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Lemvig, Jakob; Christensen, Ole

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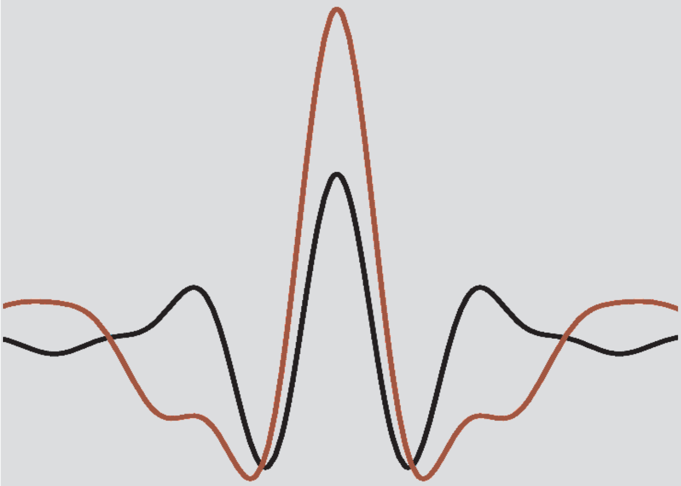

Ph.D. thesis

Jakob Lemvig

September 2008

Wavelet Frames and Their Duals

Academic advisor: Ole Christensen

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DTU Mathematics

Department of Mathematics

Wavelet Frames and Their Duals

Jakob Lemvig

Advisor: Ole Christensen

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Abstract of the Ph.D. dissertation Wavelet frames and their duals

This thesis is concerned with computational and theoretical aspects of wavelet frame analysis in higher dimensions and, in particular, with the study of so-called dual frames of wavelet frames. A frame is a system of "simple" functions or building blocks which deliver ways of analyzing signals. The signals are then represented by linear combinations of the building blocks with coefficients found by an associated frame, called a dual frame. A wavelet frame is a frame where the building blocks are stretched (dilated) and translated versions of a single function; such a frame is said to have wavelet structure. The dilation of the wavelet building blocks in higher dimension is done via a square matrix which is usually taken to be integer valued. In this thesis we step away from the "usual" integer, expansive dilation and consider more general, expansive dilations.

In most applications of wavelet frames it is essential to have a dual frame with the same structure, but this is not always the case. We explore the relationship between dual frames of a wavelet frame. We show the existence of a "nice" wavelet frame for which the canonical choice of a dual frame is not a wavelet system. At the same time, this "nice" wavelet frame has infinitely many other "nice" dual wavelet frames.

To avoid the possible lack of wavelet structure of a dual frame, we develop a construction procedure for *pairs* of dual frames which both have wavelet structure. Using this simple procedure we construct pairs of dual, bandlimited wavelet frames with good time localization and other attractive properties. Furthermore, the dual wavelet frames are constructed in such a way that we are guaranteed that both frames will have the same desirable features. The construction procedure works for any real, expansive dilation.

A quasi-affine system is a variant of the wavelet system that has been used successfully in the study of properties of wavelet systems for integer dilations. We extend the investigation of such quasi-affine systems to the class of rational, expansive dilations and introduce a new family of oversampled quasi-affine systems. We show that the wavelet system is a frame if, and only if, the corresponding family of oversampled quasi-affine systems are frames with uniform frame bounds. We also prove a similar equivalence result between pairs of dual wavelet frames and dual quasi-affine frames. We then characterize when the canonical dual frame of an oversampled quasi-affine frame is also a quasi-affine system. Finally, we uncover some fundamental differences between the integer and rational settings by exhibiting an example of a quasi-affine frame such that its wavelet counterpart is not a frame.

Resumé af ph.d.-afhandlingen Wavelet frames og deres dualer

Denne afhandling omhandler beregningsmæssige og teoretiske aspekter af wavelet-framteori i flere dimensioner og, i særdeleshed, studiet af såkaldte duale frames. En frame er et system af simple funktioner eller byggesten, som kan bruges til at analysere signaler. Signalet bliver repræsenteret ved en linearkombination af byggestenene, hvor koefficienter udregnes ved hjælp af en tilknyttet frame kaldet en dual frame. En wavelet-frame er en frame, hvor byggestenene er skalerede (dilaterede) og translaterede versioner af en enkelt funktion. Vi siger, at framen har waveletstruktur. Dilationen af wavelet-byggestenene i højere dimensioner bliver sædvanligvis udført ved en kvadratisk matrix med heltalsværdier. I denne afhandling betragtes mere generelle dilationsmatricer.

I langt de fleste anvendelser af wavelet-frames er det afgørende at være i besiddelse en dual frame med waveletstruktur, men dette er ikke altid tilfældet. Vi undersøger forholdet mellem dualer af en wavelet-frame. Vi viser, at der eksisterer wavelet-frames, for hvilke det kanoniske valg af dual ikke har waveletstruktur, men hvor der findes uendeligt mange alternative waveletdualer.

For at undgå problemer med manglende waveletstruktur af en dual frame udvikles en metode til konstruktion af *par* af duale frames, hvor begge frames har wavelet-struktur. Vi konstruerer par af duale, båndbegrænsede wavelet-frames med attraktive egenskaber. De duale wavelet-frames konstrueres endvidere således, at begge frames vil have samme gode egenskaber. Konstruktionsproceduren virker for alle reelle, ekspansive dilationsmatricer.

Quasi-affine systemer er en variation af det almindelige waveletsystem, der normalt benyttes i studiet af waveletsystemer for heltalsdilationer. Vi udvider studiet af sådanne quasi-affine systemer til klassen af rationelle, ekspansive dilationer og introducerer en ny familie af oversamplede quasi-affine systemer. Vi viser, at et waveletsystem er en frame, hvis og kun hvis den tilsvarende familie af oversamplede quasi-affine systemer er frames med uniforme framegrænser. Vi beviser også lignende ækvivalensresultater for par af duale wavelet-frames og par af duale quasi-affine frames. Desuden karakteriserer vi, hvornår den kanoniske dual af en oversamplet quasi-affin frame også er et quasi-affint system. Endeligt afdækker vi nogle fundamentale forskelle mellem den rationelle og den heltallige situation ved at give et eksempel på en quasi-affin frame, hvis tilhørende waveletsystem ikke er en frame.

Preface

This thesis is submitted in partial fulfillment of the requirements for obtaining the Ph.D. degree. The work has been carried out in the Department of Mathematics at Technical University of Denmark from August 2005 to September 2008 under the supervision of Professor Ole Christensen.

Acknowledgements

First and foremost, I thank my supervisor Ole Christensen for his guidance and advice.

During my two trips abroad I was hosted by the Department of Mathematics at the University of Oregon. I would like to thank the people at the Department of Mathematics and my friends in Eugene, the Bownik and Williams families, for their hospitality and for welcoming me. In particular, I thank my coauthor Marcin Bownik for inspiration and many discussions.

Finally, I thank Tom Høholdt and the people at DTU Mathematics.

Overview

This thesis deals with computational and theoretical aspects of wavelet frames and their duals. The thesis consists of one introductory chapter and four research papers. Two of the four papers are joint work with Marcin Bownik from University of Oregon.

Chapter 1 provides a general introduction to wavelet frame analysis and a survey of the new results presented in this thesis. The main text splits naturally into three parts: Paper I (and Appendix B in Chapter 1) on theoretical aspects of wavelet frames and their duals, Papers II and III on computational aspects of wavelet frames and dual frames and, in particular, on the construction of dual wavelet frames, and Paper IV on theoretical aspects of wavelet and quasi-wavelet frames. The introductory chapter, Chapter 1, contains two appendices: Appendix A on expansive matrices and lattices in $\mathbb{R}^n$ and Appendix B on an example of a non-biorthogonal Riesz wavelet.

The four research papers are listed below. Papers I and II have been published.

Paper I Marcin Bownik and Jakob Lemvig. *The canonical and alternate duals of a wavelet frame.* Applied and Computational Harmonic Analysis 23(2):263–272, 2007.

Paper II Jakob Lemvig. *Constructing pairs of dual bandlimited framelets with desired time localization.* Advances in Computational Mathematics doi:10.1007/s10444-008-9066-7. Appeared online May 2008.

Paper III Jakob Lemvig. *Constructing pairs of dual bandlimited frame wavelets in $L^2(\mathbb{R}^n)$.* Manuscript August 2008.

Paper IV Marcin Bownik and Jakob Lemvig. *Affine and quasi-affine frames for rational dilations.* Submitted September 2008.

A note to the reader

The papers are presented without modifications from the journal or preprint version. References to sections, theorems, equation numbers, *etc.* will therefore be references within the paper or chapter itself unless otherwise noted. References in Chapter 1 to, e.g., a theorem in one of the papers will be of the form "Theorem 2.3 in Paper II" or "Theorem II.2.3". This convention also means that the list of references is placed at the end of each chapter or paper.

Kgs. Lyngby, Jakob Lemvig
September 2008.

Contents

CHAPTER 1

Introduction

The first section, Section 1, is a brief introduction to wavelet frame analysis and the research presented in this thesis. Section 2 is a review of mathematical definitions central to the thesis. The last section of this introductory chapter is a survey of the new scientific results obtained in the four papers [6, 7, 25, 26] which are presented in the thesis as Papers I, II, III, and IV. The survey is found in Section 3.

1. Motivation

The traditional Fourier analysis yields some of the most versatile methods in engineering, and it is used in almost every branch of engineering. Wavelet analysis is a modern alternative to Fourier methods; it has its origin in mathematics, quantum physics, electrical engineering, and seismic geology.

Wavelet frames are a redundant version of the standard wavelet transform; the redundancy implies that we use more data than strictly necessary to describe the signal; the redundancy, or surplus of data, acts as our safety net in cases of corruption or loss of data.

The principal objectives in signal processing techniques encompass compression and analysis of signals by representing these in terms of convenient building blocks. In particular, we want expansions of a signal f of finite energy, i.e., $f \in L^2(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : \int_{\mathbb{R}^n} |f(x)|^2 \mathrm{d}x < \infty\}$,

$$f(x) = \sum_{k \in I} c_k f_k(x) \qquad \text{in } L^2(\mathbb{R}^n),$$

where the functions $f_k \in L^2(\mathbb{R}^n)$ are our basic building blocks. The coefficient c_k should be straightforward to calculate; in most applications it is also crucial that there are only few important coefficients $\{c_k\}$. In wavelet analysis the building blocks $\{f_k\}$ have a particular structure: they are stretched (dilated) and translated versions of a single "oscillating" function.

In the standard approach for expressing signals in terms of wavelet building blocks, one lets the building blocks form an orthonormal basis. However, the basis requirement can be so restrictive on the building blocks that we sometimes have to give up on desirable properties. One way to overcome this issue is to replace the basis approach with the more general approach of frames. Frames generalize the notion of bases in such a way that we obtain much more flexibility in the construction of our building blocks $\{f_k\}$ and more freedom in the choice of the coefficients $\{c_k\}$ and yet still so restrictive that the numerical stability of the bases approach is preserved. For frames the coefficients $\{c_k\}$ are found by using a so-called dual frame, either the canonical dual or an alternate dual frame.

One of the motivations of the research presented in this thesis was to step away from the "standard" wavelet systems with integer, expansive dilations and examine computational and theoretical aspects of wavelet systems with general expansive dilations. It is not only of theoretical interest to consider non-integer dilations since non-integer settings in some cases allow for a favourable, dense sampling of the time-frequency plane. The standard fast wavelet algorithm from multiresolution analysis breaks down for rational dilations, but Selesnick and Bayram [1] recently developed a redundant discrete frame wavelet transform based on non-integer, rational dilations, see also [22].

Frames are a generalization of orthonormal bases, hence the major reason to consider wavelet frames in place of orthonormal wavelet bases is to obtain more flexibility and freedom. The representation of a function or signal in terms of a frame involves either the canonical dual or an alternate dual. But the canonical dual of a wavelet frame need not have wavelet structure; and even worse, there might not be any (canonical or alternate) dual with wavelet structure. Since only wavelet frames with wavelet duals are useful (e.g., from the point of view that the fast wavelet algorithm will not be available for either the analysis or synthesis of the signal if there is no wavelet dual), the freedom, in these cases, ends up being deceptive. This, among other things, motivated the joint work with Marcin Bownik in Paper I on the relationship between the canonical dual and alternate duals. In particular, we show that "nice" wavelet frames can have many "nice" alternate dual wavelet frames and at the same time a canonical dual which is not even a wavelet system. Hence, when working with the canonical dual one has to pay close attention to the structure of this dual. In Paper II and III the possible lack of wavelet structure of dual frames is avoided altogether by constructing pairs of (non-canonical) dual *wavelet* frames. The generators of this pair of dual frames are given in a very explicit way and have attractive properties.

To improve results in applications involving multidimensional data the undecimated wavelet transform is sometimes preferred to the standard wavelet transform, see for example [10]. This approach adds shift invariance and redundancy to the algorithm; indeed, the associated algorithm is a frame wavelet decomposition algorithm without down sampling. The associated theoretical tool is the so-called quasi-wavelet (also called quasi-affine) system which is a shift invariant counterpart of the wavelet (also called affine) system. In Paper IV with Marcin Bownik we initiate the study of such systems and their oversampled counterpart in multiple dimensions for rational, expansive dilations. We prove equivalence results between affine and quasi-affine systems, and we characterize quasi-affine frames whose canonical dual frame takes the form of a quasi-affine system. Equivalent results on affine and quasi-affine systems are useful because they, in the study of wavelet systems, allow us to replace the dilation invariant wavelet system with the much simpler shift invariant quasi-affine system.

2. Preliminaries and notation

2.1. Frames in Hilbert spaces

We are concerned with series expansions in separable Hilbert spaces. So, let $\mathcal{H}$ be a separable Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ linear in the first entry. Our central definition is that of a frame for $\mathcal{H}$.

Definition 1. A *frame* is a countable collection of vectors $\{f_j\}_{j \in indexsetJ}$ such that there are constants $0 < C_1 \leq C_2 < \infty$ satisfying

$$C_1 \|f\|^2 \leq \sum_{j \in \mathcal{J}} |\langle f, f_j \rangle|^2 \leq C_2 \|f\|^2 \qquad \text{for all } f \in \mathcal{H}. \tag{2.1}$$

If only the upper bound in the inequality (2.1) holds, then $\{f_j\}$ is said to be a *Bessel sequence* with Bessel constant C_2.

For a Bessel sequence $\{f_j\}$, we define the frame operator of $\{f_j\}$ by

$$S \colon \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{j \in \mathcal{J}} \langle f, f_j \rangle f_j.$$

If $\{f_j\}$ is a frame, this operator is bounded, invertible, and positive. A frame $\{f_j\}$ is said to be *tight* if we can choose $C_1 = C_2$; this is equivalent to $S = C_1 I$, where I is the identity operator on $\mathcal{H}$. If furthermore $C_1 = C_2 = 1$, the sequence $\{f_j\}$ is said to be a *Parseval frame.*

Two Bessel sequences $\{f_j\}$ and $\{g_j\}$ are said to be *dual frames* if

$$f = \sum_{j \in \mathcal{J}} \langle f, g_j \rangle f_j \qquad \text{for all } f \in \mathcal{H}.$$

It can be shown that two such Bessel sequences indeed are frames, and we shall say that the frame $\{g_j\}$ is *dual* to $\{f_j\}$, and vice versa. At least one dual always exists, it is given by $\{S^{-1} f_j\}$ and called the *canonical dual.* A frame that is also a Schauder basis is called a *Riesz basis.* A frame that is not a Schauder basis is called a *redundant frame.* Redundant frames have several duals; a dual which is not the canonical dual is called an *alternate* dual.

2.2. Wavelet frames in $L^2(\mathbb{R}^n)$

Wavelet frames are frames with a dilation and translation structure in $\mathcal{H} = L^2(\mathbb{R}^n)$. Fix $n \in \mathbb{N}$, and let $f \in L^2(\mathbb{R}^n)$. The translation by $y \in \mathbb{R}^n$ is $T_y f(x) = f(x-y)$; dilation by an $n \times n$ non-singular matrix B is $D_B f(x) = |\det B|^{1/2} f(Bx)$; modulation by $b \in \mathbb{R}^n$ is $E_b f(x) = \mathrm{e}^{2\pi i \langle b, x \rangle} f(x)$. For $f \in L^1(\mathbb{R}^n)$, the Fourier transform is defined by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-2\pi i \langle \xi, x \rangle} \mathrm{d}x$$

with the usual extension to $L^2(\mathbb{R}^n)$. These four operations are unitary as operators on $L^2(\mathbb{R}^n)$, and they play a key role in wavelet analysis. The commutator relations below will be used repeatedly. For $k \in \mathbb{R}^n$, $j \in \mathbb{Z}$ and $\widetilde{B} = P^{-1} B P$ for some $P \in GL_n(\mathbb{R})$, we have

$$T_k D_B = D_B T_{Bk}, \qquad D_B \mathcal{F} = \mathcal{F} D_{(B^t)^{-1}}, \qquad D_{\widetilde{B}^j} D_P = D_P D_{B^j}. \tag{2.2}$$

The local commutant of a system of operators $\mathcal{U}$ at the point $f \in L^2(\mathbb{R}^n)$ is defined as

$$\mathcal{C}_f(\mathcal{U}) := \left\{T \in B(L^2(\mathbb{R}^n)) : TUf = UTf \quad \forall U \in \mathcal{U}\right\}.$$

A (full-rank) lattice Γ in $\mathbb{R}^n$ is a point set of the form $\Gamma = P\mathbb{Z}^n$ for some $P \in GL_n(\mathbb{R})$. The determinant of Γ is $d(\Gamma) = |\det P|$; note that the generating matrix P is not unique and $d(\Gamma)$ is independent of the particular choice of P. We refer to Appendix A.2 for more facts on lattices in $\mathbb{R}^n$.

Let $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$, let Γ be a lattice in $\mathbb{R}^n$, and let A be a fixed $n \times n$ expansive matrix, i.e., all eigenvalue λ of A satisfy $|\lambda| > 1$. The wavelet (or affine) system of unitaries $\mathcal{A}$ associated with the dilation A and translation lattice Γ is defined as $\mathcal{A} = \{D_{A^j}T_\gamma : j \in \mathbb{Z}, \gamma \in \Gamma\}$. The wavelet system $\mathcal{A}(\Psi)$ generated by Ψ is defined as

$$\mathcal{A}(\Psi) = \{\psi_{j,\gamma} : j \in \mathbb{Z}, \gamma \in \Gamma, \psi \in \Psi\}, \tag{2.3}$$

where

$$\psi_{j,\gamma} := D_{A^j}T_\gamma\psi = |\det A|^{j/2}\, \psi(A^j \cdot - \gamma) \qquad \text{for } j \in \mathbb{Z}, \gamma \in \Gamma.$$

If we need to stress the dependence of the underlying dilation matrix A and translation lattice Γ, we say that the wavelet system $\mathcal{A}(\Psi)$ is associated with (A, Γ), or we use the notation $\mathcal{A}(\Psi, A, \Gamma)$ for (2.3).

We say that Ψ is a *frame wavelet* if $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$, and say that Ψ and Φ is a *pair of dual frame wavelets* if their wavelet systems are dual frames. We usually denote the transpose of the (fixed) dilation matrix A by $B = A^t$.

A generalized multiresolution analysis (GMRA) is a sequence $\{D_{A^j}(V)\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ with the following four properties:

(a) $V \subseteq D_A(V)$,
(b) $\overline{\cup_{j\in\mathbb{Z}} D_{A^j}(V)} = L^2(\mathbb{R}^n)$,
(c) $\cap_{j\in\mathbb{Z}} D_{A^j}(V) = \{0\}$,
(d) $T_\gamma V \subseteq V$ for all $\gamma \in \Gamma$.

Whenever condition (d) is satisfied, we say that V is *shift invariant* with respect to Γ. A frame wavelet Ψ is said to be associated with a GMRA if its space of negative dilates

$$V(\Psi) := \overline{\operatorname{span}}\{\psi_{j,\gamma} : j < 0,\ \gamma \in \Gamma\} \tag{2.4}$$

satisfies conditions (a)–(d) with $V = V(\Psi)$.

Finally, the Gabor system generated by Ψ is defined as $\{E_\lambda T_\gamma \psi : \lambda \in \Lambda, \gamma \in \Gamma, \psi \in \Psi\}$ for lattices Λ and Γ in $\mathbb{R}^n$.

A note on the dilation matrix and the translation lattice

In general our only requirement on the dilation matrix $A \in GL_n(\mathbb{R})$ is that it is expansive, in other words, that it has eigenvalues strictly greater than one in absolute value (see Appendix A.1 for a list of equivalent conditions). However, we will sometimes put further restrictions on A (or Γ). In particular, we will consider the following cases: the lattice preserving dilation, i.e., $A\Gamma \subset \Gamma$, and the rank preserving dilation, i.e., the intersection $A\Gamma \cap \Gamma$ is a full-rank lattice. It is obvious that lattice preserving dilations are rank preserving.

Furthermore, it is usually not necessary to consider arbitrary translation lattices Γ, and one is often able to restrict attention to the standard translation lattice $\mathbb{Z}^n$. Indeed, for $A \in GL_n(R)$ expansive and $\Gamma = P\mathbb{Z}^n$ for some $P \in GL_n(\mathbb{R})$ consider the wavelet system $\mathcal{A}(\Psi, A, \Gamma)$. By the commutator relations (2.2), we see

$$\mathcal{A}\Big(D_P\Psi, \widetilde{A}, \mathbb{Z}^n\Big) = D_P\left(\mathcal{A}(\Psi, A, \Gamma)\right), \tag{2.5}$$

where the matrix $\widetilde{A} := P^{-1}AP$ is similar to A. Observe that the set of all matrices similar to an expansive matrix is precisely the set of all expansive matrices. Since D_P is unitary, properties such as the frame and Bessel property carry over between the two systems. Hence, in these cases it is possible to reduce studies of wavelet systems with general translation lattice to the setting of integer lattice.

Therefore, we can without loss of generality usually restrict attention to wavelet systems associated with $(A, \mathbb{Z}^n)$, i.e.,

$$\mathcal{A}(\Psi) = \{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \psi \in \Psi\},$$

as is the case in Paper IV. Moreover, whenever we take $\Gamma = \mathbb{Z}^n$, lattice preserving dilations simply mean integer dilations $A \in GL_n(\mathbb{Z})$ and rank preserving dilations simply mean rational dilations $A \in GL_n(\mathbb{Q})$. This is a simple consequence of the following two facts:

1) $A \in GL_n(\mathbb{Z}) \Leftrightarrow A\mathbb{Z}^n \subset \mathbb{Z}^n$
2) $A \in GL_n(\mathbb{Q}) \Leftrightarrow A\mathbb{Q}^n \subset \mathbb{Q}^n \Leftrightarrow A\mathbb{Z}^n \cap \mathbb{Z}^n$ has full rank.

Of course, when we reduce our study to the standard translation lattice $\mathbb{Z}^n$, we need to recall that, e.g., a result on rational dilations and translation lattice $\mathbb{Z}^n$ actually is a result on rank preserving dilations and general translation lattices Γ.

Nevertheless, in Paper III, we actually do consider the general case of wavelet systems associated with (A, Γ) for arbitrary Γ. The reason is that we, in this paper, want to construct pairs of dual wavelet frames for some *given* expansive dilation $A \in GL_n(\mathbb{R})$. Of course, we can still apply the reduction step in (2.5), but this changes the dilation matrix A (to $\widetilde{A}$).

3. Survey of the new results

The following section is a survey of the new results and their relation to known results.

3.1. Canonical and alternate duals of a wavelet frame (Paper I)

Let $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$. The canonical dual frame of a Gabor frame $\{E_\lambda T_\gamma \Psi\}$ always takes the form of a Gabor system. In other words, the canonical dual frame is of the form $\{E_\lambda T_\gamma \Phi\}$ for some $\Phi = \{\phi_1, \dots, \phi_L\} \subset L^2(\mathbb{R}^n)$. Consequently, in Gabor analysis, the frame and its canonical dual frame are always systems of functions with the same structure. This is *not* the case for wavelet frames. Indeed, Daubechies [17] and Chui and Shi [14] proved that the canonical dual of a wavelet Riesz basis need not have wavelet structure. Hence, in particular, the canonical dual frame of a wavelet frame need not be a wavelet system. In Paper I with Marcin Bownik we explore the relationship between wavelet structure of canonical and alternate dual frames of a wavelet frame.

The canonical dual of a wavelet frame $\mathcal{A}(\Psi) = \{D_{A^j} T_k \psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,\psi\in\Psi}$ is given as

$$\begin{aligned}\left\{S^{-1} D_{A^j} T_k \psi_i\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,i\in\{1,\dots,L\}} &= \left\{D_{A^j} S^{-1} T_k \psi_i\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,i\in\{1,\dots,L\}} \\ &= \left\{D_{A^j} \eta^{k,i}\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,i\in\{1,\dots,L\}},\end{aligned}$$

where S is the frame operator of $\mathcal{A}(\Psi)$, and $\{\eta^{k,i}\}$ is a family of functions, not necessarily with translation structure, indexed by $\{1,\dots,L\} \times \mathbb{Z}^n$. In the calculations above we used that the frame operator commutes with dilation; the calculations show that we only need to worry about the structure of the canonical dual on one scale, e.g., $j = 0$.

The canonical dual takes the form of a wavelet system generated by $|\Psi| = L$ functions, i.e.,

$$\begin{aligned}\left\{S^{-1} D_{A^j} T_k \psi_i\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,i\in\{1,\dots,L\}} &= \left\{D_{A^j} T_k (S^{-1}\psi_i)\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,i\in\{1,\dots,L\}} \\ &= \{D_{A^j} T_k \phi_i\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,i\in\{1,\dots,L\}},\end{aligned}$$

precisely when $T_k S^{-1}\psi = S^{-1} T_k \psi$ for all $\psi \in \Psi$ and $k \in \mathbb{Z}^n$; that is, precisely when $S^{-1} \in \mathcal{C}_\psi(\{T_k : k \in \mathbb{Z}^n\})$ for all $\psi \in \Psi$. Observe that the local commutant $\mathcal{C}_\psi(\{T_k : k \in \mathbb{Z}^n\})$ is likely to be a lot bigger than the commutant $\{T_k : k \in \mathbb{Z}^n\}'$.

One of the major open problems concerning the canonical dual of a wavelet frame is to give a characterization of those wavelet frames having a canonical dual with wavelet structure. One result in this direction is due to Bownik and Weber [8] who showed that if the canonical dual of a wavelet frame has the wavelet structure with the same number of generators, then the space of negative dilates is shift invariant:

Theorem 3.1 (Theorem 1 in [8]). *Let $A \in GL_n(\mathbb{Z})$ be expansive and $\Psi = \{\psi_1,\dots,\psi_L\} \subset L^2(\mathbb{R}^n)$. Suppose that the canonical dual of a wavelet frame $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \psi \in \Psi\}$ has a wavelet structure, i.e., it is of the form $\{\phi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \phi \in \Phi\}$ for some frame wavelet $\Phi = \{\phi_1,\dots,\phi_L\}$. Then, the space of negative dilates*

$$V(\Psi) = \overline{\operatorname{span}}\{\psi_{j,k} : j < 0,\ k \in \mathbb{Z}^n, \psi \in \Psi\}$$

is shift invariant with respect to $\mathbb{Z}^n$.

Remark 1. For a Riesz wavelet Ψ the other direction also holds, i.e., shift invariance of $V(\Psi)$ implies wavelet structure of the canonical dual.

This result gives us a necessary condition for the canonical dual of a wavelet frame to have wavelet structure, but the characterization problem is still open, even for dyadic dilation in one dimension, i.e., $A = 2$.

Now, let us take a closer look at the example of Daubechies [17] and Chui and Shi [14] that exhibits a wavelet Riesz basis whose canonical dual is not a wavelet system. Let $\psi \in L^2(\mathbb{R})$ be the generator of an orthonormal wavelet basis in $L^2(\mathbb{R})$ with dyadic dilation. Define η as a perturbation of ψ

$$\eta(x) = \psi(x) + \varepsilon 2^{1/2}\psi(2x) \equiv \psi(x) + \varepsilon D_2\psi(x) \qquad \text{for } x \in \mathbb{R}, \tag{3.1}$$

for some fixed $0 < \varepsilon < 1$. In [14] it is shown that the function η generates a wavelet Riesz basis $\{D_{2^j}T_k\eta\}_{j,k\in\mathbb{Z}}$ whose (canonical) dual is not of the form $\{D_{2^j}T_k\phi\}$ for any $\phi \in L^2(\mathbb{R})$. This argumentation can be extended to show that the canonical dual $\{S^{-1}\eta_{j,k}\}$ is not of the form

$$\{D_{2^j}T_k\phi : j,k \in \mathbb{Z}, \phi \in \Phi\}$$

for any finite set $\Phi \subset L^2(\mathbb{R})$ of generators, see Appendix B.1.

For this last statement to make sense, we need to explain precisely what is understood by a wavelet frame with a canonical dual frame with more generators than the frame itself. For a pair of dual frames $\{\psi_{j,k}\}$ and $\{\phi_{j,k}\}$ in $L^2(\mathbb{R})$ we have a representation of elements in $L^2(\mathbb{R})$ as

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \phi_{j,k}\rangle \psi_{j,k} \qquad \text{for all } f \in L^2(\mathbb{R}). \tag{3.2}$$

From this representation we observe that there is a very specific pairing of elements between the dual frames which we have to respect: the (z,l)-th element of $\{\phi_{j,k}\}$ is used to find the coefficient for the matching element in $\{\psi_{j,k}\}$ which obviously is $\psi_{z,l}$. Hence, if we want to speak of a dual frame with *more* generators than the wavelet frame itself, we need to pay close attention to this pairing (or duality) of elements. For canonical dual frames, as argued above, we only need to verify the pairing on one of the scales $j \in \mathbb{Z}$. Now, to understand what is meant by a canonical dual frame with more generators, we *lift* the pairing to another scale $j \geq 0$ or, more generally, to a sparser translation lattice. Let $\psi \in L^2(\mathbb{R})$ be the generator of a frame $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ with dyadic dilation. Suppose that the canonical dual of $\{\psi_{j,k}\}$ is not a wavelet system generated by one function. The idea is to consider the wavelet frame $\{\psi_{j,k}\}$ as a wavelet system of the form

$$\{\psi_{j,k}\}_{j,k\in\mathbb{Z}} = \left\{D_{2^j}T_{Pk}\tilde{\psi} : j,k \in \mathbb{Z}, \tilde{\psi} \in \{\psi, T\psi, \dots, T_{P-1}\psi\}\right\}$$

for some $P \in \mathbb{N}$. Now, it might happen that *this* system (that is, the right hand side system) has a canonical dual with wavelet structure as systems on the sparser translation lattice $P\mathbb{Z}$ with P generators; for further details see also page 23 in Appendix B.

Suppose for simplicity that the canonical dual frame is generated by two functions $\{\phi_1, \phi_2\}$. For this to make sense, we need to lift the duality to the translation lattice $2\mathbb{Z}$, where we match $\{D_{2^j}T_{2k}\psi\} \cup \{D_{2^j}T_{2k}(T\psi)\}$ and $\{D_{2^j}T_{2k}\phi_1\} \cup \{D_{2^j}T_{2k}\phi_2\}$ as dual frames with equal number of generators. Equivalently, we can say that we lift the duality to scale $j = 1$, where we have the well-known form of dual frames $\{\psi_{j,k}\} = \{D_{2^j}T_k(D_2\psi)\} \cup \{D_{2^j}T_k(D_2T\psi)\}$ and $\{D_{2^j}T_kD_2\phi_1\} \cup \{D_{2^j}T_kD_2\phi_2\}$. These two equivalent lifting schemes are based on the paraphrasing

$$\{\psi_{j,k}\}_{j,k\in\mathbb{Z}} = \left\{D_{2^j}T_{2k}\tilde{\psi} : j,k \in \mathbb{Z}, \tilde{\psi} \in \{\psi, T\psi\}\right\},$$

and

$$\{\psi_{j,k}\}_{j,k\in\mathbb{Z}} = \left\{D_{2^j}T_k\tilde{\psi} : j,k \in \mathbb{Z}, \tilde{\psi} \in \{D_2\psi, D_2T\psi\}\right\},$$

respectively.

Let us return to the example on the Riesz wavelet η from (3.1). The canonical dual $\{S^{-1}\eta_{j,k}\}$ is not a wavelet system generated by one function, hence we say that $\{S^{-1}\eta_{j,k}\}$ does not have wavelet structure. Since we can even say that $\{\eta_{j,k}\}$ is not a wavelet system generated by any finite number of functions, we should think of this canonical dual as being very "far from" having wavelet structure. The notion of the *period* of a wavelet frame in $L^2(\mathbb{R})$ is introduced as a measure of how "far from" we are; it tells us something about how close to or how far from the canonical dual frame is to having wavelet structure.

Definition 2. Suppose that $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R})$ is a frame wavelet associated with an integer dilation factor a, $|a| \geq 2$. The *period* of Ψ is the smallest integer $p \geq 1$ such that for all $f \in \overline{\operatorname{span}}\{T_k\psi : k \in \mathbb{Z}, \psi \in \Psi\}$,

$$T_{pk}S^{-1}f = S^{-1}T_{pk}f \qquad \text{for all } k \in \mathbb{Z},$$

where S is the frame operator of the wavelet frame generated by Ψ. If there is no such p, we say that the period of Ψ is ∞.

We note that there is no dilation operator present in the definition above simply because dilation commutes with the (inverse) frame operator. One can show that the canonical dual of $\mathcal{A}(\Psi)$ has the wavelet structure generated by $|\Psi|$ functions if, and only if, the period of Ψ is one. Moreover, in Paper I we show the following result on the relationship between the period of a wavelet frame and the number of generators of the canonical dual.

Theorem 3.2 (Proposition 2.3 in Paper I). *Suppose that $\Psi \subset L^2(\mathbb{R})$ is a frame wavelet with an integer dilation factor a, $|a| \geq 2$. For any nonnegative integer $M \in \mathbb{N}$, the following statements are equivalent:*

(i) $P(\Psi) \mid M$, i.e., the period of Ψ, denoted $P(\Psi)$, divides M.

(ii) There exist ML functions $\Phi = \{\phi_1, \ldots, \phi_{ML}\}$ such that $\{D_{a^j}T_{Mk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$ is the canonical dual of $\{D_{a^j}T_k\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi} = \{D_{a^j}T_{Mk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi_M}$, where

$$\Psi_M := \{T_m\psi : m = 0, \ldots, M-1, \psi \in \Psi\}.$$

Hence, if the period $P(\Psi)$ of a frame wavelet Ψ is finite, then the canonical dual frame is a wavelet system generated by $P(\Psi)\cdot|\Psi|$ functions, and this is the least number of generators. From Proposition 3.2 it is also obvious that any tight frame wavelet has period one.

Returning to the Riesz wavelet η from (3.1), we know that (ii) is not satisfied for any $M \in \mathbb{N}$, hence $P(\eta) = \infty$. The following result is a refinement of Theorem 3.1 and Remark 1.

Proposition 3.3 (Proposition 2 in [8]). *Let $M \in \mathbb{N}$. If Ψ is a frame wavelet and the period of Ψ divides M, then $V(\Psi)$ is shift invariant by the lattice $M\mathbb{Z}$. In addition, if Ψ is a Riesz wavelet, then the period of Ψ divides M if, and only if, $V(\Psi)$ is shift invariant by the lattice $M\mathbb{Z}$.*

From Remark 1 above we conclude that the space of negative dilates $V(\eta)$ is not shift invariant. In [18] this is verified by direct calculations. From the refinement in Proposition 3.3 we conclude that $V(\eta)$ is not even shift invariant with respect to any sublattice of $\mathbb{Z}$. This is verified by direct calculations in Appendix B.2.

We return to the main conclusion from the example of Daubechies [17] and Chui and Shi [14]: the canonical dual frame of a wavelet frame need not be a wavelet system. Since their example involved a non-biorthogonal Riesz wavelet, it has no alternate dual wavelet frames as well, and one might ask if the existence of an alternate dual frame with wavelet structure would imply wavelet structure of the canonical dual. In general, very little is known about the canonical dual frame of a wavelet frame, and this question deals with some fundamental interrelation aspects of the canonical dual and alternate duals. The main result in Paper I is a negative answer to the question:

Theorem 3.4 (Theorem 3.1 in Paper I). *For all $J \in \mathbb{N}$, there exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:*

(i) $\widehat{\psi}$ is C^∞ and compactly supported,
(ii) its canonical dual frame is not a wavelet system generated by fewer than 2^J functions,
(iii) there are infinitely many $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ form a pair of dual wavelet frames.

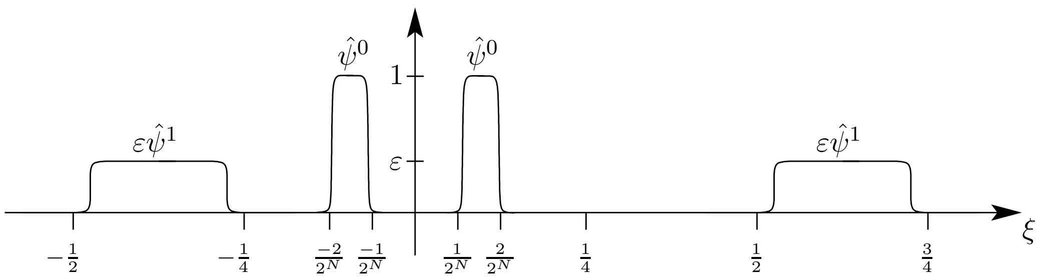

Figure 1: Sketch of the graph of a function $\hat{\psi} = \hat{\psi}^0 + \varepsilon\hat{\psi}^1$ satisfying the three conditions in Theorem 3.4 with $J = N - 3 \in \mathbb{N}$.

This claim (with $J = 1$) was asserted by Daubechies and Han [18], but the original argument in [18] uses an incorrect formula for the frame operator of a wavelet system owing to a simple change of sign mistake. This invalidates the original proof to the extent that an easy remedy appears to be doubtful. Therefore, there was a need to provide an alternative proof of Theorem 3.4. This was accomplished by Paper I. Instead of trying to work directly with the frame operator as in [18], we use a less direct approach using (the negation of) Proposition 3.3. The constructed function ψ satisfying the three conditions in Theorem 3.4 is sketched in Figure 1.

3.2. Constructions of pairs of dual wavelet frames (Paper II and III)

In the previous section we saw that duals and, in particular, the canonical dual of a wavelet frame need not have wavelet structure. In Paper II and III we therefore relegate the canonical dual to the background and develop construction procedures for *pairs* of dual (non-canonical) wavelet frames for arbitrary real, expansive dilations. This work was motivated by the existence of similar construction procedures for pairs of dual Gabor frames [12] which naturally lead to the question whether corresponding methods could be developed in the wavelet settings. We consider the one-dimensional settings $L^2(\mathbb{R})$ in Paper II and the extension to $L^2(\mathbb{R}^n)$ in Paper III.

Christensen [12] uses characterizing equations for dual Gabor frames to construct pairs of dual Gabor frames with generators given in a very explicit way. Consequently, in Paper II and III we use characterizing equations for dual wavelet frames. The existence of such equations was originally proved by Frazier, Garrigós, Wang, and Weiss [19] in the dyadic setting. Later it was extended by Bownik [2] to the setting of integer, expansive dilations and by Chui, Czaja, Maggioni, and Weiss [13] to the setting of real, expansive dilations. A proof of Theorem 3.5 can be found in Section 4 of Paper IV.

Theorem 3.5 (Theorem 4 in [13]). *Let $A \in GL_n(\mathbb{R})$ be expansive and $\Psi = \{\psi_1, \dots, \psi_L\}$, $\Phi = \{\phi_1, \dots, \phi_L\} \subset L^2(\mathbb{R}^n)$. Suppose $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ are Bessel sequences in $L^2(\mathbb{R}^n)$. Then, $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ are dual frames if, and only if,*

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \hat{\psi}_l(B^{-j}\xi)\overline{\hat{\phi}_l(B^{-j}\xi)} = 1 \qquad \text{for a.e. } \xi, \tag{3.3}$$

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} \hat{\psi}_l(B^{-j}\xi)\overline{\hat{\phi}_l(B^{-j}(\xi+\alpha))} = 0 \qquad \text{for a.e. } \xi \text{ and all } \alpha \in \mathbb{Z}^n \setminus \{0\}. \tag{3.4}$$

Characterizing equations for dual Gabor frames can be expressed in time domain while we see that equations (3.3) and (3.4) are conditions in the Fourier domain. This indicates that the construction of wavelet frames will take place in the Fourier domain as opposed to the time domain constructions in [12].

The setup will be as follows. We consider wavelet systems in the general setting with real, expansive dilation $A \in GL_n(\mathbb{R})$ and a lattice Γ in $\mathbb{R}^n$, i.e.,

$$\{D_{A^j} T_\gamma \psi\}_{j\in\mathbb{Z},\gamma\in\Gamma},$$

where the Fourier transform of ψ has compact support. Our aim is, for any given real, expansive dilation matrix A, to construct wavelet frames with attractive features and with a dual frame generator of the form

$$\phi = \sum_{j=a}^{b} c_j D_{A^j} \psi \tag{3.5}$$

for some explicitly given coefficients $c_j \in \mathbb{C}$ and $a, b \in \mathbb{Z}$. The idea behind the construction is simple: first, we make a number of assumptions of a function $\psi \in L^2(\mathbb{R}^n)$; then, we introduce ϕ in such a way that conditions (3.3) and (3.4) hold and conclude by Theorem 3.5 that ψ and ϕ generates a pair of dual wavelet frames.

Our main findings in Paper II can be stated as follows.

Theorem 3.6 (Theorem 2.3 in Paper II). *Let $d \in \mathbb{N}$, $a > 1$, and $\psi \in L^2(\mathbb{R})$. Suppose that $\widehat{\psi}$ is a real-valued function with $\operatorname{supp}\widehat{\psi} \subset \left[-a^c, -a^{c-d}\right] \cup \left[a^{c-d}, a^c\right]$ for some $c \in \mathbb{Z}$, and that*

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(a^j\xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{3.6}$$

Let $b \in \left(0, 2^{-1}a^{-c}\right]$. Then the function ψ and the function ϕ defined by

$$\phi(x) = b\psi(x) + 2b \sum_{j=1}^{d-1} a^{-j} \psi(a^{-j}x) \quad \text{for } x \in \mathbb{R}, \tag{3.7}$$

generate dual frames $\{D_{a^j} T_{bk} \psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j} T_{bk} \phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

The principal advantage of having a dual generator of the form (3.7), or more generally of the form (3.5), is that it will inherit properties from ψ preserved by dilation and linearity, e.g., vanishing moments, good time localization and regularity properties. For a more complete account of such matters we refer to Paper II, but we remark that, as a potential drawback, the wavelet frame generators will not have compact support in the time domain leading to infinite impulse response filters.

Figure 2 shows an example of a pair of generators ψ and ϕ in the Fourier domain constructed by Theorem 3.6. In Paper III we generalize and extend Theorem 3.6 to higher dimensions; we refer to Corollary 2.5 in Paper III for a generalization of Theorem 3.6.

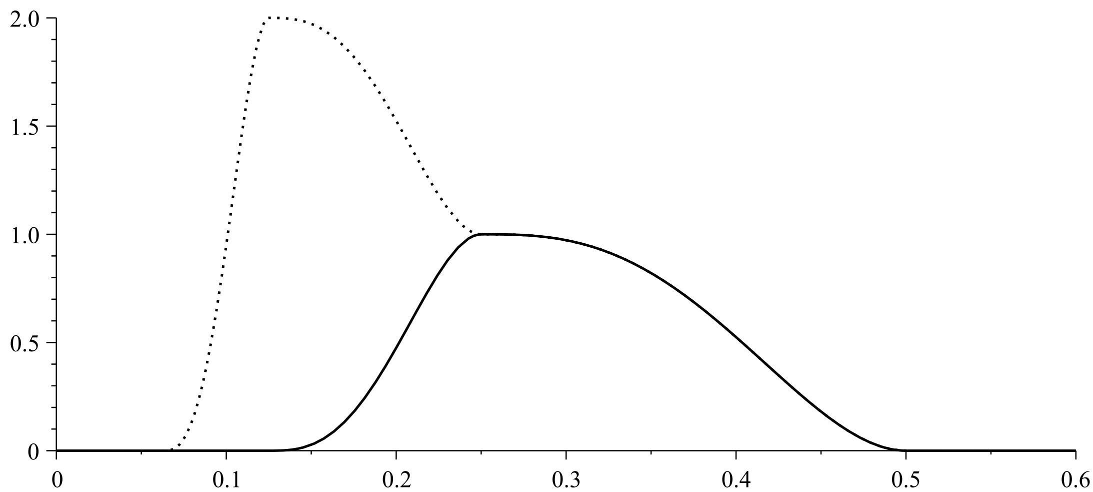

Figure 2: An example of a pair of dual generators $\hat{\psi}$ (solid line) and $\hat{\phi}$ (dotted line) in the Fourier domain (Figure 2 from Paper II).

Next, we extend the one-dimensional result on constructions of dual wavelet frames in Theorem 3.6 to higher dimensions. The extension is non-trivial since it is unclear how to determine the translation lattice Γ and how to control the support of the generators in the Fourier domain.

In order to outline the construction procedure in higher dimensions we need to introduce some notation. Let $|\cdot|_* = \langle\cdot,\cdot\rangle_*^{1/2}$ be a Hermitian norm associated with $B = A^t$ as in (vi) in Proposition A.1 and let $K \in GL_n(\mathbb{R})$ be the symmetric, positive definite matrix such that $\langle x, y\rangle_* = y^t K x$. Finally, let $\Lambda := \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, where $\{\lambda_i\}$ are the eigenvalues of K, and let $Q \in O(n)$ be such that the spectral decomposition of K is $Q^t K Q = \Lambda$. With this setup we can state the construction as follows.

Theorem 3.7 (Theorem 3.3 in Paper III). *Let $A \in GL_n(\mathbb{R})$ be expansive, $d \in \mathbb{N}_0$ and $\psi \in L^2(\mathbb{R}^n)$. Suppose that $\hat{\psi}$ is a bounded, real-valued function with $\operatorname{supp} \hat{\psi} \subset B^c(I_*) \setminus B^{c-d-1}(I_*)$ for some $c \in \mathbb{Z}$, and that*

$$\sum_{j \in \mathbb{Z}} \hat{\psi}(B^j \xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{3.8}$$

holds. Take $\Gamma = (1/2)A^c Q \sqrt{\Lambda}\mathbb{Z}^n$. Then the function ψ and the function ϕ defined by

$$\phi(x) = d(\Gamma) \left[\psi(x) + 2 \sum_{j=0}^{d} \lvert \det A \rvert^{-j} \, \psi(A^{-j}x) \right] \quad \text{for } x \in \mathbb{R}^n, \tag{3.9}$$

generate dual frames $\{D_{A^j} T_k \psi\}_{j \in \mathbb{Z}, \gamma \in \Gamma}$ and $\{D_{A^j} T_k \phi\}_{j \in \mathbb{Z}, \gamma \in \Gamma}$ for $L^2(\mathbb{R}^n)$.

The construction of redundant wavelet representations in higher dimensions is usually based on extension principles [29]. By making use of extension principles one is restricted to considering expansive dilations A with *integer* coefficients. On the other hand, the methods developed in Paper II and III work for any real, expansive dilation. The two papers contain several applications of Theorem 3.6 and 3.7. In Example II.3 and III.5 we construct pairs of dual wavelet frames generated by one smooth function with good time localization. For constructions of generators of spline type with compact support in the Fourier domain, we refer to Examples II.2, III.1 and III.4 (and Figure 2).

3.3. Affine and quasi-affine frames for rational dilations (Paper IV)

Quasi-affine systems are little known cousins of the well-studied wavelet systems also known as affine systems. Affine systems $\mathcal{A}(\psi, A, \mathbb{Z}^n)$ are dilation invariant, i.e., $\phi \in \mathcal{A}(\psi) \Rightarrow D_{A^j}\phi \in \mathcal{A}(\psi)$ for all $j \in \mathbb{Z}$, but not shift invariant. However, if the dilation A has integer entries, then one can modify the definition of affine systems to obtain shift invariant systems. This leads to the notion of quasi-affine systems

$$\mathcal{A}^q(\psi) = \left\{ \tilde{\psi}_{j,k}(x) := \begin{cases} \lvert \det A \rvert^{j/2} \, \psi(A^j x - k) : j \geq 0, \ k \in \mathbb{Z}^n \\ \lvert \det A \rvert^{j} \, \psi(A^j (x - k)) : j < 0, \ k \in \mathbb{Z}^n \end{cases} \right\},$$

which was introduced and investigated for integer, expansive dilation matrices by Ron and Shen [29]. Despite that the orthogonality of the affine system cannot be carried over to the corresponding quasi-affine system due to the oversampling of negative scales of the affine system, it turns out that the frame property is preserved. This important discovery is due to Ron and Shen [29] who proved that, for integer dilations, the affine system $\mathcal{A}(\psi)$ is a frame if, and only if, its quasi-affine counterpart $\mathcal{A}^q(\psi)$ is a frame (with the same frame bounds).

Theorem 3.8 ([29]). *Let $A \in GL_n(\mathbb{Z})$ be expansive and $\Psi \subset L^2(\mathbb{R})$. Then, $\mathcal{A}(\Psi)$ is a frame with bounds C_1, C_2 if, and only if, $\mathcal{A}^q(\Psi)$ is a frame with bounds C_1, C_2.*

Such equivalence results are useful because quasi-affine systems are shift invariant and thus much easier to study than affine systems which are dilation invariant. A proof of Theorem 3.8 can be found in Proposition 3.10 in Paper IV.

The goal of the work in Paper IV with Marcin Bownik is to extend the study of quasi-affine systems to the class of expansive rational dilations. So, let A be a fixed expansive dilation with rational entries. In [4] Bownik generalized the notion of a quasi-affine frame for rational, expansive dilations which coincides with the usual definition in the case of integer dilations. The main idea of Ron and Shen [29] is to oversample negative scales of the affine system at a rate adapted to the scale in order for the resulting system to be shift invariant. In order to define quasi-affine systems for rational, expansive dilations one needs to oversample both negative and positive scales of the affine system (at a rate proportional to the scale) which results in a quasi-affine system that in general coincides with the affine system only at the scale zero. This can easily be seen in one dimension where the quasi-affine system has a relatively simple algebraic form. Suppose that $a = p/q \in \mathbb{Q}$ is a dilation factor, where $|a| > 1$, $p, q \in \mathbb{Z}$ are relatively prime. Then, the quasi-affine system associated with a is given by

$$\mathcal{A}^q(\psi) = \left\{ \begin{array}{ll} |p|^{j/2}\,|q|^{-j}\,\psi(a^j x - q^{-j}k): & j \geq 0,\ k \in \mathbb{Z} \\ |p|^{j}\,|q|^{-j/2}\,\psi(a^j x - p^{j}k): & j < 0,\ k \in \mathbb{Z} \end{array} \right\}.$$

In the rational case it is much less clear than in the case of integer, expansive dilations (where both systems coincide at all non-negative scales), whether there is any relationship between affine and quasi-affine systems. Nevertheless, Bownik proved in [4] that the tight frame property is preserved when moving between rationally dilated affine and quasi-affine systems. This result has initially suggested that there is not much difference between integer and rational cases.

In Paper IV it is shown that this belief is largely incorrect by uncovering substantial differences between the theory of integer dilated and rationally dilated quasi-affine systems. For any rational, non-integer dilation we give an example of an affine system which is not a frame, but yet, the corresponding quasi-affine system is a frame. This kind of example does not exist for integer dilations due to Theorem 3.8.

Offhand, the equivalence result in Theorem 3.8 can seem surprising since we are dealing with two systems of functions that are quite different (at the negative scales $j < 0$). The equivalence result suggests that we have some flexibility in how the low frequency ($j < 0$) part of the system is chosen. Recall that we oversample both negative and positive scales for rational dilations. Hence, the fact that the equivalence in Theorem 3.8 does not hold for rational dilations suggests that we have less flexibility in changing high frequency ($j > 0$) parts of the system.

To understand the broken symmetry between the integer and rational settings we introduce a new class of quasi-affine systems indexed by the choice of the *oversampling lattice* Λ (see Appendix A.2 for basic facts on lattices). In short, the quasi-affine system $\mathcal{A}^q_\Lambda(\psi)$ is defined to be the smallest shift invariant system with respect to a lattice Λ, which contains all elements of the original affine system $\mathcal{A}(\psi)$. In order to make this definition meaningful we also need to renormalize the elements of $\mathcal{A}^q_\Lambda(\psi)$ at a rate corresponding to the rate of oversampling as it was done previously.

Definition 3. Let $A \in GL_n(\mathbb{Q})$ be a rational, expansive matrix, and let Λ be a rational lattice in $\mathbb{R}^n$, i.e., $\Lambda = P\mathbb{Z}^n$ with $P \in GL_n(\mathbb{Q})$. Suppose $\Psi \subset L^2(\mathbb{R}^n)$ is a finite set.

Define $\mathcal{A}^q_\Lambda(\Psi)$ the Λ-*oversampled quasi-affine* system by

$$\mathcal{A}^q_\Lambda(\Psi) = \bigcup_{j\in\mathbb{Z}} \left\{ \frac{1}{|\Lambda/(\Lambda\cap A^{-j}\mathbb{Z}^n)|^{1/2}} T_\omega D_{A^j}\Psi : \omega \in \Lambda + A^{-j}\mathbb{Z}^n) \right\}$$

When $\Lambda = \mathbb{Z}^n$ we drop the subscript Λ, and we say that $\mathcal{A}^q(\Psi) = \mathcal{A}^q_{\mathbb{Z}^n}(\Psi)$ is the standard quasi-affine system.

By definition $\mathcal{A}^q_\Lambda(\Psi)$ is shift invariant with respect to Λ. For illustration, let us display the oversampled quasi-affine system in one dimensional case with generator $\Psi = \{\psi\}$ and oversampling lattice $\Lambda = (pq)^J\mathbb{Z}$ for some $J \in \mathbb{N}_0$:

$$\mathcal{A}^q_\Lambda(\psi) = \left\{ \begin{array}{ll} |p|^{j/2}\,|q|^{-j+J/2}\,\psi(a^jx - q^{J-j}k): & j > J, k \in \mathbb{Z} \\ |a|^{j/2}\,\psi(a^jx - k): & -J \le j \le J,\ k \in \mathbb{Z} \\ |p|^{j+J/2}\,|q|^{-j/2}\,\psi(a^jx - p^{j+J}k): & j < -J,\ k \in \mathbb{Z} \end{array} \right\}.$$

Now, our main result can be stated as follows.

