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Shell model for time-correlated random advection of passive scalars

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We study a minimal shell model for the advection of a passive scalar by a Gaussian time-correlated velocity field. The anomalous scaling properties of the white noise limit are studied analytically. The effect of the time correlations are investigated using perturbation theory around the white noise limit and nonperturbatively by numerical integration. The time correlation of the velocity field is seen to enhance the intermittency of the passive scalar. [S1063-651X(99)07711-9]

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I. INTRODUCTION

The advection of a scalar observable $\theta(x,t)$ by a velocity field $\mathbf{v}$ is described in classical hydrodynamics by the linear partial differential equation (PDE)

$$\partial_t \theta + \mathbf{v}\cdot\boldsymbol{\nabla}\theta = \kappa \boldsymbol{\nabla}^2 \theta + \mathbf{f}. \tag{1.1}$$

If $\mathbf{v}$ is assumed to be the solution of the Navier-Stokes equations in a turbulent regime and the Péclet number Pe, which measures the ratio between the strength of the advective effects and the molecular diffusion κ in (1.1), is large,

$$\mathrm{Pe} \equiv \frac{Lv}{\kappa} \gg 1$$

(L and v are the characteristic length and advection velocity in the problem), and if a steady state is reached, an inertial range sets in where both the effects of the forcing f limited to the large scales and those of the molecular diffusion acting mainly on the small scales can be neglected. In the inertial range no typical scale is supposed to characterize the flow. As a consequence, the structure functions of the scalar field

$$S_p(\mathbf{r}) = \langle [\theta(\mathbf{x}+\mathbf{r}) - \theta(\mathbf{x})]^p \rangle \tag{1.2}$$

display a power law behavior in the inertial range with anomalous scaling exponents $H(p)$ [1]. The word anomalous means that the exponents $H(p)$ deviate from the linear behavior predicted by a direct scaling analysis of Eq. (1.1).

It was first realized by Kraichnan [2] that anomalous scaling can be observed in the mathematically more tractable case of the advection by a random homogeneous and isotropic Gaussian velocity field, which is delta correlated (white noise) in time and has zero average and covariance in d dimensions given by

$$\langle \mathbf{v}_i(\mathbf{x},t)\mathbf{v}_j(\mathbf{y},s)\rangle = \delta(t-s)\left[D_{i,j}(0) - D_0|\mathbf{x}-\mathbf{y}|^{\xi_{wn}}(d-1+\xi_{wn})\delta_{i,j} + \xi_{wn}\frac{(x-y)_i(x-y)_j}{|\mathbf{x}-\mathbf{y}|^2}\right].$$

The power law behavior of the covariance mimics an infinite inertial range for the velocity field. The scaling exponent ξ_{wn} is a free parameter characterizing the degree of turbulence of the advecting field. The physically meaningful values range from 0 to 2. In the first limit the effect of the random advection is just to define an effective diffusion constant [3]. In the latter case the velocity increments are smooth, as expected for a laminar flow. The choice ξ_{wn} equal to $\frac{4}{3}$ represents the scaling of the velocity field conjectured by Kolmogorov for the solution of the Navier-Stokes equation in the turbulent regime.

The hypothesis of delta correlation in time is of great mathematical advantage, for it allows one to write the equations of motion of the scalar correlations in a linear closed form. The evolution of each correlation in the inertial range is specified by a linear differential operator, the inertial operator, plus matching conditions at the boundary of the inertial range. The occurrence of anomalous scaling has been related to the existence of zero modes of the inertial operators dominating the scaling properties of higher order correlations ([3–6] and, for a recent review and more complete bibliography, [7]). The behavior of the anomaly has also been numerically measured for the fourth order structure function versus the turbulence parameter ξ_{wn} [8]. However, implementing accurate numerical experiments still remains a difficult task. Therefore, it turns out to be useful to use the shell model as a laboratory in which to test ideas and results related to the full PDE model (see [9] for a general introduction to the shell model concept). In [10,11] two different shell models advected by a delta-correlated velocity field mimicking the Kraichnan model were constructed. Anoma-

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lous scaling was observed numerically and in the simpler case [11] it was proven analytically that the anomaly of the fourth order structure function is related to the anomalous scaling of the dominant zero mode of the inertial operator.

The passive scalar advection by a white noise velocity field is a useful mathematical model, but is still very far from being a physical realistic velocity field possessing both time correlations and deviations from Gausianity. A first small step in this direction is made by investigating how the introduction of a time correlation in a Gaussian velocity field affects the statistical properties of the scalar field.

In the present paper we introduce a time-correlated velocity field in a shell model. This is done by replacing the white noise with the Ornstein-Uhlenbeck process, which provides exponentially decaying time correlations (Sec. II). We investigate the model both analytically and numerically. By means of stochastic variational calculus, which we review in Appendixes A and B, we show how to rewrite the equations of motion for the scalar correlations in integral nonclosed form. Such an operation allows evaluation of the correction to the white noise inertial operator stemming from the time-correlated velocity field. This procedure has the further advantage that it creates a nonambiguous relationship between the coupling terms for the scaling exponent ξ_{wn} of white noise advection to the scaling exponent ξ of the time-correlated velocity field (Sec. III).

The inertial operators can be expanded around the white noise limit in powers of an adimensional parameter which is interpreted as proportional to the ratio ϵ between the time correlation and the turnover time of the advecting field. We focus on the features of the steady state. There we assume that the averages over the Ornstein-Uhlenbeck process of all the observables are time-translational invariant. As a consequence, the inertial operators become linear up to any finite order in ϵ.

In the white noise case, when ϵ is equal to zero, we generalize the procedure first introduced in [11] and we show that the scaling of the zero modes of the inertial operator of any order is captured by focusing on nearest-shell interactions. The equations are closed with a scaling Ansatz (Sec. IV) by postulating that the scalar field is "close" to a multiplicative process. Furthermore, we perturb the closure scheme in order to extract the first order corrections in ϵ to the anomalous exponents for different values of ξ ranging from zero to two. The prediction of perturbation theory is an ϵ dependence (nonuniversality) of the exponents except for the second order $H(2)$ (Sec. V). The overall result is analogous to the one obtained in [12], where a Gaussian time-correlated velocity field is considered for the advection of the scalar field in Eq. (1.1): the introduction of time correlation is seen to enhance intermittency. The anomalies vanish smoothly in the laminar limit $\xi=2$.

To examine the validity of the results from the analytical calculations and explore the regime with long time correlations ($\epsilon \gg 0$), we turned to numerical experiments. The occurrence of corrections to the anomalies predicted by the perturbation theory for small values of ϵ is confirmed. However, strong nonperturbative effects set in and dominate when the expansion parameter becomes of the order of unity.

II. MODEL

The model is defined by the equations ($m=1,2,\ldots,N$)

$$\left[\frac{d}{dt}+\kappa k_m^2\right]\theta_m(t)-\delta_{1m}f(t)=i[k_{m+1}\theta^*_{m+1}(t)u^*_m(t) - k_m\theta^*_{m-1}(t)u^*_{m-1}(t)], \tag{2.1}$$

$$u_m(t)=\frac{v_m}{\epsilon\sqrt{\tau_m}}\int_0^t ds\, e^{-(t-s)/(\epsilon\tau_m)}\eta_m(s), \tag{2.2}$$

$$f(t)=\frac{\tilde{f}}{\epsilon\sqrt{\tau}}\int_0^t ds\, e^{-(t-s)/(\epsilon\tau)}\eta(s), \tag{2.3}$$

where the asterisk denotes complex conjugation and the $\eta_m(t)$'s and $\eta(t)$ are independent white noises with zero mean value and correlation:

$$\langle \eta_m(t)\eta^*_n(s)\rangle=2\delta_{mn}\delta(t-s) \quad \text{and} \quad \langle \eta(t)\eta^*(s)\rangle=2\delta(t-s). \tag{2.4}$$

The boundary conditions are $\theta_0=\theta_{N+1}=0$. The model can be regarded as a severe truncation of the equation of the passive scalar (1.1) in Fourier space. The field component θ_m is the representative of all the Fourier modes in the shell with a wave number ranging between $k_m=k_0\lambda^m$ and $k_{m+1}=k_0\lambda^{m+1}$. The parameter λ is the ratio between two adjacent scales and it is usually taken equal to two in order to identify each shell with an octave of wave numbers. The energy transfer in a turbulent flow is conjectured to occur mainly through the interactions of eddies of the same size. As a consequence the interactions in Fourier space are assumed to be local. The "localness" conjecture [1] is the motivation for the restriction to nearest neighbors of the couplings among the shells.

In the absence of external forcing and dissipation, the total "energy" of the passive field is conserved:

$$\frac{d}{dt}E=\frac{d}{dt}\sum_{m=1}^{N}|\theta_m|^2=0 \quad \text{for} \quad f(t)=\kappa=0. \tag{2.5}$$

Far from the infrared and the ultraviolet boundaries (i.e., for $1\ll m\ll N$) the conservation of energy is expected to hold approximately, giving rise to an inertial range. Equations (2.2) and (2.3) describe the random evolution according to Ornstein-Uhlenbeck (OU) processes of, respectively, the advecting and external force fields. The OU process has differentiable realizations, thus resulting in the random differential equations with multiplicative noise that specify the dynamics of the scalar θ independent of the discretization prescription.

The velocity correlations are for $t\geq s$

$$\langle u_m(t)u^*_m(s)\rangle=\frac{|v_m|^2}{\epsilon}(e^{-(t-s)/(\epsilon\tau_m)}-e^{-(t+s)/(\epsilon\tau_m)}). \tag{2.6}$$

In the limit of large t only the stationary part survives. The adimensional parameter ϵ appearing in the definition of the

OU processes (2.2) and (2.3) defines the strength of the time correlation in units of the typical times τ_m. In the white noise limit one has

$$\lim_{\epsilon\downarrow 0}\langle u_m(t)u_m^*(s)\rangle=2|v_m|^2\delta\left(\frac{t-s}{\tau_m}\right). \tag{2.7}$$

For any finite ϵ ordinary differential calculus holds true: the consistency conditions yield a Stratonovich discretization prescription when ϵ is set to zero and the recovery of the white noise advection model of [11]. Hence the factor 2 in (2.7) always cancels in computations for the δ distribution is evaluated at one of the boundaries of the domain of integration.

Information about the scaling of the correlations of the velocity field at equal times is stored in the constants v_m. We assume the power law behavior

$$|v_m|\propto k_m^{-\xi/2}. \tag{2.8}$$

Kolmogorov scaling is specified by $\xi=2/3$ while $\xi=2$ corresponds to a laminar regime. The τ_m's in Eq. (2.2) describe the typical correlation times for the random velocity field. A simple physical interpretation is to identify them with the turnover times, i.e., with the typical time rates of variation through nonlinearity of the advection field on each shell [13]:

$$\tau_m\sim\frac{1}{k_m|v_m|}\propto k_m^{-1+(\xi/2)}. \tag{2.9}$$

The scaling of the correlation times is then fully specified in terms of the parameter ξ. It is worth noting that for any ξ less than 2 the τ_m's are always decreasing functions of the wave number.

The evolution of the scalar θ is determined in the inertial range by its complex conjugate. It is useful to introduce a unified notation for the $2N$ degrees of freedom. With $\Theta=\theta\oplus\theta^*$ and $U=u\oplus u^*$ one has for the N shells

$$\frac{d}{dt}\Theta_\alpha=\sum_{\beta=1}^{2N}\left[A_{\alpha,\beta}+\sum_{\gamma=1}^{2N}B_{\alpha,\beta}^{\gamma}U_\gamma\right]\Theta_\beta+f\delta_{\alpha,1}+f^*\delta_{\alpha,N+1}, \tag{2.10}$$

with

$$A_{m,\beta}=-\kappa k_m^2\delta_{m,\beta},$$

$$A_{N+m,\beta}=-\kappa k_m^2\delta_{m,n},$$

$$B_{\alpha,\beta}^{m}=-ik_{m+1}[\delta_{\beta,m+1}\delta_{\alpha,N+m}-\delta_{\beta,m}\delta_{\alpha,N+m+1}],$$

$$B_{\alpha,\beta}^{N+m}=ik_{m+1}[\delta_{\beta,N+m+1}\delta_{\alpha,m}-\delta_{\beta,N+m}\delta_{\alpha,m}], \tag{2.11}$$

where Latin and Greek indices range respectively from 1 to N and from 1 to $2N$. The set of matrices with constant entries B^γ do not commute within each other and with the A matrix. The known sufficient condition (see, for example, [14]) to have a solution of Eq. (2.10) in an analytic exponential form is therefore not satisfied. From the geometrical point of view, noncommutativity means that the dynamics is confined to a manifold that turns into a hypersphere in $\mathcal{C}^N$ in the inertial limit (2.5).

