda Kasai financial planning model (See Cariño, Myers & Ziemba (1998)), the Towers Perrin–Tillinghast asset and liability management system (See Mulvey, Gould & Morgan (2000)), Gjensidig Nor's decision support model (See Høyland, Ranberg & Wallace (2003)) and Prometeia's model for managing insurance policies with guarantee (Consiglio, Cocco & Zenios (2002)).

In the following we summarize the most important conclusions of our work:

- Diversification pays off in particular for risk averse homebuyers or homebuyers who do not actively rebalance their mortgage portfolio. The intuition behind this is the strong negative correlation between the holding period costs of short term and long term financing.
- Rebalancing is a good idea for both risk averse and risk neutral homebuyers. Risk neutral homebuyers should start by a single mortgage and rebalance the whole outstanding debt when the embedded options are deep in the money. Risk averse homebuyers should start with a mix of fixed or capped and adjustable rate mortgages. They should then partially rebalance one of the mortgages when some profit can be locked in.
- Fixed transaction costs are important in deciding how many mortgages should be included in the homebuyers portfolio of loans. Two mortgages are often seen in the portfolio of a risk averse homebuyer even when the fixed transaction costs are present. The incremental benefits of a third mortgage do not surpass the extra fixed transac-

tion costs incurred for almost all homebuyers.

- Mortgage banks should consider tailored replications of CRMs by using plain ARMs and FRMs and hedging some risk away in the market. In this way they issue loans in few but more liquid bond series which are normally more fairly priced than the thin specialized series for funding CRMs.

- Mortgage banks should have less focus on recommending one type of mortgage for example FRMs with prices close to par to all homebuyers as a collective group. Homebuyers often do as they are told by their mortgage bank advisors and their collective preference for one particular product affects the market price of that product to homebuyers disadvantage. Instead the advisors should seek to find combinations of products whose cashflows are reasonably priced and which at the same time offer protection against adverse market movements.

8.2 Future work

We consider the following three directions as the main pointers for future work:

1. Developing a scenario generation library for the personal investor. Such a library may include:
   (a) Several stochastic processes such as interest rate models, econometric models, regression models, etc. to capture the underlying uncertainties on interest rates, household income, real state prices, stock index movements and so on.
   (b) Several discretization schemes such as moment matching, property matching, optimal discretization, Monte carlo samplings and so on.
   (c) Different pricing models for options, mortgage backed securities, etc.
2. Developing an optimal mortgage design system. The general idea here is that the mortgage banks should decide on a cashflow which is marketable and use the optimal mortgage design system to find the cheapest funding for that cashflow.
3. Developing a personal asset liability management system for the Danish household. Such a system should help the individual household with an elaborate scheme on their decisions on the two most important investments most household engage in, namely financing a house and pension investments.

Financial glossary

An alphabetical list of common financial terms used throughout the thesis are given in the following. The listing is not exhaustive. It is only meant to ease the reading of the thesis for the reader who is not familiar with finance. Most of the definitions are taken from financial glossaries on the world wide web. [1].

Annuity payments:

Annuity payments refer to any terminating stream of fixed payments over a specified period of time. Most mortgage loans have annuity payments. The annuity payment is calculated using the following formula:

$$payment = ID\left(\frac{r}{1-(1+r)^{-n}}\right).$$

[1] http://www.wikipedia.org/ and http://www.investopedia.com/.

Where:

ID = initial debt,

r = interest rate per period,

n = number of periods.

Example: You can get a $150,000 home mortgage at 7% annual interest rate for 30 years. Payments are due at the end of each month and interest is compounded monthly. The annuity payment is calculated as:

ID = 150,000, the loan amount,

r = 0.005833, interest per month (0.07 / 12),

n = 360 periods (12 payments per year for 30 years),

$$payment = 150,000(\frac{0.005833}{1 - 1.005833^{-360}}) = \$997.95.$$

This means that you should pay $997.95 (the annuity) every month in 30 years in order to pay back the mortgage.

Bullet bonds:

A bullet bond is a regular coupon paying debt instrument with a single repayment of principal on the maturity date.

Example: You invest $100,000 in a five–year bullet bond with an annual interest rate of 5%. Payments are due at the end of each year and interest is compounded yearly. You will get \$5000 at the end of year 1 to 4 and $\$100,000 + \$5000 = \$105,000$ at the end of year 5.

Buyback delivery option:

Buyback delivery option refers to the borrowers right to terminate a loan by buying back the mortgage bonds in the bond market and delivering them to the mortgage bank. In case the market price of the mortgage bond is below par (100) this option means a reduction in the size of the outstanding debt in case of prepayment. Alternatively a price above par means an increase in the size of the outstanding debt. Fixed rate mortgage bonds normally have an embedded call option with strike at par which means the mortgagor will never pay more than the value of the outstanding debt in order to terminate the loan.

Bermudan option:

A Bermudan option is a call or put option which can be exercised on prespecified days during the life of the option. Bermudan options are a hybrid of **European options**, which can only be exercised on the option expiry date, and **American options**, which can be exercised at any time during the option life time. As a consequence, under same conditions, the value of a Bermudan option is greater than (or equal to) a European option but less than (or equal to) an American option.

Call option:

A call option is a financial contract between two parties, the buyer and the seller of this type of option. Often it is simply labeled a "call". The buyer of the option has the right, but not the obligation to buy an agreed quantity of a particular commodity or financial instrument (the underlying instrument) from the seller of the option at a certain time (the expiration date) for a certain price (the strike price). The seller (or "writer") is obligated to sell the commodity or financial instrument should the buyer so decide. The buyer pays a fee (called a premium) for this right. A call option is said to be **in the money**, when the option's strike price is below the market price of the underlying asset. For a callable fixed rate covered bond with strike at par (100), if the price of the underlying non–callable covered bond is above par, then the call option is in the money. If the price of the underlying non–callable covered bond is below par, then the call option is **out of the money**. An option which is so far in the money that it is unlikely to go out of the money prior to expiration is called **deep in the money**.

Covered bonds:

Covered bonds are debt securities backed by cashflows from mortgages or public sector loans. Covered bonds have been very common in Germany for many years where they are known as Pfandbrief and can be traced back to 1769. The Danish mortgage bonds are covered bonds backed by mortgage cashflows.

Currency risk:

Currency risk is a form of risk that arises from the change in price of one currency against another. Whenever investors or companies have assets or business operations across national borders, they face currency risk if their positions are not hedged.

Default risk:

The risk that companies or individuals will be unable to pay the contractual interest or principal on their debt obligations. In other words, this is the risk that the investor will not get paid.

Derivatives:

Derivatives are financial instruments whose value is derived from the value of something else. They generally take the form of contracts under which the parties agree to payments between them based upon the value of an underlying asset or other data at a particular point in time. The main types of derivatives are futures, forwards, options, and swaps.

Embedded option:

An embedded option is an inseparable part of another financial instrument in contrast to a normal (or bare) option, which trades separately from the underlying security. A common embedded option is the call option in many covered bonds.

Floaters:

A floater is a bond or other type of debt whose coupon rate changes with market conditions (short–term interest rates). It is also known as "floating–rate debt".

Holding period costs:

The total costs associated with taking a loan for a given holding period. It includes both the cashflow payments as well as the prepayment of the outstanding debt at the horizon of the holding period.

Interest rate cap:

An interest rate cap is a derivative in which the buyer receives money at the end of each period in which an interest rate exceeds the agreed strike price. As an example a variable rate covered bond with an embedded interest rate cap of 5% guarantees the borrower the interest rate payments will never be more than 5% of the outstanding debt.

Interest rate risk:

Interest rate risk is the risk that the relative value of an interest–bearing asset, such as a loan or a bond, will worsen due to an interest rate increase. In general, as rates rise, the price of a fixed rate bond will fall, and vice versa.

Liquidity risk:

Liquidity risk arises from situations in which a party interested in trading an asset cannot do it because nobody in the market wants to trade that asset. Liquidity risk becomes particularly important to parties who are about to hold or currently hold an asset, since it affects their ability to trade. Manifestation of liquidity risk is very different from a drop of price to zero. In case of a drop of an asset's price to zero, the market is saying that the asset is worthless. However, if one party cannot find another party interested in trading the asset, this can potentially be only a problem of the market participants with finding each other. This is why liquidity risk is usually found higher in emerging markets or low–volume markets.

Mortgage:

A mortgage is a method of using property (real or personal) as security for the payment of a debt. The term mortgage refers to the legal device used for this purpose, but it is also commonly used to refer to the debt secured by the mortgage, the mortgage loan. In most jurisdictions mortgages are strongly associated with loans secured on real estate rather than other property (such as ships).

Mortgage backed security (MBS):

A mortgage backed security is a financial instrument whose cashflows are backed by the principal and interest payments of a set of mortgage loans. Payments are typically made monthly or quarterly over the lifetime of the underlying loans.

Mortgage loan:

A mortgage loan is a loan secured by real property through the use of a mortgage. The word mortgage alone, in everyday usage, is most often used to mean mortgage loan.

Mortgagor:

A mortgagor is an individual or company who borrows money to purchase a piece of real property. For most homebuyers, becoming a mortgagor is necessary for owning a home. Because the real property in question is offered as security for the loan, the lender can claim its interest in the property in the event the loan is not repaid. This decreased risk allows homebuyers to borrow funds at much lower interest rates.

Swap curve:

The swap curve identifies the relationship between swap rates at varying maturities. Swap curves are normally used as proxies for yield curves.

Swap rate:

An interest rate swap is a derivative in which one party exchanges a stream of interest payments for another party's stream of cash flows. Usually a stream of variable rates is exchanged for a fixed rate also called the swap rate.

Yield curve:

The yield curve is the relation between the interest rate (or cost of borrowing) and the time to maturity of the debt for a given borrower in a given currency.

Zero coupon bonds:

Zero coupon bonds are financial contracts that pay no periodic interest payments, or so–called "coupons". Zero coupon bonds are purchased at a discount from their value at maturity. The holder of a zero coupon bond is entitled to receive a single payment, usually of a specified sum of money at a specified time in the future. Investors earn interests via the difference between the discounted price of the bond and its par (or redemption) value, usually 100.

Part II

Papers

Appendix A

Mortgage Loan Portfolio Optimization Using Multi Stage Stochastic Programming

Published in *journal of economic dynamics and control*, February 2007, volume 31, pages 742–766.

Abstract

We consider the dynamics of the Danish mortgage loan system and propose several models to reflect the choices of a mortgagor as well as his attitude towards risk. The models are formulated as multi stage stochastic integer programs, which are difficult to solve for more than 10 stages. Scenario reduction and LP relaxation are used to obtain near optimal solutions for large problem instances. Our results show that the standard Danish mortgagor should hold a more diversified portfolio of mortgage loans, and that he should rebalance the portfolio more frequently than current practice.

A.1 Introduction

A.1.1 The Danish mortgage market

The Danish mortgage loan system is among the most complex of its kind in the world. Purchase of most properties in Denmark is financed by issuing fixed–rate callable mortgage bonds based on an annuity principle. It is also possible to raise loans, which are financed through issuing non–

callable short term bullet bonds. Such loans may be refinanced at the market rate on an ongoing basis. The proportion of loans financed by short–term bullet bonds has been increasing since 1996. Furthermore it is allowed to mix loans in a mortgage loan portfolio, but this choice has not yet become popular.

Callable mortgage bonds have a fixed coupon throughout the full term of the loan. The maturities are 10, 15, 20 or 30 years. There are two options embedded in such bonds. The borrower has a Bermudan type call option, i.e. he can redeem the mortgage loan at par at four predetermined dates each year during the lifetime of the loan. When the interest rates are low the call option can be used to obtain a new loan with less interest payment in exchange for an increase in the amount of outstanding debt. The borrower has also a delivery option. When the interest rates are high this option can be used to reduce the amount of outstanding debt, in exchange for paying higher interest rate payments. There are both fixed and variable transaction costs associated with exercising any of these options.

Non–callable short–term bullet bonds are used to finance the adjustable–rate loans. The bonds' maturities range from one to eleven years and the adjustable–rate loans are offered as 10, 15, 20 or 30–year loans. Since 1996 the most popular adjustable–rate loan has been the loan financed by the one–year bond. From 2001, however, there has been a new trend, where demand for loans financed by bullet bonds with 3 and 5–year maturities has risen substantially.

A.1.2 The mortgagor's problem

It is known on the investor side of the financial markets that investment portfolios should consist of a variety of instruments in order to decrease financial risks such as market, liquidity and currency risk while maintaining a fixed level of return. The portfolio is also rebalanced regularly to take best advantage of the moves in the market.

The portfolio diversification principle and re–balancing is, however, not common in the borrower side of the mortgage market. Most mortgagors finance their loans in one type of bond only. Besides they do not always re–balance their loan when good opportunities for this have arisen.

There are two major reasons for the mortgagors reluctance to better taking advantage of their options (that they have fully paid for) through the lifetime of the mortgage loan.

1. The complexity of the mortgage market makes it impossible for the

average mortgagor to analyze all the alternatives and choose the best.

2. The mortgage companies do not provide enough quantitative advice to the individual mortgagor. They only provide general guidelines, which are normally not enough to illuminate all different options and their consequences.

The complexity of the mortgage loan system makes it a non–trivial task to decide on an initial choice of mortgage loan portfolio and on finding a continuing plan to readjust the portfolio optimally. See e.g. Zenios (1993), Nielsen & Zenios (1996*a*), Vassiadou-Zeniou & Zenios (1996), Zenios, Holmer, McKendall & C. (1998), Zenios (1995) and Zenios & Kang (1993).

We assume in the following that the reader is familiar with the dynamics of a mortgage loan market such as the Danish one, as well as the basic ideas behind the mathematical modeling concept of stochastic programming.

The Danish mortgagor's problem has been introduced by Nielsen & Poulsen (2004). They use a two factor term structure model for generating interest rate scenarios. They have developed an approximative pricing scheme to price the mortgage instruments in all nodes of the scenario tree and on top of it have built a multi–stage stochastic program to find optimal loan strategies. The paper, however, does not describe the details necessary to have a functional optimization model, and it does not differentiate between different types of risks in the mortgage market. The main contribution of this article is to make Nielsen & Poulsen's model operational by reformulating parts of their model and adding new features to it.

We reformulate the Nielsen & Poulsen model in section A.2. In section A.3 we model different options available to the Danish mortgagor, and in section A.4 we model mortgagor's risk attitudes. Here we consider both market risk and wealth risk.

In the basic model we incorporate fixed transaction costs using binary variables. We use a non–combining binomial tree to generate scenarios in an 11 stage problem. This results in 51175 binary variables, making some versions of the problem extremely challenging to solve. Dupačová, Gröwe-Kuska & Römisch (2003) and Heitsch & Römisch (2003) have modeled the scenario reduction problem as a set covering problem and solved it using several heuristic algorithms. We review these algorithms in section A.5 and use them in our implementation to reduce the size of the problem and hereby reduce the solution times. Another approach to getting shorter solution times is proposed in section A.6, where we solve an LP–approximated version of the problem. In section A.7 we discuss and comment on our numerical results and we conclude the article with suggestions for further research in section A.8. We use GAMS (General Algebraic Modeling System) to model the problem and CPLEX 9.0 as the underlying MP and MIP solver. For scenario reduction we use the GAMS/SCENRED module (see http://www.mathematik.hu-berlin.de/ nicole/scenred/gamsscenred.html).

The obtained results show that the average Danish mortgagor would benefit from choosing more than one loan in a mortgage loan portfolio. Likewise he should readjust the portfolio more often than is the case today. The developed model and software can also be used to develop new loan products. Such products will consider the individual customer inputs such as budget constraints, risk profile, expected lifetime of the loan, etc.

Even though we consider the Danish mortgage loan market, the problem is universal and the practitioners in any mortgage loan system should be able to use the models developed in this paper, possibly with minor modifications.

A.2 The basic model

In this section we develop a risk–neutral optimization model which finds a mortgage loan portfolio with the minimum expected total payment.

We consider a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms are sequences of real–valued vectors (coupon rates and prices of mortgage backed securities) over discrete time periods $t = 0, \cdots, T$. We model this finite probability space by a scenario tree borrowing the notation from King (2002).

Consider the scenario tree in Figure (A.1). The partition of the probability atoms $\omega \in \Omega$ generated by matching path histories up to time t corresponds one–to–one with nodes $n \in \mathcal{N}_t$ at depth t in the tree.

In the scenario tree, every node $n \in \mathcal{N}$ for $1 \leq t \leq T$ has a unique parent denoted by $a(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$ for $0 \leq t \leq T-1$ has a non–empty set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_{t+1}$. The probability distribution P is modeled by attaching weights $p_n > 0$ to each leaf node $n \in \mathcal{N}_T$ so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. For each non–terminal node one has, recursively,

$$p_n = \sum_{m \in \mathcal{C}(n)} p_m \qquad \forall n \in \mathcal{N}_t, \quad t = T-1, \cdots, 0$$

and so each node receives a probability mass equal to the combined mass of the paths passing through it.

We assume that we have such a tree at hand with information on price and coupon rate for all mortgage bonds available at each node as well as the probability distribution P for the tree at hand.

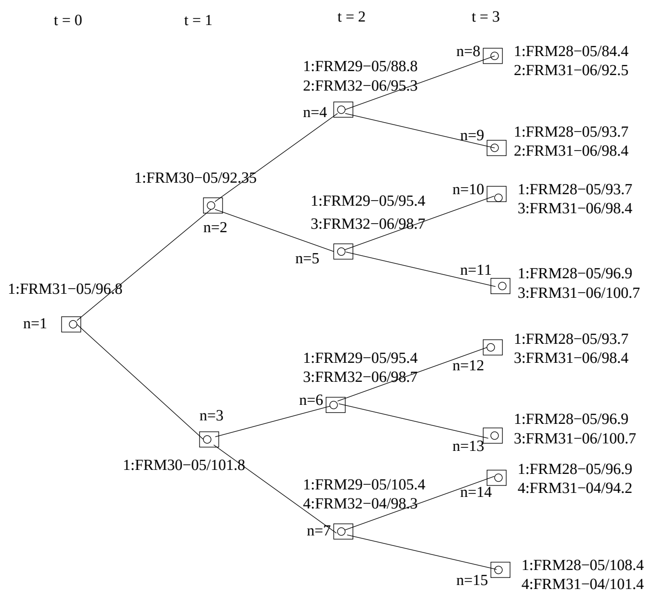


Figure A.1: A binomial scenario tree, representing our expectation of future bond prices and coupon rates. All bonds are callable fixed–rate bonds.

In the basic model we only consider fixed–rate loans, i.e. loans where the interest rate does not change during the lifetime of the loan. For the sake of demonstration we consider an example with 4 stages, $t \in \{0, 1, 2, 3\}$, and 15 decision nodes, $n \in \{1, \cdots, 15\}$, with the probability p_n for being at the node n.

We want the basic model to find an optimal portfolio of bonds from a finite number of fixed–rate bonds. Consider the 4 bonds shown in Figure (A.1). Each bond is represented as (Index:Type–Coupon/Price), so (3:FRM32–06/98.7) is a fixed–rate callable mortgage bond with maturity in 32 years, a coupon rate of 6% and a price of 98.7.

To generate bonds information we can use term structure and bond pricing theories. For an introduction to these topics see for example Hull (2003), Luenberger (1998) and Björk (1998). It is also possible to use expert knowledge to predict possible bond prices in the future. A combination of theoretical pricing and expert information can also be used to generate such scenario trees. Nielsen & Poulsen (2004) propose an approximative approach for pricing fixed rate bonds with embedded call and delivery in a scenario tree. In this paper we use the BDT model (see Black et al. (1990)) for generating an interest rate tree to represent the underlying interest rate uncertainty and estimate the prices of the mortgage backed bonds in all the nodes of the tree using the commercial pricing module RIO 4.0 developed by Scanrate Financial Systems A/S (see http://www.scanrate.dk).

Given a scenario tree with T stages and its corresponding coupon rate and price information on a set of bonds $i \in I$ we can now define the basic model.

Parameters:

p_n: The probability of being at node n.

d_t: Discount factor at time t.

IA: The initial amount of loan needed by the mortgagor.

r_{in}: Coupon rate for bond i at node n.

k_{in}: Price of bond i at node n.

$Callk_{in}$: Price of a callable bond i at node n. We have $Callk_{in} = \min\{1, k_{in}\}$ for callable bonds and $Callk_{in} = k_{in}$ for non–callable bonds.

γ: Tax reduction rate from interest rate payment.

β: Tax reduction rate from administration fees.

b: Administration fee given as a percentage of outstanding debt.

η: Transaction fee rate for sale and purchase of bonds.

m: Fixed costs associated with re–balancing.

Next we define the variables used in our model:

B_{tn}: Total net payment at node n, time t.

X_{itn}: Outstanding debt of bond i at node n, time t.

S_{itn}: Units sold of bond i at scenario n, time t.

P_{itn}: Units purchased of bond i at node n, time t.

A_{itn}: Principal payment of bond i at node n, time t.

$$L_{itn}: \begin{cases} 1 & \text{if there are any fixed costs associated with bond } i, \text{ node } n, \text{ time } t. \\ 0 & \text{otherwise.} \end{cases}$$

The multi stage stochastic integer model can now be formulated as follows:

$$\min \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} p_n \cdot d_t \cdot B_{tn} \tag{A.1}$$

$$\sum_{i \in I} k_{i1} \cdot S_{i01} \geq IA \tag{A.2}$$

$$X_{i01} = S_{i01} \qquad \forall i \in I \tag{A.3}$$

$$X_{itn} = X_{i,t-1,a(n)} - A_{itn} - P_{itn} + S_{itn} \qquad \forall i \in I, n \in \mathcal{N}_t, t = 1, \cdots, T \tag{A.4}$$

$$\sum_{i \in I} (k_{in} \cdot S_{itn}) = \sum_{i \in I} (Callk_{in} \cdot P_{itn}) \qquad \forall n \in \mathcal{N}_t, t = 1, \cdots, T \tag{A.5}$$

$$A_{itn} = X_{i,t-1,a(n)} \left[\frac{r_{i,a(n)}}{1 - (1 + r_{i,a(n)})^{-T+t-1}} - r_{i,a(n)} \right] \forall i \in I, n \in \mathcal{N}_t, t = 1, \cdots, T \tag{A.6}$$

$$B_{01} = \sum_{i \in I} \left(\eta \cdot S_{i01} + m \cdot L_{i01} \right) \tag{A.7}$$

$$\begin{aligned} B_{tn} = \sum_{i \in I} \Big(& A_{itn} + r_{i,a(n)} \cdot (1 - \gamma) X_{i,t-1,a(n)} + b \cdot (1 - \beta) X_{i,t-1,a(n)} + \\ & \eta \cdot (S_{itn} + P_{itn}) + m \cdot L_{itn} \Big) \qquad \forall n \in \mathcal{N}_t, t = 1, \cdots, T \end{aligned} \tag{A.8}$$

$$BigM \cdot L_{itn} - S_{itn} \geq 0 \qquad \forall i \in I, n \in \mathcal{N}_t, t = 0, \cdots, T \tag{A.9}$$

$$X_{itn}\ ,\ S_{itn}\ , P_{itn} \geq 0\ ,\ L_{itn} \in \{0, 1\} \qquad \forall i \in I, n \in \mathcal{N}_t, t = 0, \cdots, T \tag{A.10}$$

The objective is to minimize the weighted average payment throughout the mortgage period. The payment for all the nodes except the root is defined in equation (D.7) as the sum of principal payments, tax reduced interest payments, taxed reduced administration fees (Danish peculiarity), transaction fees for sale and purchase of bonds and finally fixed costs for establishing new mortgage loans. The principal payment is defined in equation (D.6) as an annuity payment. The payment in the root (equation A.7) is based on the transaction costs only.

The dynamics of the model are formulated in constraints (D.2) to (D.5). Constraint (D.2) makes sure that we sell enough bonds to raise an initial amount, IA, needed by the mortgagor. In equation (D.3) we initialize the outstanding debt. Equation (D.4) is the balance equation, where the outstanding debt at any child node for any bond equals the outstanding debt at the parent node minus principal payment and possible prepayment (purchased bonds), plus possible sold bonds to establish a new loan. Equation (D.5) is a cashflow equation which guarantees that the money used to prepay comes from the sale of new bonds.

Finally constraint (D.9) adds the fixed costs to the node payment, if we perform any readjustment of the mortgage portfolio. The $BigM$ constant might be set to a value slightly greater than the initial amount raised. If a too large value is used, numerical problems may arise.

A.3 Modeling mortgagor's options

The model described in section A.2 has three implicit assumptions which limit its applicability:

1. We assume that a loan portfolio is held by the mortgagor until the end of horizon.
2. We assume that all bonds are fixed–rate and callable, i.e. they can be prepaid at any time at a price no higher than 100.
3. The mortgagor is assumed to be risk–neutral.

We will relax the first two assumptions in this section and the third in the following section.

The first assumption can be easily relaxed by introducing a constant H indicating mortgagors horizon, such that $H \leq T$, where T is the maturity time of the underlying mortgage portfolio. The decision nodes represent only the first H stages, while the cashflows (principal and interest rate payments) are calculated based on a T year maturity.

These changes mean that the outstanding debt at stage $t = H$ is a positive amount which needs to be prepaid. We define this prepayment amount

(PP_{Hn}) as:

$$PP_{Hn} = \sum_i (X_{iHn} \cdot Callk_{in}) \qquad \forall n \in \mathcal{N}_H, \tag{A.11}$$

We add this equation to the model and we update the object function as follows:

$$\min \sum_{t=0}^{H} \sum_{n \in \mathcal{N}_t} p_n \cdot d_t \cdot B_{tn} + \sum_{n \in \mathcal{N}_H} p_n \cdot d_H \cdot PP_{Hn}. \tag{A.12}$$

The objective is now to minimize the weighted payments at all nodes plus the weighted prepayments at time H.

The problem with the second assumption is more subtle. Consider the scenario tree at Figure (D.5), where two adjustable–rate loans have been added to our set of loans at time 0. Loan 5 (ARM1) is an adjustable–rate loan with annual refinancing and loan 6 (ARM2) is an adjustable–rate loan with refinancing every second year

For adjustable–rate loans (ARMm–loans) the underlying m–year bond is completely refinanced every m years by selling another m–year bond. But unlike normal refinancing this kind of refinancing does not incur any extra fixed or variable transaction costs since an ARMm–loan is issued as a single loan rather than a series of bullet–bonds following each other. We model an ARMm–loan by using the same loan index for an adjustable–rate loan throughout the mortgage period. For example index 5 is used for

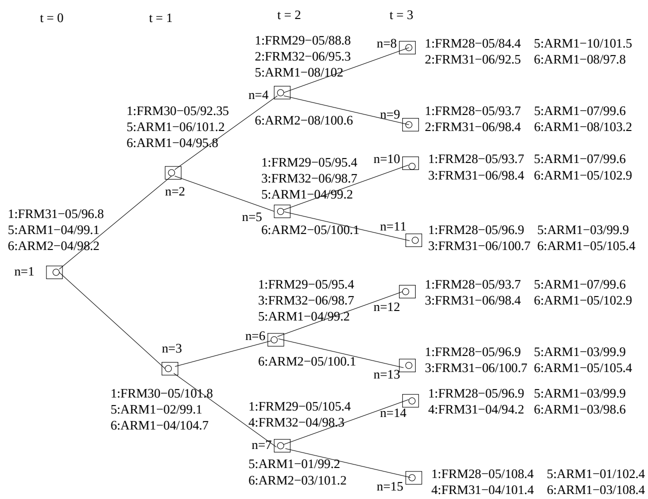


Figure A.2: A binomial scenario tree where both fixed–rate and adjustable–rate loans are considered.

the loan with annual refinancing, even though the actual bonds behind the loan change every year. Since we use the same index, the model does not register any actual sale or purchase of bonds when refinancing occurs. We should, however, readjust the outstanding debt given that the bond price is normally different from par. To take this into account we introduce the set $I' \subseteq I$ of non–callable adjustable–rate loans. For these loans we use the following balance equation instead of equation (D.4).

$$k_{in} \cdot X_{itn} = X_{i,t-1,a(n)} - A_{itn} - P_{itn} + S_{itn} \quad \forall i \in I', n \in \mathcal{N}_t, t = 1, \cdots, T. \tag{A.13}$$

Note that variables P_{itn} and S_{itn} remain 0 as long we keep an adjustable–rate loan $i \in I'$ in our mortgage portfolio. The outstanding debt in the child node is however rebalanced by multiplying the bond price.

When we consider the adjustable–rate loans we should remember that these loans are non–callable, so for prepayment purposes we have:

$$Callk_{in} = k_{in} \quad \forall i \in I', n \in \mathcal{N}_t, t = 0, \cdots, T.$$

Another issue to be dealt with is that if a bond is not available for establishing a loan at a given node, we have to set the corresponding value of k_{in} to 0 to make sure that the bond is not sold at that node in an optimal solution. For example bond (6:ARM1–04/95.8) at node 2 is not open for sale but only for prepayment.

A.4 Modeling risk

So far we have only considered a risk neutral mortgagor who is interested in the minimum weighted average of total costs. Most mortgagors however have an aversion towards risk. There are two kinds of risk which most mortgagors are aware of:

1. **Market risk**: In the mortgage market this is the risk of extra in-

terest rate payment for a mortgagor who holds an adjustable–rate loan when interest rate increases, or the risk of extra prepayment for a mortgagor with any kind of mortgage loan when the interest rate decreases so the bond price increases.

2. **Wealth risk**: In the mortgage market this is a potential risk which can be realized if the mortgagor needs to prepay the mortgage before a planned date or if he needs to use the free value of the property to take another loan. It can be measured as a deviation from an average outstanding debt at any given time during the lifetime of the loan.

We will in the following model both kinds of risk. To that end we use the ideas behind minmax optimization and utility theory with use of budget constraints.

A.4.1 The minmax criterion

An extremely risk averse mortgagor wants to pay least in the worst possible scenario. In other words if we define the maximum payment as MP then we have the following minmax criterion:

$$\min \qquad MP, \tag{A.14}$$

$$MP \geq \sum_{t=0}^{T} \sum_{n \in \mathcal{NP}_{ts}} B_{tn} \qquad \forall s \in S, \tag{A.15}$$

where $\mathcal{NP}_{ts}$ is a set of nodes defining a unique path from the root of the tree to one of the leaves. Each of these paths define a scenario $s \in S$. For the example given in Figure D.5 we have:

$$\mathcal{NP}_{t,1} = \{1, 2, 4, 8\}$$

$$\mathcal{NP}_{t,2} = \{1, 2, 4, 9\}$$

$$\ldots$$

$$\mathcal{NP}_{t,8} = \{1, 3, 7, 15\}$$

A.4.2 Utility function

Instead of minimizing costs we can define a utility function, which represents a saving and maximize it. Nielsen & Poulsen (2004) suggest a concave utility function with the same form as in Figure (A.3).

The decreasing interest for bigger savings is based on the idea that bigger savings are typically riskier than small savings. Nielsen and Poulsen use

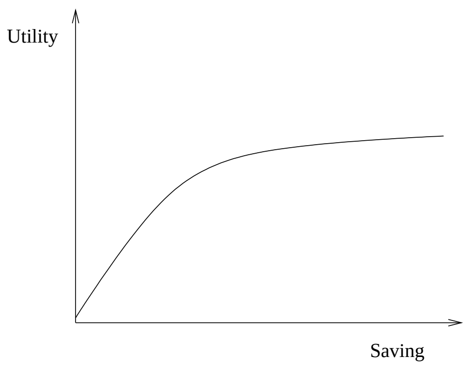


Figure A.3: A concave utility function. An increase of an already big saving is not as interesting as an increase of a smaller saving.

a logarithmic object function, which can be formulated as follows:

$$\max \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} p_n \cdot \log(d_t \cdot (B_{tn}^{max} - B_{tn})), \tag{A.16}$$

where B_{tn}^{max} is the maximum amount a mortgagor is willing to pay. Nielsen and Poulsen fix B_{tn}^{max} to a big value so that the actual payment will never rise above this level.

Adding this non–linear objective function to our stochastic binary problem makes the problem extremely challenging to solve. There are no effective general purpose solvers for solving large mixed integer non–linear programs (see Bussieck & Pruessner (2003)). There are three ways of circumventing the problem: Either we use a linear utility function in conjunction with budget constraints (mip) or relax the binary variables and solve the non–linear problem (nlp) or both (lp). We demonstrate the first approach in the following and comment on the second and third approach in section A.6.

Instead of maximizing the logarithm of the saving at each node we can simply maximize the saving: $B_{tn}^{max} - B_{tn}$. If B_{tn}^{max} is so large that the saving is always positive, then we are in effect minimizing the weighted average costs similar to the risk neutral case presented in section A.2. However if we allow the saving to be negative at times and add a penalty to the objective function whenever we get a negative saving, we can introduce risk aversion into the model. For this reason we need to have a good estimate for B_{tn}^{max}. The risk neutral model can be solved to give us these estimates. Then we can use the following objective function and budget constraints.

$$\max \sum_{t=0}^{T} \sum_{n \in \mathcal{N}_t} \Big(p_n \cdot d_t\big((B_{tn}^{max} - B_{tn}) - PR_{tn} \cdot BO_{tn}\big)\Big) \tag{A.17}$$

$$B_{tn}^{max} + BO_{tn} - B_{tn} \geq 0 \qquad \forall n \in \mathcal{N}_t, t = 0, \cdots, T \tag{A.18}$$

$$BO_{tn} \leq BO_{tn}^{max} \qquad \forall n \in \mathcal{N}_t, t = 0, \cdots, T. \tag{A.19}$$

We allow crossing the budget limit in constraint (A.18) by introducing the slack variable BO_{tn}. This value will then be penalized by a given penalty rate (PR_{tn}) in the objective function (A.17). The penalty rate can for example be a high one time interest rate for taking a bank loan. The budget overflow (BO_{tn}) is then controlled in constraint (A.19) where the overflow is not allowed to be greater than a maximum amount BO_{tn}^{max}.

A.4.3 Wealth risk aversion

So far we have only considered the market risk or the interest rate risk. In the following we will model the other important risk factor in the mortgage market, namely the wealth risk.