Theorem 3.9 (Theorem 3.9 in Paper IV). *Let $A \in GL_n(\mathbb{Q})$ be expansive and $\Psi \subset L^2(\mathbb{R}^n)$. Then, the affine system $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds C_1, C_2 if, and only if, the Λ-oversampled quasi-affine system $\mathcal{A}^q_\Lambda(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with uniform frame bounds C_1, C_2 for all integer lattices Λ.*

In the case when the dilation A is integer-valued, the class of Λ-oversampled quasi-affine systems reduces to the standard quasi-affine system $\mathcal{A}^q(\Psi)$ and its dilates. Hence, the original result of Ron and Shen [29] follows immediately from Theorem 3.9. The proof of Theorem 3.9 is influenced by the work of Hernández, Labate, Weiss, and Wilson [20, 21], where the authors obtain reproducibility characterizations of generalized shift invariant (GSI) systems including affine, wave packets, and Gabor systems. The key element of these techniques is the use of almost periodic functions which was pioneered by Laugesen [23, 24] in his work on translational averaging of the wavelet functional. Using these methods Laugesen [24, Theorem 7.1] gave another proof of the equivalence of affine and quasi-affine frames in the integer case. In this work we show that these techniques can be generalized to treat rationally dilated quasi-affine systems as well. Moreover, Laugesen [24] considered equivalence results for time-discrete wavelet systems $\mathcal{A}(\Psi)$ and time-continuous wavelet systems $\{D_{A^j}T_x\Psi\}_{j\in\mathbb{Z},x\in\mathbb{R}^n}$. The Λ-oversampled quasi-affine systems represent, in some sense, intermediate stages between these two systems. If Λ is very sparse, the oversampled quasi-affine system $\mathcal{A}^q_\Lambda(\Psi)$ will resemble the time-discrete wavelet system. If Λ, on the other hand, is very dense, then $\mathcal{A}^q_\Lambda(\Psi)$ will be close to the time-continuous wavelet system.

In Paper IV we also introduce a particularly interesting subclass of generators where the equivalence between affine and quasi-affine frames exhibits the largest degree of symmetry. This is a class of *diagonal* affine systems for which the off-diagonal functions t_α defined below vanish.

Definition 4. For a given dilation matrix A and $\Psi \subset L^2(\mathbb{R}^n)$ we introduce the family of functions $\{t_\alpha\}_{\alpha\in\mathbb{Z}^n}$ on $\mathbb{R}^n$ by:

$$t_\alpha(\xi) = \sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} \hat{\psi}(B^{-j}\xi)\overline{\hat{\psi}(B^{-j}(\xi+\alpha))} \qquad \text{for } \xi \in \mathbb{R}^n. \tag{3.10}$$

We say that the affine system $\mathcal{A}(\Psi)$ is *diagonal* if $t_\alpha(\xi) = 0$ a.e. for all $\alpha \in \mathbb{Z}^n \setminus \{0\}$.

The class of diagonal affine frames is large enough to contain all tight affine systems, but small enough to be contained in the class of affine frames having canonical duals with affine structure. By Theorem 3.12 below we see that the class of diagonal affine frames consists precisely of quasi-affine frames having a canonical dual quasi-affine frame.

Now, for diagonal generators Ψ we have "perfect" equivalence between affine and quasi-affine frames as is seen from the following result. Theorem 3.10 is an extension of [4, Theorem 3.4] on tight frames.

Theorem 3.10. *Let $A \in GL_n(\mathbb{Q})$ be expansive, $\Psi \subset L^2(\mathbb{R}^n)$ and let $C_1, C_2 > 0$ be constants. Suppose that the affine system $\mathcal{A}(\Psi)$ is diagonal. Then the following assertions are equivalent:*

(i) the affine system $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2.

(ii) the quasi-affine system $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2 for **some** *integer lattice $\Lambda_0 \subset \mathbb{Z}^n$.*

(iii) the quasi-affine system $\mathcal{A}^q_{\Lambda}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2 for **all** *integer lattices $\Lambda \subset \mathbb{Z}^n$.*

(iv)
$$C_1 \le \sum_{\psi \in \Psi} \sum_{j \in \mathbb{Z}} |\hat{\psi}(B^j \xi)|^2 \le C_2 \qquad \text{for a.e. } \xi \in \mathbb{R}^n.$$

In Section 4 of Paper IV, we investigate pairs of dual quasi-affine frames thus connecting to the theme of Paper II and III, compare Theorem 3.5 and Theorem 3.11. The theory of rationally dilated quasi-affine frames parallels quite closely that of integer dilated systems. Hence, we have a perfect equivalence between pairs of dual affine frames and pairs of dual quasi-affine frames, regardless of the choice of the oversampling lattice Λ.

Theorem 3.11 (Theorem 4.2 in Paper IV)**.** *Let $A \in GL_n(\mathbb{Q})$ be expansive. Suppose $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ are Bessel sequences in $L^2(\mathbb{R}^n)$. Then the following assertions are equivalent:*

(i) $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ are dual frames.

(ii) $\mathcal{A}^q_{\Lambda_0}(\Psi)$ and $\mathcal{A}^q_{\Lambda_0}(\Phi)$ are dual frames for **some** *integer oversampling lattice $\Lambda_0 \subset \mathbb{Z}^n$.*

(iii) $\mathcal{A}^q_{\Lambda}(\Psi)$ and $\mathcal{A}^q_{\Lambda}(\Phi)$ are dual frames for **all** *integer oversampling lattices $\Lambda \subset \mathbb{Z}^n$.*

(iv) Ψ and Φ satisfy the equations (3.3) *and* (3.4)*.*

In the integer case Theorem 3.11 was first shown by Ron and Shen [29, 30] with some decay assumptions on generators Ψ and Φ. Chui, Shi, and Stöckler [15] proved the same result without any decay assumptions, see also [2, Theorem 4.1]. Theorem 3.11 generalizes this result to the setting of rational dilations.

In Section 5 in Paper IV we characterize when the canonical dual frame of a Λ-oversampled quasi-affine frame $\mathcal{A}^q_{\Lambda}(\psi)$ is also a quasi-affine frame. In the case of integer dilations, such characterization is due to Bownik and Weber [8, Theorem 3]. Theorem 3.12 generalizes this result to the case of rational dilations. It is remarkable that the existence of the canonical quasi-affine dual frame is independent of the choice of the oversampling lattice Λ. Hence, if such canonical dual frame exists for some Λ-oversampled quasi-affine system, then it must exist for all lattices $\Lambda \subset \mathbb{Z}^n$.

Theorem 3.12 (Theorem 5.6 in Paper IV). *Let $A \in GL_n(\mathbb{Q})$ be expansive. Suppose the oversampled quasi-affine system $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ for some integer lattices $\Lambda_0 \subset \mathbb{Z}^n$. Then the canonical dual frame of $\mathcal{A}^q_{\Lambda_0}(\Psi)$ has the form $\mathcal{A}^q_{\Lambda_0}(\Phi)$ for some set of functions $\Phi \subset L^2(\mathbb{R}^n)$ with cardinality $|\Phi| = |\Psi|$ if, and only if, for all $\alpha \in \mathbb{Z}^n \setminus \{0\}$,*

$$t_\alpha(\xi) = \sum_{\psi \in \Psi} \sum_{j \in \mathbb{Z}: \alpha \in B^j \mathbb{Z}^n} \hat{\psi}(B^{-j}\xi)\overline{\hat{\psi}(B^{-j}(\xi + \alpha))} = 0 \tag{3.11}$$

Moreover, in the positive case $\mathcal{A}^q_{\Lambda}(\Psi)$ is a frame for all integer lattices $\Lambda \subset \mathbb{Z}^n$ and its canonical dual frame is $\mathcal{A}^q_{\Lambda}(\Phi)$.

This line of research connects to the theory on the canonical dual of wavelet frames which we considered in Section 3.1 (the survey on results in Paper I). We note that if Ψ generates a quasi-affine frame $\mathcal{A}^q_{\Lambda_0}(\Psi)$ for some $\Lambda_0 \subset \mathbb{Z}^n$ whose canonical dual frame has the form of a quasi-affine system, then Ψ also generates an affine frame whose canonical dual frame has affine structure; loosely speaking, this means that it is harder for a quasi-affine frame to have a canonical dual with the same structure than for an affine frame. This fact is immediate from Theorem 5.4 in Paper IV. Theorem 3.12 on canonical duals of quasi-affine frames, therefore, provides a sufficient condition for a wavelet frame having a canonical dual with wavelet structure:

Proposition 3.13. *Let $A \in GL_n(\mathbb{Q})$ be expansive and let $\Psi \subset L^2(\mathbb{R}^n)$. If $t_\alpha(\xi) = 0$ a.e. for all $\alpha \in \mathbb{Z}^n \setminus \{0\}$, then the canonical dual frame of $\mathcal{A}(\Psi)$ is of the form $\mathcal{A}(\Phi)$ for some set $\Phi \subset L^2(\mathbb{R}^n)$ with $|\Phi| = |\Psi|$.*

Proposition 3.13 tells us, in other words, that the canonical dual of diagonal affine frames has affine structure.

In the last section of Paper IV we show that, for any non-integer, rational dilation, there exist quasi-affine frames $\mathcal{A}^q_{\Lambda}(\psi)$ such that the corresponding affine system $\mathcal{A}(\psi)$ is not a frame:

Theorem 3.14 (Theorem 6.1 in Paper IV). *For each rational non-integer dilation factor $a > 1$, there exists a function $\psi \in L^2(\mathbb{R})$ such that $\mathcal{A}^q_{\Lambda}(\psi)$ is a frame for any oversampling lattice $\Lambda \subset \mathbb{Z}$, but yet, $\mathcal{A}(\psi)$ is not a frame.*

Despite that each system $\mathcal{A}^q_{\Lambda}(\psi)$ is a frame, its lower frame bound drops to zero as the lattice Λ gets sparser. Hence, this example does not contradict Theorem 3.9. Moreover, in light of Theorem 3.11, none of the quasi-affine frames $\mathcal{A}^q_{\Lambda}(\psi)$ can have a dual quasi-affine frame.

We end this the survey of Paper IV by noting that it is not possible, in general, to extend the notion of quasi-affine systems beyond rational dilations. Consider, for example, a wavelet system in $L^2(\mathbb{R})$ with dilation factor $a = \pi$. The scale $j = 0$ part $\{T_k\}_{k \in \mathbb{Z}}$ is $\mathbb{Z}$-SI while the scale $j = 1$ part $\{D_\pi T_k\}_{k \in \mathbb{Z}}$ is $\pi\mathbb{Z}$-SI. Since $\mathbb{Z} + \pi\mathbb{Z}$ is dense in $\mathbb{R}$ and therefore not a lattice, we cannot unite the two scales in a Λ-SI system for any lattice Λ in $\mathbb{R}$.

Appendix A. Some linear algebra

The following facts on expansive matrices and lattices in $\mathbb{R}^n$ will be used throughout the thesis.

A.1. Expansive matrices

An expansive matrix is a real $n \times n$ matrix with eigenvalues $|\lambda| > 1$. Matrices of this type are used as dilation matrices for wavelet systems in higher dimensions in this thesis.

If A is an expansive matrix, then so is the transpose $B = A^t$. Proposition A.1 is a collection of equivalent conditions for a (non-singular) matrix being expansive. All equivalences can be found in the literature. Since these equivalences often are stated without proof, we present a proof or a reference to a proof of each of the equivalences.

Proposition A.1. *For $B \in GL_n(\mathbb{R})$ the following assertions are equivalent:*

(i) *B is expansive, i.e., all eigenvalues λ_i of B satisfy $|\lambda_i| > 1$.*

(ii) *$\rho(B^{-1}) < 1$, where ρ denotes the spectral radius.*

(iii) $\lim_{j\to\infty} B^{-j} = 0$

(iv) $\lim_{j\to\infty} \|B^{-j}\| = 0$ *for some/all matrix norms $\|\cdot\|$.*

(v) *For any norm $|\cdot|$ on $\mathbb{R}^n$ there are constants $\lambda > 1$ and $c \geq 1$ such that*

$$|B^j x| \geq (1/c)\lambda^j |x| \qquad \text{for all } j \in \mathbb{N}_0,$$

for any $x \in \mathbb{R}^n$. Equivalently, $|B^{-j}x| \leq c\lambda^{-j} |x|$.

(vi) *There is a Hermitian norm $|\cdot|_*$ on $\mathbb{R}^n$ and a constant $\lambda > 1$ such that*

$$|B^j x|_* \geq \lambda^j |x|_* \qquad \text{for all } j \in \mathbb{N}_0,$$

for any $x \in \mathbb{R}^n$.

(vii) *$|B^j x - x| \to \infty$ for $j \to \infty$ for all $x \in \mathbb{R}^n \setminus \{0\}$ for some/all vector norms $|\cdot|$.*

(viii) *$\mathcal{E} \subset \lambda\mathcal{E} \subset B\mathcal{E}$ for some ellipsoid $\mathcal{E} = \{x \in \mathbb{R}^n : |Px| \leq 1\}$, $P \in GL_n(\mathbb{R})$ and $\lambda > 1$.*

(ix) *$\overline{\mathcal{E}} \subset B\mathcal{E}^\circ$ for some ellipsoid $\mathcal{E} = \{x \in \mathbb{R}^n : |Px| \leq 1\}$, $P \in GL_n(\mathbb{R})$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows directly from the definition of $\rho(B^{-1})$ while (ii) $\Leftrightarrow$ (iii) is a standard result. The implication (iii) $\Rightarrow$ (iv) follows by continuity of the matrix norm and (iv) $\Rightarrow$ (ii) by $\rho(B^{-1})^j \leq \|B^{-j}\| \to 0$ for $j \to \infty$.

The equivalence (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) is a result from [27]; a proof of (i) $\Leftrightarrow$ (v) can be found in [20, Lemma 5.2] and an approach to construct a Hermitian norm $|\cdot|_*$ as in (vi) can be found in [3, Lemma 2.2] and [28, Lemma 1.5.1]. We note that the only direction that requires some work is (i) $\Rightarrow$ (vi) since (vi) $\Rightarrow$ (v) follows by equivalence of norms on $\mathbb{R}^n$ with $c = C_2/C_1$, where $C_1 |x| \leq |x|_* \leq C_2 |x|$, and (v) $\Rightarrow$ (iv) follows by the estimate $\|B^{-j}\| \leq c\lambda^{-j}$ for $j \geq 0$ (take $y = B^{-j}x$ in (v)). This shows that the sequence of norms $\|B^{-j}\|$ actually decays exponentially to zero.

Assume (v) holds. Then

$$\left|B^j x - x\right| \geq \left||B^j x| - |x|\right| \geq \left|(1/c)\lambda^j |x| - |x|\right| = \left|(1/c)\lambda^j - 1\right| |x| \quad \text{for } j \geq 0,$$

for $\lambda > 1$ and $c \geq 1$. Since $|(1/c)\lambda^j - 1| \to \infty$ as $j \to \infty$, we have $|B^j x - x| \to \infty$ for any $x \neq 0$ which in turn is statement (vii). Assume (vii) holds and let (μ, v) be an eigenvalue-eigenvector pair for B. Then

$$\left|B^j v - v\right| = \left|\mu^j v - v\right| = \left|\mu^j - 1\right| |v| .$$

Since by hypothesis $v \neq 0$, we must have $|\mu^j - 1| \to \infty$ as $j \to \infty$ which implies $|\mu| > 1$. Since this must be true for any eigenvalue, we conclude that (i) holds.

The implication (i) $\Rightarrow$ (viii) is Lemma 2.2 in [3]. Assume (viii) holds. Let $P \in GL_n(\mathbb{R})$ be such that $\mathcal{E} = \{x \in \mathbb{R}^n : |Px| \leq 1\}$. Since $\mathcal{E}$ is an ellipsoid,

$$|Px|^2 = x^t K x,$$

where K is a symmetric, positive definite matrix. Define the inner product by $\langle x, y\rangle_* = x^t K y$, hence $|x|_* = |Px|$. Take $x \in \mathbb{R}^n \setminus \{0\}$, and let $y = x/|x|_*$. Thus $y \in \partial\mathcal{E}$ and $By \in \partial B(\mathcal{E})$. By $\lambda\mathcal{E} \subset B(\mathcal{E})$, we have $\lambda = \lambda |y|_* \leq |By|_*$ and thus $\lambda |x|_* \leq |Bx|_*$. For $x = 0$ this inequality is immediate. This shows that (vi) holds.

The last equivalence (viii) $\Leftrightarrow$ (ix) is trivial. □

A.2. Lattices in $\mathbb{R}^n$

A lattice Γ in $\mathbb{R}^n$ is a discrete subgroup under addition generated by integral linear combinations of n linearly independent vectors $\{p_i\}_{i=1}^n \subset \mathbb{R}^n$, i.e.,

$$\Gamma = \{z_1 p_1 + \cdots + z_n p_n : z_1, \ldots, z_n \in \mathbb{Z}\} .$$

In other words, a lattice is a finitely generated free abelian group of rank n. Yet, in other words, it is a set of points of the form $P\mathbb{Z}^n$ for a non-singular $n \times n$ matrix P. Let Γ be a lattice in $\mathbb{R}^n$. If $\Gamma = P\mathbb{Z}^n$, we say that the matrix $P \in GL_n(\mathbb{R})$ generates the lattice Γ. A generating matrix of a given lattice is only unique up to multiplication from the right by integer matrices with determinant one in absolute value; in particular, if $\Gamma = P\mathbb{Z}^n$ for some $P \in GL_n(\mathbb{R})$, then also $\Gamma = PS\mathbb{Z}^n$ for any $S \in SL_n(\mathbb{Z})$.

We mainly follow the exposition in [9]. The *determinant* of Γ is defined to be:

$$d(\Gamma) = |\det P| , \tag{A.1}$$

where $P \in GL_n(\mathbb{R})$ is a generating matrix for Γ; note that $d(\Gamma) > 0$ and $d(\mathbb{Z}^n) = 1$. The determinant $d(\Gamma)$ is independent of the particular choice of generating matrix P and equals the volume of a fundamental domain I_Γ of the lattice Γ, where

$$I_\Gamma = P([0,1)^n) = \{c_1 p_1 + \cdots + c_n p_n : 0 \leq c_i < 1 \text{ for } i = 1, \ldots, n\}$$

with p_i denoting the ith column of a generating matrix P. Note that $\mathbb{R}^n = \cup_{\gamma \in \Gamma}(\gamma + I_\Gamma)$ with the union being disjoint, and that the specific shape of I_Γ depends on the choice of the generating matrix P.

Since a generating matrix P of a lattice Γ is not unique, it is useful to have a characterization of lattices in which P does not appear. We have the following result.

Theorem A.2 (Theorem III.VI in [9]). *Let Γ be a subset of $\mathbb{R}^n$. Then, Γ is a lattice if, and only if, the following three conditions hold:*

(i) *If $x, y \in \Gamma$, then $x \pm y \in \Gamma$,*
(ii) *Γ contains n linearly independent vectors,*
(iii) *There is a constant $r > 0$ such that 0 is the only point of Γ in $B(0, r) = \{x : |x| < r\}$.*

Suppose that $\Gamma \subset \Lambda$, in other words, that Γ is a sublattice of some "denser" lattice Λ. We define the *index* of Γ in Λ as

$$D = \frac{d(\Gamma)}{d(\Lambda)}. \tag{A.2}$$

It is straightforward to verify that the index D is always a positive integer; the index D is actually the number of copies of parallelotopes I_Γ that fits inside a larger parallelotope I_Λ. If D is the index of Γ in Λ, we have from [9, §I.2.2],

$$D\Lambda \subset \Gamma \subset \Lambda. \tag{A.3}$$

Lemma A.3 (Lemma I.1 in [9]). *The index of the sublattice Γ of Λ is the order of the quotient group Λ/Γ, i.e.,*

$$|\Lambda/\Gamma| = D \equiv d(\Gamma)/d(\Lambda), \tag{A.4}$$

where $|\Lambda/\Gamma|$ is the order of the quotient group Λ/Γ.

Let $\{p_i\}_{i=1}^n$ be generators of a lattice Γ. Since $\{p_i\}$ is a basis in $\mathbb{R}^n$, there exists a unique (biorthogonal) basis $\{p_i^*\}_{i=1}^n$ such that $\langle p_i, p_j^* \rangle = \delta_{i,j}$ for $i, j = 1, \ldots, n$. The *dual lattice* of Γ is defined as

$$\Gamma^* = \{z_1 p_1^* + \cdots + z_n p_n^* : z_1, \ldots, z_n \in \mathbb{Z}\},$$

and the definition is independent of the choice of basis $\{p_i\}$. Dual lattices are sometimes called polar or reciprocal lattices.

The following result gives a representation of the dual lattice without reference to generating bases or matrices.

Lemma A.4 (Lemma I.5 in [9]). *Let $\Lambda = P\mathbb{Z}^n$ be a lattice in $\mathbb{R}^n$. Then, the dual lattice of Γ is*

$$\begin{aligned} \Gamma^* &= \{\eta \in \mathbb{R}^n : \langle \eta, \gamma \rangle \in \mathbb{Z} \text{ for } \gamma \in \Gamma\} \\ &= (P^t)^{-1}\mathbb{Z}^n. \end{aligned}$$

Furthermore, the determinants satisfy

$$d(\Gamma)d(\Gamma^*) = 1.$$

If $\Gamma \subset \Lambda$, then $\Lambda^* \subset \Gamma^*$. We refer to [9] for further basic properties of lattices.

Isomorphism theorems

Since lattices are groups, we can apply the isomorphism theorems. The second isomorphism theorem reads for lattices Γ and Λ in $\mathbb{R}^n$:

$$\Gamma/(\Gamma \cap \Lambda) \cong (\Gamma + \Lambda)/\Lambda. \tag{A.5}$$

Note that $\Gamma + \Lambda$ and $\Gamma \cap \Lambda$ are not necessarily lattices, e.g., $\pi\mathbb{Z} + \mathbb{Z}$ is dense in $\mathbb{R}$, hence not a lattice (it does not satisfy (iii) in Theorem A.2). For lattices Γ, Λ, Θ satisfying $\Gamma \subset \Lambda \subset \Theta$ the third isomorphism theorem yields

$$(\Theta/\Gamma) \,/\, (\Lambda/\Gamma) \cong \Theta/\Lambda. \tag{A.6}$$

Rational lattices

In Paper IV we consider mostly rational lattices. By a *rational* lattice Γ we understand a lattice whose points have rational coordinates, or equivalently, a lattice whose generating matrix P has rational entries. For a rational lattice Γ we define $\tilde{\Gamma}$, the *integral sublattice* of Γ, by $\tilde{\Gamma} = \mathbb{Z}^n \cap \Gamma$, and the *extended integral superlattice* of Γ by $\Gamma + \mathbb{Z}^n$. By Theorem A.2 it is straightforward to verify that $\mathbb{Z}^n \cap \Gamma$ and $\Gamma + \mathbb{Z}^n$ are indeed lattices. Since $\tilde{\Gamma} = \Gamma \cap \mathbb{Z}^n$ is a sublattice of $\mathbb{Z}^n$ with index in $\mathbb{Z}^n$ as

$$D = \frac{d(\tilde{\Gamma})}{d(\mathbb{Z}^n)} = d(\tilde{\Gamma}),$$

equation (A.3) implies

$$d(\tilde{\Gamma})\mathbb{Z}^n \subset \tilde{\Gamma} \subset \Gamma. \tag{A.7}$$

This shows that any rational lattice Γ has a integral sublattice of the form $c\mathbb{Z}^n$, where the constant $c \in \mathbb{N}$ can be taken to be $c = d(\tilde{\Gamma}) = \operatorname{vol}(I_{\tilde{\Gamma}}) = |\mathbb{Z}^n/\tilde{\Gamma}|$. Since we also have $|\Gamma/\tilde{\Gamma}| = d(\tilde{\Gamma})/d(\Gamma)$ by Lemma A.3, the above calculations show that

$$|\mathbb{Z}^n/\tilde{\Gamma}| = d(\Gamma)|\Gamma/\tilde{\Gamma}|.$$

In a similar way, we have for the extended integral superlattice of Γ

$$|(\Gamma + \mathbb{Z}^n)/\mathbb{Z}^n| = d(\Gamma + \mathbb{Z}^n)^{-1} = \operatorname{vol}(I_{\Gamma+\mathbb{Z}^n})^{-1} \in \mathbb{N}$$

and

$$|(\Gamma + \mathbb{Z}^n)/\mathbb{Z}^n|\,(\Gamma + \mathbb{Z}^n) \subset \mathbb{Z}^n.$$

For two rational lattices Γ and Λ the dual lattice of $\Gamma \cap \Lambda$ and $\Gamma + \Lambda$ are $\Gamma^* + \Lambda^*$ and $\Gamma^* \cap \Lambda^*$, respectively.

Appendix B. The dual of a non-biorthogonal Riesz wavelet

We consider a Riesz wavelets with dyadic dilation $A = 2$ in $L^2(\mathbb{R})$ defined as

$$\eta = \psi + \varepsilon D_2 \psi \qquad 0 < \varepsilon < 1, \tag{B.1}$$

where ψ is a generator of a wavelet orthonormal basis $\{\psi_{j,k} := D_{2^j} T_k \psi\}_{j,k\in\mathbb{Z}}$. This example was first considered by Chui and Shi [14] and Daubechies [16], see also Section 3.1. For any $\varepsilon < 1$ the function η will generate a wavelet Riesz basis. This can be realized by considering the wavelet system generated by the perturbation term $D_2\psi$. Obviously, the sequence $\{D_{2^j}T_k D_2\psi\} = \{D_{2^j}T_{2k}\psi\}$ is a subsequence of the orthonormal basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$, and hence a Bessel sequence with bound $C_2 \leq 1$. Therefore, by [11, Corollary 15.1.5], the sequence $\{\eta_{j,k}\}$ is a Riesz basis for any $\varepsilon \in (0,1)$ with bounds $(1 \pm \varepsilon^{1/2})^2$.

In the following section we will show that the (canonical) dual of $\{\eta_{j,k}\}$ is not a wavelet system for any finite number of generators. Observe that the (canonical) dual of the orthonormal basis $\{\psi_{j,k}\}$ is the basis itself, hence it is, in particular, a wavelet system generated by one function. Thus, by an arbitrarily small perturbation as in (B.1), the structure of the dual changes completely. Likewise, the the space of negative dilates $V(\psi)$ is shift invariant by $\mathbb{Z}$ while $V(\eta)$ is not shift invariant with respect to any sublattice of $\mathbb{Z}$. This is shown in the last section.

B.1. The structure of the dual

In [14] it is shown that the (canonical) dual of $\{D_{2^j}T_k\eta\}_{j,k\in\mathbb{Z}}$ is not of the form $\{D_{2^j}T_k\phi\}$ for any $\phi \in L^2(\mathbb{R})$. In the following we show that, in fact, the canonical dual $\{S^{-1}\eta_{j,k}\}$ is not of the form

$$\{D_{2^j}T_k\phi : j,k \in \mathbb{Z}, \phi \in \Phi\}$$

for any finite set $\Phi \subset L^2(\mathbb{R})$ of generators.

The basis elements of the wavelet Riesz basis is $\eta_{j,k} = \psi_{j,k} + \varepsilon\psi_{j+1,2k}$ for $j,k \in \mathbb{Z}$. The dual basis can easily be calculated; in [11] it is found using an operator approach. We use a different approach. As usual we let $S = S_\eta$ denote the frame operator of $\{\eta_{j,k}\}$. In order to find $\{S^{-1}\eta_{j,k}\}$ we evaluate the frame operator on $\psi_{j,k}$ for each $j,k \in \mathbb{Z}$:

$$\begin{aligned} S\psi_{j,k} &= \sum_{l,z\in\mathbb{Z}} \langle \psi_{j,k}, \eta_{l,z}\rangle \eta_{l,z} \\ &= \sum_{l,z\in\mathbb{Z}} \langle \psi_{j,k}, \psi_{l,z}\rangle \eta_{l,z} + \sum_{l,z\in\mathbb{Z}} \langle \psi_{j,k}, \varepsilon\psi_{l+1,2z}\rangle \eta_{l,z} \\ &= \eta_{j,k} + \varepsilon \sum_{l\in\mathbb{Z}, z\in 2\mathbb{Z}} \langle \psi_{j,k}, \psi_{l,z}\rangle \eta_{l-1,z/2}. \end{aligned}$$

For odd k the above calculations yield $S\psi_{j,k} = \eta_{j,k}$. Since $\{\eta_{j,k}\}$ is a frame, the frame operator is invertible, hence we find

$$S^{-1}\eta_{j,k} = \psi_{j,k} \qquad \forall j \in \mathbb{Z}, k \in 2\mathbb{Z}+1. \tag{B.2}$$

Let $n = \sup_{n\in\mathbb{N}_0}\{2^n | k\}$ for $k \in \mathbb{Z}$. For odd k we have $n = 0$, and for even, nonzero k we have $n = \max\{n \in \mathbb{N} : k/2^n \in 2\mathbb{Z}+1\} \geq 1$. For even $k \neq 0$ the above calculations show that $S\psi_{j,k} = \eta_{j,k} + \varepsilon\eta_{j-1,k/2}$ and, by application of the inverse frame operator and a rearrangement,

$$S^{-1}\eta_{j,k} = \psi_{j,k} - \varepsilon S^{-1}\eta_{j-1,k/2} \qquad \forall j \in \mathbb{Z}, k \in 2\mathbb{Z}.$$

Repeated usage of this equation gives

$$\begin{aligned} S^{-1}\eta_{j,k} &= \psi_{j,k} - \varepsilon S^{-1}\eta_{j-1,k/2} \\ &= \psi_{j,k} - \varepsilon(\psi_{j-1,k/2} - \varepsilon S^{-1}\eta_{j-2,k/4}) \\ &= \psi_{j,k} - \varepsilon\psi_{j-1,k/2} + \varepsilon^2(\psi_{j-2,k/4} - \varepsilon S^{-1}\eta_{j-3,k/8}). \end{aligned}$$

Continuing this way until the odd integer $k/2^n$, a final application of (B.2) yields

$$S^{-1}\eta_{j,k} = \psi_{j,k} - \varepsilon\psi_{j-1,k/2} + \varepsilon^2\psi_{j-2,k/4} - \cdots + (-\varepsilon)^n\psi_{j-n,k/2^n} \quad \forall j \in \mathbb{Z}, k \in 2\mathbb{Z}\setminus\{0\}, \tag{B.3}$$

where $n = \sup_{n\in\mathbb{N}}\{2^n|k\}$. For $k = 0$, we have by calculations similar to the above

$$S^{-1}\eta_{j,0} = \psi_{j,0} - \varepsilon\psi_{j-1,0} + \cdots + (-\varepsilon)^{n-1}\psi_{j-n+1,0} + (-\varepsilon)^n S^{-1}\eta_{j-n,0} \quad \forall j \in \mathbb{Z}, n \in \mathbb{N},$$

which in the limit $n \to \infty$ gives

$$S^{-1}\eta_{j,0} = \sum_{n=0}^{\infty}(-\varepsilon)^n\psi_{j-n,0} \qquad \forall j \in \mathbb{Z}, \tag{B.4}$$

by the boundedness of the (dual) Riesz basis, i.e., $\sup_{j,k}\|S^{-1}\eta_{j,k}\| < \infty$. Summarizing our findings:

$$S^{-1}\eta_{j,k} = \begin{cases} \psi_{j,k} & j \in \mathbb{Z}, k \in 2\mathbb{Z}+1, \\ \psi_{j,k} - \varepsilon\psi_{j-1,k/2} + \cdots + (-\varepsilon)^n\psi_{j-n,k/2^n} & j \in \mathbb{Z}, k \in 2\mathbb{Z}\setminus\{0\}, \\ \sum_{m=0}^{\infty}(-\varepsilon)^m\psi_{j-m,0} & j \in \mathbb{Z}, k = 0. \end{cases} \tag{B.5}$$

$$= \begin{cases} \psi_{j,k} - \varepsilon\psi_{j-1,k/2} + \cdots + (-\varepsilon)^n\psi_{j-n,k/2^n} & j \in \mathbb{Z}, k \in \mathbb{Z}\setminus\{0\}, \\ \sum_{m=0}^{\infty}(-\varepsilon)^m\psi_{j-m,0} & j \in \mathbb{Z}, k = 0. \end{cases}$$

Remark 2. If $\varepsilon > 1$, we can in general only say that $\{\eta_{j,k}\}$ is a Bessel sequence. If η in fact is a frame wavelet, then S is invertible and we can calculate the canonical dual frame explicitly as above. Note that the calculations are the same as for $\varepsilon < 1$ except when $k = 0$. For $k = 0$ we have

$$S^{-1}\eta_{j,0} = 1/\varepsilon\psi_{j+1,0} - 1/\varepsilon S^{-1}\eta_{j+1,0} \qquad \forall j \in \mathbb{Z},$$

hence in the limit

$$S^{-1}\eta_{j,0} = -\sum_{n=1}^{\infty}(-\varepsilon)^{-n}\psi_{j+n,0} \qquad \forall j \in \mathbb{Z}.$$

We note that this only holds if S is invertible, that is, if $\{\eta_{j,k}\}$ is a frame; S is well-defined since $\{\eta_{j,k}\}$ is a Bessel sequence.

The expressions for the dual basis elements in (B.5) for $k = 0$ and for $k \neq 0$ are apparently different from each other which implies, as we show below, that the dual Riesz basis cannot have wavelet structure. In [14] it is shown that there is no $\phi \in L^2(\mathbb{R})$ such that $S^{-1}\eta_{j,k} = \phi_{j,k}$ for all $j, k \in \mathbb{Z}$, and the argumentation is as follows. Assume towards a contradiction that there exists a $\phi \in L^2(\mathbb{R})$ that generates the dual frame, that is,

$$\phi_{j,k} = S^{-1}\eta_{j,k} \qquad \text{for all } j, k \in \mathbb{Z}.$$

Then, by (B.2) with $j = 0$ and $k = 1$,

$$\phi_{0,1} = \psi_{0,1} \qquad \text{or} \qquad T_1\phi = T_1\psi,$$

and therefore $\phi = \psi$. By (B.4) with $j = 0$, we have

$$\psi = \phi_{0,0} = \sum_{n=0}^{\infty}(-\varepsilon)^n \psi_{-n,0} = \psi + \sum_{n=1}^{\infty}(-\varepsilon)^n \psi_{-n,0},$$

thus

$$\sum_{n=1}^{\infty}(-\varepsilon)^n \psi_{-n,0} = 0.$$

This is contradicting the ω-independence of the orthonormal basis $\{\psi_{j,k}\}$. Recall that a sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space is said to be ω-independent if whenever $\sum_{k=1}^{\infty} c_k f_k$ is convergent and equal to zero, then necessarily $c_k = 0$ for all k. This is a strong form of linear independence. We conclude that the dual frame of $\{\eta_{j,k}\}$ cannot be generated by a single function.

We extend this argument to P functions for $P \in \mathbb{N}$, that is, we show that the dual frame of $\{\eta_{j,k}\}$ cannot be generated by P functions for any $P \in \mathbb{N}$. Towards a contradiction assume that the dual is generated by $\{\phi_0, \phi_1, \ldots, \phi_{P-1}\} \subset L^2(\mathbb{R})$ with $P < \infty$, or by Proposition 3.2, that the period is P. By [8, Corollary 7], we then have $P = 2^m$ for some $m \in \mathbb{N}$.By our assumption we are lifting the duality to the translation lattice $P\mathbb{Z}$ and pairing

$$\{D_{2^j}T_{Pk}(\eta), D_{2^j}T_{Pk}(T_1\eta), \ldots, D_{2^j}T_{Pk}(T_{P-1}\eta)\}_{j,k\in\mathbb{Z}} \tag{B.6}$$

with

$$\{D_{2^j}T_{Pk}(\phi_0), D_{2^j}T_{Pk}(\phi_1), \ldots, D_{2^j}T_{Pk}(\phi_{P-1})\}_{j,k\in\mathbb{Z}}, \tag{B.7}$$

or, equivalently (here we use that $P = 2^m$), lifting the duality to scale m and pairing

$$\{D_{2^j}T_k(D_{2^m}\eta), D_{2^j}T_k(D_{2^m}T_1\eta), \ldots, D_{2^j}T_k(D_{2^m}T_{P-1}\eta)\}_{j,k\in\mathbb{Z}} \tag{B.8}$$

with

$$\{D_{2^j}T_k(\tilde{\phi}_0), D_{2^j}T_k(\tilde{\phi}_1), \ldots, D_{2^j}T_k(\tilde{\phi}_{P-1})\}_{j,k\in\mathbb{Z}}, \tag{B.9}$$

where $\tilde{\phi}_i = D_{2^m}\phi_i$ for $i \in \{0, 1, \ldots, P-1\}$. Since (B.6) and (B.8) are simply paraphrases of $\{D_{2^j}T_k\eta\}_{j,k\in\mathbb{Z}}$, these (three) systems will have the same frame operator S.

By our assumption the P functions satisfy

$$D_{2^j}T_{Pk}\phi_i = S^{-1}D_{2^j}T_{Pk}T_i\eta, \quad \forall j,k \in \mathbb{Z}, i = 0, \ldots, P-1. \tag{B.10}$$

Since dilation commutes with the frame operator, this reduces to

$$T_{Pk}\phi_i = S^{-1}T_{Pk}T_i\eta, \quad \forall k \in \mathbb{Z},$$

which is relation (B.10) on scale $j = 0$. In general, for canonical duals, we only need to consider duality on scale $j = 0$. We conclude that our assumption is equivalent to the existence of $P = 2^m$ functions $\{\phi_0, \ldots, \phi_{P-1}\}$ satisfying

$$\phi_i = T_{-Pk}S^{-1}T_{Pk+i}\eta \qquad \text{for all } k \in \mathbb{Z}. \tag{B.11}$$

In particular, this means that the expression (B.11) of ϕ_i should be "independent" of $k \in \mathbb{Z}$, but calculating for $i = 0$ with $k = 0$ and $k = 1$ gives (using $P = 2^m$.)

$$\begin{aligned}
0 = \phi_0 - \phi_0 &= S^{-1}\eta - T_{-2^m} S^{-1} T_{2^m}\eta \\
&= \sum_{n=0}^{\infty} (-\varepsilon)^n \psi_{-n,0} - T_{-2^m}(\psi_{0,2^m} - \varepsilon\psi_{-1,2^{m-1}} + \varepsilon^2\psi_{-2,2^{m-2}} - \cdots + (-\varepsilon)^m \psi_{-m,1}) \\
&= \sum_{n=0}^{\infty} (-\varepsilon)^n \psi_{-n,0} - (\psi_{0,0} - \varepsilon\psi_{-1,0} + \varepsilon^2\psi_{-2,0} - \cdots + (-\varepsilon)^m \psi_{-m,0}) \\
&= \sum_{n=m+1}^{\infty} (-\varepsilon)^n \psi_{-n,0}. \qquad \text{(B.12)}
\end{aligned}$$

Again, this is contradicting the ω-independence of the orthonormal basis $\{\psi_{j,k}\}$. This proves that the dual Riesz basis of $\{\eta_{j,k}\}$ cannot be generated by any finite number of functions.

So, we now know that the dual Riesz basis of $\{\eta_{j,k}\}$ has no wavelet structure in the broadest sense possible: the dual is not a wavelet system generated by any finite number of functions. This leads naturally to the question of how much of or which parts of $\{S^{-1}\eta_{j,k}\}$ have wavelet structure. We specifically showed that we cannot unite $S^{-1}\eta_{0,0}$ and $S^{-1}\eta_{0,P}$ in a wavelet system. It is obvious that we cannot associate $S^{-1}\eta_{j,0}$ to other parts of $\{S^{-1}\eta_{j,k}\}$ by wavelet structure due to the infinite series in (B.4). We recall that for $i \in \{0, 1, \ldots, P-1\}$ we need to satisfy equation (B.11) for the dual generator ϕ_i to be well-defined. For any $P = 2^m$, we claim that we can satisfy equation (B.11) for $i = 1, 2, \ldots, P-1$, that is, we can satisfy equation (B.11) except for the case $i = 0$. This implies that for higher values of P a larger part of the dual frame will be associated with a wavelet system; note that the conclusion from the previous paragraph is that no value of P gives the entire dual frame wavelet structure. The claim is easily verified by the following calculations. For odd $i = 1, 3, \ldots, P-1$ we have

$$\begin{aligned}
\phi_i &= T_{-2^m k} S^{-1} T_{2^m k} T_i \eta \\
&= T_{-2^m k} S^{-1} \eta_{0,2^m k+i} = T_{-2^m k}\psi_{0,2^m k+i} = \psi_{0,i},
\end{aligned}$$

and for even nonzero $i = 2, 4, \ldots, P-2$ we have

$$\begin{aligned}
\phi_i &= T_{-2^m k}(\psi_{0,2^m k+i} - \varepsilon\psi_{-1,2^{m-1}k+i/2} + \varepsilon^2\psi_{-2,2^{m-2}k+i/2^2} - \cdots + (\varepsilon)^n \psi_{-n,2^{m-n}k+i/2^n}) \\
&= \psi_{0,i} - \varepsilon\psi_{-1,i/2} + \varepsilon^2\psi_{-2,i/2^2} - \cdots + (-\varepsilon)^n \psi_{-n,i/2^n},
\end{aligned}$$

where $n = \max_{n\in\mathbb{N}}\{2^n | i\}$ such that $i/2^n$ is odd; note that $2^{m-n}k$ is even for $k \in \mathbb{Z}$ since $n < m$. This proves the claim that equation (B.11) is satisfied for $i \neq 0$.

Let us consider the case $i = 0$. We saw in the calculations in (B.12) above that we cannot satisfy (B.11) for $k = 0$ and $k = 1$ simultaneously which, in turn, showed the non-wavelet structure of $\{S^{-1}D_{2^j}T_{Pk}\eta : j,k \in \mathbb{Z}\}$. This non-wavelet structure is not only due to the terms involving infinite series, that is $k = 0$, but also due to "most" other $k \in \mathbb{Z}$. This is seen by taking $k = 1$ and $k = 2$ in (B.11)

$$\begin{aligned} 0 = \phi_0 - \phi_0 &= T_{-2^m 2} S^{-1} T_{2^m 2} \eta - T_{-2^m} S^{-1} T_{2^m} \eta \\ &= \psi_{0,0} - \varepsilon \psi_{-1,0} + \cdots + (-\varepsilon)^m \psi_{-m,0} + (-\varepsilon)^{m+1} \psi_{-m-1,0} \\ &\quad - (\psi_{0,0} - \varepsilon \psi_{-1,0} + \cdots + (-\varepsilon)^m \psi_{-m,0}) \\ &= (-\varepsilon)^{m+1} \psi_{-m-1,0}, \end{aligned}$$

showing that we cannot satisfy (B.11) for $k = 1$ and $k = 2$ with $i = 0$. We notice that a part of the problematic set $\{S^{-1} D_{2^j} T_{Pk} \eta\}$ actually has wavelet structure in the sense that for odd $k \in \mathbb{Z}$ the set $\{S^{-1} D_{2^j} T_{Pk} \eta\}$ takes form of a wavelet system

$$\begin{aligned} \left\{ S^{-1} D_{2^j} T_{Pk} \eta : j \in \mathbb{Z}, k \in 2\mathbb{Z} + 1 \right\} &= \left\{ S^{-1} D_{2^j} T_{2Pk} T_P \eta : j \in \mathbb{Z}, k \in \mathbb{Z} \right\} \\ &= \left\{ D_{2^j} T_{2Pk} \theta : j \in \mathbb{Z}, k \in \mathbb{Z} \right\}, \end{aligned}$$

for $\theta = \psi_{0,2^m} - \varepsilon \psi_{-1,2^{m-1}} + \cdots + (-\varepsilon)^m \psi_{-m,1}$. On the other hand, for $k = \pm 2^n$ ($n \in \mathbb{N}$) the dual element $S^{-1} D_{2^j} T_{Pk} \eta$ is a linear combination of $\psi_{j,k}$ with $m + n + 1$ terms. Since the number of terms depends on n (hence on k), it is apparent that these elements cannot be united in one wavelet system.

Let us make these calculations more explicit and for this assume $P = 4$. The $i = 1, 3$ the dual sets $\{S^{-1} D_{2^j} T_{Pk} T_1 \eta\}$ and $\{S^{-1} D_{2^j} T_{Pk} T_3 \eta\}$ are wavelet systems generated by $\phi_1 = \psi_{0,1}$ and $\phi_3 = \psi_{0,3}$, respectively. Likewise, for $i = 2$ the set $\{S^{-1} D_{2^j} T_{Pk} T_2 \eta\}$ is a wavelet system generated by $\phi_2 = \psi_{0,2} - \varepsilon \psi_{-1,1}$ while (for $i = 0$) the set $\{S^{-1} D_{2^j} T_{Pk} T_0 \eta\}$ is not a wavelet system.

Remark 3. Consider the particular case when ψ is the Lemarie's wavelet, where ψ is a C^∞ function with fast decay. Obviously, these properties are inherited by the dual basis $S^{-1} \eta_{j,k}$ for all $j \in \mathbb{Z}, k \in \mathbb{Z} \setminus \{0\}$. In [14] it is shown that $S^{-1} \eta_{j,0}$ does not belong to $L^p(\mathbb{R})$ for small $p - 1 > 0$. This leads to the observation that the non-wavelet structure of the dual is *not* only due to the "non-regular" elements of the dual basis.

B.2. The space of negative dilates

The space of negative dilates $V(\eta)$ of a frame wavelet η is defined as

$$V(\eta) = \overline{\operatorname{span}} \{ D_{2^j} T_k \eta : j < 0, k \in \mathbb{Z} \} = \overline{\operatorname{span}} \Big(\bigcup_{j<0} W_j(\eta) \Big),$$

where the subspaces W_j are defined by

$$W_j(\eta) = \overline{\operatorname{span}} \{ D_{2^j} T_k \eta : k \in \mathbb{Z} \}.$$

Daubechies and Han [18] verify by direct calculations that the space of negative dilates $V(\eta)$ is not shift invariant. From the previous section we know that the period of η is $P(\eta) = \infty$ hence, by Proposition 3.3, we can conclude that $V(\eta)$ is not even shift invariant with respect to any sublattice of $\mathbb{Z}$. In the following we verify this by direct calculations.

We first show that $V_0(\eta)$ is not $\mathbb{Z}$-SI as is done in [18]. Let $X \oplus Y$ denote the orthogonal direct sum of closed subspaces $X, Y \subset L^2(\mathbb{R})$. We define

$$Y := \overline{\operatorname{span}}\{T_{2k}\psi : k \in \mathbb{Z}\}, \tag{B.13}$$

and denote the orthogonal complement to Y in $W_0(\psi)$ by Y^c, i.e., $W_0(\psi) = Y \oplus Y^c$. Let $V_j(\cdot) = D_{2^j}V(\cdot)$ for $j \in \mathbb{Z}$ so that $V_0(\cdot) = V(\cdot)$. Notice that $\langle D_{2^j}T_k\psi, D_{2^n}T_z\psi\rangle = \delta_{j,n}\delta_{k,z}$ and $V_0(\Psi) = \oplus_{j<0}W_j(\Psi)$. Since $T_1\psi \perp V_0(\Psi)$ and $T_1\psi \perp Y$ we have $T_1\psi \perp (V_0(\psi) \oplus Y)$. Every wavelet is associated with a GMRA, hence $V_0(\psi)$ is shift invariant. Therefore, since obviously $D_{2^{-1}}\psi \in V_0(\psi)$,

$$D_{2^{-1}}T_{1/2}\psi = T_1 D_{2^{-1}}\psi \in V_0(\psi).$$

By the relation

$$D_{2^j}T_k\eta = D_{2^j}T_k\psi \;+\; \varepsilon D_{2^{j+1}}T_{2k}\psi,$$

we see that

$$V_0(\eta) \subseteq \overline{\operatorname{span}}\{D_{2^j}T_k\psi : j < 0, k \in \mathbb{Z}\} \oplus \overline{\operatorname{span}}\{T_{2k}\psi : k \in \mathbb{Z}\} = V_0(\psi) \oplus Y. \tag{B.14}$$

By definition we have $D_{2^{-1}}\eta \in V_0(\eta)$. Now, let us consider a translate of this function:

$$T_1D_{2^{-1}}\eta = T_1D_{2^{-1}}\psi \;+\; \varepsilon T_1 D_{2^{-1}}D_{2^1}\psi = \underbrace{T_1D_{2^{-1}}\psi}_{\in V_0(\Psi)} \;+\; \underbrace{\varepsilon T_1\psi}_{\in Y^c} \notin V_0(\psi) \oplus Y.$$

Since $V_0(\eta)$ is a subspace of $V_0(\psi) \oplus Y$, this implies, in particular, that $T_1D_{2^{-1}}\eta \notin V_0(\eta)$. This shows non-shift invariance of $V_0(\eta)$.

We extend this argumentation to show non $M\mathbb{Z}$-shift invariance of $V_0(\eta)$ for any $M \in \mathbb{N}$.

Theorem B.1. *$V_0(\eta)$ is not shift invariant with respect to any sublattice of $\mathbb{Z}$.*

Before providing a direct proof of Theorem B.1, we analyze the argumentation of Daubechies and Han above. It is obvious that we cannot use the relation $V_0(\eta) \subset V_0(\psi) \oplus Y$ from (B.14) to show non $M\mathbb{Z}$-shift invariance of $V_0(\eta)$ for $M > 1$ since $V_0(\psi) \oplus Y$ is $2\mathbb{Z}$-SI. Hence we need a closer estimate of $V_0(\eta)$, but this is not straightforward due to the complicated structure of $V_0(\eta)$. By definition we have

$$V_0(\eta) = \overline{\operatorname{span}} \bigcup_{j<0} W_j(\eta).$$

Recall that the basis elements of the wavelet Riesz basis are $\eta_{j,k} = \psi_{j,k} + \varepsilon\psi_{j+1,2k}$ for $j, k \in \mathbb{Z}$. Hence, on a fixed scale subspace $W_{j'}(\eta)$ we have orthogonality between the elements, in other words, for fixed $j' \in \mathbb{Z}$ the elements in $\{\eta_{j',k} : k \in \mathbb{Z}\}$ are orthogonal to each other. Furthermore, the elements of scale j' from the Riesz basis are orthogonal to $\{\eta_{j,k} : j \in \mathbb{Z} \setminus \{j'-1, j'+1\}, k \in \mathbb{Z}\}$, that is, to all other scales than the coarser $j'-1$ and finer $j'+1$ scale. In general, an element $\eta_{j',k'}$ is orthogonal to all other elements of the Riesz basis $\{\eta_{j,k}\}$ except one element for odd k and two elements for even, nonzero

k. In spite of these many orthogonalities, the structure of $V_0(\eta)$ is complicated, and this is due to the interaction between the subspaces $W_j(\eta)$ and $W_{j+1}(\eta)$ on two consecutive scales.

For $J \in \mathbb{N}_0$ define the following closed subspaces of $L^2(\mathbb{R})$

$$Y_J = \bigoplus_{j=0}^{J} \overline{\operatorname{span}} \{D_{2^{-j}} T_{2^{J-j}2k}\psi : k \in \mathbb{Z}\}, \tag{B.15}$$

and

$$Z_J = \bigoplus_{k\in\mathbb{Z}} \operatorname{span} \left\{D_{2^{-j}} T_{2^{J-j}(2k+1)}\eta : j = 1, \dots, J\right\}, \tag{B.16}$$

where $Z_0 := \emptyset$. Further, let Z_J^0 denote the following subspace of Z_J

$$Z_J^0 = \bigoplus_{k\in\mathbb{Z}\setminus\{0\}} \operatorname{span} \left\{D_{2^{-j}} T_{2^{J-j}(2k+1)}\eta : j = 1, \dots, J\right\}.$$

For $J = 0, 1$ and 2, the definitions read

$$\begin{aligned}
Y_0 &= \overline{\operatorname{span}} \{T_{2k}\psi\}_{k\in\mathbb{Z}}, \\
Y_1 &= \overline{\operatorname{span}} \{D_{2^{-1}} T_{2k}\psi\}_{k\in\mathbb{Z}} \oplus \overline{\operatorname{span}} \{T_{4k}\psi\}_{k\in\mathbb{Z}}, \\
Y_2 &= \overline{\operatorname{span}} \{D_{2^{-2}} T_{2k}\psi\}_{k\in\mathbb{Z}} \oplus \overline{\operatorname{span}} \{D_{2^{-1}} T_{4k}\psi\}_{k\in\mathbb{Z}} \oplus \overline{\operatorname{span}} \{T_{8k}\psi\}_{k\in\mathbb{Z}},
\end{aligned}$$

and

$$\begin{aligned}
Z_0 &= \emptyset, \\
Z_1 &= \bigoplus_{k\in\mathbb{Z}} \operatorname{span} \{D_{2^{-1}} T_{2k+1}\eta\}, \\
Z_2 &= \bigoplus_{k\in\mathbb{Z}} \operatorname{span} \{D_{2^{-1}} T_{4k+2}\eta, D_{2^{-2}} T_{2k+1}\eta\}.
\end{aligned}$$

Notice that $Y_0 \equiv Y$ (see (B.13)) and that Y_J and Z_J are $2^{J+1}\mathbb{Z}$-SI.

In order to verify Theorem B.1 by direct calculations, we need the following two lemmas.