The complex equations (2.10) are invariant under phase transformations. Given two diagonal Hermitian $2N\times 2N$ matrices with time independent random entries

$$T\equiv\mathrm{diag}(e^{i\phi_1},\ldots,e^{i\phi_N},e^{-i\phi_1},\ldots,e^{-i\phi_N}), \tag{2.12}$$

$$S\equiv\mathrm{diag}(e^{-i(\phi_1+\phi_2)},\ldots,e^{-i(\phi_{N-1}+\phi_N)},0,e^{i(\phi_1+\phi_2)},\ldots,e^{i(\phi_{N-1}+\phi_N)},0), \tag{2.13}$$

if Θ is a realization of the solution of the equations of motion, then

$$T\Theta(U)=\Theta(SU) \tag{2.14}$$

is still a solution. The phase symmetry is the remnant of the translational invariance of the original hydrodynamical equations in real space [9]. From the phase symmetry (2.14) it follows that at stationarity the only analytic nonzero moments of the correlation are of the form

$$C_{m_1,\ldots,m_\omega}^{(2\omega)}=\langle\Pi_{i=1}^{\omega}\Theta_{m_i}\Theta_{N+m_i}\rangle\equiv\langle\Pi_{i=1}^{\omega}|\theta_{m_i}|^2\rangle. \tag{2.15}$$

In the inertial range such quantities display a power law behavior. The diagonal sector of the moments whose scaling properties are specified by the exponents $H(2\omega)$

$$C_{m,\ldots,m}^{(2\omega)}\propto k_m^{-H(2\omega)} \tag{2.16}$$

is in the shell model context, the analog of the structure functions (1.2) of the original PDE model (1.1). The exponents $H(2\omega)$'s are said to be normal if they can be derived from dimensional analysis. Under the assumption that a steady state is reached, one matches the scaling of the inertial terms in Eq. (2.1) with a power law decay of the solution

$$k_{m+1}k_m^{-(\xi/2)}\theta_{m+1}-k_mk_{m-1}^{-(\xi/2)}\theta_{m-1}\sim 0. \tag{2.17}$$

The resulting prediction is a linear behavior of the exponents versus the order ω of the diagonal correlation:

$$H(2\omega)=\omega\left(1-\frac{\xi}{2}\right). \tag{2.18}$$

The scaling argument (2.17) neglects completely the random fluctuations of the passive scalar field. Normal scaling holds if the statistics of the θ field are Gaussian. Deviations from normal scaling are then correlated with the occurrence of intermittency corrections to the Gaussian statistics. A systematic account of the fluctuations is provided by the study of the equations of motion satisfied by the moments of the scalar field.

III. EQUATIONS OF MOTION OF THE FIELD MOMENTS

In the white noise limit, ϵ equals zero; the Furutsu-Donsker-Novikov formula [1] and the delta correlation in time of the velocity ensure that the moments $C^{(2\omega)}$ are specified by the solutions of closed linear systems [10,11]. In the presence of finite time correlations, stochastic calculus of variations [15,16] allows one to write nonclosed integrodifferential equations for the correlations. A typical functional integration by parts relation is

$$\langle F(\Theta(t))U_{N+m}(t)\rangle=\int_0^t ds\langle U_{N+m}(t)U_m(s)\rangle \times\left\langle \frac{dF(\Theta(t))}{d\Theta_\alpha(t)}R_{\alpha,\beta}(t,s)B^m_{\beta,\gamma}\Theta_\gamma(s)\right\rangle, \tag{3.1}$$

where Einstein convention holds for repeated *Greek* indices. The matrix R is the fundamental solution of the homogeneous system associated with Eq. (2.10). A heuristic proof of the stochastic integration by parts formula and of Eq. (3.1) is provided in Appendixes A and B.

Let us start with the second moment of the scalar field,

$$C^{(2)}_m(t)\doteq\langle\Theta_m(t)\Theta_{N+m}(t)\rangle\equiv\langle\theta_m(t)\theta^*_m(t)\rangle. \tag{3.2}$$

From the equations of motion (2.10) one has

$$\left[\frac{d}{dt}+2\kappa k_m^2\right]C^{(2)}_m(t)-2\,\mathrm{Re}\{\langle\Theta_{N+m}(t)f(t)\rangle\}\delta_{m,1} =2\,\mathrm{Re}\{ik_{m+1}\langle U_{N+m}(t)\Theta_{N+m+1}(t)\Theta_{N+m}(t)\rangle\} -2k_m\,\mathrm{Re}\{i\langle U_{N+m-1}(t)\Theta_{N+m-1}(t)\Theta_{N+m-1}(t)\rangle\}. \tag{3.3}$$

The integration by parts formula (3.1) gives

$$\left[\frac{d}{dt}+2\kappa k_m^2\right]C^{(2)}_m(t)-2\delta_{m,1}\mathrm{Re}\int_0^t ds\langle f(t)f(s)^*\rangle \times\langle R_{N+m,N+1}(t,s)\rangle =2k^2_{m+1}\tau_m\int_0^t ds\frac{\langle U_m(t)U_{N+m}(s)\rangle}{\tau_m}\mathrm{Re}\,\mathcal{F}^{(2)}_m(t,s) -2k_m^2\tau_{m-1}\int_0^t ds\frac{\langle U_{m-1}(t)U_{N+m-1}(s)\rangle}{\tau_{m-1}} \times\mathrm{Re}\,\mathcal{F}^{(2)}_{m-1}(t,s), \tag{3.4}$$

where $m=1,\ldots,N$, Re is the real part, and

$$\mathcal{F}^{(2)}_m(t,s)\doteq G^{(2)}_{N+m+1,N+m;N+m,m+1}(t,s) -G^{(2)}_{N+m+1,N+m+1;N+m,m}(t,s) +G^{(2)}_{N+m,N+m;N+m+1,m+1}(t,s) -G^{(2)}_{N+m,N+m+1;N+m+1,m}(t,s), \tag{3.5}$$

$$G^{(2)}_{N+m,N+n;N+p,q}(t,s) \doteq\sum_{\alpha=1}^{2N}\langle\Theta_{N+p}(t)R_{N+m+1,\alpha}(t,0)R^{-1}_{\alpha,N+n}(s,0)\Theta_q(s)\rangle, \tag{3.6}$$

$$d_m\doteq|v_m|^2\tau_m\propto k_m^{-(1+\xi/2)}. \tag{3.7}$$

When a steady state is reached, the left-hand side (lhs) of Eq. (3.4) can be neglected through the whole inertial range. The rhs specifies the inertial operator of the theory. A further simplification is attained in the limit of very large shell number. For any ξ less than 2, one has

$$\lim_{m\uparrow\infty}\frac{\langle U_m(t)U_{N+m}(s)\rangle}{\tau_m}\equiv\lim_{m\uparrow\infty}\frac{\langle u_m(t)u^*_m(s)\rangle}{\tau_m}=|v_m|^2\delta(t-s) \tag{3.8}$$

independently of ϵ. At equal times the resolvent matrix R reduces to the identity. From Eqs. (2.8) and (2.9) it follows that

$$k^2_{m+1}d_m\tau_m=\lambda^2. \tag{3.9}$$

Hence for m going to infinity the inertial operator is linearized in the form

$$I(C^{(2)}_m)=2\frac{\lambda^2}{\tau_m}(C^{(2)}_{m+1}-C^{(2)}_m)-2\frac{\lambda^2}{\tau_{m-1}}(C^{(2)}_m-C^{(2)}_{m-1}). \tag{3.10}$$

The slowest decay scaling solution compatible with a zero lhs is

$$C^{(2)}_m\propto\tau_m=k_m^{-H(2)}. \tag{3.11}$$

In other words, we have proven that the scaling of the second moment is normal since it coincides with the dimensional prediction (2.18). Moreover since the result does not depend on ϵ, it is universal versus the time correlation. It is worth stressing that the derivation of Eq. (3.11) requires that each of the terms appearing in Eq. (3.10) has separately a finite nonzero limit for m going to infinity. The condition turns out not to be self-consistent when the same reasoning is applied to moments higher than the second.

An important consequence of normal scaling of the $C^{(2)}_m$'s is the Obukhov-Corrsin [17,18] law for the decay of the power spectrum $\Gamma(k)$ of the passive scalar if the Kolmogorov scaling is assumed for the advecting field:

$$\Gamma(k)=\frac{d}{dk}\sum_{k_n\leqslant k}\langle(\theta_n\theta^*_n)^2\rangle\propto k^{-(H(2)+1)}\big|_{\xi=2/3}=k^{-5/3}. \tag{3.12}$$

A second interesting limit is when ϵ tends to zero. Neglecting all nonstationary contributions to the velocity correlations the rhs of Eq. (3.4) becomes

$$I(C^{(2)}_m)=2k^2_{m+1}d_m(C^{(2)}_{m+1}-C^{(2)}_m)-2k^2_m d_{m-1}(C^{(2)}_m-C^{(2)}_{m-1}) -2k^2_{m+1}d_m\int_0^t ds\,e^{-(t-s)/(\epsilon\tau_m)}\frac{d}{ds}\mathrm{Re}\,\mathcal{F}^{(2)}_m(t,s) +k^2_m d_{m-1}\int_0^t ds\,e^{-(t-s)/(\epsilon\tau_{m-1})}\frac{d}{ds}\mathrm{Re}\,\mathcal{F}^{(2)}_{m-1}(t,s). \tag{3.13}$$

If ϵ is set exactly to zero the integral terms disappear and the white noise equations of [11] are recovered. The information about the scaling of the velocity field is absorbed in the d_m's. In a pure white noise theory it is convenient to redefine the turbulence parameter as

$$\xi_{wn}=1+\frac{\xi}{2}. \tag{3.14}$$

A Kolmogorov scaling of the velocity field corresponds to ξ_{wn} equal to $\frac{4}{3}$, which is also the value giving the Obukhov-Corrsin scaling in Eq. (3.12). The two definitions of the degree of turbulence coincide for ξ equal to two (Batchelor limit). It is natural to identify ξ_{wn} with the turbulence parameter of the Kraichnan model. The correspondence fixes the physical range of ξ between $[-2,2]$.

In the general case of the 2ωth even moment of the scalar $C^{(2\omega)}$ one has

$$\begin{aligned}I\Bigg(C^{(2\omega)}_{m_1,\ldots,m_\omega} &= \sum_{q_1,\ldots,q_\omega} I^{(2\omega;0)}_{m_1,\ldots,m_\omega,q_1,\ldots,q_\omega} C^{(2\omega)}_{q_1,\ldots,q_\omega}\\
&-\sum_{i=1}^{\omega} 2k^2_{m_i+1} d_{m_i}\int_0^t e^{-(t-s)/(\epsilon\tau_{m_i})}\frac{d}{ds}\\
&\times \mathrm{Re}\,\mathcal{F}^{(2\omega)}_{m_1,\ldots,m_i,\ldots,m_\omega}(t,s)\\
&+\sum_{i=1}^{\omega} 2k^2_{m_i} d_{m_i-1}\int_0^t ds\, e^{-(t-s)/(\epsilon\tau_{m_i-1})}\frac{d}{ds}\\
&\times \mathrm{Re}\,\mathcal{F}^{(2\omega)}_{m_1,\ldots,m_i-1,\ldots,m_\omega}(t,s).\end{aligned} \tag{3.15}$$

The multidimensional matrix $I^{(2\omega;0)}$ is the linear inertial operator of the white noise theory. The integrand functions $\mathcal{F}^{(2\omega)}_{m_1,\ldots,m_i-1,\ldots,m_\omega}(t,s)$ are given by the straightforward generalization of Eq. (3.5). The lhs, as above, is set to zero as far as the steady state features of the inertial range are concerned. Repeated integrations by parts in the large t limit generate a Laplace asymptotic expansion [19] of integral terms in the rhs, the coefficients of which are the derivatives with respect to s of the functions $\mathcal{F}^{(2\omega)}$ evaluated at equal times. When the steady state sets in we assume the latter quantities to be invariant under time translations for large t. Under such an assumption it will be proven in Sec. V that the equal time derivatives are specified at equilibrium by linear combinations of the $C^{(2\omega)}$'s. The effect of a small time correlation is therefore to generate new couplings of order ϵ in the inertial operators. The observables we focus on are the scaling exponents. As discussed in the introduction, anomalies occur in the presence of nontrivial scaling zero modes of the white noise inertial operators. It makes sense to relate the ϵ dependence of the anomalous exponents to a perturbation of the scaling zero modes derived for $\epsilon=0$. A straightforward approach to the problem calls for the solution of N^{ω} linear equations. A further source of difficulty is that exact determination of the zero eigenvectors of the inertial operators of any given order requires the matching of infrared and ultraviolet boundary conditions. In the absence of an exact diagonalization, any analytical approach must rely on closure Ansätze first to solve the white noise problem and then to yield the corrections to the zero modes by linear perturbation theory.

IV. WHITE NOISE CLOSURE

In this section we present a closure strategy to compute the $H(2\omega)$'s in the case of white noise advection. As shown in the preceding section, the second diagonal moment is normal and universal versus the time correlation. The first nontrivial zero mode problem is provided by the fourth order inertial operator $I^{(4;0)}$. In [11] it was shown that the anomalous exponent ρ_4,

$$H(4)=2H(2)-\rho_4, \tag{4.1}$$

can be extracted up to a very good accuracy from the solution of only two nonlinear algebraic equations. The stationary equations for $C^{(4)}$ in the inertial range far from the infrared and ultraviolet boundaries are given by

$$\begin{aligned}0=\frac{I^{(4;0)}_{m,n;p,q}C^{(4)}_{p,q}}{2\lambda^2}
&\equiv-\left(\frac{1}{\tau_m}+\frac{1}{\tau_{m-1}}+\frac{1}{\tau_n}+\frac{1}{\tau_{n-1}}\right)C^{(4)}_{m,n}+\frac{1}{\tau_m}C^{(4)}_{m+1,n}\\
&+\frac{1}{\tau_n}C^{(4)}_{m,n+1}+\frac{1}{\tau_{m-1}}C^{(4)}_{m-1,n}+\frac{1}{\tau_{n-1}}C^{(4)}_{m,n-1}\\
&+2\,\delta_{m,n}\left(\frac{C^{(4)}_{m,m+1}}{\tau_m}+\frac{C^{(4)}_{m,m-1}}{\tau_{m-1}}\right)-2\,\delta_{m+1,n}\frac{C^{(4)}_{m,m+1}}{\tau_m}\\
&-2\,\delta_{n,m-1}\frac{C^{(4)}_{m,m-1}}{\tau_{m-1}}.\end{aligned} \tag{4.2}$$

One recognizes two kinds of couplings in $I^{(4;0)}_{m,n;p,q}$.

