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Gabor frames with reduced redundancy

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Abstract:

Considering previous constructions of pairs of dual Gabor frames, we discuss ways to reduce the redundancy. The focus is on B-spline type windows.

1. Introduction

We will consider Gabor systems in $L^2(\mathbb{R})$, i.e., families of functions $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$, where

$$E_{mb}T_n g(x) := e^{2\pi imbx}g(x-na).$$

If there exists a constant $B>0$ such that

$$\sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_n g\rangle|^2 \le B\,||f||^2,\ \forall f\in L^2(\mathbb{R}),$$

then $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is called a Bessel sequence. If there exist two constants $A, B>0$ such that

$$A\,||f||^2 \le \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_n g\rangle|^2 \le B\,||f||^2,\ \forall f\in L^2(\mathbb{R}),$$

then $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is called a frame. If $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a frame with dual frame $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$, then

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_n h\rangle E_{mb}T_n g,\ f\in L^2(\mathbb{R}),$$

where the series expansion converges unconditionally in $L^2(\mathbb{R})$.
Our starting point is the duality condition for Gabor frames, originally due to Ron and Shen [4]. We use the version due to Janssen [3]:

Lemma 1..1 *Two Bessel sequences $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ form dual Gabor frames for $L^2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x-n/b+k)}h(x+k) \quad = \quad b\delta_{n,0} \qquad (1..1)$$

for a.e. $x\in[0,1]$.

The Bessel condition in Lemma 1..1 is always satisfied for bounded windows with compact support, see [1]. Note that if g and h have compact support, we only need to check a finite number of conditions in (1..1). In this paper we will usually choose b so small that only the condition for $n=0$ has to be verified.

2. The range $\frac{1}{2N-1} < b < \frac{1}{N}$

We first cite a result from [2]. It yields an explicit construction of dual Gabor frames:

Theorem 2..1 *Let $N\in\mathbb{N}$. Let $g\in L^2(\mathbb{R})$ be a real-valued bounded function with supp $g\subset[0,N]$, for which*

$$\sum_{n\in\mathbb{Z}} g(x-n) = 1. \qquad (2..1)$$

Let $b \in]0, \frac{1}{2N-1}]$. Consider any scalar sequence $\{a_n\}_{n=-N+1}^{N-1}$ for which

$$a_0 = b \ \text{ and } \ a_n + a_{-n} = 2b,\ n = 1, 2, \cdots N-1, \quad (2..2)$$

and define $h\in L^2(\mathbb{R})$ by

$$h(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n). \qquad (2..3)$$

Then g and h generate dual frames $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

The above result can be extended:

Corollary 2..2 *Consider any $b\le 1/N$. With g and a_n as in Theorem 2..1, the function*

$$h(x) = \left(\sum_{n=-N+1}^{N-1} a_n g(x+n)\right)\chi_{[0,N]}(x) \qquad (2..4)$$

is a dual frame generator of g.

Proof. Consider the condition (1..1) for $n=0$; only the values of $h(x)$ for $x\in[0,N]$ play a role, so since the condition holds for the function in (2..3), it also holds for the function in (2..4). □

The cut-off in (2..4) yields a non-smooth function. However, for any $b<1/N$, we might modify h slightly and obtain a smooth dual generator:
In particular, we obtain the following:

Corollary 2..3 *Consider any $b<1/N$, and take $\epsilon < 1/b-N$. With g as in Theorem 2..1, the function $h(x) = b, x\in[0,N]$ has an extension to a function of desired smoothness, supported on $[-\epsilon, N+\epsilon]$, which is a dual frame generator of g.*