Lemma B.2. *For all $J \in \mathbb{N}_0$ the following hold:*

$$V_0(\eta) \subset V_{-J}(\psi) \oplus Y_J \oplus Z_J. \tag{B.17}$$

Proof. The orthogonality of the three subspaces are obvious from the definition and the above. For $J = 0$ there is nothing left to show. Let $J = 1$. We have to show that

$$V_0(\eta) \subset V_{-1}(\psi) \oplus (\overline{\operatorname{span}} \{D_{2^{-1}} T_{2k}\psi\}_{k\in\mathbb{Z}} \oplus \overline{\operatorname{span}} \{T_{4k}\psi\}_{k\in\mathbb{Z}}) \oplus \Big(\bigoplus_{k\in\mathbb{Z}} \operatorname{span} \{D_{2^{-1}} T_{2k+1}\eta\}\Big),$$

and we note that the space on the right hand side is $4\mathbb{Z}$-SI, but not shift invariant under $2^m\mathbb{Z}$ for $m = 0, 1$. Suppose $f \in V_0(\eta)$. Then there are coefficients $\{c_{-j,k}\} \in \ell^2(\mathbb{N} \times \mathbb{Z})$ such that

$$f = \sum_{j<0} \sum_{k\in\mathbb{Z}} c_{j,k}\eta_{j,k} = \sum_{k\in\mathbb{Z}} c_{-1,k}\eta_{-1,k} + \sum_{k\in\mathbb{Z}} c_{-2,k}\eta_{-2,k} + \sum_{j<-2} \sum_{k\in\mathbb{Z}} c_{j,k}\eta_{j,k}.$$

Using $\eta_{j,k} = \psi_{j,k} + \varepsilon\psi_{j+1,2k}$ and splitting the first sum for even and odd k yields

$$\begin{aligned} f = &\sum_{k\in\mathbb{Z}} (c_{-1,2k} + \varepsilon c_{-2,k})\psi_{-1,2k} + \varepsilon \sum_{k\in\mathbb{Z}} c_{-1,2k}\psi_{0,4k} + \sum_{k\in\mathbb{Z}} c_{-1,2k+1}\eta_{-1,2k+1} \\ &+ \sum_{k\in\mathbb{Z}} c_{-2,k}\psi_{-2,k} + \sum_{j<-2}\sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k} + \varepsilon \sum_{j<-1}\sum_{k\in\mathbb{Z}} c_{j-1,k}\psi_{j,2k} \end{aligned}$$

with unconditionally convergence since $\{\psi_{j,k}\}$ is an orthonormal basis and the coefficients $\{c_{j,k}\}$ are in ℓ^2. The first two terms above belong to Y_1, the third to Z_1, and the three last terms to $V_{-1}(\psi)$, hence $f \in V_{-1}(\psi) \oplus Y_1 \oplus Z_1$. Similar calculations prove the result for $J > 1$. □

Remark 4. Note that $V_0(\eta) \subset V_j(\psi)$ for $j \geq 1$ trivially, but that $V_0(\eta)$ and $V_j(\psi)$ for $j \leq 0$ are unrelated in the sense that neither $V_0(\eta) \subset V_j(\psi)$ nor $V_0(\eta) \supset V_j(\psi)$ hold for any $j \leq 0$. In particular, $V_j(\psi) \not\subset V_0(\eta) \subset V_1(\psi)$ for $j \leq 0$.

The following fact is trivial.

Lemma B.3. *$V_{-J}(\psi)$ is $2^J\mathbb{Z}$-shift invariant.*

Proof. It follows from the shift invariance of $V_0(\psi)$ using that $f \in V_{-J}(\psi)$ if, and only if, $D_{2^J} f \in V_0(\psi)$ and that $T_k D_{2^J} = D_{2^J} T_{2^J k}$. □

Proof of Theorem B.1. By [8, Corollary 7] it suffices to show non shift invariance by lattices of the form $2^J\mathbb{Z}$ for $J \in \mathbb{N}_0$. By definition $D_{2^{-J-1}}\eta \in V_0(\eta)$. We will show that $T_{2^J} D_{2^{-J-1}}\eta = D_{2^{-J-1}} T_{1/2}\eta \notin V_0(\eta)$. From the definition of η we directly have

$$T_{2^J} D_{2^{-J-1}}\eta = T_{2^J} D_{2^{-J-1}}\psi + \varepsilon D_{2^{-J}} T_1 \psi, \tag{B.18}$$

where $T_{2^J} D_{2^{-J-1}}\psi \in V_{-J}(\psi)$ by Lemma B.3 since $D_{2^{-J-1}}\psi \in V_{-J}(\psi)$.

It is obvious that $D_{2^{-J}} T_1\psi$ is orthogonal to the subspaces $V_{-J}(\psi)$ and Y_J, and thus, in particular, to $T_{2^J} D_{2^{-J-1}}\psi$, and by $\eta_{j,k} = \psi_{j,k} + \varepsilon\psi_{j+1,2k}$, to the functions $\eta_{-j,2^{J-j}(2k+1)}$ for $j \neq J-1, J$. From $\eta_{-j,2^{J-j}(2k+1)} = \psi_{-j,2^{J-j}(2k+1)} + \varepsilon\psi_{-j+1,2^{J-j}(4k+2)}$ we see that

$$D_{2^{-J}} T_1 \psi \perp \eta_{-j,2^{J-j}(2k+1)} \quad \text{for } j = 1, \dots, J \text{ and } k \neq 0.$$

We conclude that $D_{2^{-J}} T_1\psi$ is orthogonal to the subspace $V_{-J}(\psi) \oplus Y_J \oplus Z_J^0$. Thus it follows that $T_{2^J} D_{2^{-J-1}}\eta$ is in $V_{-J}(\psi) \oplus Y_J \oplus Z_J$ if, and only if, $D_{2^{-J}} T_1\psi$ is in the orthogonal complement of $V_{-J}(\psi) \oplus Y_J \oplus Z_J^0$ in $V_{-J}(\psi) \oplus Y_J \oplus Z_J$, that is, $D_{2^{-J}} T_1\psi \in Z_J \ominus Z_J^0 = \operatorname{span}\{D_{2^{-j}} T_{2^{J-j}}\eta : j = 1, \dots, J\}$.

Now, assume that $D_{2^{-J}} T_1\psi \in Z_J \ominus Z_J^0$. Then there exist $\{\alpha_1, \dots, \alpha_J\} \in \mathbb{C}^J$ such that

$$\begin{aligned} D_{2^{-J}} T_1 \psi &= \sum_{j=1}^{J} \alpha_j D_{2^{-j}} T_{2^{J-j}} \eta \\ &= \sum_{j=1}^{J} \alpha_j D_{2^{-j}} T_{2^{J-j}} \psi + \sum_{j=0}^{J-1} \varepsilon \alpha_{j+1} D_{2^{-j}} T_{2^{J-j}} \psi, \end{aligned}$$

thus,

$$0 = \varepsilon\alpha_1\psi_{0,2^J} + \sum_{j=1}^{J-1}(\alpha_j + \varepsilon\alpha_{j+1})\psi_{-j,2^{J-j}} + (\alpha_J - 1)\psi_{-J,1}$$

and, by linear independence of the orthonormal basis $\{\psi_{j,k}\}$,

$$\varepsilon\alpha_1 = 0, \quad \alpha_j + \varepsilon\alpha_{j+1} = 0 \text{ for } j = 1, \dots, J-1, \quad \alpha_J = 1.$$

Since $\varepsilon > 0$, the first two equations implies $\alpha_j = 0$ for $j = 1, \dots, J$ contradicting $\alpha_J = 1$. We conclude that $T_{2^J}D_{2^{-J-1}}\eta$ is not in $V_{-J}(\psi) \oplus Y_J \oplus Z_J$. By Lemma B.2 it follows that $T_{2^J}D_{2^{-J-1}}\eta$ is neither in $V_0(\eta)$ since this is a subspace of $V_{-J}(\psi) \oplus Y_J \oplus Z_J$, hence $V_0(\eta)$ is not 2^J-SI. □

Remark 5. We note that the space of "negative dilates" of the dual frame $\{S^{-1}\eta_{j,k}\}$ is shift invariant since

$$\overline{\operatorname{span}}\left\{S^{-1}\eta_{j,k} : j < 0, k \in \mathbb{Z}\right\} = V_0(\psi),$$

and we see that there exists a orthonormal wavelet ψ which is associated with the GMRA given by $\{D_{2^n}\overline{\operatorname{span}}\{S^{-1}\eta_{j,k} : j < 0, k \in \mathbb{Z}\}\}_{n\in\mathbb{Z}}$. See also [5, Theorem 3.2]

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PAPER I

The canonical and alternate duals of a wavelet frame

Marcin Bownik[1] and Jakob Lemvig

Abstract. We show that there exists a frame wavelet ψ with fast decay in the time domain and compact support in the frequency domain generating a wavelet system whose canonical dual frame cannot be generated by an arbitrary number of generators. On the other hand, there exists infinitely many alternate duals of ψ generated by a single function. Our argument closes a gap in the original proof of this fact by Daubechies and Han [10].

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1. Introduction

This paper explores the relationship between canonical and alternate dual frames of a wavelet frame. One of the first results in this direction is due to Daubechies [9] and Chui and Shi [7] who proved that the canonical dual of a wavelet frame need not have a wavelet structure. Since their example involved a non-biorthogonal Riesz wavelet, it has no alternate dual wavelet frames as well.

In general, if the canonical dual of a frame wavelet has a wavelet structure, then it is quite likely that this frame wavelet has some other wavelet duals. However, the existence of dual wavelet frames does not necessarily imply that the canonical dual must have a wavelet structure. This claim was asserted by Daubechies and Han [10].

Theorem 1.1. *There exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:*

(i) $\hat{\psi}$ is C^∞ and compactly supported,

(ii) its canonical dual frame is not a wavelet system generated by a single function,

(iii) there are infinitely many $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ form a pair of dual frame wavelets.

Unfortunately, the original argument in [10] uses an incorrect formula for the frame operator of a wavelet system owing to a simple change of sign mistake. This invalidates the original proof to the extent that an easy remedy appears to be doubtful. More details about the nature of this problem can be found in Section 3.

Therefore, there is a need to provide an alternative proof of Theorem 1.1. We will use a completely different approach motivated by [5]. Instead of trying to work directly with the frame operator as in [10], we will use a less direct approach using the following result of Weber and the first author [5].

Theorem 1.2 (Theorem 1 in [5]). *Suppose that the canonical dual of a wavelet frame $\{\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k) : j,k \in \mathbb{Z}\}$ has a wavelet structure, i.e., it is of the form $\{\phi_{j,k} : j,k \in \mathbb{Z}\}$ for some frame wavelet ϕ. Then, the space of negative dilates*

$$V(\psi) := \overline{\operatorname{span}}\{\psi_{j,k} : j < 0,\ k \in \mathbb{Z}\} \tag{1.1}$$

is shift invariant (SI).

The paper is organized as follows. In Section 2 we recall some basic facts about the period of a wavelet frame. In particular, we explore the relationship between the period and the number of generators of the canonical dual of a wavelet frame. In Section 3 we give an explicit construction of a frame wavelet ψ as in Theorem 1.1. We prove that its corresponding space of negative dilates $V(\psi)$ lacks shift invariance. Consequently, by Theorem 1.2 we conclude that the canonical dual of the wavelet frame $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is not a wavelet system generated by a single function. In fact, we prove that our example can be adjusted in such a way that the canonical dual can not be generated by arbitrarily many generators, see Theorem 3.1.

Finally, we review basic definitions. A frame for a separable Hilbert space $\mathcal{H}$ is a collection of vectors $\{f_j\}_{j\in\mathcal{J}}$, indexed by a countable set, such that there are constants $0 < C_1 \le C_2 < \infty$ satisfying

$$C_1 \|f\|^2 \le \sum_{j\in\mathcal{J}} |\langle f, f_j\rangle|^2 \le C_2 \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

If the upper bound holds in the above inequality, then $\{f_j\}$ is said to be a Bessel sequence with Bessel constant C_2. The frame operator of $\{f_j\}$ is given by

$$S\colon \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{j\in\mathcal{J}} \langle f, f_j\rangle f_j.$$

This operator is bounded, invertible, and positive. A frame $\{f_j\}$ is said to be *tight* if we can choose $C_1 = C_2$; this is equivalent to $S = C_1 I$, where I is the identity operator.

Two Bessel sequences $\{f_j\}$ and $\{g_j\}$ are said to be *dual frames* if

$$f = \sum_{j\in\mathcal{J}} \langle f, g_j\rangle f_j \qquad \text{for all } f \in \mathcal{H}.$$

It can be shown that two such Bessel sequences indeed are frames, and we shall say that the frame $\{g_j\}$ is *dual* to $\{f_j\}$, and vice versa. At least one dual always exists, it is given by $\{S^{-1}f_j\}$ and called the canonical dual. Redundant frames have several duals; a dual which is not the canonical dual is called an *alternate* dual.

Let $f \in L^2(\mathbb{R})$. Define dilation operator $D_a f(x) = |a|^{1/2} f(ax)$, translation operator $T_b f(x) = f(x-b)$, and modulation operator $E_c f(x) = \mathrm{e}^{2\pi i c x} f(x)$, where $|a| > 1$, b, $c \in \mathbb{R}$. The wavelet system generated by $\Psi = \{\psi_1, \dots, \psi_L\}$, is defined as $\{\psi_{j,k}\}_{j,k\in\mathbb{Z},\psi\in\Psi}$, where $\psi_{j,k} = D_{a^j} T_k \psi$. We say that Ψ and Φ is a pair of dual frame wavelets if their wavelet systems are dual frames. As stated above the canonical dual of a wavelet frame generated by Ψ might not be a wavelet system generated by $|\Psi|$ functions. In this case, we say that the canonical dual of Ψ does not have the wavelet structure.

Given a frame wavelet Ψ, the subspaces $W_j(\Psi)$ are defined by

$$W_j(\Psi) = \overline{\operatorname{span}} \{\psi_{j,k} : k \in \mathbb{Z}, \psi \in \Psi\}, \qquad j \in \mathbb{Z}. \tag{1.2}$$

By this definition we can write the space of negative dilates, introduced in Theorem 1.2, as

$$V(\Psi) = \overline{\operatorname{span}} \bigcup_{j<0} W_j(\Psi).$$

If we have only one generator, that is $L = 1$, we shall write $V(\psi)$ instead of $V(\Psi)$. Suppose that $W \subset L^2(\mathbb{R})$ is a closed subspace. We say W is $M\mathbb{Z}$-SI, $M\mathbb{Z}$ shift invariant, or shift invariant under $M\mathbb{Z}$, $M \in \mathbb{R}$, if $T_{Mz} W \subset W$ for all $z \in \mathbb{Z}$. In the case $M = 1$, we shall say that W is shift invariant, or SI.

For $f \in L^1(\mathbb{R})$, the Fourier transform is defined by $\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi i \xi x}\mathrm{d}x$ with the usual extension to $L^2(\mathbb{R})$. Given a measurable subset $K \subset \mathbb{R}$, we define the space $\check{L}^2(K)$, which is invariant under all translations, by

$$\check{L}^2(K) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset K\}.$$

2. The period of a frame wavelet

Daubechies and Han [10] have introduced the notion of the period of a dyadic wavelet frame in $L^2(\mathbb{R})$. Weber and the first author [5] extended it to a non-dyadic situation as below.

Definition 1. Suppose that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R})$ is a frame wavelet associated with an integer dilation factor a, $|a| \geq 2$. The *period* of Ψ is the smallest integer $p \geq 1$ such that for all $f \in \overline{\operatorname{span}}\,\{T_k\psi : k \in \mathbb{Z}, \psi \in \Psi\}$,

$$T_{pk}S^{-1}f = S^{-1}T_{pk}f \qquad \text{for all } k \in \mathbb{Z},$$

where S is the frame operator of the wavelet frame generated by Ψ. If there is no such p, we say that the period of Ψ is ∞.

We remark that our convention differs from the definitions in [5, 10], where the period is said to be 0 (and not ∞) if no such p exists. The examples of non-biorthogonal Riesz wavelets by Daubechies [9] and Chui and Shi [7] mentioned in the introduction have period ∞; while any tight frame wavelet has period 1.

Following [15], the local commutant of a system of operators $\mathcal{U}$ at the point $f \in L^2(\mathbb{R})$ is defined as

$$\mathcal{C}_f(\mathcal{U}) := \left\{ T \in B(L^2(\mathbb{R})) : TUf = UTf \quad \forall\, U \in \mathcal{U} \right\}.$$

The wavelet system of unitaries is denoted by $\mathcal{A} := \{D_{a^j}T_k : j \in \mathbb{Z}, k \in \mathbb{Z}\}$. The canonical dual of a wavelet frame $\mathcal{A}(\Psi) = \{D_{a^j}T_k\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi}$ is given as

$$\begin{aligned}\left\{ S^{-1}D_{a^j}T_k\psi_i : j,k \in \mathbb{Z}, i = 1,\dots,L \right\} &= \left\{ D_{a^j}S^{-1}T_k\psi_i : j,k \in \mathbb{Z}, i = 1,\dots,L \right\} \\ &= \left\{ D_{a^j}\eta^{k,i} : j,k \in \mathbb{Z}, i = 1,\dots,L \right\},\end{aligned}$$

where S is the frame operator of $\mathcal{A}(\Psi)$, and $\{\eta^{k,i}\}$ is a family of functions, not necessarily with translation structure, indexed by $\{1,\dots,L\}\times\mathbb{Z}$. The canonical dual takes the form of a wavelet system generated by $|\Psi| = L$ functions, i.e.,

$$\begin{aligned}\left\{ S^{-1}D_{a^j}T_k\psi_i : j,k \in \mathbb{Z}, i = 1,\dots,L \right\} &= \left\{ D_{a^j}T_k(S^{-1}\psi_i) : j,k \in \mathbb{Z}, i = 1,\dots,L \right\} \\ &= \left\{ D_{a^j}T_k\phi_i : j,k \in \mathbb{Z}, i = 1,\dots,L \right\},\end{aligned}$$

precisely when $T_kS^{-1}\psi = S^{-1}T_k\psi$ for all $\psi \in \Psi$ and $k \in \mathbb{Z}$; that is, precisely when $S^{-1} \in \mathcal{C}_\psi(\{T_k : k \in \mathbb{Z}\})$ for all $\psi \in \Psi$. Equivalently, the canonical dual of $\mathcal{A}(\Psi)$ has the wavelet structure generated by $|\Psi|$ functions if, and only if, the period of Ψ is one, *c.f.* Proposition 2.3 below.

The following results from [5] will be used in the proof of Theorem 1.1. We restate them here since they were incorrectly stated in [5]. We note that these results can be thought as refinements of Theorem 1.2.

Proposition 2.1 (Proosition 2 in [5]). *Let $M \in \mathbb{N}$. If Ψ is a frame wavelet and the period of Ψ divides M, then $V(\Psi)$ is shift invariant by the lattice $M\mathbb{Z}$. In addition, if Ψ is a Riesz wavelet, then the period of Ψ divides M if, and only if, $V(\Psi)$ is shift invariant by the lattice $M\mathbb{Z}$.*

Corollary 2.2 (Corollary 5 in [5]). *If Ψ is a frame wavelet and the period of Ψ divides $|a|^J$ for some $J \geq 0$, then $D_{a^J}(V(\Psi))$ is shift invariant.*

If the period $P(\Psi)$ of a frame wavelet Ψ is finite, then the canonical dual frame is a wavelet system generated by $P(\Psi) \cdot |\Psi|$ functions, and this is the least number of generators. In this case the wavelet structure of the canonical dual frame is altered since it is based on the translation lattice $P(\Psi) \cdot \mathbb{Z}$ which is sparser than the original lattice $\mathbb{Z}$. Moreover, for any nonnegative integer M, the period of Ψ divides M if, and only if, the canonical dual is a wavelet system generated by $M \cdot |\Psi|$ functions, see the proposition below. The "only if" direction is implicitly contained in the proof of [5, Proposition 2]. For the sake of completeness we prove both directions here.

Proposition 2.3. *Suppose that $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R})$ is a frame wavelet. For any nonnegative integer $M \in \mathbb{N}$, the following statements are equivalent:*

(i) $P(\Psi) \mid M$, i.e., the period of Ψ, denoted $P(\Psi)$, divides M.

(ii) There exist ML functions $\Phi = \{\phi_1, \ldots, \phi_{ML}\}$ such that $\{D_{a^j} T_{Mk} \phi\}_{j,k \in \mathbb{Z}, \phi \in \Phi}$ is the canonical dual of $\{D_{a^j} T_k \psi\}_{j,k \in \mathbb{Z}, \psi \in \Psi} = \{D_{a^j} T_{Mk} \psi\}_{j,k \in \mathbb{Z}, \psi \in \Psi_M}$, where

$$\Psi_M := \{T_m \psi : m = 0, \ldots, M-1, \psi \in \Psi\}.$$

Proof. We note that the frame operator of $\{D_{a^j} T_k \psi\}_{j,k \in \mathbb{Z}, \psi \in \Psi}$ equals the frame operator of $\{D_{a^j} T_{Mk} \psi\}_{j,k \in \mathbb{Z}, \psi \in \Psi_M}$ since the two frames are setwise identical; we denote this operator by S.

We first prove (i) $\Rightarrow$ (ii). By assumption the period of Ψ is finite, hence the definition of the period yields the following equation.

$$T_{P(\Psi)k} S^{-1} f = S^{-1} T_{P(\Psi)k} f \quad \text{for all } k \in \mathbb{Z} \text{ and } f \in W_0(\Psi). \tag{2.1}$$

Since the period of Ψ divides M, we in particular have $P(\Psi)\mathbb{Z} \supset M\mathbb{Z}$, and the above equation gives us

$$T_{Mk} S^{-1} f = S^{-1} T_{Mk} f \quad \text{for all } k \in \mathbb{Z} \text{ and } f \in W_0(\Psi).$$

Consequently, for each $\psi \in \Psi$,

$$S^{-1} T_k \psi = S^{-1} T_{Ml}(T_m \psi) = T_{Ml} S^{-1}(T_m \psi),$$

where $k \in \mathbb{Z}$ is written as $k = Ml + m$ for $l \in \mathbb{Z}$ and $m \in \{0, 1, \ldots, M-1\}$. The last equality in the above equation shows that $S^{-1} \in C_f(\{T_{Mk} : k \in \mathbb{Z}\})$ for every $f \in \Psi_M$, so we arrive at (ii) by taking $\Phi = S^{-1}\Psi_M = \{S^{-1} T_m \psi : m = 0, \ldots, M-1, \psi \in \Psi\}$.

To prove the other direction, (ii) $\Rightarrow$ (i), we assume that the canonical dual of the system $\{D_{a^j} T_{Mk} \psi\}_{j,k \in \mathbb{Z}, \psi \in \Psi_M}$ is generated by ML functions $\Phi = \{\phi_1, \ldots, \phi_{ML}\}$. Since $|\Psi_M| = ML$, it follows that $S^{-1} \in C_\psi(\{T_{Mk} : k \in \mathbb{Z}\})$ for all $\psi \in \Psi_M$, i.e.,

$$S^{-1} T_{Mk}(T_m \psi) = T_{Mk} S^{-1}(T_m \psi) \quad \text{for all } k \in \mathbb{Z},\, m \in \{0, \ldots, M-1\}, \psi \in \Psi. \tag{2.2}$$

In this equation we replace $k \in \mathbb{Z}$ by $k + l$ with $l \in \mathbb{Z}$, whereby we obtain

$$S^{-1} T_{Mk}(T_{Ml+m} \psi) = T_{Mk} S^{-1}(T_{Ml+m} \psi) \quad \text{for all } k, l \in \mathbb{Z},\, m \in \{0, \ldots, M-1\},\, \psi \in \Psi.$$

Now since

$$W_0(\Psi) = \overline{\operatorname{span}}\{T_{Ml+m}\psi : l \in \mathbb{Z},\ m \in \{0, \dots, M-1\},\ \psi \in \Psi\},$$

we see that

$$S^{-1}T_{Mk}f = T_{Mk}S^{-1}f \qquad \text{for all } k \in \mathbb{Z},\ f \in W_0(\Psi), \tag{2.3}$$

and conclude that the period of Ψ is at most M.

To complete the proof we need to show that the period of Ψ is a divisor of M. Assume on the contrary that the period of Ψ is not a divisor of M. Then there are $q, r \in \mathbb{N} \cup \{0\}$ such that $M = qP(\Psi) + r$ and $0 < r < P(\Psi)$. We know that the period of Ψ is finite, so equation (2.1) is satisfied, and by from (2.1) and (2.3) we have

$$S^{-1}T_{P(\Psi)k_1+Mk_2}f = T_{P(\Psi)k_1+Mk_2}S^{-1}f \quad \text{for } k_1, k_2 \in \mathbb{Z},\ f \in W_0(\Psi).$$

Taking $k_1 = -qk$ and $k_2 = k$ for each $k \in \mathbb{Z}$ gives us $rk = P(\Psi)k_1 + Mk_2$. Therefore,

$$S^{-1}T_{rk}f = T_{rk}S^{-1}f \qquad \text{for all } k \in \mathbb{Z},\ f \in W_0(\Psi),$$

which contradicts the minimality of $P(\Psi)$ since $0 < r < P(\Psi)$. □

Remark 1. In the dyadic case and when M is a power of two, Proposition 2.3 reduces to [10, Proposition 2.1]. Indeed, if $M = 2^J$ for some $J \in \mathbb{N}$, then any dyadic wavelet system of the form $\{D_{2^j}T_{Mk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$ with translation with respect to the lattice $M\mathbb{Z}$, can be written as a wavelet system $\{D_{2^j}T_k\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi'}$ using the standard translation lattice $\mathbb{Z}$ and the same number of generators $|\Phi| = |\Phi'|$, see [10]. Corollary 7 in [5] states that the period of a dyadic Riesz wavelet is either a power of two or infinite. Hence, whenever a Riesz wavelet has finite period the canonical dual takes the form $\{D_{2^j}T_k\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi'}$ for some family of functions Φ', where we note that the translation is with respect to the lattice $\mathbb{Z}$.

3. Canonical dual frames without wavelet structure

In this section we will prove Theorem 1.1 by giving an example of a wavelet frame in $L^2(\mathbb{R})$ whose canonical dual does not have wavelet structure. To be precise, we will construct a family of examples, indexed by $J \in \mathbb{N}$, such that the canonical dual cannot be generated by fewer than 2^J functions. In each of these examples the wavelet itself is nice in the sense that it has compact support in the Fourier domain and fast decay in the time domain, and it has nice alternate dual frame wavelets.

Our construction is motivated by the proof of [5, Theorem 2(ii)], where Weber and the first author give an example of a frame wavelet ψ with compact support in the Fourier domain whose canonical dual cannot be generated by one function. The Fourier transform of ψ is not continuous yielding poor decay in the time domain. Furthermore, the space of negative dilates $V(\psi)$ is not $\mathbb{Z}$-SI (this is necessary in order to utilize Theorem 1.2), but it is in fact $2\mathbb{Z}$-SI, hence the canonical dual must be generated by at least two functions, *c.f.* Proposition 2.1. We modify this example so that $\hat{\psi}$ becomes C^∞ and so that the space of negative dilates becomes non $p\mathbb{Z}$-SI for $p < 2^J$ and $p \in \mathbb{N}$ for a chosen $J \in \mathbb{N}$. Hence, we have the following generalization of Theorem 1.1.

Theorem 3.1. *For all $J \in \mathbb{N}$, there exists a frame wavelet $\psi \in L^2(\mathbb{R})$ such that:*

(i) *$\hat{\psi}$ is C^∞ and compactly supported,*

(ii) *its canonical dual frame is not a wavelet system generated by fewer than 2^J function,*

(iii) *there are infinitely many $\tilde{\psi}$ such that ψ and $\tilde{\psi}$ form a pair of dual wavelet frames.*

Before providing the proof of Theorem 3.1, we will analyze the original proof of Theorem 1.1 by Daubechies and Han [10]. The key role in the argument of [10] is played by an explicit formula for the frame operator of a wavelet system.

Proposition 3.2. *Suppose that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R})$ generates a wavelet system which is a Bessel sequence. Let*

$$\mathcal{D} = \{f \in L^2(\mathbb{R}) : \hat{f} \in L^\infty(\mathbb{R}) \text{ and } \operatorname{supp} \hat{f} \subset [-R, -1/R] \cup [1/R, R] \quad \text{for some } R > 1\}.$$

Then its frame operator S is given by

$$\widehat{Sf}(\xi) = \hat{f}(\xi) \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}_l(2^j \xi)|^2 + \sum_{p \in \mathbb{Z}} \sum_{q \in 2\mathbb{Z}+1} \hat{f}(\xi + 2^{-p} q) t_q(2^p \xi) \qquad \text{for a.e. } \xi \in \mathbb{R}, \tag{3.1}$$

and for all $f \in \mathcal{D}$, where

$$t_q(\xi) = \sum_{l=1}^{L} \sum_{j=0}^{\infty} \hat{\psi}_l(2^j \xi) \overline{\hat{\psi}_l(2^j(\xi + q))} \qquad \text{for } q \in \mathbb{Z}.$$

Proposition 3.2 is implicitly contained in the book of Hernández and Weiss [16, Proposition 7.1.19]. This result can be extended to higher dimensions and more general dilations, see [4, 13, 14, 18].

Initially, the problem with the argument of Daubechies and Han appears to be very minor since the formula (2.6) of [10] lacks a negative sign which is present in $\hat{f}(\xi + 2^{-p} q)$ of (3.1). This mistake can be traced back to the proof of Lemma 2.3 in [14]. However, this change of sign has profound effects for the rest of this paper. First, it affects Lemma 3.1 in [10] by wiping out the negative signs in $2^{-j} K_1$ and $2^{-j} K_2$ of formula (3.1). Consequently, it invalidates the proof of [10, Theorem 3.3]. To see this, consider the example borrowed from the paper of Weber and the first author [5].

Example 1. Let $\psi_b \in L^2(\mathbb{R})$ be given by

$$\hat{\psi}_b = \chi_{[-1,-b] \cup [b,1]}.$$

In [5] it is shown that ψ_b is a biorthogonal Riesz wavelet whenever $1/3 \leq b \leq 1/2$. In fact, one can explicitly exhibit its dual biorthogonal wavelet ϕ_b as

$$\hat{\phi}_b = \chi_{[-1,-1/2] \cup [1/2,1]} - \chi_{[-2+2b,-1] \cup [1,2-2b]}.$$

We note that this fact is far from being obvious, since one can also show that ψ_b is **not** a frame wavelet when $1/6 < b < 1/3$, see [5, Example 2]. While ψ_b is of a slightly different form than the function considered in [10, Theorem 3.3], one could arrive at the conclusion that ψ_b is not a biorthogonal wavelet when $b = 1/3$ by following the same argument as in [10]. This stands in a direct contradiction with the above mentioned fact from [5]. In fact, this is how the change of sign mistake in [10] was uncovered by the first author.

In order to prove Theorem 3.1 we need to show two lemmas.

Lemma 3.3. *For every $N \geq 4$ and $0 < \delta < 2^{-N}$, there exists a frame wavelet ψ such that $\hat{\psi} \in C_0^\infty(\mathbb{R})$ and*

$$\begin{aligned} \hat{\psi}(\xi) \neq 0 \iff \xi \in & (-1/2, -1/4) \cup (1/2, 3/4) \\ & \cup \left(-2^{-N+1} - \delta, -2^{-N} + \delta\right) \cup \left(2^{-N} - \delta, 2^{-N+1} + \delta\right) \end{aligned} \tag{3.2}$$

$$\hat{\psi}(\xi) = \hat{\psi}(\xi - 1) \neq 0 \qquad \text{for } \xi \in (1/2, 3/4). \tag{3.3}$$

Proof. Let $\psi^0 \in L^2(\mathbb{R})$ be a frame wavelet such that $\hat{\psi}^0 \in C_0^\infty(\mathbb{R})$ and

$$\hat{\psi}^0(\xi) \neq 0 \iff \xi \in \left(-2^{-N+1} - \delta, -2^{-N} + \delta\right) \cup \left(2^{-N} - \delta, 2^{-N+1} + \delta\right),$$

where $N \geq 4$ and $0 < \delta < 2^{-N}$ as in the assumption. Let $\psi^1 \in L^2(\mathbb{R})$ be such that $\hat{\psi}^1 \in C_0^\infty(\mathbb{R})$ has support in $[-1/2, -1, 4] \cup [1/2, 3/4]$ and

$$\hat{\psi}^1(\xi) = \hat{\psi}^1(\xi - 1) \neq 0 \qquad \text{whenever } \xi \in (1/2, 3/4). \tag{3.4}$$

For any such $\psi^1 \in L^2(\mathbb{R})$ the sequence $\{D_{2^j} T_k \psi^1\}$ generates a Bessel sequence by [17, Theorem 13.0.1] or by the proof of [8, Lemma 3.4].

Define $\psi \in L^2(\mathbb{R})$ by $\psi = \psi^0 + \varepsilon\psi^1$, where $\varepsilon\psi^1$ acts as a perturbation on the wavelet frame generated by ψ^0 and ensures that ψ satisfies (3.3), see also Figure 1. Denote the frame bounds of $\{D_{2^j} T_k \psi^0\}$ by C_1 and C_2, and the Bessel bound of $\{D_{2^j} T_k \psi^1\}$ by C_0. The function $\varepsilon\psi^1$ generates a Bessel sequence with bound $\varepsilon^2 C_0$. Hence, for sufficiently small $\varepsilon > 0$, we have $\varepsilon^2 C_0 < C_1$, and by a perturbation result [6, Corollary 2.7] or [12, Theorem 3], we conclude that ψ generates a wavelet frame. By our construction $\hat{\psi}$ is in $C_0^\infty(\mathbb{R})$ and satisfies (3.2) and (3.3).

Finally, let us illustrate how one can construct two such functions ψ^0 and ψ^1. For $N \geq 4$ and $0 < \delta < 2^{-N}$, define the function η by

$$\hat{\eta} = h_\delta * \chi_{[-2^{-N+1}, -2^{-N}] \cup [2^{-N}, 2^{-N+1}]}, \tag{3.5}$$

where $h_\delta(x) = \delta^{-1} h(x/\delta)$ with $h \in C_0^\infty(\mathbb{R})$, $h \geq 0$, $\int_\mathbb{R} h(x) \mathrm{d}x = 1$, and $\operatorname{supp} h \subset [-1, 1]$. This yields $\hat{\eta} \in C^\infty$ with

$$\hat{\eta}(\xi) \neq 0 \iff \xi \in \left(-2^{-N+1} - \delta, -2^{-N} + \delta\right) \cup \left(2^{-N} - \delta, 2^{-N+1} + \delta\right).$$

By $\|\hat{\eta}\|_{L^\infty} \leq 1$ and the above, there exist constants $C_1, C_2 > 0$, such that

$$0 < C_1 \leq \sum_{j \in \mathbb{Z}} \left|\hat{\eta}(2^j \xi)\right|^2 \leq C_2 < 2 \quad \text{for all } \xi \in \mathbb{R} \setminus \{0\}.$$

Moreover, for $q \in 2\mathbb{Z}+1$,

$$t_q(\xi) := \sum_{j=0}^{\infty} \hat{\eta}(2^j\xi)\overline{\hat{\eta}(2^j(\xi+q))} = 0 \quad \text{for all } \xi \in \mathbb{R},$$

since $\hat{\eta}(2^j\cdot)$ and $\hat{\eta}(2^j(\cdot+q))$ have disjoint support for all $j \geq 0$. We define ψ^0 as a normalization of η by

$$\hat{\psi}^0(\xi) = \frac{\hat{\eta}(\xi)}{\sqrt{\sum_{j\in\mathbb{Z}} |\hat{\eta}(2^j\xi)|^2}} \quad \text{for } \xi \in \mathbb{R} \setminus \{0\}, \tag{3.6}$$

and $\hat{\psi}^0(0) = 0$. Consequently, we have $\sum_{j\in\mathbb{Z}} |\hat{\psi}^0(2^j\xi)|^2 = 1$ and $t_q(\xi) = 0$ for $\xi \in \mathbb{R}$ and $q \in 2\mathbb{Z}+1$. By [16, Theorem 7.1.6], ψ^0 generates a tight wavelet frame with frame bound 1, and it has the desired properties. For the proof of the lemma the last normalization step could be omitted since η itself generates a (non-tight) frame. However, it is included since we later want to use the fact that the ψ^0 can be chosen to be a tight frame wavelet with frame bound 1.

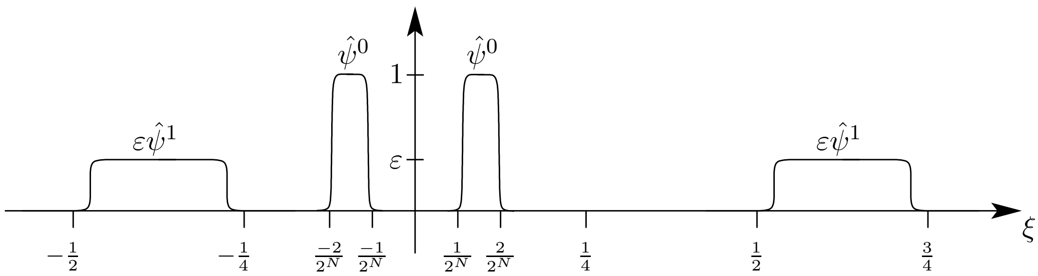

Figure 1: Sketch of the graph of $\hat{\psi} = \hat{\psi}^0 + \varepsilon\hat{\psi}^1$.

The construction of the perturbation term ψ^1 is straightforward. Let $\theta_\lambda := h_\lambda * \chi_{[1/2+\lambda, 3/4-\lambda]}$ for some $0 < \lambda < 1/8$, where h_λ is defined as above. Define ψ^1 by $\hat{\psi}^1 = \theta_\lambda + T_{-1}\theta_\lambda$. This makes $\hat{\psi}^1$ a C^∞ function with compact support in $[-1/2, -1, 4] \cup [1/2, 3/4]$, satisfying equation (3.4). This completes the proof of Lemma 3.3. ☐

Lemma 3.4. *Suppose that a function $\psi \in L^2(\mathbb{R})$ satisfies (3.2) and (3.3) for some $N \geq 4$ and $0 < \delta < 2^{-N}$. Then, the space of negative dilates $V(\psi)$ is **not** $p\mathbb{Z}$-SI for any $p < 2^{N-3}$, $p \in \mathbb{N}$.*

Proof. To prove this claim we will look at the subspaces $W_j(\psi)$ for $j \leq 0$, defined by

$$W_j(\psi) = \overline{\operatorname{span}}\{D_{2^j}T_k\psi : k \in \mathbb{Z}\}, \qquad j \in \mathbb{Z}.$$

First, consider a principal shift invariant (PSI) subspace $W_0(\psi) = \overline{\operatorname{span}}\{T_k\psi\}_{k\in\mathbb{Z}}$. By a result in [11], see also [3], this subspace can be described as

$$W_0(\psi) = \{f \in L^2(\mathbb{R}) : \hat{f} = \hat{\psi}m \quad \text{for some measurable, 1-periodic } m\}.$$

Hence, by (3.2) and (3.3) we have

$$\begin{aligned} W_0(\psi) = \Big\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-1/2,-1/4] \cup [1/2,3/4] \cup K \\ \hat{f}(\xi - 1) = \hat{f}(\xi) \quad \text{a.e. } \xi \in [1/2,3/4] \Big\}, \\ \text{where } K = \Big[-2^{-N+1} - \delta, -2^{-N} + \delta\Big] \cup \Big[2^{-N} - \delta, 2^{-N+1} + \delta\Big]. \end{aligned} \tag{3.7}$$

Applying the scaling relation $W_j(\psi) = D_{2^j} W_0(\psi)$ to (3.7) yields

$$\begin{aligned} W_j(\psi) = \Big\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset \Big[-2^{j-1}, -2^{j-2}\Big] \cup \Big[2^{j-1}, 3/2 \cdot 2^{j-1}\Big] \cup 2^j K, \\ \hat{f}(\xi - 2^j) = \hat{f}(\xi) \quad \text{a.e. } \xi \in \Big[2^{j-1}, 3/2 \cdot 2^{j-1}\Big] \Big\}. \end{aligned} \tag{3.8}$$

Therefore, each space $W_j(\psi)$, $j \in \mathbb{Z}$, can be decomposed as the orthogonal sum

$$W_j(\psi) = W_j^0 \oplus W_j^1, \qquad \text{where} \tag{3.9}$$

$$W_j^0 = \check{L}^2(2^j K), \tag{3.10}$$

$$\begin{aligned} W_j^1 = \Big\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset \Big[-2^{j-1}, -2^{j-2}\Big] \cup \Big[2^{j-1}, 3/2 \cdot 2^{j-1}\Big], \\ \hat{f}(\xi - 2^j) = \hat{f}(\xi) \quad \text{a.e. } \xi \in \Big[2^{j-1}, 3/2 \cdot 2^{j-1}\Big] \Big\}. \end{aligned} \tag{3.11}$$

Using (3.9), it is possible to describe the space of negative dilates

$$V(\psi) = \overline{\operatorname{span}}\Big(\bigcup_{j<0} W_j(\psi) \Big)$$

in the Fourier domain. However, such a description would be quite complicated owing to interactions of the spaces W_j^0 and W_j^1 at various scales $j < 0$.

Instead, we consider another space

$$\widetilde{V}(\psi) = V(\psi) \cap \check{L}^2((-\infty, -2^{-N+1}] \cup [2^{-N+2}, \infty)).$$

By (3.10) and $K \subset (-2^{-N+2}, 2^{-N+2})$, we have

$$W_j^0 \subset \check{L}^2([-2^{-N+1}, 2^{-N+2}]) \qquad \text{for } j < 0.$$

Likewise, by (3.11) we have

$$W_j^1 \subset \begin{cases} \check{L}^2([-2^{-N+1}, 2^{-N+2}]) & \text{for } j \le -N+2, \\ \check{L}^2((-\infty, -2^{-N+1}] \cup [2^{-N+2}, \infty)) & \text{for } j \ge -N+3. \end{cases}$$

Combining the last four equations with (3.9) yields

$$\widetilde{V}(\psi) = \overline{\operatorname{span}}\Big(\bigcup_{j<0} W_j(\psi) \cap \check{L}^2((-\infty, -2^{-N+1}] \cup [2^{-N+2}, \infty)) \Big) = \overline{\operatorname{span}}\Big(\bigcup_{j=-N+3}^{-1} W_j^1 \Big),$$

and further, by the orthogonality of the subspaces $W^1_{-N+3}, \ldots, W^1_{-1}$,

$$\widetilde{V}(\psi) = \bigoplus_{j=-N+3}^{-1} W^1_j.$$

Consequently, by (3.11),

$$\begin{aligned}\widetilde{V}(\psi) = \Big\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset \bigcup_{j=-N+3}^{-1} 2^j([-1/2, -1/4] \cup [1/2, 3/4]),& \\ \hat{f}(\xi - 2^{-1}) = \hat{f}(\xi) \quad \text{a.e. } \xi \in \left[2^{-2}, 3/2 \cdot 2^{-2}\right],& \\ \hat{f}(\xi - 2^{-2}) = \hat{f}(\xi) \quad \text{a.e. } \xi \in \left[2^{-3}, 3/2 \cdot 2^{-3}\right],& \\ \vdots \qquad\qquad\qquad \vdots \qquad& \\ \hat{f}(\xi - 2^{-N+3}) = \hat{f}(\xi) \quad \text{a.e. } \xi \in \left[2^{-N+2}, 3/2 \cdot 2^{-N+2}\right] \Big\}.& \qquad (3.12)\end{aligned}$$

Assume, towards a contradiction, that $V(\psi)$ is $p\mathbb{Z}$-SI for some $p < 2^{N-3}$ with $p \in \mathbb{N}$. Then, $\widetilde{V}(\psi)$ is $p\mathbb{Z}$-SI as well. Define $f \in L^2(\mathbb{R})$ by

$$\hat{f} = \chi_{I_N \cup (I_N - 2^{-N+3})}, \qquad \text{where } I_N = [2^{-N+2}, 3/2 \cdot 2^{-N+2}].$$

Then $f \in \widetilde{V}(\psi)$, and by our hypothesis we have $T_{pk} f \in \widetilde{V}(\psi)$ for all $k \in \mathbb{Z}$. Equivalently, using $\mathcal{F} T_k = E_{-k} \mathcal{F}$, we have $E_{pk} \hat{f} \in \mathcal{F}(\widetilde{V}(\psi))$ for all $k \in \mathbb{Z}$. For $k = 1$, this implies that $E_p \hat{f}(\xi) = \mathrm{e}^{2\pi i p \xi} \chi_{I_N \cup (I_N - 2^{-N+3})}(\xi) \in \mathcal{F}(\widetilde{V}(\psi))$. By (3.12),

$$\mathrm{e}^{2\pi i p(\xi - 2^{-N+3})} = \mathrm{e}^{2\pi i p \xi} \qquad \text{for a.e. } \xi \in I_N.$$

This can only be satisfied if $\mathrm{e}^{-2\pi i p 2^{-N+3}} = 1$, which contradicts the hypothesis that $1 \le p < 2^{N-3}$. This completes the proof of Lemma 3.4. □

Remark 2. A more detailed analysis shows that $V(\psi)$ is $2^{N-2}\mathbb{Z}$-SI, and it is not shift invariant by any sublattice of $\mathbb{Z}$ strictly larger than $2^{N-2}\mathbb{Z}$. Since we do not need such precise assertion, we will skip its proof.

Finally, we are ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Take any $J \in \mathbb{N}$. Suppose that ψ is a frame wavelet as in Lemma 3.3 with $N = J+3$. By Lemma 3.4 and Proposition 2.1, the period of ψ is at least 2^{N-3}. Hence, by Proposition 2.3, we need at least 2^J functions to generate the canonical dual of $\{D_{2^j} T_k \psi\}_{j,k\in\mathbb{Z}}$.

We have only left to show that the wavelet frame generated by ψ has infinitely many alternate duals that are generated by one function. For this purpose it is convenient to assume that $\psi = \psi^0 + \varepsilon\psi^1$ is of the same form as in the proof of Lemma 3.3, i.e., ψ^0 generates a tight frame with frame bound 1. Hence, the functions ψ and ψ^0 satisfy the characteristic equations

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(2^j\xi)\overline{\hat{\psi}^0(2^j\xi)} = 1, \qquad \text{a.e. } \xi \in \mathbb{R},$$

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}^0(2^j(\xi+q))} = 0, \qquad \text{a.e. } \xi \in \mathbb{R} \text{ for odd } q \in \mathbb{Z},$$

since $\hat{\psi} = \hat{\psi}^0$ on $\operatorname{supp}\hat{\psi}^0$ and since $\hat{\psi}(2^j\cdot)\hat{\psi}^0(2^j(\cdot+q)) = 0$ for all $j \geq 0$ and all odd q. We conclude that $\{\psi^0_{j,k}\}$ is a dual frame of $\{\psi_{j,k}\}$. Since $\{\psi^0_{j,k}\}$ is generated by one function, it is apparent from the above that $\{\psi^0_{j,k}\}$ must be an alternate dual.

Any function $\phi \in L^2(\mathbb{R})$ defined by $\hat{\phi} = \hat{\psi}^0 + h$, where

$$h \in \mathbb{C}^\infty(\mathbb{R}), \qquad \operatorname{supp} h \subset [-1/4, 1/2], \qquad \operatorname{supp} h \cap \operatorname{supp}\hat{\psi}^0 = \emptyset, \qquad h(0) = 0,$$

generates a Bessel sequence by [17, Theorem 13.0.1]. Since ψ and ϕ satisfy the characteristic equations above, such a ϕ is an alternate dual frame wavelet of ψ. This example demonstrates that we have infinitely many alternate duals, and completes the proof of Theorem 3.1. □

We end by putting our example in a perspective with other known results.

Remark 3. Auscher [1] proved that every "regular" orthonormal wavelet $\psi \in L^2(\mathbb{R})$ is associated with an MRA. "Regular" means that $|\hat{\psi}|$ is continuous and $\hat{\psi}(\xi) = O(|\xi|^{-1/2-\delta})$ as $|\xi| \to \infty$ for some $\delta > 0$, see [16, Corollary 7.3.16]. This fact does not hold for tight frame wavelets. In fact, Baggett et al. [2] constructed a non-MRA C^r tight frame wavelet with rapid decay for any $r \in \mathbb{N}$. Moreover, their tight frame wavelet is associated with a GMRA having the same dimension/multiplicity function as the Journé wavelet. Once we allow non-tight frame wavelets we might lose even the GMRA property. Indeed, the frame wavelet from Theorem 3.1 is an example of a non-GMRA C^∞ frame wavelet with rapid decay.

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PAPER II

Constructing pairs of dual bandlimited framelets with desired time localization in $L^2(\mathbb{R})$

Jakob Lemvig

Abstract. For sufficiently small translation parameters, we prove that any bandlimited function ψ, for which the dilations of its Fourier transform form a partition of unity, generates a wavelet frame with a dual frame also having the wavelet structure. This dual frame is generated by a finite linear combination of dilations of ψ with explicitly given coefficients. The result allows a simple construction procedure for pairs of dual wavelet frames whose generators have compact support in the Fourier domain and desired time localization. The construction is based on characterizing equations for dual wavelet frames and relies on a technical condition. We exhibit a general class of function satisfying this condition; in particular, we construct piecewise polynomial functions satisfying the condition.

Keywords. Dual frames · Framelets · Non-tight frames · Partition of unity · Bandlimited wavelets

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1. Introduction

Let $\psi \in L^2(\mathbb{R})$ be a function such that $\hat{\psi}$ is compactly supported and the functions $\xi \mapsto \hat{\psi}(a^j\xi)$, $j \in \mathbb{Z}$, form a partition of unity for some $a > 1$. We prove that for sufficiently small translation parameter b the function ψ generates a wavelet frame $\{a^{j/2}\psi(a^jx - bk) : j, k \in \mathbb{Z}\}$ with a dual wavelet frame generated by a finite linear combination of dilations of ψ. The result allows a construction procedure for pairs of dual wavelet frames generated by bandlimited functions with fast decay in the time domain where both generators are explicitly given.

The principal idea used in the proof of Theorem 2.3 comes from Christensen's construction of dual Gabor frames in [6]. Our construction is similar, but it takes place in the Fourier domain. The proof of Theorem 2.3 and the construction procedure provided by this theorem are based on the well-known characterizing equations for dual wavelet frames by Chui and Shi [8].

Our aim is to provide a construction of a pair of dual frame generators ψ and ϕ for which the functions ψ and ϕ are explicitly given in the sense that the functions or their Fourier transform are given as finite linear combinations of elementary functions. To be precise, the construction uses ψ as a starting point and defines the dual generator ϕ as a finite linear combination of dilations of ψ with explicitly given coefficients. This gives us control of the properties of both generators as opposed to using canonical duals.

The construction of redundant wavelet representations is often restricted to tight frames in order to avoid the cumbersome inversion of the frame operator. However, in this paper we consider general non-tight, non-canonical, non-dyadic dual wavelet frames. The construction of wavelet frames is usually based on the (mixed) unitary or oblique extension principle [7, 9, 12, 13]. These principles lead to dual or tight frame wavelets with many desirable features: compact support, high order of vanishing moments, high smoothness, and symmetry/antisymmetry; in particular, explicitly given spline generators are constructed from B-spline multiresolution analysis in [7, 9]. In these and similar constructions one cannot do with fewer than two generators (see [7, Theorem 9] and [9, Theorem 3.8] including the succeeding remark); in addition, higher smoothness leads to more generators or larger support of the generators. Our construction leads to frame wavelet with similar properties, the most notable difference is that the generators have compact support in the Fourier domain, not in the time domain.

Wavelet frames constructed by the unitary extension principle from a B-spline multiresolution analysis will always have one generator with only one vanishing moment yielding a wavelet system with approximation order of at most 2; this problem is circumvented in the oblique extension principle. When multiple generators are needed in our construction, all of these will share the same properties. In Examples 2 and 3 the constructed wavelet frames are generated by only *one* function, and in these cases the smoothness of the generator does not affect the size of the support (that is, in the Fourier domain).

Our construction is explicit, and it works for arbitrary real dilations, but as a drawback the wavelet frame generators will not have compact support in the time domain leading to infinite impulse response filters. In the dyadic case an efficient algorithm can be implemented by using the fast Fourier transform, see for example the fractional spline wavelet software for Matlab by Unser and Blu [3]. The idea is to perform the calculation in the Fourier domain using multiplication and periodization in place of convolution and down-sampling. For this to work, we need the frequency response of the filter coefficients (sometimes simply called filters or masks and often denoted by τ_i, m_i, or H_i), but we get this almost directly from our construction; the frequency response of *both* high pass filters (decomposition and reconstruction) can be obtained from dilations of $\hat{\psi}$. Note that this relies crucially on the fact that the dual generator ϕ is defined as a finite linear combination of dilations of ψ with explicitly given coefficients.

The paper is organized as follows. In Section 2 we prove the main result of this article, Theorem 2.3. The theorem contains a technical condition on partition of unity, and we address this problem in Example 1 where we explicitly construct functions that satisfy the condition. A note on the terminology: the functions in the "partition of unity" are not assumed to be non-negative, but can take any real value. In Example 2 we give an example of a pair of smooth, fast decaying, symmetric generators with the translation parameter being 1. The construction of dual wavelet frames using Theorem 2.3 often imposes the translation parameter to be small, e.g., smaller than 1. Consequently, we want methods to expand the range of the translation parameter, and this is the topic of Section 2.2. In Section 3 we show that the representation of functions provided by Theorem 2.3 with the explicitly given dual frame is advantageous over similar representations using tight frames or canonical dual frames. In Section 4 we present another application of Theorem 2.3 with generators in the Schwartz space. However, the construction in this example is less explicit than in the first example. We end this paper with some remarks on constructions of pairs of dual wavelet frames for the Hardy space.

We end this introduction by reviewing some basic definitions and with an observation on the canonical dual frame. A frame for a separable Hilbert space $\mathcal{H}$ is a collection of vectors $\{f_j\}_{j\in\mathcal{J}}$ with a countable index set $\mathcal{J}$ if there are constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1 \|f\|^2 \le \sum_{j\in\mathcal{J}} |\langle f, f_j\rangle|^2 \le C_2 \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

If the upper bound holds in the above inequality, then $\{f_j\}$ is said to be a Bessel sequence with Bessel constant C_2. For a Bessel sequence $\{f_j\}$ we define the frame operator by

$$S\colon \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{j\in\mathcal{J}} \langle f, f_j\rangle f_j.$$

This operator is bounded, invertible, and positive. A frame $\{f_j\}$ is said to be *tight* if we can choose $C_1 = C_2$; this is equivalent to $S = C_1 I$ where I is the identity operator. Two Bessel sequences $\{f_j\}$ and $\{g_j\}$ are said to be *dual* frames if

$$f = \sum_{j\in\mathcal{J}} \langle f, g_j\rangle f_j \quad \forall f \in \mathcal{H}.$$

It can be shown that two such Bessel sequences are indeed frames. Given a frame $\{f_j\}$, at least one dual always exists; it is called the canonical dual and is given by $\{S^{-1}f_j\}$. Only redundant frames have several duals.