(1) "Global," or "unconstrained," interactions. The indices p and q range respectively from $m-1$ to $m+1$ and from $n-1$ to $n+1$. The couplings are independent of the relative values of m and n. In this sense they are referred as global.

(2) "Purely local" interactions. These occur only for $|m-n|\leq 1$ and correspond to the terms proportional to the Kroenecker δ in Eq. (4.2).

Anomalous scaling in the inertial range is strictly related to the presence of such purely local interactions. Were these latter neglected, the fourth order moment would have a normal scaling solution

$$C^{(4)}_{m,n}\propto\frac{\tau_n}{\tau_m}\tau_m^2. \tag{4.3}$$

The idea is to capture the anomalous scaling by looking at the "renormalization" of global couplings by pure short range ones. Disregarding the boundaries, the system is invariant under a simultaneous shift of the indices. Hence, assuming a perfect index-shift invariance there are, for the mth shell, only two independent equations where δ-like terms occur:

$$0=\sum_{p,q} I^{(4;0)}_{m,m;p,q}C^{(4)}_{p,q},$$

$$0=\sum_{p,q} I^{(4;0)}_{m,m-1;p,q}C^{(4)}_{p,q}. \tag{4.4}$$

The third equation involving a purely local interaction of the mth shell with its nearest neighbors,

$$0=\sum_{p,q} I^{(4;0)}_{m+1,m;p,q}C^{(4)}_{p,q},$$

is generated from the second of Eqs. (4.4) by a simple index shift. Therefore, it is not regarded as independent. The pair (4.4) contain all the relevant information needed to extract the scaling of the fourth moment. It forms a closed system of equations independently on the shell number m as one imposes scaling relations to hold within the set of ''independent'' moments of fourth order:

$$C^{(4)}_{m+n,m+n}=z^{-n}C^{(4)}_{m,m}, \tag{4.5}$$

$$C^{(4)}_{m+n,m}=xk_{n-1}^{-H(2)}C^{(4)}_{m,m}, \tag{4.6}$$

where the integer n is taken larger than zero. As in the analysis of the interactions, the concept of independence stems from the assumption of index shift invariance: the moments of the form $C^{(4)}_{m-n,m}$ are immediately reconstructed once Eqs. (4.5) and (4.6) are given:

$$C^{(4)}_{m-n,m}=xk_{n-1}^{-H(2)}z^{n}C^{(4)}_{m,m}.$$

Let us analyze the closure Ansatz in more detail. The first equation (4.5) is a global scaling assumption of the ''diagonal'' sector of the fourth moment. Its justification lies in the very definition of an inertial range. The second scaling assumption relates the diagonal sector to the nondiagonal one via a marginal scaling. It is analogous in the present context of an operator product expansion (OPE) in statistical field theory [20]. There, renormalization group (RG) techniques are able to describe the scaling behavior of correlations of fields sampled at large real space distances one from the other. If an observable requires the evaluation of a correlation including the products of one field in two points at short distances, i.e., $\langle \phi(x-dx)\phi(x+dx)\cdots\rangle$, the RG procedure cannot be directly applied. The problem is overcome by an OPE or short distance expansion. The prescription is to rewrite the product via a Taylor expansion in terms of local composite operators sampled just at one point. Such a point is now well separated from all the others appearing in the correlation function. The original correlation is replaced by a set of correlations such that RG applies provided an extra renormalization, renormalization of composite operators (RCO), is introduced. The latter is understood by observing that in our example the first term in the Taylor expansion gives

$$\phi(x+dx)\phi(x-dx)\sim\phi(x)^2.$$

The mathematical meaning of a field is one of an operator-valued distribution. The product of two distributions at equal points, i.e., $\phi(x)^2$, requires a regularization before the cutoff is removed in order for it to make sense as a distribution. This is the content of the RCO. Finally, at leading order the relationship between the renormalized quantities reads, for the above example,

$$\langle[\phi(x+dx)\phi(x-dx)]_R\cdots\rangle\sim c(dx)\langle[\phi(x)^2]_R\cdots\rangle. \tag{4.7}$$

Roughly speaking, the small real space separations are associated with the UV behavior of the Fourier conjugated variable. In the shell model context the θ_m are representative of the scalar field variation over one octave. The moments $C^{(4)}_{m,m+n}$ correspond to the average of the product of squared increments of the scalar field at different wave numbers

$$C^{(4)}_{m,m+n}\sim\langle\phi(k_{m+n})^2\phi(k_n)^2\rangle,$$

$$\phi(k_m)\sim\left\langle\int d^Dx e^{ik\cdot x}[\theta(x)-\theta(0)]_R\right\rangle_{k_m\leqslant|k|\leqslant\lambda k_m}.$$

Equation (4.6) states then that scaling is restored for large shell separations (n going to infinity, i.e., spatial scales much smaller than k_m^{-1}) independently on the smaller wave number k_m. The analogy with the OPE is then summarized by

$$\lim_{n\uparrow\infty}\int d^Dy\, e^{ik_n\cdot y}c(y)\sim xk_{n-1}^{-H(2)}, \tag{4.8$'$}$$

where x is a renormalized constant. The insertion of the scaling Ansatz in Eq. (4.4) leaves a nonlinear system in the unknown variables z and x. By applying the definition $k_n=\lambda^n$ one gets

$$-1-\lambda^{-H(2)}+2x(1+z\lambda^{-H(2)})=0,$$

$$(1+z)\lambda^{-H(2)}+xz(-1-3\lambda^{-H(2)}\lambda^{-2H(2)}+z\lambda^{-3H(2)})=0, \tag{4.9}$$

which, after straightforward manipulation, provides z as the physical root of a second order polynomial

$$z=\frac{1+2\lambda^{-H(2)}+2\lambda^{-2H(2)}+\lambda^{-3H(2)}+\sqrt{1+4\lambda^{-H(2)}+8\lambda^{-2H(2)}-6\lambda^{-3H(2)}-4\lambda^{-5H(2)}+\lambda^{-6H(2)}}}{2(2\lambda^{-2H(2)}+\lambda^{-3H(2)}+\lambda^{-4H(2)})}. \tag{4.10}$$

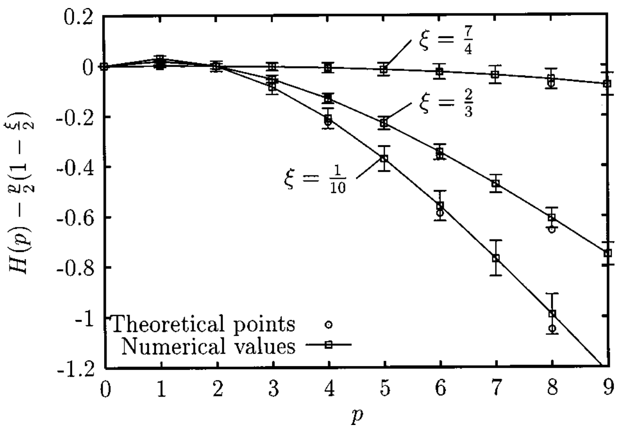

FIG. 1. Analytical prediction for the anomalous part of the scaling exponent compared with the results of the numerical experiments for different values of the turbulence degree parameter ξ. Kolmogorov scaling of the advection field corresponds to $\xi=2/3$. The dash-dotted line represents the (dimensional) normal scaling prediction. The continuous line interpolates the exponents as obtained from numerical experiment (squares). The circles are the analytical prediction from the closure Ansatz.

In terms of z, the anomaly is

$$\rho_4 = 2H(2) - \frac{\ln z}{\ln \lambda} \tag{4.11}$$

and it proves to be in fair agreement with the values obtained from the numerical solution of the exact equations (4.2) [11] and from the numerical integration of Eq. (2.1) for all the values of the turbulent exponent ξ in the physical range (see also Figs. 1 and 2). The sign of ρ_4 is always positive: the effect of the anomaly is to decrease the diagonal scaling exponent.

The procedure presented in detail for the computation of the fourth order exponent is straightforwardly extended to any higher order moment when one recognizes that, in general, two crucial observations hold.

(1) In the absence of pure short range couplings, the normal scaling prediction holds true far from the boundaries for the zero modes of the inertial operators of any order 2ω.

(2) For any fixed shell m, there is a one-to-one correspondence between the number of independent equations and moments of order 2ω.

In the case of $C^{(2\omega)}$ there are $2^{\omega-1}$ equations: for any fixed reference shell m_1, the interaction with the second index m_2 is affected by a pure short range coupling if the latter is equal to or one unit different from m_1, i.e., there are only two possible choices, and so on until the ωth index is reached. On the other hand, $2^{\omega-1}$ is the number of exponents that characterize the scaling of the 2ωth moment. The Ansatz is that the marginal scaling of the nondiagonal sector is fully specified in terms of the diagonal scaling exponents of order less than 2ω. By means of the OPE's, one is able to close the zero mode equations in terms of $2^{\omega-1}$ unknown renormalization constants and $H(2\omega)$. Analogy with a field theoretical OPE shows that the need for an infinite set of constants does not necessarily imply the nonrenormalizability of the real space theory mimicked by the shell model [21].

More concretely, the diagonal scaling exponent of the sixth moment ($\omega=3$) of the scalar field

$$C^{(6)}_{m,n,p} = \langle \Theta_{N+m}\Theta_m\Theta_{N+n}\Theta_n\Theta_{N+p}\Theta_p\rangle \equiv \langle |\theta_m|^2|\theta_n|^2|\theta_p|^2\rangle \tag{4.12}$$

according to the above criterion requires four independent equations (see Appendix D)

$$\begin{aligned}
&\sum_{p,q,r} I^{(6;0)}_{m,m,m;p,q,r} C^{(6)}_{p,q,r} = 0,\\
&\sum_{p,q,r} I^{(6;0)}_{m,m,m-1;p,q,r} C^{(6)}_{p,q,r} = 0,\\
&\sum_{p,q,r} I^{(6;0)}_{m,m-1,m-1;p,q,r} C^{(6)}_{p,q,r} = 0,\\
&\sum_{p,q,r} I^{(6;0)}_{m,m+1,m-1;p,q,r} C^{(6)}_{p,q,r} = 0.
\end{aligned} \tag{4.13}$$

The OPE-inspired closure yields

$$\begin{aligned}
C^{(6)}_{m+n,m+n,m+n} &= z^{-l} C^{(6)}_{m,m,m},\\
C^{(6)}_{m+n,m+n,m} &= x_1 k_{n-1}^{-H(4)} C^{(6)}_{m,m,m},\\
C^{(6)}_{m+n+p,m+n,m} &= x_2 k_{p-1}^{-H(2)} k_{n-1}^{-H(4)} C^{(6)}_{m,m,m},\\
C^{(6)}_{m+n,m,m} &= x_3 k_{n-1}^{-H(2)} C^{(6)}_{m,m,m}.
\end{aligned} \tag{4.14}$$

Inserting the OPE in Eq. (4.13), one gets into the algebraic system for the unknown renormalization constants (x_1, x_2, x_3) and the diagonal scaling factor z.

$$\begin{aligned}
&-1+\lambda^{\alpha}(-1+3zx_1)+3x_3=0,\\
&-\lambda^{2\alpha}zx_1+\lambda^{4\alpha+\rho_4}z^2x_1-2z(x_1-2x_2)\\
&\quad+\lambda^{\alpha}[1+z(-7x_1+4x_3)]=0,\\
&\lambda^{\alpha}(1+4x_1-6x_3)+\lambda^{2\alpha}(4zx_2-2x_3)-x_3=0,\\
&\lambda^{3\alpha+\rho_4}zx_1+\lambda^{\alpha}z(x_1-3x_2)-zx_2+\lambda^{5\alpha+\rho_4}z^2x_2\\
&\quad-\lambda^{3\alpha}z(x_2-x_3)+\lambda^{2\alpha}(-4zx_2+x_3)=0.
\end{aligned} \tag{4.15}$$

After some algebra, Eq. (4.15) reduces to a single third order polynomial specifying the physical root of z. It is worth remarking that from the functional dependence of the coefficient of Eq. (4.15) the exponent $H(6)$ depends upon the anomaly of $H(4)$. Once again, the anomaly evaluated from

$$\rho_6 = 3H(2) - \frac{\ln z}{\ln \lambda} \tag{4.16}$$

is in fair agreement with numerics (see Figs. 1 and 2) for different values of ξ.