Wealth risk is the risk that the actual outstanding debt becomes bigger than the expected outstanding debt at a given time during the lifetime of the loan. For example selling a 30–year bond at a price of 80, we have a big wealth risk given that a small fall in the interest rate can cause a considerable increase in the bond price, which means a considerable increase in the amount of the outstanding debt.

We consider the deviation from the average outstanding debt, which we define as DX_{tn}:

$$DX_{tn} = \overline{X}_t - \sum_i X_{itn}, \quad \forall n \in \mathcal{N}_t, t = 0, \cdots, T,$$

where $\overline{X}_t$ is the average outstanding debt for a given time t:

$$\overline{X}_t = \sum_{i \in I} \sum_{n \in \mathcal{N}_t} p_n \cdot X_{itn}, \quad \forall t = 0, \cdots, T.$$

A positive value of DX_{tn} means that we have a saving and a negative value means a loss as compared to the average outstanding debt $\overline{X}_t$. We introduce a surplus variable XS_{tn} to represent the amount of saving and a slack variable XL_{tn} to represent the amount of loss:

$$\big(\overline{X}_t - \sum_i X_{itn}\big) - XS_{tn} + XL_{tn} = 0 \qquad \forall n \in \mathcal{N}_t, t = 0, \cdots, T, \quad \text{(A.20)}$$

To make the model both market risk and wealth risk averse we update the objective function with weighted values of XS_{tn} and XL_{tn} as follows:

$$\max \sum_{n \in \mathcal{N}_t} \sum_{t=0}^{T} p_n \cdot d_t \Big((B_{tn}^{max} - B_{tn}) - PR_{tn} \cdot BO_{tn} + PW_n \cdot XS_{tn} - NW_n \cdot XL_{tn} \Big), \tag{A.21}$$

where PW_n is a parameter which can be used to encourage savings and NW_n is a parameter to penalize a loss as compared to the average outstanding debt. If we set $PW_n = NW_n$, it means that the model is indifferent towards wealth risk. On the other hand $PW_n < NW_n$, means that the model is wealth risk averse, since it penalizes a potential loss harder than it encourages a potential saving.

A.5 Scenario reduction

Since the number of scenarios grows exponentially as a function of time steps the stochastic binary model is no longer tractable when we have more than 10 time steps. For an 11–stage model we have the scenario tree in Figure (A.4).

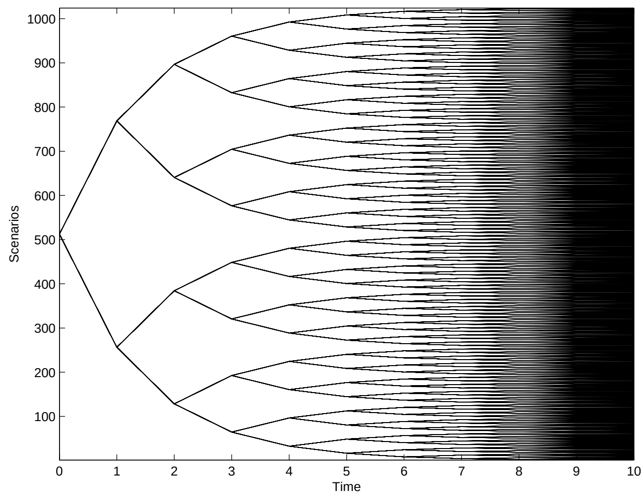


Figure A.4: A binomial scenario tree with 11 stages.

As of today there are no general purpose solvers which can solve stochastic integer problems of this size in a reasonable amount of time. Notice however that a great number of nodes in the last 3-4 time steps have such a close distance that a reduction of nodes for these time steps might not effect the first–stage results. We are in other words interested in finding a way to optimally reduce the number of scenarios. If we get the same first stage result for a reduced and a non–reduced problem, it suffices to solve the reduced problem, and then at each step resolve the problem until horizon. In that case the final result of solving any of the two problems will be the same. The reason for this is that we initially only implement the first stage solution. As the time passes by and we get more information we have to solve the new problem and implement the new first stage solution each time.

Dupačová et al. (2003) and Heitsch & Römisch (2003) have defined the scenario reduction problem (SRP) as a special set covering problem and have solved it using heuristic algorithms.

The authors behind the SCENRED articles have in cooperation with "GAMS Software GmbH" and "GAMS Development Corporation", developed a number of C++ routines, SCENRED, for optimal scenario reduction in a given scenario tree. Likewise they have developed a link, GAMS/SCENRED, which connects the GAMS program to the SCENRED module. The scenario tree in Figure A.5 is obtained after using the fast backward algorithm of the GAMS SCENRED module for a 50% relative reduction, where the relative reduction is measured as an average of node reductions for all time step. If we for example remove half of the nodes at the last time step, we get a 50% reduction for that time step only. Then we measure the reduction percentages for all other time steps in the same way. The average of these percentages corresponds to the relative reduction (see Dupačová et al. (2003) and Heitsch & Römisch (2003)). In our example the number of scenarios is reduced from 1024 to 12.

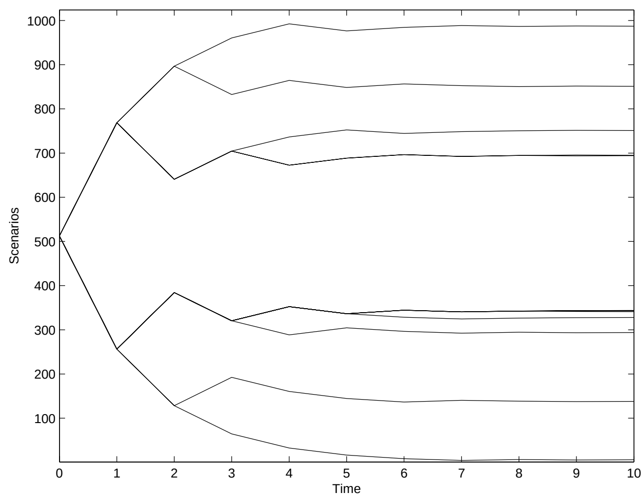


Figure A.5: A binomial scenario tree with 11 stages after a 50% scenario reduction using the fast backward algorithm of the SCENRED module in GAMS.

We use GAMS/SCENRED and SCENRED modules for scenario reduction, and compare the results with those found by solving the LP–relaxed non–reduced problem.

A.6 LP relaxation

Whenever we refinance the mortgage portfolio we need to pay a variable and a fixed transaction cost. The variable cost is $100 \cdot \eta$ percent of the sum of the sold and purchased amount of bonds and the fixed cost is simply DKK m (see constraint D.7 and D.9). The binary variables in the problem (A.1 to D.12) are due to incorporation of fixed costs m. The numeric value of these fixed costs is about DKK 2500 whereas $\eta = 0.15\%$. While the value of the variable transaction costs decreases as the time passes by, the fixed costs remain the same. Besides fixed costs are incurred per loan and not per loan portfolio which is why we cannot simply approximate the fixed costs by adding a small percentage to the variable transaction costs, even if we let this percentage increase as a function of time to adjust for the decreasing outstanding debt of the total loan portfolio. We therefore suggest an iterative updating scheme for the variable transaction costs, so that we can approximate the fixed costs without using binary variables. We do that by iteratively solving the LP problem k times as follows.

We define a ratio ψ_{itn}^{k} and initialize it to $\psi_{itn}^{0} = 0$. The ratio ψ_{itn}^{k} can then be used in the definition of a node payment (D.7) in the $k+1$st iteration as follows:

$$B_{tn} = \sum_i \Big(A_{itn} + r_{in} \cdot (1-\gamma) X_{itn} +$$
$$b \cdot (1-\beta) X_{itn} + \eta \cdot (S_{itn} + P_{itn}) + \psi_{itn}^{k+1} \cdot S_{itn} \Big) \qquad \forall n \in \mathcal{N}_t, t = 0, \cdots, T \tag{A.22}$$

Solving the LP problem at each iteration k we get S_{itn}^{*k} as the optimal value of the sold bonds at the kth iteration. Before each iteration $k > 0$, the ratio ψ_{itn}^{k} is then updated according to the following rule:

$$\psi_{itn}^{k} = \begin{cases} \frac{m}{S_{itn}^{*k}} \quad \forall i, n \in \mathcal{N}_t, t = 0, \cdots, T & \text{if } S_{itn}^{*k} > 0, \\ \psi_{itn}^{k-1} & \text{otherwise.} \end{cases} \tag{A.23}$$

This brings us to our approximation scheme for an LP relaxation of the problem:

1. Drop the fixed costs and solve the LP relaxed problem.
2. Find the ratios ψ_{itn} according to (A.23).
3. Incorporate the ratios in the model so that DKK m is added to the objective function for each purchased bond, given the same solution as the one in the last iteration is obtained. Solve the problem again.

4. Stop if the solution in iteration $k + 1$ has not changed more than α percent as compared to the solution in iteration k. Otherwise go to step 5.

5. Update ψ_{itn} according to (A.23).

6. Repeat from step 3.

Our experimental results show that for $\alpha \simeq 2\%$ we find near optimal solutions which have similar characteristics to the solutions from the original problem with the fixed costs.

A.7 Numerical results

We consider an 11 stage problem, starting with 3 callable bonds and 1 1–year bullet bond at the first stage. We then introduce 7 new bonds every 3 years. An initial portfolio of loans has to be chosen at year 0 and it may be rebalanced once a year the next 10 years. We assume that the loan is a 30–year loan and that it is prepaid fully at year 11.

The 24 callable bonds used in our test problem are seen in Table A.1. The table only presents the average coupon rates and prices for these bonds at their dates of issue. Note that only the first three bonds have already been issued, so the start prices for these three bonds are market prices on 20/02/2004, which is the date for the first stage in the stochastic program. The next 21 bonds are not issued yet, and we find their estimated prices at their future dates of issue. Since there are several states representing the uncertainty in the future we have several of these estimated start prices. In Table A.1, however, we only give an average of these prices. Besides these 24 callable bonds we use a 1–year non–callable bullet bond, bond 25. The effective interest rate on this bond is about 2% to start with. Using a BDT tree (see Black et al. (1990) Bjerksund & Stensland (1996)) with the input term structure given in Table A.2 and annual steps the effective rate can increase to 21% or decrease to slightly under 1% at the 10th year. The term structure of Table A.2 is from 20/02/2004 and is provided by the Danish mortgage bank Nykredit Realkredit A/S. The BDT tree has also been used for estimating the prices and rates of the 24 callable bonds during the lifetime of the mortgage loan using the bond pricing system Rio 4.0 (see http://www.scanrate.dk).

A practical problem arises when writing the GAMS tables containing the stochastic data. The optimization problem is a path dependent problem, whereas the BDT tree is path independent. GAMS is not well suited for such programming tasks as mapping the data from a combining binomial tree (a lattice) to a non–combining binomial tree. A general purpose programming language is better suited for this task. We have used VBA to generate the input data to the GAMS model, and we have run the GAMS models on a Sun Solaris 9 machine with a 1200 Mhz CPU, 16 GB of RAM and 4 GB of MPS.

The purpose of our tests can be summarized as the following:

1. Comparing the results of the 4 versions of our model with simple sell and hold strategies.
2. Observing the effects of using the GAMS/SCENRED module.
3. Trying our LP approximation on the problem.

For each of these objectives we consider all four versions of the model and compare the results.

A.7.1 The original stochastic MIP problem

Figure A.6 shows the solutions found for the first three stages of the problem for all four instances of our model, namely the risk neutral model, the minmax model, the model with interest rate risk aversion with budget constraints and finally the model with interest rate and wealth risk aversion with budget constraints. Notice, however, that no feasible solution could be found for the model with interest rate and wealth risk aversion with budget constraints within a time limit of 10 hours.

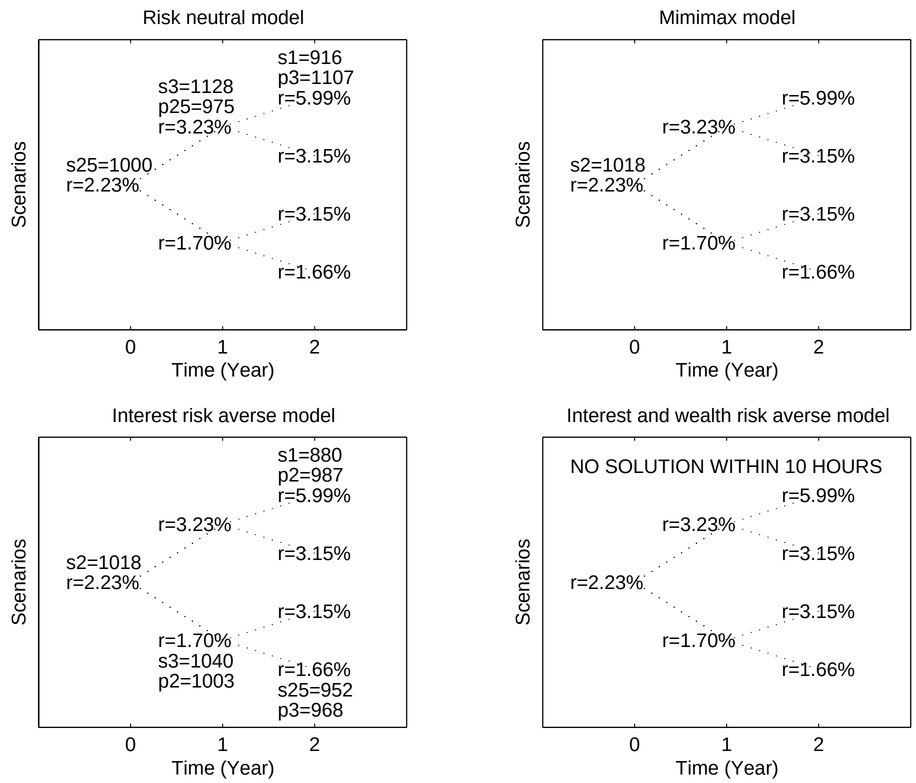


Figure A.6: Presentation of the solutions for the first 3 stages of the problem. Variable s is for sale and p for purchase and the units are given in 1000 DKK, so $s3 = 1128$ means that the mortgagor should sell approximately 1.128.000 DKK at the given node. The short rates from the BDT tree are indicated using the letter r.

A full prescription of the solution with all 11 stages will not contribute to a better understanding of the dynamics of the solution, which is why we present the solution to the first three stages of the problem only. It is though enough to give us an indication of the behaviour of each solution. In the risk neutral case we start by taking a 1–year adjustable–rate loan. If the interest rate increases after a year, the adjustable–rate loan is prepaid by taking a fixed–rate loan. Even if it means an increase in the amount of the outstanding debt, it proves to be a profitable strategy since if the rates increase again in the next stage we can reduce the amount of outstanding debt greatly by refinancing the loan to another fixed–rate loan with a higher price. The minmax strategy chooses a fixed–rate loan with a price close to par to start with. This loan is not refinanced until the 9th stage of the problem.

The risk neutral and the minmax model represent the two extreme mortgagors as far as the risk attitude is concerned. The third model reflects a mortgagor with a risk attitude between the first two mortgagors. The solution to this model guarantees that the mortgagor will not pay more than what his budget allows at any given node. Table A.3 indicates the difference in the characteristics of the solutions for the three different models.

The risk neutral model gives the lowest average total cost. The standard deviation from this average cost is, however, rather high. The minmax model has a much smaller standard deviation. This higher level of security against variation has though an average cost of about 72000 DKK. The third model has reduced the risk considerably without having increased the total average cost with more than about 7000 DKK.

We see also that these results outperform the simple sell and hold strategies (strategies 5 and 6). A traditional market risk–neutral mortgagor who chooses an ARM1 loan and keep it until horizon (year 11) is better off following either strategy 1 or 3 and a traditional market risk–averse mortgagor who chooses a fixed–rate loan and keeps it until horizon is better off following either strategy 2 or 3.

Regarding the budget constraints in model 3 and 4 we use the constants in Table A.4. Note that we are reporting these budget constraints on an aggregate level. Furthermore we define the constants PP_{Hn}^{max} and PPO_{Hn}^{max} as the target prepayment amount and maximum deviation allowed from this target respectively.

These constants are chosen after considering the average payments and the standard deviations from these in the risk neutral model.

The major problem with these solutions is the computing time taken to find near optimal solutions by CPLEX 9.0. Except for the first strategy, we cannot find solutions within 1% of a lower bound after 10 hours of CPU time. For the fourth strategy no feasible solution is found at all. Strategies 5 and 6 take a few seconds to calculate, however we do not need the optimization model for these calculations.

A.7.2 The reduced stochastic MIP problem

After reducing the number of scenarios from 1024 to 12 we get the solutions given in Figure A.7 and Table A.6.

Regarding the budget constraints in model 3 and 4 we use the constants in Table A.5.

We can see in Table A.6 that the behaviour of the solutions for the different models is similar to that of the original problem. Notice also that we get a feasible solution here for the fourth model with interest rate and wealth risk aversion.

The numeric values of the total costs for the first four strategies have however decreased considerably. Except for the risk neutral model we do not obtain the same first stage solutions as we saw for the original problem. It seems that the reduced problem gives a more optimistic view of the future as compared to the original problem. By testing the scenario

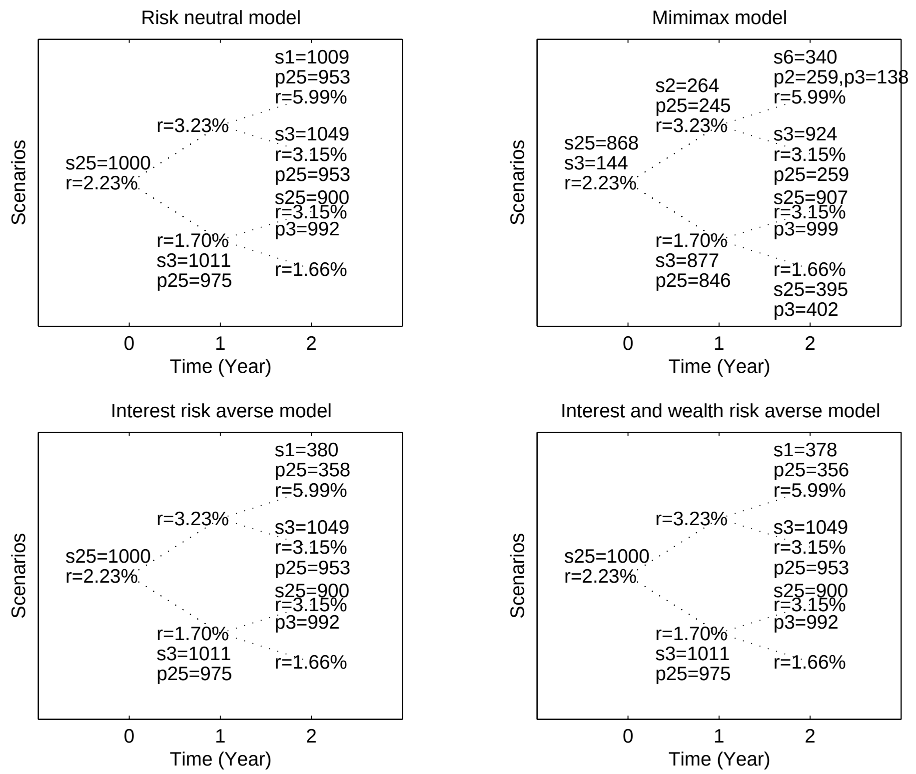


Figure A.7: Presentation of the solutions for the first 3 stages of the reduced problem. Units are given in 1000 DKK.

reduction algorithms for different levels of reduction on our problem we notice that even much less aggressive scenario reductions do not guarantee that the same initial solutions as found for the original problem are found. One explanation for this more optimistic view of the future is that since scenario reduction destroys the binomial structure of the original tree, significant arbitrage opportunities arise in parts of the new tree structure.

Another explanation is that for all levels of reduction which we have performed the reduced problem has an overweight of scenarios with lower interest rates.

We therefore need a method which 1) optimally reduces the number of scenarios while the tree remains balanced and 2) modifies bond prices in the reduced tree so that the arbitrage opportunities which are introduced as a result of scenario reduction are removed.

The question here is whether points 1 and 2 play an equally important role in getting similar first stage solutions for the original and the reduced problem. Comparing strategies 5 and 6 in Tables A.3 and A.6 indicates that performing point two might remove most of the difference between the solutions in the reduced problem as compared to the original problem. Apparently the average total costs for the ARM1 loan are slightly decreased in the reduced tree whereas the average total costs for the fixed–rate loan are slightly increased. This slight change in opposite directions can only be explained by the observation that the reduced tree has an overweight of scenarios with lower interest rates, since no trading is allowed for these two strategies and therefore the arbitrage opportunities cannot be used. We are currently working on better ways of reducing scenario trees taking into accounts points 1 and 2.

A.7.3 The reduced and LP–approximated problem

When we use our LP–approximation algorithm on this problem we get the solution as presented in Table A.7 and Figure A.8.

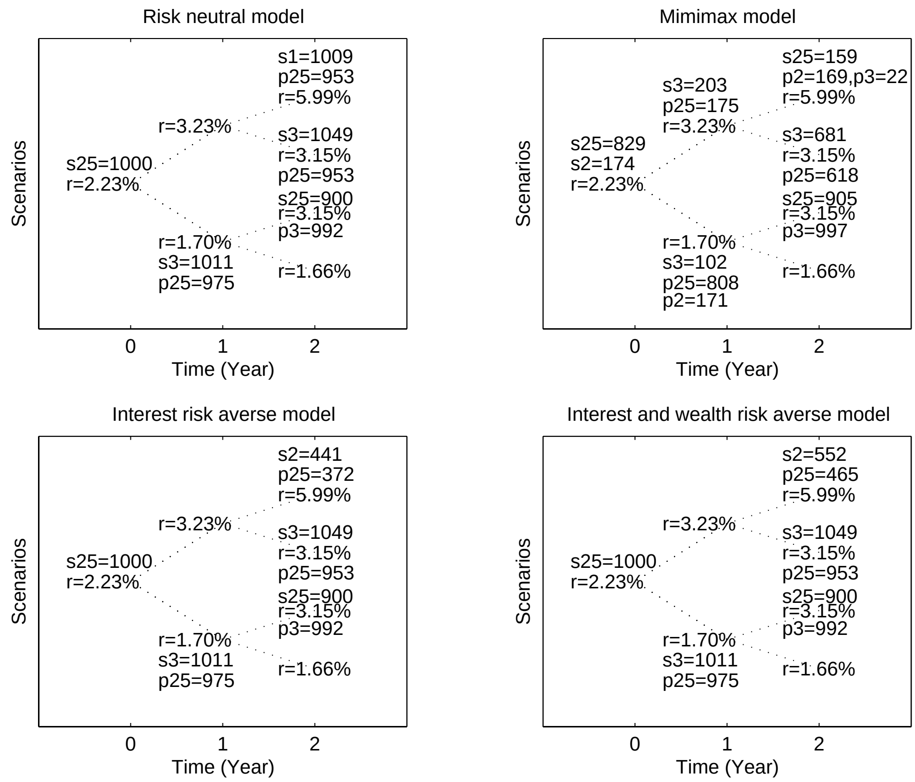


Figure A.8: Presentation of the solutions for the first 3 stages of the LP approximated reduced problem. Units are given in 1000 DKK.

The algorithm uses 10–18 runs for the different problems to find solutions which are over all less than 2% different from the solutions found in the last iteration.

It is important to point out that simply dropping the fixed costs results in solutions which deviate considerably from the problems with the fixed costs, whereas approximating the fixed costs using our algorithm gives very similar results as found by the MIP model.

A.7.4 Comments on results

The results presented in this section are in agreement with the financial arguments used in the Danish mortgage market. Even though the original problem is hard to solve we have shown that useful results can be found by solving the reduced problems. The reduced scenario trees represented a more optimistic prediction of the future, but the results found are still quite useful. In practice the mortgage portfolio manager should try several scenario trees with different risk representations as an input to the model. This way the optimization model can be used as an analytical tool for performing "what–if" analyses on a high abstraction level.

A.8 Conclusions

We have developed a functional optimization model that can be used as the basis for a quantitative analysis of the mortgagors decision options. This model in conjunction with different term structures or market expert opinions on the development of bond prices can assist market analysts in the following ways:

Decision support: Instead of calculating the consequences of the single loan portfolios for single interest rate scenarios, the optimization model allows for performing "what if" analysis on a higher level of abstraction. The analyst can provide the system with different sets of information such as the presumed lifetime of the loan, budget constraints and risk attitudes. The system then finds the optimal loan portfolio for each set of input information.

Product development: Traditionally, loan products are based on single bonds or bonds with embedded options. In some mortgage markets such as the Danish one it is allowed to mix bonds in a mortgage portfolio and there are even some standard products which are based on mixing bonds. The product P_{33} is for example a loan portfolio where 33% of the loan is financed in 3–year non–callable bonds and the rest in fixed–rate callable bonds. These mixed products are currently not popular since the rationale behind exactly this kind of mix is not well argued. The optimization model gives the possibility to tailor mixed products that, given a set of requirements, can be argued to be optimal for a certain mortgagor.

The greatest challenge in solving the presented models is on decreasing the computing times. We have experimented with scenario reduction (see Dupačová et al. (2003) and Heitsch & Römisch (2003)) and we have suggested an LP approximation method to reduce the solution times while maintaining solution quality. It is, however, an open problem to develop tailored exact algorithms such as decomposition algorithms (see Birge (1985) and Birge & Louveaux (1997)) to solve the mortgagors problem. Another approach for getting real time solutions is to investigate different heuristic algorithms or make use of parallel programming (see Nielsen & Zenios (1996*b*) and Ruszczynski (1993)) to solve the problem.

Integration of the two disciplines of mathematical finance and stochastic programming combined with use of the state of the art software has a great potential, which has not yet been realized in all financial markets in general and in mortgage companies in particular. There is a need for more detailed and operational models and high performing easy to use accompanying software to promote use of the mathematical models with special focus on stochastic programming.

Acknowledgments

We would like to thank Nykredit Realkredit A/S for providing us with the term structure and mortgage price data. We are also grateful to the referees for many good comments on an earlier version of this paper and to Rolf Poulsen for reading and commenting the revised version of the paper.

Bond nr.	rate	Average start price	Date of issue	Date of maturity
1	6%	103.06	3/10-02	3/10-35
2	5%	98.5	3/10-02	3/10-35
3	4%	89.4	3/10-02	3/10-35
4	9%	107.33	3/10-05	3/10-38
5	8%	103.16	3/10-05	3/10-38
6	7%	103.09	3/10-05	3/10-38
7	6%	100.51	3/10-05	3/10-38
8	5%	94.01	3/10-05	3/10-38
9	4%	84.55	3/10-05	3/10-38
10	3%	74.46	3/10-05	3/10-38
11	9%	105.4	3/10-08	3/10-41
12	8%	101.98	3/10-08	3/10-41
13	7%	100.3	3/10-08	3/10-41
14	6%	96.19	3/10-08	3/10-41
15	5%	89.5	3/10-08	3/10-41
16	4%	80.74	3/10-08	3/10-41
17	3%	71.32	3/10-08	3/10-41
18	9%	104.41	3/10-11	3/10-44
19	8%	100.9	3/10-11	3/10-44
20	7%	98.51	3/10-11	3/10-44
21	6%	94.07	3/10-11	3/10-44
22	5%	87.49	3/10-11	3/10-44
23	4%	79.25	3/10-11	3/10-44
24	3%	70.26	3/10-11	3/10-44

Maturity (Year)	Yield (%)	Yield Volatility (%)		Maturity (Year)	Yield (%)	Yield Volatility (%)
1	2.23%	–		16	4.87%	17.25%
2	2.35%	32.20%		17	4.93%	17.00%
3	2.73%	32.10%		18	4.99%	16.85%
4	3.08%	29.50%		19	5.05%	16.75%
5	3.41%	27.00%		20	5.11%	16.70%
6	3.68%	25.00%		21	5.16%	16.65%
7	3.92%	23.00%		22	5.21%	16.60%
8	4.12%	22.00%		23	5.25%	16.56%
9	4.30%	20.90%		24	5.29%	16.52%
10	4.44%	20.10%		25	5.34%	16.48%
11	4.56%	19.40%		26	5.37%	16.45%
12	4.62%	18.80%		27	5.40%	16.42%
13	4.68%	18.30%		28	5.43%	16.39%
14	4.74%	17.90%		29	5.46%	16.36%
15	4.80%	17.55%		30	5.49%	16.34%

Table A.2: The input term structure to the BDT model.

Loan strategy	Total costs	Std. dev.	max	min	time
1 - Risk neutral	1.281.857	92.289	1.502.042	1.004.583	276 s
2 - Minmax	1.353.713	19.729	1.374.183	1.117.084	10 h
3 - Int. rate risk averse	1.288.405	66.019	1.431.857	1.005.412	10 h
4 - Int./Wealth risk averse	No solution found within 10 h.				
5 - Loan25 (ARM1)	1.310.495	115.085	1.821.388	1.120.053	$<$ 10 s
6 - Loan2 (Fixed–rate 5%)	1.353.438	72.582	1.410.190	993.056	$<$ 10 s

Table A.3: Comparison of the four strategies for the original problem.

Constant	Definition	Value
BMAX	$\sum_{t=0}^{H-1} \sum_{n \in \mathcal{N}_t} p_n \cdot B_{tn}^{max}$	570842
PPMAX	$\sum_{n \in \mathcal{N}_H} p_n \cdot PP_{Hn}^{max}$	711015
BOMAX	$\sum_{t=0}^{H-1} \sum_{n \in \mathcal{N}_t} p_n \cdot BO_{tn}^{max}$	50000
PPOMAX	$\sum_{n \in \mathcal{N}_H} p_n \cdot PPO_{Hn}^{max}$	100000

Table A.4: Budget limits used in model 3 and 4 for the original data.

Constant	Definition	Value
BMAX	$\sum_{t=0}^{H-1} \sum_{n \in \mathcal{N}_t} p_n \cdot B_{tn}^{max}$	565915
PPMAX	$\sum_{n \in \mathcal{N}_H} p_n \cdot PP_{Hn}^{max}$	601983
BOMAX	$\sum_{t=0}^{H-1} \sum_{n \in \mathcal{N}_t} p_n \cdot BO_{tn}^{max}$	50000
PPOMAX	$\sum_{n \in \mathcal{N}_H} p_n \cdot PPO_{Hn}^{max}$	35000

Table A.5: Budget limits used in model 3 and 4 for the reduced data.

Model type	Total costs	Std. dev.	max	min	time
1 - Risk neutral	1.169.173	49.765	1.274.079	1.064.525	12 s
2 - Minmax	1.187.938	0.00	1.187.938	1.187.938	52.2 s
3 - Int. rate risk averse	1.171.926	24.270	1.229.897	1.136.655	300 s
4 - Int./Wealth risk averse	1.172.479	25.610	1.229.742	1.128.412	300 s
5 - Loan25 (ARM1)	1.301.237	120.958	1.560.244	1.129.983	$<$ 1 s
6 - Loan2 (Fixed–rate 5%)	1.356.228	59.356	1.410.190	1.249.483	$<$ 1 s

Table A.6: Comparison of the four strategies for the reduced problem.

Model type	Total costs	Std. dev.	max	min	time
1 - Risk neutral	1.169.147	49.775	1.274.078	1.064.524	25 s
2 - Minmax	1.179.654	11.150	1.185.795	1.154.602	22 s
3 - Int. rate risk averse	1.172.364	26.436	1.239.168	1.130.196	28 s
4 - Int./Wealth risk averse	1.174.038	29.128	1.249.520	1.131.185	44 s
5 - Loan25 (ARM1)	1.301.237	120.958	1.560.244	1.129.983	–
6 - Loan2 (Fixed–rate 5%)	1.356.228	59.356	1.410.190	1.249.483	–

Table A.7: Comparison of the four strategies for the reduced problem with LP approximation.

Appendix B

Financial Giffen Goods: Examples and Counterexamples

Accepted by *European Journal of Operational Research (2007, in press).*

Abstract

In the basic Markowitz and Merton models, a stock's weight in efficient portfolios goes up if its expected rate of return goes up. Put differently, there are no financial Giffen goods. By an example from mortgage choice we illustrate that for more complicated portfolio problems Giffen effects do occur.

Keywords: Finance, portfolio choice, Giffen good, mortgage planning.

JEL code: G11 **Subject category:** IE13

B.1 Introduction

A Giffen good is one for which demand goes down if its price goes down. At first, it is counter-intuitive that such goods exist at all. But most introductory text-books in economics will tell you that they do; some with stories about potatoes and famine in Ireland, some with first order conditions for constrained optimization. In this note we study similar effects

— by which we mean a negative relation between expected return and demand — in portfolio choice models. Surprising dependence on expected rates of return is not uncommon in finance. In complete models, option prices do not depend on the stock's growth rate. And quite generally call-option prices increase with the interest rate; immediately you would think that cash-flows are discounted harder, but in fact the replicating strategy which entails a short position in the bank-account becomes more expensive, and hence the call-option does too.

We first show that in the basic Markowitz mean/variance model, there are no Giffen goods; if a stock's expected rate of return goes up, its weight in any efficient portfolio goes up. This seems a text-book comparative statics result. We have, however, only been able to find it indirectly stated, for instance one could view it as a corollary or lemma related to the Harmony Theorem from Luenberger (1998, Section 7.8). So we give a simple proof. We then look at Merton's dynamic investment framework. In its basic version demand for any asset depends positively on its expected rate of return, but if a subsistence level is included, demand for the risk-free asset may fall with the interest rate.

Skeptics would say that Giffen goods exist in *and only in* economic text-books. We end the paper by illustrating that it is not so. Our example uses a generalized version of the multi-stage stochastic programming framework from Rasmussen & Clausen (2007) and shows that some – completely rational – mortgagors react to lower costs of long-term financing (reflecting a smaller market price of risk) by using more short-term financing.

