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Collapse arrest and soliton stabilization in nonlocal nonlinear media

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We investigate the properties of localized waves in cubic nonlinear materials with a symmetric nonlocal nonlinear response of arbitrary shape and degree of nonlocality, described by a general nonlocal nonlinear Schrödinger type equation. We prove rigorously by bounding the Hamiltonian that nonlocality of the nonlinearity prevents collapse in, e.g., Bose-Einstein condensates and optical Kerr media in all physical dimensions. The nonlocal nonlinear response must be symmetric and have a positive definite Fourier spectrum, but can otherwise be of completely arbitrary shape and degree of nonlocality. We use variational techniques to find the soliton solutions and illustrate the stabilizing effect of nonlocality.

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I. INTRODUCTION

Collapse is a fundamental physical phenomenon well known in the theory of waves in nonlinear media. It refers to the situation when strong contraction or self-focusing of a wave leads to a catastrophic increase or blowup of its amplitude after a finite time or propagation distance (see [1–3] for reviews). Wave collapse has been observed in plasma waves [4] (the famous Langmuir wave collapse), electromagnetic waves or laser beams [5] (also called self-focusing), Bose-Einstein condensates (BEC's) or matter waves [6], and even capillary-gravity waves on deep water [7]. The effect of collapse appears also in astrophysics, where the gravitational attraction plays the same role as the self-focusing of electromagnetic waves, tending to compress stars of sufficient mass, eventually leading to their collapse into a black hole [8].

Typically the contraction must be able to act freely in two or more dimensions to be strong enough to generate a collapse. Moreover, the so-called norm, which is the power for electromagnetic and plasma waves, the atom density for BEC's, and the mass for stars, must be above a certain critical value for a collapse to occur. Most commonly the collapse has been discussed in the context of the nonlinear Schrödinger (NLS) equation, which is a universal model for dispersive (or diffractive) weakly nonlinear physical systems [9]. The NLS equation models, e.g., all systems mentioned above, in which a wave collapse has been predicted and verified experimentally.

The collapse singularity is an artifact of the model and signals the limit of its validity. Close to the singularity, when the amplitude is extremely high and the temporal and spatial scales are extremely short, different physical processes will come into play [1–3]. A common effect is nonlinear dissipation, such as two-photon absorption of electromagnetic waves and inelastic two- and three-body recombination for matter waves, which efficiently absorbs the collapsing part of the wave. Thus collapse acts as an *effective nonlinear loss mechanism*, as is well known in, e.g., Langmuir turbulence [11] and BEC's [12]. Effects such as discreteness [13], nonparaxiality [14], and saturation of the nonlinearity (see [3] for references to all the different types of saturation, such as exponential, threshold, logarithmic, etc.) will also completely eliminate the possibility of a collapse singularity appearing. In contrast, effects such as weak linear loss [3], temperature fluctuations [15], and spatial incoherence [16], cannot eliminate the collapse, only change the critical value of the norm. In any case the collapse effect represents a *strong mechanism for energy localization*, which it is important to study to understand the properties of a given physical system.

The inherent nonlocal character of the nonlinearity has attracted considerable interest as a means of eliminating collapse and stabilizing multidimensional solitary waves. Nonlocality appears naturally in optical systems with a thermal [17] or diffusive [18] type of nonlinearity. Nonlocality is also known to influence the propagation of electromagnetic waves in plasmas [19–22] and plays an important role in the theory of BEC's, where it accounts for the finite-range many-body interaction [12,23–25].

In this work we consider NLS equations with a general nonlocal form of the nonlinearity. Turitsyn proved the absence of collapse for three particular shapes of the nonlocal nonlinear response [26]. The analysis of the collapse conditions for general response functions is difficult and has been carried out only numerically [24]. However, in many systems, such as BEC's, one has no knowledge of the particular response function. Furthermore, the degree of nonlocality is the relative width of the response function and the wave packet and thus it changes dynamically when the wave packet spreads or contracts. Therefore it is important to maintain the generality of the nonlocal response function in the model. Here we prove rigorously that nonlocality eliminates collapse in all physical dimensions for arbitrary shapes of the nonlocal response, as long as the response function is symmetric and has a positive definite Fourier spectrum.