For $f \in L^2(\mathbb{R})$, we define the dilation operator by $D_a f(x) = a^{1/2} f(ax)$ and the translation operator by $T_b f(x) = f(x-b)$ where $1 < a < \infty$ and $b \in \mathbb{R}$. We say that $\{D_{a^j} T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ is the wavelet system generated by ψ where $a > 1$ and $b > 0$. In the following we use the index set $(j,k) \in \mathbb{Z}\times\mathbb{Z}$ whenever a sequence is stated without index set. If $\{D_{a^j} T_{bk}\psi\}$ is a frame for $L^2(\mathbb{R})$, the generator ψ is termed a framelet or frame wavelet. For $f \in L^1(\mathbb{R})$ the Fourier transform is defined by $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) \mathrm{e}^{-2\pi i \xi x} \mathrm{d}x$ with the usual extension to $L^2(\mathbb{R})$. Given a measurable set $K \subset \mathbb{R}$ we define the Paley-Wiener space $\check{L}^2(K)$, which is invariant under all translations, by $\check{L}^2(K) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset K\}$.

2. Construction of dual wavelet frames

Our main result, Theorem 2.3, is obtained from the following result by Chui and Shi [8]. The result is stated in the last two lines of Section 4 on page 263 in their article.

Theorem 2.1. *Let $a > 1$, $b > 0$, and $\psi, \tilde{\psi} \in L^2(\mathbb{R})$. Suppose the two wavelet systems $\{D_{a^j} T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j} T_{bk}\tilde{\psi}\}_{j,k\in\mathbb{Z}}$ form Bessel families. Then $\{D_{a^j} T_{bk}\psi\}$ and $\{D_{a^j} T_{bk}\tilde{\psi}\}$ will be dual frames if the following conditions hold*

$$\sum_{j\in\mathbb{Z}} \overline{\hat{\psi}(a^j\xi)}\hat{\tilde{\psi}}(a^j\xi) = b \qquad \text{a.e. } \xi \in \mathbb{R}, \tag{2.1}$$

$$\hat{\tilde{\psi}}(\xi)\overline{\hat{\psi}(\xi+q)} = 0 \qquad \text{a.e. } \xi \in \mathbb{R} \text{ for } 0 \neq q \in b^{-1}\mathbb{Z}. \tag{2.2}$$

The conditions (2.1) and (2.2) are also necessary when $a > 1$ is such that a^j is irrational for all positive integers j, see [8, p. 263]. For this reason the above conditions are often refereed to as *characterizing equations* for such irrational dilations. The result in Theorem 2.1 follows from the general result of characterizing equations for dual wavelet frames [8, Theorem 2].

The next result, Lemma 2.2, gives a sufficient condition for a wavelet system to be a Bessel sequence. Its proof can be found in [5, Theorem 11.2.3].

Lemma 2.2. *Let $a > 1$, $b > 0$, and $f \in L^2(\mathbb{R})$. Suppose that*

$$C_2 = \frac{1}{b} \sup_{|\xi|\in[1,a]} \sum_{j,k\in\mathbb{Z}} \left|\hat{f}(a^j\xi)\hat{f}(a^j\xi + k/b)\right| < \infty.$$

Then the affine system $\{D_{a^j} T_{bk} f\}$ is a Bessel sequence with bound C_2.

Theorem 2.1 and Lemma 2.2 are all we need to prove our main result, Theorem 2.3. The main result contains the technical condition (2.3) on ψ. In the example following the proof of the main result, Example 1, we explicitly construct functions satisfying this condition.

Theorem 2.3. *Let $n \in \mathbb{N}$, $a > 1$, and $\psi \in L^2(\mathbb{R})$. Suppose that $\hat{\psi} \in L^\infty(\mathbb{R})$ is a real-valued function with $\operatorname{supp}\hat{\psi} \subset [-a^c, -a^{c-n}] \cup [a^{c-n}, a^c]$ for some $c \in \mathbb{Z}$, and that*

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(a^j\xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{2.3}$$

Let $b \in (0, 2^{-1}a^{-c}]$. Then the function ψ and the function ϕ defined by

$$\phi(x) = b\psi(x) + 2b\sum_{j=1}^{n-1} a^{-j}\psi(a^{-j}x) \quad \text{for } x \in \mathbb{R}, \tag{2.4}$$

generate dual frames $\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{bk}\phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof. By assumption the function $\hat{\psi}$ is compactly supported in $\mathbb{R}\setminus\{0\}$; the same holds for $\hat{\phi}$ since, by the definition in (2.4) and the linearity of the Fourier transform,

$$\hat{\phi}(\xi) = b\hat{\psi}(\xi) + 2b\sum_{j=1}^{n-1}\hat{\psi}(a^j\xi).$$

An application of Lemma 2.2 shows that the functions ψ and ϕ generate wavelet Bessel sequences.

To conclude that ψ and ϕ generate dual wavelet frames we will show that conditions (2.1) and (2.2) in Theorem 2.1 hold. By a^j-dilation periodicity of the sum in condition (2.1) it is sufficient to verify this condition on the intervals $[-a,-1]$ and $[1,a]$. On these two intervals, only finitely many terms in the sum (2.3) are nonzero since $\hat{\psi}$ has compact support; in particular, only the terms $j = c-n, c-n+1, \dots, c-1$ contribute which follows from the support of the dilations of $\hat{\psi}$:

$$\begin{aligned}
&\operatorname{supp}\hat{\psi}(a^{c-n}\cdot) \subset [-a^n,-1]\cup[1,a^n],\\
&\operatorname{supp}\hat{\psi}(a^{c-n+1}\cdot) \subset [-a^{n-1},-1/a]\cup[1/a,a^{n-1}],
\end{aligned}$$

and continuing to

$$\operatorname{supp}\hat{\psi}(a^{c-1}\cdot) \subset [-a,-a^{-n+1}]\cup[a^{-n+1},a].$$

For $|\xi| \in [1,a]$, by the assumption, we have

$$\begin{aligned}
1 &= \Big(\sum_{j\in\mathbb{Z}}\hat{\psi}(a^j\xi)\Big)^2 = \Big(\sum_{j=c-n}^{c-1}\hat{\psi}(a^j\xi)\Big)^2 \qquad (2.5)\\
&= \Big[\hat{\psi}(a^{c-n}\xi) + \hat{\psi}(a^{c-n+1}\xi) + \cdots + \hat{\psi}(a^{c-1}\xi)\Big]^2\\
&= \hat{\psi}(a^{c-n}\xi)\Big[\hat{\psi}(a^{c-n}\xi) + 2\hat{\psi}(a^{c-n+1}\xi) + \cdots + 2\hat{\psi}(a^{c-1}\xi)\Big]\\
&\quad + \hat{\psi}(a^{c-n+1}\xi)\Big[\hat{\psi}(a^{c-n+1}\xi) + 2\hat{\psi}(a^{c-n+2}\xi) + \cdots + 2\hat{\psi}(a^{c-1}\xi)\Big]\\
&\quad + \cdots + \hat{\psi}(a^{c-1}\xi)\Big[\hat{\psi}(a^{c-1}\xi)\Big]\\
&= \frac{1}{b}\sum_{j=c-n}^{c-1}\hat{\psi}(a^j\xi)\hat{\phi}(a^j\xi) = \frac{1}{b}\sum_{j\in\mathbb{Z}}\overline{\hat{\psi}(a^j\xi)}\hat{\phi}(a^j\xi),
\end{aligned}$$

hence ψ and ϕ satisfy condition (2.1).

To realize that ψ and ϕ satisfy equation (2.2) as well, we note $\operatorname{supp}\hat{\psi}(\cdot \pm q) \subset \bar{B}(\mp q, a^c)$ and $\operatorname{supp}\hat{\phi} \subset \big[-a^c, -a^{c-2n+1}\big]\cup\big[a^{c-2n+1}, a^c\big] \subset \bar{B}(0,a^c)$ where $\bar{B}(x,r) =$

$[x-r, x+r]$ denotes the closed ball with center at x and radius r. The two functions above will have disjoint support modulo null sets whenever $|q| \geq 2a^c$. Consequently, by choosing the translation parameter $b \leq 2^{-1}a^{-c}$, the two functions in condition (2.2) will have disjoint support for all $q \in b^{-1}\mathbb{Z} \setminus \{0\}$ since $\min |b^{-1}\mathbb{Z} \setminus \{0\}| = 1/b \geq 2a^c$, and the condition will be trivially satisfied. □

Whenever $n = 1$ in Theorem 2.3 above, we have $\phi = b\psi$ by equation (2.4), thus ψ generates a tight frame with bound b. In this case, i.e., $n = 1$, the choices of ψ are very limited since functions ψ satisfying the conditions in Theorem 2.3 with $n = 1$ must be of the form $\hat{\psi} = \chi_{a^c S}$, where $S = [-1, -1/a] \cup [1/a, 1]$. As a consequence, interesting constructions using Theorem 2.3 are restricted to $n > 1$. For $n > 1$, the dual frames generated by ψ and ϕ will be non-canonical.

The important thing to note about the definition of ϕ in (2.4) is that ϕ will inherit properties from ψ that are preserved by linearity and dilation, e.g., $\hat{\phi}$ will have compact support because $\hat{\psi}$ has this property. This holds also for properties such as smoothness, symmetry, fast decay, and vanishing moments up to some order. If ψ (or $\hat{\psi}$) can be written in terms of elementary functions, the same will hold for ϕ (or $\hat{\phi}$). These observations naturally lead to a review of the properties generally possessed by the dual generators we construct. As mentioned above, all non-trivial applications of Theorem 2.3 involve $n > 1$, $n \in \mathbb{N}$. We will furthermore assume that $\hat{\psi} \in L^2(\mathbb{R})$ is even, explicitly given, and, when mentioned, a C^r-function for some $r \in \mathbb{N} \cup \{0\}$. In this situation the resulting pair of dual generators has the following properties:

- Explicit and similar form: $\hat{\psi}$ and $\hat{\phi}$ are of similar form, e.g., piecewise polynomial of the same order (see Example 2) unlike the situation for the canonical dual (see Section 3). A similar construction procedure for tight frames gives "less" explicitly given generators (see Section 3).

- Compact support in Fourier domain of both ψ and ϕ.

- Fast decay in time domain. For $\hat{\psi} \in C_0^r(\mathbb{R})$ the generating function ψ will satisfy $\lim_{|x|\to\infty} x^r \psi(x) = 0$, that is, $\psi(x) = \mathrm{o}(x^{-r})$ as $|x| \to \infty$. The dual generator ϕ has the same properties.

- High order of vanishing moments. In general for $\hat{\psi} \in C_0^r(\mathbb{R})$ the generator ψ will have vanishing moments up to order $r \in \mathbb{N} \cup \{0\}$ since

$$0 = \frac{d^m \hat{\psi}}{d\xi^m}(0) = (-2\pi i)^m \int_{\mathbb{R}} x^m \psi(x) \mathrm{d}x \quad \text{for } m = 0, \ldots, r.$$

  And again, the same holds for the dual generator ϕ.

- Symmetry: $\hat{\psi}$ and $\hat{\phi}$ are even and real functions and so are ψ and ϕ.

- Frequency overlap between scales for increased stability and non-semiorthogonality: For all $j, k \in \mathbb{Z}$ there is a $j' \neq j$ and a $k' \in \mathbb{Z}$ so that $\langle D_{a^j} T_{bk} \psi, D_{a^{j'}} T_{bk'} \psi \rangle \neq 0$. The same holds for the dual generator ϕ.

- Generalized multiresolution structure [1] (see also Section 2.3). The two generators can be associated with the same GMRA with identical core subspace, the Paley-Wiener space $\check{L}^2(K)$ with $K = \cup_{j<0}(a^j \operatorname{supp} \hat{\psi}) \subset [-a^{c-1}, a^{c-1}]$, hence both generators can be associated with the same scaling function. These types of dual wavelet frames are called sibling frames in [7]. Furthermore, the GMRA provides arbitrarily large approximation order [10].

To make Theorem 2.3 applicable, we need to show how to construct functions that satisfy the technical condition (2.3) in the theorem. It is important that this construction is explicit because one of the key features of the theorem is that the dual generator is *explicitly* given in terms of dilations of ψ. In Example 1 we construct a dyadic partition of unity, that is, we construct a function $g \in L^2(\mathbb{R})$ satisfying

$$\sum_{j\in\mathbb{Z}} g(2^j x) = 1 \quad \text{for a.e. } x \in \mathbb{R}. \tag{2.6}$$

This corresponds to condition (2.3) for dyadic dilation $a = 2$; a generalization of the construction to arbitrary real dilation parameter $a > 1$ is straightforward (replace every occurrence of "2" with "a"). As we shall see a very general class of functions satisfy the condition (see also Example 3).

Example 1. For any $m \in \mathbb{Z}$, any $\delta > 0$ smaller than or equal to $2^m/3$, and a bounded function f on $[2^m - \delta, 2^m + \delta]$ satisfying $f(2^m - \delta) = 0$ and $f(2^m + \delta) = 1$, we define

$$h_1(x) = \begin{cases} f(x) & x \in \bar{B}(2^m, \delta), \\ 1 & x \in \left(2^m + \delta, 2^{m+1} - 2\delta\right), \\ 1 - f(x/2) & x \in \bar{B}(2^{m+1}, 2\delta), \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Any such $h_1 \in L^2(\mathbb{R})$ will be continuous if f is continuous, and it will satisfy:

$$\sum_{j\in\mathbb{Z}} h_1(2^j x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

We use the same approach to construct $h_2 \in L^2(\mathbb{R})$ satisfying:

$$\sum_{j\in\mathbb{Z}} h_2(2^j x) = \begin{cases} 0 & \text{for } x \ge 0, \\ 1 & \text{for } x < 0, \end{cases}$$

and define $g = h_1 + h_2$. This gives us the dyadic partition of unity almost everywhere.

The function f above could be chosen as any polynomial satisfying $f(2^m - \delta) = 0$ and $f(2^m + \delta) = 1$; this will make g continuous. If we also let the polynomial f satisfy $f'(2^m - \delta) = f'(2^m + \delta) = 0$, then $g \in C^1(\mathbb{R})$. Continuing this way, we can make g as smooth as desired while still keeping g piecewise polynomial.

In the next example we apply the ideas from the above example to Theorem 2.3 and construct dual wavelet frames with dyadic dilation and translation parameter $b = 1$; actually, any $b \in (0, 1]$ can be used, but we take $b = 1$ for simplicity.

Example 2. Let f be a continuous function on the interval $[1/4, 1/2]$ satisfying $f(1/4) = 1$ and $f(1/2) = 0$. For example f can be any of the functions below:

$$f(x) = 2 - 4x, \tag{2.8a}$$
$$f(x) = 8(24x^2 - 8x + 1)(2x - 1)^2, \tag{2.8b}$$
$$f(x) = -16(320x^3 - 192x^2 + 42x - 3)(2x - 1)^3, \tag{2.8c}$$
$$f(x) = 32(4480x^4 - 3840x^3 + 1280x^2 - 192x + 11)(2x - 1)^4, \tag{2.8d}$$
$$f(x) = \tfrac{1}{2} + \tfrac{1}{2}\cos \pi(4x - 1). \tag{2.8e}$$

In definitions (2.8b) and (2.8e) the function f satisfy $f'(1/4) = f'(1/2) = 0$, in definition (2.8c) this also holds for the second derivative, and in (2.8d) even for the third derivative. As in Example 1 define $\psi \in L^2(\mathbb{R})$ by:

$$\hat{\psi}(\xi) = \begin{cases} 1 - f(2\,|\xi|) & \text{for } |\xi| \in [1/8, 1/4]\,, \\ f(|\xi|) & \text{for } |\xi| \in (1/4, 1/2]\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

This way $\hat{\psi}$ becomes a dyadic partition of unity with $\operatorname{supp} \hat{\psi} \subset [-1/2, -1/8] \cup [1/8, 1/2]$, so we can apply Theorem 2.3 with $c = -1$, $n = 2$, and $b = 1$. Following Theorem 2.3 we define the dual generator $\phi \in L^2(\mathbb{R})$ by:

$$\hat{\phi}(\xi) = \begin{cases} 2[1 - f(4\,|\xi|)] & \text{for } |\xi| \in [1/16, 1/8]\,, \\ 1 + f(2\,|\xi|) & \text{for } |\xi| \in (1/8, 1/4]\,, \\ f(|\xi|) & \text{for } |\xi| \in (1/4, 1/2]\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

whereby ψ and ϕ generate dual frames $\{D_{2^j} T_k \psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{2^j} T_k \phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. The translation parameter in these wavelet systems is set to $b = 1$, and each wavelet frame is generated by only one function.

Suppose we let $\hat{\psi} \in L^2(\mathbb{R})$ be piecewise polynomial as defined by equations (2.8a) to (2.8d). Then $\hat{\psi} \in C^r(\mathbb{R})$ with $r = 0, 1, 2, 3$, respectively. Further, the generators ψ and ϕ will be real and even, and $\hat{\psi}$ and $\hat{\phi}$ will be piecewise polynomial and have compact support with $\operatorname{supp} \hat{\psi} \subset [-1/2, -1/8] \cup [1/8, 1/2]$ and $\operatorname{supp} \hat{\phi} \subset [-1/2, -1/16] \cup [1/16, 1/2]$. We have a greater number of vanishing moments and faster decay than indicated by the review of properties above: ψ and ϕ will have $r+1$ vanishing moments and decay as $\mathrm{O}(x^{-r-2})$ as $|x| \to \infty$, e.g., using (2.8b) we have $\hat{\psi}, \hat{\phi} \in C^1(\mathbb{R})$, and ψ and ϕ with vanishing moments up to order 2, and $\psi(x) = \mathrm{O}(x^{-3})$ and $\phi(x) = \mathrm{O}(x^{-3})$, see Figures 1 and 2. The explicit form of ψ and hence ϕ are easily found; in general, they are finite linear combination of sine and cosine of the form $\sin(2\pi\alpha x)/(\pi x)^n$ and $\cos(2\pi\alpha x)/(\pi x)^n$ for integer $n \geq 2 + r$ and $\alpha \in \mathbb{Q}$.

We end the example with some notes on the numerical aspects and the multiresolution structure. We claim that $C_1 = 1/2$ and $C_2 = 1$ are frame bounds for $\{D_{2^j} T_k \psi\}$, that $C_1 = 7/2$ and $C_2 = 5$ are frame bounds for the dual frame $\{D_{2^j} T_k \phi\}$, and that

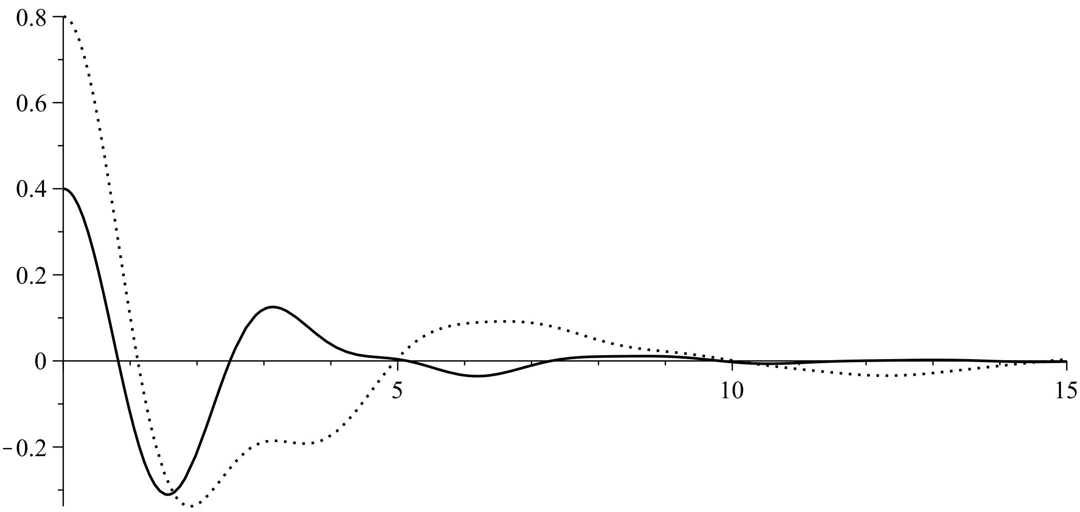

Figure 1: A pair of dual generators ψ (solid line) and ϕ (dotted line) in the time domain with f as in (2.8b).

this holds for any f from equations (2.8); even more, the frame bounds hold for *any* f satisfying $0 \leq f(x) \leq 1$ for $x \in [1/4, 1/2]$. To prove the claim observe that

$$\sum_{k \neq 0} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}(2^j \xi) \hat{\psi}(2^j \xi + k) \right| = 0, \qquad \text{for } \xi \in \mathbb{R},$$

by the support of $\hat{\psi}$. This reduces the frame bound estimates in [5, Theorem 11.2.3] to

$$C_1 = \inf_{|\xi| \in [1/4, 1/2]} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}(2^j \xi) \right|^2, \quad C_2 = \sup_{|\xi| \in [1/4, 1/2]} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}(2^j \xi) \right|^2,$$

where C_1 and C_2 are a lower and upper frame bound of $\{D_{2^j} T_k \psi\}$, respectively. For $|\xi| \in [1/4, 1/2]$ we have, by the definition (2.9),

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = f(|\xi|)^2 + (1 - f(|\xi|))^2 = 1 - 2f(|\xi|) + 2f(|\xi|)^2,$$

and thus,

$$C_1 = \min_{x \in [\alpha, \beta]} 1 - 2x + 2x^2 = 1/2, \quad C_2 = \max_{x \in [\alpha, \beta]} 1 - 2x + 2x^2,$$

with $\alpha := \min_{1/4 \leq x \leq 1/2} f(x)$ and $\beta := \max_{1/4 \leq x \leq 1/2} f(x)$. Since $0 \leq f(x) \leq 1$ for $x \in [1/4, 1/2]$, we have $\alpha = 0$ and $\beta = 1$, hence $C_2 = 1$, and this proves the claim for $\{D_{2^j} T_k \psi\}$; similar calculations will show the claim for the dual frame. In particular, we see that the condition number C_2/C_1 does not depend on the smoothness of the generators, and that the condition number of the dual frame $\{D_{2^j} T_k \phi\}$ is smaller than the condition number of $\{D_{2^j} T_k \psi\}$ and the condition number of the canonical dual frame.

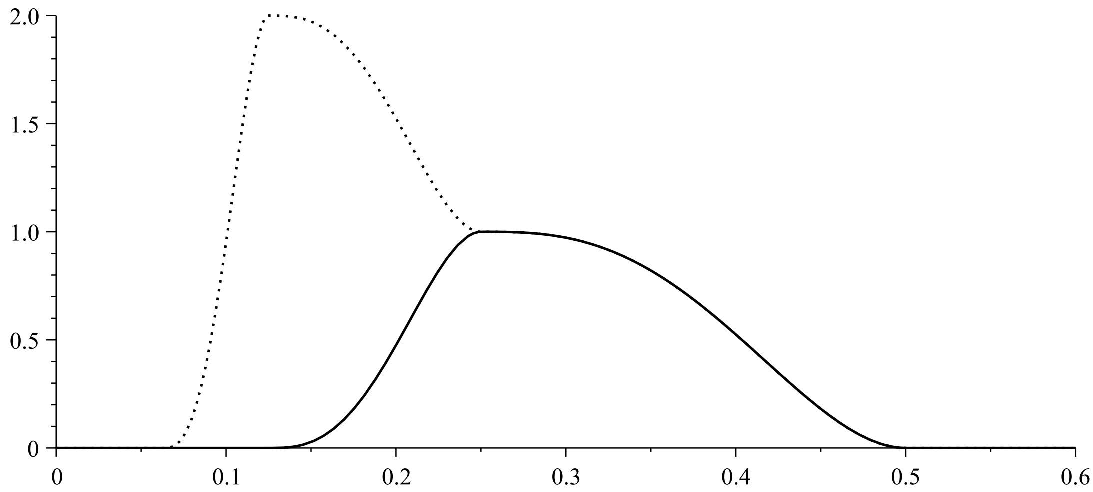

Figure 2: A pair of dual generators $\hat{\psi}$ (solid line) and $\hat{\phi}$ (dotted line) in the Fourier domain with f as in (2.8b).

The core subspace of the GMRA is the Paley-Wiener space $V_0 = \check{L}^2([-1/4, 1/4])$. The function $\eta \in L^2(\mathbb{R})$ defined by $\hat{\eta} = \chi_{[-1/4,1/4]}$ is a generator for V_0, that is, $\overline{\text{span}}\{T_k\eta\}_{k\in\mathbb{Z}} = V_0$, and $\{T_k\eta\}_{k\in\mathbb{Z}}$ is a tight frame with frame bound 1 for V_0. We note that this frame contains twice as many elements as "necessary" in the sense that $\{T_{2k}\eta\}_{k\in\mathbb{Z}}$ and $\{T_{2k+1}\eta\}_{k\in\mathbb{Z}}$ are orthogonal bases for V_0. Obviously, we can take the refinable symbol $H_0 \in L^2(\mathbb{T})$ to be the 1-periodic extension of $H_0 = \chi_{[-1/8,1/8]}$ so that $\hat{\eta}(2\xi) = H_0(\xi)\hat{\eta}(\xi)$ for $\xi \in \mathbb{R}$; note that the choice of H_0 is not unique, and by letting $H_0 = \chi_{[-3/8,1/4)\cup[-1/8,1/8)\cup[1/4,3/8)}$ we obtain a quadrature mirror filter since $H_0(0) = 1$ and $|H_0(\xi)|^2 + |H_0(\xi + 1/2)|^2 = 1$. The refinable symbol H_0 is sometimes called a low pass filter or mask. As wavelet symbol (high pass filter) for the decomposition H_d and reconstruction H_r we can take $H_d = \hat{\psi}(2\cdot)$ and $H_r = \hat{\phi}(2\cdot)$ extending them to 1-periodic functions; these symbols obviously satisfy $\hat{\psi}(2\xi) = H_d(\xi)\hat{\eta}(\xi)$ and $\hat{\phi}(2\xi) = H_r(\xi)\hat{\eta}(\xi)$.

2.1. An alternative definition of the dual generator

The following result resembles Theorem 2.3, but it gives an alternative way of defining ϕ; note the change from $\psi(a^{-j}x)$ in (2.4) to $\psi(a^j x)$ in (2.11). The result follows from the symmetry of the calculations in (2.5).

Proposition 2.4. *Let $n \in \mathbb{N}$ and $a > 1$. Suppose $\psi \in L^2(\mathbb{R})$ is as in Theorem 2.3. Let $b \in \big(0, a^{-c}(1 + a^{n-1})^{-1}\big]$. Then the function ψ and the function ϕ defined by*

$$\phi(x) = b\psi(x) + 2b\sum_{j=1}^{n-1} a^j \psi(a^j x) \quad \text{for } x \in \mathbb{R} \tag{2.11}$$

generate dual frames $\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{bk}\phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof. The functions $\hat{\psi}$ and $\hat{\phi}$ satisfy condition (2.1). This follows from calculations similar to those in (2.5): We start by factoring out $\hat{\psi}(a^{c-1}\xi)$ instead of $\hat{\psi}(a^{c-n}\xi)$, then $\psi(a^{c-2}\xi)$ and continue in a similar way. To see that condition (2.2) is satisfied, we note that $\operatorname{supp}\hat{\phi} \subset \left[-a^{c+n-1}, -a^{c-n}\right] \cup \left[a^{c-n}, a^{c+n-1}\right]$ since $\operatorname{supp}\hat{\phi}(a^{-n+1}\cdot) \subset \left[-a^{c+n-1}, -a^{c-1}\right] \cup \left[a^{c-1}, a^{c+n-1}\right]$. The two functions in (2.2) will have disjoint support modulo null sets whenever $|q| \geq a^c + a^{c+n-1} = a^c(1 + a^{n-1})$. □

The choice of the translation parameter b is more restrictive in Proposition 2.4 than in Theorem 2.3 since the support of $\hat{\phi}$ defined by (2.11) is larger than when defined by (2.4). Note that $b \in \left(0, a^{-c}(1 + a^{n-1})^{-1}\right]$ can be replaced by the simpler, but more restrictive, $b \in (0, a^{-c-n}]$ in the case $a \geq 2$.

2.2. Expanding the range of the translation parameter

The construction of dual wavelet frames from Theorem 2.3 often imposes the translation parameter b to be small, e.g., $b < 1$. Hence, it would be interesting to know in which cases we can take $b = 1$. For the sake of simplicity let $a = 2$ for a moment, and assume that ψ satisfies the assumptions of Theorem 2.3. Obviously, we can take $b = 1$ if the support of $\hat{\psi}$ is contained in $[-1/2, 1/2]$, that is, if $c \leq -1$; this is exactly what we used in Example 2. If $c \geq 0$, we need, in order to achieve $b = 1$, to apply Theorem 2.3 to $\hat{\psi}(2^{c+1}\cdot)$ in place of $\hat{\psi}$. This dilated version of ψ will still be a dyadic partition of unity and $\operatorname{supp}\hat{\psi}(2^{c+1}\cdot) \subset [-1/2, 1/2]$. Moreover, we have the following result.

Corollary 2.5. *Let $n \in \mathbb{N}$ and $a > 1$. Suppose $\psi \in L^2(\mathbb{R})$ is as in Theorem 2.3. Let $b \in \left(0, 2^{-1}a^{-c}\right]$. Then the function $\tilde{\psi} := D_b\psi$ and the function $\tilde{\phi} := D_b\phi$, where ϕ is defined as in (2.4), generate dual frames $\{D_{a^j}T_k\tilde{\psi}\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_k\tilde{\phi}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.*

Proof. The result basically follows from an application of the identity

$$D_bT_{bk} = T_kD_b, \tag{2.12}$$

and the fact that dilation preserves the frame property and the duality of (wavelet) frames since it is a unitary operator on $L^2(\mathbb{R})$. By assumption $\{D_{a^j}T_{bk}\psi\}$ and $\{D_{a^j}T_{bk}\phi\}$ are dual frames for $b \in \left(0, 2^{-1}a^{-c}\right]$. The identity (2.12) yields,

$$D_bD_{a^j}T_{bk}\psi = D_{a^j}T_k(D_b\psi),$$

hence $\{D_{a^j}T_k\tilde{\psi}\}$ is a frame as a unitary image of a wavelet frame where $\tilde{\psi} = D_b\psi$. The same conclusion holds for $\{D_{a^j}T_k\tilde{\phi}\}$. For all $f \in L^2(\mathbb{R})$, we have

$$f = D_b(D_b^*f) = \sum_{j,k\in\mathbb{Z}} \langle f, D_bD_{a^j}T_{bk}\phi\rangle\, D_bD_{a^j}T_{bk}\psi = \sum_{j,k\in\mathbb{Z}} \left\langle f, D_{a^j}T_k\tilde{\phi}\right\rangle D_{a^j}T_k\tilde{\psi},$$

and conclude that duality is preserved. □

Another approach (for obtaining $b = 1$) makes use of multigenerated wavelet systems. In the following result the constructed dual wavelet frames are generated by m functions again sharing the properties of the starting point function ψ; in particular, if ψ has vanishing moments up to some order, then so will every function in the generator sets Ψ and Φ.

Corollary 2.6. *Let $n \in \mathbb{N}$ and $a > 1$. Suppose $\psi \in L^2(\mathbb{R})$ is as in Theorem 2.3. Let $m \in \mathbb{N}$ and $b \in (0, 2^{-1}a^{-c}m]$. Then the functions $\Psi = \{\psi, T_{b/m}\psi, \ldots, T_{(m-1)b/m}\psi\}$ and the functions $\Phi = \{\phi, T_{b/m}\phi, \ldots, T_{(m-1)b/m}\phi\}$, where ϕ is defined as in (2.4), generate dual frames $\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi}$ and $\{D_{a^j}T_{bk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$ for $L^2(\mathbb{R})$.*

Proof. Let $m \in \mathbb{N}$. For b so that $0 < b/m \leq 2^{-1}a^{-c}$, the functions ψ and ϕ, where ϕ is defined as in (2.4), generate dual frames $\{D_{a^j}T_{bk/m}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{bk/m}\phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. Note that $(m^{-1}\mathbb{Z}) \,/\, \mathbb{Z} = \{0, 1, \ldots, m-1\}$, and define:

$$\Psi = \left\{\psi, T_{b/m}\psi, T_{2b/m}\psi, \ldots, T_{(m-1)b/m}\psi\right\}.$$

It follows immediately that $\{D_{a^j}T_{b/mk}\psi\}_{j,k\in\mathbb{Z}} = \{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi}$. Similarly, we have for ϕ that $\{D_{a^j}T_{b/mk}\phi\}_{j,k\in\mathbb{Z}} = \{D_{a^j}T_{bk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$, where

$$\Phi := \left\{\phi, T_{b/m}\phi, T_{2b/m}\phi, \ldots, T_{(m-1)b/m}\phi\right\}.$$

We conclude $\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z},\psi\in\Psi}$ and $\{D_{a^j}T_{bk}\phi\}_{j,k\in\mathbb{Z},\phi\in\Phi}$ are dual frames for $L^2(\mathbb{R})$ for $b/m \leq 2^{-1}a^{-c}$, that is, for $b \leq 2^{-1}a^{-c}m$. □

It follows from the corollary that, in the dyadic case, we can always obtain $b = 1$ by using 2^{c+1} generators.

2.3. On the generalized multiresolution structure

We end this section with a closer study of the GMRA structure of ψ and ϕ. To this end, let $\psi \in L^2(\mathbb{R})$ satisfy the assumptions in Theorem 2.3. We consider the subspaces $W_j^b(\psi) := \overline{\operatorname{span}}\{D_{a^j}T_{bk}\psi : k \in \mathbb{Z}\}$. Let $\tilde{\psi} = D_b\psi$ be the generator of frame $\{D_{a^j}T_k\tilde{\psi}\}$, see Corollary 2.5. From the identity $T_{bk} = D_{b^{-1}}T_kD_b$ we have $W_0^b(\psi) = D_{b^{-1}}W_0^1(\tilde{\psi})$ where $W_j^1(\tilde{\psi}) = \overline{\operatorname{span}}\left\{D_{a^j}T_k\tilde{\psi} : k \in \mathbb{Z}\right\}$. By [10, Theorem 2.14],

$$W_0^1(\tilde{\psi}) = \left\{f \in L^2(\mathbb{R}) : \hat{f} = m\hat{\tilde{\psi}} \text{ for some measurable, 1-periodic } m\right\}$$

and further, using $\operatorname{supp}\hat{\tilde{\psi}} \subset [-1/2, 1/2]$,

$$W_0^1(\tilde{\psi}) = \left\{f \in L^2(\mathbb{R}) : \operatorname{supp}\hat{f} \subset \operatorname{supp}\hat{\tilde{\psi}}\right\} = \check{L}^2(\operatorname{supp}\hat{\tilde{\psi}}),$$

hence $W_0^b(\psi) = \check{L}^2(\operatorname{supp}\hat{\psi})$ by the above, and by dilation, $W_j^b(\psi) = \check{L}^2(a^j\operatorname{supp}\hat{\psi})$. We conclude that the space of negative dilates, also called the core subspace, associated with ψ is given by

$$V_0(\psi) = \overline{\operatorname{span}}\Big(\bigcup_{j<0} W_j^b(\psi)\Big) = \check{L}^2(K), \qquad K = \bigcup_{j<0}(a^j\operatorname{supp}\hat{\psi}) \subset \left[-a^{c-1}, a^{c-1}\right],$$

which is a subspace invariant under all translations. It is straightforward to see $V_0(\psi) = V_0(\phi)$; we will denote this space by V_0. A function $\eta \in L^2(\mathbb{R})$ is said to generate V_0 if $\overline{\operatorname{span}}\{T_{bk}\eta\}_{k\in\mathbb{Z}} = V_0$, and we have that η generates V_0 if, and only if, $\operatorname{supp}\hat{\eta} = K$ (see

[10]). If we further require $\{T_{bk}\eta\}_{k\in\mathbb{Z}}$ to be a frame for V_0, then $\hat{\eta}$ cannot be continuous hence η will be poorly localized in time. This drawback follows from a result in [2]; indeed, the sum $\sum_{k\in\mathbb{Z}}|\hat{\eta}((\xi+k)/b)|^2$ reduces to $|\hat{\eta}(\xi/b)|^2$ for $\xi \in [-1/2,1/2]$ since $b \le 2^{-1}a^{-c}$ implies $ba^{c-1} \le 1/(2a) \le 1/2-\varepsilon$ for some $\varepsilon > 0$ hence $\operatorname{supp}\hat{\eta}(\cdot/b) = bK \subset [-ba^{c-1}, ba^{c-1}] \subset [-1/2+\varepsilon, 1/2-\varepsilon]$. Now, the conclusion follows from [2, Theorem 3.4]. We note that the constructed wavelet frame will not necessarily be a frame for a fixed dilation level subspace $W_j(\psi)$ of $L^2(\mathbb{R})$. This situation is similar to that of the unitary and oblique extension principles, but in contrast to frame multiresolution analysis.

3. Dual frames versus tight frames

In Theorem 2.3 we explicitly *construct* the dual frame. One might ask why we do not use the canonical dual frame, or why we do not use the characterizing equations for tight frames to formulate a similar construction procedure of tight frames. In the following we will show that these approaches have some disadvantages compared to Theorem 2.3.

For a wavelet frame $\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z}}$, the canonical dual frame is given by

$$\left\{S^{-1}D_{a^j}T_{bk}\psi : j,k\in\mathbb{Z}\right\} = \left\{D_{a^j}S^{-1}T_{bk}\psi : j,k\in\mathbb{Z}\right\},$$

where S is frame operator of $\{D_{a^j}T_{bk}\psi\}_{j,k\in\mathbb{Z}}$. In general the canonical dual need *not* have the structure of a wavelet system, and this is one reason to avoid working with canonical dual frames. However, as we show below, the canonical dual of all wavelet frames considered in this paper will be of wavelet structure, hence the canonical dual could be used in the synthesis process in the frame wavelet transform. The problem with this approach is that it is difficult to control which properties the canonical dual frame inherits from the frame since the application of the inverse frame operator can destroy desirable properties. We give an example of this issue in the following.

Let $\psi \in L^2(\mathbb{R})$ be as in the assumptions of Theorem 2.3. Then $\hat{\psi}(\xi)\overline{\hat{\psi}(\xi+b^{-1}k)} = 0$ for $k \in \mathbb{Z}\setminus\{0\}$, and consequently, by [11, Proposition 7.1.19] in the dyadic case and a simple generalization of parts of the proof of the proposition in the general case, the associated frame operator is the Fourier multiplier given by

$$\widehat{Sf}(\xi) = \Big(\sum_{j\in\mathbb{Z}}\left|\hat{\psi}(a^j\xi)\right|^2\Big)\hat{f}(\xi) \quad \text{for a.e. } \xi\in\mathbb{R}, \tag{3.1}$$

for all $f \in L^2(\mathbb{R})$ with $C_1 \le \sum_{j\in\mathbb{Z}}|\hat{\psi}(a^j\xi)|^2 \le C_2$ and C_1, C_2 as frame bounds for $\{D_{a^j}T_{bk}\psi\}$. Since S is a Fourier multiplier, it commutes with all translations, that is, $ST_r = T_rS$ for all $r\in\mathbb{R}$, and the same holds for the inverse frame operator, hence the canonical dual takes the form

$$\left\{D_{a^j}T_{bk}(S^{-1}\psi) : j,k\in\mathbb{Z}\right\},$$

which is a wavelet frame generated by $S^{-1}\psi$. Moreover, the canonical dual generator is given by

$$\widehat{S^{-1}\psi}(\xi) = \frac{\hat{\psi}(\xi)}{\sum_{j\in\mathbb{Z}}\left|\hat{\psi}(a^j\xi)\right|^2} \quad \text{for a.e. } \xi\in\mathbb{R}, \tag{3.2}$$

Since $\operatorname{supp}\hat{\psi} \subset [-a^c, -a^{c-n}] \cup [a^{c-n}, a^c]$ for some $c \in \mathbb{Z}$ and $n \in N$, we conclude, by equation (3.2), $\operatorname{supp}\widehat{S^{-1}\psi} = \operatorname{supp}\hat{\psi}$ and

$$\widehat{S^{-1}\psi}(\xi) = \frac{\hat{\psi}(\xi)}{\sum_{|j|<n}\left(\hat{\psi}(a^j\xi)\right)^2} \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{3.3}$$

This implies, among other things, that $\hat{\psi}$ and $\widehat{S^{-1}\psi}$ will have the same regularity. But it also implies that choosing $\hat{\psi}$ to be piecewise linear will not make the canonical dual generator $S^{-1}\psi$ piecewise linear (in the Fourier domain, that is) owing to the denominator in (3.3). This is unlike the situation in Example 2 where a piecewise polynomial $\hat{\psi}$ by Theorem 2.3 gave a dual generator $\hat{\phi}$ piecewise polynomial of the same order, e.g., a piecewise linear $\hat{\psi}$ gave a piecewise linear $\hat{\phi}$. In general the denominator in (3.3) makes the expression for the canonical dual generator "less" explicit. The price we pay for using the non-canonical dual is a slightly larger support (in the Fourier domain) of the dual generator.

Since the construction of wavelet frames by Theorem 2.3 is based on characterizing equations for dual wavelet frames, it would be natural to look for a similar way of constructing tight frames from their characterizing equations. In a naive approach to such a construction one would need to choose $\psi \in L^2(\mathbb{R})$ so that $\hat{\psi}$ is real and the family $\xi \mapsto (\hat{\psi}(a^j\xi))^2$, $j \in \mathbb{Z}$, form a partition of unity and to choose a sufficiently small translation parameter (so that all terms in the series in the so-called "t_q-equations" become zero owing to disjoint support). Following the ideas from Example 1 we take $\psi \in L^2(\mathbb{R})$ as (extending $\hat{\psi}$ to an even function):

$$\hat{\psi}(\xi) = \begin{cases} f(\xi) & \xi \in \bar{B}(a^m, \delta), \\ 1 & \xi \in \left(a^m + \delta, a^{m+1} - a\delta\right), \\ \sqrt{1 - (f(\xi/a))^2} & \xi \in \bar{B}(a^{m+1}, a\delta), \\ 0 & \xi \in [0, \infty) \setminus \left[a^m - \delta, a^{m+1} + a\delta\right]. \end{cases}$$

for any $m \in \mathbb{Z}$, any $\delta > 0$ smaller than or equal to $a^m/3$, and a bounded function f on $[a^m - \delta, a^m + \delta]$ satisfying $f(a^m - \delta) = 0$, $f(a^m + \delta) = 1$, and $|f| \le 1$. The important thing to note with this approach is that $\hat{\psi}$ does not inherit properties from f in opposition to the situation in Example 1, e.g., taking f to be linear does not make $\hat{\psi}$ piecewise linear because of the square root in the expression above; moreover, it is well known that the property of being a smooth (non-negative) function need not be preserved when taking square roots.

4. Another application of Theorem 2.3

In Examples 1 and 2 we constructed dual wavelet frames in a rather explicit way. The following construction is less explicit. In the first part of the example below we construct a C^∞ function on $\mathbb{R}$ with compact support satisfying the technical condition (2.6), and in the second part we apply Theorem 2.3 to the constructed function.

Example 3 (Part I). Let $f \in C^\infty(\mathbb{R})$ be defined as

$$f(x) = \begin{cases} \mathrm{e}^{-1/x} & x > 0, \\ 0 & x \le 0, \end{cases}$$

and choose positive constants $R > r > 0$ so that

$$\exists \delta > 0 : \bigcup_{j\in\mathbb{Z}} 2^j[r+\delta, R-\delta] = [0,\infty)\,, \tag{4.1}$$

holds, e.g., take $r = 1/8$ and $R = 1/2$. We define $f_1(x) = f(x-r)f(R-x)$ for $x \in \mathbb{R}$, hence $\operatorname{supp} f_1 \subset [r, R]$ and $f_1 \in C_0^\infty(\mathbb{R})$, and we introduce a symmetric version of f_1, denoted f_2, in order to get a dyadic partition of unity of the negative as well as the positive real line.

$$f_2(x) = \begin{cases} f_1(x) & \text{for } x > 0, \\ f_1(-x) & \text{for } x \leq 0. \end{cases} \tag{4.2}$$

The function w will be used to normalize f_2:

$$w(x) = \sum_{j\in\mathbb{Z}} f_2(2^j x).$$

For a fixed $x \in \mathbb{R}$ this sum only has finitely many nonzero terms. Obviously, w is a 2^j-dyadic periodic function and, by (4.1) and the definition of f_1, it is also bounded away from 0 and ∞:

$$\exists c, C > 0 : c < w(x) < C \quad \text{for all } x \in \mathbb{R}\setminus\{0\},$$

hence we can define a function $g \in C_0^\infty(\mathbb{R})$ by

$$g(x) = \frac{f_2(x)}{w(x)} \quad \text{for } x \in \mathbb{R}\setminus\{0\}, \quad \text{and,} \quad g(0) = 0. \tag{4.3}$$

This g will be a dyadic partition of unity; the calculations are straightforward:

$$\sum_{j\in\mathbb{Z}} g(2^j x) = \sum_{j\in\mathbb{Z}} \frac{f_2(2^j x)}{w(2^j x)} = \sum_{j\in\mathbb{Z}} \frac{f_2(2^j x)}{w(x)} = \frac{\sum_{j\in\mathbb{Z}} f_2(2^j x)}{\sum_{k\in\mathbb{Z}} f_2(2^k x)} = 1.$$

The construction of g looks indeed less explicit than the piecewise polynomial partition of unity in Example 1 primarily because g is normalized by an infinite series w. This situation improves by noticing that, in practice, the series w reduce to a finite sum since $\operatorname{supp} g = \operatorname{supp} f_2 \subset [-R, -r] \cup [r, R]$. For example, if we let $r = 1/8$ and $R = 1/2$, we can do with three terms $g(x) = f_2(x)/\sum_{j=-1}^{1} f_2(2^j x)$ for all $x \in \mathbb{R}\setminus\{0\}$.

Remark 1. 1. Note that the mirroring step (4.2) introducing f_2 also makes g symmetric. But it is obvious from the example that we can carry out the construction for the positive part of the real line only to get a dyadic partition of the unity on the positive real line, and, then, by the same approach (but with different choices of r and R), for the negative real line. This way g will not be symmetric.

2. In place of f one could choose any function in $C_0^\infty(\mathbb{R})$ having the same support as f. In place of f_1 one could take any characteristic function $f_1 = \chi_{[2^n, 2^{n+1}]}$ for some $n \in \mathbb{N}$ convolved with a smooth $h_\delta \in C_0^\infty(\mathbb{R})$ for a sufficiently small $\delta > 0$, where $h_\delta(x) = \delta^{-1} h(\delta^{-1} x)$, and $\operatorname{supp} h \subset [-1, 1]$, $h \geq 0$, $\int h \, \mathrm{d}\mu = 1$, and $h \in C_0^\infty(\mathbb{R})$. Then $\operatorname{supp} h_\delta \subset [-\delta, \delta]$ and $\operatorname{supp} h_\delta * f_1 \subset [2^n - \delta, 2^{n+1} + \delta]$.

Example 3 (Part II). We take $r = 1/8$ and $R = 1/2$ in Example 3 and set $\hat{\psi} = f_2 / \sum_{j=-1}^{1} f_2(2^j \cdot)$ where f_2 is given by (4.2), hence

$$\hat{\psi}(\xi) = \begin{cases} \frac{e^{(1/8-\xi)^{-1}} e^{(\xi-1/2)^{-1}}}{e^{(1/8-\xi)^{-1}} e^{(\xi-1/2)^{-1}} + e^{(1/8-2\xi)^{-1}} e^{(2\xi-1/2)^{-1}}} & \xi \in (1/8, 1/4), \\ 1 & \xi = 1/4, \\ \frac{e^{(1/8-\xi)^{-1}} e^{(\xi-1/2)^{-1}}}{e^{(1/8-\xi/2)^{-1}} e^{(\xi/2-1/2)^{-1}} + e^{(1/8-\xi)^{-1}} e^{(\xi-1/2)^{-1}}} & \xi \in (1/4, 1/2), \\ 0 & \xi \in \mathbb{R}_+ \setminus (1/8, 1/2), \end{cases}$$

and symmetrically for the negative real line. Applying this to Theorem 2.3 with $n = 2$, $c = -1$, and $b = 1$ yields a pair of dual wavelet generators with $\hat{\psi}, \hat{\phi} \in C^\infty(\mathbb{R})$, where $\hat{\phi}$ is defined as in (2.4), and $\operatorname{supp} \hat{\psi} \subset [-1/2, -1/8] \cup [1/8, 1/2]$ and $\operatorname{supp} \hat{\phi} \subset [-1/2, -1/16] \cup [1/16, 1/2]$. The generators are smooth, rapidly decaying, symmetric dual framelets with vanishing moments of infinite order. It is clear that both belong to the Schwartz space, but it is also clear, from the equation above, that ψ and ϕ are not explicitly given in the time domain.

5. The Hardy space

A similar construction procedure for dual wavelet frames holds for the Hardy space $H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [0, \infty)\}$. The result in Corollary 2.1 can easily be transformed from $L^2(\mathbb{R})$ settings to the Hardy space $H^2(\mathbb{R})$. Indeed, we only need to replace the right hand side b in equation (2.1) by $b\chi_{[0,\infty)}(\xi)$. In [4, Theorem 1.3] such a transformation is carried out for a similar result on tight wavelet frames [8, Theorem 1]. The analogue version of Theorem 2.3 for the Hardy space is as follows. Let $n \in \mathbb{N}$ and $a > 1$. Suppose for $\psi \in H^2(\mathbb{R})$ that $\hat{\psi}$ is a real-valued function with $\operatorname{supp} \hat{\psi} \subset [a^{c-n}, a^c]$ for some $c \in \mathbb{Z}$ and that

$$\sum_{j \in \mathbb{Z}} \hat{\psi}(a^j \xi) = \chi_{[0,\infty)}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}.$$

Let $b \in (0, a^{-c}]$; actually, we could even let $b \in \big(0, a^{-c}(1 - a^{-2n+1})^{-1}\big]$. Then ψ and ϕ defined by (2.4) generate dual frames for $H^2(\mathbb{R})$. We note that, in the Hardy space, the choice of translation parameter becomes less restrictive than for $L^2(\mathbb{R})$. This owes to the fact that $\hat{\psi}$ and $\hat{\phi}$ have smaller support since they are zero on the negative real line.

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PAPER III

Constructing pairs of dual bandlimited frame wavelets in $L^2(\mathbb{R}^n)$

Jakob Lemvig

Abstract. Given a real, expansive dilation matrix we prove that any bandlimited function $\psi \in L^2(\mathbb{R}^n)$, for which the dilations of its Fourier transform form a partition of unity, generates a wavelet frame for certain translation lattices. Moreover, there exists a dual wavelet frame generated by a finite linear combination of dilations of ψ with explicitly given coefficients. The result allows a simple construction procedure for pairs of dual wavelet frames whose generators have compact support in the Fourier domain and desired time localization. The construction relies on a technical condition on ψ, and we exhibit a general class of function satisfying this condition.

Keywords. Real, expansive dilation · Bandlimited wavelets · Dual frames · Non-tight frames · Partition of unity

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1. Introduction

For $A \in GL_n(\mathbb{R})$ and $y \in \mathbb{R}^n$, we define the dilation operator on $L^2(\mathbb{R}^n)$ by $D_A f(x) = |\det A|^{1/2} f(Ax)$ and the translation operator by $T_y f(x) = f(x-y)$. Given a $n \times n$ real, expansive matrix A and a lattice of the form $\Gamma = P\mathbb{Z}^n$ for $P \in GL_n(\mathbb{R})$, we consider wavelet systems of the form

$$\{D_{A^j} T_\gamma \psi\}_{j \in \mathbb{Z}, \gamma \in \Gamma},$$

where the Fourier transform of ψ has compact support. Our aim is, for any given real, expansive dilation matrix A, to construct wavelet frames with good regularity properties and with a dual frame generator of the form

$$\phi = \sum_{j=a}^{b} c_j D_{A^j} \psi \tag{1.1}$$

for some explicitly given coefficients $c_j \in \mathbb{C}$ and $a, b \in \mathbb{Z}$. This will generalize and extend the one-dimensional results on constructions of dual wavelet frames in [16, 19] to higher dimensions. The extension is non-trivial since it is unclear how to determine the translation lattice Γ and how to control the support of the generators in the Fourier domain. This will be done by considering suitable norms in $\mathbb{R}^n$ and non-overlapping packing of ellipsoids in lattice arrangements.

The construction of redundant wavelet representations in higher dimensions is usually based on extension principles [7, 8, 10, 11, 12, 13, 15, 17, 18]. By making use of extension principles one is restricted to considering expansive dilations A with integer coefficients. Our constructions work for any real, expansive dilation. Moreover, in the extension principle the number of generators often increases with the smoothness of the generators. We will construct pairs of dual wavelet frames generated by one smooth function with good time localization.

It is a well-known fact that a wavelet frame need not have dual frames with wavelet structure. In [21] frame wavelets with compact support and explicit analytic form are constructed for real dilation matrices. However, no dual frames are presented for these wavelet frames. This can potentially be a problem because it might be difficult or even impossible to find a dual frame with wavelet structure. Since we exhibit *pairs* of dual wavelet frames, this issue is avoided.

The principal importance of having a dual generator of the form (1.1) is that it will inherit properties from ψ preserved by dilation and linearity, e.g., vanishing moments, good time localization and regularity properties. For a more complete account of such matters we refer to [16].

In the rest of this introduction we review basic definitions. A frame for a separable Hilbert space $\mathcal{H}$ is a countable collection of vectors $\{f_j\}_{j \in \mathcal{J}}$ for which there are constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1 \|f\|^2 \le \sum_{j \in \mathcal{J}} |\langle f, f_j \rangle|^2 \le C_2 \|f\|^2 \qquad \text{for all } f \in \mathcal{H}.$$

If the upper bound holds in the above inequality, then $\{f_j\}$ is said to be a Bessel sequence with Bessel constant C_2. For a Bessel sequence $\{f_j\}$ we define the frame operator by

$$S \colon \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{j \in \mathcal{J}} \langle f, f_j \rangle f_j.$$

This operator is bounded, invertible, and positive. A frame $\{f_j\}$ is said to be *tight* if we can choose $C_1 = C_2$; this is equivalent to $S = C_1 I$ where I is the identity operator. Two Bessel sequences $\{f_j\}$ and $\{g_j\}$ are said to be *dual* frames if

$$f = \sum_{j \in \mathcal{J}} \langle f, g_j \rangle f_j \quad \forall f \in \mathcal{H}.$$

It can be shown that two such Bessel sequences are indeed frames. Given a frame $\{f_j\}$, at least one dual always exists; it is called the canonical dual and is given by $\{S^{-1} f_j\}$. Only a frame, which is not a basis, has several duals.