B.2 The Markowitz Model

Consider a model with n risky assets with expected rate of return vector μ and invertible covariance matrix Σ, and put $\mathbf{1}^\top = (1, \ldots, 1)$. The mean/variance efficient portfolios are found by solving

$$\max_w w^\top \mu - \frac{1}{2}\gamma w^\top \Sigma w \text{ st } w^\top \mathbf{1} = 1,$$

for different values of risk-aversion γ. This is a slight but convenient reparametrization of traditional formulations (e.g. Huang & Litzenberger (1988, Chapter 3)). The optimal portfolios are

$$\widehat{w} = \gamma^{-1}\Sigma^{-1}\left(\mu - \eta(\gamma; \mu, \Sigma)\mathbf{1}\right)$$

where $\eta(\gamma; \mu, \Sigma) = (\mathbf{1}^\top\Sigma^{-1}\mu - \gamma)/\mathbf{1}^\top\Sigma^{-1}\mathbf{1}$ can be interpreted as the expected rate of return on $\widehat{w}$'s zero-beta portfolio.

A sensible definition of a Giffen good is an asset, say the i'th, for which $\partial\widehat{w}_i/\partial\mu_i < 0$ for some γ, this meaning that when the asset's expected rate of return goes up, its weight in some optimal portfolio goes down. Let us show that there are no such assets. To do this we look at the problem with the modified expected return vector $\mu + \alpha e_i$, where $\alpha \in \mathbb{R}$ and e_i is the i'th unit vector. The optimal portfolio in this case we can write as

$$\widehat{w}(\alpha) = \widehat{w} + \alpha h,$$

where $h = \gamma^{-1}(\Sigma^{-1}e_i - \frac{e_i^\top \Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1})$. Showing that $\partial\widehat{w}_i/\partial\mu_i > 0$ amounts to proving positivity of the i'th coordinate of h, which we can write as

$$e_i^\top h = \gamma^{-1}\left(e_i^\top \Sigma^{-1} e_i - \frac{(e_i^\top \Sigma^{-1}\mathbf{1})^2}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}\right).$$

Because Σ^{-1} is strictly positive definite and symmetric, $x^\top \Sigma^{-1} y$ defines an inner product, and strict positivity of the term in parenthesis on the right hand side of the equation above follows immediately from the Cauchy-Schwartz inequality.

The inclusion of a risk-free asset is handled in the same way with η replaced by the risk-free rate of return because the risk-free asset is any portfolio's zero-beta portfolio.

With this result we can easily prove the Harmony Theorem from Luenberger (1998, Section 7.8) — or equivalently answer the question posed in the title of Zhang (2004) — that says that a newly introduced $(n+1)$'st asset (or "project") will be in positive demand (or: "attractive") precisely if there is strict inequality in the CAPM-like expression

$$\mu_{n+1} - r > \frac{\mathrm{cov}(r_{n+1}, r_M)}{\mathrm{var}(r_M)}(\mu_M - r), \tag{B.1}$$

where M denotes the market (or tangent) portfolio, and rs with subscripts are (stochastic) rates of returns. It is well-known, see Constantinides & Malliaris (1995, Theorem 4) but it dates back to Roll (1977), that a portfolio w is mean/variance efficient precisely if for any individual asset i we have

$$\mu_i - r = \frac{\mathrm{cov}(r_i, r_w)}{\mathrm{var}(r_w)}(\mu_w - r).$$

For the portfolio $(w_M^\top, 0)^\top$ the n first necessary equations hold because the market portfolio is efficient in the old economy, and we see that the new asset is in 0-demand if equality holds in (B.1). Now the absence of Giffen tells us that if there is strict inequality as stated, the $(n+1)$'st asset has strictly positive weight in the new market portfolio.

B.3 The Merton Model

Another classic portfolio model is Merton's dynamic investment framework, see Merton (1990, Chapter 5). In its simplest case, an agent invests his wealth, W, in either a risk-free asset with rate of return r or a risky asset whose price follows a Geometric Brownian motion. Suppose the agent maximizing expected utility cares only about terminal wealth, $W(T)$, and has a utility function with constant relative risk aversion,

$$U(W(T)) = \frac{W(T)^{1-\gamma}}{1-\gamma}.$$

It is optimal for this agent to invest a fixed fraction,

$$\pi = \frac{\mu - r}{\gamma\sigma^2},$$

of wealth in the risky asset, So if the expected rate of return of an asset (be that risky or risk-free) goes up, that asset gets higher weight in any agent's portfolio. Further, by combining 2-fund separation with the Markowitz analysis from the previous section, the same conclusion is reached in a model with n rather than just one risky asset.

An extension (that was actually considered in Merton's original paper; see Merton (1990, Section 5.6)) is a utility function of the form

$$\widetilde{U}(W(T)) = \frac{(W(T) - \bar{W})^{1-\gamma}}{1-\gamma},$$

where $\bar{W}$ is some minimal required wealth; a subsistence level. Assuming initial wealth is greater than $e^{-rT}\bar{W}$ (otherwise the problem is ill-posed), the optimal strategy is to buy $e^{-rT}\bar{W}$ units of the risk-free asset and invest the rest of the wealth according to the Merton-rule from above. Thus the optimal fraction invested at time t in the risky asset is

$$\widetilde{\pi}(t) = \frac{W(t) - e^{-r(T-t)}\bar{W}}{W(t)} \frac{\mu - r}{\gamma\sigma^2},$$

so that

$$\frac{\partial \widetilde{\pi}(0)}{\partial r} = \frac{1}{\gamma \sigma^2} \left(\frac{e^{-rT} \bar{W}}{W(0)} (T(\mu - r) + 1) - 1 \right).$$

From this we see that we can have $\partial \widetilde{\pi}(0)/\partial r > 0$ (for instance if $e^{-rT} \bar{W}/W(0) = 1/2$, $T = 30$ and $\mu - r = 0.05$), so the percentage of initial wealth invested in the risky asset goes up, and hence the investment in the risk-free asset goes down when the risk-free rate of return goes up. The intuition behind is that if the return of the risk-free asset goes up, you need less of it to ensure survival, and you have more money to do what you like, rather than what you have to.

B.4 A Mortgage Choice Model

A way to quantify mortgage planning — for many people the largest financial decisions, they ever make — as a portfolio optimization problem suitable for modern OR techniques is to study

$$\text{minimize}_{\phi} \ (1 - \gamma) \times \mathbf{E}(X(\phi)) + \gamma \times \mathbf{ES}_{\beta}(X(\phi)),$$

where:

— $X(\phi)$ is the (cumulative discounted) payments from the mortgagor's dynamic portfolio strategy, ϕ, and $\mathbf{ES}_{\beta}(X) = \mathbf{E}(X | X \geq q_{\beta})$ denotes expected shortfall (also called tail or conditional value-at-risk) based on the β-quantile q_β.

— The minimization is done subject to

  — a stochastic interest rate model discretized by paths through trees, each node having a universe of securities.

  — portfolio and cash-flow constraints, transaction and mortgage origination costs as well as re-balancing constraints.

This multi-stage stochastic programming problem is an extension of the models considered in Rasmussen & Clausen (2007), and it has some appealing features of both intuitive and technical natures:

— It takes into account both reward (low expected payments) and risk (large, extreme payments), it does so based on the coherent risk-measure (as defined by Artzner et al. (1999)) expected shortfall, and it allows us to explicitly control the trade-off between risk and reward (varying γ gives an efficient frontier, just like in the Markowitz model).

— As shown by Rockafeller & Uryasev (2000), expected shortfall gives rise to a piece-wise linear objective function. This means even large instances of the problem can be solved efficiently using standard software such as GAMS and CPLEX.

For all technical details and analysis of this generalized model see Rasmussen & Zenios (2007).

To model the stochastic behaviour of interest rates, we use a Vasicek model

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dZ(t),$$

where Z is a Brownian motion. To specify the full yield curve dynamics, a market price of risk is needed. We parameterize this by λ, that technically shifts the stationary mean of r to $\theta + \lambda$ under the risk-neutral measure, but more tellingly, determines the typical difference between long and short rates. This represents the fundamental trade-off in the mortgagor's problem: Short rates are typically lower than long rates, but with short-term financing, he doesn't know how much he will have to pay.

Table B.1 shows the composition of the initial optimal portfolios for two different values of the market price of risk. These two values correspond to calibration to observed Danish yield curves in October 2004 and February 2005, as depicted in Figure B.1.

	Optimal initial loan portfolio compositions			
	October 2004		February 2005	
Mortgagor risk aversion (γ)	Fixed rate callable	Full yearly refinancing	Fixed rate callable	Full yearly refinancing
0	0%	100%	0%	100%
1/4	20%	80%	15%	85%
1/2	36%	64%	37%	63%
3/4	46%	54%	49%	51%
1	68%	32%	74%	26%

Table B.1: Optimal initial loan portfolio compositions for various mortgagors facing the yield curves shown in Figure B.1. (We used the 90%-quantile for expected shortfall, 2% discounting, 1.5% transactions costs, a 7-year horizon and 6 stages.)

We first note that only the 1-year adjustable-rate bond and the 30-year callable, fixed-rate bond are used in the optimal portfolios, although the numerical algorithm allowed for a larger universe of mortgage products (about 10 at each node). Row-wise comparisons in Table B.1 give no surprises. The risk-neutral mortgagor uses full short-term financing and as risk-aversion rises more long-term financing is used. Note, however, that because short rates were historically low and the yield curve quite steep, even very risk-averse mortgagors use a significant amount (one-third to one quarter) of short-term financing. Comparing the columns tells us what a lowering of the market price of risk parameter can do to optimal portfolios. The very risk-averse mortgagor uses a larger proportion (74% compared to 68%) of long-term financing, and the risk-neutral mortgagor does not care. But for a moderately risk-averse mortgagor ($\gamma = 1/4$), the lowered market price of risk, which makes short-term financing relatively less attractive, causes him to use more short-term financing (up to 85% compared to 80% before). Although a more complicated model, the intuition is again that this mortgagor uses long-term financing initially not because he wants to, but because he has to, and lower long rates — still higher than typical short rates — make the necessity cheaper; like the Irish potatoes.

B.5 Conclusion

In this note we first analyzed sensitivity to expected returns in two textbook models for optimal portfolio choice (Markowitz and Merton) and showed that the relation is as one would think; (for any asset) higher expected return raises demand (from any investor). We then demonstrated by examples — the most interesting being from mortgage planning — that this is not a general result. Let us end by a couple of remarks on extensions and future research.

While we think our definition of a Giffen good is quite sensible, it very much takes a "comparative statics" view-point, that is: Differentiate the optimal solution wrt. to a specific expected return parameter. One can investigate other parameter derivatives and may find surprises. But we think there is a limit to how far this analysis can be taken before running into the Lucas critique: Sensitives from a static model may tell you nothing about effects in a truly dynamic model. If you want to know how people react to a change, you must build a model where they take such changes into their optimization considerations.

We see the use of stochastic programming techniques in financial engineering as very promising. The framework can be used as we did here to analyze individual mortgagors' problems, but it can also be "reversed" to put together structured products that are optimal (in a precise quantitative sense) for investors or mortgagors. Huang, Kai, Fabozzi & Fukushima (2007) look at such a case and with the liberalization of capital markets enforced by new rules from the European Union much more work is needed in that direction.

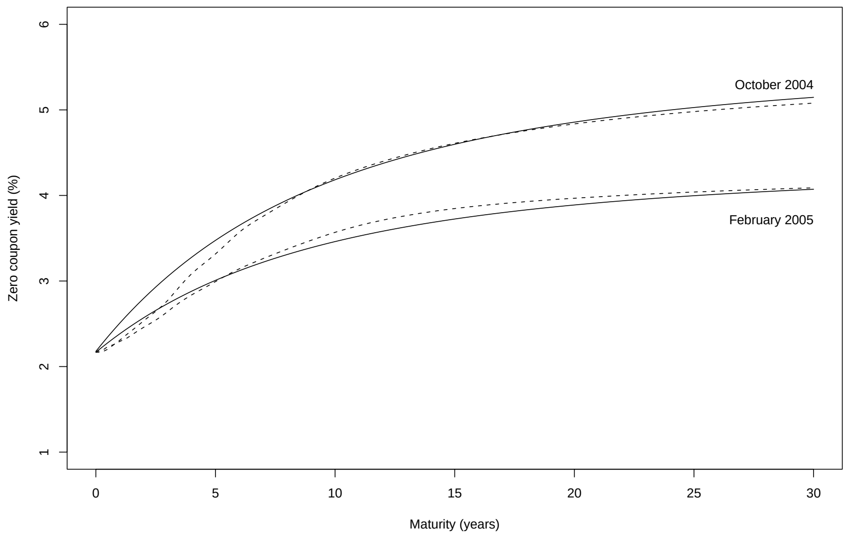


Figure B.1: Danish yield curves from October 2004 and February 2005; full curves are calibrated model curves, dotted lines are observations. The estimated Vasicek model parameters $(\theta, \kappa, \sigma) = (0.042, 0.2, 0.01)$ are held fixed and only the calibrated market price of risk, λ, differs from October 2004 (0.017) to February 2005 (0.004). In October the difference between the 30-year and the 1-year rate is 2.8%; in February it is 1.8%.

APPENDIX C

Well ARMed and FiRM: Diversification of mortgage loans for homeowners

Published in *The Journal of Risk*, Fall 2007, volume 10, pages 67–84.

Abstract

Individual homeowners are offered today a wide range of mortgage options for financing the purchase of a house. Usually, homeowners are also granted an option to repay the mortgage loan, and in some countries—such as Denmark—it is particularly efficient to do so as market conditions change or the homeowner's situation warrants it. And while, traditionally, a single mortgage loan would serve borrower needs, today it appears that a portfolio of loans may satisfy much better the mortgage needs of the individual and his or her appetite for risk. In this paper we develop a model for the diversification of mortgage loans of a homeowner and apply it to data from the Danish market. Even in the presence of mortgage origination costs it is shown that most risk averse homeowners will do well to consider a diversified portfolio of both fixed (FRM) and adjustable (ARM) rate mortgages. This is particularly so if one takes, unavoidably, a long term perspective in financing the purchase of a home through a mortgage loan.

C.1 Introduction

What is in a name? Everything is in the name, the marketing guru will tell us, and the names used to pitch the hottest subset of the old-fashioned mortgages tell a lot: Interest-Only (I-O), OptionARM and Pick-a-Payment have been added to the traditional shopping list of fixed (FRM) or adjustable (ARM) rate mortgages.

In essence all these mortgage products aim at satisfying the same underlying need of borrowers: offering them a loan to achieve the immediate home purchasing goals, with payment terms that can be adapted as the family earnings change—usually with an upward adjustment especially for young home buyers—while offering some protection from market changes to both borrowers and lenders. The issue, especially for first time buyers, has been one of trading off the lower rate and payment on an ARM with the interest rate risk of the ARM, or going for the higher initial payments of a FRM and lower risk when the FRM is kept to maturity. With an I-O the initial payments can be reduced substantially, but future payments will increase significantly when the interest-only clause expires and principal payments must be made, especially if there have been significant changes in interest rates since the loan was issued.

According to some surveys (Real Estate Center at the Wharton School, University of Pennsylvania, "Could risky mortgage lending practices prick the housing bubble?", Web newsletter, 2002) innovative mortgage products account for half of the new mortgages written in the USA, up from less than 10% in 2001. The innovations in the Danish market have not been as exotic as their US counterparts, but they have been particularly simple and efficient to promote. The seven specialized mortgage banks that operate in Denmark fund the loans by issuing bonds in the capital markets. The terms of the bonds are identical to the mortgage loan they fund, with the mortgage bank adding a markup on the market yield of bonds with maturity comparable to the loan. Given the readily available information of market yields all banks offer the same markup rate, and this mechanism is equivalent to the borrower issuing his or her own bonds. Furthermore, all borrowers pay the same rate on the same type of loan issued on a given date. (This is possible, even accounting for differences in credit worthiness of the borrowers, as most loans require a 20% down payment which adequately covers a wide range of credit risky borrowers. However, borrowers with very poor credit or without initial endowment are not served in this market.)

Given the simplicity with which market rates are transformed into mortgage loans, on any given day a borrower can be offered a range of fixed-rate mortgages with 15, 20 or 30 year maturity, and adjustable rate mortgages with adjustment periods ranging from one to ten years, and these products reach a wide market segment. The total amount of mortgage loans issued in Denmark—this includes origination of new loans plus the refinancing of old loans—in 2005 totals almost 100 billion EURO. This represents a record increase of 25% from the year before, and corresponds to half of the Danish gross domestic product. Fixed and adjustable rate mortgages account for 50% of the mortgage market each. Innovative products such as I-O and ARM with a cap are quite popular. About 30% of the total ARMs in the market come with a cap, while about 30% of the total amount of mortgage loans in the private and summer house market are I-O.

The refinancing activity in this market is also noteworthy. About one third of all outstanding loans, for a total of 70 billion EURO, were refinanced in 2005. The issue of new loans reached 20 billion EURO, corresponding to a growth of 10% to the total amount of mortgage loans in the Danish market which sets a new record.

Some form of protection from either market changes or changes in family conditions—job loss, births, deaths or divorce—comes in the form of an early prepayment option. Additional protection from interest rate risk is offered through caps on the rate adjustment of ARMs or with the purchase of fixed rate mortgages. But then one has to deal with the higher rates associated with a FRM and accept the risk that rates will decline while payments on the loan remain locked. Furthermore, in the Danish market, early termination of a FRM requires prepayment of the mortgage at prevailing market prices which represents significant interest rate risk for homeowners with issued loans with prices below par, except those few who keep the mortgage to maturity.

C.2 Are there diversification benefits from portfolios of mortgage loans?

In the context introduced above it is somewhat surprising that the question of diversification of homeowner mortgage loan has not received to date any attention. This is primarily due to the transaction costs involved in obtaining more than one mortgage loans—mortgage origination fees in Denmark stand currently at 300 EUR. But still, in the efficient Danish market there has been an interest in selling more than one product to each homeowner. Indeed, it can be easily observed that a combination of mortgage loans could provide lower average total payments during the life of a mortgage, and with less variability as interest rates change. Figure C.1 illustrates the changes in total payment for FRMs and an ARM with increasing interest rates. It is seen from this figure that the sensitivity of FRMs and ARMs to interest rate changes have opposite signs, and hence a properly balanced portfolio of both types of loans could provide better protection than either mortgage alone.

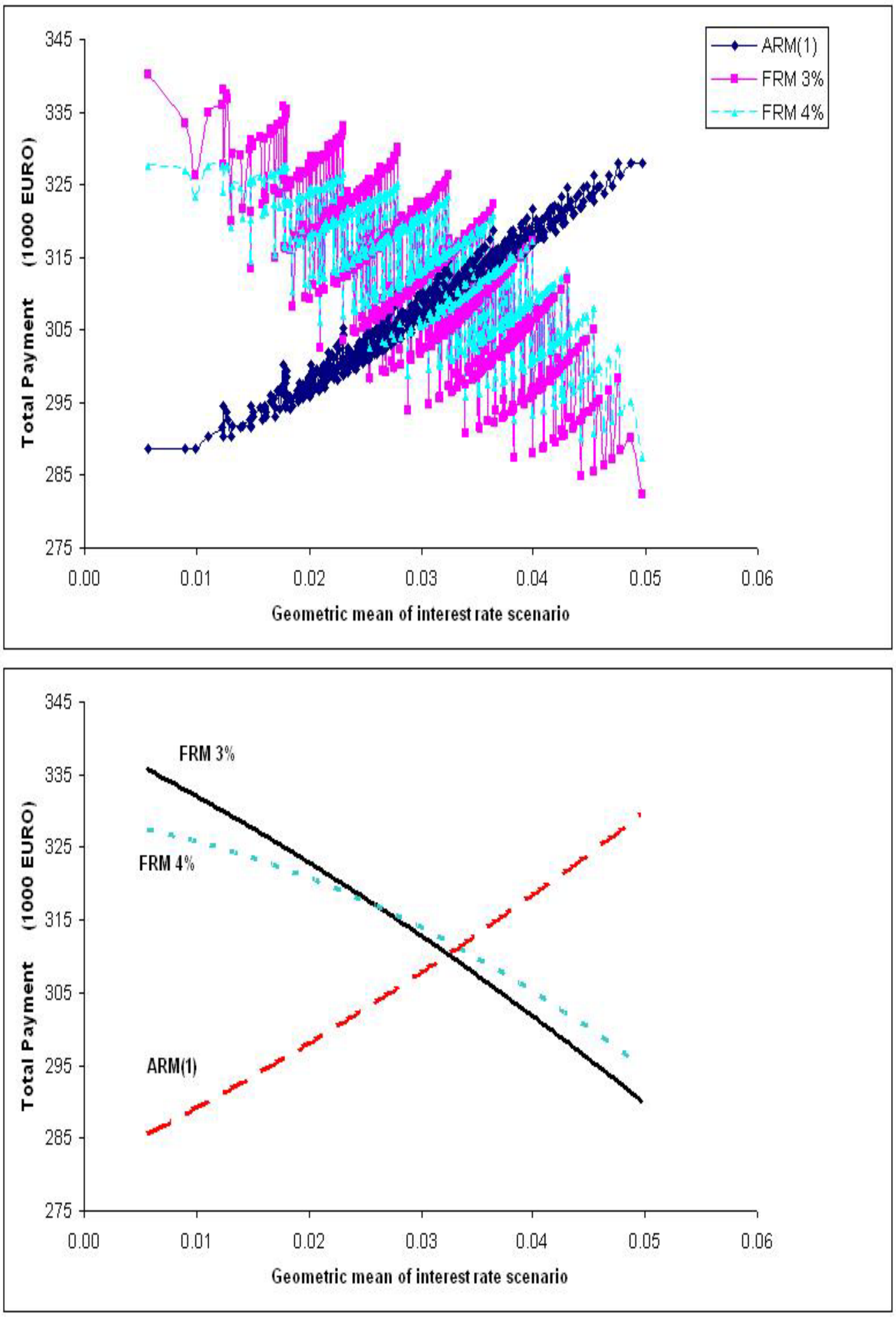


Figure C.1: The total expected payment of 30-year fixed (3% and 4% FRM) and adjustable (ARM(1) with annually adjusted rates) rate mortgages have opposite sensitivities to changing interest rates. (Top figure shows all simulated data for a six-year planning horizon; bottom figure

The same conclusion is further highlighted from the analysis of the historical performance of mortgage loans in the Danish market during the period 1995–2005. Figure C.2 shows the mean payment and the risk of the payment for various typical mortgages during this period. Risk is measured by the Conditional Value-at-Risk (CVaR) at the 90% confidence level, see, e.g., Rockafellar and Uryasev (2000) or Jobst and Zenios (2001). The differences in mean payments and CVaR, coupled with the negative correlation of the FRM and ARM payments suggest that interest rate risk can be diversified by holding portfolios of mortgages.

The structuring of diversified portfolios is the topic of this paper. But first some more explanations are in order for the operations of the Danish mortgage market.

C.3 Some explanations on Danish mortgages

The Danish mortgage banks are highly specialized institutions whose line of business is, first, to collect the investments from the investors of mortgage backed securities, and, second, to pool the investments together and issue mortgage loans to house buyers. The great volume of housing trade—the outstanding mortgage debt corresponds to one half of the

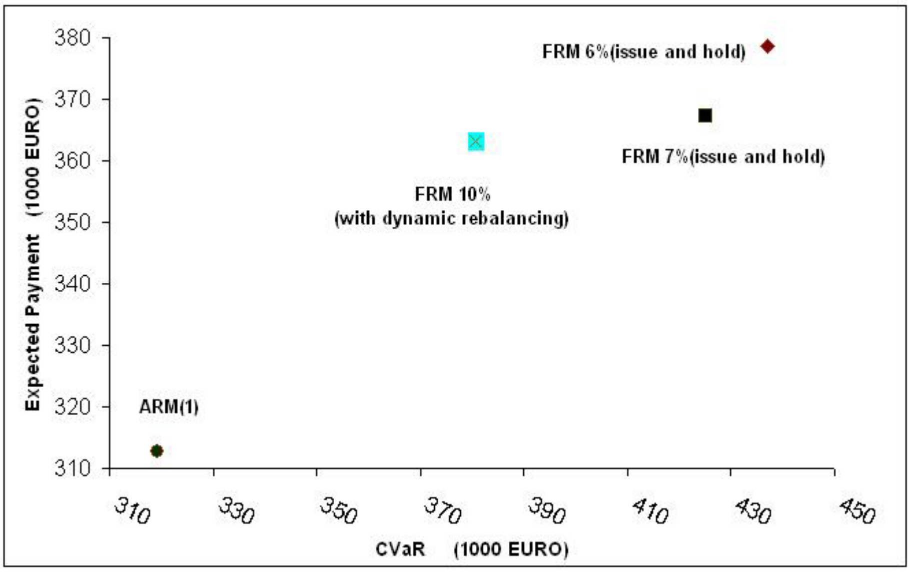


Figure C.2: Expected payment and risk of payments (measured by Conditional Value-at-Risk, CVaR) for Fixed (FRM) and Adjustable (ARM) Rate Mortgages in the Danish mortgage market during the period 1995–2005.

gross domestic product of the country—the efficiency of the one-stop-shop process of mortgage origination by specialized banks, together with a 200-year history of no default from the mortgage banks[1] result in cheap loans for prospective house buyers. As investors are not exposed to default risk the Danish mortgage backed securities are rated AAA, and banks simply add to the market bond yields a markup rate which currently stands at 0.55%.

A unique feature to the Danish mortgage market is the "balance principle" prescribing that the payments made by the mortgagor are exactly the payments received by the investor. In effect, Danish mortgagors are trading directly mortgage bonds and may exercise all the options. A Danish FRM has a call option typically with strike price at 100 and a buy–back delivery option embedded on the underlying bonds. This has in particular an impact on loans with long maturities, as small movements in interest rates result in big movements in the prices of FRMs and, hence, have a direct impact on the amount of outstanding debt for the Danish mortgagor.

Until 1995 FRMs were the only kind of mortgage backed securities which were traded in the Danish market. Since then the mortgage market has

[1] House owners may fail to pay their liabilities, but there has yet not been an incident of default when it comes to payments to investors via the mortgage banks.

been growing fast and a number of new products have been introduced. The two most popular products have been the adjustable rate mortgage loans with varying adjustment intervals, and capped ARM loans where the interest rate cannot grow higher than a predetermined level.

All these loans can be issued with Interest-Only payments for a grace period of up to 10 years, although after 10 years it is possible to refinance the loan with another 10 years of I-O payments and as a result the outstanding debt is not being reduced during this period.[2]

The features of either FRM or ARM, together with the flexibility for refinancing the loans, imply that the total payments on the mortgage during the life of the loan is highly uncertain. While a mortgage owner will finance a loan in ways that are consistent with his or her short term financial capabilities, in the long run the payments made and the outstanding debt will be determined by the changing interest rates. A simulation model can be used to project the total payments, including the value of the outstanding debt if the loan is refinanced before maturity; see Mulvey and Zenios (1994) on the use of simulation models for capturing correlations of fixed income securities. Payment projections are made based

[2]This practice has been challenged in the article by the Real Estate Center mentioned above, who questions whether rolling over debt over long horizons could be a ticking time bomb for the mortgage markets.

on an underlying interest rate process—we use a variation of the Vasicek model in our work; see Appendix C.7—for all types of loans. The same process is also used to estimate the mortgage security market price for the outstanding debt and to determine the exercise of any options. The result is a distribution of net payments for different types of loans that can then be used to combine loans and obtain an optimized, diversified, portfolio. The simulation results for FRM and ARM given next further highlight the potential diversification effects of optimized portfolios.

C.3.1 Fixed rate mortgages

A FRM pays a fixed annual rate for the duration of the loan—normally 15, 20 or 30 years. In addition to the fixed rate there is also a price associated with a FRM, which is based on the amount paid by the investor to the mortgage bank upon loan origination. In particular, the interest rate and principal payment calculations are not based on the amount paid by the investor, but on the face value of the FRM. For example if the price of a FRM is 96.8 then for every 96.8 EUR that the mortgagor receives he will owe the investor 100 EUR.

As a result, although the interest rate payments on a FRM are fixed the overall payment is not constant due to the fact that the price of a FRM changes with the general level of interest rates change and the Danish mortgagor has a buy back delivery option on the FRM, which means that the mortgage can be prepaid at any time at the prevailing market price. Hence, unless the loan is kept until maturity—an unlikely situation—the borrower does not know with certainty the overall payments. This situation is illustrated using simulations in Figure C.3 (top), for a 30-year 3% FRM, which is prepaid after six years.

It is worth pointing out that, to our knowledge, no mortgage banks outside Denmark offer this buy back delivery option. In all other cases we are aware of, should the mortgagor wish to prepay the mortgage loan then payment is due of the original loan, or any remaining part thereof. Most mortgage banks across the world of course offer a call option, so the mortgagor may prepay the mortgage at a predetermined price, usually par. The buy back delivery (call) option of the Danish mortgages introduces an asymmetry in the payment distribution which is illustrated in Figure C.3 (bottom) for a 30-year 4% FRM with a price close to strike, when it is also prepaid after six years under different interest rate scenarios.

Comparing the payments of the two FRMs we observe that the FRM with the price closest to par has a smaller volatility of payments but a higher

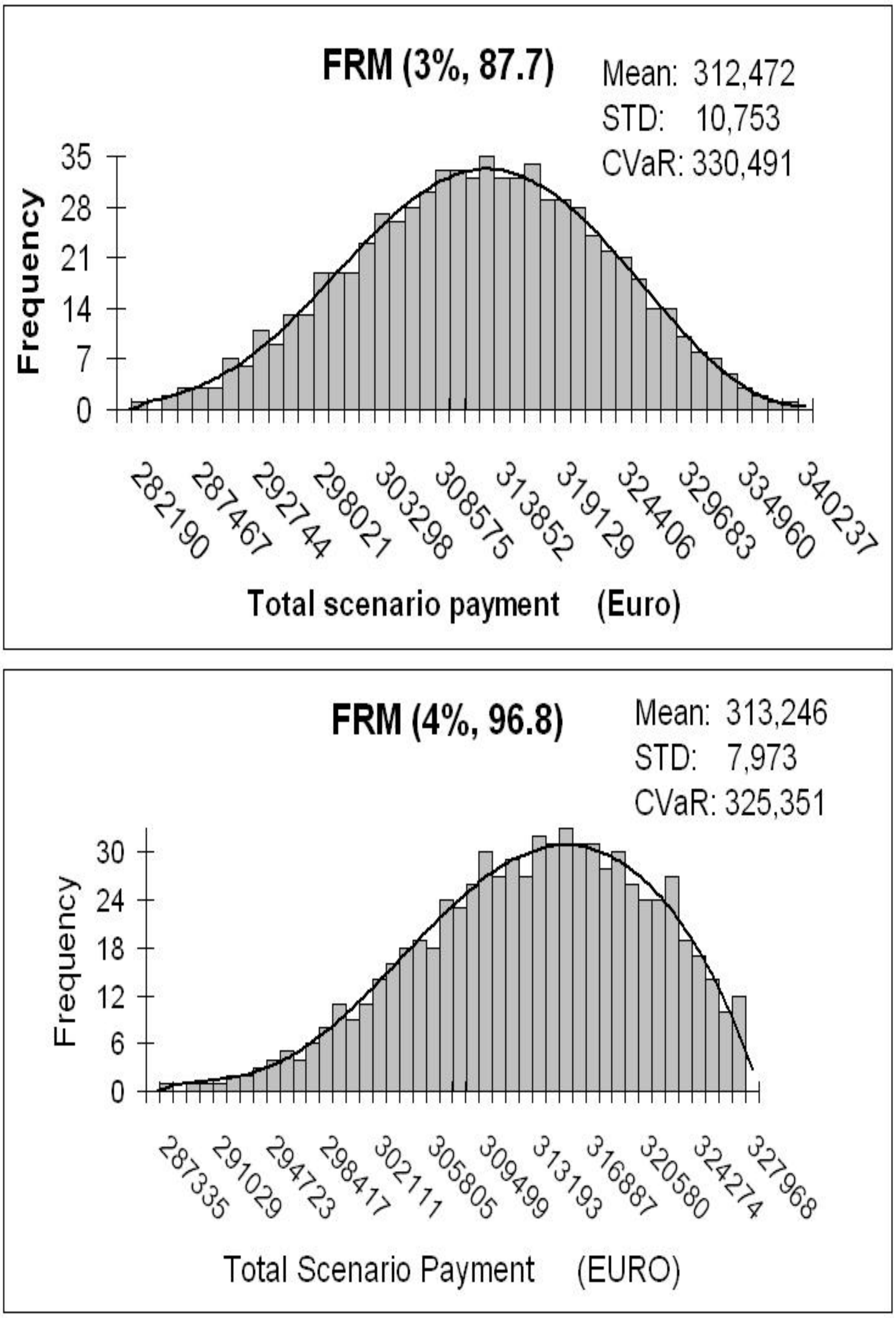


Figure C.3: Distribution of total payments, including interest payments and principal prepayment after six years, of 30-year fixed rate mortgages with different rates: 3% (top) and 4% (bottom).

mean. Increased volatility is the price to pay for the upper bound on the payment. Some mortgagors are willing to pay higher mean payments in order not to worry about very high payments that might occur if the initial price of the FRM is below par and interest rates drop. Others may prefer the low original payment today to get into a new home, in expectation of higher income in the future. As a rule of thumb house buyers are not advised to issue FRM with prices below 95.

C.3.2 Adjustable rate mortgages

Adjustable rate mortgages have both a varying rate and a varying price and the distribution of Figure C.4 illustrates the net payments of an annually adjusted 30-year ARM(1) over a holding period of six years.