II. GENERAL MODEL

We consider the evolution of a wave field $u=u(\vec{r})=u(\vec{r},\tau)$ described by the general nonlocal NLS equation

$$i\frac{\partial u}{\partial \tau}+\nabla^2 u - V(\vec{r})u + N(I)u = 0, \tag{1}$$

where $V=V(\vec{r})$ is an external (linear) confining potential, $I=I(\vec{r})=I(\vec{r},\tau)=|u|^2$, τ is the evolution coordinate and $\vec{r}=(r_1,r_2,r_3)$ spans a D-dimensional "transverse" coordinate space. By virtue of being a confining potential, $V(\vec{r})$ has a finite global minimum, which can be set to zero without loss of generality due to the gauge invariance of Eq. (1). Thus $V(\vec{r})>0$. The nonlinear term $N=N(I)$ is represented in the general nonlocal form

$$N(I)=\int R(\vec{r}\,'-\vec{r})I(\vec{r}\,')d\vec{r}\,', \tag{2}$$

where the integral $\int d\vec{r}$ is over all transverse dimensions. We consider only response functions $R(\vec{r})$ that are real (i.e., no nonlinear loss or gain) and symmetric (e.g., excluding the asymmetric Raman response—see [28] and references therein). We further assume the response function to be localized or L^1 integrable like all physically reasonable response functions. In this case Eq. (1) may always be rescaled so that

$$\int R(\vec{r})d\vec{r}=1 \tag{3}$$

without any lack of generality. In media with, e.g., a thermal or diffusive type of nonlinearity the response profile is an exponential function, $R(\vec{r})=(1/2\sigma)\exp(|\vec{r}|/\sigma)$ [17,18], where σ determines the degree of nonlocality.

Because the response function is real Eq. (1) conserves the power (in optics) or number of atoms (for BEC's) P,

$$P=\int I d\vec{r} \tag{4}$$

for localized waves. Because the response function is also symmetric Eq. (1) conserves the Hamiltonian H,

$$H=\int\left[|\nabla u|^2+VI-\frac{1}{2}NI\right]d\vec{r}. \tag{5}$$

In optics u is the envelope of the electric field with intensity I and V represents a guiding structure (waveguide). Here Eq. (2) represents a general phenomenological model for self-focusing Kerr-like media, in which the change in the refractive index induced by an optical beam involves a transport process. This includes heat conduction in materials with a thermal nonlinearity [17] or diffusion of molecules or atoms in atomic vapors [18]. A nonlocal response in the form (2) appears naturally due to many-body interaction processes in the description of BEC's [12,24,25,27], if the assumption of a zero-range interaction potential is relaxed [27]. For BEC's with a negative scattering length Eq. (1) is the nonlocal Gross-Pitaevskii (GP) equation for the collective wave function u (τ is time), with I representing the density of atoms and V representing the magnetic trap.

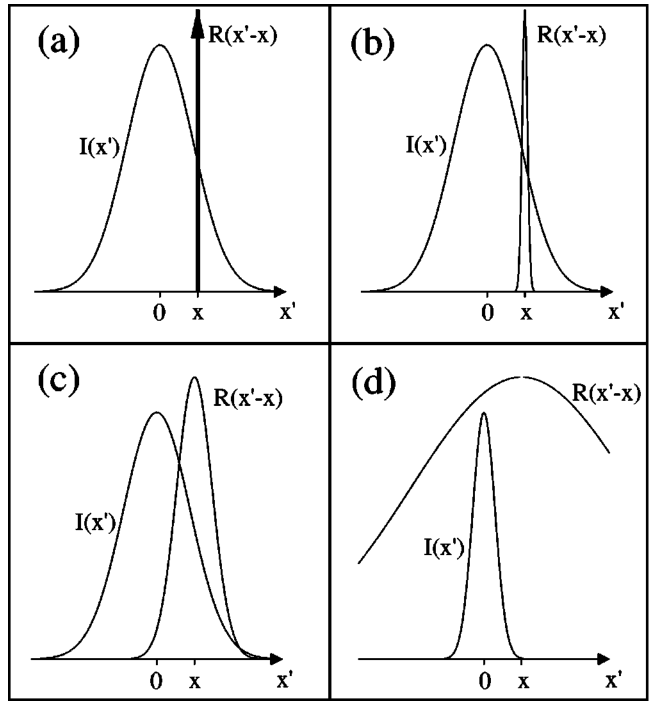

FIG. 1. Degrees of nonlocality, as given by the relative width of the response function R and the intensity profile I in the x plane. Shown is the local (a), the weakly nonlocal (b), the general (c), and the strongly nonlocal (d) response.

III. SIMPLE KNOWN LIMITS

In the limit when the response function is a delta function, $R(\vec{r})=\delta(|\vec{r}|)$, the nonlinear response is local [see Fig. 1(a)] and simply given by

$$N(I)=I, \tag{6}$$

as in local optical Kerr media described by the standard NLS equation and in BEC's described by the standard GP equation. In this local limit multidimensional optical beams with a power higher than a certain critical value will experience unbounded self-focusing and *collapse* after a finite propagation distance [1–3]. It is also well known that BEC's will collapse when the total number of atoms is larger than a critical number [12].