For $f \in L^1(\mathbb{R}^n)$ the Fourier transform is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-2\pi i \langle \xi, x \rangle} \mathrm{d}x$ with the usual extension to $L^2(\mathbb{R}^n)$.

Sets in $\mathbb{R}^n$ are, in general, considered equal if they are equal up to sets of measure zero. The boundary of a set E is denoted by ∂E, the interior by E°, and the closure by $\overline{E}$. Let $B \in GL_n(\mathbb{R})$. A *multiplicative tiling set* E for $\{B^j : j \in \mathbb{Z}\}$ is a subset of positive measure such that

$$\left| \mathbb{R}^n \setminus \cup_{j \in \mathbb{Z}} B^j(E) \right| = 0 \quad \text{and} \quad \left| B^j(E) \cap B^l(E) \right| = 0 \quad \text{for } l \neq j. \tag{1.2}$$

In this case we say that $\{B^j(E) : j \in \mathbb{Z}\}$ is an almost everywhere partition of $\mathbb{R}^n$, or that it tiles $\mathbb{R}^n$. A multiplicative tiling set E is *bounded* if E is a bounded set and $0 \notin \overline{E}$. By B-dilative periodicity of a function $f \colon \mathbb{R}^n \to \mathbb{C}$ we understand $f(x) = f(Bx)$ for a.e. $x \in \mathbb{R}^n$, and by a B-dilative partition of unity we understand $\sum_{j \in \mathbb{Z}} f(B^j x) = 1$; note that the functions in the "partition of unity" are not assumed to be non-negative, but can take any real or complex value.

A (full-rank) lattice Γ in $\mathbb{R}^n$ is a point set of the form $\Gamma = P\mathbb{Z}^n$ for some $P \in GL_n(\mathbb{R})$. The determinant of Γ is $d(\Gamma) = |\det P|$; note that the generating matrix P is not unique, and that $d(\Gamma)$ is independent of the particular choice of P.

2. The general form of the construction procedure

Fix the dimension $n \in \mathbb{N}$. We let $A \in GL_n(\mathbb{R})$ be expansive, i.e., all eigenvalues of A have absolute value greater than one, and denote the transpose matrix by $B = A^t$. For any such dilation A, we want to construct a pair of functions that generate dual wavelet frames for some translation lattice. Our construction is based on the following result which is a consequence of the characterizing equations for dual wavelet frames by Chui, Czaja, Maggioni, and Weiss [6, Theorem 4].

Theorem 2.1. *Let $A \in GL_n(\mathbb{R})$ be expansive, let Γ be a lattice in $\mathbb{R}^n$, and let $\Psi = \{\psi_1, \dots, \psi_L\}$, $\tilde{\Psi} = \{\tilde{\psi}_1, \dots, \tilde{\psi}_L\} \subset L^2(\mathbb{R}^n)$. Suppose that the two wavelet systems $\{D_{A^j} T_\gamma \psi_l : j \in \mathbb{Z}, \gamma \in \Gamma, l = 1, \dots, L\}$ and $\{D_{A^j} T_\gamma \tilde{\psi}_l : j \in \mathbb{Z}, \gamma \in \Gamma, l = 1, \dots, L\}$ form Bessel families. Then $\{D_{A^j} T_\gamma \psi_l\}$ and $\{D_{A^j} T_\gamma \tilde{\psi}_l\}$ will be dual frames if the following conditions hold*

$$\sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \hat{\tilde{\psi}}_l(B^j \xi) \overline{\hat{\psi}_l(B^j \xi)} = d(\Gamma) \qquad \text{a.e. } \xi \in \mathbb{R}^n, \tag{2.1}$$

$$\sum_{l=1}^{L} \hat{\tilde{\psi}}_l(\xi) \overline{\hat{\psi}_l(\xi + \gamma)} = 0 \qquad \text{a.e. } \xi \in \mathbb{R}^n \text{ for } \gamma \in \Gamma^* \setminus \{0\}. \tag{2.2}$$

Proof. By $\xi = B^j\omega$ for $j \in \mathbb{Z}$, condition (2.2) becomes

$$\sum_{l=1}^{L} \hat{\tilde{\psi}}_l(B^j\omega)\overline{\hat{\psi}_l(B^j\omega+\gamma)} = 0 \quad \text{a.e. } \omega \in \mathbb{R}^n \text{ for } \gamma \in \Gamma^* \setminus \{0\}. \tag{2.3}$$

We use the notation as in [6], thus $\Lambda(A,\Gamma) = \{\alpha \in \mathbb{R}^n : \exists (j,\gamma) \in \mathbb{Z}\times\Gamma^* : \alpha = B^{-j}\gamma\}$ and $I_{A,\Gamma}(\alpha) = \{(j,\gamma) \in \mathbb{Z}\times\Gamma^* : \alpha = B^{-j}\gamma\}$. Since $I_{A,\Gamma}(\alpha) \subset \mathbb{Z}\times(\Gamma^*\setminus\{0\})$ for any $\alpha \in \Lambda(A,\Gamma)\setminus\{0\}$, equation (2.3) yields

$$\frac{1}{d(\Gamma)} \sum_{(j,\gamma)\in I_{A,\Gamma}(\alpha)} \sum_{l=1}^{L} \hat{\tilde{\psi}}_l(B^j\omega)\overline{\hat{\psi}_l(B^j(\omega+B^{-j}\gamma))} = 0 \quad \text{a.e. } \omega \in \mathbb{R}^n$$

for $\alpha \neq 0$. By $I_{A,\Gamma}(0) = \mathbb{Z}\times\{0\}$, we can rewrite (2.1) as

$$\frac{1}{d(\Gamma)} \sum_{(j,\gamma)\in I_{A,\Gamma}(0)} \sum_{l=1}^{L} \hat{\tilde{\psi}}_l(B^j\omega)\overline{\hat{\psi}_l(B^j(\omega+B^{-j}\gamma))} = 1 \quad \text{a.e. } \omega \in \mathbb{R}^n,$$

using that $B^{-j}\gamma = 0$ for all $j \in \mathbb{Z}$. Gathering the two equations displayed above yields

$$\frac{1}{d(\Gamma)} \sum_{(j,\gamma)\in I_{A,\Gamma}(\alpha)} \sum_{l=1}^{L} \hat{\tilde{\psi}}_l(B^j\omega)\overline{\hat{\psi}_l(B^j(\omega+B^{-j}\gamma))} = \delta_{\alpha,0} \quad \text{a.e. } \omega \in \mathbb{R}^n,$$

for all $\alpha \in \Lambda(A,\Gamma)$. The conclusion follows now from [6, Theorem 4]. □

The following result, Lemma 2.2, gives a sufficient condition for a wavelet system to form a Bessel sequence; it is an extension of [3, Theorem 11.2.3] from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^n)$.

Lemma 2.2. *Let $A \in GL_n(\mathbb{R})$ be expansive, Γ a lattice in $\mathbb{R}^n$, and $\phi \in L^2(\mathbb{R}^n)$. Suppose that, for some set $M \subset \mathbb{R}^n$ satisfying $\cup_{l\in\mathbb{Z}} B^l(M) = \mathbb{R}^n$,*

$$C_2 = \frac{1}{d(\Gamma)} \sup_{\xi\in M} \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma^*} \left|\hat{\phi}(B^j\xi)\hat{\phi}(B^j\xi+\gamma)\right| < \infty. \tag{2.4}$$

Then the wavelet system $\{D_{A^j}T_\gamma\phi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ is a Bessel sequence with bound C_2. Further, if also

$$C_1 = \frac{1}{d(\Gamma)} \inf_{\xi\in M} \left(\sum_{j\in\mathbb{Z}} \left|\hat{\phi}(B^j\xi)\right|^2 - \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma^*\setminus\{0\}} \left|\hat{\phi}(B^j\xi)\hat{\phi}(B^j\xi+\gamma)\right| \right) > 0, \tag{2.5}$$

holds, then $\{D_{A^j}T_\gamma\phi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds C_1 and C_2.

Proof. The statement follows directly by applying Theorem 3.1 in [5] on generalized shift invariant systems to wavelet systems. In the general result for generalized shift invariant systems [5, Theorem 3.1], the supremum/infimum is taken over $\mathbb{R}^n$, but because of the B-dilative periodicity of the series in (2.4) and (2.5) for wavelet systems, it suffices to take the supremum/infimum over a set $M \subset \mathbb{R}^n$ that has the property that $\cup_{l\in\mathbb{Z}} B^l(M) = \mathbb{R}^n$ up to sets of measure zero. □

Theorem 2.1 and Lemma 2.2 are all we need to prove the following result on pairs of dual wavelet frames.

Theorem 2.3. *Let $A \in GL_n(\mathbb{R})$ be expansive and $\psi \in L^2(\mathbb{R}^n)$. Suppose that $\hat{\psi}$ is a bounded, real-valued function with $\operatorname{supp}\hat{\psi} \subset \cup_{j=0}^{d} B^{-j}(E)$ for some $d \in \mathbb{N}_0$ and some bounded multiplicative tiling set E for $\{B^j : j \in \mathbb{Z}\}$, and that*

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(B^j\xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{2.6}$$

Let $b_j \in \mathbb{C}$ for $j = -d, \dots, d$ and let $\overline{m} = \max\{j : b_j \neq 0\}$ and $\underline{m} = -\min\{j : b_j \neq 0\}$. Take a lattice Γ in $\mathbb{R}^n$ such that

$$\Big(\bigcup_{j=0}^{d} B^{-j}(E) + \gamma\Big) \cap \bigcup_{j=-\underline{m}}^{\overline{m}+d} B^{-j}(E) = \emptyset \quad \text{for all } \gamma \in \Gamma^* \setminus \{0\}, \tag{2.7}$$

and define the function ϕ by

$$\phi(x) = d(\Gamma) \sum_{j=-\underline{m}}^{\overline{m}} b_j \left|\det A\right|^{-j} \psi(A^{-j}x) \quad \text{for } x \in \mathbb{R}^n. \tag{2.8}$$

If $b_0 = 1$ and $b_j + b_{-j} = 2$ for $j = 1, 2, \dots, d$, then the functions ψ and ϕ generate dual frames $\{D_{A^j}T_\gamma\psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ and $\{D_{A^j}T_\gamma\phi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ for $L^2(\mathbb{R}^n)$.

Proof. On the Fourier side, the definition in (2.8) becomes

$$\hat{\phi}(\xi) = d(\Gamma) \sum_{j=-\underline{m}}^{\overline{m}} b_j \hat{\psi}(B^j\xi).$$

Since $\hat{\psi}$ by assumption is compactly supported in a "ringlike" structure bounded away from the origin, this will also be the case for $\hat{\phi}$. This property implies that ψ and ϕ will generate wavelet Bessel sequences. The details are as follows. The support of $\hat{\psi}$ and $\hat{\phi}$ is

$$\operatorname{supp}\hat{\psi} \subset \bigcup_{j=0}^{d} B^{-j}(E), \qquad \operatorname{supp}\hat{\phi} \subset \bigcup_{j=-\underline{m}}^{\overline{m}+d} B^{-j}(E). \tag{2.9}$$

Note that $0 \le \underline{m}, \overline{m} \le d$. The sets $\{B^j(E) : j \in \mathbb{Z}\}$ tiles $\mathbb{R}^n$, whereby we see that

$$\left|\operatorname{supp}\hat{\psi}(B^j\cdot) \cap B^{-d}(E)\right| = 0 \quad \text{for } j < 0 \text{ and } j > d, \tag{2.10}$$

and,

$$\left|\operatorname{supp}\hat{\phi}(B^j\cdot) \cap B^{-d}(E)\right| = 0 \quad \text{for } j < -\underline{m} \text{ and } j > \overline{m} + d. \tag{2.11}$$

Since $\underline{m}, \overline{m} \ge 0$, condition (2.7) implies that $\hat{\psi}(B^j\xi)\hat{\psi}(B^j\xi + \gamma) = 0$ for $j \ge 0$ and $\gamma \in \Gamma^* \setminus \{0\}$. Therefore, using (2.10), we find that

$$\sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma^*} \left|\hat{\psi}(B^j\xi)\hat{\psi}(B^j\xi + \gamma)\right| = \sum_{j=0}^{d} \Big(\hat{\psi}(B^j\xi)\Big)^2 < \infty \qquad \text{for } \xi \in B^{-d}(E).$$

An application of Lemma 2.2 with $M = B^{-d}(E)$ shows that ψ generates a Bessel sequence. Similar calculations using (2.11) will show that ϕ generates a Bessel sequence; in this case the sum over $\gamma \in \Gamma^*$ will be finite, but it will in general have more than one nonzero term.

To conclude that ψ and ϕ generate dual wavelet frames we will show that conditions (2.1) and (2.2) in Theorem 2.1 hold. By B-dilation periodicity of the sum in condition (2.1), it is sufficient to verify this condition on $B^{-d}(E)$. For $\xi \in B^{-d}(E)$ we have by (2.10),

$$\begin{aligned}\frac{1}{d(\Gamma)}\sum_{j\in\mathbb{Z}}\overline{\hat{\psi}(B^j\xi)}\hat{\phi}(B^j\xi) &= \frac{1}{d(\Gamma)}\sum_{j=0}^{d}\hat{\psi}(B^j\xi)\hat{\phi}(B^j\xi)\\ &= \hat{\psi}(\xi)\left[b_0\hat{\psi}(\xi)+b_1\hat{\psi}(B\xi)+\cdots+b_d\hat{\psi}(B^d\xi)\right]\\ &\quad+\hat{\psi}(B\xi)\left[b_{-1}\hat{\psi}(\xi)+b_0\hat{\psi}(B\xi)+\cdots+b_{d-1}\hat{\psi}(B^d\xi)\right]+\cdots\\ &\quad+\hat{\psi}(B^d\xi)\left[b_{-d}\hat{\psi}(\xi)+\cdots+b_{-1}\hat{\psi}(B^{d-1}\xi)+b_0\hat{\psi}(B^d\xi)\right],\end{aligned}$$

and further, by an expansion of these terms,

$$\begin{aligned}&= \sum_{j,l=0}^{d} b_{l-j}\hat{\psi}(B^j\xi)\hat{\psi}(B^l\xi)\\ &= b_0\sum_{j=0}^{d}\hat{\psi}(B^j\xi)^2+\sum_{\substack{j,l=0\\ j>l}}^{d}(b_{j-l}+b_{l-j})\hat{\psi}(B^j\xi)\hat{\psi}(B^l\xi).\end{aligned}$$

Using that $b_0 = 1$ and $b_{j-l} + b_{l-j} = 2$ for $j \neq l$ and $j, l = 0, \ldots, d$, we arrive at

$$\begin{aligned}\frac{1}{d(\Gamma)}\sum_{j\in\mathbb{Z}}\overline{\hat{\psi}(B^j\xi)}\hat{\phi}(B^j\xi) &= \sum_{j=0}^{d}\hat{\psi}(B^j\xi)^2+\sum_{\substack{j,l=0\\ j>l}}^{d}2\hat{\psi}(B^j\xi)\hat{\psi}(B^l\xi)\\ &= \Big(\sum_{j=0}^{d}\hat{\psi}(B^j\xi)\Big)^2 = \Big(\sum_{j\in\mathbb{Z}}\hat{\psi}(B^j\xi)\Big)^2 = 1,\end{aligned}$$

exhibiting that ψ and ϕ satisfy condition (2.1).

By (2.9) we see that condition (2.7) implies that the functions $\hat{\phi}$ and $\hat{\psi}(\cdot+\gamma)$ will have disjoint support for $\gamma \in \Gamma^* \setminus \{0\}$, hence (2.2) is satisfied. □

Remark 1. The use of the parameters b_j in the definition of the dual generator together with the condition $b_{-j} + b_j = 2$ was first seen in the work of Christensen and Kim [4] on pairs of dual Gabor frames.

We can restate Theorem 2.3 for wavelet systems with standard translation lattice $\mathbb{Z}^n$ and dilation $\widetilde{A} = P^{-1}AP$, where $P \in GL_n(\mathbb{R})$ is so that $\Gamma = P\mathbb{Z}^n$. The result follows directly by an application of the relations $D_{\widetilde{A}^j}D_P = D_P D_{A^j}$ for $j \in \mathbb{Z}$ and $D_P T_{Pk} = T_k D_P$ for $k \in \mathbb{Z}^n$, and the fact that D_P is unitary as an operator on $L^2(\mathbb{R}^n)$.

Corollary 2.4. *Suppose ψ, $\{b_j\}$, A and Γ are as in Theorem 2.3. Let $P \in GL_n(\mathbb{R})$ be such that $\Gamma = P\mathbb{Z}^n$, and let $\widetilde{A} = P^{-1}AP$. Then the functions $\tilde{\psi} = D_P\psi$ and $\tilde{\phi} = D_P\phi$, where ϕ is defined in (2.8), generate dual frames $\{D_{\widetilde{A}^j}T_k\tilde{\psi}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ and $\{D_{\widetilde{A}^j}T_k\tilde{\phi}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ for $L^2(\mathbb{R}^n)$.*

The following Example 1 is an application of Theorem 2.3 in $L^2(\mathbb{R}^2)$ for the quincunx matrix. In particular, we construct a partition of unity of the form (2.6) for the quincunx matrix.

Example 1. The quincunx matrix is defined as

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and its action on $\mathbb{R}^2$ corresponds to a counter clockwise rotation of 45 degrees and a dilation by $\sqrt{2}I_{2\times 2}$. Define the tent shaped, piecewise linear function g by

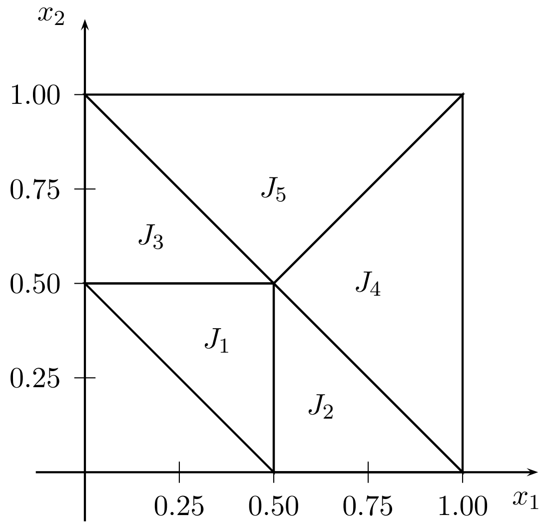

Figure 1: Sketch of the triangular domains J_i, $i = 1, 2, 3, 4, 5$.

$$g(x_1, x_2) = \begin{cases} -1 + 2x_1 + 2x_2, & \text{for } (x_1, x_2) \in J_1, \\ 2x_2, & \text{for } (x_1, x_2) \in J_2, \\ 2x_1, & \text{for } (x_1, x_2) \in J_3, \\ 2 - 2x_1, & \text{for } (x_1, x_2) \in J_4, \\ 2 - 2x_2, & \text{for } (x_1, x_2) \in J_5, \\ 0 & \text{otherwise,} \end{cases}$$

where the sets J_i are the triangular domains sketched in Figure 1. Note that the value at "the top of the tent" is $g(1/2, 1/2) = 1$. Define $\hat{\psi}$ as a mirroring of g in the x_1 axis and the x_2 axis:

$$\hat{\psi}(\xi_1,\xi_2) = \begin{cases} g(\xi_1,\xi_2) & \text{for } (\xi_1,\xi_2) \in [0,\infty) \times [0,\infty)\,, \\ g(\xi_1,-\xi_2) & \text{for } (\xi_1,\xi_2) \in [0,\infty) \times (-\infty,0)\,, \\ g(-\xi_1,\xi_2) & \text{for } (\xi_1,\xi_2) \in (-\infty,0) \times [0,\infty)\,, \\ g(-\xi_1,-\xi_2) & \text{for } (\xi_1,\xi_2) \in (-\infty,0) \times (-\infty,0)\,. \end{cases}$$

Since the transpose B of the quincunx matrix also corresponds to a rotation of 45 degrees (but clockwise) and a dilation by $\sqrt{2}I_{2\times 2}$, we see that $\sum_{j\in\mathbb{Z}} \hat{\psi}(B^j\xi) = 1$.

We are now ready to apply Theorem 2.3 with $E = [-1,1]^2 \setminus B^{-1}([-1,1]^2) = [-1,1]^2 \setminus I_1$ and $d = 2$; the set E is the union of the domians J_4 and J_5 and their mirrored versions. We choose $b_{-2} = b_{-1} = 0$ and $b_1 = b_2 = 2d(\Gamma)$, hence $\underline{m} = 0$ and $\overline{m} = 2$. Therefore,

$$\bigcup_{j=0}^{d} B^{-j}(E), \bigcup_{j=-\underline{m}}^{\overline{m}+d} B^{-j}(E) \subset [-1,1]^2\,,$$

that shows that we can take $\Gamma^* = 2\mathbb{Z}^2$ or $\Gamma = 1/2\mathbb{Z}^2$, since $([-1,1]^2 + \gamma) \cap [-1,1]^2 = \emptyset$ whenever $0 \neq \gamma \in 2\mathbb{Z}^2$. Defining the dual generator according to (2.14) yields

$$\phi(x) = (1/4)\psi(x) + (1/4)\psi(A^{-1}x) + (1/8)\psi(A^{-2}x); \tag{2.12}$$

using that $d(\Gamma) = 1/4$, and we remark that $\hat{\phi}$ is a piecewise linear function since this is the case for $\hat{\psi}$. The conclusion from Theorem 2.3 is that ψ and ϕ generate dual frames $\{D_{A^j}T_{k/2}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{A^j}T_{k/2}\phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R}^2)$.

The frame bounds can be found using Lemma 2.2 since the series (2.4) and (2.5) are finite sums on E; for $\{D_{A^j}T_{k/2}\psi\}$ one finds $C_1 = 4/3$ and $C_2 = 4$.

When the result on constructing pairs of dual wavelet frames is written in the generality of Theorem 2.3, it is not always clear how to choose the set E and the lattice Γ. In Example 1 we showed how this can be done for the quincunx dilation matrix and constructed a pair of dual frame wavelets. In Section 3 and Theorem 3.3 we specify how to choose E and Γ for general dilations. The issue of exhibiting functions ψ satisfying the condition (2.6) is addressed in Section 4.

In one dimension, however, it is straightforward to make good choices of E and Γ as is seen by the following corollary of Theorem 2.3. The corollary unifies the construction procedures in Theorem 2 and Proposition 1 from [16] in a general procedure.

Corollary 2.5. *Let $d \in \mathbb{N}_0$, $a > 1$, and $\psi \in L^2(\mathbb{R})$. Suppose that $\hat{\psi}$ is a bounded, real-valued function with $\operatorname{supp}\hat{\psi} \subset [-a^c, -a^{c-d-1}] \cup [a^{c-d-1}, a^c]$ for some $c \in \mathbb{Z}$, and that*

$$\sum_{j\in\mathbb{Z}} \hat{\psi}(a^j\xi) = 1 \quad \textit{for a.e. } \xi \in \mathbb{R}. \tag{2.13}$$

Let $b_j \in \mathbb{C}$ for $j = -d, \dots, d$, let $m = -\min\{j : \{b_j \neq 0\}\}$, and define the function ϕ by

$$\phi(x) = \sum_{j=-m}^{d} b_j a^{-j}\psi(a^{-j}x) \quad \textit{for } x \in \mathbb{R}. \tag{2.14}$$

Let $b \in \left(0, a^{-c}(1+a^m)^{-1}\right]$. If $b_0 = b$ and $b_j + b_{-j} = 2b$ for $j = 1, 2, \dots, d$, then ψ and ϕ generate dual frames $\{D_{a^j} T_{bk}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j} T_{bk}\phi\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Proof. In Theorem 2.3 for $n = 1$ and $A = a$ we take $E = [-a^c, -a^{c-1}] \cup [a^{c-1}, a^c]$ as the multiplicative tiling set for $\{a^j : j \in \mathbb{Z}\}$. The assumption on the support of $\hat{\psi}$ becomes

$$\operatorname{supp}\hat{\psi} \subset \bigcup_{j=0}^{d} a^{-j}(E) = [-a^c, -a^{c-d-1}] \cup [a^{c-d-1}, a^c].$$

Moreover, since

$$\bigcup_{j=0}^{d} a^{-j}(E) \subset [-a^c, a^c], \qquad \bigcup_{j=-m}^{2d} a^{-j}(E) \subset \left[-a^{c+m}, a^{c+m}\right],$$

and

$$([-a^c, a^c] + \gamma) \cap \left[-a^{c+m}, a^{c+m}\right] = \emptyset \quad \text{for } |\gamma| \geq a^c + a^{c+m} = a^c(1+a^m),$$

the choice $\Gamma^* = b^{-1}\mathbb{Z}$ for $b^{-1} \geq a^c(1+a^m)$ satisfies equation (2.7). This corresponds to $\Gamma = b\mathbb{Z}$ for $0 < b \leq a^{-c}(1+a^m)^{-1}$. ☐

The assumptions in Corollary 2.5 imply that $m \in \{0, 1, \dots, d\}$; we note that in case $m = 0$, the corollary reduces to [16, Theorem 2].

3. A special case of the construction procedure

We aim for a more automated construction procedure than what we have from Theorem 2.3, in particular, we therefore need to deal with good ways of choosing E and Γ. The basic idea in this automation process will be to choose E as a dilation of the difference between I_* and $B^{-1}(I_*)$, where I_* is the unit ball in a norm in which the matrix $B = A^t$ is expanding "in all directions"; we will make this statement precise in Section 3.1. This idea is instrumental in the proof of Theorem 3.3.

3.1. Some results on expansive matrices

We need the following well-known equivalent conditions[1] for a (non-singular) matrix being expansive.

Proposition 3.1. *For $B \in GL_n(\mathbb{R})$ the following assertions are equivalent:*

(i) B is expansive, i.e., all eigenvalues λ_i of B satisfy $|\lambda_i| > 1$.

(ii) For any norm $|\cdot|$ on $\mathbb{R}^n$ there are constants $\lambda > 1$ and $c \geq 1$ such that

$$|B^j x| \geq 1/c\lambda^j \, |x| \qquad \text{for all } j \in \mathbb{N}_0,$$

for any $x \in \mathbb{R}^n$.

(iii) There is a Hermitian norm $|\cdot|_*$ on $\mathbb{R}^n$ and a constant $\lambda > 1$ such that

$$|B^j x|_* \geq \lambda^j \, |x|_* \qquad \text{for all } j \in \mathbb{N}_0,$$

for any $x \in \mathbb{R}^n$.

[1] See Proposition A.1 in Chapter 1 for a more extensive list of equavalent conditions and a proof.

(iv) $\mathcal{E} \subset \lambda\mathcal{E} \subset B\mathcal{E}$ *for some ellipsoid* $\mathcal{E} = \{x \in \mathbb{R}^n : |Px| \leq 1\}$, $P \in GL_n(\mathbb{R})$ *and* $\lambda > 1$.

By Proposition 3.1 we have that for a given expansive matrix B, there exists a scalar product with the induced norm $|\cdot|_*$ so that

$$|Bx|_* \geq \lambda\, |x|_* \quad \text{for } x \in \mathbb{R}^n,$$

holds for some $\lambda > 1$. We say that $|\cdot|_*$ is a norm associated with the expansive matrix B. Note that such a norm is not unique; we will follow the construction as in the proof of [2, Lemma 2.2], so let c and λ be as in (ii) in Proposition 3.1 for the standard Euclidean norm with $1 < \lambda < |\lambda_i|$ for $i = 1, \ldots, n$, where λ_i are the eigenvalues of B. For $k \in \mathbb{N}$ satisfying $k > 2\ln c/\ln\lambda$ we introduce the symmetric, positive definite matrix $K \in GL_n(\mathbb{R})$:

$$K = I + (B^{-1})^t B^{-1} + \cdots + (B^{-k})^t B^{-k}. \tag{3.1}$$

The scalar product associated with B is then defined by $\langle x, y\rangle_* = x^t K y$. It might not be effortless to estimate c and λ for some given B, but it is obvious that we just need to pick $k \in \mathbb{N}$ such that $B^t K B - \lambda^2 K$ becomes positive semi-definite for some $\lambda > 1$ since this corresponds to $\langle KBx, Bx\rangle \geq \lambda^2 \langle Kx, x\rangle$, that is, $|Bx|_*^2 \geq \lambda^2\, |x|_*^2$ for all $x \in \mathbb{R}^n$.

We let I_* denote the unit ball in the Hermitian norm $|\cdot|_* = |K^{1/2}\cdot|$ associated wth B, i.e.,

$$I_* = \{x \in \mathbb{R}^n : |x|_* \leq 1\} = \left\{x \in \mathbb{R}^n : |K^{1/2}x| \leq 1\right\} = \left\{x \in \mathbb{R}^n : x^t K x \leq 1\right\}, \tag{3.2}$$

and we let O_* denote the annulus

$$O_* = I_* \setminus B^{-1}(I_*).$$

The ringlike structure of O_* is guaranteed by the fact that B is expanding in all directions in the $|\cdot|_*$ norm, i.e.,

$$I_* \subset \lambda I_* \subset B(I_*), \qquad \lambda > 1, \tag{3.3}$$

which is (iv) in Proposition 3.1. We note that by an orthogonal substitution I_* takes the form $\{x \in \mathbb{R}^n : \mu_1\tilde{x}_1^2 + \cdots + \mu_n\tilde{x}_n^2 \leq 1\}$ where μ_i are the positive eigenvalues of K and $x = Q\tilde{x}$ with $Q \in O(n)$ comprising of the ith eigenvector of K as the ith column. The annulus O_* is a bounded multiplicative tiling set for $\{B^j : j \in \mathbb{Z}\}$. This is a consequence of the following result.

Lemma 3.2. *Let* $B \in GL_n(\mathbb{R})$ *be an expansive matrix. For* $x \neq 0$ *there is a unique* $j \in \mathbb{Z}$ *so that* $B^j x \in O_*$; *that is,*

$$\mathbb{R}^n \setminus \{0\} = \bigcup_{j\in\mathbb{Z}} B^j(O_*) \qquad \textit{with disjoint union.} \tag{3.4}$$

Proof. From equation (3.3) we know that $\{B^l(I_*)\}_{l\in\mathbb{Z}}$ is a nested sequence of subsets of $\mathbb{R}^n$, thus

$$B^l(I_*) \setminus B^{l-1}(I_*) = B^l(O_*), \quad l \in \mathbb{Z},$$

are disjoint sets. Since $\left|B^{-j}x\right|_* \le \lambda^{-j}\left|x\right|_*$ and $\left|B^j x\right|_* \ge \lambda^j \left|x\right|_*$ for $j \ge 0$ and $\lambda > 1$, we also have

$$\begin{aligned}\bigcup_{m=-l+1}^{l} B^m(O_*) &= B^l(I_*) \setminus B^{-l}(I_*) = \left\{x \in \mathbb{R}^n : |B^{-l}x|_* \le 1 \text{ and } |B^l x|_* > 1\right\} \\ &\supset \left\{x \in \mathbb{R}^n : \lambda^{-l}\left|x\right|_* \le 1 \text{ and } \lambda^l \left|x\right|_* > 1\right\} = \left\{x \in \mathbb{R}^n : \lambda^{-l} < \left|x\right|_* \le \lambda^l\right\}.\end{aligned}$$

Taking the limit $l \to \infty$ we get (3.4). □

Example 2. Let the following dilation matrix be given

$$A = \begin{pmatrix} 3 & -3 \\ 1 & 0 \end{pmatrix}. \tag{3.5}$$

Here we are interested in the transpose matrix $B = A^t$ with eigenvalues $\mu_{1,2} = 3/2 \pm i\sqrt{3}/2$, hence B is an expansive matrix with $|\mu_{1,2}| = \sqrt{3} > 1$. The dilation matrix B is not expanding in the standard norm $|\cdot|_2$ in $\mathbb{R}^n$, i.e., $I_2 \not\subset B(I_2)$, as shown by Figure 2. In order to have B expanding the unit ball we need to use the Hermitian norm from

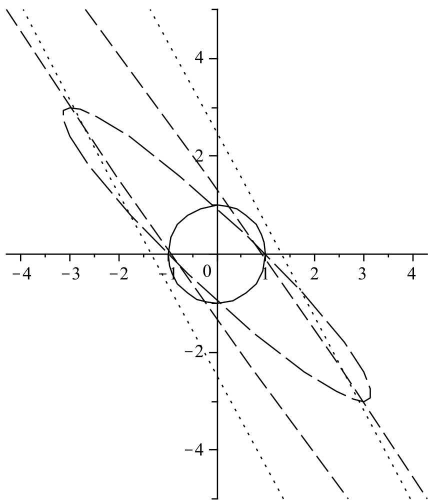

Figure 2: Boundaries of the sets I_2, $B(I_2)$, $B^2(I_2)$, and $B^3(I_2)$ marked by solid, long dashed, dashed, and dotted lines, respectively. Note that $I_2 \setminus B(I_2)$ is non-empty, and even $I_2 \setminus B^2(I_2)$ is non-empty.

(iii) in Proposition 3.1 associated with B. In (3.1) we take $k = 2$ so that the real, symmetric, positive definite matrix K is

$$K = I + (B^{-1})^t B^{-1} + (B^{-2})^t B^{-2} = \begin{pmatrix} 28/9 & 16/9 \\ 16/9 & 8/3 \end{pmatrix},$$

and let $\langle x, y\rangle_* := x^t K y$. The choice $k = 2$ suffices since it makes $B^t K B - \lambda^2 K$ semi-positive definite for $\lambda = 1.03$ and thus

$$|Bx|_* \geq \lambda \, |x|_* \, , \qquad x \in \mathbb{R}^2,$$

holds for $\lambda = 1.03$.

Figure 3 and 4 illustrate that B indeed expands the Hermitian norm unit ball I_* in all directions. We also remark that the Hermitian norm with $k = 1$ will not make the

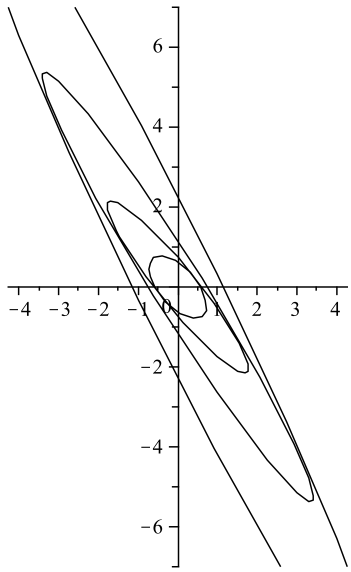

Figure 3: The unit ball I_* in the Hermitian norm $|\cdot|_*$ associated with B and its dilations $B(I_*), B^2(I_*), B^3(I_*)$. Only the boundaries are marked.

dilation matrix B expanding in $\mathbb{R}^n$; in this case we have a situation similar to Figure 2.

3.2. A crude lattice choice

Let us consider the setup in Theorem 2.3 with the set $E = B^c(O_*)$ for some $c \in \mathbb{Z}$, where the norm $|\cdot|_* = |K^{1/2}\cdot|$ is associated with B. Let μ be the smallest eigenvalue of K such that $\ell = \sqrt{1/\mu}$ is the largest semi-principal axis of the ellipsoid I_*, i.e., $\ell = \max_{x \in I_*} |x|_2$. Then we can take any lattice $\Gamma = P\mathbb{Z}^n$, where P is a non-singular matrix satisfying

$$\|P\|_2 \leq \frac{1}{\ell \, \|A^c\|_2 \, (1 + \|A^{\underline{m}}\|_2)}, \tag{3.6}$$

as our translation lattice in Theorem 2.3. To see this, recall that we are looking for a lattice Γ^* such that, for $\gamma \in \Gamma^* \setminus \{0\}$,

$$\operatorname{supp} \hat{\phi} \cap \operatorname{supp} \hat{\psi}(\cdot \pm \gamma) = \emptyset. \tag{3.7}$$

For our choice of E we find that $\operatorname{supp} \hat{\phi} \subset B^{c+\underline{m}}(I_*)$ and $\operatorname{supp} \hat{\psi} \subset B^c(I_*)$. Since

$$|B^{c+\underline{m}} x|_2 \leq \|B^{c+\underline{m}}\|_2 \, |x|_2 \leq \|B^{c+\underline{m}}\|_2 \, \ell \qquad \text{for any } x \in I_*,$$

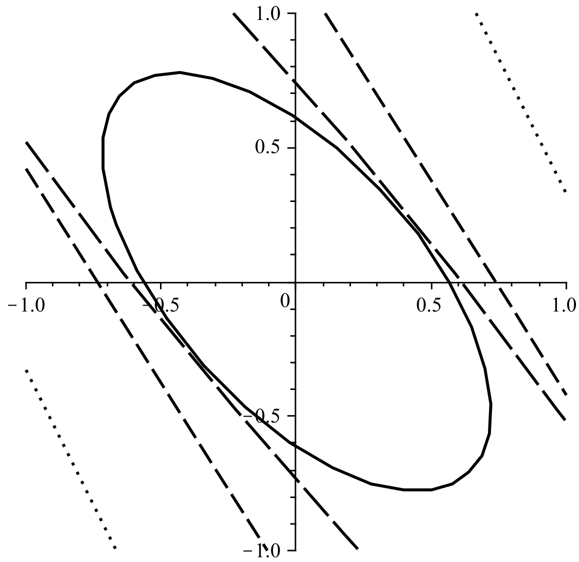

Figure 4: A zoom of Figure 3. Boundaries of the sets I_*, $B(I_*)$, $B^2(I_*)$, and $B^3(I_*)$ marked by solid, long dashed, dashed, and dotted lines, respectively.

and similar for $B^c x$, we have the situation in (3.7) whenever $|\gamma|_2 \geq \ell(\|A^c\|_2 + \|A^{c+\underline{m}}\|_2)$. Here we have used that for the 2-norm $\|A\|_2 = \|B\|_2$. For $z \in \mathbb{Z}^n$ we have

$$|z|_2 \leq \|P^t\|_2 \, |(P^t)^{-1}z|_2 = \|P\|_2 \, |(P^t)^{-1}z|_2,$$

therefore, by $|z|_2 \geq 1$ for $z \neq 0$, we have

$$\left|(P^t)^{-1}z\right|_2 \geq \frac{1}{\|P\|_2} \qquad \text{for } z \in \mathbb{Z} \setminus \{0\}.$$

Now, by assuming that P satisfies (3.6), we have

$$|\gamma|_2 = |(P^t)^{-1}z|_2 \geq 1/\|P\|_2 \geq \ell \, \|A^c\|_2 \, (1 + \|A^{\underline{m}}\|_2) \geq l(\|A^c\|_2 + \|A^{c+\underline{m}}\|_2)$$

for $0 \neq \gamma = (P^t)^{-1}z \in (P^t)^{-1}\mathbb{Z}^n = \Gamma^*$, hence the claim follows.

A lattice choice based on (3.6) can be rather crude, and produces consequently a wavelet system with unnecessarily many translates. From equation (3.6) it is obvious that any lattice $\Gamma = P\mathbb{Z}^n$ with $\|P\|$ sufficiently small will work as translation lattice for our pair of generators ψ and ϕ. Hence, the challenging part is to find a sparse translation lattice whereby we understand a lattice Γ with large determinant $d(\Gamma) := |\det P|$. In the dual lattice system this corresponds to a dense lattice Γ^* with small volume $d(\Gamma^*)$ of the fundamental parallelotope I_{Γ^*} since $d(\Gamma)d(\Gamma^*) = 1$. In Theorem 3.3 in the next section we make a better choice of the translation lattice compared to what we have from (3.6).

Using a crude lattice approach as above, we can easily transform the translation lattice to the integer lattice if we allow multiple generators. We pick a matrix P that satisfies condition (3.6) and whose inverse is integer valued, i.e., $Q := P^{-1} \in GL_n(\mathbb{Z})$. The conclusion from Theorem 2.3 is that $\{D_{A^j}T_{Q^{-1}k}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ and $\{D_{A^j}T_{Q^{-1}k}\phi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ are dual frames. The order of the quotient group $Q^{-1}\mathbb{Z}^n/\mathbb{Z}^n$ is $|\det Q|$, so let $\{d_i : i = 1,\dots,|\det Q|\}$ denote a complete set of representatives of the quotient group, and define

$$\Psi = \{T_{d_i}\psi : i = 1,\dots,|\det Q|\}\,, \quad \Phi = \{T_{d_i}\phi : i = 1,\dots,|\det Q|\}\,.$$

Since $\{D_{A^j}T_{Q^{-1}k}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n} = \{D_{A^j}T_k\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n,\psi\in\Psi}$ and likewise for the dual frame, the statement follows.

3.3. A concrete version of Theorem 2.3

We list some standing assumptions and conventions for this section.

General setup. We assume $A \in GL_n(\mathbb{R})$ is expansive. Let $|\cdot|_* = \langle\cdot,\cdot\rangle_*^{1/2}$ be a Hermitian norm as in (iii) in Proposition 3.1 associated with $B = A^t$ and let $K \in GL_n(\mathbb{R})$ be the symmetric, positive definite matrix such that $\langle x, y\rangle_* = y^tKx$. Let $\Lambda := \mathrm{diag}(\lambda_1,\dots,\lambda_n)$, where $\{\lambda_i\}$ are the eigenvalues of K, and let $Q \in O(n)$ be such that the spectral decomposition of K is $Q^tKQ = \Lambda$.

The following result is a special case of Theorem 2.3, where we, in particular, specify how to choose the translation lattice Γ. Since we in Theorem 3.3 define Γ, it allows for a more automated construction procedure.

Theorem 3.3. *Let A, K, Q, Λ be as in the general setup. Let $d \in \mathbb{N}_0$ and $\psi \in L^2(\mathbb{R}^n)$. Suppose that $\hat\psi$ is a bounded, real-valued function with $\mathrm{supp}\,\hat\psi \subset B^c(I_*) \setminus B^{c-d-1}(I_*)$ for some $c \in \mathbb{Z}$, and that (2.6) holds. Take $\Gamma = (1/2)A^cQ\sqrt{\Lambda}\mathbb{Z}^n$. Then the function ψ and the function ϕ defined by*

$$\phi(x) = d(\Gamma)\left[\psi(x) + 2\sum_{j=0}^{d}|\det A|^{-j}\,\psi(A^{-j}x)\right] \quad \text{for } x \in \mathbb{R}^n, \tag{3.8}$$

generate dual frames $\{D_{A^j}T_\gamma\psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ and $\{D_{A^j}T_\gamma\phi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ for $L^2(\mathbb{R}^n)$

Remark 2. Note that $d(\Gamma) = 2^{-n}\,|\det A|^c\,(\lambda_1\cdots\lambda_n)^{1/2}$ and $\sqrt{\Lambda} = \mathrm{diag}(\sqrt{\lambda_1},\dots,\sqrt{\lambda_n})$.

Proof. The annulus O_* is a bounded multiplicative tiling for the dilations $\{B^j : j \in \mathbb{Z}\}$ by Lemma 3.2, hence this is also the case for $B^c(O_*)$ for $c \in \mathbb{Z}$. The support of $\hat\psi$ is $\mathrm{supp}\,\hat\psi \subset B^c(I_*) \setminus B^{c-d-1}(I_*) = \cup_{j=0}^{d}B^{c-j}(O_*)$. Therefore we can apply Theorem 2.3 with $E = B^c(O_*)$, $b_j = 2$ and $b_{-j} = 0$ for $j = 1,\dots,d$ so that $\underline{m} = 0$ and $\overline{m} = d$. The only thing left to justify is the choice of the translation lattice Γ. We need to show that condition (2.7) with $\underline{m} = 0$ and $\overline{m} = d$ in Theorem 2.3 is satisfied by $\Gamma^* = 2B^cQ\Lambda^{-1/2}\mathbb{Z}^n$. By the orthogonal substitution $x = Q\tilde{x}$ the quadratic form x^tKx of equation (3.2) reduces to

$$\lambda_1\tilde{x}_1^2 + \cdots + \lambda_n\tilde{x}_n^2,$$

where $\lambda_i > 0$, hence in the $\tilde{x} = Q^tx$ coordinates I_* is given by

$$\tilde{I}_* = \left\{\tilde{x} \in \mathbb{R}^n : \left(\frac{\tilde{x}_1}{1/\sqrt{\lambda_1}}\right)^2 + \cdots + \left(\frac{\tilde{x}_n}{1/\sqrt{\lambda_n}}\right)^2 < 1\right\}$$

which is an ellipsoid with semi axes $\frac{1}{\sqrt{\lambda_1}}, \dots, \frac{1}{\sqrt{\lambda_n}}$. Therefore, in the $\tilde{x}$ coordinates,

$$(\tilde{I}_* + \gamma) \cap \tilde{I}_* = \emptyset \qquad \text{for } 0 \neq \gamma \in 2\Lambda^{-1/2}\mathbb{Z}^n,$$

or, in the x coordinates,

$$(I_* + \gamma) \cap I_* = \emptyset \qquad \text{for } 0 \neq \gamma \in 2Q\Lambda^{-1/2}\mathbb{Z}^n.$$

By applying B^c to this relation it becomes

$$(B^c(I_*) + \gamma) \cap B^c(I_*) = \emptyset \qquad \text{for } 0 \neq \gamma \in \Gamma^* = 2B^cQ\Lambda^{-1/2}\mathbb{Z}^n, \tag{3.9}$$

whereby we see that condition (2.7) is satisfied with $\underline{m} = 0$ and $\Gamma^* = 2B^cQ\Lambda^{-1/2}\mathbb{Z}^n$. The dual lattice of Γ^* is $\Gamma = 1/2A^{-c}Q\Lambda^{1/2}\mathbb{Z}^n$. It follows from Theorem 2.3 that ψ and ϕ generate dual frames for this choice of the translation lattice. □

The frame bounds for the pair of dual frames $\{D_{A^j}T_\gamma\psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ and $\{D_{A^j}T_\gamma\phi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ in Theorem 3.3 can be given explicitly as

$$C_1 = \frac{1}{d(\Gamma)} \inf_{\xi\in B^{c-d}(O_*)} \sum_{j=0}^{d} \left(\hat{\psi}(B^j\xi)\right)^2, \qquad C_2 = \frac{1}{d(\Gamma)} \sup_{\xi\in B^{c-d}(O_*)} \sum_{j=0}^{d} \left(\hat{\psi}(B^j\xi)\right)^2,$$

and

$$C_1 = \frac{1}{d(\Gamma)} \inf_{\xi\in B^{c-d}(O_*)} \sum_{j=-d}^{d} \left(\hat{\phi}(B^j\xi)\right)^2, \qquad C_2 = \frac{1}{d(\Gamma)} \sup_{\xi\in B^{c-d}(O_*)} \sum_{j=-d}^{d} \left(\hat{\phi}(B^j\xi)\right)^2,$$

respectively. The frame bounds do not depend on the specific structure of Γ, but only on the determinant of Γ; in particular, the condition number C_2/C_1 is independent of Γ.

To verify these frame bounds, we note that equation (3.9) together with the fact $\operatorname{supp}\hat{\psi}$, $\operatorname{supp}\hat{\phi} \subset B^c(I_*)$ imply that

$$\hat{\psi}(\xi)\hat{\psi}(\xi+\gamma) = \hat{\phi}(\xi)\hat{\phi}(\xi+\gamma) = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}^n \text{ and } \gamma \in \Gamma^* \setminus \{0\}.$$

Therefore, by equations (2.10) and (2.11) with $E = B^c(O_*)$, $\underline{m} = 0$ and $\overline{m} = d$, we have

$$\sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma^*} \left|\hat{\psi}(B^j\xi)\hat{\psi}(B^j\xi+\gamma)\right| = \sum_{j\in\mathbb{Z}} \left|\hat{\psi}(B^j\xi)\right|^2 = \sum_{j=0}^{d} \left(\hat{\psi}(B^j\xi)\right)^2,$$

and

$$\sum_{j\in\mathbb{Z}} \sum_{\gamma\in\Gamma^*} \left|\hat{\phi}(B^j\xi)\hat{\phi}(B^j\xi+\gamma)\right| = \sum_{j\in\mathbb{Z}} \left|\hat{\phi}(B^j\xi)\right|^2 = \sum_{j=-d}^{d} \left(\hat{\phi}(B^j\xi)\right)^2,$$

for $\xi \in B^{c-d}(O_*)$. The stated frame bounds follow from Lemma 2.2.

Example 3. Let A and K be as in Example 2. The eigenvalues of K are $\lambda_1 = (26 + 2\sqrt{65})/9 \approx 4.7$ and $\lambda_2 = (26 - 2\sqrt{65})/9 \approx 1.1$. Let the normalized (in the standard norm) eigenvectors of K be columns of $Q \in O(2)$ and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2)$, hence $Q^t K Q = \Lambda$. By the orthogonal transformation $x = Q\tilde{x}$ the Hermitian norm unit ball I_* becomes

$$\tilde{I}_* = \left\{ \tilde{x} \in \mathbb{R}^2 : \left(\frac{\tilde{x}_1}{1/\sqrt{\lambda_1}} \right)^2 + \left(\frac{\tilde{x}_2}{1/\sqrt{\lambda_2}} \right)^2 < 1 \right\} \subset I_2$$

which is an ellipse with semimajor axis $1/\sqrt{\lambda_2} \approx 0.95$ and semiminor axis $1/\sqrt{\lambda_1} \approx 0.46$. Since $\Lambda^{-1/2} = \operatorname{diag}(1/\sqrt{\lambda_1}, 1/\sqrt{\lambda_2})$, we have

$$\left| (\tilde{I}_* + \gamma) \cap \tilde{I}_* \right| = 0 \qquad \text{for } 0 \neq \gamma \in 2\Lambda^{-1/2}\mathbb{Z}^2.$$

By the orthogonal substitution back to x coordinates, we get

$$|(I_* + \gamma) \cap I_*| = 0 \qquad \text{for } 0 \neq \gamma \in 2Q\Lambda^{-1/2}\mathbb{Z}^2.$$

Suppose that $\hat{\psi}$ is a bounded, real-valued function with $\operatorname{supp}\hat{\psi} \subset B^c(I_*) \backslash B^{c-d-1}(I_*)$ for $c = 1$ that satisfies the B-dilative partition (2.6). Since $c = 1$ we need to take $\Gamma^* = 2B^1 Q\Lambda^{-1/2}\mathbb{Z}^2$ and $\Gamma = 1/2A^{-1}Q\Lambda^{1/2}\mathbb{Z}^2$, see Figure 5 and 6.

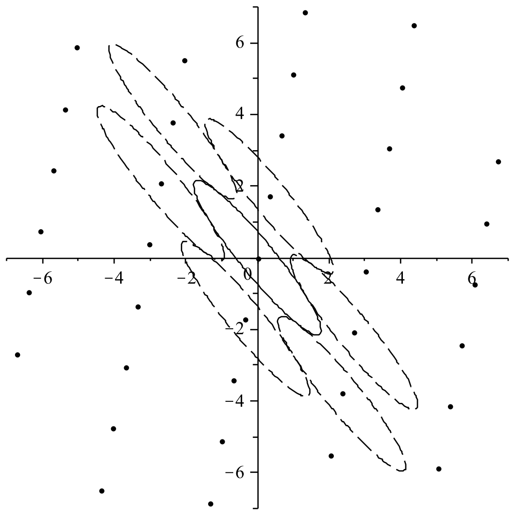

Figure 5: The dual lattice $\Gamma^* = 2B^c Q\Lambda^{-1/2}\mathbb{Z}^2$ for $c = 1$ is shown by dots, and the boundary of the set $B^c(I_*)$ by a solid line. Boundaries of the set $B^c(I_*)$ translated to several different $\gamma \in \Gamma^* \setminus \{0\}$ are shown with dashed lines. Recall that $\operatorname{supp}\hat{\psi}, \operatorname{supp}\hat{\phi} \subset B^c(I_*)$, hence $\operatorname{supp}\hat{\phi} \cap \operatorname{supp}\hat{\psi}(\cdot + \gamma) = \emptyset$ for $\gamma \in \Gamma^* \setminus \{0\}$.

3.4. An alternative lattice choice

Let the setup up and assumptions be as in Theorem 3.3, except for the lattice Γ which we want to choose differently. As in Section 3.2 the dual lattice Γ^* needs to satisfy (3.7) for $\gamma \in \Gamma^* \setminus \{0\}$. We want to choose Γ^* as dense as possible since this will make the translation lattice Γ as sparse as possible and the wavelet system with as few translates

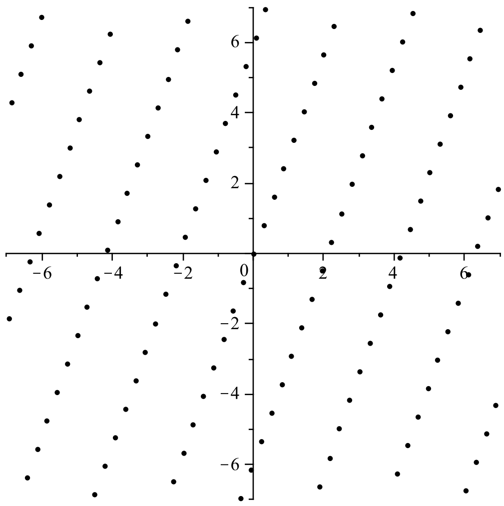

Figure 6: The translation lattice $\Gamma = (1/2)A^c Q\Lambda^{1/2}\mathbb{Z}^2$ for $c = 1$.

as possible. Since $\operatorname{supp}\hat{\psi}, \operatorname{supp}\hat{\phi} \subset B^c(I_*)$, we are looking for lattices Γ^* that packs the ellipsoids $B^c(I_*) + \gamma$, $\gamma \in \Gamma^*$, in a non-overlapping, optimal way. By the coordinate transformation $\hat{x} = \Lambda^{-1/2}Q^t B^{-c}x$, the ellipsoid $B^c(I_*)$ turns into the standard unit ball I_2 in $\mathbb{R}^n$. This calculations are as follows.

$$\begin{aligned} B^c(I_*) &= \left\{ B^c x : |x|_*^2 \leq 1 \right\} = \left\{ x : |K^{1/2}B^{-c}x|_2^2 \leq 1 \right\} \\ &= \left\{ x : \left| K^{1/2}B^{-c}B^c Q\Lambda^{-1/2}\hat{x} \right|_2^2 \leq 1 \right\} \\ &= \left\{ x : \left\langle \hat{x}, \Lambda^{-1/2}Q^t K Q\Lambda^{-1/2}\hat{x} \right\rangle_2 \leq 1 \right\} = \left\{ x : |\hat{x}|_2^2 \leq 1 \right\}, \end{aligned}$$

and we arrive at a standard sphere packing problem with lattice arrangement of non-overlapping unit n-balls. The proportion of the Euclidean space $\mathbb{R}^n$ filled by the balls is called the density of the arrangement, and it is this density we want as high as possible.