For an ARM(1) with annually adjustable rate the price is almost constant and close to par, but as the re-adjustment period increases—in Denmark up to 10 years for ARM(10)—the price may vary considerably as the general level of interest rates changes similarly to a FRM with maturity of 10 years. In contradistinction to FRMs, however, most ARMs have no embedded call options and their price might increase to such extremes which makes it impossible for the mortgagor to prepay the loan, should he decide to quit the loan before the horizon of the fixed rate term in

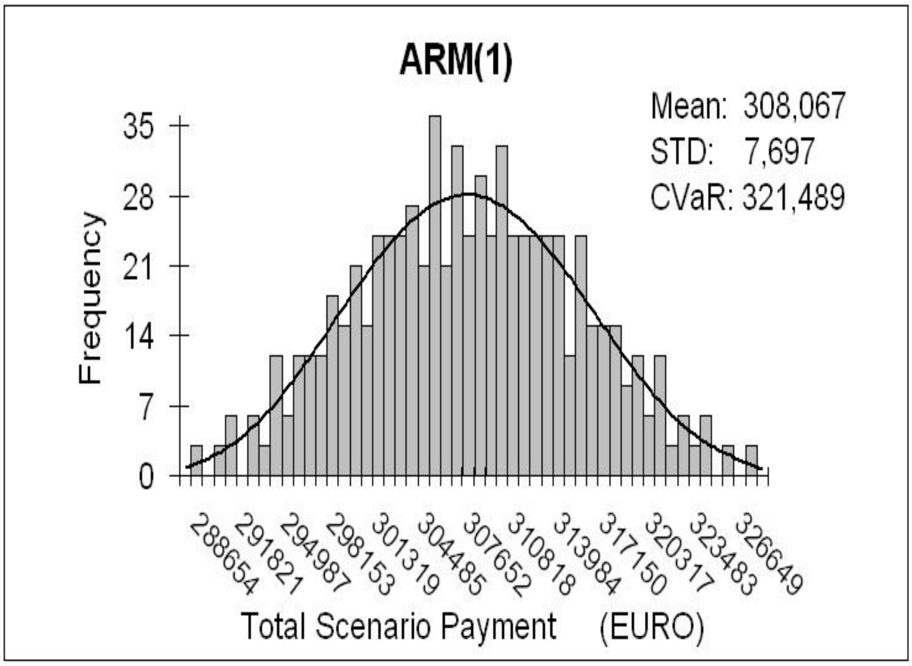


Figure C.4: Distribution of total payments, including interest payments and principal prepayment after six years, of a 30-year adjustable rate mortgage ARM(1) with annual adjustments.

question.

The interaction between rates and prices, and the uncertainty surrounding the timing for selling the house—due to changing family conditions—makes it difficult to choose an ARM for a particular mortgagor. Nielsen and Poulsen (2004) and Rasmussen and Clausen (2006) proposed models for structuring mortgage loans for homeowners. However, these models focus on a single product and do not explicitly introduce a risk measure that bring to the surface the diversification issues. A comparison of the distribution of payments for a 30-year ARM(1) with annual adjustments and a 30-year 4% FRM shown in Figure C.5, together with the negative correlations of FRM and ARM shown earlier, further highlights the fact that a combination of both types of mortgages should reduce both the average payment and the volatility of payments, and in addition impose a limit on the upside potential for high payments in the future.

A model for diversifying mortgage loans is introduced next.

C.4 A diversification model

The optimization model specifies portfolios that trade off the net present value of the total mortgage payment against a risk measure of these pay-

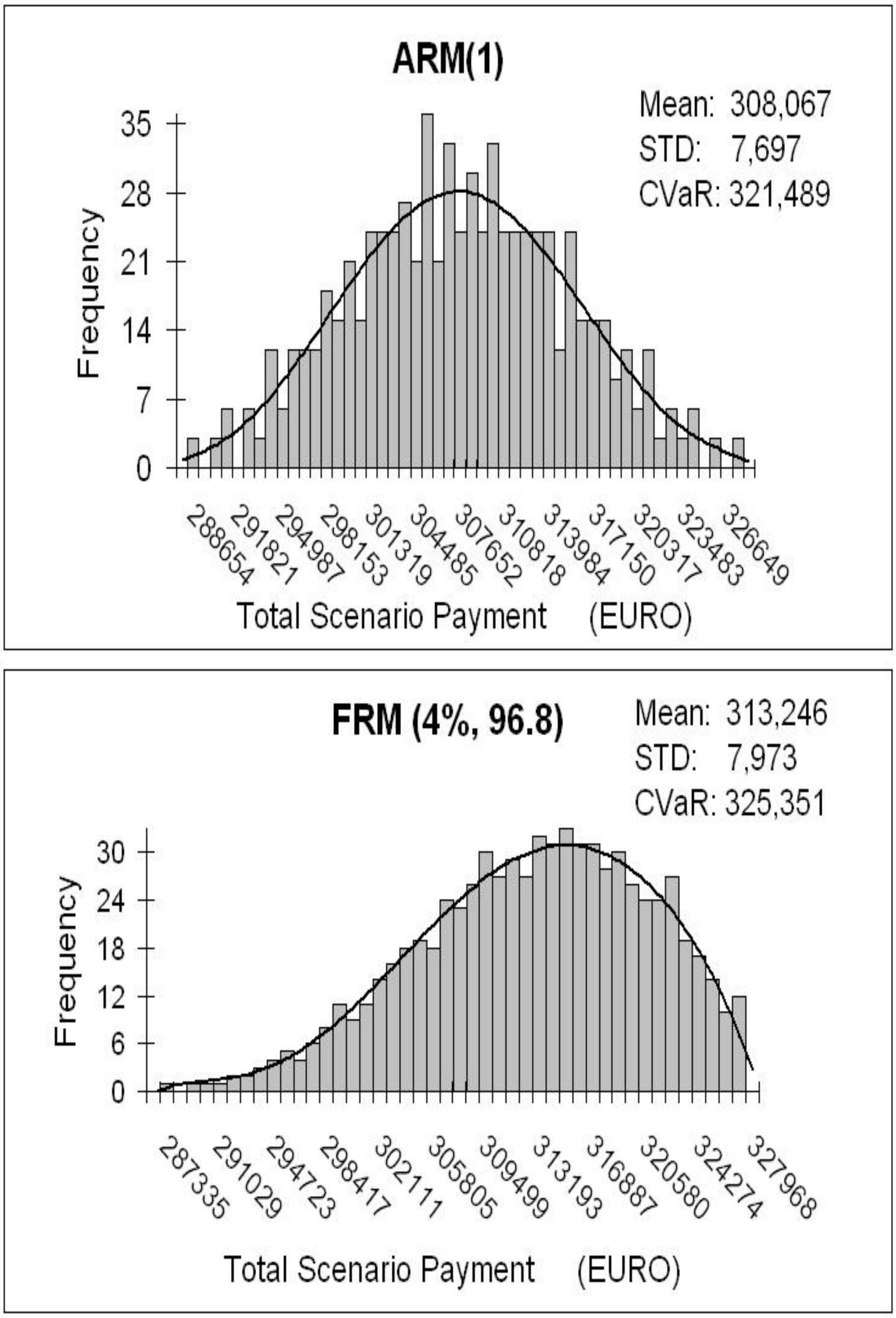


Figure C.5: Comparing the distribution of total payments, including interest payments and principal prepayment after six years, of a 30-year adjustable rate mortgage with annual adjustments and a 30-year fixed rate mortgage.

ments. The risk measure we adopt in this paper is that of Conditional Value-at-Risk (CVaR) that has both the theoretical properties of "coherence" (Artzner et al., 1999) **?** and is also well suited for diversifying portfolios of assets with skewed distributions (Jobst and Zenios, 2001).

We are given a set of scenarios $l \in \Omega$ obtained from the simulation model (see Appendix C.7) and a set of mortgage loans $i \in U$, and the following parameters generated by the simulation model for each scenario:

p^l**,** the probability associated with scenario l,

d^l**,** discount factor under scenario l,

K_i^l**,** the call price of loan i under scenario l,

r_i^l**,** coupon rate for loan i under scenario l.

CF_i^l**,** the net present value of payments from one unit of loan i under scenario l, including interest and principal payments as well as any fees,

PP_i^l**,** the net present value of prepayments from one unit of loan i under scenario l including any retirement of the debt at prevailing market prices.

The following are given input data, relating to features of the problem:

IA, the initial amount to be borrowed in order to finance the house purchase,

P_i, price of loan i at origination time,

c, variable transaction costs (in percentage),

c_f, fixed costs associated with mortgage origination or refinancing.

Finally, we define the model variables:

y_i, units sold of loan i,

ζ, Value-at-Risk (VaR) at the $100\alpha\%$ confidence level,

$\mathrm{CVaR}(y;\alpha)$, conditional Value-at-Risk of a portfolio with loans $y = (y_i)_{i \in U}$ at the $100\alpha\%$ confidence level,

y^l_+, amount of payment under scenario l exceeding the VaR level ζ,

$$Z_i = \begin{cases} 1 & \text{if any amount of loan } i \text{ is originated.} \\ 0 & \text{otherwise.} \end{cases}$$

The optimization model can now be formulated as follows using the linear programming formulation of Rockafellar and Uryasev (2000), where we use λ to denote the degree of risk aversion, ranging from 1 for high risk aversion and 0 for no risk aversion (see also Zenios, 2006):

$$
\begin{aligned}
\text{Minimize} \quad & (1-\lambda)\Big[\sum_{i\in U}\sum_{l\in\Omega} p^l(CF_i^l + PP_i^l)y_i\Big] + \lambda \text{CVaR}(y;\alpha) && && \text{(C.1)}\\
\text{subject to} \quad & \sum_{i\in U} P_i y_i \geq IA + \sum_{i\in U}\big(cy_i + c_f Z_i\big) && && \text{(C.2)}\\
& MZ_i - y_i \geq 0 && \text{forall } i\in U && \text{(C.3)}\\
& y_+^l \geq \Big[\sum_{i\in U}(CF_i^l + PP_i^l)y_i\Big] - \zeta && \text{forall } l\in\Omega && \text{(C.4)}\\
& \text{CVaR}(y;\alpha) = \zeta + \frac{\sum_{l\in\Omega} p^l y_+^l}{1-\alpha} && && \text{(C.5)}\\
& y_i, \zeta, y_+^l \geq 0, \quad Z_i \in \{0,1\} && \text{forall } i\in U, l\in\Omega && \text{(C.6)}
\end{aligned}
$$

The objective function (C.1) trades off the net present value of total payments (including prepayments) against the risk measure as given by CVaR. Constraint (C.2) makes sure that we originate enough loans to buy the house at a cost IA and pay any transaction costs and the fixed mortgage origination costs. Constraint (C.3) sets the binary variable Z_i to 1 indicating that fixed mortgage origination costs need to be incurred, if any amount of loan i is chosen in the portfolio of loans, where M is a large constant to account for the maximum allowable loan.

Constraints (C.4) and (C.5) together define the CVaR of the portfolio at the $100\alpha\%$ confidence level, see Rockafellar and Uryasev (2000) or Zenios (2006). Finally we have the non-negativity constraints (C.6).

We applied this model to build diversified portfolios of mortgages. First, simulations are employed to develop scenarios of cashflow payments and outstanding principal for both FRM and ARM and then the optimization model is run on the set Ω, and for different values of the risk aversion parameter λ. The results are shown in Figure C.6, together with the performance of the individual mortgage loans available to our investor; the benefits from the diversified portfolio become apparent. We observe that the ARM(1) appears as the sole mortgage on the portfolio of only the least risk averse investors, but as risk aversion increases the portfolios diversify into ARM(5) and FRMs as well.

We go a step further, however, and show on the same figure the performance of the loan of a homeowner who follows a *dynamic strategy* of rebalancing his or her single FRM as market conditions change. This is clearly a better strategy than issuing and holding a single mortgage throughout and it is, indeed, the strategy pursued by most homeowners who chose FRMs. But even so, we observe from the results of this figure that the dynamic policy reduces the expected payments, but it does so by assuming higher risks, and it is dominated by the diversified portfolios.

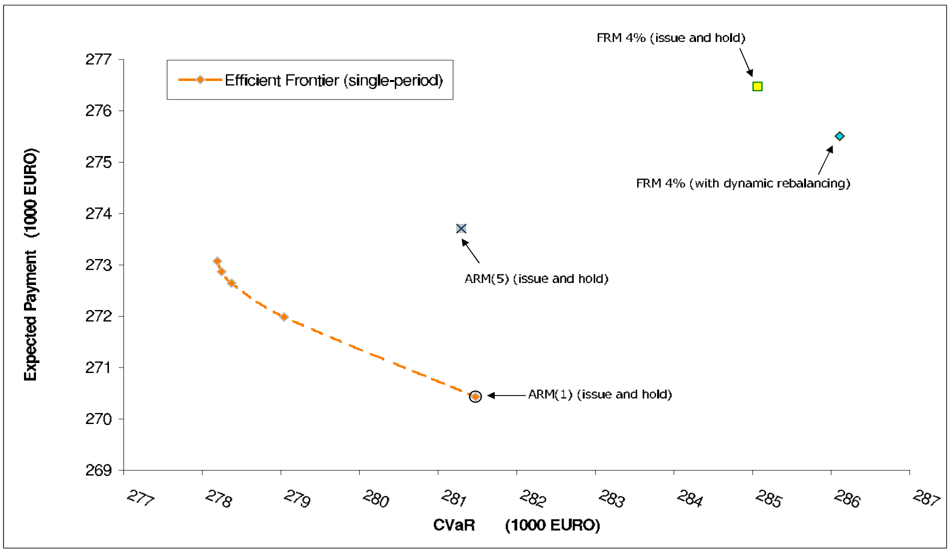


Figure C.6: The efficient frontier of diversified portfolios of mortgage loans is shown together with performance of individual mortgages in the mean/CVaR space, and the performance of dynamic strategies for rebalancing a single mortgage loan.

Here, we may rest our case, having demonstrated the validity of the diversification approach for portfolios of mortgage loans. However, an interesting question has been raised that prompts us to further modelling investigations: If a dynamic strategy of rebalancing a single mortgage loan has some advantages over the issue-and-hold strategy, could it be the case that a dynamic portfolio optimization model would do even better than the model of this section that is defined over a single period, with allowing the possibility of dynamic rebalancing at some future intermediate stages? The answer is affirmative as we see in the next section.

C.5 Taking a long term perspective

The long time horizon of the mortgage decision, and the ability of the mortgage owner to rebalance the loan as market (or family) conditions warrant it, begs for the application of dynamic multi-period portfolio optimization strategies using multi-stage stochastic programming. Such programs have a long history in the optimization literature (see, e.g., Birge and Louveaux (1997) or Censor and Zenios (1997)) and have been gaining prominence in the risk management literature since the eighties (Ziemba (2003), Zenios and Ziemba (2006)). The extension of the model above into a multi-stage setting is developed in Rasmussen and Zenios

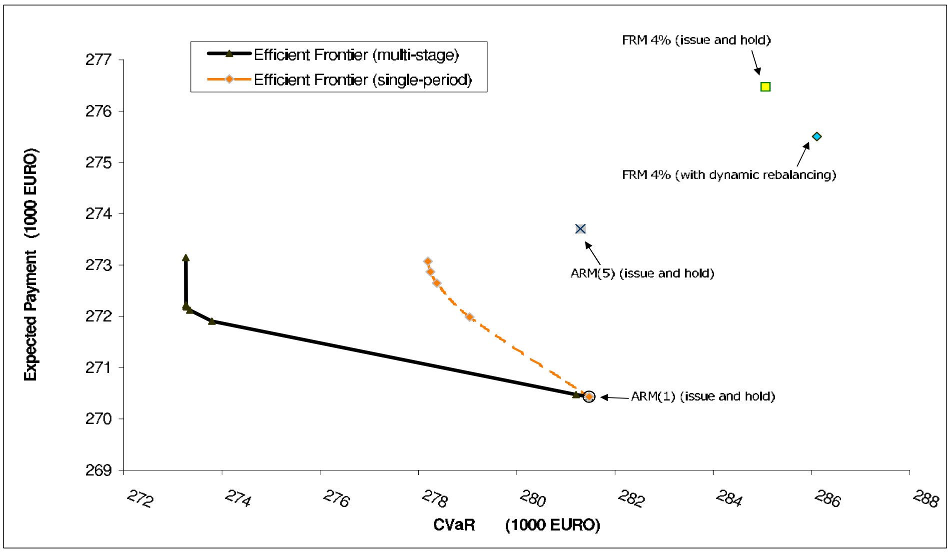
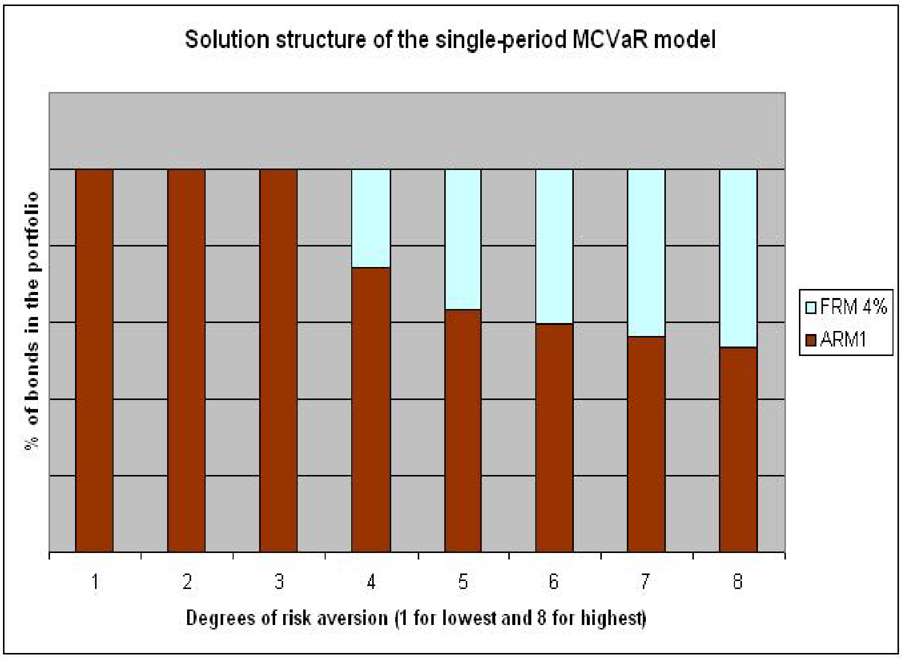


Figure C.7: Significant improvements in the performance of diversified portfolios of mortgage loans are realized with the use of a multi-stage model over the single-period model.

(2006), where a five-period, four-stage model is developed allowing for refinancing the loan at years one, two, three and five, and maturity at year seven.

The application of the multi-stage model for optimizing diversified portfolios leads to significant improvements in performance as witnessed from the results of Figure D.6, leading to the simultaneous reduction of both the expected net payments and the risk of the payments as measured by CVaR

The structure of the diversified portfolios obtained with both the single-period and the multi-stage model are shown in Figure C.8. We observe that for investors with low risk aversion, ARMs is the predominant class of mortgage loans no matter which optimization model is used. However, as risk aversion increases we observe a gradual shift towards the class of FRMs and while this trend is common with both models there is significantly more reliance on FRMs for the investor using the multi-stage model. This is so, since with the multi-stage model we can rebalance the portfolio of FRMs at the appropriate time for each scenario. The optimal strategy recommended by the multi-stage model is essentially equivalent to synthesizing an ARM with optimal timing for rate re-adjustment at intermediate stages, at one, two, three or five years, depending on the scenario of interest rates. This finding points out that it is worth designing more complex ARM structures that will lock in a rate for pre-specified periods that may depend on the prevailing rates. To do so, however, a legal construct is required so that the synthesized ARM will be presented as a single loan.



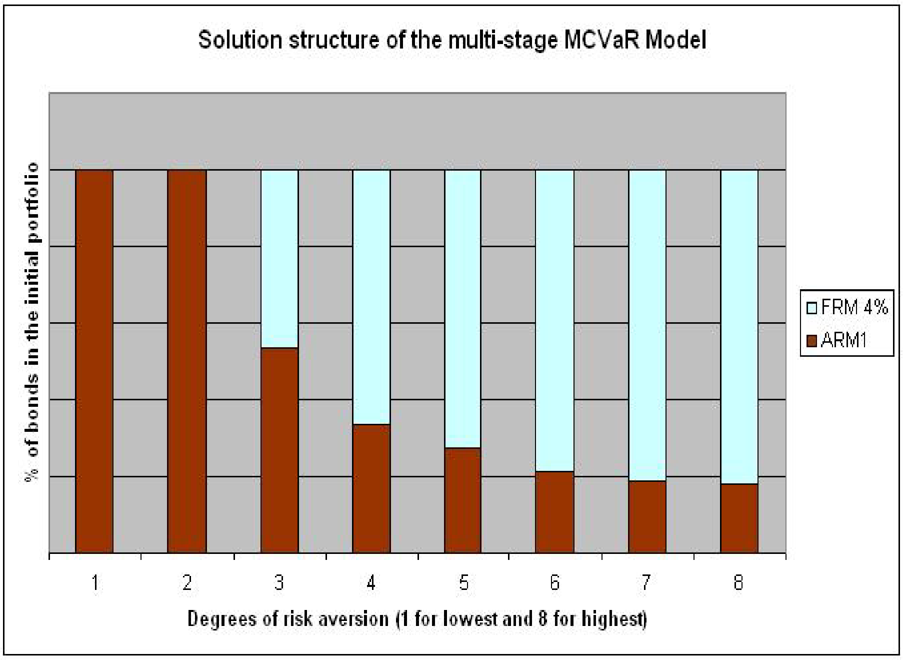


Figure C.8: The composition of the diversified portfolio in the aggregate categories of fixed and adjustable mortgages when using both the single-period and the multi-stage models.

C.6 Two interesting observations

Finally we use the model to offer answers to two questions that are often raised in the context of mortgage management.

C.6.1 The effect of mortgage origination costs

First, we consider the effect of fixed transaction costs for loan origination and the adverse effect this has in rebalancing mortgage loans. Indeed, the arguments against portfolios of loans of mortgages is based on the assumption that the origination costs will be prohibitively high. Mortgage origination costs, in Denmark, include a fee of 1.5% on the required loan paid to the bank upfront for all and any mortgage loans obtained from the bank, a 0.5% penalty for refinancing a loan with a different mortgage and a 300 EUR administration fee for originating every new mortgage. The largest of these fees (1.5% on the loan amount) is a sunk cost, and does not affect our decision to refinance a mortgage, assuming we stay with the same bank. The rebalancing proportional cost of 0.5% is akin to the transaction costs for any asset management problem and it has been included in all our previous runs. What is left unexamined is the cost for originating new mortgages over and above the original loan. We run the portfolio optimization model with and without the 300 EUR

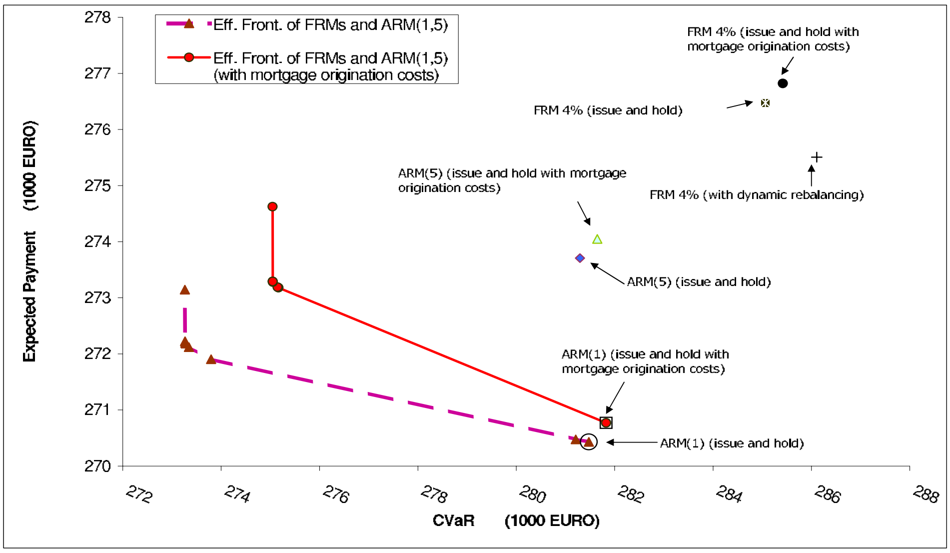


Figure C.9: Efficient frontiers of diversified portfolios of loans with and without the mortgage origination costs.

mortgage origination costs. The results are shown in Figure C.9. While we note that the performance of the diversified portfolio deteriorates when origination costs are included, it is still the case that they do much better in mean/CVaR space than any of the individual mortgages, and they also outperform the dynamic strategy of rebalancing a single FRM. Hence, loan diversification pays even when accounting for the higher costs of originating multiple loans.

C.6.2 Designing new mortgage products

Armed with the portfolio diversification models we can analyze the effect of new mortgage products on the homeowners' portfolios. This has clear implications for the introduction of new products in the market.

Naturally, as we add more products in the market the diversified portfolios will improve in performance. Or, at least, they will not perform worse as the optimization model will simply ignore any new products that do not contribute to the diversification. Indeed, Figure C.10 clearly shows the improvements in efficient frontier as new instruments are added in the universe of mortgage backed securities, although the improvements are diminishing when adding more than three new securities.

What happens, however, if a bank wishes to issue only one type of ARM(t) with some to-be-determined period t for rate readjustment, to complement a diversified portfolio of FRM and ARM(1)? We run the model by introducing—one at a time—ARM(2), ARM(5) and ARM(7). The results are shown in Figure D.9, where we observe significant improvements in the performance of the diversified portfolios when we add an ARM(2) or ARM(5) to a portfolio with FRM and ARM(1), but things would deteriorate for almost all levels of risk aversion if an ARM(7) were introduced instead. This analysis provides guidance as to the best mortgage prod-

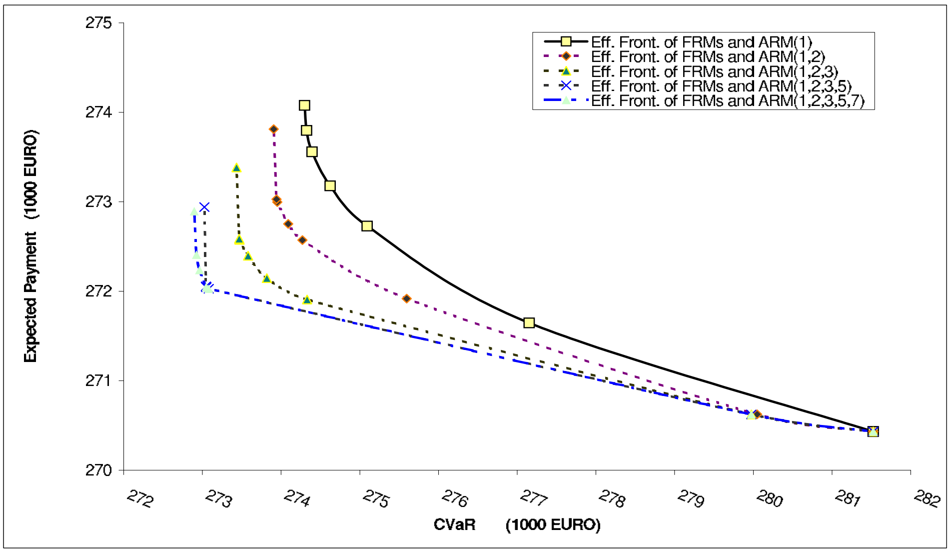


Figure C.10: Expanding the universe of mortgage loans available to home-owners improves the performance of diversified portfolios, although the improvements come at diminishing rate when adding more than three securities.

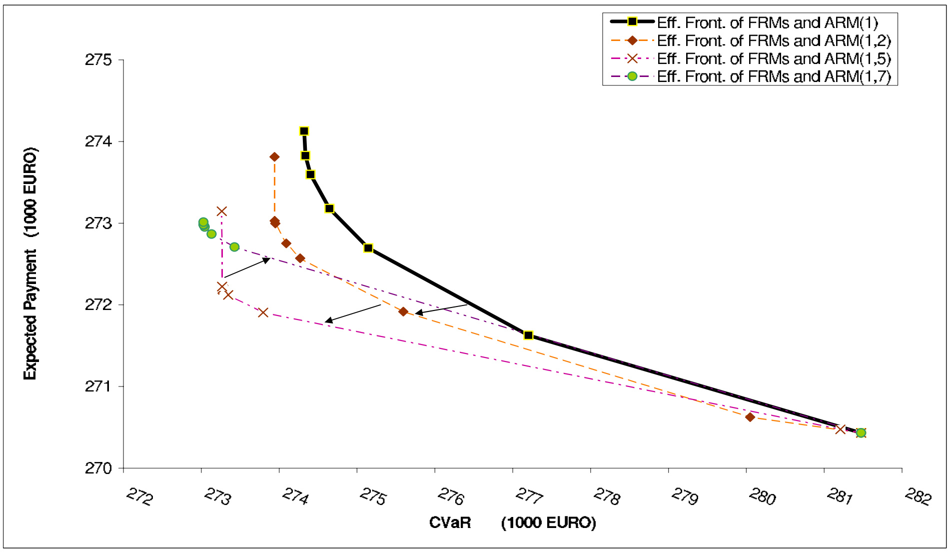


Figure C.11: Designing new mortgage loans: Adding new products to the existing portfolio has the optimum positive impact for an ARM(2) and ARM(5) but the diversification effects are diminished for ARM(7).

ucts to be introduced, to maximize the diversification benefits for the homeowners.

C.7 Conclusions

We have shown that well diversified portfolios of mortgage loans can better serve the needs of homeowners, in both financing the purchase of a home and staying within acceptable risk profiles. This conclusion is robust in the sense that it holds true even in the presence of transaction costs and for short and long horizons alike. From the models developed we have seen that the multi-stage stochastic programming approach is particularly well suited for this type of problems. However, even a single-period model such as the mean/CVaR optimization that has been gaining widespread acceptance in risk management serves well the needs of this problem. The models also shed some insights on the introduction of new mortgages in the market.

Finally a word on potential extensions that are possible with the modelling setup we introduced. In this paper we only considered the interest rate risk. However, the scenario tree can be extended to represent house price and income dynamics in order to capture the wealth risk of the home owner as well.

Appendix 1: The simulation model

We use a variation of the Vasicek interest rate model (Jensen and Poulsen, 2002) as the underlying stochastic process to generate estimates of future short rates

$$dr(t) = \kappa(\theta_P - r(t))dt + \sigma dW_p(t),$$

where $r(t)$ is the short rate, W is a Brownian motion and κ, θ and σ are model parameters controlling the height of the interest rate jumps, the long run mean level of interest rates and the volatility of the interest rates. The model is given under real–world probability measure P. This can be shifted to the risk free measure Q using the transformation

$$\theta_Q = \theta_P + \pi, \qquad \pi \in \mathbb{R}$$

where π is the risk premium, so the Vasicek model under the risk free probability measure Q becomes

$$dr(t) = \kappa(\theta_Q - r(t))dt + \sigma dW_Q(t).$$

The expected short rates are then found from

$$E_Q[r(t)] = r_0 \cdot \exp(-\kappa t) + \theta_Q(1 - \exp(-\kappa t)).$$

We discretize this short rate process and estimate future rates and prices for mortgage backed securities using the pricing method of Nielsen and Poulsen (2004).

Appendix D

Optimal Mortgage Loan Diversification

Working paper availabe at http://www2.imm.dtu.dk/ kmr/.

Abstract

Homebuyers in several countries may finance the purchase of their properties using different variants of either adjustable–rate mortgages (ARMs) or fixed–rate mortgages (FRMs). The variety and complexity of these loan products poses a risk management task for mortgage bank advisors to recommend the right mortgage loan strategy for the individual mortgagor; almost all mortgage banks advise their customers to take a single loan product. This argument is often justified by the fact that trade frictions make it unattractive to hold a portfolio of loans as a private home owner. Even with transaction costs, however, we show in this paper that most mortgagors with some degree of risk aversion benefit from holding a mortgage portfolio. To do so we develop a multistage Mean–Conditional Value at Risk (MCVaR) model to consider the risk of the mortgage payment frequency function explicitly using a coherent risk measure. In addition to the diversification benefits we also show that the multistage model produces superior results as compared to single period models and that the solutions are robust with regards to changes in uncertainty parameters

in particular for risk averse mortgagors. Finally, we show how the model can be used to calculate fair premia for adjustable rate mortgages with interest rate guarantees (caps) which are becoming increasingly popular as a hybrid product between the existing ARM and FRM mortgages.

Keywords: Mortgage loans products, CVaR modeling, stochastic programming.

D.1 Introduction

Most homebuyers across the world finance the purchase of their houses by taking a mortgage loan. Mortgage banks in several countries offer several mortgage products with different payment schemes and risk profiles. This complicates the job of the mortgage advisor who has to account for the credit worthiness of the mortgagor and its effect on the amount and type of mortgage loan that should be granted. This paper deals with modeling the mortgage choice problem to account for some of the most significant uncertainties of this problem.

Even though the model in this paper can be applied to any mortgage market (perhaps with some modifications) the cases considered are based on the Danish mortgage market. The Danish mortgage banks are highly specialized institutions whose main focus is, on the one hand, to collect investments from investors in mortgage–backed securities, and on the other hand, to pool the investments together and issue mortgage loans to home buyers. The Danish mortgage–backed security legislation requires equal payments on the investor and the mortgagor side – "the balance principle". This law in effect limits the financial risks assumed by the mortgage banks to credit–default risk on the mortgagor side and mortgage–bond–liquidity risk on the investor side.

A feature unique to the Danish mortgage market is that mortgagors, via the mortgage banks, are virtually trading mortgage bonds and may exercise all the embedded options in the underlying bonds. For a fixed–rate mortgage (FRM) this includes a call option of Bermudan type with a strike price of 100 and a buy–back delivery option which gives the mortgagor the right to redeem the mortgage at the actual market price of the underlying bond. These two features provide security on the mortgagor side. In the case of falling interest rates the mortgagor can exercise the call option and refinance the existing FRM, with a new FRM with lower coupon payments. Furthermore for rising interest rates the mortgagor can reduce the outstanding debt by prepaying the mortgage at a market price lower than the original issuance price. This has an impact on FRMs with large durations. Small movements in interest rates result in big movements in prices of FRMs and this has an important impact on the debt–free value of the property. The buy–back may be refinanced either by selling the house or by taking a new loan with an underlying FRM with higher coupon payments, or an adjustable–rate mortgage (ARM). The exercise of the buy–back delivery option is useful in case the mortgagor needs to move or in case he or she believes the interest rates will fall again in the near future. These features of the FRM, and the security they offer, however come at a price; the effective interest rate payments are often considerably higher than those of adjustable–rate mortgages.