With increasing width of the response function $R(\vec{r})$ the wave intensity in the vicinity of the point $\vec{r}$ also contributes to the nonlinear response at that point. In case of weak nonlocality, when $R(\vec{r})$ is much narrower than the width of the beam [see Fig. 1(b)], one can expand $I(\vec{r}\,')$ around $\vec{r}\,'=\vec{r}$ and obtain the simplified model

$$N(I)=I+\gamma\nabla^2 I,\quad \gamma=\frac{1}{2}\int r^2R(r)d\vec{r}, \tag{7}$$

where the small positive definite parameter γ measures the relative width of the nonlocal response. The diffusion type model (7) of the nonlocal nonlinearity is a model in its own right in plasma physics, where γ can take any sign [19,20]. It was also applied to BEC's [25], nonlinear optics [29], and energy transfer in biomolecules [30]. In weakly nonlocal media with $N(I)=I+\gamma\nabla^2 I$ it is straightforward to show that collapse cannot occur. This was first done for plasmas [20], and later for BEC's [25].

In the limit of a strongly nonlocal response much broader than the characteristic width of the wave function [see Fig. 1(d)], one can instead expand the response function and obtain (to lowest order)

$$N(I)\approx P(R_0+R_2r^2), \tag{8}$$

where $R_0=R(\vec{0})$ and $R_2=\frac{1}{2}\nabla^2R(\vec{0})$. The evolution of optical beams in such a strongly nonlocal medium was considered in [31]. Since this relation is linear, the highly nonlinear effect of collapse cannot occur.

So in the two extreme limits of a weakly and highly nonlocal nonlinear response the collapse is prevented. For arbitrary degree of nonlocality it is difficult to prove anything rigorously. Just saying that the dynamics is described by either the weakly nonlocal model (7) or the linear oscillator model (8), which both have no collapse, is not enough. First of all the degree of nonlocality is the *relative width* of the response function and the wave packet and thus it changes dynamically when the wave packet spreads or contracts. Thus, the system may dynamically switch state, e.g., from being in the local limit (6) to the highly nonlocal limit (8). Furthermore, as is well known from studies of general NLS equations, the typical singularity is a so-called blowup featuring the amplitude locally going to infinity on a broad background localized structure [1]. Such a two-scale field distribution, which was also recently observed in BEC's [24], is clearly described by neither of the two simple limiting systems.

IV. PROOF OF ABSENCE OF COLLAPSE AND SOLITON STABILITY

The stabilizing effect of nonlocality of an arbitrary degree was proven by Turitsyn for three specific examples, including Coulomb interaction $[R(\vec{r})=1/|\vec{r}|]$ [26]. Turitsyn bounded the Hamiltonian from below for fixed power, which proves that a collapse cannot occur and that the soliton solutions are stable in the Liapunov sense. Here we consider the general case of *arbitrarily shaped, nonsingular response functions* and prove rigorously that the Hamiltonian is bounded from below in all dimensions.

Introducing the D-dimensional Fourier transform (denoted with a tilde)

$$\tilde{I}(\vec{k})=\int I(\vec{r})\exp(i\vec{k}\cdot\vec{r})d\vec{r} \tag{9}$$

and its inverse

$$I(\vec{r})=\frac{1}{(2\pi)^D}\int \tilde{I}(\vec{k})\exp(-i\vec{k}\cdot\vec{r})d\vec{k}, \tag{10}$$

it is straightforward to show that for $N(I)$ given by Eq. (2) the following relations hold:

$$|\tilde{I}(\vec{k})|=\left|\int I(\vec{r})e^{i\vec{k}\cdot\vec{r}}d\vec{r}\right|\leqslant\int I d\vec{r}=P, \tag{11}$$

$$\int NId\vec{r}=\frac{1}{(2\pi)^D}\int \tilde{R}(\vec{k})|\tilde{I}(\vec{k})|^2d\vec{k}. \tag{12}$$

With these relations we can bound the Hamiltonian by conserved quantities, which is necessary for employing standard Liapunov stability theory [32] (first applied by Rosen [33] for the standard NLS equation). For response functions with a finite degree of nonlocality $R_0<\infty$ and a positive definite spectrum $\tilde{R}(\vec{k})\geqslant 0$, we obtain the following bound of the Hamiltonian:

$$R_0<\infty,\ \ \tilde{R}(\vec{k})\geqslant 0:\ \ H\geqslant||\nabla u||_2^2-\frac{R_0}{2}P^2, \tag{13}$$

where $||u||_p^p\equiv\int|u|^pd\vec{r}>0$ and we have used that $V(\vec{r})>0$. In the local limit when $R(\vec{r})=\delta(\vec{r})$ and thus $R_0=\infty$, the well-known properties of the standard NLS equation apply [1,2,9].