In Appendix E, the same steps are performed in the case of the eighth moment $C^{(8)}_{m,n,p,q}$. The analytical predictions for

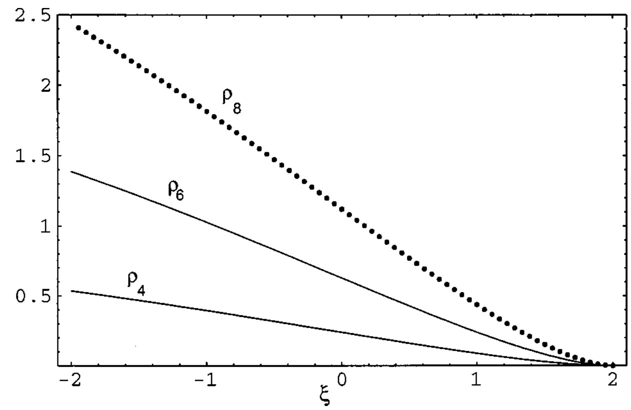

FIG. 2. Prediction of the closure Ansatz for the anomalies in the scaling exponents of the fourth, ρ_4, the sixth, ρ_6, and the eighth, ρ_8, moments of the scalar field versus the turbulence parameter ξ. In all cases, the anomalies are a decreasing function of ξ going smoothly to zero as the Batchelor limit ξ equal to 2 is approached. The anomaly of the eighth moment is obtained as the numerical solution of a sixth order polynomial.

the anomalous exponents are summarized in Fig. 2. In all cases the anomalies are decreasing functions of the turbulence parameter ξ vanishing smoothly when the laminar limit (ξ equal two) is approached. The anomaly of the fourth order moment can be compared with the results of numerical experiments for the fourth order structure function of the Kraichnan model [8]. There, the adopted turbulence parameter is ξ_{wn}. For values of ξ_{wn} of order one, i.e., from the Kolmogorov scaling up to the Batchelor limit, one indeed observes the same monotonically decreasing behavior with values of the anomaly of the same order as those found in the shell model. For lower values of ξ_{wn} the anomaly in the Kraichnan model displays a maximum before decreasing to zero for ξ_{wn} equal to zero, i.e., when ξ tends to -2. No sign of such behavior is observed in the shell model. The discrepancy might be an artifact of the shell model, which was originally designed to mimic the supposed local-in-scale character of the nonlinear interactions in a turbulent flow [9] and fails to describe a regime where strong nonlocal effects become important.

On a phenomenological level, the energy transfer in the inertial range of the turbulent field is related to the occurrence of a cascade mechanism, as conjectured by Richardson [22]. The conservation of energy in the inertial range imposes that the force occurring on large real space scales is transferred to small scales (i.e., large wave numbers) before dissipating. A mathematical description of the cascade is provided by multiplicative stochastic processes [23]. Multiplicative modeling has been shown to account for most of the features observed in real and synthetic turbulence [24,9]. In the present case, the idea of a multiplicative structure is incorporated into the hypothesis that the scaling of the nondiagonal sector of a given moment of order 2ω is reconstructed once the scaling of the lower moments is known. Such an assumption, together with the analysis of the couplings in the inertial operator of order 2ω, yields with fair accuracy the scaling exponents of the model without having to resort to an exact diagonalization of the inertial operator.

V. PERTURBATIVE ANALYSIS

Let us now turn to the time-correlated case. The idea is to evaluate the scaling behavior of the dominant zero modes of the inertial operators (3.15) linearized up to first order in ϵ by perturbing the white noise closure Ansatz.

The first order corrections in ϵ to the inertial operators are obtained by truncating the integration in parts to the terms linear in ϵ:

$$2k_{m_i+1}^2 d_{m_i}\int_0^t e^{-(t-s)/(\epsilon\tau_{m_i})}\frac{d}{ds}\mathrm{Re}\,\mathcal{F}^{(2\omega)}_{m_1,\ldots,m_i,\ldots,m_\omega}(t,s)$$
$$=2\epsilon\lambda^2\frac{d}{ds}\mathrm{Re}\,\mathcal{F}^{(2\omega)}_{m_1,\ldots,m_i,\ldots,m_\omega}(t,s)|_{s=t}$$
$$+O(\epsilon^2,\epsilon e^{-t/(\epsilon\tau_{m_i})}). \tag{5.1}$$

The use of Eq. (3.9) in the rhs stresses that the effective adimensional expansion parameter is $\epsilon\lambda^2$: the range of reliability of first order perturbation theory is compressed to $\epsilon \leqslant \lambda^{-2}/10$. As mentioned in Sec. III, in the limit t going to infinity, one expects the time quantities to be stationary. In such a case, the derivative with respect to the variable s can be interchanged with the derivative with respect to t and one can use the equations of motion to evaluate Eq. (5.1). A direct differentiation with respect to s is consistently taken with respect to the system of stochastic differential equations conjugated by time reversal of equations (2.1)–(2.3). The latter operation in general requires knowledge of the probability density of the forward in time problem. In the stationary limit, the time reversal operation for the OU process reduces to the inversion of the sign of the drift term as in the deterministic case. After slightly more lengthy algebra, the result is equal to the differentiation with respect to t with opposite sign.

The computations in the general case are very cumbersome (see Appendixes C, D, and E). It is convenient to exemplify the procedure in the simpler case of the second order correlation. There are four contributions to $\mathrm{Re}\,\mathcal{F}^{(2)}_m$:

$$\frac{d}{dt}G^{(2)}_{N+m+1,N+m;N+m,m+1}(t,s)|_{t=s}=0,$$

$$\frac{d}{dt}G^{(2)}_{N+m+1,N+m+1;N+m,m}(t,s)|_{t=s}$$
$$=-\kappa k_{m+1}^2 C^{(2)}_m(t)+\langle\dot{\Theta}_{N+m}(t)\Theta_m(t)\rangle, \tag{5.2}$$

$$\frac{d}{dt}G^{(2)}_{N+m,N+m+1;N+m,m+1}(t,s)|_{t=s}=0,$$

$$\frac{d}{dt}G^{(2)}_{N+m,N+m;N+m+1,m+1}(t,s)|_{t=s}$$
$$=-\kappa k_m^2 C^{(2)}_m(t)+\langle\dot{\Theta}_{N+m+1}(t)\Theta_{m+1}(t)\rangle.$$

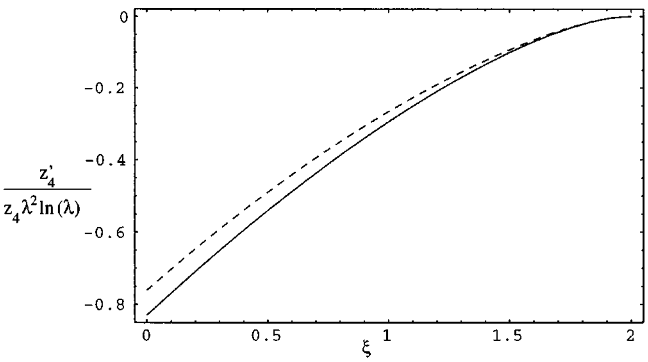

FIG. 3. First order corrections to $H(4)$ with (dashed line) and without inclusion of second neighbor interactions are plotted versus the turbulence exponent. The inclusion of second neighbor couplings increases the intensity of the anomaly.

By definition

$$\langle \dot{\Theta}_{N+m}(t)\Theta_m(t)\rangle = \frac{1}{2}\frac{d}{dt}\langle |\theta_m(t)|^2\rangle,$$

$$\langle \dot{\Theta}_{N+m+1}(t)\Theta_{m+1}(t)\rangle = \frac{1}{2}\frac{d}{dt}\langle |\theta_{m+1}(t)|^2\rangle.$$

The time derivative of Re $\mathcal{F}^{(2)}_{m-1}$ is derived by a simple index shift. Since the terms nondiagonal in the resolvent R indices are zero, the second moment inertial operator is not affected by first order perturbation theory. The result is not surprising. The second moment has only one free index. Hence at any order of perturbation theory only global coupling can be generated, which is forced by the symmetries of the model to be consistent with a normal scaling of the zero mode. Moreover, the Θ_m components of the scalar evolve only through the coupling with their complex conjugated Θ_{N+m}'s: their variation is a second order effect in ϵ. The only possible nonzero corrections are viscous and can be consistently neglected.

Let us now draw the general picture when ω is larger than one. Once again, the phase symmetries (2.14), and the fact that when Re $\mathcal{F}^{(2\omega)}_{m_1,\ldots,m_i,\ldots,m_\omega}$ is known all other terms are yielded by index shift or exchange operations, prevent the corrections to the global couplings from affecting the scaling properties: the resulting global sectors of inertial operators have a normal scaling zero mode. This is in agreement with the observation made in [12], where the dependence of the scaling exponents on the time correlation for generalized models of passive scalar advection is predicted to appear only through anomalies. The corrections to the purely short range couplings are therefore the relevant ones. They occur in two ways. On one hand, new terms of order ϵ show up in the purely self- and nearest-neighbor interactions. On the other hand, terms proportional to $\delta_{m_i,m_j\pm 2}$ appear. The latter are the most dangerous, for they in principle perturb the logic of the white noise closure by introducing new independent equations, hence the need for more renormalization constants in the nondiagonal sector of the moments. Nevertheless, one can argue *a priori* in the spirit of the renormalization group [25], only the nearest neighbors interactions are relevant for scaling. Hence first order corrections can be obtained allowing an ϵ dependence in the renormalization constant of the white noise closure and determining the first order coefficient of their Taylor expansion. Moreover, for ω larger than 2, such a strategy is already able to take into account the corrections due to the purely second neighbor interactions.

Let us analyze in more detail the case of $C^{(4)}_{m,n}$. The white noise closure is perturbed by introducing an ϵ dependence into the renormalization constants

$$C^{(4)}_{m+n,m+n} = z(\epsilon)^{-n} C^{(4)}_{m,m}, \tag{5.3}$$

$$C^{(4)}_{m+n,m} = x(\epsilon)\lambda^{-H(2)}_{n-1} C^{(4)}_{m,m}. \tag{5.4}$$

The marginal scaling in the nondiagonal sector in Eq. (5.4) is assumed to stay universal as it is for $C^{(2)}$ while the ϵ dependence is stored in the prefactor. The diagonal exponent is then determined up to first order as

$$H(4,\epsilon) = \frac{\ln(z)}{\ln(\lambda)} + \epsilon\lambda^2 \frac{z'}{\lambda^2 z \ln(\lambda)}, \tag{5.5}$$

where z' is the derivative of z at $\epsilon = 0$ yielded by the perturbative solution of the system

$$\sum_{p,q} [I^{(4;0)}_{m,m;p,q} + \epsilon I^{(4;1)}_{m,m;p,q}] C^{(4)}_{p,q}(\epsilon) = 0,$$

$$\sum_{p,q} [I^{(4;0)}_{m,m-1;p,q} + \epsilon I^{(4;1)}_{m,m-1;p,q} C^{(4)}_{p,q}(\epsilon) = 0. \tag{5.6}$$

The correction to ρ_4 due to time correlation increases the anomaly leading to a slower decay of the diagonal moment. For negative z' (see Fig. 3) the overall anomaly is

$$\rho_4(\epsilon) = (2-\xi) - \frac{\ln(z)}{\ln(\lambda)} + \left| \epsilon\lambda^2 \frac{z'}{\lambda^2 z \ln(\lambda)} \right|. \tag{5.7}$$

In the range of reliability of first order perturbation theory the effect is very small: for $\epsilon\lambda^2 \approx O(10^{-1})$ the prediction is a correction amounting to the 3% of the white noise exponent $H(4)$. The perturbative scheme just proposed does not take into account the emergence of pure second neighbor interactions. In order to weight their relevance for the diagonal scaling and simultaneously to check the hypothesis of normal scaling for the marginal scaling in Eq. (5.4), one can relax the closure in order to encompass the equation

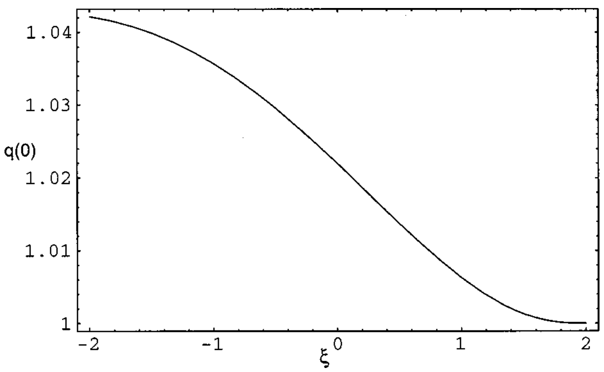

FIG. 4. Renormalization constant $q(0)$ is plotted versus ξ. It remains close to one through the entire physical range, proving self-consistent the conjecture of normal scaling for the nondiagonal sector of the fourth moment. The result stresses that the renormalization of nearest neighbor interactions provides an accurate framework in which to extract the scaling exponent.

$$\sum_{p,q} [I^{(4;0)}_{m,m-2;p,q} + \epsilon I^{(4;1)}_{m,m-2;p,q}] C^{(4)}_{p,q}(\epsilon) = 0, \tag{5.8}$$

which describes the independent (in the sense stated above) second neighbor interaction. Consistency with the white noise theory compels the latter equation to decouple when ϵ is set to zero. The requirement is satisfied if the closure is chosen in the form

$$C^{(4)}_{m+n,m} = x(\epsilon) q(\epsilon)^{[(n-1)(n-2)]/2} k_{(n-1)} - H(2) C^{(4)}_{m,m}. \tag{5.9}$$

The prefactor $q(\epsilon)^{[(n-1)(n-2)]/2}$ forces $q(0)$ to be a function of the white noise renormalization constants. Were the white noise closure exact, it would fix the value of $q(0)$ to one.

In Fig. 4 $q(0)$ is plotted versus ξ: through the entire physical range it stays close to one with a maximal deviation on the order of 4% for ξ equal to -2. Moreover, as shown in Fig. 3, the time-correlation-induced correction to $H(4,\epsilon)$ when Eq. (5.8) is included has the same qualitative behavior and is quantitatively very close to the nearest neighbor prediction. The result is an *a posteriori* check of the robustness of the closure approach. It confirms that first order corrections can be extracted within the logical scheme of the zero order one. It follows that the equations specifying the zero modes of the inertial operator acting on the sixth moment (see Appendix D) can be closed by assuming

$$\begin{aligned} C^{(6)}_{m+n,m+n,m+n} &= z(\epsilon)^{-l} C^{(6)}_{m,m,m}, \\ C^{(6)}_{m+n,m+n,m} &= x_1(\epsilon) k_{n-1}^{-H(4,\epsilon)} C^{(6)}_{m,m,m,m}, \\ C^{(6)}_{m+n+p,m+n,m} &= x_3(\epsilon) k_{p-1}^{-H(2)} k_{n-1}^{-H(4,\epsilon)} C^{(6)}_{m,m,m,m}, \\ C^{(6)}_{m+n,m,m,m} &= x_2(\epsilon) k_{n-1}^{-H(2)} C^{(6)}_{m,m,m,m}. \end{aligned} \tag{5.10}$$

The exponent $H(4,\epsilon)$ is known perturbatively from Eq. (5.5), while $H(2)$ is universal. With the same rationale (Appendix D) one can evaluate $H(8,\epsilon)$.