Since the mid 1990's the Danish mortgage market has been growing rapidly and a number of new mortgage products have been introduced in addition to FRMs. The two most popular products have been ARM with varying adjustment intervals, and the *capped rate mortgages* (CRM) where the interest rate cannot grow higher than a predetermined level (cap). All these loans may be issued with or without principal payments (interest payments only) for a period of up to 10 years. The interest–only period is renewable after the initial 10 years.

ARMs are financed by issuing underlying bullet bonds with maturities of 1 to 10 years. For example an ARM1 is a mortgage with annual interest rate adjustments, whereas the rate of an ARM2 is readjusted every other year, etc. Since there are no embedded options available with an ARM, the average payments of an ARM are lower than an FRM but payment volatility is considerably greater for very long horizons. CRMs are financed by a variable rate security with an embedded cap. The interest rate follows a 6–month CIBOR (Copenhagen Inter Bank Offered Rate) plus a premium. Should the CIBOR plus premium increase to a level higher than the interest rate guarantee level, the rate will be fixed at the guarantee level. Should the CIBOR plus premium fall below the guarantee level again, the CRM's rate will follow accordingly.

The extra features of the CRM come at a price, i.e. higher interest rate payments than alternative floating rate products such as an ARM1. For more details on the workings of the Danish mortgage market see Svenstrup & Willemann (2005). For a review of recent innovations in the mortgage–backed security market see Piskorski & Tchistyi (2006).

It must be evident by now that it is a non–trivial task to advise a mortgagor on choices of mortgage loan. Indeed wrong advice together with unfavorable market behavior might result in financial ruin for a large pool of mortgagors and this in turn will have unprecedented macro–economic effects such as a devaluation of the housing market. The research interest in this problem is well justified.

Nielsen & Poulsen (2004) design a trinomial scenario tree using an underlying two–factor model of interest rates for pricing existing and synthetic mortgage bonds. Furthermore they introduce a stochastic programming model to find the optimal initial loan strategy and to advise the mortgagor on optimal readjustments along the way. Their optimization model, however, does not include a risk measure and the effects of fixed–mortgage origination costs were ignored. **?** further develop this model to include fixed–mortgage origination costs and budget constraints. Their conclusion is that a mortgagor with budget constrains benefits from choosing an initial portfolio of an ARM and a FRM, given that there are only these two types of products to choose from. The budget constraints provide indirect means for risk control, but no explicit risk measure is considered in this paper either.

An explicit risk measure for this class of problems was introduced by **?** who develop a single period stochastic programming model to trade off the present value of average mortgage payments against the Conditional Value at Risk (CVaR[1]) value. They use a Mean/CVaR efficient frontier to show that diversified mortgage loan strategies outperform single mortgage loan strategies; Figure D.1 highlights their findings which speak strongly

[1]For a review of CVaR as a coherent risk measure see **?**, Rockafellar & Uryasev (2000) and Zenios (2007).

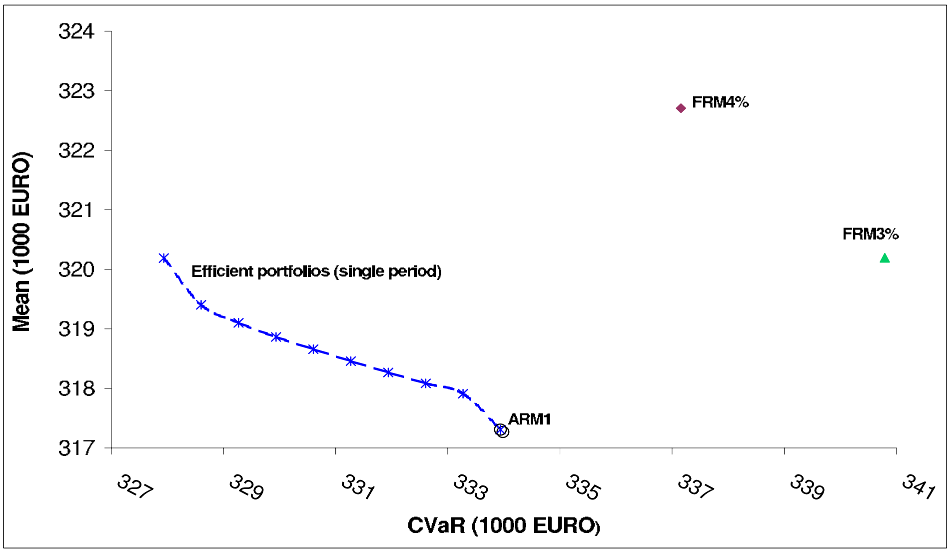


Figure D.1: For a mortgagor with a seven year horizon a mix of variable and fixed–rate mortgages provide low payments and low risk, here measured by the 10% CVaR value.

in favor of diversification.

In this paper we develop a multistage version of our earlier model and show that improved results can be obtained by introducing dynamic trading into the model. It will be seen that the budget–constrained model of **?** is subsumed by the bilinear Mean/CVaR minimizing model. Furthermore, we consider CRMs as part of our universe of loans and suggest a simple approach to determine whether the cap option comes at a fair price for a given mortgagor with a certain risk appetite.

D.2 Single mortgage strategies

Today the advisors in the Danish mortgage market recommend homebuyers to take single mortgage loans only. Until 2005 this included only ARMs or FRMs. Since 2005 CRMs have been among the favorites of the Danish advisors as well. When comparing the cashflows of these loans only the first year payments are quantified. This leaves out important information on the uncertain cashflows from year 1 on. In this section we suggest a scenario analysis approach which gives a quantitative comparison of different loans across a number of representative scenarios.

We generate an event tree with a seven–year horizon, using the one factor Vasicek model; see Appendix D.6 for details of the interest rate model and Nielsen & Poulsen (2004) for its discrete implementation. Price calculations are performed using the RIO application which is a specialized commercial system for pricing Danish mortgage–backed securities, (see www.scanrate.dk).

D.2.1 FRMs and ARMs

Even though the interest rate payments on an FRM are fixed, the overall payment distribution is not. This is due to the fact that the price of an FRM changes as the general level of interest rates changes and the (Danish) mortgagor has a buy–back delivery option, meaning that the mortgagor can prepay the mortgage at any time at the market price. So, unless the mortgagor keeps the FRM until maturity, the overall payments remain uncertain. Figure D.2 shows the density functions for the total payments of two 30–year FRMs, given that the mortgages are repaid after seven years.

It is noteworthy that at present no mortgage banks outside Denmark offer this buy–back delivery option to an FRM mortgagor. Should the FRM mortgagor repay the mortgage loan early, it occurs at par. Most mortgage banks across the world, however, offer a call option, so the mortgagor may repay the mortgage early at most at a predetermined price (usually par). The call option introduces an asymmetry in the payment density functions of FRMs with prices close to par as seen in Figure D.2. The density function of the 4% FRM has a longer left tail than the right tail due to the call option at par.

Comparing the payments of two FRMs with different rates and prices it is easily seen that the FRM 4% has a smaller volatility but a higher average payment that compensates for the upper bound on the payments due to the embedded call option. Most mortgagors are willing to make higher payments on average in order to avoid the very high payments that might occur if the initial price of the FRM is significantly below par and interest rates fall.

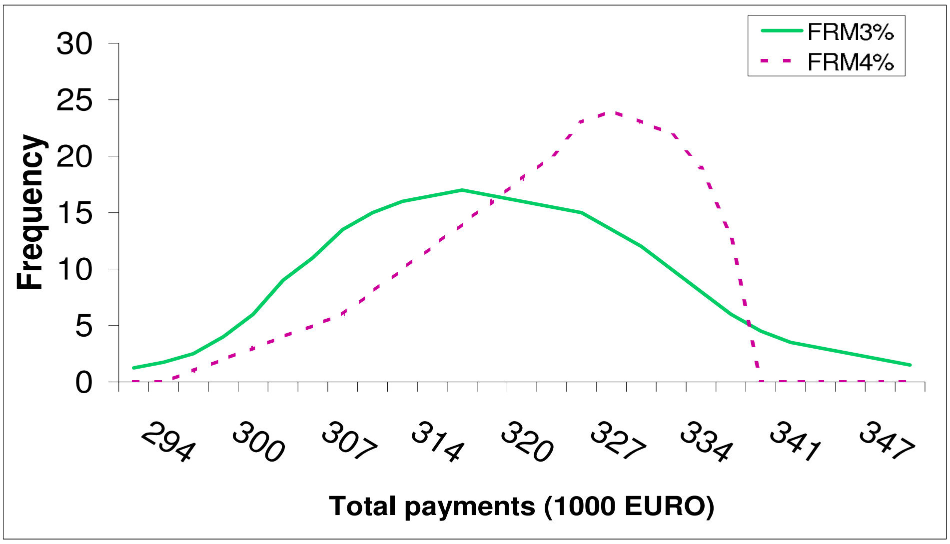
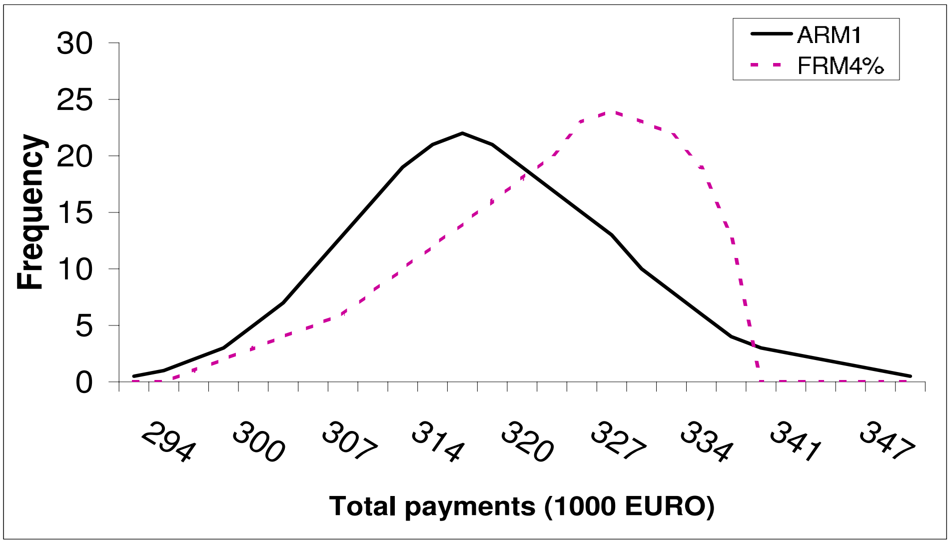


Figure D.2: The density of total payments with early repayment at year 7, of two 30–year FRMs with different coupon rates.

Adjustable–rate mortgages (ARMs) have both a varying rate and a varying price, resulting in uncertain payments. Figure D.3 (top) shows that ARM1 has not only a lower average payment than the FRM 4% but it also offers a lower risk (shorter right tail). Comparing the payments of ARM1 and FRM 4% for each scenario, however, you will notice a very strong negative correlation, see Figure D.3 (bottom), and this has implication if the mortgage must be prepaid.



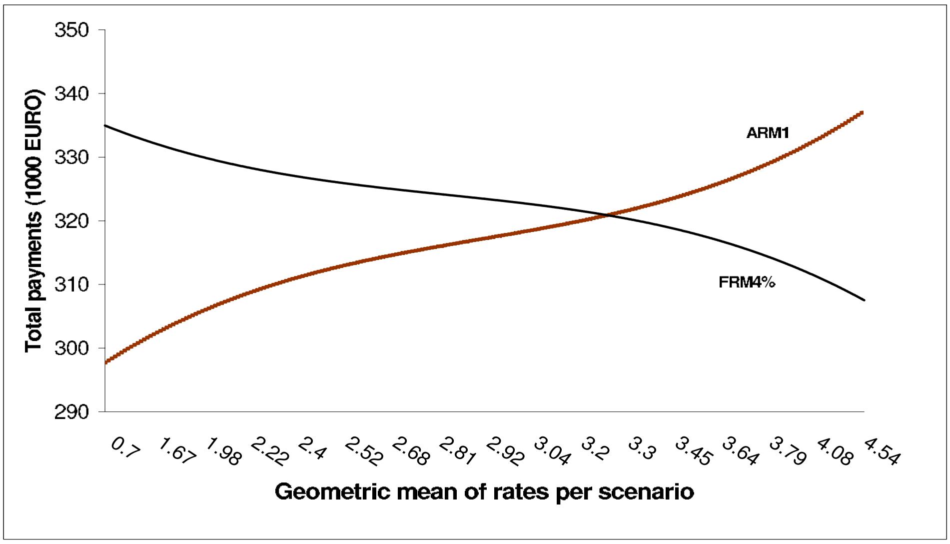


Figure D.3: Comparison of payment densities (top) and scenario payments (bottom) for an ARM1 and an FRM4%. In the bottom figure the scenarios are ranked according to the geometric average of the short rates on the scenario paths.

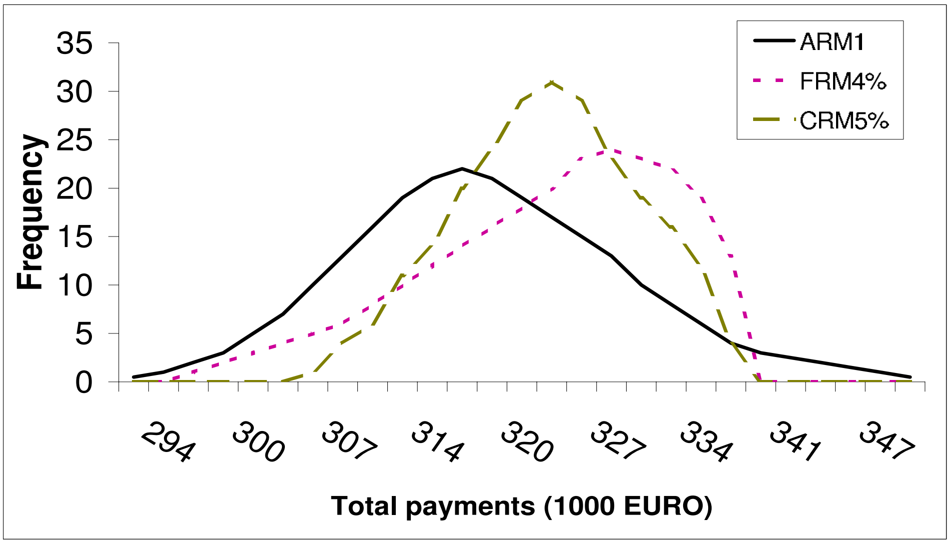
The high payment scenarios for an FRM occur when interest rates are decreasing, so that the mortgagor is both paying high interest rates and a high repayment. These same low interest rate scenarios are obviously low payment scenarios for an ARM. The scenarios in which the interest rates increase, however, are low payment scenarios for FRMs, due to low repayments, whereas they are high payment scenarios for ARMs due to upward adjustment of interest rates. There is, in other words, a negative correlation between the payment of FRMs and ARMs making it potentially attractive to hold a diversified portfolio of mortgages. The interesting question now raised is how big the amount of the loan should be before the diversification can pay off, considering transaction and mortgage–origination costs, and what is a good mix for a mortgagor with a certain risk attitude or limited budget. We will answer these questions in the rest of this paper.

D.2.2 CRMs

In 2005 one of the leading Danish mortgage banks released a new mortgage product under the commercial name "garantilån", where the loan starts as a floating rate loan with an agreed cap level (the guarantee level). If interest rates increase so that the floating rate reaches the guarantee level, then the loan is transferred into a fixed–rate loan with the guarantee rate as the coupon for the rest of the loan's lifetime. The mortgagor pays a premium on top of the underlying floating level for the optionality embedded in this mortgage construction. Other Danish banks responded by introducing similar products and in particular a competing bank introduced a product where an additional feature was built into the loan so that if the interest rates fall below cap again then the mortgagor's coupon payments will decrease accordingly. This type of loan has the commercial name "RenteMax" and its popularity inspired other mortgage banks to provide similar constructions. The main feature of these products is that they offer the best of the two worlds to customers; they are a hybrid of an ARM and FRM. But the extra optionality comes at a cost and this cost might be too high for a mortgagor with a short horizon. In Figure D.4 (top) we compare payment densities for an ARM1, an FRM4% and CRM5%.[2]

The payment density of the CRM5% is shifted to the right as compared to that of an ARM1 with the exception of the right tail. This indicates that the mortgagor pays an extra premium without getting anything in exchange for most scenarios but that for extreme scenarios the cap is exercised and very large payments can be avoided. In Figure D.4 (bottom)

[2]CRM5% is a Capped Rate Mortgage with a 5% cap. In this paper we only consider CRMs of type “RenteMax”.



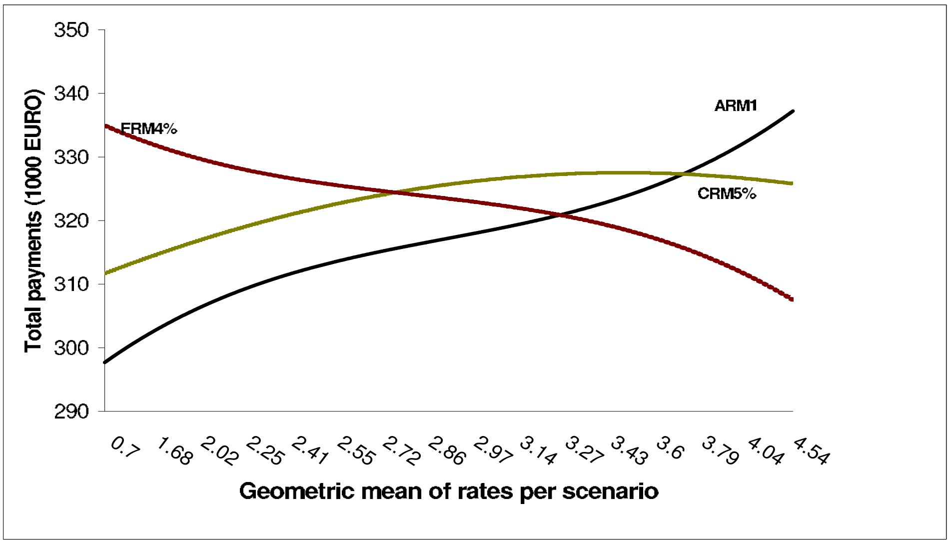


Figure D.4: Comparison of payment densities (top) and scenario payments (bottom) for an ARM1, FRM4% and a CRM5%.

it can further be seen that CRMs are in part negatively correlated with ARMs and in part with FRMs. Hence there is a potential that a combination of the three mortgage types may offer more diversification than a combination of two.

Our preliminary studies, however, indicate that the cap option of the CRM5% as released in February 2005 is not very beneficial for a mortgagor with a horizon of up to 7 years, i.e. if the mortgagor is willing to assume that interest rates will stay within the range estimated by the Vasicek model. Even for a mortgagor with a longer horizon (20 to 30 years), it is a good idea to consider the alternative of making a tailored replication of a CRM product using plain ARMs and FRMs, where the bond series are more liquid and therefore more fairly priced than the CRM.

D.3 The mortgage choice model

In this section we will first introduce the event tree notation and then develop the mortgage choice model. See Zenios for event trees, and Nielsen & Poulsen for event tree scenarios for mortgage products.

D.3.1 Presentation of mortgage rate and price scenarios

To present the mortgage rates and price scenarios we use the notion of an event tree. An event tree is a directed graph $\mathcal{G} = (\Sigma, \mathcal{E})$ where nodes Σ denote time and state, and links $\mathcal{E}$ indicate possible transitions between states as time evolves. At each time t we have one or more states, $s \in \Sigma_t$, representing the underlying stochastic variables. There is a unique path of states, $s \in \mathcal{E}_t^l$, from the root to any one of the leaf states, where $l \in \Sigma_\tau$ denotes a scenario.

An example of such an event tree is seen in Figure D.5. Any node of the tree is populated with a number of loans, each with a set of specific data (LoanID:LoanName–Rate/Price) connected to them.

Note that throughout this paper we operate with two time horizons, namely the mortgage maturity and the mortgage early repayment horizons. Set $\mathcal{T}$ includes all the time periods up to the maturity of the loans $\mathcal{T} = \{0, \dots, \tau, \dots, T\}$.

T is, in other words, the length of the loans considered in the model and it is needed to determine the cashflows of the loans in question. Most loans are, however, repaid early at sometime, $\tau \leq T$ which is why we only need to estimate a scenario tree of length τ.

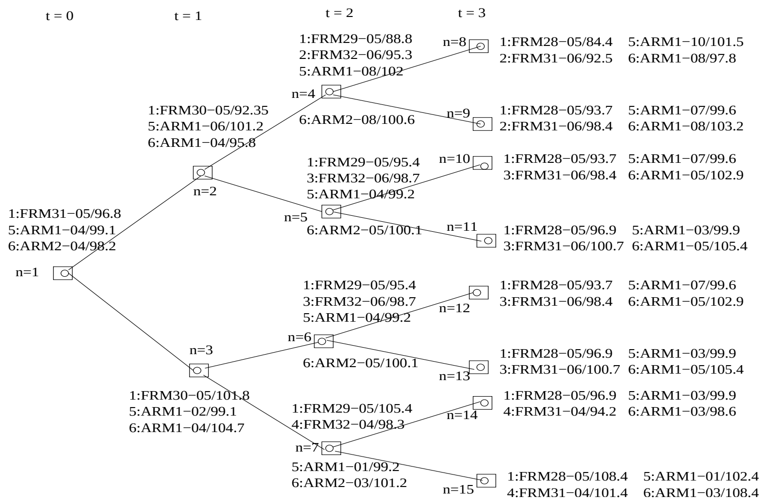


Figure D.5: A binomial event tree, representing the uncertainty on bond prices and coupon rates.

In the event tree, every state $s \in \Sigma_t$, for $1 \leq t \leq \tau$, has a unique parent denoted by $s^- \in \Sigma_{t-1}$, and every state $s \in \Sigma_t$ for $0 \leq t \leq \tau - 1$ has a non–empty set $s^+ \in \Sigma_{t+1}$ of child states. The probability distribution P is modeled by attaching weights $p_t^s > 0$ to each leaf node Σ_τ so that $\sum_{s \in \Sigma_\tau} p_t^s = 1$. For each non–terminal node one has, recursively,

$$p_t^s = \sum_{s^+ \in \Sigma_{t+1}} p_{t+1}^{s^+}, \qquad \text{for all } s \in \Sigma_t, t = \tau - 1, \cdots, 0,$$

and so each node receives a probability mass equal to the combined mass of the paths passing through it.

At every node of this tree we need estimates of interest rates and prices associated with any mortgage loan in the considered market. In order to obtain these estimates we need a stochastic process to represent the uncertainties on the dynamics of interest rates and we need pricing methods to determine mortgage loan prices consistently with the estimated term structures of interest rates.

We use a one–factor Vasicek model as the underlying stochastic process for the interest rates (see Appendix D.6). The model is discretized in a trinomial fashion as described in Nielsen & Poulsen (2004). Mortgage–backed securities are then priced using pricing system RIO; See www.scanrate.dk.

D.3.2 A dynamic stochastic mortgage choice model

The model in this section finds an efficient portfolio of loans that trades off the expected net present value (NPV) of total payments against the Conditional Value at Risk (CVaR) of the payments.

Given an event tree with τ stages and its corresponding mortgage loan rate and price information on a set of loans $i \in I$ we define the following parameters:

p_t^s, probability at state s, time t,

d_t^s, discount factor at state s, time t,

P_0, current price of loan raised initially by the mortgagor,

r_{ti}^s, coupon rate for loan i at state s, time t,

P_{ti}^s, price of loan i at state s, time t,

K_{ti}^s, call price of loan i at state s, time t. We have $K_{ti}^s = \min\{1, P_{ti}^s\}$ for callable loans and $K_{ti}^s = P_{ti}^s$ for non–callable loans,

γ, tax reduction rate from interest rate and administration fee payments,

c_a, administration costs (in percent),

c, variable transaction costs (in percent),

c_f, fixed costs associated with mortgage origination and re–balancing,

λ, degree of risk aversion; 1 for very high, and 0 for no risk aversion,

α, confidence level for the Value at Risk (VaR),

M, a big constant.

Next we define the variables used in our model:

z_{ti}^s, outstanding debt of loan i at state s, time t,

y_{ti}^s, units sold (originated) of loan i at state s, time t,

x_{ti}^s, units bought back of loan i at state s, time t,

$$Z_{ti}^s = \begin{cases} 1, & \text{if loan } i \text{ is originated at state } s, \text{ time } t, \\ 0 & \text{otherwise,} \end{cases}$$

A_{ti}^s, principal payment of loan i at state s, time t,

CF_t^s, total net payment at state s, time t,

ζ, Value–at–Risk (VaR) at the $100\alpha\%$ confidence level,

$\mathrm{CVaR}(y;\alpha)$, Conditional Value–at–Risk of a portfolio with loans $y = (y_i)_{i\in U}$ at the $100\alpha\%$ confidence level,

y_+^l, amount of payment under scenario l exceeding the VaR level ζ.

We are now ready to formulate the multistage stochastic model for the mortgage choice problem. The objective is to trade off the total expected present value of payments (repayment included) against the Conditional Value–at–Risk (CVaR) of the payments as weighted by λ:

$$\min \quad (1-\lambda)\Big[\sum_{t=1}^{\tau}\sum_{s\in\Sigma_t} p_t^s d_t^s CF_t^s + \sum_{s\in\Sigma_\tau} p_\tau^s d_\tau^s PP_\tau^s\Big] + \lambda \mathrm{CVaR}(y;\alpha) \tag{D.1}$$

We also need to make sure that we sell enough bonds to raise an initial amount, P_0, to buy the house and pay the mortgage–origination costs as follows:

$$\sum_{i\in U} P_{0i}^0 y_{0i}^0 \geq P_0 + \sum_{i\in U}\Big(c y_{0i}^0 + c_f Z_{0i}^0\Big), \tag{D.2}$$

In eqn. (D.3) we initialize the outstanding debt:

$$z_{0i}^0 = y_{0i}^0, \qquad \text{for all } i \in U. \tag{D.3}$$

Eqn. (D.4) is the balance equation, where the outstanding debt at any child node for any bond equals the outstanding debt at the parent node minus principal payment and possible repayment (buying back), plus possible origination of new bonds to establish a new loan.

$$z_{ti}^s = z_{t-1,i}^{s^-} - A_{ti}^s - x_{ti}^s + y_{ti}^s, \qquad \text{for all } i \in U, s \in \Sigma_t, t = 1,\ldots,\tau. \tag{D.4}$$

Eqn. (D.5) is a cashflow equation which guarantees that the money used to repay existing mortgages (in case of re–adjustments), plus the transaction fees for sale and purchase of bonds, and fixed costs for establishing new mortgage loans come from the sale of new bonds:

$$\sum_{i\in U}(P^s_{ti}y^s_{ti}) = \sum_{i\in U}\left(K^s_{ti}x^s_{ti} + c(y^s_{ti}+x^s_{ti}) + c_f Z^s_{ti}\right), \quad \text{for all } s\in\Sigma_t, t=1,\dots,\tau. \tag{D.5}$$

The principal payment is defined in eqn. (D.6) as an annuity payment.

$$A^s_{ti} = z^{s^-}_{t-1,i}\left[\frac{r^{s^-}_{t-1,i}(1+r^{s^-}_{t-1,i})^{-T+t-1}}{1-(1+r^{s^-}_{t-1,i})^{-T+t-1}}\right], \quad \text{for all } i\in U, s\in\Sigma_t, t=1,\dots,\tau. \tag{D.6}$$

The total net payment at each node, CF^s_t, is defined as the sum of principal payments, interest payments net of tax and administration fees in eqn. (D.7), whereas the total net prepayment amount for each leaf node is defined in eqn. (D.8).

$$CF_t^s = \sum_{i \in U} \big(A_{ti}^s + (1-\gamma)(r_{t-1,i}^{s^-} + c_a) z_{t-1,i}^s\big), \qquad \text{for all } s \in \Sigma_t, t = 1, \ldots, \tau, \tag{D.7}$$

$$PP_\tau^s = \sum_{i \in U} (z_{\tau i}^s K_{\tau i}^s), \qquad \text{for all } s \in \Sigma_\tau. \tag{D.8}$$

The next constraint uses the binary variables Z_{ti}^s to ensure that the fixed cost indicator is set to 1 in case of re–financing along the way.[3]

$$MZ_{ti}^s - y_{ti}^s \geq 0, \qquad \text{for all } i \in U, s \in \Sigma_t, t = 0, \ldots, \tau. \tag{D.9}$$

Constraints (D.10) and (D.11) together define the VaR and CVaR at the $100\alpha\%$ confidence level using the standard linear programming formulation (See Rockafellar & Uryasev and Zenios).

$$y_+^l \geq \Big[\Big(\sum_{t=0}^{\tau} \sum_{s \in \mathcal{E}_t^l} d_t^s CF_t^s\Big) + d_\tau^l PP_\tau^l\Big] - \zeta, \qquad \text{for all } l \in \Sigma_\tau, \tag{D.10}$$

$$\mathrm{CVaR}(\mathrm{y}; \alpha) = \zeta + \frac{\sum_{\mathrm{l} \in \Sigma_\tau} \mathrm{p}_\tau^\mathrm{l} \mathrm{y}_+^\mathrm{l}}{1-\alpha}. \tag{D.11}$$

[3]The constant M might be set to a value slightly greater than the initial amount raised; If a too large value is used, numerical problems may arise.

Finally non–negativity and binary constraints are introduced:

$$z_{ti}^{s},\ y_{ti}^{s},\ x_{ti}^{s},\ y_{+}^{l},\ \zeta \geq 0\ , Z_{ti}^{s} \in \{0,1\} \ \text{ for all } i \in U, l \in \Sigma_{\tau}, s \in \Sigma_{t}, t = 0, \ldots, \tau. \tag{D.12}$$

D.3.3 Generalization of the Rasmussen and Clausen models

? introduce a family of models which together cover risk preferences among mortgagors. Their risk–neutral model (minimizing average payments across scenarios) and the minmax model (minimizing the maximum payment) represent the two poles of risk preferences considered in their paper. Their budget–constrained models are then used to find mortgage strategies resulting in cashflows between the two poles. One of the main contributions of our paper is to consider an explicit risk measure (CVaR) and thereby generalize the models of **?** within a common model framework.

The two parameters (λ and α) can be used to generate any solution found in **?**. By setting $\lambda = 0$ in the objective function eqn. (D.1), the model turns into a risk–neutral model. For $\lambda = 1$ and $\alpha = 1$ the model turns into a minmax model. The budget–constrained model takes as input a predefined maximum budget level that the mortgagor does not wish to violate either for any given single scenario or for several scenarios. The model has a hard and a soft constraint which ensure that the mortgagor's wishes are respected. The soft budget constraint may be violated but if this occurs a penalty is incurred, whereas the hard budget constraint may by no means be violated. In our model the confidence level α implicitly decides the Value at Risk (VaR) level ζ which may be interpreted as a budget constraint. The level of λ corresponds to the penalty level.

The main advantages of using our Mean/CVaR model as compared to the budget–constrained model are the following:

1. Reasonable budget constraint levels are often hard to find. An inappropriate choice of budget levels may result in either infeasible problems or leave out interesting regions of the solution domain.

2. CVaR is a widely acceptable risk measure and its use is becoming increasingly popular, while it has the property of being coherent.

3. The Mean/CVaR models are easier to solve since they do not use hard constraints which usually add to the non–convexity of the problem.

D.4 Model testing and validity

Stochastic Programming models are discretizations of the uncertain world, hence in testing such models we should be concerned about convergence of solutions for different discretizations. Likewise we should test for robustness with respect to errors in the parameters representing uncertainty.

D.4.1 Convergence of solutions

Mortgagors pay[4] for having the right to re–balance their mortgage portfolios, so it is crucial that the model facilitates this option. Ideally we would like to have as many stages in the model as in real life, for instance in a quarterly or yearly basis. However given our trinomial discretization of the interest–rate model and the path–dependent nature of the problem we need to limit the number of stages to less than 6, i.e. 729 scenarios or 1093 nodes. Bigger problems are not computationally tractable on a standard personal computer within reasonable time. A relevant question to investigate is the necessary number of stages and decision nodes in order to obtain best possible solutions given the underlying model of uncertainty.

[4] Options embedded in the mortgage backed securities have a price. The price is not paid upfront but it is either recalculated as an extra premium on top of interest rate payments or as a higher initial outstanding debt resulting on higher future payments

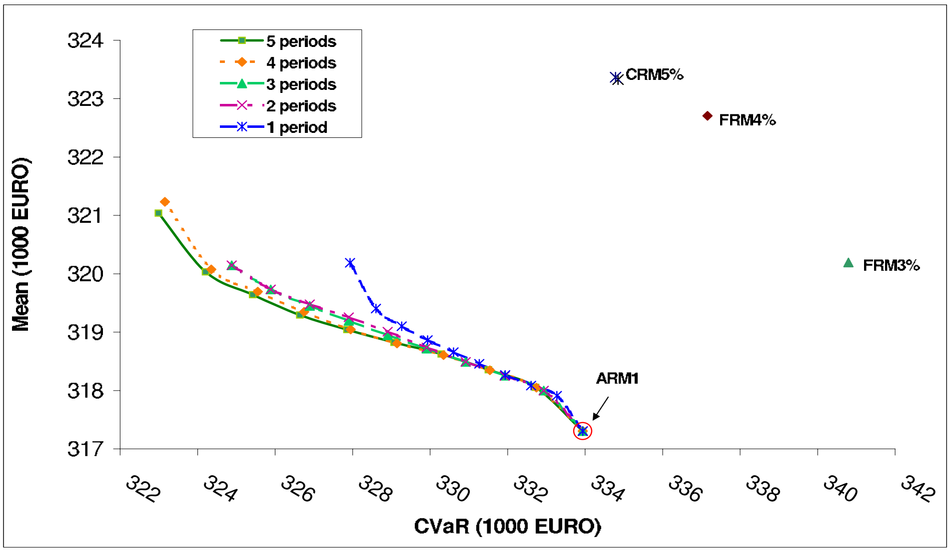


Figure D.6: As more decision stages are added to the problem the solution quality is improved. The improvement is, however, marginal after adding three extra decision stages.