Taking Γ as in Theorem 3.3 corresponds to a square packing of the unit n-balls $I_2 + k$ by the lattice $2\mathbb{Z}^n$, i.e., $k \in 2\mathbb{Z}^n$. The density of this packing is $V_n 2^{-n}$, where V_n is the volume of the n-ball: $V_{2n} = \pi^n/(n!)$ and $V_{2n+1} = (2^{2n+1}n!\pi^n)/(2n+1)!$. This is not the densest packing of balls in $\mathbb{R}^n$ since there exists a lattice with density bigger than $1.68n2^{-n}$ for each $n \neq 1$ [9]; a slight improvement of this lower bound was obtained in [1] for $n > 5$. Moreover, the densest lattice packing of hyperspheres is known up to dimension 8, see [20]; it is precisely this dense lattice we want to use in place of $2\mathbb{Z}^n$ (at least whenever $n \leq 8$).

In $\mathbb{R}^2$ Lagrange proved that the hexagonal packing, where each ball touches 6 other balls in a hexagonal lattice, has the highest density $\pi/\sqrt{12}$. Hence using $P\mathbb{Z}^2$ with

$$P = \begin{pmatrix} 2 & 0 \\ 1 & \sqrt{3} \end{pmatrix}$$

instead of $2\mathbb{Z}^2$ improves the packing by a factor of

$$\frac{\pi/\sqrt{12}}{\pi/2^2} = 4/\sqrt{12} = 2/\sqrt{3}.$$

It is easily seen that this factor equals the relation between the area of the fundamental parallelogram of the two lattices $|\det 2I_{2\times 2}| \,/\, |\det P|$. In Figure 5 we see that each ellipse only touches 4 other ellipses corresponding to the square packing $2\mathbb{Z}^n$; in the optimal packing each ellipse touch 6 others. In $\mathbb{R}^3$ Gauss proved that the highest density is $\pi/\sqrt{18}$ obtained by the hexagonal close and face-centered cubic packing; here each ball touches 12 other balls.

4. Dilative partition of unity

With Theorem 3.3 at hand the only issue left is to specify how to construct functions satisfying the partition of unity (2.6) for any given expansive matrix. In the two examples of this section we outline possible ways of achieving this.

4.1. Constructing a partition of unity

As usual we fix the dimension $n \in \mathbb{N}$ and the expansive matrix $B \in GL_n(\mathbb{R})$. In the examples in this section we construct functions satisfying the assumptions in Theorem 3.3, that is, a real-valued function $g \in L^2(\mathbb{R}^n)$ with $\operatorname{supp} g \subset B^c(I_*) \setminus B^{c-d-1}(I_*)$ for some $c \in \mathbb{Z}$ and $d \in \mathbb{N}_0$ so that the B-dilative partition

$$\sum_{j\in\mathbb{Z}} g(B^j\xi) = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{4.1}$$

holds.

In the construction we will use that the radial coordinate of the surface of the ellipsoid $\partial B^j(I_*)$, $j \in \mathbb{Z}$, can be parametrized by the $n-1$ angular coordinates $\theta_1, \dots, \theta_{n-1}$. The radial coordinate expression will be of the form $h(\theta_1, \dots, \theta_{n-1})^{-1/2}$ for some positive, trigonometric function h, where h is bounded away from zero and infinity with the specific form of h depending on the dimension n and the length and orientation of the ellipsoid axes.

We illustrate this with the following example in $\mathbb{R}^4$. We want to find the radial coordinate r of the ellipsoid

$$\left\{x \in \mathbb{R}^4 : (x_1/\ell_1)^2 + (x_2/\ell_2)^2 + (x_3/\ell_3)^2 + (x_4/\ell_4)^2 = 1\right\}, \qquad \ell_i > 0,\ i = 1,2,3,4,$$

as a function the angular coordinates θ_1, θ_2 and θ_3. We express $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ in the hyperspherical coordinates $(r, \theta_1, \theta_2, \theta_3) \in \{0\} \cup \mathbb{R}_+ \times [0,\pi] \times [0,\pi] \times [0, 2\pi)$ as follows:

$$x_1 = r\cos\theta_1, \qquad x_2 = r\sin\theta_1\cos\theta_2,$$
$$x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, \qquad x_4 = r\sin\theta_1\sin\theta_2\sin\theta_3.$$

Then we substitute x_i, $i = 1, \dots, 4$, in the expression above and factor out r^2 to obtain $r^2 f(\theta_1, \theta_2, \theta_3) = 1$, where

$$\begin{aligned} f(\theta_1,\theta_2,\theta_3) = {} & \ell_1^{-2}\cos^2\theta_1 + \ell_2^{-2}\sin^2\theta_1\cos^2\theta_2 \\ & + \ell_3^{-2}\sin^2\theta_1\sin^2\theta_2\cos^2\theta_3 + \ell_4^{-2}\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3. \end{aligned} \tag{4.2}$$

The conclusion is that $r = r(\theta_1, \theta_2, \theta_3) = f(\theta_1, \theta_2, \theta_3)^{-1/2}$.

Example 4. For $d = 1$ in Theorem 3.3 we want $g \in C_0^s(\mathbb{R}^n)$ for any given $s \in \mathbb{N} \cup \{0\}$. The choice $d = 1$ will fix the "size" of the support of g so that $\operatorname{supp} g \subset B^c(I_*) \setminus B^{c-2}(I_*)$ for some $c \in \mathbb{Z}$. Now let $r_1 = r_1(\theta_1, \dots, \theta_{n-1})$ and $r_2 = r_2(\theta_1, \dots, \theta_{n-1})$ denote the radial coordinates of the surface of the ellipsoids $\partial B^{c-1}(I_*)$ and $\partial B^c(I_*)$ parametrized by $n-1$ angular coordinates $\theta_1, \dots, \theta_{n-1}$, respectively.

Let f be a continuous function on the annulus $S = \overline{B^c(O_*)}$ satisfying $f|_{\partial B^{c-1}(I_*)} = 1$ and $f|_{\partial B^c(I_*)} = 0$. Using the parametrizations r_1, r_2 of the surfaces of the two ellipsoids and fixing the $n-1$ angular coordinates we realize that we only have to find a continuous function $f : [r_1, r_2] \to \mathbb{R}$ of one variable (the radial coordinate) satisfying $f(r_1) = 1$ and $f(r_2) = 0$. For example the general function $f \in C^0(S)$ of d variables can be any of the functions below:

$$f(x) = f(r, \theta_1, \dots, \theta_{n-1}) = \frac{r_2 - r}{r_2 - r_1}, \tag{4.3a}$$

$$f(x) = f(r, \theta_1, \dots, \theta_{n-1}) = \frac{(r_2 - r)^2}{(r_2 - r_1)^3}(2(r - r_1) + r_2 - r_1), \tag{4.3b}$$

$$f(x) = f(r, \theta_1, \dots, \theta_{n-1}) = \tfrac{1}{2} + \tfrac{1}{2}\cos \pi(\tfrac{r - r_1}{r_2 - r_1}), \tag{4.3c}$$

where $r = |x| \in [r_1, r_2]$, $\theta_1, \dots, \theta_{n-2} \in [0, \pi]$, and $\theta_{n-1} \in [0, 2\pi)$; recall that $r_1 = r_1(\theta_1, \dots, \theta_{n-1})$ and $r_2 = r_2(\theta_1, \dots, \theta_{n-1})$. In definitions (4.3b) and (4.3c) the function f even belongs to $C^1(S)$.

Define $g \in L^2(\mathbb{R})$ by:

$$g(x) = \begin{cases} 1 - f(Bx) & \text{for } x \in B^{c-1}(I_*) \setminus B^{c-2}(I_*), \\ f(x) & \text{for } x \in B^c(I_*) \setminus B^{c-1}(I_*), \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

This way g becomes a B-dilative partition of unity with $\operatorname{supp} g \subset B^c(I_*) \setminus B^{c-2}(I_*)$, so we can apply Theorem 3.3 with $\hat{\psi} = g$ and $d = 2$.

We can simplify the expressions for the radial coordinates r_1, r_2 of the surface of the ellipsoids $\partial B^{c-1}(I_*)$ and $\partial B^c(I_*)$ from the previous example by a suitable coordinate change. The idea is to transform the ellipsoid $B^{c-1}(I_*)$ to the standard unit ball I_2 by a first coordinate change $\tilde{x} = \Lambda^{1/2} Q^t B^{-c+1} x$. This will transform the outer ellipsoid $B^c(I_*)$ to another ellipsoid. A second and orthogonal coordinate transform $\hat{x} = Q_\prime^t \tilde{x}$ will make the semiaxes of this new ellipsoid parallel to the coordinate axes, leaving the standard unit ball I_2 unchanged. Here $Q_\prime$ comes from the spectral decomposition of $A^{-1}B^{-1}$, i.e., $A^{-1}B^{-1} = Q_\prime^t \Lambda_\prime Q_\prime$. In the $\hat{x}$ coordinates $r_1 = 1$ is a constant and $r_2 = f^{-1/2}$ with f of the form (4.2) for $n = 4$ and likewise for $n \neq 4$.

In the construction in Example 4 we assumed that $d = 1$. The next example works for all $d \in \mathbb{N}$; moreover, the constructed function will belong to $C_0^\infty(\mathbb{R}^n)$.

Example 5. For sufficiently small $\delta > 0$ define $\Delta_1, \Delta_2 \subset \mathbb{R}^n$ by

$$\Delta_1 = B^{c-d-1}(I_*) + \mathbf{B}(0, \delta),$$
$$\Delta_2 + \mathbf{B}(0, \delta) = B^c(I_*).$$

This makes $\Delta_2 \setminus \Delta_1$ a subset of the annulus $B^c(I_*) \setminus B^{c-d-1}(I_*)$; it is exactly the subset, where points less than δ in distance from the boundary have been removed, or in other words

$$\Delta_2 \setminus \Delta_1 + \mathbf{B}(0,\delta) = B^c(I_*) \setminus B^{c-d-1}(I_*).$$

For this to hold, we of course need to take $\delta > 0$ sufficiently small, e.g., such that $\Delta_1 \subset r\Delta_1 \subset \Delta_2$ holds for some $r > 1$.

Let $h \in C_0^\infty(\mathbb{R}^n)$ satisfy $\operatorname{supp} h = \mathbf{B}(0,1)$, $h \geq 0$, and $\int h \,\mathrm{d}\mu = 1$, and define $h_\delta = \delta^{-d} h(\delta^{-1}\cdot)$. By convoluting the characteristic function on $\Delta_2 \setminus \Delta_1$ with h_δ we obtain a smooth function living on the annulus $B^c(I_*) \setminus B^{c-d-1}(I_*)$. So let $p \in C_0^\infty(\mathbb{R}^n)$ be defined by

$$p = h_\delta * \chi_{\Delta_2 \setminus \Delta_1},$$

and note that $\operatorname{supp} p = B^c(I_*) \setminus B^{c-d-1}(I_*)$ since $\operatorname{supp} h_\delta = \mathbf{B}(0,\delta)$. Normalizing the function p in a proper way will give us the function g we are looking for. We will normalize p by the function w:

$$w(x) = \sum_{j\in\mathbb{Z}} p(B^j x).$$

For a fixed $x \in \mathbb{R}^n \setminus \{0\}$ this sum has either d or $d+1$ nonzero terms, and w is therefore bounded away from 0 and ∞:

$$\exists c, C > 0 : c < w(x) < C \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\},$$

hence we can define a function $g \in C_0^\infty(\mathbb{R}^n)$ by

$$g(x) = \frac{p(x)}{w(x)} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}, \quad \text{and,} \quad g(0) = 0. \tag{4.5}$$

The function g will be an almost everywhere B-dilative partition of unity as is seen by using the B-dilative periodicity of w:

$$\sum_{j\in\mathbb{Z}} g(B^j x) = \sum_{j\in\mathbb{Z}} \frac{p(B^j x)}{w(B^j x)} = \sum_{j\in\mathbb{Z}} \frac{p(B^j x)}{w(x)} = \frac{1}{w(x)} \sum_{j\in\mathbb{Z}} p(B^j x) = 1.$$

Since p is supported on the annulus $B^c(I_*) \setminus B^{c-d-1}(I_*)$, we can simplify the definition in (4.5) to get rid of the infinite sum in the denominator; this gives us the following expression

$$g(x) = p(x) / \sum_{j=-d}^{d} p(B^j x) \qquad \text{for } x \in \mathbb{R}^n \setminus \{0\}.$$

We can obtain a more explicit expression for p by the following approach. Let $r_1 = r_1(\theta_1, \dots, \theta_{n-1})$ and $r_2 = r_2(\theta_1, \dots, \theta_{n-1})$ denote the radial coordinates of the surface of the ellipsoids $\partial B^{c-d-1}(I_*)$ and $\partial B^c(I_*)$ parametrized by $n-1$ angular coordinates $\theta_1, \dots, \theta_{n-1}$, respectively. Finally, let $p \in C_0^\infty(\mathbb{R}^n)$ be defined by

$$p(x) = \eta(|x| - r_1)\,\eta(r_2 - |x|), \quad \text{with } r_1 = r_1(\theta_1, \dots, \theta_{n-1}) \text{ and } r_2 = r_2(\theta_1, \dots, \theta_{n-1})$$

where $\theta_1, \dots, \theta_{n-1}$ can be found from x, and

$$\eta(x) = \begin{cases} \mathrm{e}^{-1/x} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

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PAPER IV

Affine and quasi-affine frames for rational dilations

Marcin Bowník[1] and Jakob Lemvig

Abstract. In this paper we extend the investigation of quasi-affine systems, which were originally introduced by Ron and Shen [20] for integer, expansive dilations, to the class of rational, expansive dilations. We show that an affine system is a frame if, and only if, the corresponding family of quasi-affine systems are frames with uniform frame bounds. We also prove a similar equivalence result between pairs of dual affine frames and dual quasi-affine frames. Finally, we uncover some fundamental differences between the integer and rational settings by exhibiting an example of a quasi-affine frame such that its affine counterpart is not a frame.

Keywords. Wavelets · Affine systems · Quasi-affine systems · Rational dilations · Shift invariant systems · Oversampling

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1. Introduction

Quasi-affine systems are little known cousins of well-studied affine systems also known as wavelet systems. Let A be an expansive dilation matrix, i.e., $n \times n$ real matrix with all eigenvalues $|\lambda| > 1$. The affine system generated by a function $\psi \in L^2(\mathbb{R}^n)$ is

$$\mathcal{A}(\psi) = \left\{ \psi_{j,k}(x) := |\det A|^{j/2}\, \psi(A^j x - k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^n \right\}. \tag{1.1}$$

The affine systems are dilation invariant, but not shift invariant. However, if the dilation A has integer entries, that is $A\mathbb{Z}^n \subset \mathbb{Z}^n$, then one can modify the definition of affine systems to obtain shift invariant systems. This leads to the notion of a quasi-affine system

$$\mathcal{A}^q(\psi) = \left\{ \tilde{\psi}_{j,k}(x) := \begin{cases} |\det A|^{j/2}\, \psi(A^j x - k) : j \geq 0,\ k \in \mathbb{Z}^n \\ |\det A|^{j}\, \psi(A^j (x - k)) : j < 0,\ k \in \mathbb{Z}^n \end{cases} \right\}, \tag{1.2}$$

which was introduced and investigated for integer, expansive dilation matrices by Ron and Shen [20]. Despite that the orthogonality of the affine system cannot be carried over to the corresponding quasi-affine system due to the oversampling of negative scales of the affine system, it turns out that the frame property is preserved. This important discovery is due to Ron and Shen [20] who proved that the affine system $\mathcal{A}(\psi)$ is a frame if, and only if, its quasi-affine counterpart $\mathcal{A}^q(\psi)$ is a frame (with the same frame bounds). Furthermore, quasi-affine systems are shift invariant and thus much easier to study than affine systems which are dilation invariant.

The goal of this work is to extend the study of quasi-affine systems to the class of expansive rational dilations. Let A be an expansive dilation with rational entries, that is $A\mathbb{Q}^n \subset \mathbb{Q}^n$. The first author [3] generalized the notion of a quasi-affine frame for rational, expansive dilations which coincides with the usual definition in the case of integer dilations. The main idea of Ron and Shen [20] is to oversample negative scales of the affine system at a rate adapted to the scale in order for the resulting system to be shift invariant, i.e., $\phi \in \mathcal{A}^q(\psi) \Rightarrow T_k\phi \in \mathcal{A}^q(\psi)$ for all $k \in \mathbb{Z}^n$. In order to define quasi-affine systems for rational expansive dilations one needs to oversample both negative and positive scales of the affine system (at a rate proportional to the scale) which results in a quasi-affine system that in general coincides with the affine system only at the scale zero. This can easily be seen in one dimension where the quasi-affine system has a relatively simple algebraic form. Suppose that $a = p/q \in \mathbb{Q}$ is a dilation factor, where $|a| > 1$, $p, q \in \mathbb{Z}$ are relatively prime. Then, the quasi-affine system associated with a is given by

$$\mathcal{A}^q(\psi) = \left\{ \begin{array}{ll} |p|^{j/2}\, |q|^{-j}\, \psi(a^j x - q^{-j} k) : & j \geq 0,\ k \in \mathbb{Z} \\ |p|^{j}\, |q|^{-j/2}\, \psi(a^j x - p^j k) : & j < 0,\ k \in \mathbb{Z} \end{array} \right\}. \tag{1.3}$$

In the rational case it is much less clear than in the case of integer, expansive dilations (where both systems coincide at all non-negative scales), whether there is any relationship between affine and quasi-affine systems. Nevertheless, the first author proved in [3] that the tight frame property is preserved when moving between rationally dilated affine and quasi-affine systems. This result has initially suggested that there is not much difference between integer and rational cases.

In this work we show that this belief is largely incorrect by uncovering substantial differences between the theory of integer dilated and rationally dilated quasi-affine systems. For any rational, non-integer dilation we give an example of an affine system which is not a frame, but yet, the corresponding quasi-affine system is a frame. This kind of example does not exist for integer dilations due to the above mentioned result of Ron and Shen.

To understand the broken symmetry between the integer and rational case we introduce a new class of quasi-affine systems indexed by the choice of the oversampling lattice $\Lambda \subset \mathbb{Z}^n$. In short, the quasi-affine system $\mathcal{A}^q_\Lambda(\psi)$ is defined to be the smallest shift invariant system with respect to a lattice Λ, i.e., $\phi \in \mathcal{A}^q_\Lambda(\psi) \Rightarrow T_\lambda \phi \in \mathcal{A}^q_\Lambda(\psi)$ for $\lambda \in \Lambda$, which contains all elements of the original affine system $\mathcal{A}(\psi)$. In order to make this definition meaningful we also need to renormalize the elements of $\mathcal{A}^q_\Lambda(\psi)$ at a rate corresponding to the rate of oversampling as it was done previously. Again, this is best illustrated in one dimension. We take $\Lambda = (pq)^J\mathbb{Z}$ for $J \in \mathbb{N}_0$ since this particular choice gives the oversampled quasi-affine system $\mathcal{A}^q_\Lambda(\psi)$ a nice algebraic form:

$$\mathcal{A}^q_\Lambda(\psi) = \left\{ \begin{array}{ll} |p|^{j/2}\,|q|^{-j+J/2}\,\psi(a^j x - q^{J-j}k): & j > J, k \in \mathbb{Z} \\ |a|^{j/2}\,\psi(a^j x - k): & -J \le j \le J,\ k \in \mathbb{Z} \\ |p|^{j+J/2}\,|q|^{-j/2}\,\psi(a^j x - p^{j+J}k): & j < -J,\ k \in \mathbb{Z} \end{array} \right\}, \tag{1.4}$$

see Example 3. Then our main result can be stated as follows.

Theorem 1.1. *The affine system $\mathcal{A}(\psi)$ is a frame for $L^2(\mathbb{R}^n)$ if, and only if, every Λ-oversampled quasi-affine system $\mathcal{A}^q_\Lambda(\psi)$ is a frame with uniform frame bounds for all $\Lambda \subset \mathbb{Z}^n$.*

In the case when the dilation A is integer-valued, the class of Λ-oversampled quasi-affine systems reduces to the standard quasi-affine system $\mathcal{A}^q(\psi)$ and its dilates, see Example 2. Hence, the original result of Ron and Shen [20] follows immediately from Theorem 1.1. The proof of Theorem 1.1 is influenced by the work of Hernández, Labate, Weiss, and Wilson [13, 14], where the authors obtain reproducibility characterizations of generalized shift invariant (GSI) systems including affine, wave packets, and Gabor systems. The key element of these techniques is the use of almost periodic functions which was pioneered by Laugesen [17, 18] in his work on translational averaging of the wavelet functional. Using these methods Laugesen [18] gave another proof of the equivalence of affine and quasi-affine frames in the integer case. In this work we show that these techniques can be generalized to treat rationally dilated quasi-affine systems as well.

In the next part of the paper we investigate more subtle frame properties of quasi-affine systems. We characterize when the canonical dual frame of a Λ-oversampled quasi-affine frame $\mathcal{A}^q_\Lambda(\psi)$ is also a quasi-affine frame. In the case of integer dilations, such characterization is due to the first author and Weber [5]. Theorem 1.2 generalizes this result to the case of rational dilations. It is remarkable that the existence of the canonical quasi-affine dual frame is independent of the choice of the oversampling lattice Λ. Hence, if such canonical dual frame exists for some Λ-oversampled quasi-affine system, then it must exist for all lattices $\Lambda \subset \mathbb{Z}^n$.

Theorem 1.2. *Suppose the quasi-affine system $\mathcal{A}^q_{\Lambda_0}(\psi)$ is a frame for $L^2(\mathbb{R}^n)$ for some lattice $\Lambda_0 \subset \mathbb{Z}^n$. Then, the canonical dual frame of $\mathcal{A}^q_{\Lambda_0}(\psi)$ is of the form $\mathcal{A}^q_{\Lambda_0}(\phi)$ for some $\phi \in L^2(\mathbb{R}^n)$ if, and only if, for all $\alpha \in \mathbb{Z}^n \setminus \{0\}$,*

$$t_\alpha(\xi) := \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} \hat{\psi}(B^{-j}\xi)\overline{\hat{\psi}(B^{-j}(\xi+\alpha))} = 0. \tag{1.5}$$

In this case, $\mathcal{A}^q_\Lambda(\phi)$ is the canonical dual frame of $\mathcal{A}^q_\Lambda(\psi)$ for all lattices $\Lambda \subset \mathbb{Z}^n$.

We also investigate pairs of dual quasi-affine frames. Here, the theory of rationally dilated quasi-affine frames parallels quite closely that of integer dilated systems. Hence, we have a perfect equivalence between pairs of dual affine frames and pairs of dual quasi-affine frames, regardless of the choice of the oversampling lattice Λ.

Theorem 1.3. *Suppose that $\mathcal{A}(\psi)$ and $\mathcal{A}(\phi)$ are Bessel sequences in $L^2(\mathbb{R}^n)$. Then the following are equivalent:*

(i) $\mathcal{A}(\psi)$ and $\mathcal{A}(\phi)$ are dual frames,

(ii) $\mathcal{A}^q_{\Lambda_0}(\psi)$ and $\mathcal{A}^q_{\Lambda_0}(\phi)$ are dual frames for ***some*** *oversampling lattice $\Lambda_0 \subset \mathbb{Z}^n$,*

(iii) $\mathcal{A}^q_\Lambda(\psi)$ and $\mathcal{A}^q_\Lambda(\phi)$ are dual frames for ***all*** *oversampling lattices $\Lambda \subset \mathbb{Z}^n$.*

Theorem 1.3 points at a location of the broken symmetry in the equivalence between affine and quasi-affine frames in the rational non-integer case. If such non-equivalence exists, then it can only exhibit itself for quasi-affine frames which do not have a dual quasi-affine frame. The last section of this work is devoted to showing that such phenomena does indeed exist. For any non-integer rational dilation factor we give an example of a quasi-affine frame $\mathcal{A}^q_\Lambda(\psi)$ such that the corresponding affine system $\mathcal{A}(\psi)$ is not a frame.

Theorem 1.4. *For each rational non-integer dilation factor $a > 1$, there exists a function $\psi \in L^2(\mathbb{R})$ such that $\mathcal{A}^q_\Lambda(\psi)$ is a frame for any oversampling lattice $\Lambda \subset \mathbb{Z}$, but yet, $\mathcal{A}(\psi)$ is not a frame.*

Despite that each system $\mathcal{A}^q_\Lambda(\psi)$ is a frame, its lower frame bound drops to zero as the lattice Λ gets sparser. Hence, this example does not contradict Theorem 1.1. Moreover, in the light of Theorem 1.3, none of the quasi-affine frames $\mathcal{A}^q_\Lambda(\psi)$ can have a dual quasi-affine frame.

We end this introduction by reviewing some basic definitions. A *frame sequence* is a countable collection of vectors $\{f_j\}_{j\in\mathcal{J}}$ such that there are constants $0 < C_1 \le C_2 < \infty$ satisfying, for all $f \in \overline{\operatorname{span}}\{f_j\}$,

$$C_1 \|f\|^2 \le \sum_{j\in\mathcal{J}} |\langle f, f_j\rangle|^2 \le C_2 \|f\|^2 .$$

If $\overline{\operatorname{span}}\{f_j\} = \mathcal{H}$ for a separable Hilbert space $\mathcal{H}$, we say that the frame sequence $\{f_j\}_{j\in\mathcal{J}}$ is a *frame* for $\mathcal{H}$; if the upper bound in the above inequality holds, but not necessarily the lower bound, the sequence $\{f_j\}$ is said to be a *Bessel sequence* with Bessel constant C_2. For a Bessel sequence $\{f_j\}$, we define the frame operator of $\{f_j\}$ by

$$S\colon \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{j\in\mathcal{J}} \langle f, f_j\rangle f_j.$$

If $\{f_j\}$ is a frame, this operator is bounded, invertible, and positive. A frame $\{f_j\}$ is said to be *tight* if we can choose $C_1 = C_2$; this is equivalent to $S = C_1 I$, where I is the identity operator. If furthermore $C_1 = C_2 = 1$, the sequence $\{f_j\}$ is said to be a *Parseval frame.*

Two Bessel sequences $\{f_j\}$ and $\{g_j\}$ are said to be *dual frames* if

$$f = \sum_{j\in\mathcal{J}} \langle f, g_j\rangle f_j \qquad \text{for all } f \in \mathcal{H}.$$

It can be shown that two such Bessel sequences indeed are frames, and we shall say that the frame $\{g_j\}$ is *dual* to $\{f_j\}$, and vice versa. At least one dual always exists, it is given by $\{S^{-1}f_j\}$ and called the canonical dual.

Let $f \in L^2(\mathbb{R}^n)$ for some fixed $n \in \mathbb{N}$. The translation by $y \in \mathbb{R}^n$ is $T_y f(x) = f(x-y)$; dilation by an $n\times n$ non-singular matrix B is $D_B f(x) = |\det B|^{1/2} f(Bx)$. These two operations are unitary as operators on $L^2(\mathbb{R}^n)$. Let $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$ and let A be a fixed $n \times n$ expansive matrix, i.e., all eigenvalue λ of A satisfy $|\lambda| > 1$. The affine system of unitaries $\mathcal{A}$ associated with the dilation A is defined as $\mathcal{A} = \{D_{A^j}T_\gamma : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$, and the affine system $\mathcal{A}(\Psi)$ generated by Ψ is defined as

$$\mathcal{A}(\Psi) = \{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, \psi \in \Psi\},$$

where $\psi_{j,k} = D_{A^j}T_\gamma\psi$ for $j \in \mathbb{Z}, k \in \mathbb{Z}^n$. We say that Ψ is a frame wavelet if $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$, and say that Ψ and Φ is a pair of dual frame wavelets if their wavelet systems are dual frames. The transpose of the (fixed) dilation matrix A is denoted by $B = A^t$.

Following [12], the local commutant of a system of operators $\mathcal{U}$ at the point $f \in L^2(\mathbb{R}^n)$ is defined as

$$\mathcal{C}_f(\mathcal{U}) := \Big\{T \in B(L^2(\mathbb{R}^n)) : TUf = UTf \quad \forall\, U \in \mathcal{U}\Big\}.$$

For $f \in L^1(\mathbb{R}^n)$, the Fourier transform is defined by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-2\pi i\langle\xi,x\rangle}\mathrm{d}x$$

with the usual extension to $L^2(\mathbb{R}^n)$. We will frequently prove our results on the following subspace of $L^2(\mathbb{R}^n)$

$$\mathcal{D} = \Big\{f \in L^2(\mathbb{R}^n) : \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp}\hat{f} \text{ is compact in } \mathbb{R}^n \setminus \{0\}\Big\}, \tag{1.6}$$

and extend the result by density arguments.

2. Generalized shift invariant systems, lattices and oversampling

In this section we review some fundamental properties of lattices, shift invariant systems, oversampling of shift invariant systems, mixed dual Gramians, and generalized shift invariant systems.

2.1. Lattices in $\mathbb{R}^n$

A lattice Γ in $\mathbb{R}^n$ is a discrete subgroup under addition generated by integral linear combinations of n linearly independent vectors $\{p_i\}_{i=1}^n \subset \mathbb{R}^n$, i.e.,

$$\Gamma = \{z_1 p_1 + \cdots + z_n p_n : z_1, \ldots, z_n \in \mathbb{Z}\}.$$

In other words, it is a set of points of the form $P\mathbb{Z}^n$ for a non-singular $n \times n$ matrix P. Let Γ be a lattice in $\mathbb{R}^n$. If $\Gamma = P\mathbb{Z}^n$, we say that the matrix $P \in GL_n(\mathbb{R})$ generates the lattice Γ. A generating matrix of a given lattice is only unique up to multiplication from the right by integer matrices with determinant one in absolute value; in particular, if $\Gamma = P\mathbb{Z}^n$ for some $P \in GL_n(\mathbb{R})$, then also $\Gamma = PS\mathbb{Z}^n$ for any $S \in SL_n(\mathbb{Z})$. The *determinant* of Γ is defined to be:

$$d(\Gamma) = |\det P|, \tag{2.1}$$

where $P \in GL_n(\mathbb{R})$ is a generating matrix for Γ; note that $d(\Gamma) > 0$ and $d(\mathbb{Z}^n) = 1$. The determinant $d(\Gamma)$ is independent of the particular choice of generating matrix P and equals the volume of a fundamental domain I_Γ of the lattice Γ, where

$$I_\Gamma = P([0,1)^n) = \{c_1 p_1 + \cdots + c_n p_n : 0 \leq c_i < 1 \text{ for } i = 1, \ldots, n\}$$

with p_i denoting the ith column of a generating matrix P. Note that $\mathbb{R}^n = \cup_{\gamma \in \Gamma}(\gamma + I_\Gamma)$ with the union being disjoint, and that the specific shape of I_Γ depends on the choice of the generating matrix P.

Suppose that $\Gamma \subset \Lambda$, in other words, that Γ is a sublattice of some "denser" lattice Λ. We define the *index* of Γ in Λ as

$$D = \frac{d(\Gamma)}{d(\Lambda)}. \tag{2.2}$$

The index D is always a positive integer; it is actually the number of copies of parallelotopes I_Γ that fits inside a larger parallelotope I_Λ. If D is the index of Γ in Λ, we have from [6, §I.2.2],

$$D\Lambda \subset \Gamma \subset \Lambda, \tag{2.3}$$

and, from [6, Lemma I.1],

$$\#\{\Lambda/\Gamma\} = D \equiv d(\Gamma)/d(\Lambda), \tag{2.4}$$

where $\#\{\Lambda/\Gamma\}$ is the order of the quotient group Λ/Γ. As illustrated in the following, these simple relations are often very useful. Suppose Γ is a *rational* lattice, i.e., the points of the lattice have rational coordinates or, equivalently, the entries of a generating matrix P are rational. In this situation we define $\tilde{\Gamma}$, the *integral sublattice* of Γ, by $\tilde{\Gamma} = \mathbb{Z}^n \cap \Gamma$, and the *extended integral superlattice* of Γ by $\Gamma + \mathbb{Z}^n$. Using the characterization of lattices in [6, Theorem III.VI], it is straightforward to show that these point sets actually are lattices. Thus $\tilde{\Gamma} = \Gamma \cap \mathbb{Z}^n$ is a sublattice of $\mathbb{Z}^n$ with index in $\mathbb{Z}^n$ as

$$D = \frac{d(\tilde{\Gamma})}{d(\mathbb{Z}^n)} = d(\tilde{\Gamma}),$$

and consequently,

$$d(\tilde{\Gamma})\mathbb{Z}^n \subset \tilde{\Gamma} \subset \Gamma. \tag{2.5}$$

This shows that any rational lattice Γ has a integral sublattice of the form $c\mathbb{Z}^n$, where the constant $c \in \mathbb{N}$ can be taken to be $c = d(\tilde{\Gamma}) = \mathrm{vol}\,(I_{\tilde{\Gamma}}) = \#\{\mathbb{Z}^n/\tilde{\Gamma}\}$. Since we also have $\#\{\Gamma/\tilde{\Gamma}\} = d(\tilde{\Gamma})/d(\Gamma)$, the above calculations show that

$$\#\{\mathbb{Z}^n/\tilde{\Gamma}\} = \#\{\Gamma/\tilde{\Gamma}\}d(\Gamma).$$

In a similar way, we have for the extended integral superlattice of Γ

$$\#\{(\Gamma + \mathbb{Z}^n)/\mathbb{Z}^n\} = d(\Gamma + \mathbb{Z}^n)^{-1} = \mathrm{vol}\,(I_{\Gamma+\mathbb{Z}^n})^{-1} \in \mathbb{N}$$

and

$$\#\{(\Gamma + \mathbb{Z}^n)/\mathbb{Z}^n\}(\Gamma + \mathbb{Z}^n) \subset \mathbb{Z}^n.$$

The *dual lattice* of Γ is given by

$$\Gamma^* = \{\eta \in \mathbb{R}^n : \langle \eta, \gamma \rangle \in \mathbb{Z} \text{ for } \gamma \in \Gamma\}, \tag{2.6}$$

thus if $\Gamma = P\mathbb{Z}^n$, then $\Gamma^* = (P^t)^{-1}\mathbb{Z}^n$. The determinants of dual lattices satisfy the following relation

$$d(\Gamma)d(\Gamma^*) = 1.$$

If $\Gamma \subset \Lambda$, then $\Lambda^* \subset \Gamma^*$. For rational lattices Γ and Λ the dual lattice of $\Gamma \cap \Lambda$ and $\Gamma + \Lambda$ are $\Gamma^* + \Lambda^*$ and $\Gamma^* \cap \Lambda^*$, respectively. Dual lattices are sometimes called polar or reciprocal lattices. We refer to [6] for further basic properties of lattices.

2.2. Shift invariant systems

Definition 1. Suppose that Γ is a (full-rank) lattice in $\mathbb{R}^n$, i.e., $\Gamma = P\mathbb{Z}^n$ for some $n \times n$ non-singular matrix P. A closed subspace $W \subset L^2(\mathbb{R}^n)$ is said to be *shift invariant* (SI) with respect to the lattice Γ or simply Γ-SI, if $f \in W$ implies $T_\gamma f \in W$ for all $\gamma \in \Gamma$. Given a countable family $\Phi \subset L^2(\mathbb{R}^n)$ and a lattice Γ we define the Γ-SI system $E^\Gamma(\Phi)$ and the Γ-SI subspace $S^\Gamma(\Phi)$ by

$$E^\Gamma(\Phi) = \{T_\gamma \phi : \phi \in \Phi, \gamma \in \Gamma\}, \quad S^\Gamma(\Phi) = \overline{\mathrm{span}}\, E^\Gamma(\Phi).$$

We will need the following result on oversampling of shift invariant frame sequences; in case the frame sequence is actually a frame for all of $L^2(\mathbb{R}^n)$ assertion (i) below reduces to [14, Theorem 3.3]. Our proof is more elementary than [14, Theorem 3.3] and is included to illustrate how well behaved shift invariant systems are under oversampling.

Proposition 2.1. *Let Γ, Γ' be lattices in $\mathbb{R}^n$ and $\Phi, \Psi \subset L^2(\mathbb{R}^n)$ countable sets of the same cardinality. Suppose that $\Gamma \subset \Gamma'$ and $S^\Gamma(\Phi) = S^{\Gamma'}(\Phi)$. Then the following assertions hold:*

(i) If $E^\Gamma(\Phi)$ is a frame sequence with bounds C_1, C_2, then

$$\frac{1}{\#\{\Gamma'/\Gamma\}^{1/2}} E^{\Gamma'}(\Phi)$$

is a frame sequence with bounds C_1, C_2.

(ii) *Suppose that* $S^{\Gamma}(\Phi) = S^{\Gamma}(\Psi) = S^{\Gamma'}(\Psi)$. *If* $E^{\Gamma}(\Phi)$ *and* $E^{\Gamma}(\Psi)$ *are dual frames for* $S^{\Gamma}(\Phi)$, *then*

$$\frac{1}{\#\{\Gamma'/\Gamma\}^{1/2}} E^{\Gamma'}(\Phi) \quad \text{and} \quad \frac{1}{\#\{\Gamma'/\Gamma\}^{1/2}} E^{\Gamma'}(\Psi)$$

are dual frames for $S^{\Gamma}(\Phi)$.

Proof. To prove (i) assume that there are constant $C_1, C_2 > 0$ such that

$$C_1 \|f\|^2 \le \sum_{\phi\in\Phi} \sum_{\gamma\in\Gamma} |\langle f, T_\gamma \phi\rangle|^2 \le C_2 \|f\|^2 \qquad \text{for all } f \in S^{\Gamma}(\Phi)\,.$$

Let $\{d_1, \ldots, d_q\}$ be a complete set of representatives of the quotient group Γ'/Γ. For each d_r, $r = 1, \ldots, q$, we then have

$$C_1 \|f\|^2 \le \sum_{\phi\in\Phi} \sum_{\gamma\in\Gamma} |\langle T_{-d_r} f, T_\gamma \phi\rangle|^2 \le C_2 \|f\|^2 \qquad \text{for all } f \in S^{\Gamma}(\Phi)$$

using the isometry of the translation operator, i.e., $\|T_{-d_r} f\| = \|f\|$, and the Γ'-SI of $S^{\Gamma'}(\Phi) = S^{\Gamma}(\Phi)$. Adding these q inequalities yield

$$qC_1 \|f\|^2 \le \sum_{\phi\in\Phi} \sum_{r=1}^{q} \sum_{\gamma\in\Gamma} |\langle f, T_{d_r+\gamma} \phi\rangle|^2 \le qC_2 \|f\|^2\,,$$

and thus,

$$C_1 \|f\|^2 \le \sum_{\phi\in\Phi} \sum_{\gamma\in\Gamma'} |\langle f, q^{-1/2} T_\gamma \phi\rangle|^2 \le C_2 \|f\|^2\,.$$

Since $q = \#\{\Gamma'/\Gamma\}$, assertion (i) is proved.

Let Φ and Ψ be indexed by $\mathcal{I}$, i.e., $\Phi = \{\phi_i\}_{i\in\mathcal{I}}$ and $\Psi = \{\psi_i\}_{i\in\mathcal{I}}$. By our assumption we have

$$f = \sum_{i\in\mathcal{I}} \sum_{\gamma\in\Gamma} \langle f, T_\gamma \phi_i\rangle T_\gamma \psi_i \qquad \text{for all } f \in S^{\Gamma}(\Phi) = S^{\Gamma}(\Psi)\,,$$

hence, in particular,

$$\|f\|^2 = \sum_{i\in\mathcal{I}} \sum_{\gamma\in\Gamma} \langle f, T_\gamma \phi_i\rangle \langle T_\gamma \psi_i, f\rangle\,.$$

Using the same techniques as in the proof of (i) we arrive at

$$f = \sum_{i\in\mathcal{I}} \sum_{\gamma\in\Gamma'} \langle f, q^{-1/2} T_\gamma \phi_i\rangle\, q^{-1/2} T_\gamma \psi_i \qquad \text{for all } f \in S^{\Gamma}(\Phi) = S^{\Gamma'}(\Phi)\,.$$

By (i) the sequences $q^{-1/2} E^{\Gamma'}(\Phi)$ and $q^{-1/2} E^{\Gamma'}(\Psi)$ are Bessel sequences, and (ii) is proved. □

As an immediate consequence of Proposition 2.1 we have the following useful fact for SI frame sequences spanning all of $L^2(\mathbb{R}^n)$.

Corollary 2.2. *Let Γ be a lattice. If $E^{\Gamma}(\Phi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2, then, for any superlattice Γ' of Γ, i.e., $\Gamma \subset \Gamma'$,*

$$\frac{1}{\#\{\Gamma'/\Gamma\}^{1/2}} E^{\Gamma'}(\Phi)$$

is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2.

Corollary 2.2 is [14, Theorem 3.3] stated in terms of lattices rather than in terms of lattice generating matrices. In the matrix version the condition $\Gamma \subset \Gamma'$ becomes the less transparent, but equivalent, condition $C^{-1}RC \in GL_n(\mathbb{Z})$, where $\Gamma = C\mathbb{Z}^n$ and $\Gamma' = R^{-1}C\mathbb{Z}^n$ for $R, C \in GL_n(\mathbb{R})$, i.e., $E^{\Gamma}(\Phi) = \{T_{Ck}\phi : k \in \mathbb{Z}^n, \phi \in \Phi\}$ and $E^{\Gamma'}(\Phi) = \{T_{R^{-1}Ck}\phi : k \in \mathbb{Z}^n, \phi \in \Phi\}$.

2.3. Oversampling SI systems

Following [3] we introduce the notion of oversampling a SI system by a rational lattice.

Definition 2. Let Γ, Λ be rational lattices in $\mathbb{R}^n$, i.e., lattices with generating matrices in $GL_n(\mathbb{Q})$. Suppose $\Phi \subset L^2(\mathbb{R}^n)$ is a countable set. Define $O^{\Gamma}_{\Lambda}(\Phi)$, the oversampling of $E^{\Gamma}(\Phi)$ by a rational lattice $\Lambda \subset \mathbb{Q}^n$, as

$$O^{\Gamma}_{\Lambda}(\Phi) = E^{\Gamma+\Lambda}\Big(\frac{1}{\#\{\Lambda/(\Lambda\cap\Gamma)\}^{1/2}}\Phi\Big).$$

By definition $O^{\Gamma}_{\Lambda}(\Phi)$ is always SI with respect to Λ, and if $\Lambda \subset \Gamma$, no oversampling occurs, and the oversampled system $O^{\Gamma}_{\Lambda}(\Phi) = E^{\Gamma}(\Phi)$. Moreover,

$$\begin{aligned} O^{\Gamma}_{\Lambda}(\Phi) &\equiv \left\{\frac{1}{\#\{\Lambda/(\Lambda\cap\Gamma)\}^{1/2}} T_{\omega}\phi : \phi \in \Phi, \omega \in \Gamma+\Lambda\right\} \\ &= \left\{\frac{1}{\#\{\Lambda/(\Lambda\cap\Gamma)\}^{1/2}} T_{d+\gamma}\phi : \phi \in \Phi, d \in [\Lambda/(\Lambda\cap\Gamma)], \gamma \in \Gamma\right\} \\ &\equiv \frac{1}{\#\{\Lambda/(\Lambda\cap\Gamma)\}^{1/2}} \bigcup_{d\in[\Lambda/(\Lambda\cap\Gamma)]} T_d\left(E^{\Gamma}(\Phi)\right), \end{aligned}$$

where the union runs over representatives of distinct cosets of the group $\Lambda/(\Lambda\cap\Gamma)$. Indeed, the penultimate equality is a consequence of the fact that by choosing representatives of cosets of $(\Gamma+\Lambda)/\Gamma$ in Λ, we also have representatives of $\Lambda/(\Lambda\cap\Gamma)$. Likewise, choosing the representatives of cosets of $(\Gamma+\Lambda)/\Lambda$ to be in Γ yields representatives of $\Gamma/(\Lambda\cap\Gamma)$, hence

$$O^{\Gamma}_{\Lambda}(\Phi) = \frac{1}{\#\{\Lambda/(\Lambda\cap\Gamma)\}^{1/2}} \bigcup_{d\in[\Gamma/(\Lambda\cap\Gamma)]} T_d\left(E^{\Lambda}(\Phi)\right). \tag{2.7}$$

2.4. Mixed dual Gramians

Let Λ be a lattice in $\mathbb{R}^n$, and let I_{Λ^*} denote a fundamental domain of Λ^*. Define the isometric, isomorphism $\mathcal{J}$ between $L^2(\mathbb{R}^n)$ and $L^2(I_{\Lambda^*}, \ell^2(\Lambda^*))$ by

$$\mathcal{J}f\colon I_{\Lambda^*} \to \ell^2(\Lambda^*), \quad \mathcal{J}f(\xi) = \left\{\hat{f}(\xi+\lambda)\right\}_{\lambda\in\Lambda^*} \quad \text{for } f \in L^2(\mathbb{R}^n). \tag{2.8}$$

Sequences of the form $\mathcal{J}f(\xi)$ are called fibers of $\ell^2(\Lambda^*)$ parametrized by the base space $\xi \in I_{\Lambda^*}$. Let $\{f_i\}_{i\in\mathcal{I}}$ and $\{g_i\}_{i\in\mathcal{I}}$ be countable collections of functions in $L^2(\mathbb{R}^n)$. By generalizing [1, Theorem 2.3], we have that $E^\Lambda(\{f_i\})$ is a frame (or Bessel sequence) in $L^2(\mathbb{R}^n)$ if, and only if, $\{d(\Lambda^*)^{1/2}\mathcal{J}f_i(\xi)\}_{i\in\mathcal{I}}$ is a frame (or Bessel sequence) in $\ell^2(\Lambda^*)$ for a.e. $\xi \in I_{\Lambda^*}$ with bounds being preserved. From this fact it is straightforward to verify that $E^\Lambda(\{f_i\})$ and $E^\Lambda(\{g_i\})$ are dual frames if, and only if, $\{d(\Lambda^*)^{1/2}\mathcal{J}f_i(\xi)\}_{i\in\mathcal{I}}$ and $\{d(\Lambda^*)^{1/2}\mathcal{J}g_i(\xi)\}_{i\in\mathcal{I}}$ are dual frames for a.e. $\xi \in I_{\Lambda^*}$.

Now, assume that $E^\Lambda(\{f_i\})$ and $E^\Lambda(\{g_i\})$ are Bessel sequences. For a fixed $\xi \in I_{\Lambda^*}$ set $t_i = d(\Lambda^*)^{1/2}\mathcal{J}f_i(\xi)$ and $u_i = d(\Lambda^*)^{1/2}\mathcal{J}g_i(\xi)$ for $i \in \mathcal{I}$. The synthesis operators for the fibers $\{t_i\}$ and $\{u_i\}$ are defined by

$$T\colon \ell^2(I) \to \ell^2(\Lambda^*), \quad T(\{c_i\}) = \sum_{i\in\mathcal{I}} c_i t_i,$$
$$U\colon \ell^2(I) \to \ell^2(\Lambda^*), \quad U(\{c_i\}) = \sum_{i\in\mathcal{I}} c_i u_i,$$

respectively. The analysis operators are the adjoint operators, and one finds

$$T^*(a) = \{\langle a, t_i\rangle\}_{i\in\mathcal{I}}, \qquad U^*(a) = \{\langle a, u_i\rangle\}_{i\in\mathcal{I}},$$

for $a = \{a_\lambda\}_{\lambda\in\Lambda^*} \in \ell^2(\Lambda^*)$. The fibers $\{t_i\}$ and $\{u_i\}$ being dual frames in $\ell^2(\Lambda^*)$ means in terms of the analysis and synthesis operators that

$$TU^* = I_{\ell^2(\Lambda^*)}, \qquad \text{or}, \qquad UT^* = I_{\ell^2(\Lambda^*)},$$

where $I_{\ell^2(\Lambda^*)}$ is the identity operator on $\ell^2(\Lambda^*)$. This fact is obvious.

The mixed dual Gramian $\widetilde{G} = \widetilde{G}(\xi)$ is defined as $\widetilde{G} = UT^*$. In the standard basis $\{e_k\}_{k\in\Lambda^*}$ of $\ell^2(\Lambda^*)$ the mixed dual Gramian acts by $\langle \widetilde{G}e_k, e_l\rangle = \sum_{i\in\mathcal{I}} t_i(k)\overline{u_i(l)}$, so

$$\widetilde{G} = \left(d(\Lambda^*) \sum_{i\in\mathcal{I}} \hat{f}_i(\xi+k)\overline{\hat{g}_i(\xi+l)} \right)_{k,l\in\Lambda^*}. \tag{2.9}$$

By the above, the SI systems $E^\Lambda(\{f_i\})$ and $E^\Lambda(\{g_i\})$ are dual frames if, and only if, $\widetilde{G}(\xi) = I_{\ell^2(\Lambda^*)}$ for a.e. $\xi \in I_{\Lambda^*}$.

The following result is a generalization of [3, Lemma 2.5]. Lemma 2.3 says that the mixed dual Gramian of a pair of oversampled SI systems is in one part a rescaling of the original mixed dual Gramian, whereas in the other part it has zero entries.

Lemma 2.3. *Let Γ and Λ be lattices, and let $\Psi = \{\psi_i\}_{i\in\mathcal{I}}$ and $\Phi = \{\phi_i\}_{i\in\mathcal{I}}$ be countable sets in $L^2(\mathbb{R}^n)$. Suppose $O^\Gamma_\Lambda(\Psi)$ and $O^\Gamma_\Lambda(\Phi)$ are Bessel sequences. Then the mixed dual Gramian of $O^\Gamma_\Lambda(\Psi)$ and $O^\Gamma_\Lambda(\Phi)$ is given for $k, l \in \Lambda^*$ as*

$$\widetilde{G}(\xi)_{k,l} = \begin{cases} d(\Gamma^*)\sum_{i\in\mathcal{I}} \hat{\psi}_i(\xi+k)\overline{\hat{\phi}_i(\xi+l)} & \text{if } k-l \in \Gamma^*\cap\Lambda^*, \\ 0 & \text{if } k-l \in \Lambda^*\setminus\Gamma^*. \end{cases} \tag{2.10}$$

Proof. We paraphrase the oversampled systems $O^\Gamma_\Lambda(\Psi)$ and $O^\Gamma_\Lambda(\Phi)$ using (2.7) which yields

$$O^\Gamma_\Lambda(\Psi) = E^\Lambda(\Psi'), \quad \text{where } \Psi' = \bigcup_{d\in[\Gamma/(\Lambda\cap\Gamma)]} \left\{ \frac{1}{\#\{\Lambda/(\Lambda\cap\Gamma)\}^{1/2}} T_d\Psi \right\},$$

and

$$O^{\Gamma}_{\Lambda}(\Phi) = E^{\Lambda}(\Phi'), \quad \text{where } \Phi' = \bigcup_{d\in[\Gamma/(\Lambda\cap\Gamma)]} \left\{ \frac{1}{\#\{\Lambda/(\Lambda\cap\Gamma)\}^{1/2}} T_d\Phi \right\}.$$

Hence, by (2.9),

$$\begin{aligned} d(\Lambda^*)^{-1}\widetilde{G}(\xi)_{k,l} &= \frac{1}{\#\{\Lambda/(\Lambda\cap\Gamma)\}} \sum_{i\in\mathcal{I}} \sum_{d\in[\Gamma/(\Lambda\cap\Gamma)]} \widehat{T_d\psi_i}(\xi+k)\overline{\widehat{T_d\phi_i}(\xi+l)} \\ &= \frac{1}{\#\{\Lambda/(\Lambda\cap\Gamma)\}} \left(\sum_{d\in[\Gamma/(\Lambda\cap\Gamma)]} \mathrm{e}^{-2\pi i\langle k-l,d\rangle} \right) \sum_{i\in\mathcal{I}} \hat{\psi}_i(\xi+k)\overline{\hat{\phi}_i(\xi+l)}. \end{aligned}$$

Using Lemma 3.6 and $\#\{\Gamma/(\Lambda\cap\Gamma)\}/\#\{\Lambda/(\Lambda\cap\Gamma)\} = d(\Lambda)/d(\Gamma) = d(\Gamma^*)/d(\Lambda^*)$ this yields (2.10). □

2.5. Generalized shift invariant systems

Generalized shift invariant system were introduced and studied in the work of Hernández, Labate, and Wilson [13], and independently by Ron and Shen [23].

Definition 3. For a collection of functions $\{g_p\}_{p\in\mathcal{P}}$, a *generalized shift invariant* (GSI) system is defined as

$$\bigcup_{p\in\mathcal{P}} E^{\Gamma_p}(g_p), \tag{2.11}$$

where $\{\Gamma_p\}_{p\in\mathcal{P}}$ is a countable collection of lattices in $\mathbb{R}^n$. The Γ_p-SI system $E^{\Gamma_p}(g_p)$ is said to be the pth *layer* of the GSI system.

Letting $\Phi = \{g_p\}_{p\in\mathcal{P}}$ and $\Gamma = \Gamma_p$ for each $p \in \mathcal{P}$ in (2.11) for a GSI system, we recover the SI system $E^{\Gamma}(\Phi)$. Moreover, a GSI system is SI if there exists a (sparse) lattice Γ so that $\Gamma \subset \Gamma_p$ for each $p \in \mathcal{P}$. Furthermore, if $C_p \in GL_n(\mathbb{R})$ is chosen such that $\Gamma_p = C_p\mathbb{Z}^n$ for each $p \in \mathcal{P}$, then the GSI system in (2.11) takes the form

$$\left\{ T_{C_pk}g_p : k \in \mathbb{Z}^n, p \in \mathcal{P} \right\}. \tag{2.12}$$

We will use the following results about GSI systems from [13]. Here, we state the results from [13] in terms of lattices in $\mathbb{R}^n$ rather than in terms of (2.12) and matrices $\{C_p\}$. The reason behind this convention is that a matrix C_p satisfying $\Gamma_p = C_p\mathbb{Z}^n$ is not unique and most of our conditions simplify when stated in terms of lattices rather than matrices.