In Fig. 5 the analytic predictions for the corrections to the scaling exponents are summarized. In all cases the corrections are negative, i.e., they carry a positive contribution to

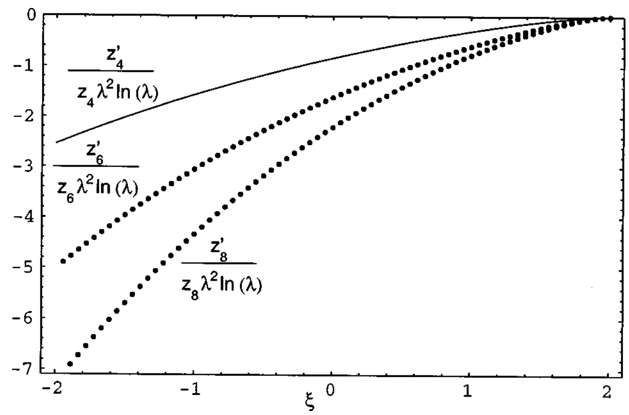

FIG. 5. First order correction to $H(4)$ (continuous line), $H(6)$ and $H(8)$ versus the turbulence parameter ξ. In the last two cases the corresponding linear systems are solved numerically. In all cases the corrections are derived by perturbing the white noise closure renormalization constants. The effect of time correlation is seen to add a negative correction to the scaling exponents, highlighting an increase of intermittency.

$\rho_{2\omega}$. The corrections increase with ω, the rate of growth being slightly slower than the $\Delta(2\omega) \propto \omega(\omega-1)\Delta(4)/2$ predicted in [12] for time-correlation-generalized PDE Kraichnan models.

Within the range of first order perturbation theory, the overall effect of time correlation is seen to enhance intermittency. An intuitive understanding of the phenomenon might be obtained by interpreting the time correlation as a mechanism by which to increase the probability of coherent fluctuations of the scalar field. The latter are rare events felt in the tail of the probability density of the scalar field as extreme deviations from the Gaussian behavior of the typical events.

VI. NUMERICAL EXPERIMENTS

Resorting to numerical experiments has a double motivation. On one hand, they can be used to test the predictions from the first order perturbation theory. On the other hand, they provide a broader scenario of the features of the model beyond the grasp of perturbative approaches. The first task is far from being easy because a quantitative check of perturbation theory requires measurements of the scaling exponents within an accuracy smaller than 2%.

The main feature of the inertial range is the conservation of the scalar energy. From the analytical point of view, this is seen in the noncommutativity of the terms associated with the multiplicative noise, B^{γ} in Eq. (2.10). This property rules out the use of a simple Euler scheme, which can be applied in the case of delta-correlated noise. In the case of white noise advection the multiplicative structure of the noise Eq. (2.10), which is interpreted in the Stratonovich sense, can be mapped into the corresponding Itô equations. The advantage is that the diagonal nonzero average part of the noise is explicitly turned into an effective drift term [26]. The nondiagonal terms in the Taylor expansion of the scalar field Θ are of the order three halves in dt, which are neglected in the

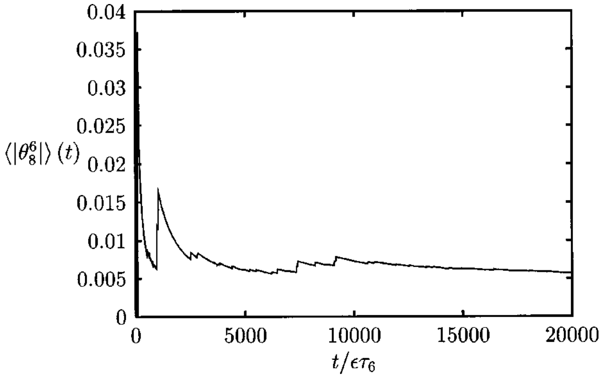

FIG. 6. The convergence of the sixth order structure function for $\epsilon=1$. Shown is $\langle|\theta_m^6|\rangle(t)$ for $m=8$. The fast upward changes and slow downward relaxations reveal the intermittent nature of the signal $\theta_{m=8}^6(t)$.

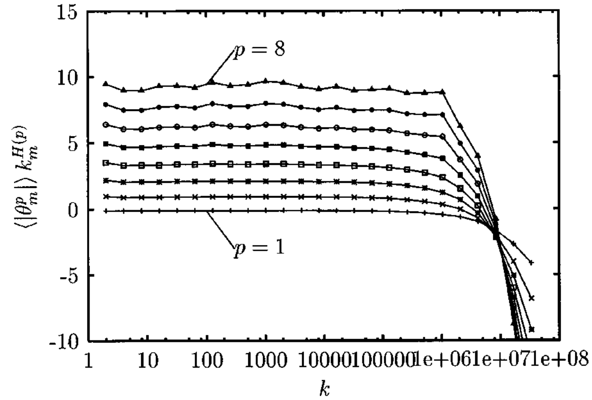

FIG. 7. An example of the scaling of the structure functions for $\epsilon=2.0$. The plot shows the structure functions normalized by the fitted scaling $k_m^{H(p)}$ to make the scaling regime appear as horizontal lines. The lower line is for $p=1$, and the upper line is $p=8$. Each line is offset to make it possible to distinguish the lines from each other.

Euler scheme. This procedure becomes meaningless for a time-correlated noise. There, ordinary calculus holds and in the Taylor expansion of Θ both diagonal and nondiagonal products of the noise are of the same order in dt. Moreover, the algorithm to be used must tend smoothly to a white noise limit, so that the same relative error is preserved independently of the value of ϵ.

Following Burrage and Burrage [27], a reliable way out of the stated difficulties is to adopt the Trotter-Lie-Magnus formula to integrate the equations of motion to first order. For each time increment dt, Eq. (2.10) is solved in exponential form. Fast matrix exponentiation algorithms are provided by the package EXPOKIT [28].

To generate the correlated noise, the exact method described by Miguel and Toral [29] is used. This method ensures that the noise is accurate down to the limit $\epsilon\to 0$.

The time scale relevant to measuring the convergence of the solution is the slowest time scale in the system, namely, the eddy-turnover time of the first shell estimated as the maximum between $\epsilon\tau_1$ and τ_1. As shown in Fig. 6, more than $N_\tau = 10\,000$ eddy-turnover times are needed to achieve a converged solution for the sixth order structure function. The time step is set by the fastest time scale in the system, which is the one with the largest shell $\epsilon\tau_M$. The number of iterations ($\mathcal{N}$) needed to achieve convergence is then for ϵ less than one:

$$\mathcal{N}(\text{iterations}) = \frac{N_\tau \tau_M}{\epsilon\tau_1} = \frac{N_\tau}{\epsilon}\lambda^{(M-1)(1-\xi/2)}, \qquad (6.1)$$

which shows that the number of iterations needed grows like $1/\epsilon$, making it difficult to get close to the white noise limit using the same algorithm.

The scaling of the diagonal moments of higher order has been extracted by means of extended self-similarity [30], where the pth order structure function is plotted versus the second order one, which is assumed to be normal. The scaling is found as the average slope of the logarithmic derivatives in the inertial range.

We considered a system with 25 shells with wave numbers increasing as powers of $\lambda=2$, with viscosity $\kappa=5\times10^{-9}$. This choice ensures that there are several shells in the dissipative range. We focused on the results for ξ equal $\frac{2}{3}$ (Kolmogorov scaling).

In Fig. 7, the normalized structure functions $\langle|\theta_m^p|\rangle k_m^{H(p)}$ are shown. The quality of the scaling is demonstrated by the fact that the moments show scaling over a wide range of scales.

A summary of the numerical experiments is given in Fig. 8, where the scaling exponents are plotted versus the order of the moments of the scalar field for different values of ϵ. It is evident that the anomaly grows as the time correlation increases.

When turning to the interpretation of the results in more detail, the uncertainty in the extraction of the scaling has to be kept in mind. For the sixth moment this uncertainty turned out to be on the order of 4%. The changes in the scaling between different values of ϵ is also on the order of a few percent. This seems to exclude a proper resolution in the numerics to compare the results with the analytical predictions from the perturbation analysis. However, the results for different values of ϵ can still be compared with some confidence, since the relative uncertainty between the different

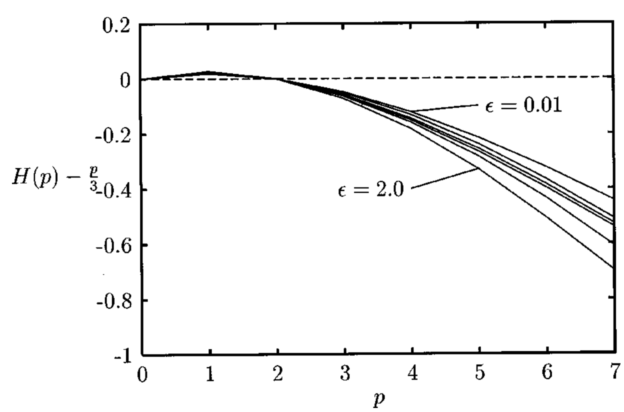

FIG. 8. Anomalous part of the structure functions $H(p)-p/3$ as a function of p for $\epsilon=0.01$ to $\epsilon=2$. The lines correspond to (from the top): $\epsilon=0.01$, 0.02, 0.10, 0.25, 1.0, and 2.0. The dashed line corresponds to normal scaling.

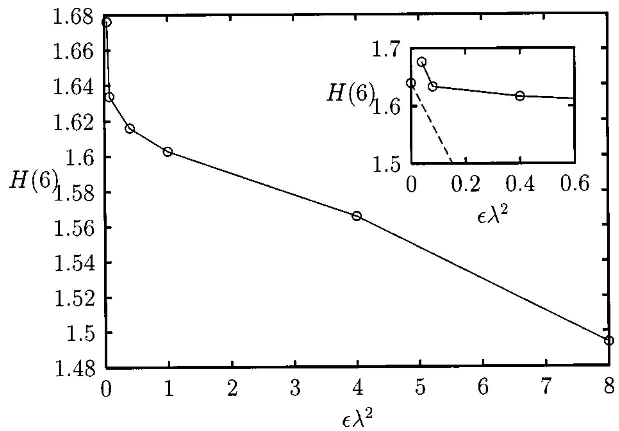

FIG. 9. Scaling of the sixth order structure function versus $\epsilon\lambda^2$. The inset shows an enlargement of the perturbative range $\epsilon\lambda^2 \ll 1$ where the analytical prediction from first order perturbation theory can be compared with the numerical experiments.

runs is much smaller than the absolute uncertainty. This means that the slope of, e.g., the sixth order structure functions versus ϵ will be well resolved, while the absolute values can be shifted up and down a few percent.

In Fig. 9 the analytical prediction of the exponents is compared with the result of the numerics. The theoretical prediction for small ϵ (see inset of Fig. 9) is below the numerical points, which is due to the absolute uncertainty as explained above. The slope is the same for the analytical calculation and the numerics, giving credibility to the results of the perturbation analysis. It should be noted that the effect of time correlation on the anomaly is quantitatively quite small even in the nonperturbative range when ϵ is equal to one ($\epsilon\lambda^2=4$).

The global picture provided by the numerical experiments is that $H(2\omega)$ is seen to be a nonlinear function of ϵ which, after rapid initial decrease in the perturbative range, displays a much slower rate of variation. An interesting question is whether or not there is a limiting value of the scaling of the structure function as $\epsilon \gg 1$. However, the quality of the numerics does not allow us to answer this question.

VII. CONCLUSION

We have presented a shell model for the advection of a passive scalar by a velocity field which is exponentially correlated in time. We developed a systematic procedure to calculate the exponents of the correlation of the diagonal moments (the structure functions). For the delta-correlated velocity we find good agreement between analytical and numerical calculations up to the eighth order. We presented an analytical perturbative theory to compute the correction to the scaling exponents due to the exponentially correlated velocity field.

The occurrence of anomalies in the exponents of the diagonal moments of the scalar and their nonuniversality versus the intensity $\epsilon\lambda^2$ of the time correlation is related to the presence of pure short range couplings in the corresponding inertial operator, which provide for nontrivial scaling of the zero modes. In the absence of such short range couplings, as is the case for the second moment, normal scaling would take place independently of the value of $\epsilon\lambda^2$.

The behavior of the anomalous exponents in the nonperturbative regime was studied numerically. This was found to be a nonlinear monotonic function of $\epsilon\lambda^2$, decreasing at a rate much slower than in the perturbative regime. It is thus clear that the addition of the time correlation to the advecting velocity field enhances the anomalous scaling. The anomaly found in the present study is still much smaller than that found when the passive scalar is driven by a turbulent velocity field driven by Navier-Stokes turbulence or by a shell model for the velocity field [13]. This indicates that the non-Gaussian nature of the real turbulent velocity field plays a significant role in the strong anomalous scaling observed for real passive scalars.