Proof. The choice $a_n = b,\ n = -N+1,\ldots,N-1$, leads to

$$\sum_{n=-N+1}^{N-1} a_n g(x+n) = b,\ x \in [0,N].$$

Given $\epsilon < 1/b - N$ and any functions $\phi_1 : [-\epsilon, 0[\to \mathbb{R}$ and $\phi_2 :]N, N+\epsilon] \to \mathbb{R}$, the function

$$h(x) = \begin{cases} \phi_1(x), & x \in [-\epsilon, 0[, \\ \sum_{n=-N+1}^{N-1} a_n g(x+n) = b, & x \in [0,N], \\ \phi_2, & x \in]N, N+\epsilon], \\ 0, & x \notin [-\epsilon, N+\epsilon], \end{cases}$$

will satisfy (1..1); in fact, for $n \neq 0$, the support of the functions $g(\cdot \pm n/b)$ and h are disjoint, and for $n = 0$ we are (for all relevant values of x) back at the function in (2..4). The functions ϕ_1 and ϕ_2 can be chosen such that the function h has the desired smoothness. □

The assumptions in Theorem 2..1 are tailored to B-splines, defined inductively by

$$B_1 := \chi_{[0,1]}, \quad B_{N+1} := B_N * B_1.$$

Direct calculations shows that

$$B_2(x) = \begin{cases} x & \text{if } x \in [0,1], \\ 2-x & \text{if } x \in [1,2], \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B_3(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \in [0,1], \\ -x^2+3x-\frac{3}{2} & \text{if } x \in [1,2], \\ \frac{1}{2}x^2-3x+\frac{9}{2} & \text{if } x \in [2,3], \\ 0 & \text{otherwise.} \end{cases}$$

In general, the functions B_N are $(N-2)-$times differentiable piecewise polynomials (explicit expressions are known). Furthermore, supp $B_N = [0,N]$, and the partition of unity condition (2..1) is satisfied.
In case $g = B_N$, the dual generators in Theorem 2..1 are splines, of the same smoothness as B_N itself. By compressing the function $\sum_{n=-N+1}^{N-1} a_n g(x+n)$ from the interval $[-N+1, 0]$ to $[-\epsilon, 0]$ and from $[N, 2N-1]$ to $[N, N+\epsilon]$ we obtain a dual in (2..3) with the same features:

Example 2..4 For the B-spline $B_3(x)$ and $b = 1/5$, Theorem 2..1 yields the symmetric dual

$$h_3(x) = \frac{1}{5}\begin{cases} 1/2\,x^2 + 2\,x + 2, & x \in [-2,-1[, \\ -1/2\,x^2 + 1, & x \in [-1,0[, \\ 1, & x \in [0,3[, \\ -1/2\,x^2 + 3\,x - 7/2, & x \in [3,4[, \\ 1/2\,x^2 - 5x + 25/2, & x \in [4,5[, \\ 0, & x \notin [0,5[. \end{cases} \tag{2..5}$$

See Figure 1.
Now, for $b = 1/4$, we can use Corollary 2..3 for $\epsilon < 4 - 3 = 1$. Taking $\epsilon = 1/2$, we compress the function

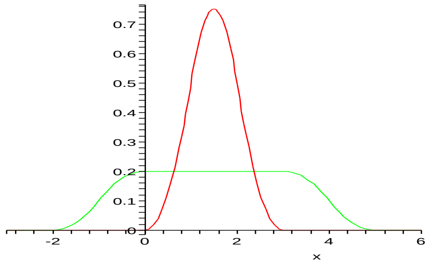

Figure 1: B_3 and the dual generator h_3 in (2..5).

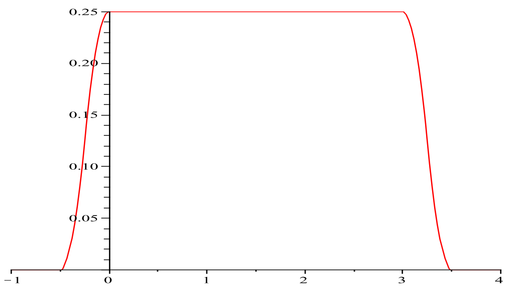

Figure 2: The function h in (3..13)..