Figure D.6 compares the mean/CVaR efficient frontiers found by one to five period models for our example with a seven–year horizon. As seen in Figure D.6 the efficient frontiers tend to converge for four to five re–balancing stages with period lengths of one to two years. Likewise, as seen in Figure D.7 the first stage solutions converge after only two to three stages. This behavior is expected since only some of the improvement in the efficient frontier is due to the structure of the first stage solutions and the rest of the improvement comes from extra re–balancing activities resulting from adding extra decision stages.

D.4.2 Stability

The solutions found by the stochastic program are dependent on the parameters of the stochastic process used to generate the scenario tree. So we need to study to what extent changes in the parameters depicting uncertainty have an influence on the solutions found. The stochasticity in the model comes from the underlying interest–rate model, so we are interested in observing changes in the solution structure as a result of changes in the interest–rate model parameters. The three parameters of the Vasicek model include: (i) The long–run mean value of the interest rate θ; (ii) the volatility of the interest rate σ; (iii) the dispersion of interest rates at each step κ.

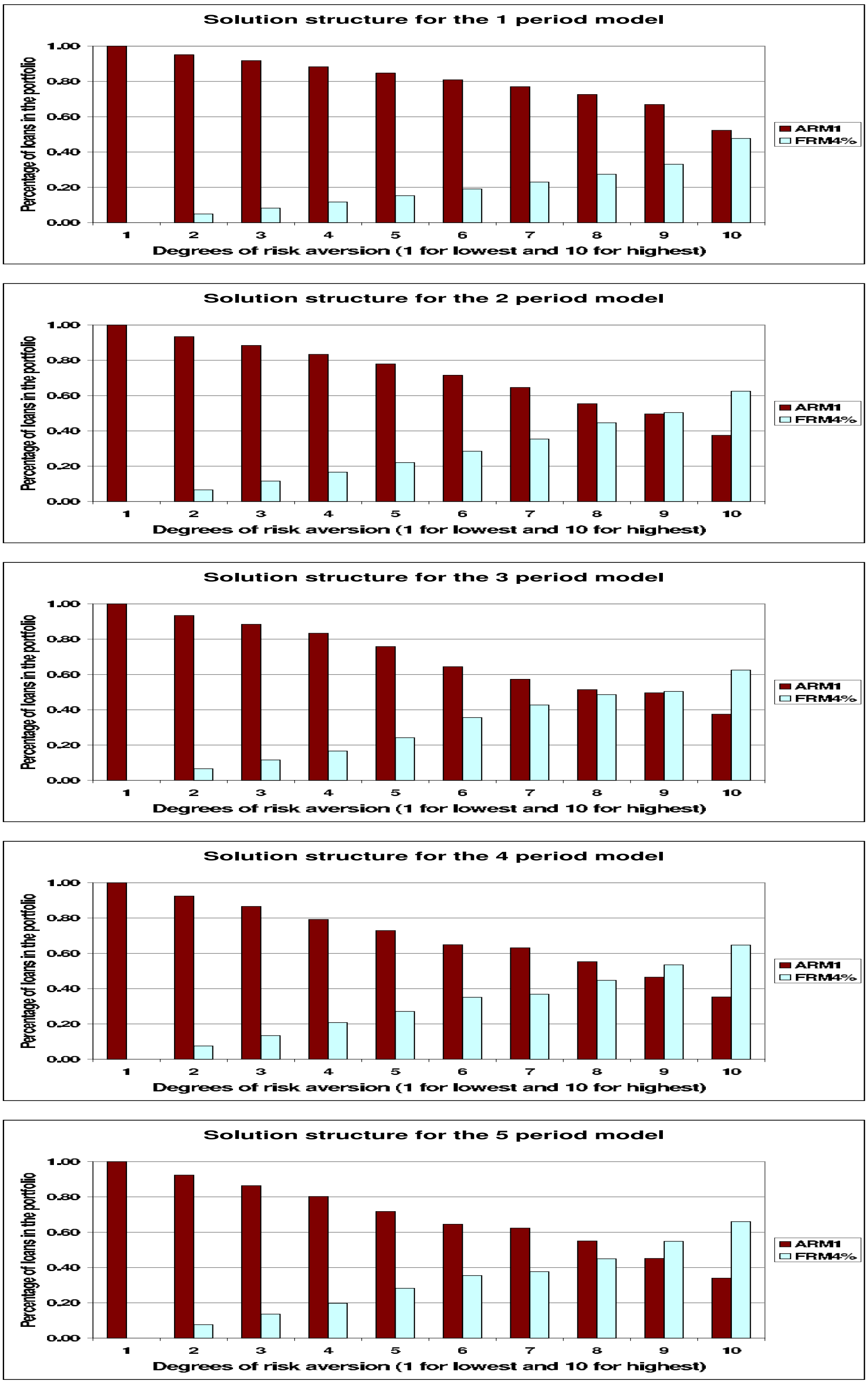


Figure D.7: First stage solutions for different degrees of risk aversion and increasing number of decision stages.

We perturb the two most significant parameters, namely the long–run mean θ and the volatility σ; the calibrated parameters based on historical time series were 0.042 and 0.010 respectively. We generate 100 different scenario trees based on uniformly random θ values in the interval of $[0.032; 0.052]$ and σ values in the interval $[0.008; 0.012]$. The mean/CVaR model is then run for all 100 scenario trees and the results are analyzed.

By studying the first stage solutions we find that the risk–averse mortgage strategies, i.e. those found for high λ values in the objective function are robust with regards to parameter uncertainty. Figure D.8 gives the intuition behind this finding. The model chooses to combine mortgage loans for high risk aversion no matter what the parameters are, whereas single mortgage loans are chosen for the mortgagor with low degree of risk aversion. As a result the diversified portfolio is more robust with respect to changes in parameters. This is an additional argument why we should choose a portfolio of mortgage loans rather than a single mortgage product, as is normal practice today.

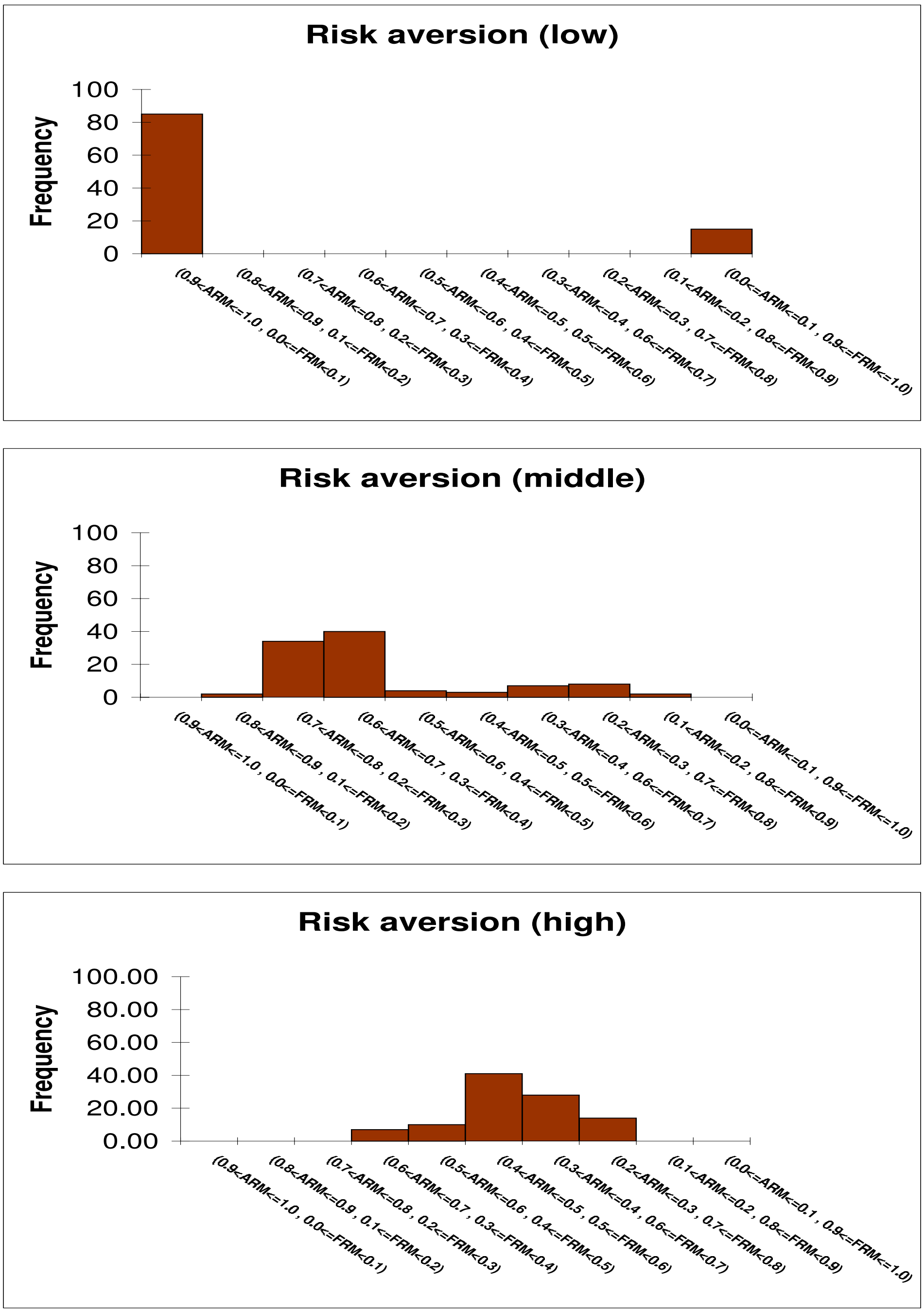


Figure D.8: The first stage solutions are very sensitive to changes in uncertainty parameters for a model with little or no embedded risk aversion whereas they become more robust as the degree of risk aversion increases.

D.5 Analysis of the model applications

We now use the model to analyze the underlying application of mortgage choice. In particular we investigate the use of the model for the advice offered to individual homeowners, for developing new products, and for estimating fair premia for CRMs. Throughout this section a five–period model with four rebalancing stages at years 1,2,3 and 5, and prepayment horizon at year seven is used.

D.5.1 Personal advice

Consider the bottom graph in Figure D.7. Ten different first stage (initial) solutions are represented. Except in the case of the risk–neutral mortgagor (minimizing the average payments only) the optimal solution involves mixing an ARM1 with FRM4%. For the more risk–averse mortgagors a greater part of the mix comes from the fixed–rate mortgage. Each of these solutions corresponds to a Mean/CVaR point on the five–period frontier of Figure D.6. The main lesson here is that diversification pays off for individual mortgagors regardless of existence of fixed mortgage origination costs. Considering Figures D.6 and D.7 together it also becomes clear that rebalancing pays off as well, regardless of both fixed and variable transaction costs. It is noteworthy, however, that diversification and rebalancing are not usually relevant for small mortgage loans (below 100,000 EURO) or for very short horizons (under 3 years). These issues were further exemplified in **?** even with the use of a simpler single period model.

D.5.2 Product development

So far we have used the model for finding optimal mortgage strategies based on FRMs with different coupon payments and ARM1. We can, however, easily add new products as input in order to quantify the value added by the new product. This is particularly useful before launching a new product. A synthetic equivalent of the new product may be tested within the model framework in order to find out whether the product adds value to certain segments of the market. The marketing of the new product may then be concentrated on these segments only.

Figure D.9 illustrates this use of the model. Starting with our standard products (ARM1 and FRMs) we add new mortgage products one at a time and observe their effect on the Mean/CVaR efficient frontier. For any mortgagor with a 7–year repayment horizon, adding an ARM2 (adjustable–rate mortgage with bi–annual rate adjustments) does not add much value, whereas adding ARM3 and in particular ARM4 makes some contribution. Adding ARM5 adds more value for the very risk–averse mortgagors, but it adds less value for mortgagors with some appetite for risk.

A noteworthy observation was made when we continued the experiments adding the more exotic products CRM5% and CRM6%. These new products had no influence on the original efficient frontier indicating too high premia on their embedded cap options. At their current prices these new products do not add value to homebuyers. In the next section we use the model to estimate fair premia so that the new products become attractive for mortgagors.

D.5.3 Deciding fair premia for CRMs

CRMs are designed so that they follow the six–month CIBOR rate with a fixed premium on top of that for the embedded cap option. The premium for the CRM5% in November 2006 was for instance 0.8%. We have already observed that CRMs add no value to the existing mortgage products – at least for a mortgagor with a seven–year horizon. In Figure D.10 we illustrate how to use our model in order to find out fair premia for a given CRM for mortgagors with different risk appetites.

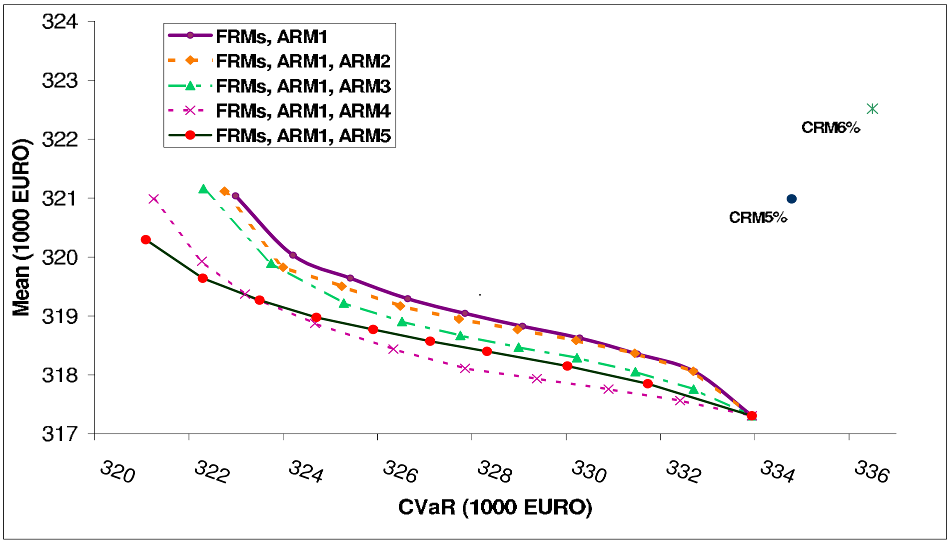


Figure D.9: Adding one mortgage at at time to the existing universe of mortgages.

We add a CRM5% to the existing universe of ARM1 and FRMs, but we remove the premium of 0.8% (so that we follow the 6–month CIBOR rate with a 5% cap) in order to make sure the CRM5% is chosen in the efficient frontier. As expected such a loan would significantly improve the efficient frontier. Notice, however, that even without any premium on CRM5% the model still suggests diversification (initially with FRM4%) in order to reduce the CVaR. Only the point furthest right on the frontier corresponds to a strategy of holding only CRM5%.

Then we add some small premium in incremental steps of 0.1% and rerun the model, until the CRM5% is no longer chosen as part of the optimal solution. This happens at around 0.4% for the risk neutral mortgagor

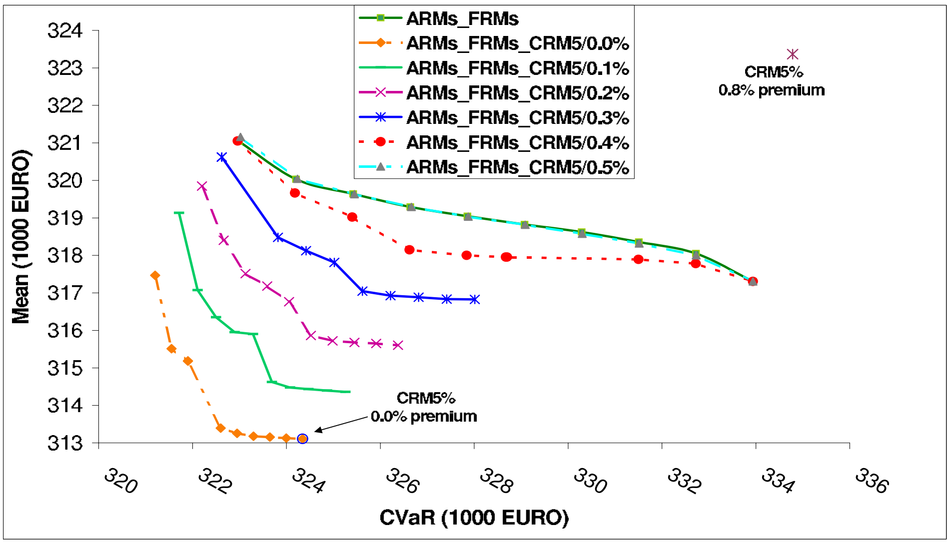


Figure D.10: We introduce a CRM5% without any premium to begin with and then increase the premium in small increments until the CRM is no longer part of the efficient frontier.

and surprisingly for the very risk averse mortgagor as well.[5] For all other mortgagors the CRM5% is not attractive anymore at a premium of about 0.5%. Hence, a fair premium for the CRM5% would be 0.4–0.5%.

[5]The intuition behind this is that the features of CRMs appeal to mortgagors with a medium risk appetite, so exactly these mortgagors would accept a higher premium for this product.

D.6 Conclusion

In this paper we showed that diversification and re–balancing of the mortgage loans pay off for a typical mortgagor. Building on the single period model of **?** we observed that adding stages implies more diversification in the initial portfolio. This is in contrast to the existing practice where mortgagors hold one type of mortgage loan only. We exemplified how the results of the multistage Mean/CVaR model may be used for advanced analysis prior to mortgage choice recommendations as well as for product development.

Finally we used the model to show that CRMs, as offered today, are not attractive. Hence a reduction in premia for CRMs with shorter horizons and using CRMs as a component of the mortgage portfolio is recommended.

Considering income and house price dynamics would add some insights for even more individualized advice. Likewise extensive studies for different repayment horizons, different initial loan values and different mortgage loan combinations are needed to establish some rules of thumb which could be used on a daily basis for personal advice on the mortgage choice. These issues remain as future work.

Appendix 1: The interest rate model

We use a variant of the Vasicek interest rate model as used in Jensen & Poulsen (2002) the underlying stochastic process to generate estimates of future short rates:

$$dr(t) = \kappa(\theta^P - r(t))dt + \sigma dW^p(t),$$

where $r(t)$ is the short rate, W is a brownian motion and κ, θ and σ are the Vasicek model's parameters controlling the height of the interest rate jumps, the long run mean level of interest rates and the volatility of the interest rates. The model is given under real–world probability measure P. This can be shifted to the risk free measure Q using the transformation:

$$\theta^Q = \theta^P + \pi, \qquad \pi \in \mathcal{R}$$

where π is the risk premium, so the Vasicek model under the risk free probability measure Q becomes:

$$dr(t) = \kappa(\theta^Q - r(t))dt + \sigma dW^Q(t)$$

The expectation of short rates are then found as follows:

$$E^Q(r(t)) = r_0 \cdot \exp(-\kappa t) + \theta^Q(1 - \exp(-\kappa t))$$

Appendix E

Yield curve event tree construction for multi stage stochastic programming models

Working paper availabe at http://www2.imm.dtu.dk/ kmr/.

Abstract

Dynamic stochastic programming (DSP) provides an intuitive framework for modelling of financial portfolio choice problems where market frictions are present and dynamic re–balancing has a significant effect on initial decisions. The application of these models in practice, however, is limited by the quality and size of the event trees representing the underlying uncertainty. Most often the DSP literature assumes existence of "appropriate" event trees without defining and examining qualities that must be met (ex–ante) in such an event tree in order for the results of the DSP model to be reliable. Indeed defining a universal and tractable framework for fully "appropriate" event trees is in our opinion an impossible task. A problem specific approach to designing such event trees is the way ahead. In this paper we propose a number of desirable properties which should be present in an event tree of yield curves. Such trees may then be used to represent the underlying uncertainty in DSP models of fixed income risk and portfolio management.

Keywords: Interest rate modeling, scenario tree generation.

E.1 Introduction

One of the main sources of uncertainty in analyzing risk and return properties of a portfolio of fixed income securities is the stochastic behavior in the evolution of the shape of the term structure of the interest rates (yield curve). This uncertainty is sometimes referred to as shape risk, see for example Zenios (2007). Shape risk refers to the risk that interest rates with different maturities change in different ways as the time goes by. Figure E.1 shows how the Danish yield curves have changed in the period 1995 to 2006.

We can see that the short rates have been more volatile than the long rates. We also observe that a simple parallel shift assumption does not hold; yield curves evolve in more complicated manners. Capturing the dynamics of yield curves in a multi period scenario tree is the purpose of this paper.

Dynamic stochastic programming (DSP) provides a flexible framework for portfolio and risk management problems. Trade frictions such as fixed costs, tax affects and limits on borrowing and short sale of assets can

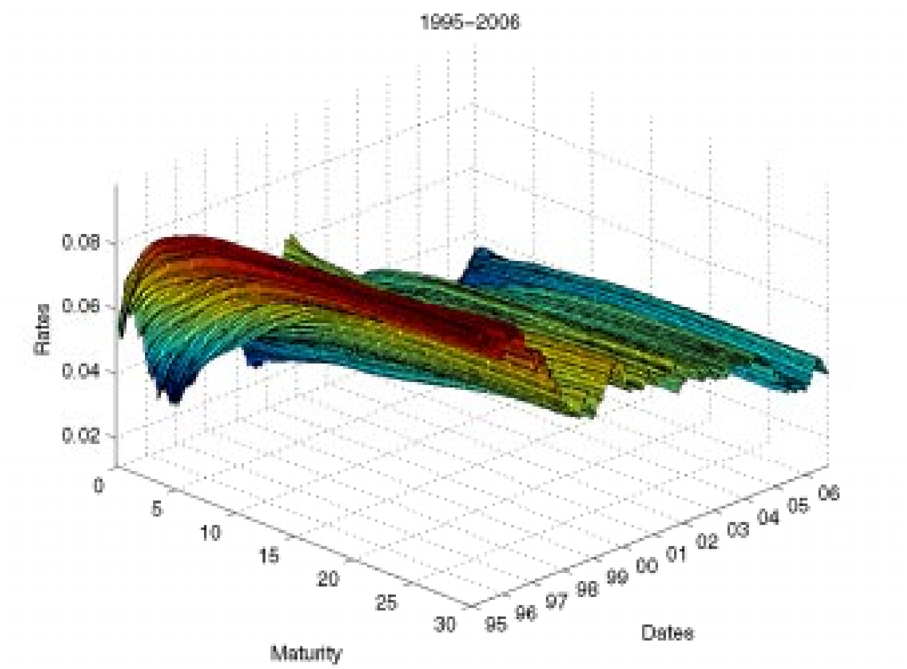


Figure E.1: Historical data on Danish yield curves for the period 1995 to 2006.

be incorporated in such models. Portfolio readjustments may as well be captured. This is in particular important for fixed income securities due to the usually long term perspectives of such investments. Finally no assumptions on the underlying uncertainty are required. This means that for example heavy tails which play an important role in extreme event considerations can be accounted for. But it also means that special care needs to be taken when it comes to modelling the underlying uncertainty.

The event trees should be consistent with historical data as well as internally consistent with regards to the mechanisms governing the dynamics of the uncertain variables (see Ziemba 2001). **?** Such consistency criteria include for example the no arbitrage conditions (see Klaassen 2002).

We suggest the following guidelines for generating an event tree of yield curves:

1. The distance between the underlying continuous interest rate process and the discretized event tree should be minimized.

2. The event tree should match the underlying continuous process both globally, i.e. for any given future period as well as locally, i.e. for any subtree of the event tree.

3. The actual levels of the generated scenarios should be realistic, for example the tree should not include any negative interest rates, or many extreme scenarios.

4. The volatilities of the interest rates of different maturity should be consistent with the implied volatilities of a market benchmark.

5. There should be no arbitrage opportunities in any of the subtrees of the event tree.

6. Types of changes in the shape of the yield curve in future nodes of the event tree should reflect those observed historically from an economical regime which is assumed similar to the one the event tree is built for.

7. The model should be mean reversive.

8. No volatility clumping; Volatility clumping refers to the case where a period of high volatility is followed by another period of high volatility. Volatility clumping is observable in the equity market, but empirical studies have shown that there is no volatility clumping for the interest rates.

There is a vast amount of literature on interest rate modelling. These models can in general be categorized as being discrete or continuous, normal or a log–normal, 1–factor or multi–factor and finally either more theoretically or more empirically inclined. What all such models have in common is the fact that they have been originally developed either for estimating current prices of interest rate sensitive assets, or for prediction purposes. None of the standard models therefore are designed in order to construct yield curve event trees fulfilling criteria 1 to 8 at the same time.

In this paper, we propose an overall framework for building a yield curve event tree and testing whether or not the consistency criteria are respected. The rest of this paper is organized as follows:

In section E.2 we perform factor analysis (also known as principal component analysis) in order to identify the most significant factors in capturing yield curve variability. Then in section E.3 we describe a simple 3–factor vector auto regressive model with lag 1 (VAR1) representing the underlying stochastic process. A non–linear discretization model of the stochastic process is then suggested in section E.4. In section E.5 we outline an approximative approach for solving the discretization model. In section E.6 we argue why a simple 1–factor interest rate model such as the Vasicek model is not appropriate for stochastic programming applications and why the proposed 3–factor model provides more reliable solutions. Finally we conclude the paper in section E.7.

E.2 Factor analysis of yield curves

Factor analysis is a statistical technique to detect the most important sources of variability among observed random variables. It may be used on historic time series of a multidimensional random variable to decide factors ordered after how much variability they explain. In linear algebraic terms it is an orthogonal linear transformation that transforms data to a new coordinate system in such a way that the greatest source of variance lies on the first factor, the second largest on the second factor and so on. It is used for reducing the dimensionality of a data set while keeping its characteristics. This is done by keeping only the main factors while ignoring the ones that only explain an insignificant proportion of the variance.

Litterman & Scheinkman (1991) and Knez, Litterman & Scheinkman (1994) use factor analysis to show that three factors explain – at a minimum – 96% of the variability on several American zero coupon yield curves in the period 1985 to 1988. Dahl (1994) shows similar results for the Danish data in the 1980's and Bertocchi, Giacometti & Zenios (2005) repeat the experiments for American and Italian data during 1990's again with similar results.

These findings are used by some practitioners to improve duration hedging (immunization) by factor based duration hedging (factor immunization). The main shortcoming of these hedging techniques is that they are myopic and do not consider the re–balancing effects in long term fixed income portfolio investments. Rather than using factor analysis for shape risk hedging, we use factor analysis as a means of finding a sufficient number of factors to be used as the underlying factors of uncertainty for the proposed interest rate model of this paper. We perform factor analysis on the Danish yield curves for the period 1995–2006. Like in earlier works we find that 3 factors are enough to capture almost all variability (99.99%) for the Danish yield curves. Figure E.2 shows the factor loadings as a function of maturities in years.

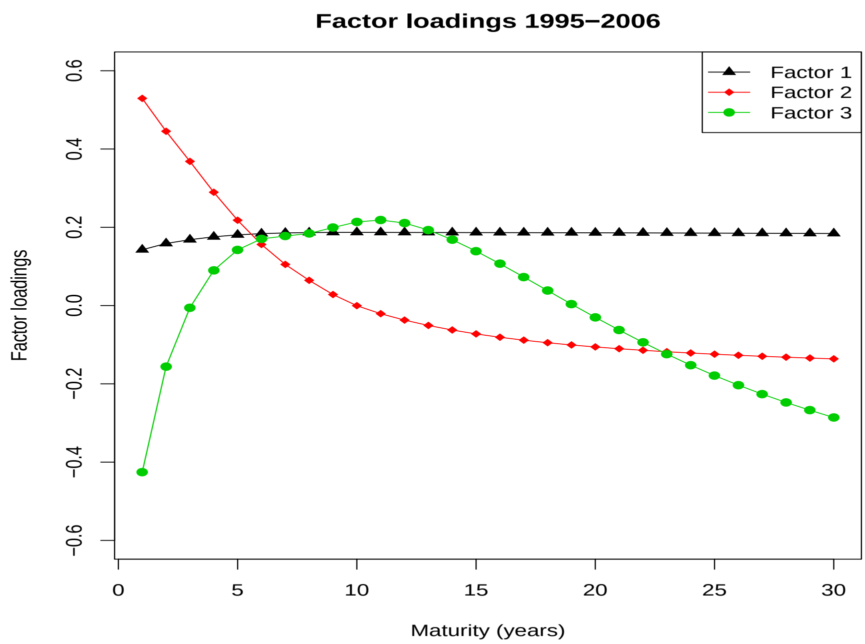


Figure E.2: Factor loadings of the Danish yield curves for the period 1995 to 2006.

The first factor explains almost 95% of all variability. It can be interpreted as a slight change of slope for interest rates with maturities up to approximately 6 years together with a parallel shift for the rest of the curve. The second factor, explaining 4.7% of the variability, corresponds to a change of slope for the whole curve. However the slope change for the first 10 years is much more pronounced. Finally the third factor corresponds to a change of curvature in the yield curves. This factor explains only about 0.3% of the total variability.

From a statistical viewpoint we could suffice with level and slope as the main sources of variability. Nevertheless we do not reject the third factor, curvature, due to its economical appeal; changes of curvature are observed now and then, and a model not being able to represent those changes properly has a potential weakness of not capturing important movements in the interest rate market.

Inspired by the results found in this section we define the three factors which we want to use in our interest rate model as follows:

1. Level: An arbitrary rate such as the one year rate,Y_1, may be used as a proxy for level.

2. Slope: A good proxy for the slope would be $Y_{20} - Y_1$ where Y_{20} stands for the 20 year rate. This expression is an approximation of the average slope of the yield curve. The 20 year bond is chosen

as the long rate here, since we observe in our historical data, that almost all yield curves flatten at about this maturity.

3. Curvature: The expression $Y_6 - (\omega Y_1 + (1-\omega)Y_{20})$, with Y_6 as the 6 year rate, may be used as a proxy for the curvature. ω is the weight corresponding to the proportion of the distance in between the middle of long rates. It is chosen so that the curvature would be zero if the curve is a straight line, negative if the curve is convex and positive if the curve is concave.

In the rest of this paper we use level, slope and curvature defined as above as the factors of the interest rate model in question.

E.3 A vector autoregressive model of interest rates

A vector autoregressive model with lag 1 (VAR1) may be defined as:

$$x_{t+1} = \mu + A(x_t - \mu) + \epsilon_{t+1}$$

where x_t is an $n \times n$ matrix, μ is an $n \times 1$ vector and $\epsilon_{t+1} \sim \mathcal{N}_n(\overline{0}, \Omega)$ and Ω is an $n \times n$ matrix. In this formulation of the VAR1 model, μ is interpreted as the long term drift. A and μ are deterministic parameters which need to be calibrated based on historical data.

The conditional mean and covariance for the error term ϵ_{t+1} are given as:

$$E[\epsilon_{t+1}|x_t] = 0$$

$$E[\epsilon_{t+1}\epsilon_{t'+1}|x_t] = \Omega$$

Given the state of an uncertain variable at time x_t, the purpose of the model is to predict the state of the variable at time $t+1$, i.e. x_{t+1}. Based on the findings of the previous section we define the vector x_t as the proxies for level, slope and curvature $(l_t, s_t, c_t)^{\mathrm{T}}$ of the yield curves.

An example of the VAR1 model with 3 factors looks like:

$$l_{t+1} = \mu_l + a_{ll}(l_t - \mu_l) + a_{ls}(s_t - \mu_s) + a_{lc}(c_t - \mu_c) + \epsilon_{l,t+1}$$

$$s_{t+1} = \mu_s + a_{sl}(l_t - \mu_l) + a_{ss}(s_t - \mu_s) + a_{sc}(c_t - \mu_c) + \epsilon_{s,t+1}$$

$$c_{t+1} = \mu_c + a_{cl}(l_t - \mu_l) + a_{cs}(s_t - \mu_s) + a_{cc}(c_t - \mu_c) + \epsilon_{c,t+1}$$

To estimate the parameters of the VAR1 model (μ, A, Ω) we can use the parameter estimation for a general linear regression model of the form:

$$y_i = \alpha + \beta x_i + \epsilon_i, \qquad \text{for all } i = 1, \cdots, n$$

Or in matrix form:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

This can be rewritten as:

$$Y = \overline{X}\delta + \varepsilon$$

The VAR1 model can be rewritten in this form. Now we may use standard least square estimators as follows:

$$\hat{\delta} = (\overline{X}^{\mathrm{T}}\overline{X})^{-1}\overline{X}^{\mathrm{T}}Y$$

which minimizes the sum of least squares in the expression $||Y - \overline{X}\delta||^2$.

The estimator for the residuals (ε) is given as:

$$res = Y - \overline{X}\hat{\delta}$$

$$\hat{\Omega} = res^{\mathrm{T}} res/(n-1)$$

The estimator $\hat{\delta}$ is then decomposed into μ and A from the VAR1 model and the estimator $\hat{\Omega}$ can be directly used as the estimator for Ω in the VAR1 model.

The VAR1 model so far may only be used for one–period predictions (same interval length as in the historical time series). But it may easily be extended to predict k periods ahead:

$$x_{t+k} = \mu + A^k(x_t - \mu) + \epsilon_{t+k}$$

where $\epsilon_{t+k} \sim \mathcal{N}_n(\overline{0}, \sum_{i=1}^{k} A^{i-1}\Omega(A^{i-1})^{\mathrm{T}})$

The reasons for choosing a VAR1 model as the underlying model of interest rate uncertainty are the following:

1. One can choose any factors or any number of factors to describe the variability. This gives us maximum flexibility with respect to our observations from a factor analysis of interest rates.

2. Time step flexibility. Varying time steps can be easily implemented.

3. Mean reversion is built into the VAR1 model.

The VAR1 model is discrete in time but continuous in states, so in order to use the model as a scenario generator for stochastic programs we need to discretize it in states as well. This can be done using a moment matching model (See Høyland & Wallace (2001)). We propose a yield curve scenario discretization model in the next section.