The inequality (13) is the main result of this article. It shows that, for all symmetric response functions with a positive definite Fourier spectrum and a finite value at the center, the Hamiltonian is bounded from below by the conserved quantity $-\frac{1}{2}R_0P^2$, or conversely, that the gradient norm $||\nabla u||_2^2$ is bounded from above by the conserved quantity $H+\frac{1}{2}R_0P^2$. According to standard Liapunov theory this represents a rigorous proof that a collapse with the wave-amplitude locally going to infinity cannot occur in BEC's, plasma, or optical Kerr media with a nonlocal nonlinear response, for any physically reasonable response function with a positive definite spectrum.

V. ILLUSTRATION THROUGH THE VARIATIONAL APPROACH

The stabilizing effect of the nonlocality can be further illustrated by the properties of the stationary solutions of Eq. (1). As a simple example we consider nonlocal optical bulk Kerr media with a Gaussian response

$$R(\vec{r}\,'-\vec{r})=\left(\frac{1}{\pi\sigma^2}\right)^{D/2}\exp\left(-\frac{|\vec{r}\,'-\vec{r}|^2}{\sigma^2}\right). \tag{14}$$

The ground-state stationary solutions are then radially symmetric bell shaped, nodeless solutions of the form $u(\vec{r},z)$

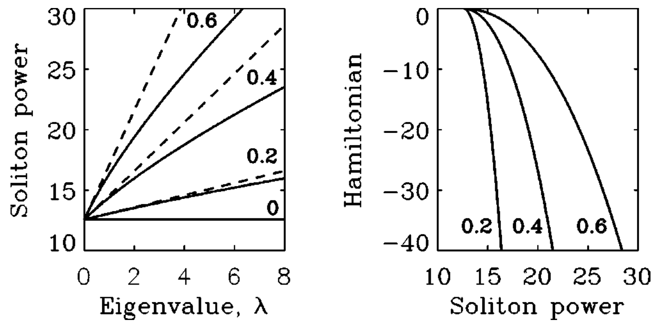

FIG. 2. 2D variational results with Gaussian response and trial function. Left: Soliton power (solid) versus eigenvalue λ for different degrees of nonlocality, $\sigma=0$, 0.2, 0.4, and 0.6. Dashed lines show the weakly nonlocal approximation. Right: Corresponding Hamiltonian versus power diagrams.

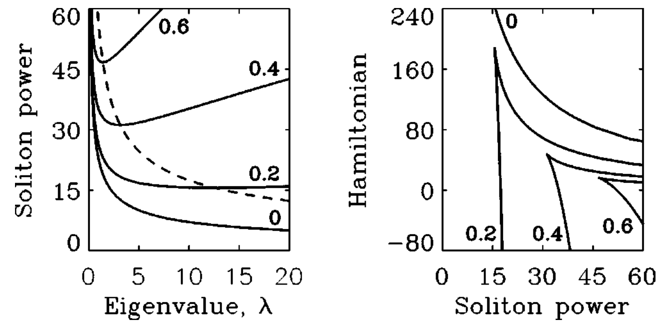

FIG. 3. 3D variational results with Gaussian response and trial function. Left: Soliton power (solid) versus eigenvalue λ for different degrees of nonlocality, $\sigma=0$, 0.2, 0.4, and 0.6. Dashed lines show the threshold power (17). Right: Corresponding Hamiltonian versus power diagrams.

$=\phi(r)\exp(i\lambda z)$, where the profile $\phi(r)$ is found from the Euler-Lagrange equations for the Lagrangian

$$L=\int\left[\lambda\phi^2+|\nabla\phi|^2-\frac{1}{2}N(\phi^2)\phi^2\right]d\vec{r}. \tag{15}$$