h_3 in (2..5) from $[-2, 0]$ to $[-1/2, 0]$ and from $[3, 5]$ to $[3, 31/2]$ and obtain the dual

$$h(x) = \frac{1}{4}\begin{cases} 1/2\,(4x)^2 + 2\,(4x) + 2, & x \in [-1/2,-1/4[, \\ -1/2\,(4x)^2 + 1, & x \in [-1/4,0[, \\ 1, & x \in [0,3[, \\ -1/2\,(4(x-3)+3)^2 + 3\,(4(x-3)+3) - 7/2, & x \in [3,3+1/4[, \\ 1/2\,(4(x-3)+3)^2 - 5(4(x-3)+3) + 25/2, & x \in [3+1/4,3+1/2[, \\ 0, & x \notin [-1/2,3+1/2[. \end{cases}$$

$$= \frac{1}{4}\begin{cases} 8\,x^2 + 8\,x + 2, & x \in [-1/2,-1/4[, \\ -8\,x^2 + 1, & x \in [-1/4,0[, \\ 1, & x \in [0,3[, \\ -8\,x^2 + 48\,x - 71, & x \in [3,3+1/4[, \\ 8\,x^2 - 56\,x + 98, & x \in [3+1/4,3+1/2[, \\ 0, & x \notin [-1/2,3+1/2[. \end{cases}$$

See Figure 2. □

3. B_2 and $1/2 < b < 1$

In the following discussion, we consider dual windows associated with a Gabor frame $\{E_{mb}T_nB_2\}_{m,n\in\mathbb{Z}}$ generated by the B-spline B_2. The arguments can be extended to general functions supported on $[0,2]$. Take any function h with values specified only on $[0,2]$ and such that

$$\sum_{k\in\mathbb{Z}} B_2(x+k)h(x+k) = 1,\ x \in [0,1]. \tag{3..1}$$

In fact, due to the support of B_2, only the values for $h(x)$ for $x \in [0,2]$ play a role for that condition. We know that for any $b \leq 1/2$ the function generates – up to a certain scalar multiple – a dual of g.

Now consider any $1/2 < b < 1$; that is, we have $1 < 1/b < 2$.

Lemma 3..1 *Assume that $h(x),\ x \in [0,2]$ is chosen such that* (3..1) *is satisfied. The the following hold:*

(i) If

$$\sum_{k\in\mathbb{Z}} B_2(x-1/b+k)h(x+k) = 0,\ x\in\mathbb{R}, \quad (3..2)$$

and

$$\sum_{k\in\mathbb{Z}} B_2(x+1/b+k)h(x+k) = 0,\ x\in\mathbb{R}, \quad (3..3)$$

then

$$B_2(x-1/b)h(x) + B_2(x-1/b+1)h(x+1) = 0,$$

$$x \in [1/b, 2], \quad (3..4)$$

$$B_2(x+1/b-1)h(x-1) + B_2(x+1/b)h(x) = 0$$

$$x \in [0, 2-1/b]. \quad (3..5)$$

These equations determine $h(x)$ for

$$x \in [-1, 1-1/b] \cup [1+1/b, 3].$$

(ii) If $h(x)$ for $x \in [-1,1-1/b]\cup[1+1/b,3]$ is chosen such that (3..4) *and* (3..5) *are satisfied, and*

$$h(x) = 0, x \notin [0,2]\cup[-1,1-1/b]\cup[1+1/b,3],$$

then (3..2) *and* (3..3) *hold.*

Proof. We consider (3..2) for $x \in [1,2]$, and split into two cases:

For $x \in [1, 1/b]$, (3..2) yields that

$$0 = B_2(x-1/b+1)h(x+1) + B_2(x-1/b+2)h(x+2); \quad (3..6)$$

the equation only involve $h(x)$ for

$$x \in [2, 1+1/b] \cup [3, 2+1/b].$$

For $x \in [1/b, 2]$, (3..2) yields that

$$0 = B_2(x-1/b)h(x) + B_2(x-1/b+1)h(x+1);$$

since $h(x)$ is known, this implies that

$$h(x+1) = \frac{-B_2(x-1/b)h(x)}{B_2(x-1/b+1)},\ x \in [1/b, 2],$$

that is,

$$h(x) = \frac{-B_2(x-1/b-1)h(x-1)}{B_2(x-1/b)},\ x \in [1/b+1, 3].$$

Similarly, considering (3..3) for

$$x \in [0,1] = [0, 2-1/b] \cup [2-1/b, 1]$$

leads to (3..5) and

$$B_2(x+1/b-2)h(x-2) + B_2(x+1/b-1)h(x-1)$$

$$= 0,\ x \in [2-1/b, 1]; \quad (3..7)$$

the equation (3..7) only involves $h(x)$ for

$$x \in [-1/b, -1] \cup [1-1/b, 0],$$

and (3..5) implies that

$$h(x-1) = \frac{-B_2(x+1/b)h(x)}{B_2(x+1/b-1)},\ x \in [0, 2-1/b],$$

i.e.,

$$h(x) = \frac{-B_2(x+1/b+1)h(x+1)}{B_2(x+1/b)},\ x \in [-1, 1-1/b].$$

For the proof of (ii), the condition

$$h(x) = 0, x \notin [0,2]\cup[-1,1-1/b]\cup[1+1/b,3],$$

implies that (3..6) and (3..7) are satisfied. By construction, (3..2) and (3..3) are satisfied. □

Lemma 3..1 shows that if we want that (3..1), (3..2), and (3..3) hold for some $b \in]1/2, 1]$, then h in general will take values outside $[0, 2]$. However, the proof shows that we under certain circumstances can find a solution h having support in $[0, 2]$. In that case, the support will actually be a subset of $[0, 2]$:

Corollary 3..2 *Let $b \in]1/2, 1]$. Assume that supp $h \subseteq [0, 2]$ and that* (3..1) *and* (3..2) *holds. Then*

$$h(x) = 0,\ x \in [0, 2-1/b] \cup [1/b, 2]. \quad (3..8)$$

Proof. According to the proof of Lemma 3..1, we obtain that $h(x) = 0$ on $[1/b+1, 3]$ by requiring that $h(x) = 0$ for $x \in [1/b, 2]$; and we obtain that $h(x) = 0$ on $[-1, 1-1/b]$ by requiring that $h(x) = 0$ for $x \in [0, 2-1/b]$. □

If supp $h \subseteq [0, 2]$, the condition (3..8) implies that h at most can be nonzero on the interval $[2-1/b, 1/b]$ having length $2/b - 2$. In order for (3..1) to hold, this interval must have length at least 1; thus, we need to consider b such that $2/b - 2 \geq 1$, i.e., $b \leq 2/3$. Note that if $b \leq 2/3$, then $2/b \geq 3$: that is, because B_2 and h are supported on $[0, 2]$, Janssen's duality conditions in (1..1) are automatically satisfied for $n = \pm 2, \pm 3, \ldots$.

Corollary 3..3 *Consider $b \in]1/2, 2/3]$. Then there exists a function h with supp $h \subseteq [0, 2]$ such that* (3..1) *and* (3..2) *hold; and $bh(x)$ is a dual generator of B_2 for these values of b.*

Proof. For $x \in [0, 2-1/b] \cup [1/b, 2]$, let $h(x) = 0$. For $x \in [0, 1]$, the equation (3..1) means that

$$xh(x) + (1-x)h(x+1) = 1.$$

This implies that

$$\begin{aligned} xh(x) &= 1, \ x \in [1/b - 1, 1], \\ (1-x)h(x+1) &= 1, \ x \in [0, 2-1/b]; \end{aligned}$$

that is,

$$h(x) = \frac{1}{x}, \ x \in [1/b - 1, 1], \tag{3..9}$$

and

$$h(x) = \frac{1}{2-x}, \ x \in [1, 3-1/b]. \tag{3..10}$$

Finally, for $x \in [2-1/b, 1/b-1]$ and $x \in [3-1/b, 1/b]$, choose $h(x)$ such that

$$xh(x) + (1-x)h(x+1) = 1.$$

By construction, $bh(x)$ is a dual generator. □

For $b = 3/5$ we will now explicitly construct a continuous dual generator h of B_2 with support in $[0, 2]$. Putting Corollary 3..2, (3..9), and (3..10) together, we can state a result about how a dual window supported on $[0, 2]$ must look like on parts of $[0, 2]$:

Lemma 3..4 *For $b = 3/5$, every dual generator of B_2 with support in $[0, 2]$ has the form*

$$h(x) = \begin{cases} 0 & \text{if } x \le 1/3; \\ \frac{1}{x} & \text{if } x \in [2/3, 1]; \\ \frac{1}{2-x} & \text{if } x \in [1, 4/3]; \\ 0 & \text{if } x \ge 5/3. \end{cases}$$

That is, we only have freedom on the definition of h on $]1/3, 2/3[\cup]4/3, 5/3[$.

Note that on $[2/3, 4/3]$, the function h is symmetric around $x = 1$. We will now show that it is possible to define h on $]1/3, 2/3[\cup]4/3, 5/3[$ in such a way that h becomes symmetric around $x = 1$.
First, we note that this form of symmetry means that

$$h(1-x) = h(1+x), \ x \in]1/3, 2/3[. \tag{3..11}$$

Put together with the duality condition, we thus require that

$$xh(x) = 1 - (1-x)h(1-x), \ x \in]1/3, 2/3[. \tag{3..12}$$

The condition (3..12) shows that must define $h(1/2) = 1$. Now, taking any continuous function h defined on $[1/3, 1/2]$ with the properties that $h(1/3) = 0$ and $h(1/2) = 1$, the condition (3..12) shows how to define $h(x)$ on $]1/2, 2/3[$; and, finally, the condition (3..11) shows how to define h on $]4/3, 5/3[$ such that the resulting function is a symmetric dual generator.

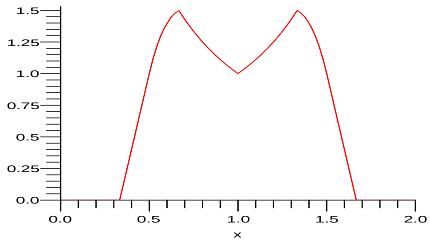

Figure 3: The function h in (3..13)..

Put

$$h(x) = 6x - 2, \ x \in [1/3, 1/2].$$

Then, for $x \in [1/2, 2/3]$,

$$\begin{aligned} h(x) &= \frac{1 - (1-x)h(1-x)}{x} \\ &= \frac{-6x^2 + 10x - 3}{x}. \end{aligned}$$

The condition $h(1+x) = h(1-x), \ x \in]1/3, 2/3[$ can also be expressed as $h(x) = h(2-x), \ x \in]4/3, 5/3[$. Thus, for $x \in [4/3, 3/2]$ we arrive at

$$h(x) = h(2-x) = \frac{-6x^2 + 14x - 7}{2-x}, \ x \in [4/3, 3/2];$$

while, for $x \in [3/2, 5/3]$,

$$h(x) = h(2-x) = 6(2-x) - 2 = 10 - 6x.$$

We have arrived at the following conclusion:

Lemma 3..5 *For $b = 3/5$, the function*

$$h(x) = \begin{cases} 0 & \text{if } x \le 1/3; \\ 6x-2 & \text{if } x \in [1/3, 1/2]; \\ \frac{-6x^2+10x-3}{x} & \text{if } x \in [1/2, 2/3]; \\ \frac{1}{x} & \text{if } x \in [2/3, 1]; \\ \frac{1}{2-x} & \text{if } x \in [1, 4/3]; \\ \frac{-6x^2+14x-7}{2-x} & \text{if } x \in [4/3, 3/2]; \\ 10-6x & \text{if } x \in [3/2, 5/3]; \\ 0 & \text{if } x \ge 5/3 \end{cases} \tag{3..13}$$

is a continuous symmetric dual generator of B_2.

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