E.4 Scenario generation and event tree construction

In DSP literature for fixed income securities often simple models of interest rates are used to represent the underlying interest rate uncertainty. In several applications lattice structures are either blown up into unique paths or sampled from so that to account for the path dependency of DSP problems. One immediate problem with such approaches is that the uncertainty space is not covered as efficiently as possible. This is due to the recombining structure of the original trees together with the fact that only a very coarse time step discretization is possible due to the curse of dimensionality when the recombining trees are blown up.

Others (Nielsen and Poulsen 2004, etc.) have used continuous interest rate models. Such models are either continuous both in time and state, or discrete in time and continuous in states. Discretizing in time is normally straight forward; it is a question of reformulating a differential equation into a difference equation. Discretizing in state, however, is often a more challenging issue. A number of nodes (in our case including yield curve information) have to be generated for each time point to give a discrete representation of the continuous distribution. There is no general consensus as to the best way of doing this discretization. In one stream of research the main focus is on generating discrete distributions which mimic the underlying continuous distribution as closely as possible. This is either done by sampling (see Shtilman and Zenios 1993) , or moment matching approaches (Høyland and Wallace 2001). In the other stream of research the aim is not necessarily to get the closest discrete representation of the continuous distribution, but rather finding a discrete representation which results in a closer approximation to the "true" optimal solution of the stochastic program in question. Here the "true" optimal solution refers to the solution we would get, if we were able to solve the stochastic program using the underlying continuous process directly. Indeed if we were able to do that, there would be no need to discretize the process in the first place, but it can be shown (See Pflug 2001) that in general if the discrete process has the smallest distance (using the transport metric) to the underlying continuous process, then the SP solutions found will be guaranteed to be within certain bounds of the "true" SP solutions. (See also Pflug 2001, Pflug and Hochreiter 2002, Pennanen 2004, Romisch and Heitsch 2003) **?**. Although theoretically appealing, the guaranteed bounds are in many cases too large in order to have any practical interest, (See Wallace and Kaut 2003) . Comparison and further development of specialized models and solution algorithms for these two streams of scenario discretization approaches is the subject of future research.

An extensive comparative study of different yield curve scenario generation approaches is outside the scope of this paper. Instead we propose a yield curve scenario generation model which abides by the criteria 1 to 8 mentioned earlier in this paper. Note that the following model is single period. It can be extended to a multi–period model with some minor changes.

We define the following sets, parameters and variables:

Sets:

f: Set of factors (level, slope and curvature), f' is alias for f.

i: Set of zero coupon bonds (zcb's).

i': A subset of the set i corresponding to the zcb–rates which define the three factors. We have chosen i' to be the set of 1, 6 and 20 year zero coupon bonds.

j: Set of parameters of the Nelson Siegel function; 0 to 3.

t: Set of time points.

s: Set of scenarios.

Parameters:

$Mean_f$: The mean value for factor f. This value comes from the VAR1 model.

$Covar_{f,f'}$: The covariance matrix of the error term taken from the VAR1 model.

$Skewness_f$: Skewness of factor f. Assumed to be zero based on the normality assumption of the VAR1 model.

τ_i^t: Time to maturity for zcb i at time t.

PP_i^{parent}: Prices of the zero coupon bonds at the root, The prices are calculated using initial rates: $PP_i^{\text{parent}} = e^{-r_i \tau_i^{\text{parent}}}$.

ψ^{Const}: The martingale probability; assumed equal for all scenarios. It is found from the equation $PP_{i''}^{\text{parent}} = \sum_s \psi^{\text{Const}}$ where bond i'' matures exactly at the children nodes of the tree with a price of 1.

Variables:

$x_{f,s}$: A future estimate of factor f in scenario s given by the VAR1 model.

$E(x)_f$: The expected value of factor f over all scenarios.

$\sigma(x)_{f,f'}$: The covariance matrix of factors across all scenarios.

$E3(x)_f$: The skewness of factors across all scenarios.

$Y_{i',s}^{(VAR1)}$: The 3 yields comprising the 3 factors at scenario s.

$NSY_{i',s}$: The 3 yields comprising the 3 factors at scenario s as given by the Nelson Siegel function.

$\varphi_{s,j}$: Parameter j of the Nelson Siegel function at scenario s.

$R_{i,s}$: The entire yield curve given by the Nelson Siegel function at scenario s.

$CP_{i,s}$: Price of bond i at scenario s.

The overall objective of the optimization model is to match the moments of the underlying stochastic process (the VAR1 model) as closely as possible. At the same time the parameters of the Nelson Siegel function should be found so that the yields resulting from Nelson Siegel are as close as possible to those found by the VAR1 model. We need Nelson Siegel (or some other yield curve smoothing function) in order to get the rest of the yield curve, since the VAR1 model is based on 3 yields only.

The objective function is to minimize sums of least squares corresponding to the overall objective of the model:

$$\text{Minimize } \sum_f (E(x)_f - Mean_f)^2 + \sum_f \sum_{f'} (\sigma(x)_{f,f'} - Covar_{f,f'})^2 + \sum_f (E3(x)_f - Skewness_f)^2 + \sum_s \sum_{i'} (Y_{i',s}^{(VAR1)} - NSY_{i',s})^2 \tag{E.1}$$

The moments of the discrete scenarios as found by the optimization model are defined in Equations E.2 to E.4:

$$E(x)_f = \sum_s p_s x_{f,s} \qquad \text{for all } f \tag{E.2}$$

$$\sigma(x)_{f,f'} = \sum_s p_s (x_{f,s} - E(x)_f)(x_{f',s'} - E(x)_{f'}) \qquad \text{for all } f, f' \tag{E.3}$$

$$E3(x)_f = \frac{\sum_s (x_{f,s} - E(x)_f)^3}{(\sum_s (x_{f,s} - E(x)_f)^2)^{3/2}} \qquad \text{for all } f \tag{E.4}$$

In Equation E.5 the 3 yields corresponding to the 3 underlying maturities used in the VAR1 model are found by the Nelson Siegel model. Note that the final term of the objective function requires that $NSY_{i',s}$ should be as close as possible to the 3 yields found by the VAR1 model. So Equation E.5 in interaction with the objective function calibrates the parameters of the Nelson Siegel function. These parameteres are used in Equation E.6 to decide the entire yield curve at each scenario.

$$NSY_{i',s} = \varphi_{s,0} + \varphi_{s,1} e^{-\varphi_{s,3}\tau_{i'}^{\text{parent}}} + \varphi_{s,2}\tau_{i'}^{\text{parent}} e^{-\varphi_{s,3}\tau_{i'}} \qquad \text{for all } i', s \tag{E.5}$$

$$R_{i,s} = \varphi_{s,0} + \varphi_{s,1} e^{-\varphi_{s,3}\tau_i} + \varphi_{s,2}\tau_i^{\text{parent}} e^{-\varphi_{s,3}\tau_i^{\text{parent}}} \qquad \text{for all } i, s \tag{E.6}$$

The VAR1 model is defined in terms of factors and not yields. Equations E.7 to E.9 find the yields corresponding to the factors estimated by the VAR1 model at each scenario.

$$Y_{1,s}^{(VAR1)} = x_{1,s} \qquad \text{for all } s \tag{E.7}$$

$$Y_{20,s}^{(VAR1)} = x_{2,s} + Y_{1,s}^{(VAR1)} \qquad \text{for all } s \tag{E.8}$$

$$Y_{6,s}^{(VAR1)} = \frac{5}{19} Y_{20,s}^{(VAR1)} + \frac{14}{19} Y_{1,s}^{(VAR1)} + x_{3,s} \qquad \text{for all } s \tag{E.9}$$

The main reason to define the yield curve discretization process as an optimization model is that it enables us to add constraints which give the user a degree of control over the outcome. One such constraint may be forcing a lower bound on interest rates, for instance not allowing negative rates:

$$R_{i,s} \geq 0 \qquad \text{for all } i, s \tag{E.10}$$

Another condition may be not to allow arbitrage in the interest rates. In Equations E.11 and E.12 we introduce a more restrict condition than the no arbitrage condition, namely we require that martingale probabilities should be equal across all scenarios:

$$CP_{i,s}^{\text{child}} = e^{-R_{i,s}\tau_i^{\text{child}}} \qquad \text{for all } i, s \tag{E.11}$$

$$PP_i^{\text{parent}} = \sum_s \psi^{\text{Const}} CP_{i,s}^{\text{child}} \qquad \text{for all } i \tag{E.12}$$

The model E.1 through E.12 gives the user a great degree of flexibility over the outcome of the discretization process. Subjective expert opinion is integrated with objective econometrical and financial theory. The model, however, is non–linear, non–convex and as such has several local minima. Solving such a problem fall into the realm of global optimization. The general purpose global solvers are as of yet underdeveloped. Specialization of existing algorithms is therefore needed for solving this problem to optimality. This is outside the scope of the current paper. Instead we propose an approximative approach to find reasonable solutions in the next section.

E.5 An approximative solution approach

The approximation is in dividing the model into three parts and solving them in a serial manner instead of solving the entire problem in one go:

1. First we solve a model comprising of the objective function less the 4th term with constraints E.2 to E.4. This model results in discretized factors matching the first 3 moments of the underlying VAR1 model one period ahead. We also add constraints E.7 through E.10 to guarantee no negative rates.
2. Then we solve a second model where the objective function is made of the 4th term and the only constraint is Equation E.5. Finding the parameters of the Nelson Siegel model we now simply use Equation E.6 to find the entire yield curves for each scenario.
3. Finally we apply Equations E.11 and E.12 to remove arbitrage.

The two sub models are non–linear non–convex themselves but it is possible to find optimal solutions to these problems using standard non–linear solvers which is what we have done using GAMS/ConOpt[1].

[1] GAMS/CONOPT is a non linear problem (NLP) solver available for use with General Algebraic Modeling System (GAMS). See http://www.gams.com/solvers/solvers.htm

Wasn't it due to the no–arbitrage conditions then solving the two models separately would corresponded to solving the entire problem. We therefore compare the yield scenarios before removing arbitrage with those after arbitrage removal, See Figures E.3 to E.6. The scenarios in the left are before the arbitrage removal part of the approximative algorithm has been applied. The scenarios in the right are after arbitrage removal. The smaller the change is between the left hand side and the right hand side scenarios the closer the results of the approximative approach will be to that of solving the entire problem.

The first 2 figures are from August 2005 when the initial term structure is rather steep (the stippled curve). In these cases we note that there is very little difference between the rates before and after arbitrage removal, meaning that the approximative approach generates near optimal solutions for the entire model. In the last 2 figures the starting point is May 2007 when the initial yield curve is essentially flat. In this case we note a considerable difference between the rates before and after removal of arbitrage. In both cases, however, the solutions found may be used as initial solutions for solving the entire problem.

We leave solving the entire problem as future work. Instead we replace the Nelson Siegel function with an affine function developed for our 3–

factor VAR1 model of interest rates (See Poulsen 2007) . It is known from interest rate theory that Nelson Siegel does not produce arbitrage free curves in any continuous model. Given that, there is little hope that the discretized models will be arbitrage free regardless of the number of scenarios generated. The affine function is, however, constructed arbitrage free in the continuous setting. So the hope is that by adding scenarios we will satisfy the no–arbitrage condition in the discrete scenarios as well. The graphs in the bottom of Figures E.3 to E.6 are the result of an affine smoothing of the yield curves. Again the yield curve scenarios before and after removing of arbitrage are considered.

In the rest of this work we use the scenario trees based on the affine model. In the next section we will compare interest rate scenarios generated by our VAR1 model with the well known 1–factor Vasicek model.

E.6 Vasicek versus VAR1 for event tree construction

A central theme in this paper is to convince the reader that simple 1–factor interest rate models do not capture the dynamics of historic rates as indicated by a factor analysis of historic interest rates. Even though that does not necessarily have an influence on how well such models are in pricing fixed income securities here and now, that does have an impact on estimates of prices of assets in future nodes. That is why using simple models of interest rate as the underlying source of uncertainty in a stochastic program might result in misleading solutions to the asset allocation and risk management problems that are formulated based on such interest rate scenario trees. How wrong the solutions of such stochastic programs will be is problem dependent and need to be studied for individual applications. In this section we show how we can get a graphical feel of how well an interest rate scenario tree mirrors what we expect interest rates to behave based on the criteria mentioned in the introductory part of this work.

Figures E.7 to E.9 show interest rate trees for 1, 6 and 20 year maturities starting on the 1th of May 2007 and running over 5 years once using the 1–factor Vasicek model as the underlying source of uncertainty and twice using our VAR1 model. The only difference between the VAR1 representations is the manner in which discretization takes place. We use our approximative discretization approach described in the last section iteratively to the future nodes of the tree to produce these multi period tree structures.

It is obvious from the figures that the trees using the Vasicek model have almost no volatility for the long rates. Looking at historic yield curves in Figure E.10 this seems very unrealistic. On the other hand the VAR1 trees branched in a 4–4–4–4 fashion seem to produce overly large volatilities for all maturities. This is better seen in Figure E.11 where we only consider the yield curves 5 years from May 2007. The initial yield curve is presented using a solid line. Note, however, that in the Vasicek model the initial yield curve is not the observed curve but reproduced by the model. By only looking at these graphs there is little room for suspicion left as for the insufficiency of a 1–factor Vasicek model in capturing future dynamics of interest rate, in particular the long rates.

Obviously we do not wish for our model of choice to reproduce historical yield curves exactly. That said, it is desired that the model captures characteristics seen in historic data. Our VAR1 model with a 16–4–2–2 discretization seems to produce a good approximation to the real world data from 1995–2006 as seen in Figure E.12. Whether or not this is a good historical period which characteristics to mimic is a subjective question, but it is a subjective question at a high level of abstraction; we do not choose how the yield curves should exactly look like, but we make a decision as to which historic period we believe gives rise to a good approximation of future yield curve scenarios.

E.7 Conclusions

We have set up a number of qualitative conditions with which a yield curve scenario generation method should comply. We have shown that the 1–factor Vasicek model, even though suitable for option pricing, is unable to capture future dynamics of interest rate, which disqualifies this model as a source of uncertainty for stochastic programs. We have tailored a 3–factor VAR1 model using the 3 factors, level, slope and curvature, describing over 99% of variability in historical interest rates and we have introduced a discretzation scheme on top of that. We have presented graphs which give the user a feel of whether or not the scenarios generated are representative of what is observed in historical data as well as what is prescribed by econometrical and interest rate theory. Our VAR1 model with a 16–4–2–2 discretization gives rise to a reasonable representation of uncertainty over a 5–year period with a modest number of scenarios, 256. The three major types of yield curve shifts are present in representative quantities and the volatility of the last 10 years historic data is captured properly. There is also reversion towards the long term drifts. No negative rates or extremely low rates are observed. There are, however, some gaps in between the extreme scenarios and the main bulk of scenarios in the high end of the scale in particular for long rates. The gap can be closed if we generate more scenarios for example 32–4–4–4, but this results in

2048 scenarios which is probably about the highest number of scenarios most realistic linear stochastic programming applications can handle on a standard pc. Given that the stochastic programming problems we have in mind have 0–1 constraints we find the trees of approximately 200–300 scenarios more appealing. Whether or not this leads to serious solution deficiencies as compared to using 2000-3000 scenarios is subject of future work. We need special purpose algorithms and/or parallel routines to perform the comparison. Super computers may as well provide sufficient computing power for these tests. Our preliminary trials on LP–relaxed version of our optimization problems at hand show, however, that the first stage solution structures stabilize already at about 200–300 scenarios despite the gaps in between the high extreme scenarios and the main bulk of scenarios. Another idea that we leave to future work is trying another moment matching approach where the first four moments (kurtosis being the fourth) are matched simultaneously at each period conditioned on the root, and that only the first 2 or 3 moments are matched for the sub–trees in between the periods. Likewise applying the ideas of Pflug (2001) and Hochreiter and Pflug (2006) on optimal discretization to our problem remain as future work.

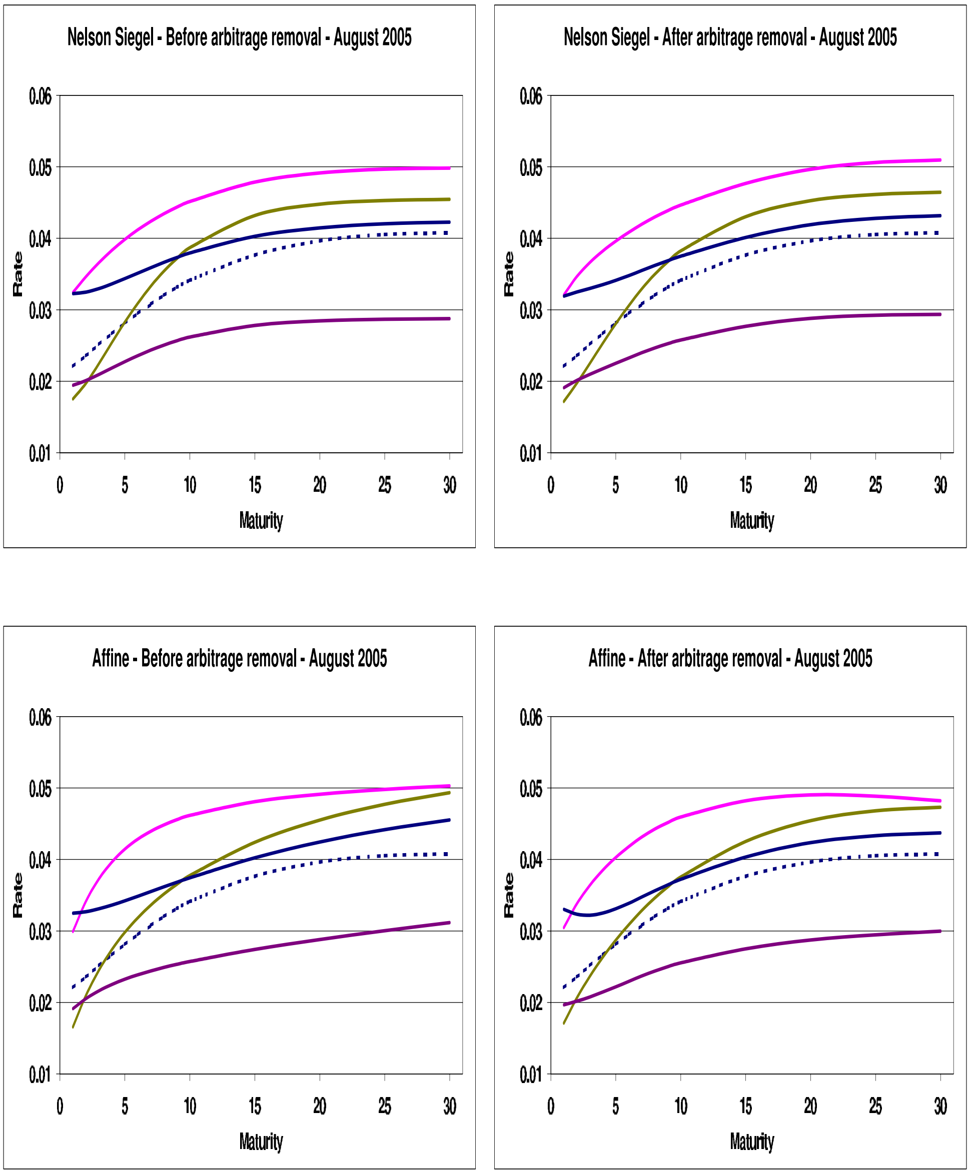


Figure E.3: Each graph includes the observed yield curve on the 1th of August 2005 (the stippled curve). Four yield curve scenarios one year ahead are included as well. In the top figures the Nelson Siegel method is used to smooth the curves. In the bottom figures an affine function is used. Figures to the left are before removing arbitrage from the yield curves and figures to the right are after removal of arbitrage.

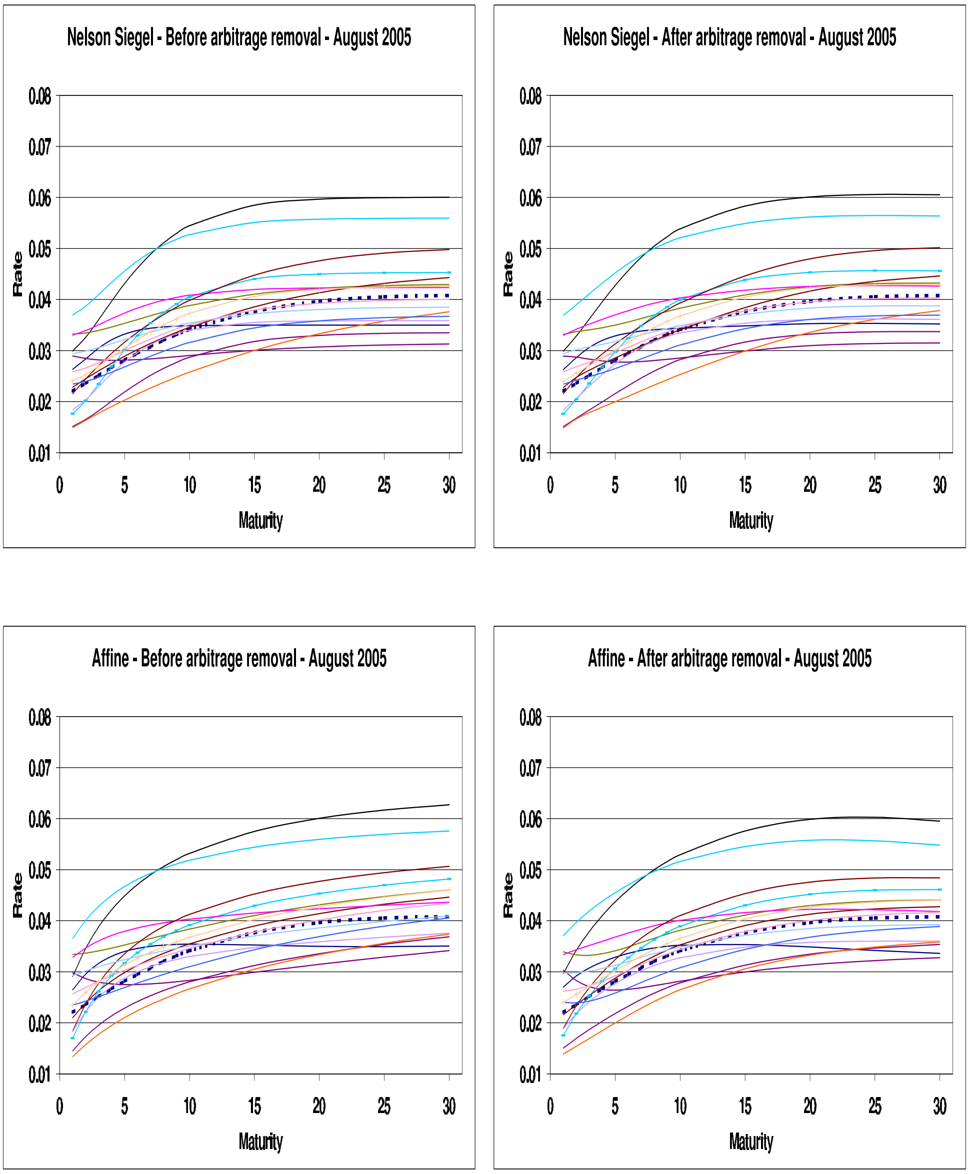


Figure E.4: Each graph includes the observed yield curve on the 1th of August 2005 (the stippled curve). 16 yield curve scenarios one year ahead are included as well. In the top figures the Nelson Siegel method is used to smooth the curves. In the bottom figures an affine function is used. Figures to the left are before removing arbitrage from the yield curves and figures to the right are after removal of arbitrage.

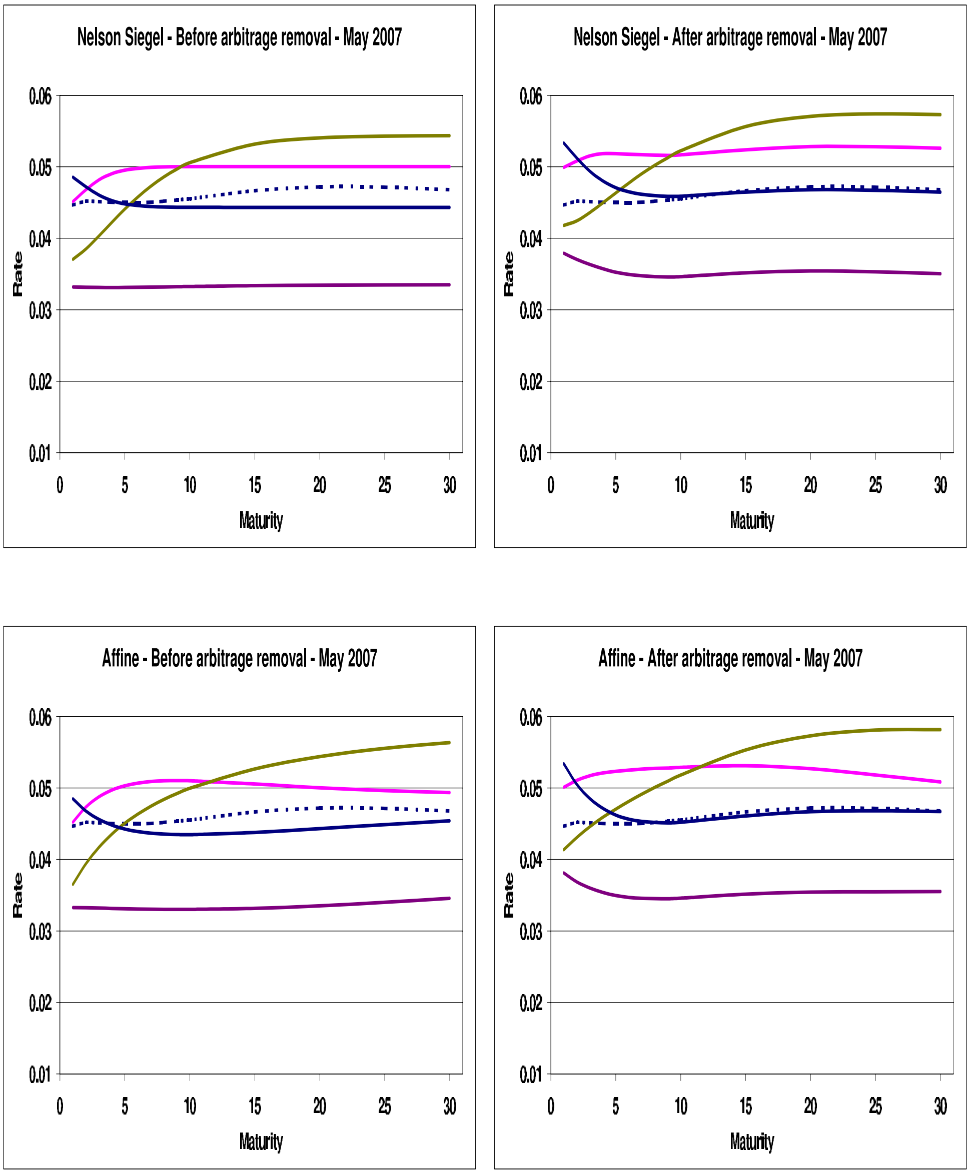


Figure E.5: Each graph includes the observed yield curve on the 1th of May 2007 (the stippled curve). Four yield curve scenarios one year ahead are included as well. In the top figures the Nelson Siegel method is used to smooth the curves. In the bottom figures an affine function is used. Figures to the left are before removing arbitrage from the yield curves and figures to the right are after removal of arbitrage.

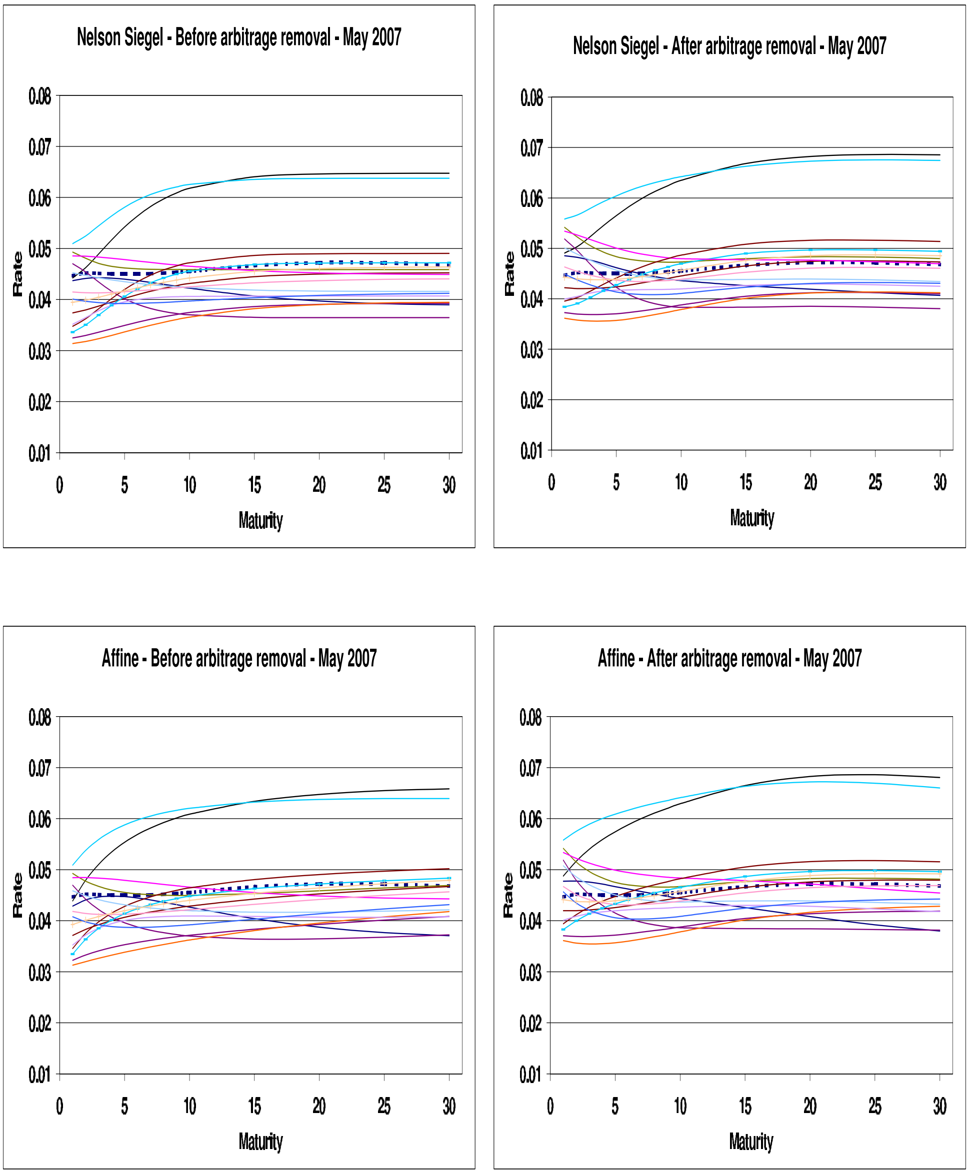


Figure E.6: Each graph includes the observed yield curve on the 1th of May 2007 (the stippled curve). 16 yield curve scenarios one year ahead are included as well. In the top figures the Nelson Siegel method is used to smooth the curves. In the bottom figures an affine function is used. Figures to the left are before removing arbitrage from the yield curves and figures to the right are after removal of arbitrage.

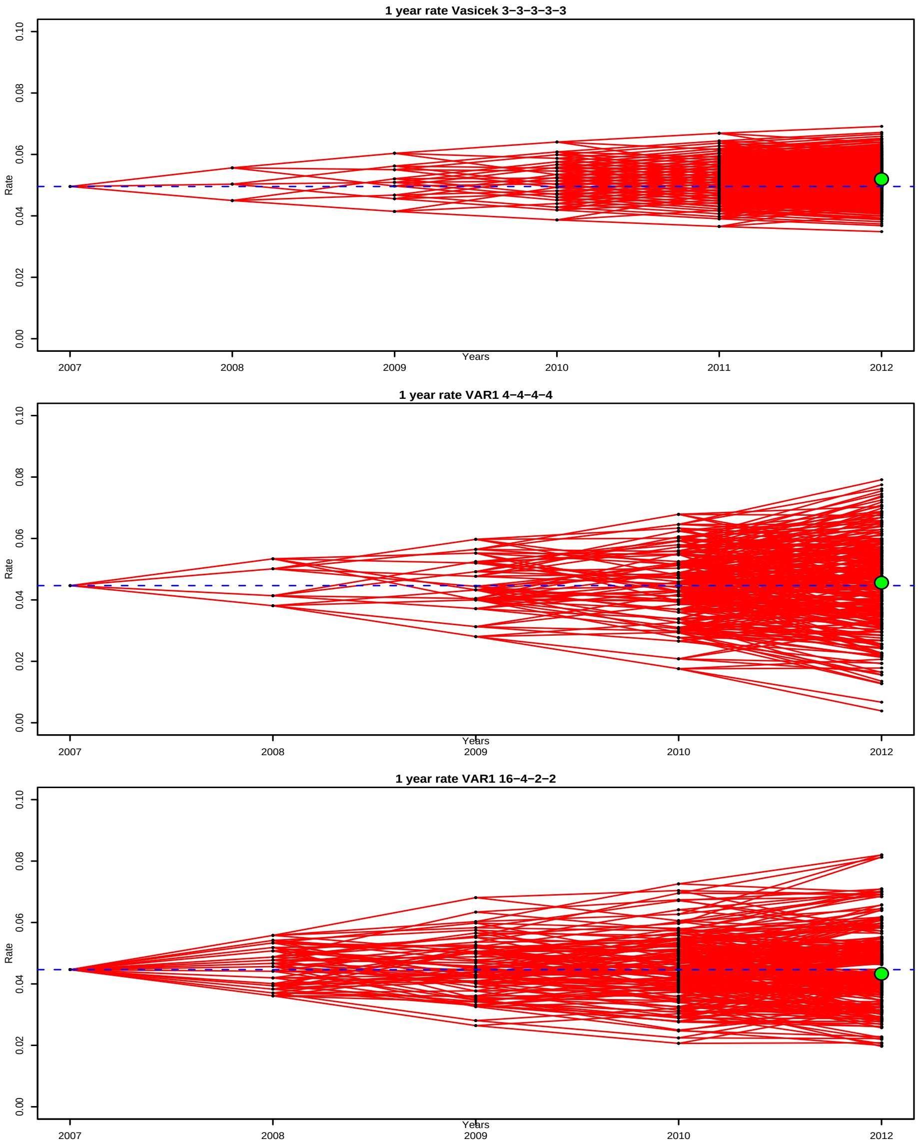


Figure E.7: Scenario trees for 1–year rates over 5 years as produced by a 1–factor Vasicek model with a 3–3–3–3–3 discretization (top), our 3–factor VAR1 model with a 4–4–4–4 discretization (middle) and our 3–factor VAR1 model with a 16–4–2–2 discretization (down). The green circle shows the average level of scenarios. Note that there is a jump from year 3 to year 5 in the VAR1 trees.