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APPENDIX A: STOCHASTIC INTEGRATION BY PARTS FORMULA

A heuristic proof of the stochastic integration by parts formula is provided. For a rigorous treatment, see [15,16]. Let ζ_t be a stochastic process whose realizations are defined as the solution of the Itô stochastic differential equation (SDE):

$$\dot{x}_t = b(x_t,t) + \sigma(x_t,t)\,\eta_t, \quad x_t|_{t=0} = x_0, \tag{A1}$$

where η_t is white noise. Let ζ_t^ϵ be the stochastic process specified by

$$\dot{x}_t = b(x_t,t) + \epsilon h(x_t,t)\sigma(x_t,t) + \sigma(x_t,t)\,\eta_t \quad x_t|_{t=0} = x_0. \tag{A2}$$

For equal ϵ the two SDE's coalesce: Eq. (A2) can be derived from Eq. (A1) under the variation of the white noise $\eta_t \to \eta_t + h(x_t,t)$. The integration by parts formula states that for any smooth functional f the following identity holds:

$$\left\langle \frac{d}{d\epsilon} f(\zeta_t^\epsilon) \right\rangle_{\zeta_t^\epsilon} \Bigg|_{\epsilon=0} = \left\langle f(\zeta_t) \int_0^t ds\, h(\zeta_s,s) \right\rangle_{\zeta_t}, \tag{A3}$$

where $\langle\,\rangle_{\zeta_t}$ denotes the expectation values with respect to the measure induced by ζ_t. In order to prove it, let us observe that the transition probability density for Eq. (A2) can be written formally as a path integral (Itô discretization):

$$p_{\eta^{\epsilon}}(x,t|x_0,0)=\int_{x_0}^{x_t=x}\mathcal{D}x\, e^{-S_{\zeta}(x,t|x_0,0)+\int_0^t dt'\{[\dot{x}_{t'}-b(x_{t'})/\sigma(x_{t'},t')]\epsilon h(x_{t'},t')-(\epsilon^2/2)h^2(x_{t'},t')\}},$$
$$S_{\zeta}(x,t|x_0,0)=\int_0^t\frac{dt'}{2}\left[\frac{\dot{x}_{t'}-b(x_{t'})}{\sigma(x'_t,t')}\right]^2. \quad \text{(A4)}$$

If one introduces the functional

$$M(\zeta_t^{\epsilon})=e^{-\int_0^t dt'\{[\dot{x}'_t-b(x'_t,t')/\sigma(x'_t,t')]\epsilon h(x'_t,t')-(\epsilon^2/2)h^2(x'_t,t')\}}, \quad \text{(A5)}$$

one has by construction

$$\frac{d}{d\epsilon}\langle M(\zeta_t^{\epsilon})f(\zeta_t^{\epsilon})\rangle_{\zeta_t^{\epsilon}}=0. \quad \text{(A6)}$$

To each realization of the solutions of Eq. (A2) there is a corresponding mapping $\eta_t\rightarrow x_t=x(t,\eta_t,\epsilon)$. Hence the last equality can be rewritten as the white noise average:

$$\frac{d}{d\epsilon}\langle M(x(t,\eta_t,\epsilon))f(x(t,\eta_t,\epsilon))\rangle_{\eta_t}=0, \quad \text{(A7)}$$

which implies Eq. (A3) when ϵ is set to zero. The derivative

$$\frac{d}{d\epsilon}f(\zeta_t^{\epsilon})|_{\epsilon=0}=D\zeta_t\partial_{\zeta_t}f(\zeta_t) \quad \text{(A8)}$$

is a Fréchet derivative. The dynamics of the stochastic process $D\zeta_t$ is linear once the realizations x_t of ζ_t are known:

$$y_t\equiv Dx_t$$
$$\dot{y}_t=y_t\partial_{x_t}[b(x_t,t)+\sigma(x_t,t)\eta_t]+h(x_t,t)\sigma(x_t,t). \quad \text{(A9)}$$

It is worth noting that for $b=0$, $\sigma=h=1$, the integration by parts formula (A3) reduces to

$$t\langle\partial_{w_t}f(w_t)\rangle=\langle f(w_t)w_t\rangle, \quad \text{(A10)}$$

which is the Gaussian integration by parts formula (see, e.g., [1]) applied to the Wiener process $\mathcal{N}(0,t)$.

The generalization to a multidimensional complex case proceeds straightforwardly by introducing $2N$ variational parameters $\{\epsilon_i,\epsilon_i^*\}_{i=1}^{2N}$ and applying the definitions

$$\langle\eta_m(t)\eta_n^*(s)\rangle=2\delta_{mn}\delta(t-s) \quad \text{(A11)}$$

for the white noise correlations.

APPENDIX B: STOCHASTIC INTEGRATION BY PARTS FOR THE OU PROCESS

As in the above appendix, we limit ourselves to the real case, the generalization to the complex case being straightforward. Functional differentiation is formally derived from a Fréchet derivative with $h(x_t,t)=\delta(t-s)$, where s is a parameter specifying the time when the white noise is perturbed. The variation is assumed to be nonanticipating (causal):

$$\lim_{s\uparrow t}\int_0^t ds'\,\delta(s-s')=0. \quad \text{(B1)}$$

Let us consider the system of SDE's

$$\dot{x}_m=b_m(x)+\sum_{n=1}^{2}\sigma_{m,n}(x)c_n(t), \quad \text{(B2)}$$

where c is the colored noise:

$$c_n(t)=\int_0^t ds'\frac{e^{-(t-s')/(\epsilon\tau_n)}}{\epsilon\sqrt{\tau_n}}\eta_n(s'). \quad \text{(B3)}$$

Functional differentiation gives

$$\frac{d}{dt}(D_l^s x_m)=\sum_{k=1}^{N}D_l^s x_k\left[\partial_k b_m(x)+\partial_k\sum_{n=1}^{N}\sigma_{m,n}(x)\int_0^t ds'\frac{e^{-(t-s')/(\epsilon\tau_n)}}{\epsilon\sqrt{\tau_n}}\eta_n(s')\right]+\frac{e^{-(t-s)/(\epsilon\tau_l)}}{\epsilon\sqrt{\tau_l}}\sigma_{m,l}(x). \quad \text{(B4)}$$

The functional derivative is fully specified when its form is known at the time s when the variation of the white noise occurs. The latter is determined by the causality requirement

$$\frac{d}{dt}(D_l^s c_n(t))=\left[\partial_t\frac{e^{-(t-s)/(\epsilon\tau_n)}}{\epsilon\sqrt{\tau_n}}+\frac{1}{\epsilon\sqrt{\tau_n}}\delta(t-s)\right]\delta_{n,l}, \quad \text{(B5)}$$

which implies that the variation of the colored noise is nonzero *only* immediately after the instantaneous kick

$$D_l^s c_n(t)=\frac{e^{-(t-s)/(\epsilon\tau_n)}}{\epsilon\sqrt{\tau_n}}\delta_{n,l}, \quad \forall t\geqslant s. \quad \text{(B6)}$$

By differentiating (B4) one finds

$$\frac{d^2}{dt^2}(D_l^s x_m) = \frac{e^{-(t-s)/(\epsilon\tau_l)}}{\epsilon\sqrt{\tau_l}}\sigma_{m,l}(x)\,\delta(t-s) + \text{smooth terms.} \tag{B7}$$

From the last equation it emerges that for $t=s$

$$\frac{d}{dt}(D_l^s x_m)|_{t=s} = \frac{1}{\epsilon\sqrt{\tau_n}}\sigma_{m,l}(x). \tag{B8}$$

Consistency with Eq. (B4) then requires that the variation of the x's associated with a nonanticipating variation of the white noise at time s fulfills the initial condition

$$D_l^s x_m(s)|_{t=s} = 0. \tag{B9}$$

The integration by parts formula (A3) for a smooth functional $O(x)$ is

$$\langle O(x_t)c_n(t)\rangle = \int_0^t ds'\frac{e^{-(t-s')/(\epsilon\tau_n)}}{\epsilon\sqrt{\tau_n}}\sum_{l=1}^{N}\langle D^{s'}x_l\partial_{x_l}O(x_t)\rangle. \tag{B10}$$

The variation is the solution of the linear problem (B4) of which we define R to be the fundamental solution. It follows that

$$\langle O(x_t)c_n(t)\rangle = \int_0^t ds\frac{e^{-(t-s)/(\epsilon\tau_n)}}{\epsilon}\int_s^t ds'\frac{e^{-(s'-s)/(\epsilon\tau_n)}}{\epsilon} \times\sum_{l=1}^{N}\sum_{m=1}^{N}\langle[\partial_{x_l}O(x_t)]R_{l,m}(t,s')\sigma_{m,n}(x_s')\rangle. \tag{B11}$$

Finally, inverting the order of integration one obtains

$$\langle O(x_t)c_n(t)\rangle = \int_0^t ds'\frac{e^{-(t-s')/(\epsilon\tau_n)} - e^{-(t+s')/(\epsilon\tau_n)}}{2\epsilon} \times\sum_{l=1}^{N}\sum_{m=1}^{N}\langle[\partial_{x_l}O(x_t)]R_{l,m}(t,s')\sigma_{m,n}(x_s')\rangle. \tag{B12}$$

This proves the real version of formula (3.1).

APPENDIX C: THE FOURTH ORDER CORRELATION TO FIRST ORDER

The inertial operator acting on the fourth moment $C_{m,n}^{(4)}(t)$ is in the large time limit

$$\begin{aligned}\Phi_{\text{rhs}} = {}& I_{m,n;p,q}^{(4,0)}C_{p,q}^{(4)}\\ &-2k_{m+1}^2 d_m\int_0^t ds\, e^{-(t-s)/(\epsilon\tau_m)}\frac{d}{ds}\text{Re}\,\mathcal{F}_{m,n}^{(4)}(t,s)\\ &+k_m^2 d_{m-1}\int_0^t ds\, e^{-(t-s)/(\epsilon\tau_{m-1})}\frac{d}{ds}\text{Re}\,\mathcal{F}_{m-1,n}^{(4)}(t,s)\\ &-k_{n+1}^2 d_n\int_0^t ds\, e^{-(t-s)/(\epsilon\tau_n)}\frac{d}{ds}\text{Re}\,\mathcal{F}_{n,m}^{(4)}(t,s)\\ &+k_n^2 d_{n-1}\int_0^t ds\, e^{-(t-s)/(\epsilon\tau_{n-1})}\frac{d}{ds}\text{Re}\,\mathcal{F}_{n-1,m}^{(4)}(t,s).\end{aligned} \tag{C1}$$

The bidimensional matrix $I_{m,n;p,q}^{(4,0)}$ is the white noise linear inertial operator. The corrections to the white noise theory are generated by the time derivative at equal times of the integrand function $\text{Re}\,\mathcal{F}_{n,m}^{(4)}$:

$$\begin{aligned}\mathcal{F}_{m,n}^{(4)}(t,s) \doteq {}& \langle\Theta_{N+m}(t)\Theta_{N+n}(t)\Theta_n(t)R_{N+m+1,N+m}(t,s)\Theta_{m+1}(s)]\rangle\\ &-\langle\Theta_{N+m}(t)\Theta_{N+n}(t)\Theta_n(t)R_{N+m+1,N+m+1}(t,s)\Theta_m(s)]\rangle\\ &+\langle\Theta_{N+m+1}(t)\Theta_{N+n}(t)\Theta_n(t)R_{N+m,N+m}(t,s)\Theta_{m+1}(s)]\rangle\\ &-\langle\Theta_{N+m+1}(t)\Theta_{N+n}(t)\Theta_n(t)R_{N+m,N+m+1}(t,s)\Theta_m(s)]\rangle\\ &+\langle\Theta_{N+m}(t)\Theta_{N+m+1}(t)\Theta_{N+n}(t)R_{n,N+m}(t,s)\Theta_{m+1}(s)]\rangle\\ &-\langle\Theta_{N+m}(t)\Theta_{N+m+1}(t)\Theta_{N+n}(t)R_{n,N+m+1}(t,s)\Theta_m(s)]\rangle\\ &+\langle\Theta_{N+m}(t)\Theta_{N+m+1}(t)\Theta_n(t)R_{N+n,N+m}(t,s)\Theta_{m+1}(s)]\rangle\\ &-\langle\Theta_{N+m}(t)\Theta_{N+m+1}(t)\Theta_n(t)R_{N+n,N+m+1}(t,s)\Theta_m(s)]\rangle.\end{aligned} \tag{C2}$$