Theorem 2.4 (Theorem 2.1 in [13]). *Let $\mathcal{P}$ be a countable set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{\Gamma_p\}_{p\in\mathcal{P}}$ a collection of lattices in $\mathbb{R}^n$. Assume the local integrability condition (LIC):*

$$L(f) := \sum_{p\in\mathcal{P}} \sum_{m\in\Gamma_p^*} \int_{\operatorname{supp}\hat{f}} \left|\hat{f}(\xi+m)\right|^2 d(\Gamma_p^*)\,|\hat{g}_p(\xi)|^2\,\mathrm{d}\xi < \infty \quad \text{for all } f \in \mathcal{D}. \tag{2.13}$$

Then the GSI system $\cup_{p\in\mathcal{P}}E^{\Gamma_p}(g_p)$ is a Parseval frame for $L^2(\mathbb{R}^n)$ if, and only if,

$$\sum_{p\in\mathcal{P}} d(\Gamma_p^*)\hat{g}_p(\xi)\overline{\hat{g}_p(\xi+\alpha)} = \delta_{\alpha,0} \qquad \text{for a.e. } \xi \in \mathbb{R}^n \tag{2.14}$$

for each $\alpha \in \cup_{i\in\mathcal{P}}\Gamma_p^*$.

The fact that LIC, in general, is necessary can be found in [4, Example 3.2]. Recall the relation between the determinants of dual lattices $d(\Gamma_p^*) = 1/d(\Gamma_p)$.

Proposition 2.5 (Proposition 2.4 in [13]). *Let $\mathcal{P}$ be a countable set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{\Gamma_p\}_{p\in\mathcal{P}}$ a collection of lattices in $\mathbb{R}^n$. Assume that the LIC given by (2.13) holds. Then, for each $f \in \mathcal{D}$, the function*

$$w(x) = \sum_{p\in\mathcal{P}} \sum_{k\in\Gamma_p} |\langle T_x f, T_k g_p\rangle|^2 \tag{2.15}$$

is a continuous function that coincides pointwise with the absolutely convergent series

$$w(x) = \sum_{p\in\mathcal{P}} \sum_{m\in\Gamma_p^*} \hat{w}_p(m) \mathrm{e}^{2\pi i\langle m,x\rangle}\,, \tag{2.16}$$

where

$$\hat{w}_p(m) = d(\Gamma_p^*) \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi+m)}\,\overline{\hat{g}_p(\xi)}\hat{g}_p(\xi+m)\,\mathrm{d}\xi. \tag{2.17}$$

The function w in (2.16) is an almost periodic function. In case the GSI system from Proposition 2.5 is a Γ-SI system for some lattice Γ, the function w is actually Γ-periodic, and can thus be considered as a regular Fourier series on the fundamental parallelopiped I_Γ.

Proposition 2.6 (Proposition 4.1 in [13]). *Let $\mathcal{P}$ be a countable set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(\mathbb{R}^n)$ and $\{\Gamma_p\}_{p\in\mathcal{P}}$ a collection of lattices in $\mathbb{R}^n$. If the GSI system $\cup_{p\in\mathcal{P}} E^{\Gamma_p}(g_p)$ is a Bessel sequence with bound $C_2 > 0$, then*

$$\sum_{p\in\mathcal{P}} |\hat{g}_p(\xi)|^2 / d(\Gamma_p) \le C_2 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{2.18}$$

The following result is a generalization of Proposition 5.6 in [13]. The result states that the local integrability condition for affine systems $\mathcal{A}(\psi)$ is equivalent with local integrability of a Calderón sum (2.19), hence the name of the condition.

Proposition 2.7. *Let $A \in GL_n(\mathbb{R})$ be expansive and $\psi \in L^2(\mathbb{R}^n)$. Then,*

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(B^{-j}\xi)\right|^2 \in L^1_{loc}(\mathbb{R}^n \setminus \{0\}), \tag{2.19}$$

if, and only if,

$$\begin{aligned} L(f) &= \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j m)|^2 \,|\det A^j|\, |\mathcal{F} D_{A^j}\psi(\xi)|^2 \,\mathrm{d}\xi \\ &= \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi + B^j m)|^2 \,|\hat{\psi}(B^{-j}\xi)|^2 \,\mathrm{d}\xi < \infty \quad \text{for all } f \in \mathcal{D}. \end{aligned} \tag{2.20}$$

In the proof of Proposition 2.7 we use the following elementary lattice counting lemma.

Lemma 2.8. *Let $B \in GL_n(\mathbb{R})$ be expansive and $R > 0$. Then, there exists $C > 0$ such that*

$$\#\{(\xi + B^j\mathbb{Z}^n) \cap \mathbf{B}(0,R)\} \leq C \max(1, |\det B|^{-j}) \qquad \text{for any } j \in \mathbb{Z},\ \xi \in \mathbb{R}^n. \tag{2.21}$$

Proof. Since the matrix B is expansive, there exists $J \in \mathbb{Z}$ such that

$$\mathbf{B}(0,\sqrt{n}) \subset B^{-j}(\mathbf{B}(0,R)) \qquad \text{for all } j \leq J. \tag{2.22}$$

For the same reason, once J is fixed, there exists $R_0 > 0$ such that

$$B^{-j}(\mathbf{B}(0,R)) \subset \mathbf{B}(0,R_0) \qquad \text{for all } j > J. \tag{2.23}$$

Let

$$K_j = \{k \in \mathbb{Z}^n : \xi + B^j k \in \mathbf{B}(0,R)\} = \{k \in \mathbb{Z}^n : B^{-j}\xi + k \in B^{-j}(\mathbf{B}(0,R))\}.$$

Using (2.22) and (2.23)

$$\bigcup_{k \in K_j} (B^{-j}\xi + k + [0,1]^n) \subset B^{-j}(\mathbf{B}(0,R)) + \mathbf{B}(0,\sqrt{n}) \subset \begin{cases} 2B^{-j}(\mathbf{B}(0,R)) & \text{for } j \leq J, \\ \mathbf{B}(0, R_0 + \sqrt{n}) & \text{for } j > J. \end{cases}$$

Thus,

$$\#K_j = \left| \bigcup_{k \in K_j} (B^{-j}\xi + k + [0,1]^n) \right| \leq \begin{cases} c_n (2R)^n |\det B|^{-j} & \text{for } j \leq J, \\ c_n (R_0 + \sqrt{n})^n & \text{for } j > J, \end{cases}$$

where $c_n = |\mathbf{B}(0,1)|$. This immediately implies (2.21). □

Proof of Proposition 2.7. Assume (2.19). Let $f \in \mathcal{D}$ and choose $R > 1$ such that

$$\operatorname{supp} \hat{f} \subset \left\{ \xi \in \mathbb{R}^n : \frac{1}{R} < |\xi| < R \right\}.$$

Since the matrix B is expansive, there exists a constant $K \in \mathbb{N}$ such that, each trajectory $\{B^j\xi\}_{j\in\mathbb{Z}}$ hits the above annulus at most K times. Thus,

$$\#\Big\{ j \in \mathbb{Z} : \xi \in B^{-j}(\operatorname{supp} \hat{f}) \Big\} \leq K.$$

On the other hand, by Lemma 2.8 we have that, for any $\xi \in \mathbb{R}^n$,

$$\#\{(\xi + B^j\mathbb{Z}^n) \cap \operatorname{supp} \hat{f}\} \leq C \max(1, |\det B|^{-j}).$$

Combining the last two estimates

$$\begin{aligned} L(f) &\leq \sum_{j \in \mathbb{Z}} \|\hat{f}\|_\infty^2 C \max(1, |\det B|^{-j}) \int_{\operatorname{supp} \hat{f}} \left| \hat{\psi}(B^{-j}\xi) \right|^2 \mathrm{d}\xi \\ &\leq \|\hat{f}\|_\infty^2 C \sum_{j \geq 0} \int_{\operatorname{supp} \hat{f}} \left| \hat{\psi}(B^{-j}\xi) \right|^2 \mathrm{d}\xi + \|\hat{f}\|_\infty^2 C \sum_{j<0} \int_{B^{-j}(\operatorname{supp} \hat{f})} \left| \hat{\psi}(\xi) \right|^2 \mathrm{d}\xi \\ &\leq \|\hat{f}\|_\infty^2 C \int_{\operatorname{supp} \hat{f}} \sum_{j \geq 0} \left| \hat{\psi}(B^{-j}\xi) \right|^2 \mathrm{d}\xi + \|\hat{f}\|_\infty^2 CK \int_{\mathbb{R}^n} \left| \hat{\psi}(\xi) \right|^2 \mathrm{d}\xi < \infty. \end{aligned}$$

The last inequality is a consequence of (2.19) and $\psi \in L^2(\mathbb{R}^n)$.

Conversely, if $L(f) < \infty$ for all $f \in \mathcal{D}$, then in particular by choosing $\hat{f} = \chi_E$ for a compact set $E \subset \mathbb{R}^n \setminus \{0\}$ we have

$$\int_E \sum_{j\in\mathbb{Z}} \left|\hat{\psi}(B^{-j}\xi)\right|^2 \,\mathrm{d}\xi = \sum_{j\in\mathbb{Z}} \int_E \left|\hat{\psi}(B^{-j}\xi)\right|^2 \,\mathrm{d}\xi \le L(f) < \infty.$$

Since the set E was arbitrarily chosen, the validity of (2.19) follows. □

Remark 1. One should add that (2.19) and thus (2.20) hold if, and only if, the Bessel-like condition holds on the dense subspace $\mathcal{D}$,

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi_{j,k}\rangle\right|^2 < \infty \qquad \text{for all } f \in \mathcal{D}. \tag{2.24}$$

Indeed, this fact is a consequence of [2, Lemma 3.1] which holds for real expansive dilations.

3. Oversampling affine systems into quasi-affine systems

In this section we show that the frame property is preserved when going from affine to quasi-affine systems. To characterize under what conditions we can also go from quasi-affine to affine systems, we introduce a new family of oversampled quasi-affine systems. We then show that an affine system is a frame if, and only if, the corresponding family of quasi-affine systems are frames with uniform frame bounds.

3.1. Properties of quasi-affine systems

For a rational lattice Λ we introduce the notion of a Λ-oversampled quasi-affine system.

Definition 4. Let $A \in GL_n(\mathbb{Q})$ be a rational, expansive matrix, and let Λ be rational lattice in $\mathbb{R}^n$, i.e., $\Lambda = P\mathbb{Z}^n$ with $P \in GL_n(\mathbb{Q})$. Suppose $\Psi \subset L^2(\mathbb{R}^n)$ is a finite set. Define $\mathcal{A}^q_\Lambda(\Psi)$ the Λ-*oversampled quasi-affine* system by

$$\mathcal{A}^q_\Lambda(\Psi) = \bigcup_{j\in\mathbb{Z}} O_\Lambda^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi)\,.$$

When $\Lambda = \mathbb{Z}^n$ we often drop the subscript Λ, and we say that $\mathcal{A}^q(\Psi) = \mathcal{A}^q_{\mathbb{Z}^n}(\Psi)$ is the standard quasi-affine system.

By definition $\mathcal{A}^q_\Lambda(\Psi)$ is SI with respect to Λ. Note that we need to assume that the dilation A and the lattice Λ are rational in order to guarantee lattice structure of $A^{-j}\mathbb{Z}^n + \Lambda$ for each $j \in \mathbb{Z}$. If $\Lambda = \mathbb{Z}^n$, we recover the usual quasi-affine system, i.e., $\mathcal{A}^q_\Lambda(\Psi) = \mathcal{A}^q(\Psi)$, introduced in [3].

We will use the following notation throughout this paper. The translation lattice for the affine system at scale $j \in \mathbb{Z}$ is denoted by $\Gamma_j = A^{-j}\mathbb{Z}^n$; its Λ-sublattice is $\widetilde{\Gamma}_j = A^{-j}\mathbb{Z}^n \cap \Lambda$ and its Λ-extended superlattice is $\mathsf{K}_j = A^{-j}\mathbb{Z}^n + \Lambda$. Note that K_j is the translation lattice for the Λ-oversampled quasi-affine system at scale $j \in \mathbb{Z}$. Finally, for $J \in \mathbb{N}$, let

$$\mathsf{M}_J = \bigcap_{|j|\le J} \Gamma_j \equiv \bigcap_{|j|\le J} A^j\mathbb{Z}^n,$$

and note that M_J is an integral lattice. Summarizing, we will use the following lattices together with their dual lattices:

$$\Gamma_j = A^{-j}\mathbb{Z}^n, \qquad \Gamma_j^* = B^j\mathbb{Z}^n, \tag{3.1}$$
$$\widetilde{\Gamma}_j = A^{-j}\mathbb{Z}^n \cap \Lambda, \qquad \widetilde{\Gamma}_j^* = B^j\mathbb{Z}^n + \Lambda^*, \tag{3.2}$$
$$\mathsf{K}_j = A^{-j}\mathbb{Z}^n + \Lambda, \qquad \mathsf{K}_j^* = B^j\mathbb{Z}^n \cap \Lambda^*, \tag{3.3}$$
$$\mathsf{M}_J = \bigcap_{|j|\le J} A^j\mathbb{Z}^n, \qquad \mathsf{M}_J^* = \underset{|j|\le J}{+} B^j\mathbb{Z}^n = B^{-J}\mathbb{Z}^n + \dots + B^J\mathbb{Z}^n. \tag{3.4}$$

Let $\Psi, \Phi \subset L^2(\mathbb{R}^n)$ be finite sets. For $j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R}^n)$ define the affine functionals

$$K_j(f) = \sum_{g \in E^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi)} |\langle f, g\rangle|^2, \qquad N(f,\Psi) = \sum_{j\in\mathbb{Z}} K_j(f) = \sum_{g\in\mathcal{A}(\Psi)} |\langle f, g\rangle|^2, \tag{3.5}$$

and quasi-affine functionals

$$K^q_{\Lambda,j}(f) = \sum_{g \in O_\Lambda^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi)} |\langle f, g\rangle|^2, \quad N^q_\Lambda(f,\Psi) = \sum_{j\in\mathbb{Z}} K^q_{\Lambda,j}(f) = \sum_{g\in\mathcal{A}^q_\Lambda(\Psi)} |\langle f, g\rangle|^2. \tag{3.6}$$

Whenever unambiguous, we drop the reference to the set of generators and simply write $N(f)$ and $N^q_\Lambda(f)$.

Before going deeper into our investigation we illustrate the notion of a quasi-affine system in a few specific situations.

Example 1. Let $J \in \mathbb{N}$ and consider the quasi-affine system obtained by oversampling with respect to $\mathsf{M}_J = \cap_{|j|\le J} A^j\mathbb{Z}^n$ introduced above. Since $A^{-j}\mathbb{Z}^n + \mathsf{M}_J = A^{-j}\mathbb{Z}^n$ and $A^{-j}\mathbb{Z}^n \cap \mathsf{M}_J = \mathsf{M}_J$ for $|j| \le J$, we see that

$$O_{\mathsf{M}_J}^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi) = E^{A^{-j}\mathbb{Z}^n + \mathsf{M}_J}\Big(\#\{\mathsf{M}_J/\mathsf{M}_J\}^{-1/2} D_{A^j}\Psi\Big) = E^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi), \tag{3.7}$$

for $|j| \le J$. Hence with this oversampling lattice, the scales $|j| \le J$ for the affine system

$$\mathcal{A}(\Psi) = \bigcup_{j\in\mathbb{Z}} E^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi)$$

and the M_J-oversampled quasi-affine system

$$\mathcal{A}^q_{\mathsf{M}_J}(\Psi) = \bigcup_{j\in\mathbb{Z}} O_{\mathsf{M}_J}^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi)$$

coincide.

Example 2. Suppose $A \in GL_n(\mathbb{Z})$ is integer valued. Let $\Lambda = A^l\mathbb{Z}^n$ for some $l \in \mathbb{Z}$. Then the Λ-oversampled quasi-affine system is just a dilated version of standard quasi-affine system (1.2). To be precise, we have the following relation:

$$\mathcal{A}^q_{A^l\mathbb{Z}^n}(\Psi) = D_{A^{-l}}(\mathcal{A}^q(\Psi)). \tag{3.8}$$

To see this note that

$$A^{-j}\mathbb{Z}^n + \Lambda = A^{-j}\mathbb{Z}^n + A^l\mathbb{Z}^n = \begin{cases} A^l\mathbb{Z}^n, & j < -l, \\ A^{-j}\mathbb{Z}^n, & j \geq -l, \end{cases}$$

and that

$$\#\{A^l\mathbb{Z}^n/(A^l\mathbb{Z}^n \cap A^{-j}\mathbb{Z}^n)\} = \begin{cases} \#\{A^l\mathbb{Z}^n/A^{-j}\mathbb{Z}^n\} = \frac{d(\mathbb{Z}^n)}{d(A^{j+l}\mathbb{Z}^n)} = \frac{1}{|\det A^{j+l}|}, & j < -l, \\ \#\{A^l\mathbb{Z}^n/A^l\mathbb{Z}^n\} = 1, & j \geq -l, \end{cases}$$

whereby we have

$$\mathcal{A}^q_{A^l\mathbb{Z}^n}(\Psi) = \bigcup_{j\geq -l} E^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi) \cup \bigcup_{j<-l} E^{A^l\mathbb{Z}^n}\Big(|\det A|^{(j+l)/2} D_{A^j}\Psi\Big).$$

Recall that

$$\mathcal{A}^q(\Psi) = \bigcup_{j\geq 0} E^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi) \cup \bigcup_{j<0} E^{\mathbb{Z}^n}\Big(|\det A|^{j/2} D_{A^j}\Psi\Big),$$

and the validity of (3.8) follows by $D_{A^{-l}}T_k = T_{A^l k}D_{A^{-l}}$ and a change of variables.

Example 3. The quasi-affine system has a relatively simple algebraic form in one dimension. Suppose $a = p/q \in \mathbb{Q}$ is a dilation factor, where $|a| > 1$, $p, q \in \mathbb{Z}$ are relatively prime. Let $\Lambda \subset \mathbb{Z}$ be a lattice. For simplicity, we assume that $\Lambda = p^{J_1}q^{J_2}r\mathbb{Z}$ for some $J_1, J_2 \in \mathbb{N}_0, r \in \mathbb{N}$, where pq and r are relatively prime. Then, the quasi-affine system $\mathcal{A}^q_\Lambda(\Psi)$ associated with a is given by

$$\mathcal{A}^q_\Lambda(\Psi) = \{\tilde{\psi}_{j,k} : j, k \in \mathbb{Z},\ \psi \in \Psi\}.$$

Here, for $\psi \in L^2(\mathbb{R})$ and $j, k \in \mathbb{Z}$, we set

$$\tilde{\psi}_{j,k}(x) = \begin{cases} |a|^{j/2}\,|q|^{(J_2-j)/2}\,\psi(a^j x - q^{J_2-j}k) & \text{if } j > J_2, \\ |a|^{j/2}\,\psi(a^j x - k) & \text{if } -J_1 \leq j \leq J_2, \\ |a|^{j/2}\,|p|^{(j+J_1)/2}\,\psi(a^j x - p^{j+J_1}k) & \text{if } j < -J_1. \end{cases} \tag{3.9}$$

Note that the above convention for $\tilde{\psi}_{j,k}$ in the case when $\Lambda = \mathbb{Z}$ becomes the rationally dilated quasi-affine system (1.3) introduced by the first author in [3]. In particular, if the dilation factor a is an integer, this is the original quasi-affine system of Ron and Shen [20]. To show (3.9) note that

$$a^{-j}\mathbb{Z} + \Lambda = a^{-j}\mathbb{Z} + p^{J_1}q^{J_2}r\mathbb{Z} = \begin{cases} p^{-j}(q^j\mathbb{Z} + p^{J_1+j}q^{J_2}r\mathbb{Z}) = p^{-j}q^{\min(j,J_2)}\mathbb{Z} & \text{for } j \geq 0 \\ q^j(p^{-j}\mathbb{Z} + p^{J_1}q^{J_2-j}r\mathbb{Z}) = q^j p^{\min(-j,J_1)}\mathbb{Z} & \text{for } j < 0 \end{cases}$$

$$= \begin{cases} p^{-j}q^{J_2}\mathbb{Z} & \text{for } j > J_2, \\ a^{-j}\mathbb{Z} & \text{for } -J_1 \leq j \leq J_2, \\ p^{J_1}q^j\mathbb{Z} & \text{for } j < -J_1. \end{cases}$$

Hence, one needs to oversample at a rate $|q|^{j-J_2}$ if $j > J_2$ (or $|p|^{-J_1-j}$ if $j < -J_1$) to obtain the quasi-affine system $\mathcal{A}^q_\Lambda(\Psi)$ from the affine system $\mathcal{A}(\Psi)$. Note that in the intermediate range $-J_1 \le j \le J_2$, no oversampling is required and both systems coincide at these scales. Also note that the choice $J_1 = J_2$ corresponds to oversampling by M_{J_1}, see Example 1.

Remark 2. Let Λ be a rational lattice, and consider the Λ-oversampled quasi-affine system $\mathcal{A}^q_\Lambda(\Psi)$. By definition this system is Λ-SI. Take a rational superlattice Λ' of Λ, i.e., $\Lambda \subset \Lambda'$. Then the further oversampled system $\mathcal{A}^q_{\Lambda'}(\Psi)$ is obviously Λ'-SI; moreover, it can be written in term of $\mathcal{A}^q_\Lambda(\Psi)$ as

$$\mathcal{A}^q_{\Lambda'}(\Psi) = \frac{1}{\#\{\Lambda'/\Lambda\}^{1/2}} \bigcup_{d\in[\Lambda'/\Lambda]} T_d\left(\mathcal{A}^q_\Lambda(\Psi)\right).$$

By Corollary 2.2 we have the following useful result for oversampled quasi-affine frames:

Lemma 3.1. *Let $A \in GL_n(\mathbb{Q})$. Suppose $\Lambda \subset \Lambda'$ for rational lattices Λ, Λ'. Then if $\mathcal{A}^q_\Lambda(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2, then $\mathcal{A}^q_{\Lambda'}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2.*

3.2. Affine and quasi-affine systems as GSI systems

Since affine and quasi-affine systems are GSI systems, the results from Section 2.5 can be applied to these systems, see [13, 14]. We restate some of these results in terms of lattices in $\mathbb{R}^n$. The quasi-affine system $\mathcal{A}^q_\Lambda(\Psi)$ introduced above can be expressed as a GSI system (2.11) by taking $\mathcal{P} = \{(j,l) : j \in \mathbb{Z}, l = 1, \dots, L\}$ and

$$\Gamma_p = \Gamma_{(j,l)} = A^{-j}\mathbb{Z}^n + \Lambda \tag{3.10}$$

$$g_p(x) = g_{(j,l)}(x) = \#\{\Lambda/(\Lambda \cap A^{-j}\mathbb{Z}^n)\}^{-1/2} D_{A^j}\psi_l(x) \tag{3.11}$$

for all $p \in \mathcal{P}$.

By applying Proposition 2.6 to affine and quasi-affine systems we immediately have the following result, see also [3, Proposition 4.5].

Proposition 3.2. *Suppose that $\Psi \subset L^2(\mathbb{R}^n)$ and that either of the following holds:*

(a) $A \in GL_n(\mathbb{R})$ is expansive and $\mathcal{A}(\Psi)$ is a Bessel sequence with bound C_2,

(b) $A \in GL_n(\mathbb{Q})$ is expansive and $\mathcal{A}^q_\Lambda(\Psi)$ is a Bessel sequence with bound C_2 for some rational lattice Λ.

Then,

$$\sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^j\xi)|^2 \le C_2 \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{3.12}$$

For the Λ-oversampled quasi-affine systems we have the following result on the quasi-affine functional w^q_Λ defined below.

Proposition 3.3. *Let $A \in GL_n(\mathbb{Q})$ be expansive, $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$, and let Λ be a rational lattice. Suppose that each $\psi \in \Psi$ satisfies condition (2.19). Then, for each $f \in \mathcal{D}$, the Λ-periodic function*

$$w^q_\Lambda(x) = \sum_{g \in \mathcal{A}^q_\Lambda(\Psi)} |\langle T_x f, g\rangle|^2 = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k \in \mathsf{K}_j} \left|\langle T_x f, d_j T_k D_{A^j} \psi_l\rangle\right|^2, \tag{3.13}$$

where $d_j = \#\{\Lambda/(\Lambda \cap A^{-j}\mathbb{Z}^n)\}^{-1/2}$ and K_j is given by (3.3), is a continuous function that coincides pointwise with the (Λ-periodic) absolutely convergent series

$$w^q_\Lambda(x) = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mu \in \mathsf{K}^*_j} b_{j,l}(\mu) \mathrm{e}^{2\pi i \langle \mu, x\rangle}, \tag{3.14}$$

where

$$b_{j,l}(\mu) = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi+\mu)}\,\overline{\hat{\psi}_l(B^{-j}\xi)}\hat{\psi}_l(B^{-j}(\xi+\mu))\,\mathrm{d}\xi. \tag{3.15}$$

Proof. The result follows by an application of Proposition 2.5 to quasi-affine systems. In order to apply Proposition 2.5 we need to verify the LIC condition (2.13) for quasi-affine systems, i.e., that

$$L^q_\Lambda(f) := \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{\mu \in \mathsf{K}^*_j} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi+\mu)|^2 \, |\hat{\psi}_l(B^{-j}\xi)|^2 \,\mathrm{d}\xi < \infty \tag{3.16}$$

holds for $f \in \mathcal{D}$. Since each $\psi \in \Psi$ satisfies condition (2.19), Proposition 2.7 tells us that the LIC condition for affine systems is satisfied, i.e., that $L(f) < \infty$. Finally, the estimate in (3.16) follows by

$$L^q_\Lambda(f) \le \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \int_{\operatorname{supp}\hat{f}} |\hat{f}(\xi+B^j m)|^2 \, |\hat{\psi}_l(B^{-j}\xi)|^2 \,\mathrm{d}\xi \equiv L(f) < \infty,$$

where we have used that $\mathsf{K}^*_j \subset B^j\mathbb{Z}^n$ for all $j \in \mathbb{Z}$. Consequently, the expression in (3.15) follows directly from (2.17) by

$$1/d(\mathsf{K}^*_j) = d(\mathsf{K}_j) = \frac{\left|\det A^{-j}\right|}{\#\{\Lambda/(\Lambda \cap A^{-j}\mathbb{Z}^n)\}}.$$

□

Proposition 3.4 below states a similar result for affine systems. The result is a generalization of [14, Proposition 2.8], where the Bessel condition on $\mathcal{A}(\Psi)$ is relaxed by (2.19). Proposition 3.4 is a direct consequence of Propositions 2.5 and 2.7.

Proposition 3.4. *Let $A \in GL_n(\mathbb{R})$ be expansive and $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$. Suppose that each $\psi \in \Psi$ satisfies condition (2.19). Then, for each $f \in \mathcal{D}$, the function*

$$w(x) = \sum_{g \in \mathcal{A}(\Psi)} |\langle T_x f, g\rangle|^2 = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \left|\langle T_x f, D_{A^j} T_\gamma \psi_l\rangle\right|^2, \tag{3.17}$$

is an almost periodic function that coincides pointwise with the absolutely convergent series

$$w(x) = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} c_{j,l}(m) \mathrm{e}^{2\pi i\langle B^j m, x\rangle}, \tag{3.18}$$

where

$$c_{j,l}(m) = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi + B^j m)}\, \overline{\hat{\psi}_l(B^{-j}\xi)}\hat{\psi}_l(B^{-j}(\xi + B^j m))\, \mathrm{d}\xi. \tag{3.19}$$

Remark 3. As noted in [14] the sum over $j \in \mathbb{Z}$ in Proposition 3.4 can be replaced by a sum over a smaller set $j \in \mathcal{J} \subset \mathbb{Z}$. The same holds for Proposition 3.3.

The series representing w and w_Λ^q are very similar. By a change of variables, (3.14) becomes

$$w_\Lambda^q(x) = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n \cap B^{-j}\Lambda^*} c_{j,l}(m) \mathrm{e}^{2\pi i\langle B^j m, x\rangle}\,, \tag{3.20}$$

where the coefficients $c_{j,l}(m)$ are given by (3.19). Since $\mathbb{Z}^n \cap B^{-j}\Lambda^* \subset \mathbb{Z}^n$ for all $j \in \mathbb{Z}$, we can consider the series for w_Λ^q in (3.20) as the series representing w in (3.18) with some coefficients set to zero; exactly those coefficients $c_{j,l}(m)$ for which $m \in \mathbb{Z}^n \setminus B^{-j}\Lambda^*$. We stress that this connection holds without any assumptions on the rational lattice Λ, e.g., there is no assumption on Λ being integer valued.

3.3. From affine to quasi-affine systems

The frame property carries over when moving from affine to Λ-oversampled quasi-affine systems for any rational lattice Λ. This statement is the main result of this section and is contained in Theorem 3.5.

Theorem 3.5. *Let $A \in GL_n(\mathbb{Q})$ be expansive, $\Psi \subset L^2(\mathbb{R}^n)$, and let Λ be any rational lattice in $\mathbb{R}^n$. If the affine system $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds C_1, C_2, then the Λ-oversampled quasi-affine system $\mathcal{A}_\Lambda^q(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds C_1, C_2.*

The following lemma, which is needed in the proof of Theorem 3.5, is a consequence of [15, Lemma 23.19].

Lemma 3.6. *Suppose* K, M *are lattices in* $\mathbb{R}^n$ *such that* $\mathsf{K} \subset \mathsf{M}$. *Then, for* $m \in \mathsf{K}^*$,

$$\frac{1}{\#\{\mathsf{M}/\mathsf{K}\}} \sum_{d\in[\mathsf{M}/\mathsf{K}]} e^{2\pi i\langle m,d\rangle} = \begin{cases} 1 & m \in \mathsf{M}^*, \\ 0 & m \in \mathsf{K}^* \setminus \mathsf{M}^*. \end{cases} \tag{3.21}$$

The proof of Theorem 3.5 relies on the following key result on translational averaging of affine functionals.

Lemma 3.7. *Let $A \in GL_n(\mathbb{Q})$ be expansive, $\Psi \subset L^2(\mathbb{R}^n)$, and let Λ be an* integral *lattice in $\mathbb{R}^n$. For each $J \in \mathbb{N}$ define*

$$\mathsf{M}_J = \bigcap_{|j|\le J} A^j \mathbb{Z}^n.$$

Suppose the affine system $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$. Then

$$N^q_{\Lambda}(f) = \lim_{J\to\infty} \frac{1}{\#\{(\mathsf{M}_J+\Lambda)/\mathsf{M}_J\}} \sum_{d\in[(\mathsf{M}_J+\Lambda)/\mathsf{M}_J]} N(T_d f) \qquad \text{for } f \in \mathcal{D}, \tag{3.22}$$

where $\mathcal{D}$ is given by (1.6), *N by (3.5) and N^q_{Λ} by (3.6).*

Proof. Let $\Psi = \{\psi_1, \dots, \psi_L\}$. For $f \in \mathcal{D}$, by (3.20),

$$N^q_{\Lambda}(f) = w^q_{\Lambda}(0) = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n \cap B^{-j}\Lambda^*} c_{j,l}(m), \tag{3.23}$$

where $c_{j,l}(m)$ are given in equation (3.19). So fix $J \in \mathbb{N}$ and let $\{d_1, \dots, d_{s(J)}\}$ be a complete set of representative of the quotient group $(\mathsf{M}_J + \Lambda)/\mathsf{M}_J$ so that $s(J)$ is the order of the group. We want to express $N^q_{\Lambda}(f)$ as an average of $N(T_{d_r} f)$ over $r = 1, \dots, s(J)$, thus we consider

$$\begin{aligned} \frac{1}{s(J)} \sum_{r=1}^{s(J)} N(T_{d_r} f) &= \frac{1}{s(J)} \sum_{r=1}^{s(J)} \sum_{l=1}^{L} \sum_{|j|\le J} \sum_{m\in\mathbb{Z}^n} c_{j,l}(m)\, \mathrm{e}^{2\pi i \left\langle B^j m, d_r \right\rangle} \\ &\quad + \frac{1}{s(J)} \sum_{r=1}^{s(J)} \sum_{l=1}^{L} \sum_{|j|> J} \sum_{m\in\mathbb{Z}^n} c_{j,l}(m)\, \mathrm{e}^{2\pi i \left\langle B^j m, d_r \right\rangle} \\ &=: I_1(J) + I_2(J), \end{aligned} \tag{3.24}$$

which follows by (3.18). By absolute convergence of the sum above, we conclude that $I_2(J) \to 0$ as $J \to \infty$. Assume that the following identity holds.

$$I_1(J) = \sum_{l=1}^{L} \sum_{|j|\le J} \sum_{m\in\mathbb{Z}^n\cap B^{-j}\Lambda^*} c_{j,l}(m). \tag{3.25}$$

Taking the limit $J \to \infty$ in (3.24) and using equation (3.23) yield

$$\begin{aligned} \lim_{J\to\infty} \frac{1}{s(J)} \sum_{r=1}^{s(J)} N(T_{d_r} f) = \lim_{J\to\infty} \left(I_1(J) + I_2(J)\right) &= \lim_{J\to\infty} \sum_{l=1}^{L} \sum_{|j|\le J} \sum_{m\in\mathbb{Z}^n\cap B^{-j}\Lambda^*} c_{j,l}(m) \\ &= N^q_{\Lambda}(f). \end{aligned}$$

Hence, to complete the proof we have only left to show (3.25). Taking $\mathsf{K} = \mathsf{M}_J$ and $\mathsf{M} = \mathsf{M}_J + \Lambda$ in Lemma 3.6 gives us for all $\tilde{m} \in \mathsf{M}_J^*$:

$$\sum_{r=1}^{s(J)} e^{2\pi i \langle \tilde{m}, d_r\rangle} = \begin{cases} s(J) & \tilde{m} \in \mathsf{M}_J^* \cap \Lambda^*, \\ 0 & \tilde{m} \in \mathsf{M}_J^* \setminus \Lambda^*. \end{cases} \tag{3.26}$$

Fix $j \in \mathbb{Z}$ with $|j| \le J$. Take $\tilde{m} = B^j m$. Obviously, $\tilde{m} \in \mathsf{M}_J^* \cap \Lambda^*$ precisely when $m \in B^{-j}\mathsf{M}_J^* \cap B^{-j}\Lambda^*$, and $\tilde{m} \in \mathsf{M}_J^* \setminus \Lambda^*$ precisely when $m \in B^{-j}\mathsf{M}_J^* \setminus B^{-j}\Lambda^*$. Since

$$B^{-j}\mathsf{M}_J^* = \underset{-J-j\le l\le J-j}{+} B^l \mathbb{Z}^n \supset \mathbb{Z}^n,$$

we conclude from equation (3.26) that, for all $m \in \mathbb{Z}^n$,

$$\sum_{r=1}^{s(J)} e^{2\pi i \langle B^j m, d_r\rangle} = \begin{cases} s(J) & m \in \mathbb{Z}^n \cap B^{-j}\Lambda^*, \\ 0 & m \in \mathbb{Z}^n \setminus B^{-j}\Lambda^*, \end{cases} \tag{3.27}$$

and this holds for all $|j| \le J$. Using these relations we arrive at:

$$\begin{aligned} I_1(J) \equiv & \sum_{l=1}^{L} \sum_{|j|\le J} \sum_{m\in\mathbb{Z}^n} c_{j,l}(m) \frac{1}{s(J)} \sum_{r=1}^{s(J)} \mathrm{e}^{2\pi i \langle B^j m, d_r\rangle} \\ = & \sum_{l=1}^{L} \sum_{|j|\le J} \sum_{m\in\mathbb{Z}^n\cap B^{-j}\Lambda^*} c_{j,l}(m), \end{aligned}$$

which completes the proof of the lemma. □

Proof of Theorem 3.5. Assume that the affine system $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2. It suffices to prove that $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a frame for integer lattices Λ_0, i.e., $\Lambda_0 \subset \mathbb{Z}^n$, which follows from the fact that any rational lattice Λ has an integral sublattice of the form $c\mathbb{Z}^n$ for some $c \in \mathbb{N}$, e.g., take $c = d(\Lambda \cap \mathbb{Z}^n)$, see equation (2.5). Hence, if we prove that $\mathcal{A}^q_{c\mathbb{Z}^n}(\Psi)$ is a frame with bounds C_1, C_2, then, by applying Lemma 3.1, $\mathcal{A}^q_{\Lambda}(\Psi)$ is a frame with the frame bounds being preserved.

So let Λ_0 be an integral lattice. By our hypothesis there are constants $C_1, C_2 > 0$ so that

$$C_1 \|f\|^2 \le N(f) \le C_2 \|f\|^2 \qquad \forall f \in L^2(\mathbb{R}^n).$$

Fix $J \in \mathbb{N}$ and consider M_J introduced above. For each representative $d \in [\mathsf{M}_J + \Lambda_0)/\mathsf{M}_J]$ we have

$$C_1 \|f\|^2 \le N(T_d f) \le C_2 \|f\|^2 \qquad \forall f \in L^2(\mathbb{R}^n),$$

where we have used that $\|T_x f\| = \|f\|$ for $x \in \mathbb{R}^n$. Adding these equations for each representative d yields:

$$\#\{(\mathsf{M}_J + \Lambda_0)/\mathsf{M}_J\} C_1 \|f\|^2 \le \sum_{d\in[(\mathsf{M}_J+\Lambda_0)/\mathsf{M}_J]} N(T_d f) \le \#\{(\mathsf{M}_J + \Lambda_0)/\mathsf{M}_J\} C_2 \|f\|^2 .$$

By taking the limit $J \to \infty$, we have

$$C_1 \|f\|^2 \le \lim_{J\to\infty} \frac{1}{\#\{(\mathsf{M}_J + \Lambda_0)/\mathsf{M}_J\}} \sum_{d\in[(\mathsf{M}_J+\Lambda_0)/\mathsf{M}_J]} N(T_d f) \le C_2 \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$. Since Λ_0 is an integer lattice, we can apply Lemma 3.7. This gives us

$$C_1 \|f\|^2 \le N^q_\Lambda(f) \le C_2 \|f\|^2$$

for $f \in \mathcal{D}$. Extending these inequalities to all of $L^2(\mathbb{R}^n)$ by a standard density argument completes the proof. □

Remark 4. The special case of Theorem 3.5 in one dimension with $\Lambda = \mathbb{Z}$ was first shown in [14, Theorem 2.18]. In fact, [14, Theorem 2.18] is stated for quasi-affine systems obtained by oversampling with respect to the lattice $\Lambda = s^{-1}\mathbb{Z}$, where s is relatively prime to p and q, and $a = p/q$ is a dilation factor. In this case the quasi-affine system $\mathcal{A}^q_\Lambda(\Psi)$ takes a nice algebraic form:

$$\mathcal{A}^q_{s^{-1}\mathbb{Z}}(\Psi) = \left\{ \begin{array}{ll} |p|^{j/2}\,|q|^{-j}\,|s|^{-1/2}\,\psi(a^j x - s^{-1}q^{-j}k) : & j \geq 0,\ k \in \mathbb{Z} \\ |p|^{j}\,|q|^{-j/2}\,|s|^{-1/2}\,\psi(a^j x - s^{-1}p^{j}k) : & j < 0,\ k \in \mathbb{Z} \end{array} \right\}.$$

Hence, the above system is obtained by further oversampling of the standard quasi-affine system $\mathcal{A}^q_{\mathbb{Z}}(\Psi)$ given by (1.3). However, our Theorem 3.5 holds for oversampling with respect to any rational lattice Λ, such as in (1.4) or in Example 3. The sparser the lattice Λ is, the better result we have due to Lemma 3.1 on oversampling of quasi-affine systems.

3.4. From quasi-affine to affine systems

When moving from quasi-affine to affine systems the frame property only carries over if we impose stronger conditions on the set of generators. Hence, we have only the following partial converse of Theorem 3.5.

Theorem 3.8. *Let $A \in GL_n(\mathbb{Q})$ be expansive and $\Psi \subset L^2(\mathbb{R}^n)$. If M_J-oversampled quasi-affine system $\mathcal{A}^q_{\mathsf{M}_J}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with uniform frame bounds C_1, C_2 for all $J \in \mathbb{N}$, where M_J is given by* (3.4)*, then the affine system $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds C_1, C_2.*

Proof. Assume that

$$C_1 \|f\|^2 \leq N^q_{\mathsf{M}_J}(f) \leq C_2 \|f\|^2 \qquad \text{for all } f \in \mathcal{D}$$

holds for all $J \in \mathbb{N}$. Since scale j of the affine system and the M_J-oversampled quasi-affine system agrees whenever $|j| \leq J$, we have by (3.7),

$$K_j(f) = K^q_{\mathsf{M}_J,j}(f) \qquad \text{for all } |j| \leq J,\ f \in L^2(\mathbb{R}^n).$$

Thus, for $J \in \mathbb{N}$,

$$\sum_{|j|\leq J} K_j(f) = \sum_{|j|\leq J} K^q_{\mathsf{M}_J,j}(f) \leq C_2 \|f\|^2.$$

Letting $J \to \infty$ yields

$$N(f) = \lim_{J\to\infty} \sum_{|j|\leq J} K_j(f) \leq \limsup_{J\to\infty} \sum_{|j|\leq J} K^q_{\mathsf{M}_J,j}(f) \leq C_2 \|f\|^2,$$

whereby we conclude that $\mathcal{A}(\Psi)$ is a Bessel sequence with bound C_2. Likewise for the lower bound:

$$C_1 \|f\|^2 \leq \sum_{|j|\leq J} K^q_{\mathsf{M}_J,j}(f) + \sum_{|j|>J} K^q_{\mathsf{M}_J,j}(f) = \sum_{|j|\leq J} K_j(f) + \sum_{|j|>J} K^q_{\mathsf{M}_J,j}(f). \tag{3.28}$$

Suppose that

$$\lim_{J\to\infty} \sum_{|j|>J} K^q_{\mathsf{M}_J,j}(f) = 0 \qquad \text{for } f \in \mathcal{D}. \tag{3.29}$$

Then, by equation (3.28),

$$C_1 \|f\|^2 \le \lim_{J\to\infty} \sum_{|j|\le J} K_j(f) = N(f) \qquad \text{for } f \in \mathcal{D}.$$

Since $\mathcal{A}(\Psi)$ satisfies the upper bound, we can extend this inequality to all of $L^2(\mathbb{R}^n)$ by a density argument, hence the affine system $\mathcal{A}(\Psi)$ satisfies the lower bound with constant C_1.

To complete the proof we need to verify (3.29). We have already showed that $\mathcal{A}(\Psi)$ is a Bessel sequence, so by Proposition 3.4 the series in (3.18) converges absolutely and

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} |c_{j,l}(m)| < \infty,$$

where $c_{j,l}(m)$ is given by (3.19). Therefore, by (3.20) and Remark 3,

$$\begin{aligned}\sum_{|j|>J} K^q_{\mathsf{M}_J,j}(f) &\equiv \sum_{|j|>J} \sum_{l=1}^{L} \sum_{k\in\mathsf{K}_j} |\langle f, d_j T_k D_{A^j}\psi_l\rangle|^2 \le \sum_{|j|>J} \sum_{l=1}^{L} \sum_{m\in\mathbb{Z}^n\cap B^{-j}\mathsf{M}_J^*} |c_{j,l}(m)| \\ &\le \sum_{|j|>J} \sum_{l=1}^{L} \sum_{m\in\mathbb{Z}^n} |c_{j,l}(m)| \to 0 \quad \text{as } J \to \infty.\end{aligned}$$

This shows (3.29) and completes the proof of Theorem 3.8. ☐

The following result combines Theorems 3.5 and 3.8 in a more conceptually transparent and less technical form.

Theorem 3.9. *Let $A \in GL_n(\mathbb{Q})$ be expansive and $\Psi \subset L^2(\mathbb{R}^n)$. Then, the affine system $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with frame bounds C_1, C_2 if, and only if, the Λ-oversampled quasi-affine system $\mathcal{A}^q_\Lambda(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with uniform frame bounds C_1, C_2 for all integer lattices Λ.*

3.5. Recovering known equivalence results

We end this section by illustrating the general nature of Theorems 3.5 and 3.8. In particular, we will show that the well known equivalence result of Ron and Shen [20] for affine and quasi-affine frames for integer dilation $A \in GL_n(\mathbb{Z})$ is a simple consequence of these results. Moreover, we have the following generalization of their result.

Proposition 3.10. *Let $A \in GL_n(\mathbb{Z})$ be expansive and $\Psi \subset L^2(\mathbb{R})$. Then the following assertions are equivalent:*

(i) $\mathcal{A}(\Psi)$ is a frame with bounds C_1, C_2,

*(ii) $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a frame with bounds C_1, C_2 for **some** oversampling lattice $\Lambda_0 \subset \mathbb{Z}^n$,*

*(iii) $\mathcal{A}^q_\Lambda(\Psi)$ is a frame with bounds C_1, C_2 for **all** oversampling lattices $\Lambda \subset \mathbb{Z}^n$.*

Proof. By Theorem 3.5, we are only left to prove (ii) $\Rightarrow$ (i), but this will follow from an application of Theorem 3.8. From Lemma 3.1 we have that $\mathcal{A}^q(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2. Recall the identity

$$\mathcal{A}^q_{A^l\mathbb{Z}^n}(\Psi) = D_{A^{-l}}(\mathcal{A}^q(\Psi)) \qquad \text{for } l \in \mathbb{Z}$$

from Example 2. This tells us, by unitarity of the dilation operator, that $\mathcal{A}^q_{A^l\mathbb{Z}^n}(\Psi)$ is a frame with (uniform) bounds C_1, C_2 for each $l \in \mathbb{Z}$. Since A has integer entries, we have

$$\mathsf{M}_J \equiv \bigcap_{|j|\leq J} A^j\mathbb{Z}^n = A^J\mathbb{Z}^n \qquad \text{for } J \in \mathbb{N},$$

and the conclusion follows from Theorem 3.8. □

4. Dual affine and quasi-affine frames

The goal of this section is to prove the equivalence between pairs of dual affine and quasi-affine frames in the setting of rational dilations. To achieve this we will use well-studied fundamental equations of affine systems.

Definition 5. Suppose that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$ and $\Phi = \{\phi_1, \dots, \phi_L\} \subset L^2(\mathbb{R}^n)$ are such that

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}(|\hat{\psi}_l(B^{-j}\xi)|^2 + |\hat{\phi}_l(B^{-j}\xi)|^2) < \infty \qquad \text{for a.e. } \xi. \tag{4.1}$$

We say that a pair (Ψ, Φ) satisfies the *fundamental equations* if

$$\tilde{t}_0(\xi) := \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\hat{\psi}_l(B^{-j}\xi)\overline{\hat{\phi}_l(B^{-j}\xi)} = 1 \quad \text{for a.e. } \xi, \tag{4.2}$$

$$\tilde{t}_\alpha(\xi) := \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n}\hat{\psi}_l(B^{-j}\xi)\overline{\hat{\phi}_l(B^{-j}(\xi+\alpha))} = 0 \quad \text{for a.e. } \xi \text{ and all } \alpha \in \mathbb{Z}^n \setminus \{0\}. \tag{4.3}$$

Remark 5. Note that the assumption (4.1) is made to guarantee that the series in (4.2) converges absolutely, and hence the Calderón condition (4.2) is meaningful. On the other hand, the series (4.3) converges absolutely for a.e. ξ without any assumptions (apart from $\Psi, \Phi \subset L^2(\mathbb{R}^n)$, that is). Indeed, for any $\psi \in L^2(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n}\sum_{j\leq J}\left|\hat{\psi}(B^{-j}(\xi+\alpha))\right|^2 \mathrm{d}\xi = \int_{\mathbb{R}^n}\sum_{j\leq J}|\det A|^j\,|\hat{\psi}(\xi)|^2\mathrm{d}\xi = \frac{|\det A|^{J+1}}{|\det A|-1}\,\|\psi\|^2 < \infty \tag{4.4}$$

for any $J \in \mathbb{N}$. Since the dilation B is expansive, for any $\alpha \neq 0$, there exists $J \in \mathbb{N}$ such that $j \in \mathbb{Z}$ and $\alpha \in B^j\mathbb{Z}^n$ implies that $j \leq J$. Hence, by $2\,|zw| \leq |z|^2 + |w|^2$ for

$z, w \in \mathbb{C}$,

$$\sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} |\hat{\psi}_l(B^{-j}\xi)\overline{\hat{\phi}_l(B^{-j}(\xi+\alpha))}| \le \tfrac{1}{2} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} |\hat{\psi}_l(B^{-j}\xi)|^2 + \tfrac{1}{2} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} |\hat{\phi}_l(B^{-j}(\xi+\alpha))|^2 < \infty \qquad \text{for a.e. } \xi \in \mathbb{R}^n.$$

The last inequality is a consequence of (4.4).

We will need the following result which was originally proved by Frazier, Garrigós, Wang, and Weiss [11] in the dyadic setting. Later it was extended by the first author [2] to the setting of integer, expansive dilations and by Chui, Czaja, Maggioni, and Weiss [8] to the setting of real, expansive dilations. We include an alternative proof of Theorem 4.1 for the sake of completeness and since its techniques will be used later.

Theorem 4.1. *Let $A \in GL_n(\mathbb{R})$ be expansive. Suppose that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$ and $\Phi = \{\phi_1, \dots, \phi_L\} \subset L^2(\mathbb{R}^n)$ are such that*

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}(|\hat{\psi}_l(B^{-j}\xi)|^2 + |\hat{\phi}_l(B^{-j}\xi)|^2) \in L^1_{loc}(\mathbb{R}^n \setminus \{0\}).$$

Then, the affine systems $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ form a weak pair of frames, i.e.,

$$\|f\|^2 = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle f, D_{A^j}T_k\psi_l\rangle\langle D_{A^j}T_k\phi_l, f\rangle \qquad \text{for all } f \in \mathcal{D}, \tag{4.5}$$

if, and only if, the fundamental equations (4.2) *and* (4.3) *hold.*

Proof. The proof is based on Proposition 3.4 on affine systems and the idea of polarization as in [18, Section 8]. By our assumption on Ψ and Φ, we can define

$$N(f,\Psi,\Phi) = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle f, D_{A^j}T_k\psi_l\rangle\langle D_{A^j}T_k\phi_l, f\rangle \quad \text{for } f \in \mathcal{D}, \tag{4.6}$$

where the multiple series converge absolutely. This follows immediately by Remark 1 and

$$2\,|\langle f, D_{A^j}T_k\psi_l\rangle\langle D_{A^j}T_k\phi_l, f\rangle| \le |\langle f, D_{A^j}T_k\psi_l\rangle|^2 + |\langle D_{A^j}T_k\phi_l, f\rangle|^2.$$

By the polarization identity

$$\bar{z}w = \frac{1}{4}\sum_{p=1}^{4} i^p\,|i^p z + w|^2 \qquad \text{for } z, w \in \mathbb{C},$$

we have

$$N(f,\Psi,\Phi) = \frac{1}{4}\sum_{p=1}^{4} i^p N(f,\Theta_p), \quad \text{where } \Theta_p = \{\theta_{l,p}\}_{l=1}^{L},\ \theta_{l,p} = i^p\psi_l + \phi_l \tag{4.7}$$

for $f \in \mathcal{D}$.

Since, for $p = 1, 2, 3, 4$,

$$\begin{aligned}\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\theta}_{l,p}(B^{-j}\xi)\right|^2 &\equiv \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\left|i^p\hat{\psi}_l(B^{-j}\xi)+\hat{\phi}_l(B^{-j}\xi)\right|^2\\ &\le 2\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}(|\hat{\psi}_l(B^{-j}\xi)|^2+|\hat{\phi}_l(B^{-j}\xi)|^2) \in L^1_{\text{loc}}(\mathbb{R}^n\setminus\{0\}),\end{aligned}$$

we can apply Proposition 3.4 to Θ_p for each p. This yields

$$N(T_xf,\Theta_p) = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n} b_{j,l,p}(m)\mathrm{e}^{2\pi i\langle B^jm,x\rangle},$$

where

$$b_{j,l,p}(m) = \int_{\mathbb{R}^n}\hat{f}(\xi)\overline{\hat{f}(\xi+B^jm)}\,\overline{\hat{\theta}_{l,p}(B^{-j}\xi)}\hat{\theta}_{l,p}(B^{-j}(\xi+B^jm))\,\mathrm{d}\xi, \tag{4.8}$$

for $l = 1,\dots,L, j \in \mathbb{Z}, m \in \mathbb{Z}^n$, and the integral in (4.8) converges absolutely. By the polarization identity

$$\bar{z_1}w_2 = \frac{1}{4}\sum_{p=1}^{4} i^p\overline{(i^pz_1+w_1)}(i^pz_2+w_2) \qquad \text{for } z_1,z_2,w_1,w_2\in\mathbb{C},$$

we have

$$\begin{aligned}&\frac{1}{4}\sum_{p=1}^{4} i^p\overline{\hat{\theta}_{l,p}(B^{-j}\xi)}\hat{\theta}_{l,p}(B^{-j}(\xi+B^jm))\\ &\quad\equiv\frac{1}{4}\sum_{p=1}^{4} i^p\,\overline{(i^p\hat{\psi}_l(B^{-j}\xi)+\hat{\phi}_l(B^{-j}\xi))}\left(i^p\hat{\psi}_l(B^{-j}(\xi+B^jm))+\hat{\phi}_l(B^{-j}(\xi+B^jm))\right)\\ &\quad=\overline{\hat{\psi}_l(B^{-j}\xi)}\hat{\phi}_l(B^{-j}(\xi+B^jm)).\end{aligned}$$

Therefore, by (4.7),

$$\tilde{w}(x) := N(T_xf,\Psi,\Phi) = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n}\tilde{c}_{j,l}(m)\mathrm{e}^{2\pi i\langle B^jm,x\rangle}, \tag{4.9}$$

where

$$\tilde{c}_{j,l}(m) = \frac{1}{4}\sum_{p=1}^{4} i^p b_{j,l,p}(m) = \int_{\mathbb{R}^n}\hat{f}(\xi)\overline{\hat{f}(\xi+B^jm)}\,\overline{\hat{\psi}_l(B^{-j}\xi)}\hat{\phi}_l(B^{-j}(\xi+B^jm))\,\mathrm{d}\xi.$$

By a change of summation order, using absolute convergence of the series in (4.9), we have

$$\tilde{w}(x) = \sum_{\alpha\in\cup_{j\in\mathbb{Z}}B^j\mathbb{Z}^n}\tilde{c}_\alpha\mathrm{e}^{2\pi i\langle\alpha,x\rangle}, \tag{4.10}$$

where

$$\tilde{c}_\alpha = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi+\alpha)} \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} \overline{\hat{\psi}_l(B^{-j}\xi)}\hat{\phi}_l(B^{-j}(\xi+\alpha))\,\mathrm{d}\xi$$
$$= \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi+\alpha)\,\tilde{t}_\alpha(\xi)}\,\mathrm{d}\xi, \qquad \text{for } \alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n. \tag{4.11}$$

Assume that the affine systems $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ form a weak pair of frames. Using $\|T_x f\| = \|f\|$, this implies that the almost periodic function $\tilde{w}(x)$ from (4.9) is constant. To be precise: $\tilde{w}(x) = \|f\|^2$. By uniqueness of coefficients for Fourier series of almost periodic functions [13, Lemma 2.5], this only happens if, for $\alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n$,

$$\tilde{c}_0 = \|f\|^2 \qquad \text{and} \qquad \tilde{c}_\alpha = 0 \quad \text{for } \alpha \neq 0. \tag{4.12}$$

By (4.11), this shows that

$$\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \tilde{t}_0(\xi)\,\mathrm{d}\xi = \|f\|^2 = \|\hat{f}\|^2, \qquad \text{for all } f \in \mathcal{D}.$$

Since $\mathcal{D}$ is dense in $L^2(\mathbb{R}^n)$, this implies further that $\tilde{t}_0(\xi) = 1$ for a.e. $\xi \in \mathbb{R}^n$ showing that the first fundamental equation (4.2) holds.