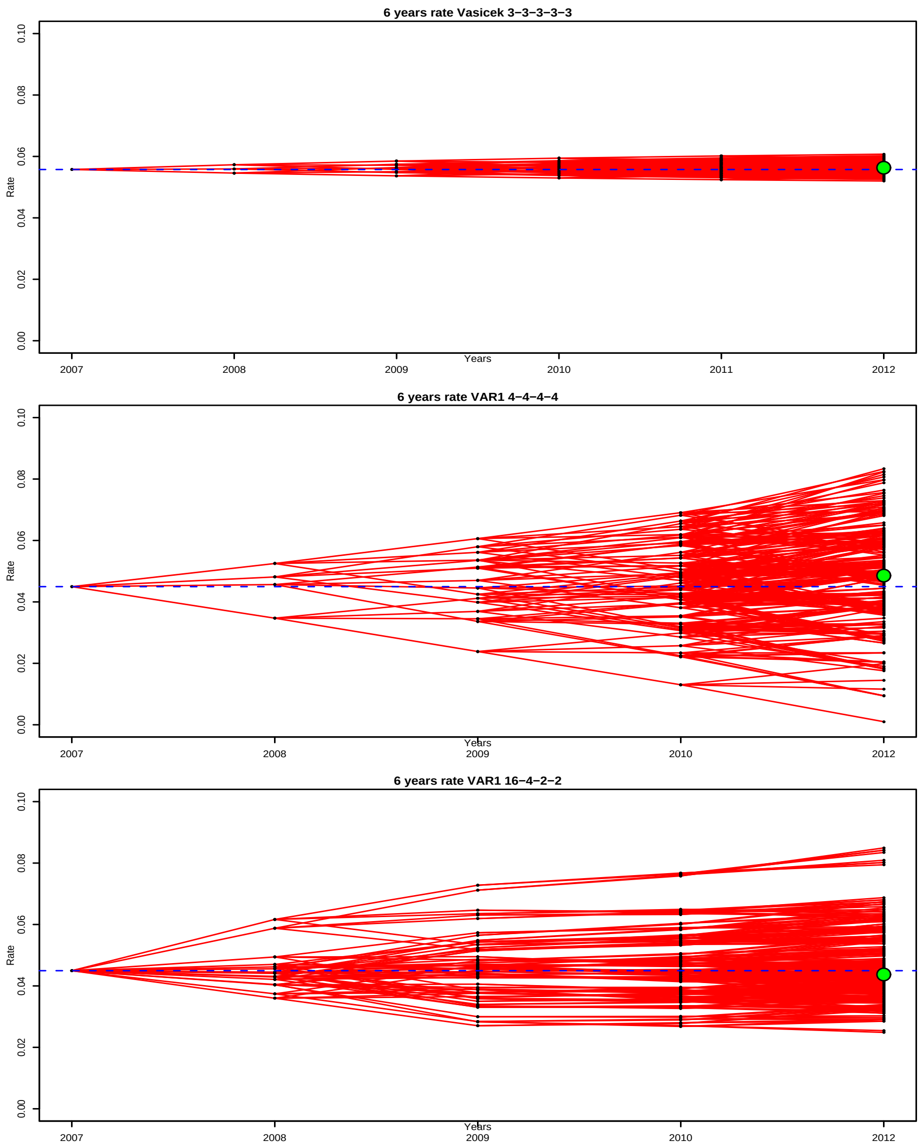


Figure E.8: Scenario trees for 6–year rates over 5 years as produced by a 1–factor Vasicek model with a 3–3–3–3–3 discretization (top), our 3–factor VAR1 model with a 4–4–4–4 discretization (middle) and our 3–factor VAR1 model with a 16–4–2–2 discretization (down). The green circle shows the average level of scenarios. Note that there is a jump from year 3 to year 5 in the VAR1 trees.

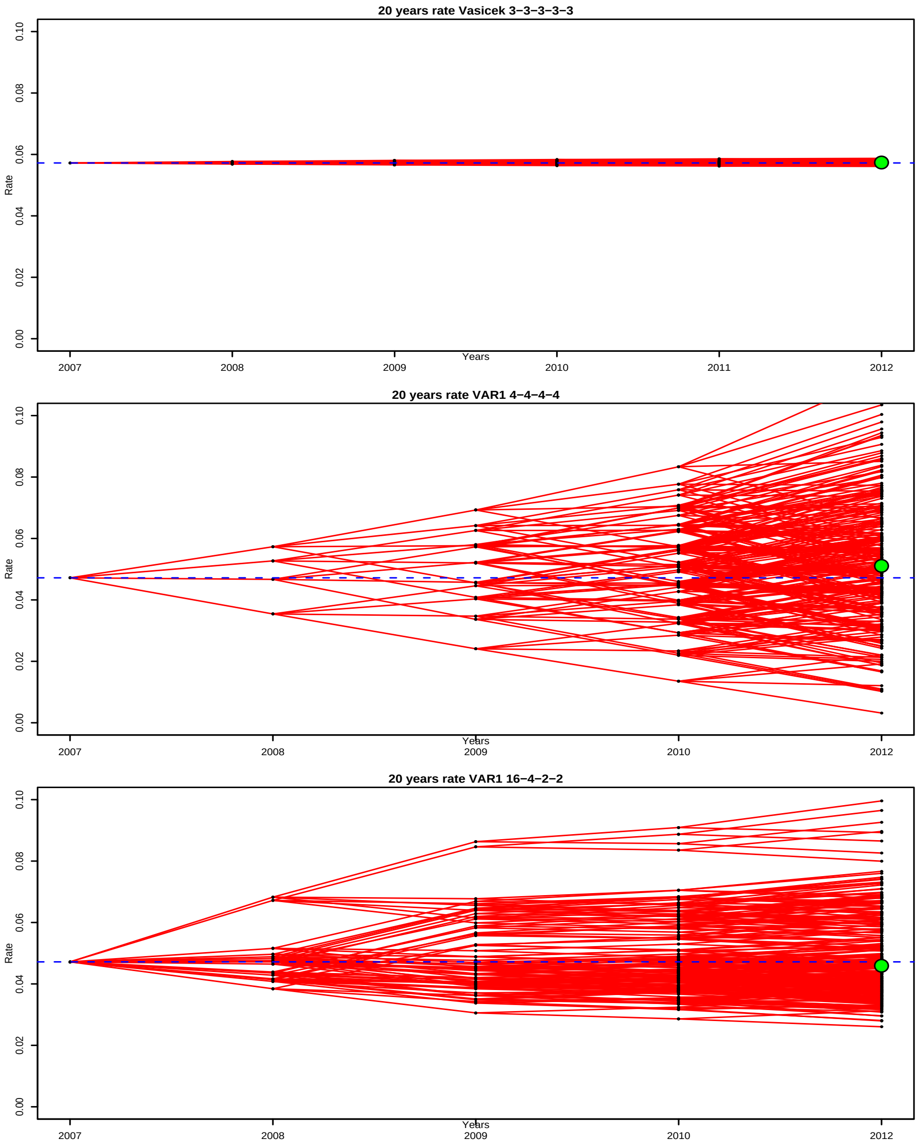
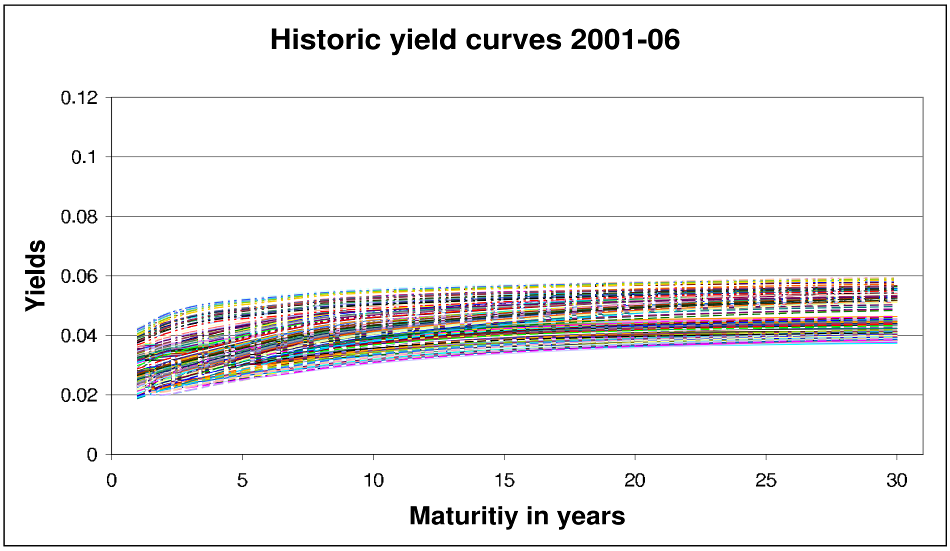


Figure E.9: Scenario trees for 20–year rates over 5 years as produced by a 1–factor Vasicek model with a 3–3–3–3–3 discretization (top), our 3–factor VAR1 model with a 4–4–4–4 discretization (middle) and our 3–factor VAR1 model with a 16–4–2–2 discretization (down). The green circle shows the average level of scenarios. Note that there is a jump from year 3 to year 5 in the VAR1 trees.



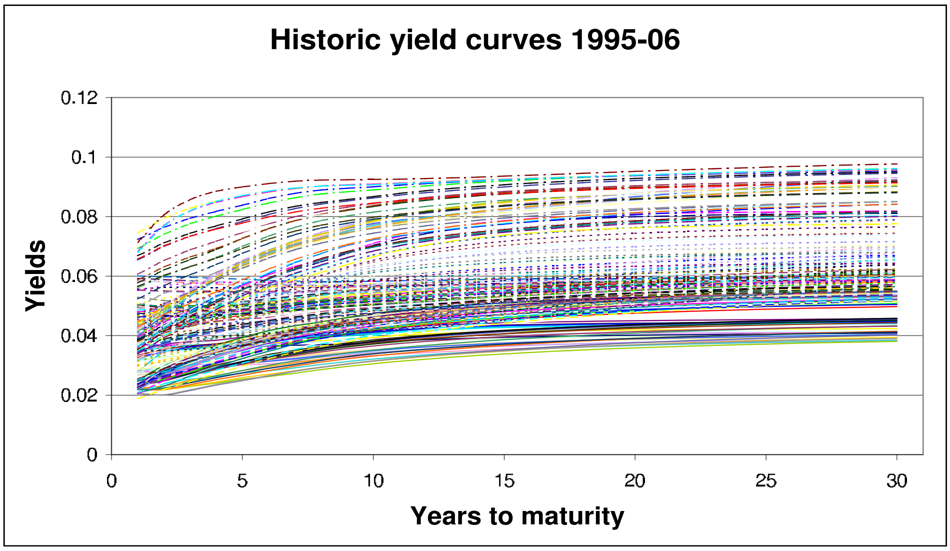
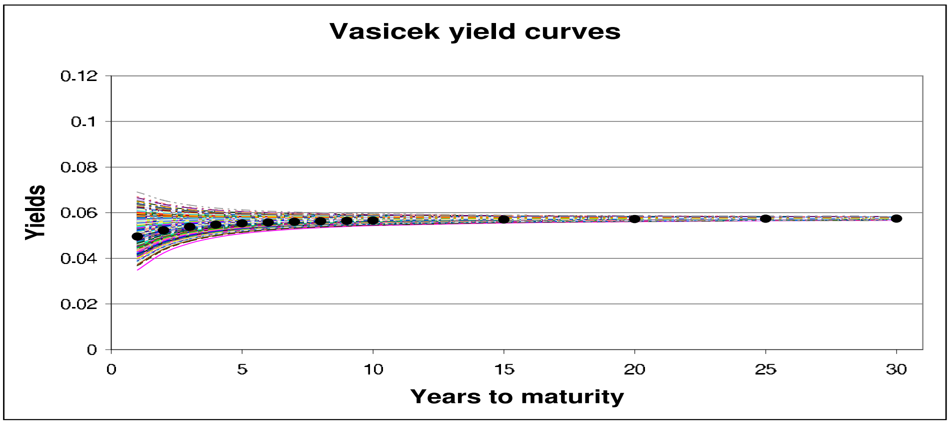
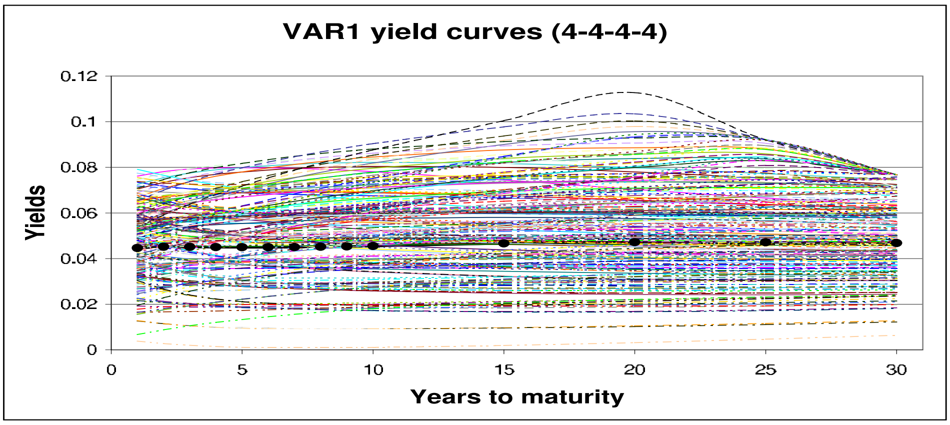


Figure E.10: Historic yield curves from 2001 to 2006 (top) and from 1995 to 2006 (down).





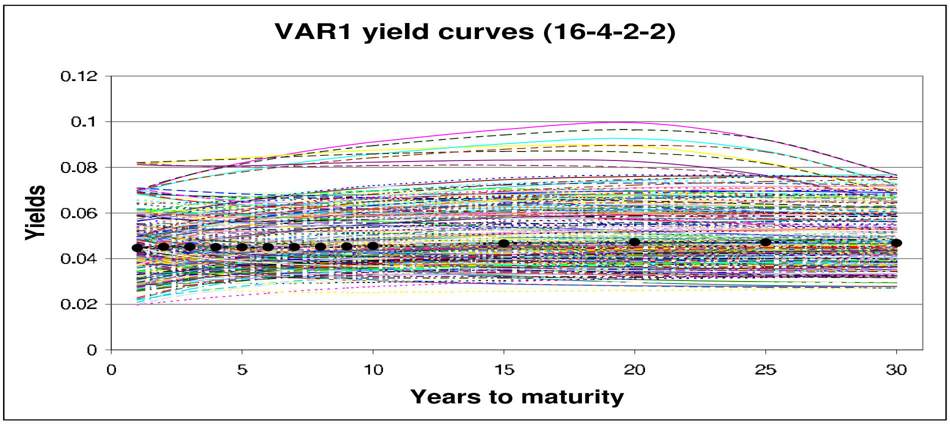
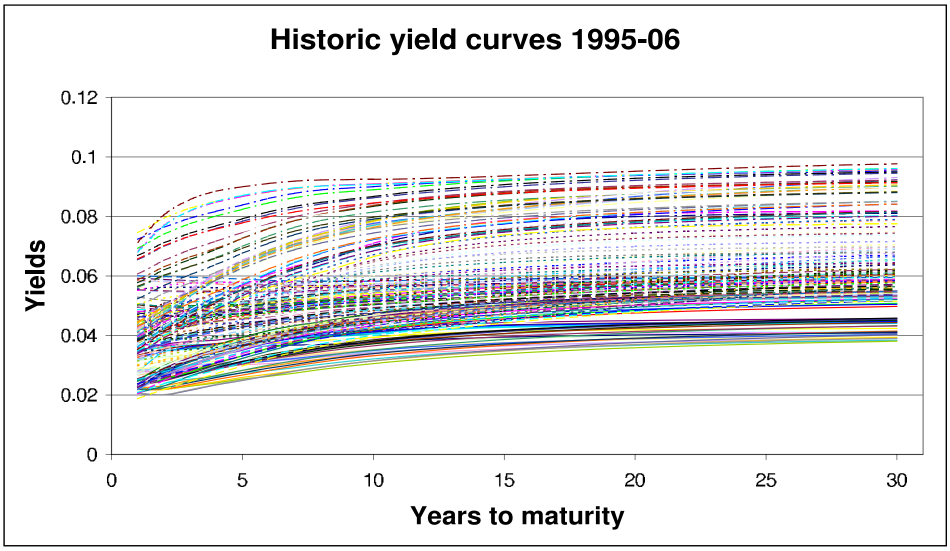


Figure E.11: Yield curves generated 5 years from now (May 2007) using the 1–factor Vasicek model with a 3–3–3–3–3 discretization (top), our VAR1 model with a 4–4–4–4 discretization (middle) and our VAR1 model with a 16–4–2–2 discretization (down). The initial yield curve is also presented using solid lines for comparison.



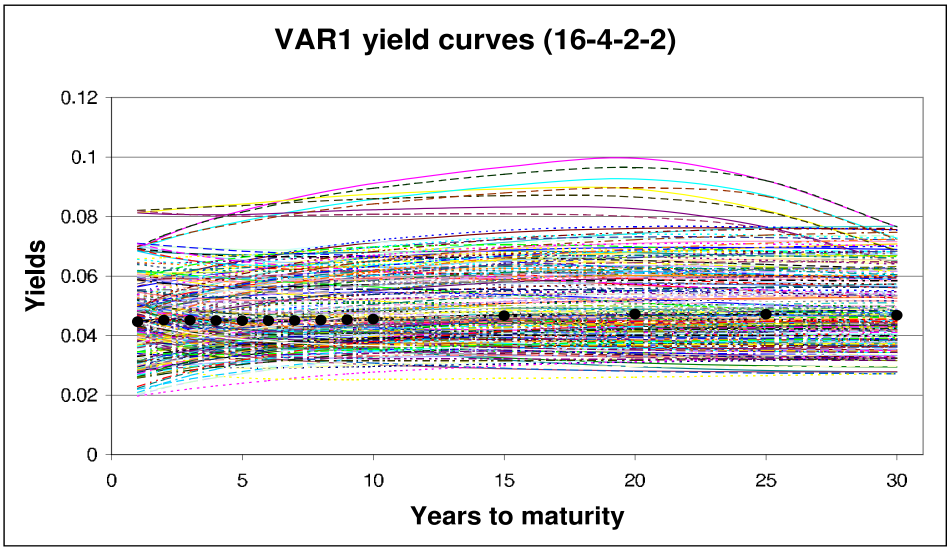


Figure E.12: Comparison of the historic yield curves from 1995 to 2006 (top) with Yield curves generated 5 years from now (May 2007) using our VAR1 model with a 16–4–2–2 discretization (down).

Appendix 1: No arbitrage arguments

It is well known in option pricing literature that pricing trees should preclude arbitrage opportunities. No arbitrage arguments go hand in hand with the risk neutral probabilities. Existence of positive risk neutral probabilities is equivalent to no arbitrage. Non positive risk neutral probabilities mean on the other hand that it is possible to go short in one asset and long in another and receive a risk free positive cashflow. It is common practice to use the no arbitrage argument for pricing purposes. The intuition is that the price of one cashflow should be used as a reference for pricing the cashflows of other assets under similar market uncertainty scenarios. This no arbitrage argument is accompanied by the assumption of rational investors and complete markets.

Some researchers within stochastic programming such as Klaassen (2002) suggest that scenario trees in stochastic programs should in the same manner as pricing trees abide by the no arbitrage conditions. The arguments for why this is a good idea are two fold:

1. If the optimization models allow the investor to enter both short and long positions then the optimization model is unbounded if arbitrage possibilities exist in the tree.

2. Even though it is not allowed to enter both short and long positions, or if due to frictions such as transaction costs or trade limitations the arbitrage possibilities would not result in unboundedness they would introduce some bias in the result.

While the arguments sound reasonable at a first reading there is no principal truth about them. The counter argument is the following:

What if the purpose of an optimization model is indeed to detect assumed arbitrage possibilities or biases in an incomplete market? Then by removing the arbitrage possibilities ex ante we are essentially removing the grounds for our optimization. Consider the extreme case when the scenario tree is made of one single scenario. This corresponds to an extremely speculative situation when the decision maker has a very clear (deterministic) opinion as to future market movements. Such a scenario almost always includes arbitrage possibilities. Yet optimization over this single scenario makes perfect sense as long as the investor has made a clear assumption as to what particular market movement he or she is willing to bet on. The point is that removing arbitrage just for the sake of precluding unboundedness or bias is not necessarily a good idea. For our speculative investor it would basically mean removing all the fun and excitement there is in betting.

In our opinion in optimization models one should make a careful decision as where to preclude arbitrage possibilities and where to allow them.

We argued in this paper that when generating a tree of interest rates one should preclude arbitrage possibilities. This basically means that we are not willing to bet on zero coupon bonds. Once the tree of interest rates is built, however, we suggest using "state of the art" pricing software (or ones favorite method for that matter) on any path of tree in order to get asset prices which are consistent with market data and our interest rate expectations. Now removing possible arbitrage from the asset prices in such a tree, by for example adding some high interest scenarios in different parts of the tree, amounts to not being loyal to our original subjective expectations on market uncertainty.

These arguments are possibly the most controversial ones in this paper, but we believe that they are at the same time an important contribution to the debate of whether or not the no arbitrage arguments should be abided by blindly in the scenario generation part of stochastic optimization.

Appendix 2: Separation of the interest rate and mortgage bond pricing models

In our papers we explicitly preclude arbitrage in the scenario trees of term structures of interest rate, i.e. we do not allow arbitrage as far as the zero coupon bonds are concerned. We then use observed mortgage bond prices in the root node and calculate future mortgage bond prices for every single path of the tree based on the pricing model of our choice. This means that we obtain realistic mortgage bond prices at every path of the scenario tree, but that arbitrage opportunities in the prices of the mortgage bonds may occur across some of the subtrees of the scenario tree. Our optimization models do not allow the mortgagor to take advantage of the possible arbitrage opportunities. The existence of arbitrage in mortgage bond prices will at worst mean that the decisions on the mortgage choice would be biased as compared to the situation where we remove these arbitrage possibilities either by changing future mortgage bond prices or by producing another term structure scenario tree. This would, in effect, amount to merging the term structure tree and the pricing model together.

In our opinion the separation of the term structure tree and the mortgage bond pricing is necessary to reflect the existing mismatch in between our subjective interest rate expectations and the observed market prices of the option elements of the mortgages. One may say that the no arbitrage case for mortgage bond prices is a special case of our setting with separate term structure trees and mortgage bond pricing. It is in this separation that we introduce subjectiveness in the choice of a term structure tree. Whether we choose one term structure tree according to a set of criteria or another term structure tree according to another set of criteria is a subjective choice. Which of the trees produce superior results in a real world setting is left as future work. In our opinion, however, it is more important to generate realistic term structure and mortgage bond price scenarios (with no arbitrage in the zero coupon bonds) than removing the mortgage bond price arbitrage by for example introducing some unrealistic interest rate scenarios.

Bibliography

Artzner, P., Delbaen, F., Eber, J.-M. & Heath, D. (1999), 'Coherent Measures of Risk', *Mathematical Finance* **9**, 203–228.

Bertocchi, M., Giacometti, R. & Zenios, S. A. (2005), 'Risk Factor Analysis and Portfolio Immunization in the Corporate Bond Market.', *European Journal of Operational Research* **161(2)**, 348–363.

Birge, J. R. (1985), 'Decomposition and partitioning methods for multi-stage stochastic linear programs.', *Operations Research* **33**(5), 989–1007.

Birge, J. R. & Louveaux, F. (1997), *Introduction to Stochastic Programming.*, Springer-Verlag New York, Inc.

Bjerksund, P. & Stensland, G. (1996), 'Implementation of the Black–Derman–Toy Interest Rate Model.', *The Journal of Fixed Income.* **6**(2), 67–75.

Björk, T. (1998), *Arbitrage Theory in Continuous Time.*, Oxford University Press.

Black, F., Derman, E. & Toy, W. (1990), 'A One–Factor Modelof Interest Rates and its Application to Treasury Bond Options.', *Financial Analysts Journal.* **January–February**, 24–32.

Brigo, D. & Mercurio, F. (2006), *Interest Rate Models: Theory and Practice – With Smile, Inflation and Credit.*, Springer–verlag, Berlin Heidelberg.

Bussieck, M. & Pruessner, A. (2003), Mixed–Integer Nonlinear Programming. Overview for GAMS Development Corporation.

Cariño, D. R., Myers, D. H. & Ziemba, W. T. (1998), 'Concepts, Technnical Issues and Uses of the Russell-Yasuda Kasai Financial Planning Model.', *Operations Research.* **46**(4), 450–462.

Censor, Y. & Zenios, S. A. (1997), *Parallel Optimization: Theory, Algorithms, and Applications.*, Oxford University Press, New York, NY. Series on Numerical Mathematics and Scientific Computing.

Chan, K. C., Karolyi, G. A., Longstaff, F. A. & Sanders, A. B. (1992), 'The Volatility of Short Term Interest Rates: An Empirical Comparison of Alternative Models of the Term structure of Interest Rates.', *Journal of Finance.* **47**, 1209–1227.

Consiglio, A., Cocco, F. & Zenios, S. A. (2002), The PROMETEIA Model for Managing Insurance Policies with Guarantees. To appear.

Constantinides, G. & Malliaris, A. G. (1995), *Portfolio Theory*, Vol. 9 of *Handbooks in Operations Research and Management Science: Finance*, Elsevier, chapter 1.

Cox, J. C., Ingersoll, J. E. & Ross, S. A. (1985), 'A Theory of the Term Structure of Interest Rates.', *Econometrica.* **53**, 385–407.

Dahl, H. (1994), 'A Flexible Approach to Interest–Rate Risk Management.', *Financial Optimization.* pp. 189–209.

Dupačová, J., Gröwe-Kuska, N. & Römisch, W. (2003), 'Scenario reduction in stochastic programming: An approach using probability metrics.', *Mathematical Programming.* **Ser. A**(95), 493–511.

Heitsch, H. & Römisch, W. (2003), 'Scenario reduction algorithms in stochastic programming.', *Computational Optimization and Applications.* **24**, 187–206.

Hochreiter, R. & Pflug, G. C. (2002), Scenario tree generation as a multidimensional facility location problem. AURORA TR2002-33.

Huang, C.-f. & Litzenberger, R. L. (1988), *Foundations of Financial Economics*, North-Holland.

Huang, D., Kai, Y., Fabozzi, F. J. & Fukushima, M. (2007), 'An optimal design of collateralized mortgage obligation with pac-companion structure using dynamic cash reserve', *European Journal of Operational Research* **177**, 1134–1152.

Hull, J. C. (2003), *Options, Futures and Other Derivatives, Fifth Edition.*, Prentice hall international.

Hull, J. C. & White, A. D. (1993), 'One–Factor Interest–Rate Models and the Valuation of Interest–Rate Derivative Securities.', *Journal of Financial and Quantitative Analysis.* **28**, 235–254.

Høyland, K., Ranberg, E. & Wallace, S. (2003), 'Developing and Implementing a Stochastic Decision-Support Model Within an Organisational Context: Part 1- The Model.', *The Journal of Risk Finance.* pp. 55–60.

Høyland, K. & Wallace, S. W. (2001), 'Generating Scenario Trees for Multistage Decision Problems.', *Management Science* **47**, 295–307.

James, J. & Webber, N. (2000), *Interest Rate Modelleing.*, John Wiley and Sons, Ltd.

Jamshidian, F. (1995), 'A simple Class of Square Root Interest Rate Models.', *Applied Mathematical Finance* **2**, 61–72.

Jensen, B. & Poulsen, R. (2002), 'Transition Densities of Diffusion Processes: Numerical Comparison of Approximation Techniques.', *Journal of Derivatives.* **9**(4), 18–32.

Kall, P. & Wallace, S. W. (1994), *Stochastic Programming.*, John Wiley & Sons, Chichester. Available from World Wide Web: ./bib/ka-wal-94.pdf.

Kaut, M. & Wallace, S. W. (2003), Evaluation of scenario generation methods for stochastic programming. Working paper, University of Molde.

King, A. J. (2002), 'Duality and martingales: a stochastic programming perspective on contingent claims.', *Mathematical Programming.* **Ser. B**(91), 543–562.

Klaassen, P. (2002), 'Comment on "Generating Scenario Trees for Multistage Decision Problems".', *Management Science.* **48**, 1512–1516.

Knez, P. J., Litterman, R. & Scheinkman, J. (1994), 'Explorations Into Factors Explaining Money Market Returns.', *Journal of Finance* **49(5)**, 1861–1882.

Litterman, R. & Scheinkman, J. (1991), 'Common Factors Affecting Bond Returns.', *The journal of fixed income.* **June**, 54–61.

Luenberger, D. (1998), *Investment Science*, Oxford.

Merton, R. C. (1990), *Continuous-Time Finance*, Blackwell.

Mulvey, J. M., Gould, G. & Morgan, C. (2000), 'An Asset and Liability Management System for Towers Perrin-Tillinghast.', *Interfaces.* **30**(1), 96–114.

Mulvey, J. M. & Zenios, S. A. (1994), 'Capturing the correlations of fixed–income instruments.', *Management Science.* **40.**, 1329–1342.

Nielsen, S. S. & Poulsen, R. (2004), 'A Two–Factor, Stochastic Programming Model of Danish Mortgage–Backed Securities.', *Journal of Economic Dynamics and Control.* **28**(7), 1267–1289.

Nielsen, S. S. & Zenios, S. A. (1996*a*), 'A stochastic programming model for funding single premium deferred annuities.', *Mathematical Programming.* **75**, 177–200.

Nielsen, S. S. & Zenios, S. A. (1996*b*), 'Solving Multistage Stochastic Network Programs on Massively Parallel Computers.', *Mathematical Programming.* **73**, 227–250.

Pennanen, T. (2004), Epi–convergent discretizations of multistage stochastic programs via integration quadratures. Working paper, Helsinki school of economics.

Pflug, G. C. (2001), 'Scenario tree generation for multiperiod financial optimization by optimal discretization.', *Math. Program., Ser. B* **89**, 251–271.

Piskorski, T. & Tchistyi, A. (2006), Optimal mortgage design. Available at SSRN: http://ssrn.com/abstract=971223.

Poulsen, R. (2007), A Gaussian Factor Model. Working paper.

Rasmussen, K. M. & Clausen, J. (2007), 'Mortgage loan portfolio optimization using multi-stage stochastic programming', *Journal of Economic Dynamics and Control* **31**, 742–766.

Rasmussen, K. M. & Zenios, S. A. (2006), Optimal Mortgage Loan Diversification. Available at http://www.imm.dtu.dk/Forskning/Soeg_publikationer.aspx.

Rasmussen, K. M. & Zenios, S. A. (2007), 'Well ARMed and FiRM: Diversification of mortgage loans for homeowners.', *The Journal of Risk* **Fall**, 1–18.

Rockafellar, R. T. & Uryasev, S. (2000), 'Optimization of conditional Value–at–Risk.', *The Journal of Risk* **2**(3), 21–41.

Rockafeller, R. T. & Uryasev, S. (2000), 'Optimization of conditional value-at-risk', *Journal of Risk* **2**, 21–41.

Roll, R. (1977), 'A critique of the asset pricing theory's tests: Part I', *Journal of Financial Economics* **4**, 129–176.

Ruszczynski, A. (1993), 'Parallel decomposition of multistage stochastic programming problems.', *Mathematical Programming.* **58**, 201–228.

Shtilman, M. S. & Zenios, S. A. (1993), 'Constructing optimal samples from a binomial lattice.', *J. Inf. Optim.* **14**, 1–23.

Svenstrup, M. & Willemann, S. (2005), Reforming Housing Finance – Perspectives from Denmark. Working paper. Available at: http://www.scanrate.dk/cms/view/Document.aspx?wid=2040&ElementID=636.

Vassiadou-Zeniou, C. & Zenios, S. A. (1996), 'Robust optimization models for managing callable bond portfolios.', *European Journal of Operational Research.* **91**, 264–273.

Zenios, S. A. (1993), 'A Model for Portfolio Management with Mortgage–backed Securities.', *Annals of Operations Research.* **43**, 337–356.

Zenios, S. A. (1995), 'Asset/liability management under uncertainty for fixed–income securities.', *Annals of Operations Research.* **59**, 77–97.

Zenios, S. A. (2007), *Practical Financial Optimization: Decision making for financial engineers.*, Blackwell, Oxford, UK.

Zenios, S. A., Holmer, M. R., McKendall, R. & C., V.-Z. (1998), 'Dynamic models for fixed–income portfolio management under uncertainty.', *Journal of Economic Dynamics and Control.* **22**, 1517–1541.

Zenios, S. A. & Kang, P. (1993), 'Mean–absolute deviation portfolio optimization for mortgage–backed securities.', *Annals of Operations Research.* **45**, 433–450.

Zenios, S. A. & Ziemba, W. T. (2006), *Asset and Liability Management. Handbook in Finance.*, North–Holland, Amsterdam, The Netherlands. (To appear).

Zhang, B. (2004), 'What kind of new asset will push up the CML?', *Insurance: Mathematics and Economics* **34**, 539–545.

Ziemba, W. T. (2003), *The Stochastic Programming Approach to Asset, Liability and Wealth Management.*, The Research Foundation of AIMR, Association for Investment Management and Research, Charlottesville, VA.