After a double integration by parts neglecting viscous contributions, one gets into

$$\sum_{p,q} \frac{(I^{(4;0)}_{m,n;p,q}+\epsilon I^{(4;1)}_{m,n;p,q})}{2} C^{(4)}_{q,p}$$
$$=\left(-\frac{\lambda^2}{\tau_{-1+m}}-\frac{\lambda^2}{\tau_m}-\frac{\lambda^2}{\tau_n}-\frac{\lambda^2}{\tau_{-1+n}}+\frac{7\epsilon\lambda^4}{\tau_{-1+m}}+\frac{7\epsilon\lambda^4}{\tau_m}+\frac{7\epsilon\lambda^4}{\tau_{-1+n}}+\frac{7\epsilon\lambda^4}{\tau_n}\right)C^{(4)}_{m,n}$$
$$+\left(\frac{\lambda^2}{\tau_m}-\frac{\epsilon\lambda^4}{\tau_{1+m}}-\frac{2\epsilon\lambda^4}{\tau_{-1+n}}-\frac{2\epsilon\lambda^4}{\tau_n}-\frac{7\epsilon\lambda^4}{\tau_m}\right)C^{(4)}_{m+1,n}+\left(\frac{\lambda^2}{\tau_n}-\frac{\epsilon\lambda^4}{\tau_{1+n}}-\frac{2\epsilon\lambda^4}{\tau_{-1+m}}-\frac{2\epsilon\lambda^4}{\tau_m}-\frac{7\epsilon\lambda^4}{\tau_n}\right)C^{(4)}_{m,n+1}$$
$$+\left(\frac{\lambda^2}{\tau_{-1+m}}-\frac{\epsilon\lambda^4}{\tau_{-2+m}}-\frac{2\epsilon\lambda^4}{\tau_{-1+n}}-\frac{2\epsilon\lambda^4}{\tau_n}-\frac{7\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(4)}_{m-1,n}+\left(\frac{\lambda^2}{\tau_{-1+n}}-\frac{\epsilon\lambda^4}{\tau_{-2+n}}-\frac{2\epsilon\lambda^4}{\tau_{-1+m}}-\frac{2\epsilon\lambda^4}{\tau_m}-\frac{7\epsilon\lambda^4}{\tau_{-1+n}}\right)C^{(4)}_{m,n-1}$$
$$+\left(\frac{2\epsilon\lambda^4}{\tau_m}+\frac{2\epsilon\lambda^4}{\tau_n}\right)C^{(4)}_{m+1,n+1}+\left(\frac{2\epsilon\lambda^4}{\tau_{-1+m}}+\frac{2\epsilon\lambda^4}{\tau_{-1+n}}\right)C^{(4)}_{m-1,n-1}+\left(\frac{2\epsilon\lambda^4}{\tau_m}+\frac{2\epsilon\lambda^4}{\tau_{-1+n}}\right)C^{(4)}_{m+1,n-1}$$
$$+\left(\frac{2\epsilon\lambda^4}{\tau_{-1+m}}+\frac{2\epsilon\lambda^4}{\tau_n}\right)C^{(4)}_{m-1,n+1}+\frac{\epsilon\lambda^4}{\tau_{-2+m}}C^{(4)}_{m-2,n}+\frac{\epsilon\lambda^4}{\tau_{1+m}}C^{(4)}_{m+2,n}+\frac{\epsilon\lambda^4}{\tau_{-2+n}}C^{(4)}_{m,n-2}+\frac{\epsilon\lambda^4}{\tau_{1+n}}C^{(4)}_{m,n+2}$$
$$+\delta_{n,m}\left[\left(\frac{2\lambda^2}{\tau_m}-\frac{2\epsilon\lambda^4}{\tau_{1+m}}-\frac{4\epsilon\lambda^4}{\tau_{-1+m}}-\frac{34\epsilon\lambda^4}{\tau_m}\right)C^{(4)}_{m+1,m}+\left(\frac{2\lambda^2}{\tau_{-1+m}}-\frac{2\epsilon\lambda^4}{\tau_{-2+m}}-\frac{4\epsilon\lambda^4}{\tau_m}-\frac{34\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(4)}_{m,m-1}\right.$$
$$+\left(\frac{4\epsilon\lambda^4}{\tau_{-1+m}}+\frac{4\epsilon\lambda^4}{\tau_m}\right)C^{(4)}_{m,m}+\left(\frac{4\epsilon\lambda^4}{\tau_{-1+m}}+\frac{4\epsilon\lambda^4}{\tau_m}\right)C^{(4)}_{m+1,m-1}+\frac{4\epsilon\lambda^4}{\tau_m}C^{(4)}_{m+1,m+1}+\frac{4\epsilon\lambda^4}{\tau_{-1+m}}C^{(4)}_{m-1,m-1}+\frac{2\epsilon\lambda^4}{\tau_{1+m}}C^{(4)}_{m+2,m}$$
$$\left.+\frac{2\epsilon\lambda^4}{\tau_{-2+m}}C^{(4)}_{m,m-2}\right]+\delta_{n,m+1}\left[\left(-\frac{2\lambda^2}{\tau_m}+\frac{3\epsilon\lambda^4}{\tau_{-1+m}}+\frac{3\epsilon\lambda^4}{\tau_{1+m}}+\frac{34\epsilon\lambda^4}{\tau_m}\right)C^{(4)}_{m+1,m}-\frac{4\epsilon\lambda^4}{\tau_m}C^{(4)}_{m,m}+\left(\frac{3\epsilon\lambda^4}{\tau_{-1+m}}+\frac{\epsilon\lambda^4}{\tau_m}\right)C^{(4)}_{m,m-1}\right.$$
$$\left.-\left(\frac{3\epsilon\lambda^4}{\tau_{-1+m}}+\frac{\epsilon\lambda^4}{\tau_m}\right)C^{(4)}_{m+1,m-1}-\frac{4\epsilon\lambda^4}{\tau_m}C^{(4)}_{m+1,m+1}+\left(\frac{\epsilon\lambda^4}{\tau_m}+\frac{3\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(4)}_{m+2,m+1}-\left(\frac{\epsilon\lambda^4}{\tau_m}+\frac{3\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(4)}_{m+2,m}\right]$$
$$+\delta_{n,m-1}\left[\left(-\frac{2\lambda^2}{\tau_{-1+m}}+\frac{3\epsilon\lambda^4}{\tau_{-2+m}}+\frac{3\epsilon\lambda^4}{\tau_m}+\frac{34\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(4)}_{m,m-1}-\frac{4\epsilon\lambda^4}{\tau_{-1+m}}C^{(4)}_{m,m}+\left(\frac{\epsilon\lambda^4}{\tau_{-1+m}}+\frac{3\epsilon\lambda^4}{\tau_m}\right)C^{(4)}_{m+1,m}\right.$$
$$\left.-\left(\frac{\epsilon\lambda^4}{\tau_{-1+m}}+\frac{3\epsilon\lambda^4}{\tau_m}\right)C^{(4)}_{m+1,m-1}-\frac{4\epsilon\lambda^4}{\tau_{-1+m}}C^{(4)}_{m-1,m-1}+\left(\frac{3\epsilon\lambda^4}{\tau_{-2+m}}+\frac{\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(4)}_{m-1,m-2}-\left(\frac{3\epsilon\lambda^4}{\tau_{-2+m}}+\frac{\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(4)}_{m,m-2}\right]$$
$$+\delta_{n,m+2}\left[\left(-\frac{3\epsilon\lambda^4}{\tau_m}-\frac{\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(4)}_{m+1,m}\left(-\frac{\epsilon\lambda^4}{\tau_m}-\frac{3\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(4)}_{m+2,m+1}+\left(\frac{\epsilon\lambda^4}{\tau_m}+\frac{\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(4)}_{m+2,m}\right]$$
$$+\delta_{n,m-2}\left[\left(-\frac{\epsilon\lambda^4}{\tau_{-2+m}}-\frac{3\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(4)}_{m,m-1}+\left(-\frac{3\epsilon\lambda^4}{\tau_{-2+m}}-\frac{\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(4)}_{m-1,m-2}+\left(\frac{\epsilon\lambda^4}{\tau_{-2+m}}+\frac{\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(4)}_{m,m-2}\right]. \tag{C3}$$

The diagonal scaling exponent is derived up to first order in ϵ resorting to linear perturbation theory. If pure second neighbor interactions are taken into account, the constant $q(0)$ is specified by

$$q(0)=\frac{1-(1+\lambda^{-2H(2)}+\lambda^{-H(2)})z(0)}{z(0)+z(0)^2\lambda^{-3H(2)}}. \tag{C4}$$

The result is approximately equal to one for all ξ ranging between [0,2]. The first order correction $z'(0)$ is extracted from the solution of the linear system:

$$(4\lambda^{2+H(2)}+4\lambda^2 z(0))x'(0)+4\lambda^2 x(0)z'(0)$$
$$=-4\lambda^{4+2H(2)}x(0)+4\lambda^{4-H(2)}x(0)z(0)+4\lambda^{2+2[1-H(2)]}x(0)z(0)^2$$
$$+2\lambda^4(9-4x(0)\quad+[4-22x(0)]z(0))+2\lambda^{4+H(2)}(4+[9-24x(0)]-4x(0)z(0)),$$

$$[\lambda^2(-3-\lambda^{-H(2)}-\lambda^{H(2)})z(0)+\lambda^{2-2H(2)}z(0)^2]x'(0)+[\lambda^2+(-3-\lambda^{-H(2)}-\lambda^{H(2)})x(0)\lambda^2+2\lambda^{2-2H(2)}x(0)z(0)]z'(0)$$
$$=-13\lambda^4-3\lambda^{4+H(2)}+3\lambda^{4+H(2)}(2+\lambda^{-H(2)})x(0)-13\lambda^4z(0)-3\lambda^{4-H(2)}z(0)$$
$$+\lambda^4(42-2\lambda^{-2H(2)}+7\lambda^{-H(2)}+9\lambda^{H(2)}+q(0))x(0)z(0)$$
$$+\lambda^4x(0)z(0)^2[1-10\lambda^{-2H(2)}+(-1+2q(0))\lambda^{-3H(2)}+(3+2q(0))\lambda^{-H(2)}]$$
$$+\lambda^{4-4H(2)}q(0)x(0)z(0)^3,$$

$$\lambda^{2-H(2)}x(0)[1-2z(0)(1-q(0)+3\lambda^{-2H(2)}q(0)z(0)+\lambda^{-2H(2)}+\lambda^{-H(2)})]z'(0)$$
$$+\lambda^{2-H(2)}z(0)[1-z(0)(1+\lambda^{-2H(2)}+\lambda^{-H(2)}-q(0)$$
$$+\lambda^{-2H(2)}q(0)z(0))]x'(0)+\lambda^{2-H(2)}z(0)^2[x(0)+\lambda^{-2H(2)}x(0)z(0)]q'(0)$$
$$=2\lambda^{4+H(2)}(-1+\lambda^{-3H(2)}-4\lambda^{-2H(2)}-2\lambda^{-H(2)})x(0)z(0)+\lambda^{4-H(2)}z(0)^2+6\lambda^{2+2(1-H(2))}x(0)z(0)^2$$
$$-2\lambda^{4-4H(2)}q(0)x(0)z(0)^2-\lambda^4(1+q(0))x(0)z(0)^2-\lambda^{4-3H(2)}(-7+2q(0))x(0)z(0)^2$$
$$+\lambda^{4-H(2)}(2-7q(0)+q(0)^3)x(0)z(0)^2+2\lambda^{4-H(2)}(1+\lambda^{-2H(2)})x(0)z(0)^3$$
$$-\lambda^{4-H(2)}(2+\lambda^{-4H(2)}+7\lambda^{3H(2)}+\lambda^4+2\lambda^4(1+\lambda^{-H(2)})z(0)+2\lambda^{H(2)})q(0)x(0)z(0)^3$$
$$+2\lambda^{4-2H(2)}(1+\lambda^{-3H(2)})q(0)^3x(0)z(0)^3+\lambda^{4-6H(2)}q(0)^3x(0)z(0)^4. \tag{C5}$$

APPENDIX D: THE INERTIAL OPERATOR FOR THE SIXTH MOMENT OF THE CORRELATION UP TO FIRST ORDER

Under the hypothesis that pure second neighbor interactions do not require new equations to specify the diagonal scaling for small values of ϵ, there are only four equations describing how global coupling is renormalized by relevant pure short range interactions. Given the mth shell, one has

$$0=\sum_{p,q,r}[I^{(6;0)}_{m,m,m;p,q,r}+\epsilon I^{(6;1)}_{m,m,m;p,q,r}]C^{(6);1}_{p,q,r}$$
$$=-\left(\frac{3\lambda^2}{\tau_{-1+m}}+\frac{3\lambda^2}{\tau_m}-\frac{45\epsilon\lambda^4}{\tau_{-1+m}}-\frac{45\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{m,m,m}+\left(\frac{9\lambda^2}{\tau_m}-\frac{9\epsilon\lambda^4}{\tau_{1+m}}-\frac{36\epsilon\lambda^4}{\tau_{-1+m}}-\frac{207\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{m,m,1+m}$$
$$+\left(\frac{9\lambda^2}{\tau_{-1+m}}-\frac{9\epsilon\lambda^4}{\tau_{-2+m}}-\frac{36\epsilon\lambda^4}{\tau_m}-\frac{207\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{m,m,-1+m}+\left(\frac{72\epsilon\lambda^4}{\tau_{-1+m}}+\frac{72\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{-1+m,m,1+m}$$
$$+\frac{72\epsilon\lambda^4}{\tau_m}C^{(6)}_{1+m,1+m,m}+\frac{72\epsilon\lambda^4}{\tau_{-1+m}}C^{(6)}_{-1+m,-1+m,m}+\frac{9\epsilon\lambda^4}{\tau_{1+m}}C^{(6)}_{m,m,2+m}+\frac{9\epsilon\lambda^4}{\tau_{-2+m}}C^{(6)}_{m,m,-2+m},$$