For a nonzero α we have by (4.11) and (4.12),

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi+\alpha)\,\tilde{t}_\alpha(\xi)}\,\mathrm{d}\xi = 0, \qquad \text{for all } f \in \mathcal{D},$$

for $\alpha \in \left(\cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n\right) \setminus \{0\}$. In particular, this equality holds for $\alpha \in \mathbb{Z}^n \setminus \{0\}$. We need to show that $\tilde{t}_\alpha = 0$ almost everywhere for $\alpha \in \mathbb{Z}^n \setminus \{0\}$. The conclusion is almost immediate from Du Bois-Reynold's lemma that says that for local integrable functions u on $\mathbb{R}^n$ satisfying $\int uv = 0$ for all $v \in C_0^\infty$ we have $u = 0$. We fix $\alpha \in \mathbb{Z}^n \setminus \{0\}$, and let $I_{\mathbb{Z}^n}$ denote a fundamental domain of $\mathbb{Z}^n$. For arbitrary $l \in \mathbb{Z}^n$ we consider the translated parallelepiped $I_l = I_{\mathbb{Z}^n} + l \subset \mathbb{R}^n$ and define f by

$$\hat{f}(\xi) = \begin{cases} 1 & \text{for } \xi \in I_l, \\ \overline{\tilde{t}_\alpha(\xi)} & \text{for } \xi+\alpha \in I_l, \\ 0 & \text{otherwise.} \end{cases}$$

This definition makes sense since $\cup_{l\in\mathbb{Z}^n} I_l = \mathbb{R}^n$ and $(I_l - \alpha) \cap I_l = \emptyset$ for $\alpha \in \mathbb{Z}^n \setminus \{0\}$. Furthermore, since $\tilde{t}_\alpha$ is bounded by Remark 5, we have $f \in \mathcal{D}$. Consequently,

$$0 = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi+\alpha)\,\tilde{t}_\alpha(\xi)}\,\mathrm{d}\xi = \int_{I_l} 1\,\tilde{t}_\alpha(\xi)\,\overline{\tilde{t}_\alpha(\xi)}\,\mathrm{d}\xi = \int_{I_l} \left|\tilde{t}_\alpha(\xi)\right|^2 \mathrm{d}\xi,$$

which implies that $\tilde{t}_\alpha(\xi)$ vanishes almost everywhere for $\xi \in I_l$. Since $l \in \mathbb{Z}^n$ was arbitrarily chosen we deduce that $\tilde{t}_\alpha(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^n$. This shows that the second fundamental equation (4.3) holds.

Conversely, assume that the fundamental equations (4.2) and (4.3) hold. Equation (4.3) states that $\tilde{t}_\alpha(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^n$ for $\alpha \in \mathbb{Z}^n \setminus \{0\}$. By a change of variables $\gamma = B^l\xi$ and $\beta = B^l\alpha\,(l \in \mathbb{Z})$, this implies $\tilde{t}_\beta(\gamma) = 0$ for $\beta \in B^l\mathbb{Z}^n \setminus \{0\}$. Since this holds for all $l \in \mathbb{Z}$, we conclude $\tilde{t}_\alpha = 0$ almost everywhere for $\alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n \setminus \{0\}$. Hence, by (4.11), $\tilde{c}_\alpha = 0$ for $\alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n \setminus \{0\}$. Therefore, $\tilde{w}(x) = \tilde{c}_0 = \|f\|^2$ for all $x \in \mathbb{R}^n$ so, in particular,

$$N(f, \Psi, \Phi) \equiv \tilde{w}(0) = \|f\|^2 \qquad \text{for all } f \in \mathcal{D}.$$

We conclude that the affine systems $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ form a weak pair of frames. $\square$

We are now able to prove the characterization of dual affine and quasi-affine frames in terms of fundamental equations using the theory of mixed dual Gramians of Ron and Shen [19, 21, 23]. An alternative proof using the ideas of polarization of affine functionals is presented at the end of this section. In the integer case Theorem 4.2 was first shown by Ron and Shen [20, 22] with some decay assumptions on generators Ψ and Φ. Chui, Shi, and Stöckler [9] proved the same result without any decay assumptions, see also [2, Theorem 4.1]. Theorem 4.2 generalizes this result to the setting of rational dilations.

Theorem 4.2. *Let $A \in GL_n(\mathbb{Q})$ be expansive. Suppose $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ are Bessel sequences in $L^2(\mathbb{R}^n)$. Then the following assertions are equivalent:*

(i) $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ are dual frames.

*(ii) $\mathcal{A}^q_{\Lambda_0}(\Psi)$ and $\mathcal{A}^q_{\Lambda_0}(\Phi)$ are dual frames for **some** integer oversampling lattice $\Lambda_0 \subset \mathbb{Z}^n$.*

*(iii) $\mathcal{A}^q_{\Lambda}(\Psi)$ and $\mathcal{A}^q_{\Lambda}(\Phi)$ are dual frames for **all** integer oversampling lattices $\Lambda \subset \mathbb{Z}^n$.*

(iv) Ψ and Φ satisfy the fundamental equations (4.2) & (4.3).

Proof. The local integrability condition in Theorem 4.1 is satisfied by Proposition 3.2 since $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ are assumed to be Bessel sequences. Furthermore, weak duality (4.5) of two Bessel sequences implies "strong" duality [2, Lemma 2.7], i.e., that $\mathcal{A}(\Psi)$ and $\mathcal{A}(\Phi)$ are dual frames. Hence, by Theorem 4.1, we have (i) $\Leftrightarrow$ (iv); this equivalence is well-known, even for real dilations [8, Theorem 4].

The proof of the equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) is based on the approach used in [3, Theorem 3.4]. Let $\widetilde{G}_j(\xi)_{k,l}$ denote the mixed dual Gramian of $O^{A^{-j}\mathbb{Z}^n}_\Lambda(D_{A^j}\Psi)$ and $O^{A^{-j}\mathbb{Z}^n}_\Lambda(D_{A^j}\Phi)$ for $j \in \mathbb{Z}$, see Section 2.4. By Lemma 2.3 with $\Gamma = A^{-j}\mathbb{Z}^n$, this mixed dual Gramian is given as

$$\widetilde{G}_j(\xi)_{k,l} = \begin{cases} |\det A|^j \sum_{l=1}^{L} \widehat{D_{A^j}\psi}(\xi+k)\overline{\widehat{D_{A^j}\phi}(\xi+l)} & k-l \in \Gamma^* \cap \Lambda^*, \\ 0 & k-l \in \Lambda^* \setminus \Gamma^*, \end{cases}$$
$$= \begin{cases} \sum_{l=1}^{L} \hat{\psi}_l(\xi+k)\overline{\hat{\phi}_l(\xi+l)} & k-l \in B^j\mathbb{Z}^n \cap \Lambda^*, \\ 0 & k-l \in \Lambda^* \setminus B^j\mathbb{Z}^n, \end{cases}$$

for $k, l \in \Lambda^*$. The mixed dual Gramian of $\mathcal{A}^q_\Lambda(\Psi)$ and $\mathcal{A}^q_\Lambda(\Phi)$ is found by additivity of the jth layer mixed dual Gramian $\widetilde{G}_j(\xi)$ as

$$\widetilde{G}(\xi)_{k,l} = \sum_{j\in\mathbb{Z}} \widetilde{G}_j(\xi)_{k,l}$$
$$= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} \hat{\psi}_l(B^{-j}(\xi+k))\overline{\hat{\phi}_l(B^-j(\xi+l))} \times \begin{cases} 1 & k-l \in B^j\mathbb{Z}^n \cap \Lambda^*, \\ 0 & k-l \in \Lambda^* \setminus B^j\mathbb{Z}^n, \end{cases}$$

for $k,l \in \Lambda^*$. We only consider $k,l \in \Lambda^*$ so $k-l \in \Lambda^*$ is trivially satisfied. Thus, we arrive at the following expression for the mixed dual Gramian:

$$\widetilde{G}(\xi)_{k,l} = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}:k-l\in B^j\mathbb{Z}^n} \hat{\psi}_l(B^{-j}(\xi+k))\overline{\hat{\phi}_l(B^-j(\xi+l))} \equiv \tilde{t}_{l-k}(\xi+k). \tag{4.13}$$

Assume (ii) holds. This implies that the mixed dual Gramian $\widetilde{G}(\xi)$ is the identity operator on $\ell^2(\Lambda_0^*)$ for a.e. $\xi \in I_{\Lambda_0^*}$, hence $\widetilde{G}(\xi)_{k,l} = \delta_{k,l}$ for a.e. $\xi \in I_{\Lambda_0^*}$. By equation (4.13), for $\alpha \in \Lambda_0^*$,

$$\delta_{\alpha,0} = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} \hat{\psi}_l(B^{-j}\xi)\overline{\hat{\phi}_l(B^{-j}(\xi+\alpha))} \equiv \tilde{t}_\alpha(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{4.14}$$

This implies (iv) since $\mathbb{Z}^n \subset \Lambda_0^*$.

Assume (iv) holds. We will show that this implies (iii), i.e., that $\widetilde{G}(\xi)_{k,l} = \delta_{k,l}$ for a.e. $\xi \in I_{\Lambda^*}$ and all $k,l \in \Lambda^*$, where Λ is any integer lattice satisfying $\Lambda \subset \mathbb{Z}^n$. By a change of variables, we see that $\tilde{t}_\alpha(\xi) = 0$ for a.e. ξ and all $\alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n \setminus \{0\}$. If $\alpha \in \Lambda^* \setminus \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n$, then obviously $\tilde{t}_\alpha = 0$, hence equation (4.14) holds for $\alpha \in \Lambda^*$. This shows that the mixed dual Gramian $\widetilde{G}(\xi)$ is the identity operator on $\ell^2(\Lambda^*)$ for a.e. $\xi \in I_{\Lambda_0^*}$ which is equivalent to assertion (iii).

The last implication (iii) $\Rightarrow$ (ii) is obvious. □

It is possible to give an alternative proof of Theorem 4.2 using the ideas of polarization from the proof of Theorem 4.1. Since the equivalence (i) $\Leftrightarrow$ (iv) in Theorem 4.2 is well-known, we will only (re)prove (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) here.

Another proof of Theorem 4.2. Let $\Lambda \subset \mathbb{Z}^n$. For $f \in \mathcal{D}$, we define the Λ-periodic function $\tilde{w}_\Lambda^q(x)$ by

$$\tilde{w}_\Lambda^q(x) = N_\Lambda^q(T_x f, \Psi, \Phi) = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in \mathsf{K}_j} \langle T_x f, d_j T_k D_{A^j}\psi_l\rangle \langle d_j T_k D_{A^j}\phi_l, T_x f\rangle, \tag{4.15}$$

where $d_j = \#\{\Lambda/(\Lambda \cap A^{-j}\mathbb{Z}^n)\}^{-1/2}$ and K_j is given by (3.3). The series in (4.15) converges absolutely since $\mathcal{A}_\Lambda^q(\Psi)$ and $\mathcal{A}_\Lambda^q(\Phi)$ are Bessel sequences. Applying polarization identities as in the proof of Theorem 4.1 yields

$$\tilde{w}_\Lambda^q(x) = \sum_{\alpha\in\cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n\cap\Lambda^*} \tilde{c}_\alpha e^{2\pi i\langle\alpha,x\rangle}, \tag{4.16}$$

where the coefficients $\{\tilde{c}_\alpha\}$ are given in (4.11).

Assume (ii) holds. It is well-known that under the Bessel condition the weak duality of frames is equivalent to the duality of frames, see for example [7, Theorem 5.6.2]. Hence, (ii) is equivalent to $N_{\Lambda_0}^q(f,\Psi,\Phi) = \|f\|^2$ for all $f \in L^2(\mathbb{R}^n)$. Since $\|T_x f\| = \|f\|$, this implies that $\tilde{w}_{\Lambda_0}^q(x) = \|f\|^2$. By uniqueness of coefficients of the Fourier series of $\tilde{w}_{\Lambda_0}^q$, this happens only when

$$\tilde{c}_\alpha = \|f\|^2 \delta_{\alpha,0} \qquad \text{for } \alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n \cap \Lambda^*.$$

Following the proof of Theorem 4.1, we immediately have that this implies $\tilde{t}_\alpha = \delta_{\alpha,0}$ almost everywhere for $\alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n \cap \Lambda^*$. In particular, since $\mathbb{Z}^n \subset \Lambda^*$, we have $\tilde{t}_\alpha(\xi) = \delta_{\alpha,0}$ for a.e. ξ and $\alpha \in \mathbb{Z}^n$. This is precisely assertion (iv).

Assume (iv) holds. By a change of variables, this implies that $\tilde{t}_\alpha(\xi) = \delta_{\alpha,0}$ for a.e. ξ and all $\alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n$. Therefore,

$$\tilde{c}_\alpha = \|f\|^2 \delta_{\alpha,0} \qquad \text{for } \alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n,$$

and we note that these equations are independent of Λ. Hence, by (4.16), for any $\Lambda \subset \mathbb{Z}^n$,

$$N^q_\Lambda(f, \Psi, \Phi) = \tilde{w}^q_\Lambda(0) = \tilde{c}_0 = \|f\|^2 \qquad \text{for all } f \in \mathcal{D}.$$

By a density argument, this equality holds for all $f \in L^2(\mathbb{R}^n)$, and assertion (iii) follows. □

Remark 6. It is apparent from the proof above that the equivalence of (ii), (iii), and (iv) in Theorem 4.2 holds under the weaker assumption that $\mathcal{A}^q_{\Lambda_0}(\Psi)$ and $\mathcal{A}^q_{\Lambda_0}(\Phi)$ are Bessel sequences in $L^2(\mathbb{R}^n)$ for some $\Lambda_0 \subset \mathbb{Z}^n$.

5. Diagonal affine systems

In this section we study a particularly interesting subclass of generators where the equivalence between affine and quasi-affine frames exhibits the largest degree of symmetry. This is a class of diagonal affine systems for which the off-diagonal functions t_α defined below vanish. We show that the class of diagonal affine frames consists precisely of quasi-affine frames having a canonical dual quasi-affine frame. This extends a result of Weber and the first author [5] from the setting of integer dilations to that of rational dilations.

Definition 6. For a given dilation matrix A and $\Psi \subset L^2(\mathbb{R}^n)$ we introduce the family of functions $\{t_\alpha\}_{\alpha\in\mathbb{Z}^n}$ on $\mathbb{R}^n$ by:

$$t_\alpha(\xi) = \sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} \hat{\psi}(B^{-j}\xi)\overline{\hat{\psi}(B^{-j}(\xi+\alpha))} \qquad \text{for } \xi \in \mathbb{R}^n. \tag{5.1}$$

In particular,

$$t_0(\xi) = \sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^j\xi)|^2.$$

We say that the affine system $\mathcal{A}(\Psi)$ is *diagonal* if $t_\alpha(\xi) = 0$ a.e. for all $\alpha \in \mathbb{Z}^n \setminus \{0\}$.

Note that the series in (5.1) converges absolutely for a.e. ξ in light of Remark 5. In addition, if $\Psi \subset L^2(\mathbb{R}^n)$ generates an affine Bessel sequence $\mathcal{A}(\Psi)$ with bound C_2, or a quasi-affine Bessel sequence $\mathcal{A}^q_\Lambda(\Psi)$ for some lattice Λ, then each t_α is well defined and essentially bounded in light of Proposition 3.2 and

$$\begin{aligned}\sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} |\hat{\psi}(B^{-j}\xi)\overline{\hat{\psi}(B^{-j}(\xi+\alpha))}| &\le \tfrac{1}{2}\sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} |\hat{\psi}(B^{-j}\xi)|^2 \\ &\quad + \tfrac{1}{2}\sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} |\hat{\psi}(B^{-j}(\xi+\alpha))|^2 \le C_2.\end{aligned}$$

Now, with the extra assumption $t_\alpha(\xi) = 0$ a.e. for $\alpha \in \mathbb{Z}^n \setminus \{0\}$, we have the following equivalence result.

Theorem 5.1. *Let $A \in GL_n(\mathbb{Q})$ be expansive, $\Psi \subset L^2(\mathbb{R}^n)$ and let $C_1, C_2 > 0$ be constants. Suppose that the affine system $\mathcal{A}(\Psi)$ is diagonal. Then the following assertions are equivalent:*

(i) the affine system $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2.

(ii) the quasi-affine system $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2 for ***some*** *integer lattice $\Lambda_0 \subset \mathbb{Z}^n$.*

(iii) the quasi-affine system $\mathcal{A}^q_{\Lambda}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2 for ***all*** *integer lattices $\Lambda \subset \mathbb{Z}^n$.*

(iv)
$$C_1 \le \sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^j\xi)|^2 \le C_2 \qquad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Proof. Let $\Lambda \subset \mathbb{Z}^n$ be a lattice in $\mathbb{R}^n$. For fixed $f \in \mathcal{D}$, let w and w^q_Λ be the functions introduced in (3.13) and (3.17). By a change of summation order, using absolute convergence of the series, these functions can be written as

$$w(x) = \sum_{\alpha\in\cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n} c_\alpha\, \mathrm{e}^{2\pi i\langle\alpha,x\rangle}, \qquad w^q_\Lambda(x) = \sum_{\alpha\in\cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n\cap\Lambda^*} c_\alpha\, \mathrm{e}^{2\pi i\langle\alpha,x\rangle}, \tag{5.2}$$

where

$$\begin{aligned} c_\alpha &= \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi+\alpha)} \sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} \overline{\hat{\psi}(B^{-j}\xi)}\hat{\psi}(B^{-j}(\xi+\alpha))\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi+\alpha)\, t_\alpha(\xi)}\,\mathrm{d}\xi, \qquad \text{for } \alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n. \end{aligned} \tag{5.3}$$

Our standing assumption in this theorem is that $t_\alpha(\xi) = 0$ a.e. for $\alpha \in \mathbb{Z}^n \setminus \{0\}$. By a change of variables, this implies $t_\alpha(\xi) = 0$ a.e. for $\alpha \in \cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n \setminus \{0\}$. Thus the expressions in (5.2) reduce to

$$w(x) = w^q_\Lambda(x) = c_0 = \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 t_0(\xi)\mathrm{d}\xi \qquad \text{for all } x \in \mathbb{R}^n,$$

hence w and w^q_Λ are equal and constant functions of x. Therefore

$$N(f) = w(0) = w^q_\Lambda(0) = N^q_\Lambda(f)$$

for $f \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $L^2(\mathbb{R}^n)$, we find that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). Note that (i) $\Rightarrow$ (iii) also follows directly from Theorem 3.5.

We will verify that (i) $\Leftrightarrow$ (iv). In terms of the t_α-functions, assertion (iv) reads, $C_1 \le t_0(\xi) \le C_2$ almost everywhere. By the above and an application of the Plancherel theorem, assertion (i) is equivalent to

$$C_1\langle\hat{f},\hat{f}\rangle \le \langle t_0\hat{f},\hat{f}\rangle \le C_2\langle\hat{f},\hat{f}\rangle \qquad \text{for } f \in L^2(\mathbb{R}^n). \tag{5.4}$$

This implies that

$$C_1 \le t_0(\xi) \le C_2 \qquad \text{for a.e. } \xi \in \mathbb{R}^n,$$

which, on the other hand, clearly implies (5.4). ∎

As a corollary we have the following converse of Theorem 3.5.

Corollary 5.2. *Let $A \in GL_n(\mathbb{Q})$ be expansive, $\Psi \subset L^2(\mathbb{R}^n)$, and let $\mathcal{A}(\Psi)$ be diagonal. Suppose that the Λ_0-oversampled quasi-affine system $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2 for some integer lattice $\Lambda_0 \subset \mathbb{Z}^n$. Then, the affine system $\mathcal{A}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ with bounds C_1, C_2.*

As a direct consequence of Theorem 3.5 and Corollary 5.2 we generalize the equivalence of affine and quasi-affine Parseval frames due to the first author [3, Theorem 3.4], see also [14, Theorem 2.17].

Theorem 5.3. *Suppose $A \in GL_n(\mathbb{Q})$ is expansive and $\Psi \subset L^2(\mathbb{R}^n)$. Then the following assertions are equivalent:*

(i) the affine system $\mathcal{A}(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^n)$

*(ii) the quasi-affine system $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^n)$ for **some** integer lattice $\Lambda_0 \subset \mathbb{Z}^n$*

*(iii) the quasi-affine system $\mathcal{A}^q_{\Lambda}(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^n)$ for **all** integer lattices $\Lambda \subset \mathbb{Z}^n$*

Proof. The implication (i) $\Rightarrow$ (iii) is a special case of Theorem 3.5, and (iii) $\Rightarrow$ (ii) is obvious. Proposition 3.2 and the proof of Proposition 3.3 show that the local integrability condition (3.16) for the quasi-affine system is satisfied, hence we can apply Theorem 2.4 to $\mathcal{A}^q_{\Lambda}(\Psi)$. By equations (2.14), (3.10) and (3.11) this implies that $t_\alpha = 0$ for $\alpha \in \mathbb{Z}^n \setminus \{0\}$, hence the affine system is diagonal. An application of Corollary 5.2 gives us (ii) $\Rightarrow$ (i). $\square$

5.1. Canonical dual quasi-affine frames

Our next aim is to characterize when the canonical dual of a quasi-affine frame is also a quasi-affine frame. To achieve this we need the following result resembling [5, Proposition 1].

Theorem 5.4. *Let $A \in GL_n(\mathbb{Q})$ be expansive. Suppose the $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a frame for some $\Lambda_0 \subset \mathbb{Z}^n$, which has a dual quasi-affine frame $\mathcal{A}^q_{\Lambda_0}(\Phi)$. Then, for any $S \in B(L^2(\mathbb{R}^n))$ we have*

$$S \in \mathcal{C}_\psi(\mathcal{A}^q_{\Lambda_0}) \text{ for all } \psi \in \Psi \quad \Leftrightarrow \quad S \in \{D_A, T_\lambda : \lambda \in \Lambda_0\}'$$

Note that we need to assume a much stronger hypothesis than the assumption of [5, Proposition 1] saying that the quasi-affine system $\mathcal{A}^q_{\mathbb{Z}^n}(\Psi)$ is complete in $L^2(\mathbb{R}^n)$.

Proof. The fact that $\mathcal{A}^q_{\Lambda_0}(\Psi)$ and $\mathcal{A}^q_{\Lambda_0}(\Phi)$ are dual frames implies that the fundamental equations (4.2) and (4.3) hold, see Remark 6. By Theorem 4.1, the affine system $\mathcal{A}(\Psi)$ is complete in $L^2(\mathbb{R}^n)$.

Suppose that $S \in \mathcal{C}_\psi(\mathcal{A}^q_{\Lambda_0})$. Since the quasi-affine system $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is Λ_0-SI, S must commute with translations T_λ, $\lambda \in \Lambda_0$. Likewise, since the affine system $\mathcal{A}(\Psi)$ is a part of the quasi-affine system $\mathcal{A}^q_{\Lambda_0}(\Psi)$ (up to normalizing constants), $S \in \mathcal{C}_\psi(\mathcal{A})$. Since the affine system $\mathcal{A}(\Psi)$ is complete in $L^2(\mathbb{R}^n)$ and $\mathcal{A}(\Psi)$ is dilation-invariant, S must commute with the dilation operator D_A.

Conversely, if $S \in \{D_A, T_\lambda : \lambda \in \Lambda_0\}'$, then clearly S belongs to the local commutant $\mathcal{C}_\psi(\mathcal{A}^q_{\Lambda_0})$ for any choice of $\psi \in L^2(\mathbb{R}^n)$. $\square$

Remark 7. Note that if $S \in \{D_A, T_\lambda : \lambda \in \Lambda_0\}'$, then S commutes with all translation T_λ, $\lambda \in \mathbb{R}^n$. Indeed, by $T_{A^j\lambda} = D_{A^{-j}} T_\lambda D_{A^j}$, S must commute with $T_{A^j\lambda}$ for $j \in \mathbb{Z}$ and $\lambda \in \Lambda_0$. Since A is expansive, $\cup_{j\in\mathbb{Z}} A^j \Lambda_0$ is dense in $\mathbb{R}^n$. Hence, by continuity of $x \mapsto T_x f$ for $f \in L^2(\mathbb{R}^n)$, we have $S \in \{D_A, T_\lambda : \lambda \in \mathbb{R}^n\}'$. In fact, we have the following lemma which is a straightforward generalization of [5, Lemma 2].

Lemma 5.5. *Let $A \in GL_n(\mathbb{R})$ be expansive, Λ a lattice, and $S \in B(L^2(\mathbb{R}^n))$. Then, $S \in \{D_A, T_\lambda : \lambda \in \Lambda\}'$ if, and only if, S is a B-dilation periodic Fourier multiplier, i.e., there exists a function $s \in L^\infty(\mathbb{R}^n)$ such that*

$$\widehat{Sf}(\xi) = s(\xi)\hat{f}(\xi) \quad \text{for a.e. } \xi,$$

where $s(\xi) = s(B\xi)$ for a.e. ξ.

Proof. Assume $S \in \{D_A, T_\lambda : \lambda \in \Lambda\}'$. By $T_{A^j\lambda} = D_{A^{-j}} T_\lambda D_{A^j}$, S commutes with $T_{A^j\lambda}$ for $j \in \mathbb{Z}$ and $\lambda \in \Lambda$, i.e.,

$$ST_k = T_k S \qquad \text{for } k \in \cup_{j\in\mathbb{Z}} A^j \Lambda. \tag{5.5}$$

The union $\cup_{j\in\mathbb{Z}} A^j \Lambda$ is dense in $\mathbb{R}^n$ since A is expansive. For $x \in \mathbb{R}^n$ take $\{k_n\}_{n\in\mathbb{N}}$ from $\cup_{j\in\mathbb{Z}} A^j \Lambda$ such that $k_n \to x$. By continuity of $x \mapsto T_x f$ for $f \in L^2(\mathbb{R}^n)$, $k_n \to x$ implies $T_{k_n} f \to T_x f$ in the L^2 norm, i.e., $T_{k_n} \to T_x$ in the strong operator topology. Hence, by equation (5.5), we have $ST_x = T_x S$, proving that S is a Fourier multiplier. Finally, by $D_A S = S D_A$ and

$$\mathcal{F}\, D_A S f(\xi) = \int_{\mathbb{R}^n} D_A S f(x) \mathrm{e}^{-2\pi i x\cdot\xi} \mathrm{d}x = \left|\det A\right|^{-1/2} s(B^{-1}\xi)\hat{f}(B^{-1}\xi),$$

and

$$\mathcal{F}\, S D_A f(\xi) = s(\xi) \int_{\mathbb{R}^n} D_A f(x) \mathrm{e}^{-2\pi i x\cdot\xi} \mathrm{d}x = \left|\det A\right|^{-1/2} s(\xi)\hat{f}(B^{-1}\xi),$$

we have B-periodicity of the symbol s.

Conversely, assume S is a Fourier multiplier with a B-dilation periodic symbol. The operator S commutes with all translations by the Fourier multiplier property and with dilations D_A by the B-dilation periodicity of the symbol and the two displayed equations above. □

Theorem 5.6. *Let $A \in GL_n(\mathbb{Q})$ be expansive. Suppose the oversampled quasi-affine system $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a frame for $L^2(\mathbb{R}^n)$ for some integer lattices $\Lambda_0 \subset \mathbb{Z}^n$. Then the canonical dual frame of $\mathcal{A}^q_{\Lambda_0}(\Psi)$ has the form $\mathcal{A}^q_{\Lambda_0}(\Phi)$ for some set of functions $\Phi \subset L^2(\mathbb{R}^n)$ with cardinality $\#\Phi = \#\Psi$ if, and only if,*

$$t_\alpha(\xi) = \sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}:\alpha\in B^j\mathbb{Z}^n} \hat{\psi}(B^{-j}\xi)\overline{\hat{\psi}(B^{-j}(\xi+\alpha))} = 0 \quad \text{for all } \alpha \in \mathbb{Z}^n \setminus \{0\}. \tag{5.6}$$

Moreover, in the positive case $\mathcal{A}^q_{\Lambda}(\Psi)$ is a frame for all integer lattices $\Lambda \subset \mathbb{Z}^n$ and its canonical dual frame is $\mathcal{A}^q_{\Lambda}(\Phi)$.

Proof. Let $S^q_{\Lambda_0}$ be the frame operator of the quasi-affine system $\mathcal{A}^q_{\Lambda_0}(\Psi)$. Since $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a frame, equation (3.12) is satisfied, hence the expression for $w^q_{\Lambda_0}$ in (5.2) holds for $f \in \mathcal{D}$.

Assume that the canonical dual of $\mathcal{A}^q_{\Lambda_0}(\Psi)$ has the form $\mathcal{A}^q_{\Lambda_0}(\Phi)$, i.e., $S^q_{\Lambda_0} \in \mathcal{C}_\psi(\mathcal{A}^q_{\Lambda_0})$ for all $\psi \in \Psi$. By Theorem 5.4 and Remark 7, $S^q_{\Lambda_0} \in \{D_A, T_\lambda : \lambda \in \mathbb{R}^n\}'$, hence

$$w^q_{\Lambda_0}(x) = \left\langle S^q_{\Lambda_0} T_x f, T_x f \right\rangle = \left\langle T_x S^q_{\Lambda_0} f, T_x f \right\rangle = \left\langle S^q_{\Lambda_0} f, f \right\rangle \qquad \forall x \in \mathbb{R}^n,$$

which shows that $w^q_{\Lambda_0}$ is constant for every $f \in \mathcal{D}$.

For each $f \in \mathcal{D}$ we express $w^q_{\Lambda_0}$ as the Λ_0-periodic Fourier series (5.2). Such a Fourier series is identically constant if, and only if,

$$c_\alpha \equiv \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi+\alpha)\, t_\alpha(\xi)}\, \mathrm{d}\xi = 0 \quad \text{for all } \alpha \in \Big(\bigcup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n \cap \Lambda_0^*\Big) \setminus \{0\},$$

by the uniqueness of the Fourier coefficients. In particular, this equality holds for $\alpha \in \mathbb{Z}^n \setminus \{0\}$ since $\mathbb{Z}^n \subset \Lambda_0^*$. Fix $\alpha \in \mathbb{Z}^n \setminus \{0\}$. Let $I_{\Lambda_0^*}$ denote a fundamental domain of Λ_0^* and, for $l \in \Lambda_0^*$, let $I_l = I_{\Lambda_0^*} + l$. Define f by

$$\hat{f}(\xi) := \begin{cases} 1 & \text{for } \xi \in I_l, \\ \overline{t_\alpha(\xi)} & \text{for } \xi + \alpha \in I_l, \\ 0 & \text{otherwise.} \end{cases}$$

Since t_α is bounded by the Bessel bound C_2, we have $f \in \mathcal{D}$. Now,

$$0 = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi+\alpha)\, t_\alpha(\xi)}\mathrm{d}\xi = \int_{I_l} t_\alpha(\xi)\overline{t_\alpha(\xi)}\mathrm{d}\xi = \int_{I_l} |t_\alpha(\xi)|^2\, \mathrm{d}\xi,$$

for each $l \in \Lambda_0^*$. Since $\cup_{l\in\Lambda_0^*} I_l = \mathbb{R}^n$ we deduce that $t_\alpha(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^n$, and the theorem is half proved.

Conversely, assume $t_\alpha(\xi) = 0$ for $\alpha \in \mathbb{Z}^n \setminus \{0\}$. Then $t_\alpha(\xi) = 0$ for $\alpha \in (\cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n) \setminus \{0\}$ by a change of variables. In particular, $t_\alpha(\xi) = 0$ for $\alpha \in (\cup_{j\in\mathbb{Z}} B^j\mathbb{Z}^n \cap \Lambda_0^*) \setminus \{0\}$, hence $w^q_{\Lambda_0}(x) = c_0$ for every $x \in \mathbb{R}^n$, i.e., $w^q_{\Lambda_0}$ is constant on $\mathbb{R}^n$ for every $f \in \mathcal{D}$. Therefore, for every $x \in \mathbb{R}^n$,

$$\left\langle S^q_{\Lambda_0} T_x f, T_x f \right\rangle = w^q_{\Lambda_0}(x) = w^q_{\Lambda_0}(0) = \left\langle S^q_{\Lambda_0} f, f \right\rangle \qquad \text{for } f \in \mathcal{D}.$$

This equality extends to all $f \in L^2(\mathbb{R}^n)$ by a density argument, hence

$$\left\langle \left(T_{-x} S^q_{\Lambda_0} T_x - S^q_{\Lambda_0}\right) f, f \right\rangle = 0 \qquad \text{for } f \in L^2(\mathbb{R}^n).$$

We conclude that $S^q_{\Lambda_0} T_x = T_x S^q_{\Lambda_0}$ for all $x \in \mathbb{R}^n$, in other words, $S^q_{\Lambda_0}$ is a Fourier multiplier:

$$\widehat{S^q_{\Lambda_0} f}(\xi) = s(\xi)\hat{f}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n \text{ and all } f \in L^2(\mathbb{R}^n), \tag{5.7}$$

for some symbol $s \in L^\infty(\mathbb{R}^n)$. We claim the symbol of $S^q_{\Lambda_0}$ is

$$s(\xi) = t_0(\xi) = \sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(B^j\xi)\right|^2 .$$

This function is obviously a B-dilation periodic function, that is, $s(\xi) = s(B\xi)$. By Proposition 3.2 the function is bounded by the upper frame bound $s(\xi) \le C_2$ for a.e. ξ, so $s \in L^\infty(\mathbb{R}^n)$. By the Plancherel theorem, we see

$$w^q_{\Lambda_0}(0) = \langle S^q_{\Lambda_0} f, f\rangle = \langle \widehat{S^q_{\Lambda_0} f}, \hat{f}\rangle \qquad \text{for all } f \in \mathcal{D},$$

and, by (5.3) with $\alpha = 0$, that

$$c_0 = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi)} \sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(B^j\xi)\right|^2 \,\mathrm{d}\xi.$$

Since $w^q_{\Lambda_0}(x) = c_0$ for all $x \in \mathbb{R}^n$, we have, in particular,

$$\langle \widehat{S^q_{\Lambda_0} f}, \hat{f}\rangle = w^q_{\Lambda_0}(0) = c_0 = \langle s\hat{f}, \hat{f}\rangle \qquad \text{for all } f \in \mathcal{D}.$$

Therefore, s is a B-dilation periodic symbol of $S^q_{\Lambda_0}$ implying that $S^q_{\Lambda_0}$ commutes with D_A, see Lemma 5.5. The frame operator $S^q_{\Lambda_0}$ belongs therefore to $\{D_A, T_\lambda : \lambda \in \Lambda_0\}'$. As a result we find that $(S^q_{\Lambda_0})^{-1} \in \mathcal{C}_\psi(\mathcal{A}^q_{\Lambda_0})$ for $\psi \in \Psi$. This is equivalent to the canonical dual of $\mathcal{A}^q_{\Lambda_0}(\Psi)$ having the quasi-affine structure with the same number of generators. ☐

6. Broken symmetry between the integer and rational case

The goal of this section is to illustrate fundamental differences between integer and rational cases. That is, a mere fact that a quasi-affine system is a frame does not imply that an affine system must be a frame as well. This kind of phenomenon cannot happen for integer dilations where we have a perfect equivalence of the frame property between affine and quasi-affine systems. Moreover, this cannot happen for Parseval frames due to Theorem 5.3, or more generally, for affine frames having duals by Theorem 4.2. Moreover, Theorem 6.1 shows the optimality of our results. That is, the assumption of uniformity of frame bounds of quasi-affine systems in Theorem 3.8 cannot be removed in general.

Theorem 6.1. *Let $1 < a \in \mathbb{Q} \setminus \mathbb{Z}$ be a rational non-integer dilation factor. Then, there exists a function $\psi \in L^2(\mathbb{R})$ such that $\mathcal{A}^q_\Lambda(\psi)$ is a frame for any oversampling lattice $\Lambda \subset \mathbb{Z}$, but yet, $\mathcal{A}(\psi)$ is not a frame.*

Remark 8. In the light of Theorem 3.8, the frame bounds of the quasi-affine systems $\mathcal{A}^q_\Lambda(\psi)$ are not uniform for all lattices $\Lambda \subset \mathbb{Z}$. In fact, we will see that the lower frame bound of $\mathcal{A}^q_\Lambda(\psi)$ drops to 0 as a lattice Λ gets sparser and sparser. Consequently, in the limiting case, when no oversampling is present, we obtain an affine system $\mathcal{A}(\psi)$ which is not a frame due to the failure of the lower frame bound.

We will need the following well-known result, see [16, Theorem 13.0.1] or the proof of [10, Lemma 3.4]

Theorem 6.2. *Suppose that* $\psi \in L^2(\mathbb{R})$ *is such that* $\hat{\psi} \in L^\infty(\mathbb{R})$ *and*

$$\begin{aligned} \hat{\psi}(\xi) =& O(|\xi|^\delta) \qquad \text{as } \xi \to 0, \\ \hat{\psi}(\xi) =& O(|\xi|^{-1/2-\delta}) \qquad \text{as } |\xi| \to \infty, \end{aligned}$$

for some $\delta > 0$*. Then the affine system* $\mathcal{A}(\psi)$ *is a Bessel sequence.*

We define the space $\check{L}^2(K)$, invariant under all translations, by

$$\check{L}^2(K) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset K\}$$

for measurable subsets K of $\mathbb{R}$.

Proof of Theorem 6.1. Choose $\delta > 0$ so that $\frac{1}{a(a+1)} < \delta < \frac{1}{a^2+1}$. Define $\psi \in L^2(\mathbb{R})$ as $\hat{\psi} = \mathbf{1}_{(-a^2\delta,-\delta)\cup(\delta,a^2\delta)}$. First, we shall show that the affine system $\mathcal{A}(\psi)$ is not a frame. To achieve this we will follow the idea from [5, Example 2]. We will need the following standard identity, which can be shown by the periodization argument

$$\sum_{k\in\mathbb{Z}} |\langle f, T_k\psi\rangle|^2 = \int_0^1 \Big| \sum_{k\in\mathbb{Z}} \hat{f}(\xi+k)\overline{\hat{\psi}(\xi+k)} \Big|^2 \mathrm{d}\xi \qquad \text{for any } f \in L^2(\mathbb{R}). \tag{6.1}$$

Let $K_\delta = (1 - a^2\delta, a^2\delta)$. By the restriction on δ, we have

$$K_\delta \subset (\delta, a^2\delta) \subset (\delta, 1-\delta) \qquad \text{and} \qquad K_\delta - 1 \subset (-a^2\delta, -\delta) \subset (-1+\delta, -\delta).$$

Hence, by a direct calculation using (6.1) we have for any $f \in L^2(\mathbb{R})$

$$\sum_{k\in\mathbb{Z}} |\langle f, T_k\psi\rangle|^2 = \int_{K_\delta} |\hat{f}(\xi-1) + \hat{f}(\xi)|^2 \mathrm{d}\xi + \int_{(\delta,1-a^2\delta)} |\hat{f}(\xi)|^2 \mathrm{d}\xi + \int_{(a^2\delta,1-\delta)} |\hat{f}(\xi-1)|^2 \mathrm{d}\xi. \tag{6.2}$$

In particular, by restricting (6.2) to a subspace $\check{L}^2(L_\delta)$, where

$$L_\delta = (-\infty, -1+\delta) \cup (K_\delta - 1) \cup (-\delta, \delta) \cup K_\delta \cup (1-\delta, \infty),$$

we obtain a convenient formula

$$\sum_{k\in\mathbb{Z}} |\langle f, T_k\psi\rangle|^2 = \int_{K_\delta} |\hat{f}(\xi-1) + \hat{f}(\xi)|^2 \mathrm{d}\xi \qquad \text{for any } f \in \check{L}^2(L_\delta). \tag{6.3}$$

For any natural number N and sufficiently small $\varepsilon = \varepsilon(N) > 0$, we define a function $f_N \in L^2(\mathbb{R})$ by

$$\hat{f}_N = \sum_{k=0}^{N} (\mathbf{1}_{I_k^+} - \mathbf{1}_{I_k^-}), \tag{6.4}$$

where

$$I_k^+ = \Big(\frac{a^{-k}}{a+1} - \varepsilon, \frac{a^{-k}}{a+1}\Big), \; I_k^- = \Big(-\frac{a^{-k}}{a+1} - \varepsilon, -\frac{a^{-k}}{a+1}\Big). \tag{6.5}$$

Intuitively, one might think of $\hat{f}_N$ as a linear combination of point masses

$$\varepsilon \sum_{k=0}^{N} (\delta_{a^{-k}/(a+1)} - \delta_{-a^{-k}/(a+1)}).$$

We claim that

$$D_{a^j} f_N \in \check{L}^2(L_\delta) \qquad \text{for all } j \in \mathbb{Z}. \tag{6.6}$$

Indeed, (6.6) follows immediately from

$$a^j(I_k^+ \cup I_k^-) \subset \begin{cases} (-\delta, \delta) & j \le k-1, \\ (K_\delta - 1) \cup K_\delta & j = k, k+1, \\ (-\infty, -1+\delta) \cup (1-\delta, \infty) & j \ge k+2, \end{cases}$$

for $k = 0, \dots, N$ and for sufficiently small $\varepsilon = \varepsilon(N) > 0$, i.e.,

$$0 < \varepsilon < \min\Big\{ a^{-N+1}\Big(\delta - \frac{1}{a(a+1)}\Big), a^{-N-2}\Big(\frac{a^2}{a+1} - 1 + \delta\Big)\Big\}.$$

Let S be the frame operator corresponding to the affine system $\mathcal{A}(\psi)$. Note that by Theorem 6.2, S is bounded. Our goal is to show that S is not bounded from below. Combining (6.3)–(6.6) we have

$$\begin{aligned} \|Sf_N\|^2 &= \sum_{j\in\mathbb{Z}} \sum_{z\in\mathbb{Z}} |\langle f_N, D_{a^j} T_z \psi\rangle|^2 = \sum_{j\in\mathbb{Z}} \sum_{z\in\mathbb{Z}} |\langle D_{a^j} f_N, T_z \psi\rangle|^2 \\ &= \sum_{j=0}^{N+1} a^{-j} \int_{K_\delta} |\hat{f}_N(a^{-j}(\xi-1)) + \hat{f}_N(a^{-j}\xi)|^2 \mathrm{d}\xi = 4\varepsilon. \end{aligned}$$

Here, we used that for $\xi \in K_\delta$

$$\hat{f}_N(a^{-j}(\xi-1)) + \hat{f}_N(a^{-j}\xi) = \begin{cases} \mathbf{1}_{I_0^+}(\xi) - \mathbf{1}_{I_0^-}(\xi-1) & j = 0, \\ 0 & j = 1, \dots, N, \\ \mathbf{1}_{a^{N+1}I_N^+}(\xi) - \mathbf{1}_{a^{N+1}I_N^-}(\xi-1) & j = N+1. \end{cases}$$

The presence of cancellations at scales $j = 1, \dots, N$ is due to translation-dilation linkage of the quadruple of points $\{\pm a/(a+1), \pm 1/(a+1)\}$. On the other hand,

$$\|f_N\|^2 = \|\hat{f}_N\|^2 = 2\varepsilon(N+1).$$

Since N is arbitrary, this shows that the frame operator S is not bounded from below. Consequently, $\mathcal{A}(\psi)$ is not a frame.

Next, we will show that $\mathcal{A}^q_\Lambda(\psi)$ is a frame for any choice of lattice $\Lambda \subset \mathbb{Z}$. Since $\mathcal{A}(\psi)$ is a Bessel sequence, Theorem 3.5 yields that $\mathcal{A}^q_\Lambda(\psi)$ is a Bessel sequence as well. Hence, it remains to establish the lower frame bound for $\mathcal{A}^q_\Lambda(\psi)$.

Let $a = p/q$, where $p, q \in \mathbb{N}$ are relatively prime, and $l \in \mathbb{N}$ be such that $\Lambda = l\mathbb{Z}$. Let

$$J_1 = \max\{j \in \mathbb{N}_0 : p^j \text{ divides } l\}, \qquad J_2 = \max\{j \in \mathbb{N}_0 : q^j \text{ divides } l\}.$$

Take any $j \in \mathbb{Z}$. Then, we have the equality of lattices $a^{-j}\mathbb{Z} + \Lambda = a^{-j}\mathbb{Z} \iff l$ is an integer multiple of a^{-j}. Clearly, this is equivalent to l being divisible by q^j if $j > 0$ or l divisible by p^{-j} if $j < 0$. Therefore,

$$a^{-j}\mathbb{Z} + \Lambda = a^{-j}\mathbb{Z} \iff -J_1 \le j \le J_2. \tag{6.7}$$

Consequently,

$$a^{-j}\mathbb{Z} + \Lambda = \frac{1}{c_j} a^{-j}\mathbb{Z} \qquad \text{for some } c_j \ge 2, \text{ where } j < -J_1 \text{ or } j > J_2. \tag{6.8}$$

The properties (6.7) and (6.8) enable us to identify the quasi-affine system $\mathcal{A}^q_\Lambda(\psi)$. At the scales $-J_1 \le j \le J_2$, the quasi-affine system $\mathcal{A}^q_\Lambda(\psi)$ coincides with the affine system $\mathcal{A}(\psi)$. However, outside of this finite range of scales the quasi-affine system is obtained by oversampling the affine system at a rate $c_j \ge 2$. This will lead to a simple form of the frame operator S^q_Λ of the quasi-affine system $\mathcal{A}^q_\Lambda(\psi)$.

Indeed, suppose that $j < -J_1$ or $j > J_2$. By Definition 4 and (6.8) the quasi-affine system $\mathcal{A}^q_\Lambda(\psi)$ at the scale j is

$$O^{a^{-j}\mathbb{Z}}_\Lambda(D_{a^j}\psi) = E^{a^{-j}\mathbb{Z}+\Lambda}\Big(\frac{1}{|\Lambda/(\Lambda \cap a^{-j}\mathbb{Z})|^{1/2}} D_{a^j}\psi\Big) = E^{a^{-j}/c_j\mathbb{Z}}((c_j)^{-1/2} D_{a^j}\psi).$$

Hence,

$$\begin{aligned}\sum_{g \in O^{a^{-j}\mathbb{Z}}_\Lambda(D_{a^j}\psi)} |\langle f, g\rangle|^2 &= \frac{1}{c_j}\sum_{k\in\mathbb{Z}} |\langle f, T_{a^{-j}k/c_j} D_{a^j}\psi\rangle|^2 \\ &= \sum_{k\in\mathbb{Z}} \frac{1}{c_j a^j}\Big|\int_{\mathbb{R}} \hat{f}(\xi)\hat{\psi}(a^{-j}\xi) e^{2\pi i k\xi/(a^j c_j)} \mathrm{d}\xi\Big|^2 = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\hat{\psi}(a^{-j}\xi)|^2 \mathrm{d}\xi.\end{aligned}$$

The last step is a consequence of the fact that $\operatorname{supp}\hat{\psi}(a^{-j}\cdot) \subset (-a^j, a^j)$ and that $c_j \ge 2$. Combining this with (6.7) yields

$$\begin{aligned}\|S^q_\Lambda f\|^2 &= \sum_{j=-J_1}^{J_2}\sum_{k\in\mathbb{Z}} |\langle f, D_{a^j}T_k\psi\rangle|^2 + \Big(\sum_{j<-J_1} + \sum_{j>J_2}\Big)\int_{(-a^{j+2}\delta, -a^j\delta)\cup(a^j\delta, a^{j+2}\delta)} |\hat{f}(\xi)|^2 \mathrm{d}\xi \\ &\ge \sum_{j=-J_1}^{J_2} |\langle D_{a^{-j}} f, T_k\psi\rangle|^2 + \Big(\int_{|\xi|<a^{-J_1+1}\delta} + \int_{|\xi|>a^{J_2+1}\delta}\Big)|\hat{f}(\xi)|^2 \mathrm{d}\xi. \end{aligned}\tag{6.9}$$

By (6.2),

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} |\langle D_{a^{-j}} f, T_k\psi_b\rangle|^2 \\ &= a^j\int_{K_\delta} |\hat{f}(a^j(\xi-1)) + \hat{f}(a^j\xi)|^2 \mathrm{d}\xi + a^j\int_\delta^{1-a^2\delta} |\hat{f}(a^j\xi)|^2 \mathrm{d}\xi + a^j\int_{a^2\delta}^{1-\delta} |\hat{f}(a^j(\xi-1))|^2 \mathrm{d}\xi. \\ &\qquad = \int_{a^j K_\delta} |\hat{f}(\xi - a^j) + \hat{f}(\xi)|^2 \mathrm{d}\xi + \int_{a^j(a^2\delta-1,-\delta)\cup a^j(\delta,1-a^2\delta)} |\hat{f}(\xi)|^2 \mathrm{d}\xi. \end{aligned}\tag{6.10}$$

Take any $f \in L^2(\mathbb{R})$ with $\|f\| = 1$ and let $\eta = \|S^q_\Lambda(f)\|^2$. By equations (6.9) and (6.10), $\int_Z |\hat{f}|^2 \leq \eta$, where

$$Z = \{\xi : |\xi| < a^{-J_1+1}\delta\} \cup \{\xi : |\xi| > a^{J_2+1}\delta\} \cup \bigcup_{j=-J_1}^{J_2} \{\xi : a^j\delta < |\xi| < a^j(1-a^2\delta)\}.$$

Using (6.10) one can show that

$$\begin{aligned} I_j &:= \int_{a^j\delta<|\xi|<a^{j+1}\delta} |\hat{f}(\xi)|^2 \mathrm{d}\xi \\ &\leq 2\int_{a^{j+1}\delta<|\xi|<a^{j+2}\delta} |\hat{f}(\xi)|^2\mathrm{d}\xi + 2\int_{a^jK_\delta} |\hat{f}(\xi-a^j)+\hat{f}(\xi)|^2\mathrm{d}\xi + \int_{a^j\delta<|\xi|<a^j(1-a^2\delta)} |\hat{f}(\xi)|^2\mathrm{d}\xi \\ &\leq 2\int_{a^{j+1}\delta<|\xi|<a^{j+2}\delta} |\hat{f}(\xi)|^2\mathrm{d}\xi + 2\eta. \end{aligned} \tag{6.11}$$

Thus, we have a bound $I_j \leq 2(I_{j+1} + \eta)$. Combining this with the fact that $I_{J_2+1} \leq \eta$ yields $I_j \leq 6 \cdot 2^{J_2-j}\eta$ for $j \leq J_2$. Consequently,

$$\|f\|^2 \leq \int_Z |\hat{f}(\xi)|^2\mathrm{d}\xi + \sum_{j=-J_1+1}^{J_2} I_j \leq 6 \cdot 2^{J_1+J_2}\eta.$$

This proves that the frame operator S^q_Λ of $\mathcal{A}^q_\Lambda(\psi)$ is bounded from below by a constant depending only on J_1 and J_2, thus completing the proof of Theorem 6.1. □

Remark 9. By Theorem 3.8, the frame bounds of the quasi-affine systems $\mathcal{A}^q_\Lambda(\psi)$ are not uniform for all $\Lambda \subset \mathbb{Z}$. More precisely, the lower frame bound of $\mathcal{A}^q_\Lambda(\psi)$ must approach 0 for some choice of sparser and sparser lattices Λ. By analyzing the proof of Theorem 6.1 it is not difficult to show that this happens for the family of lattices $\Lambda_J = (pq)^J\mathbb{Z}$ as $J \to \infty$. This is due to the fact that in this case the quasi-affine system $\mathcal{A}^q_\Lambda(\psi)$ coincides with the affine system $\mathcal{A}(\psi)$ at the scales $-J \leq j \leq J$ and the same argument as in the first part of the proof of Theorem 6.1 applies.

Theorem 6.1 says that the lower frame bound is not preserved in general when we move from a quasi-affine system $\mathcal{A}^q_\Lambda(\Psi)$ to the corresponding affine system $\mathcal{A}(\Psi)$ for rational non-integer dilations. It is not known whether the same could happen with the upper bound. This leads to the following open problem.

Question 1. Let $\Psi \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{Q})$. Suppose that $\mathcal{A}^q_{\Lambda_0}(\Psi)$ is a Bessel sequence for some oversampling lattice $\Lambda_0 \subset \mathbb{Z}^n$. Is $\mathcal{A}(\Psi)$ necessarily a Bessel sequence?

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