$$0=\sum_{p,q,r}[I^{(6;0)}_{m,m,m-1;p,q,r}+\epsilon I^{(6;1)}_{m,m,m-1;p,q,r}]C^{(6)}_{p,q,r}$$
$$=\left(\frac{\lambda^2}{\tau_{-1+m}}-\frac{5\epsilon\lambda^4}{\tau_m}-\frac{23\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{m,m,m}+\left(\frac{10\epsilon\lambda^4}{\tau_{-1+m}}+\frac{15\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{m,m,1+m}$$
$$-\left(\frac{\lambda^2}{\tau_{-2+m}}+\frac{2\lambda^2}{\tau_m}+\frac{7\lambda^2}{\tau_{-1+m}}-\frac{17\epsilon\lambda^4}{\tau_{-2+m}}-\frac{40\epsilon\lambda^4}{\tau_m}-\frac{169\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{m,m,-1+m}$$
$$+\left(\frac{4\lambda^2}{\tau_m}-\frac{4\epsilon\lambda^4}{\tau_{1+m}}-\frac{8\epsilon\lambda^4}{\tau_{-2+m}}-\frac{36\epsilon\lambda^4}{\tau_{-1+m}}-\frac{96\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{-1+m,m,1+m}+\left(\frac{4\lambda^2}{\tau_{-1+m}}-\frac{8\epsilon\lambda^4}{\tau_m}-\frac{12\epsilon\lambda^4}{\tau_{-2+m}}-\frac{124\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{-1+m,-1+m,m}$$
$$+\left(\frac{8\epsilon\lambda^4}{\tau_{-1+m}}+\frac{8\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{-1+m,-1+m,1+m}+\frac{8\epsilon\lambda^4}{\tau_m}C^{(6)}_{1+m,1+m,-1+m}+\frac{8\epsilon\lambda^4}{\tau_{-1+m}}C^{(6)}_{-1+m,-1+m,-1+m}$$
$$+\left(\frac{12\epsilon\lambda^4}{\tau_{-1+m}}+\frac{24\epsilon\lambda^4}{\tau_{-2+m}}\right)C^{(6)}_{-2+m,-1+m,m}+\left(\frac{\lambda^2}{\tau_{-2+m}}-\frac{\epsilon\lambda^4}{\tau_{-3+m}}-\frac{4\epsilon\lambda^4}{\tau_m}-\frac{6\epsilon\lambda^4}{\tau_{-1+m}}-\frac{17\epsilon\lambda^4}{\tau_{-2+m}}\right)C^{(6)}_{m,m,-2+m}$$
$$+\left(\frac{8\epsilon\lambda^4}{\tau_{-2+m}}+\frac{8\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{-2+m,m,1+m}+\frac{4\epsilon\lambda^4}{\tau_{1+m}}C^{(6)}_{-1+m,m,2+m}+\frac{\epsilon\lambda^4}{\tau_{-3+m}}C^{(6)}_{m,m,-3+m},$$

$$\sum_{p,q,r}[I^{(6;0)}_{m,m-1,m-1;p,q,r}+\epsilon I^{(6;1)}_{m,m-1,m-1;p,q,r}]C^{(6)}_{p,q,r}$$

$$=\frac{8\epsilon\lambda^4}{\tau_{-1+m}}C^{(6)}_{m,m,m}+\left(\frac{4\lambda^2}{\tau_{-1+m}}-\frac{8\epsilon\lambda^4}{\tau_{-2+m}}-\frac{12\epsilon\lambda^4}{\tau_m}-\frac{124\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{m,m,-1+m}+\left(\frac{12\epsilon\lambda^4}{\tau_{-1+m}}+\frac{24\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{-1+m,m,1+m}$$

$$-\left(\frac{\lambda^2}{\tau_m}+\frac{2\lambda^2}{\tau_{-2+m}}+\frac{7\lambda^2}{\tau_{-1+m}}-\frac{17\epsilon\lambda^4}{\tau_m}-\frac{40\epsilon\lambda^4}{\tau_{-2+m}}-\frac{169\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{-1+m,-1+m,m}$$

$$+\left(\frac{\lambda^2}{\tau_m}-\frac{4\epsilon\lambda^4}{\tau_{-2+m}}-\frac{6\epsilon\lambda^4}{\tau_{-1+m}}-\frac{17\epsilon\lambda^4}{\tau_m}-\frac{\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(6)}_{-1+m,-1+m,1+m}$$

$$+\left(\frac{\lambda^2}{\tau_{-1+m}}-\frac{5\epsilon\lambda^4}{\tau_{-2+m}}-\frac{23\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{-1+m,-1+m,-1+m}+\left(\frac{8\epsilon\lambda^4}{\tau_{-2+m}}+\frac{8\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{m,m,-2+m}$$

$$+\left(\frac{15\epsilon\lambda^4}{\tau_{-2+m}}+\frac{10\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{-1+m,-1+m,-2+m}+\left(\frac{4\lambda^2}{\tau_{-2+m}}-\frac{4\epsilon\lambda^4}{\tau_{-3+m}}-\frac{8\epsilon\lambda^4}{\tau_m}-\frac{36\epsilon\lambda^4}{\tau_{-1+m}}-\frac{96\epsilon\lambda^4}{\tau_{-2+m}}\right)C^{(6)}_{-2+m,-1+m,m}$$

$$+\frac{\epsilon\lambda^4}{\tau_{1+m}}C^{(6)}_{-1+m,-1+m,2+m}+\left(\frac{8\epsilon\lambda^4}{\tau_{-2+m}}+\frac{8\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{-2+m,-1+m,1+m}+\frac{8\epsilon\lambda^4}{\tau_{-2+m}}C^{(6)}_{-2+m,-2+m,m}+\frac{4\epsilon\lambda^4}{\tau_{-3+m}}C^{(6)}_{-3+m,-1+m,m},$$

$$\sum_{p.q,r}[I^{(6;0)}_{m,m+1,m-1;p,q,r}+\epsilon I^{(6;1)}_{m,m+1,m-1;p,q,r}]C^{(6)}_{p,q,r}$$

$$=\left(\frac{2\epsilon\lambda^4}{\tau_{-1+m}}+\frac{2\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{m,m,m}+\left(\frac{\lambda^2}{\tau_{-1+m}}-\frac{2\epsilon\lambda^4}{\tau_{1+m}}-\frac{12\epsilon\lambda^4}{\tau_m}-\frac{22\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{m,m,1+m}$$

$$+\left(\frac{\lambda^2}{\tau_m}-\frac{2\epsilon\lambda^4}{\tau_{-2+m}}-\frac{12\epsilon\lambda^4}{\tau_{-1+m}}-\frac{22\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{m,m,-1+m}+\left(\frac{3\epsilon\lambda^4}{\tau_{-1+m}}+\frac{6\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{1+m,1+m,m}$$

$$+\left(\frac{3\epsilon\lambda^4}{\tau_m}+\frac{6\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{-1+m,-1+m,m}-\left(\frac{\lambda^2}{\tau_{-2+m}}+\frac{\lambda^2}{\tau_{1+m}}+\frac{4\lambda^2}{\tau_{-1+m}}+\frac{4\lambda^2}{\tau_m}-\frac{18\epsilon\lambda^4}{\tau_{-2+m}}-\frac{18\epsilon\lambda^4}{\tau_{1+m}}-\frac{84\epsilon\lambda^4}{\tau_{-1+m}}-\frac{84\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{-1+m,m,1+m}$$

$$+\left(\frac{\lambda^2}{\tau_m}-\frac{2\epsilon\lambda^4}{\tau_{-2+m}}-\frac{3\epsilon\lambda^4}{\tau_{-1+m}}-\frac{3\epsilon\lambda^4}{\tau_{1+m}}-\frac{20\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{1+m,1+m,-1+m}$$

$$+\left(\frac{\lambda^2}{\tau_{-1+m}}-\frac{2\epsilon\lambda^4}{\tau_{1+m}}-\frac{3\epsilon\lambda^4}{\tau_{-2+m}}-\frac{3\epsilon\lambda^4}{\tau_m}-\frac{20\epsilon\lambda^4}{\tau_{-1+m}}\right)C^{(6)}_{-1+m,-1+m,1+m}$$

$$+\left(\frac{3\epsilon\lambda^4}{\tau_m}+\frac{6\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(6)}_{-1+m,1+m,2+m}+\left(\frac{3\epsilon\lambda^4}{\tau_{-1+m}}+\frac{6\epsilon\lambda^4}{\tau_{-2+m}}\right)C^{(6)}_{-2+m,-1+m,1+m}$$

$$+\left(\frac{2\epsilon\lambda^4}{\tau_{-1+m}}+\frac{2\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(6)}_{-1+m,-1+m,2+m}+\left(\frac{2\epsilon\lambda^4}{\tau_{-1+m}}+\frac{2\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(6)}_{m,m,2+m}+\left(\frac{2\epsilon\lambda^4}{\tau_{-2+m}}+\frac{2\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{m,m,-2+m}$$

$$+\left(\frac{2\epsilon\lambda^4}{\tau_{-2+m}}+\frac{2\epsilon\lambda^4}{\tau_m}\right)C^{(6)}_{1+m,1+m,-2+m}+\left(\frac{\lambda^2}{\tau_{-2+m}}-\frac{\epsilon\lambda^4}{\tau_{-3+m}}-\frac{2\epsilon\lambda^4}{\tau_{1+m}}-\frac{3\epsilon\lambda^4}{\tau_{-1+m}}-\frac{8\epsilon\lambda^4}{\tau_m}-\frac{18\epsilon\lambda^4}{\tau_{-2+m}}\right)C^{(6)}_{-2+m,m,1+m}$$

$$+\left(\frac{\lambda^2}{\tau_{1+m}}-\frac{\epsilon\lambda^4}{\tau_{2+m}}-\frac{2\epsilon\lambda^4}{\tau_{-2+m}}-\frac{3\epsilon\lambda^4}{\tau_m}-\frac{8\epsilon\lambda^4}{\tau_{-1+m}}-\frac{18\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(6)}_{-1+m,m,2+m}+\left(\frac{2\epsilon\lambda^4}{\tau_{-2+m}}+\frac{2\epsilon\lambda^4}{\tau_{1+m}}\right)C^{(6)}_{-2+m,m,2+m}$$

$$+\frac{\epsilon\lambda^4}{\tau_{2+m}}C^{(6)}_{-1+m,m,3+m}+\frac{\epsilon\lambda^4}{\tau_{-3+m}}C^{(6)}_{-3+m,m,1+m}. \tag{D1}$$

APPENDIX E: THE INERTIAL OPERATOR FOR THE EIGHTH MOMENT OF THE CORRELATION UP TO FIRST ORDER

The set of independent equations is finally given as

$$\begin{aligned}
&\sum_{p,q,r,s}[I^{(8;0)}_{m,m,m,m;p,q,r,s}+\epsilon I^{(8;1)}_{m,m,m,m;p,q,r,s}]C^{(8)}_{p,q,r,s}=0,\\
&\sum_{p,q,r,s}[I^{(8,0)}_{m,m,m,m-1;p,q,r,s}+\epsilon I^{(8;1)}_{m,m,m,m-1;p,q,r,s}]C^{(8)}_{p,q,r,s}=0,\\
&\sum_{p,q,r,s}[I^{(8,0)}_{m,m,m-1,m-1;p,q,r,s}+\epsilon I^{(8;1)}_{m,m,m-1,m-1;p,q,r,s}]C^{(8)}_{p,q,r,s}=0,\\
&\sum_{p,q,r,s}[I^{(8,0)}_{m,m,m+1,m-1;p,q,r,s}+\epsilon I^{(8;1)}_{m,m,m+1,m-1;p,q,r,s}]C^{(8)}_{p,q,r,s}=0,\\
&\sum_{p,q,r,s}[I^{(8,0)}_{m,m-1,m-1,m-1;p,q,r,s}+\epsilon I^{(8;1)}_{m,m-1,m-1,m-1;p,q,r,s}]C^{(8)}_{p,q,r,s}=0,\\
&\sum_{p,q,r,s}[I^{(8,0)}_{m,m+1,m-1,m-1;p,q,r,s}+\epsilon I^{(8;1)}_{m,m+1,m-1,m-1;p,q,r,s}]C^{(8)}_{p,q,r,s}=0,\\
&\sum_{p,q,r,s}[I^{(8,0)}_{m,m+1,m+1,m-1;p,q,r,s}+\epsilon I^{(8;1)}_{m,m+1,m+1,m-1;p,q,r,s}]C^{(8)}_{p,q,r,s}=0,\\
&\sum_{p,q,r,s}[I^{(8,0)}_{m,m+1,m-1,m-2;p,q,r,s}+\epsilon I^{(8;1)}_{m,m+1,m-1,m-2;p,q,r,s}]C^{(8)}_{p,q,r,s}=0.
\end{aligned} \tag{E1}$$

The closure is provided again assuming scaling for all the possible conditioned expectation values with respect to a given shell. It follows that

$$\begin{aligned}
C^{(8)}_{m+n,m+n,m+n,m+n}&=z(\epsilon)^{-l}C^{(8)}_{m,m,m,m},\\
C^{(8)}_{m+n,m+n,m+n,m}&=y_1(\epsilon)k_{n-1}^{-H(6,\epsilon)}C^{(8)}_{m,m,m,m},\\
C^{(8)}_{m+n,m+n,m,m}&=y_2(\epsilon)k_{n-1}^{-H(4,\epsilon)}C^{(8)}_{m,m,m,m},\\
C^{(8)}_{m+n,m,m,m}&=y_3(\epsilon)k_{n-1}^{-H(2)}C^{(8)}_{m,m,m,m},\\
C^{(8)}_{m+n+p,m+n,m,m}&=y_4(\epsilon)k_{p-1}^{-H(2)}k_{n-1}^{-H(4,\epsilon)}C^{(8)}_{m,m,m,m},\\
C^{(8)}_{m+n+p,m+n,m+n,m}&=y_5(\epsilon)k_{p-1}^{-H(2)}k_{n-1}^{-H(6,\epsilon)}C^{(8)}_{m,m,m,m},\\
C^{(8)}_{m+n+p,m+n+p,m+n,m}&=y_6(\epsilon)k_{p-1}^{-H(4,\epsilon)}k_{n-1}^{-H(6,\epsilon)}C^{(8)}_{m,m,m,m},\\
C^{(8)}_{m+n+p+q,m+n+p,m+n,m}&=y_7(\epsilon)k_{q-1}^{-H(2)}k_{p-1}^{-H(4,\epsilon)}k_{n-1}^{-H(6,\epsilon)}C^{(8)}_{m,m,m,m}.
\end{aligned} \tag{E2}$$

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