To capture the main physical effects we use the approximate variational technique with a Gaussian trial profile $\phi(r)=\alpha\exp[-(r/\beta)^2]$, in view of the fact that the Gaussian profile is an exact solution in the strongly nonlocal limit with $N(I)$ given by the parabolic potential (8). Inserting this ansatz into the Lagrangian (15), with N given by the general expression (2), the Euler-Lagrange equations give the amplitude $\alpha^2=(\lambda+D/\beta^2)(2+2\sigma^2/\beta^2)^{D/2}$ and width $\beta^2=[4-D+\sqrt{(4-D)^2+16\lambda\sigma^2}]/(2\lambda)$. In Fig. 2 we show the power $P_s=(\pi/2)^{D/2}\alpha^2\beta^D$ and Hamiltonian of the stationary solutions in two dimensions (2D). The dashed lines give the results of the weakly nonlocal approximation with N given by Eq. (7), from which $\alpha^2=4\lambda$ and $\beta^2=2/\lambda+2\sigma^2$ is found, resulting in the power

$$P_s=4\pi(1+\sigma^2\lambda), \tag{16}$$

where 4π is the (λ-independent) power of the Gaussian approximation to the soliton solution of the standard 2D NLS equation, recovered in the local limit $\sigma=0$.

In the 2D NLS equation the collapse is critical and the stationary solutions are "marginally stable" with $dP_s/d\lambda=0$ [1,2,9]. Typically, perturbations act against the self-focusing, with several effects, such as nonparaxiality and saturability, completely eliminating collapse [3]. This is also the case with nonlocality, as evidenced from Fig. 2 and the simplified expression (16), which shows that any finite width of the response function (nonzero value of σ) implies that $dP_s/d\lambda$ becomes positive definite. According to the (necessary) Vakhitov-Kolokolov (VK) criterion [34] the soliton solutions therefore (possibly) become linearly stable.

For small λ the soliton width β decreases as $1/\lambda$. Thus the assumption of weak nonlocality, i.e., that the soliton is much wider than the response function, applies only to sufficiently small values of λ satisfying $\lambda\sigma^2\ll 1$, which is also clearly seen from Fig. 2. The accuracy of the assumption of weak nonlocality is further discussed in Ref. [35] in terms of modulational instability.

The 3D case shown in Fig. 3 is more interesting, because the nonlinearity is much stronger than in 2D. The collapse in the local 3D NLS equation is so-called supercritical [1,2,9,10]. Again the soliton width β decreases as $1/\lambda$, so a threshold width should exist, below which the nonlocality is not strong enough to stabilize the soliton. This is exactly what is observed in Fig. 3: For $\lambda<\lambda^{\text{th}}$ the solitons are still linearly unstable with $dP_s/d\lambda<0$, but above threshold the nonlocality is strong enough to bend the curve and make $dP_s/d\lambda>0$, i.e., the solitons become linearly stable according to the VK criterion. From the definition $dP_s(\lambda^{\text{th}})/d\lambda=0$ the variational results give $\lambda^{\text{th}}=1/(2\sigma^2)$, corresponding to a threshold in the soliton power (dashed curve in Fig. 3) and width

$$P_s^{\text{th}}=(5\pi)^{3/2}5\sigma/4,\quad \beta^{\text{th}}=2\sigma, \tag{17}$$

which are both proportional to the degree of nonlocality σ. Thus, sufficiently broad and high-power solitons are stable. In the Hamiltonian versus power diagram in Fig. 3 the lower (upper) branches correspond to stable (unstable) solutions while the threshold is represented by the cusp [36]. This stable solution branch was recently found numerically in the context of BEC's with a nonlocal negative scattering potential [25]. It corresponds to high-density, self-bound states of the condensate.

VI. CONCLUSION

In conclusion we studied the properties of localized wave packets in nonlocal NLS equations. We have presented a simple, but *rigorous proof* that nonlocality of an *arbitrary shape* eliminates collapse in *all physical dimensions*. The only requirement is that the nonlocal response function should have a positive definite Fourier spectrum, as do most physically reasonable response functions.

We also demonstrated that multidimensional soliton solutions of the NLS equation may be stabilized by the nonlocality. This opens a new interesting discussion as to what is actually observed in collapse experiments in nonlocal systems. It seems clear that it all comes down to oscillations between opposite extreme states and how strong and rapid they are. Such oscillations were recently found to occur in BEC's through numerical and variational studies [24].

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[1] J. J. Rasmussen and K. Rypdal, Phys. Scr. **33**, 481 (1986).
[2] L. Bergé, Phys. Rep. **303**, 259 (1998).
[3] Y. S. Kivshar and D. E. Pelinovsky, Phys. Rep. **331**, 117 (2000).
[4] A. Y. Wong and P. Y. Cheung, Phys. Rev. Lett. **52**, 1222 (1984).
[5] E. Garmire, R. Y. Chiao, and C. H. Townes, Phys. Rev. Lett. **16**, 347 (1966).
[6] C. A. Sackett, J. M. Gerton, M. Welling, and R. C. Hulet, Phys. Rev. Lett. **82**, 876 (1999).
[7] P. P. Banerjee, A. Korpel, and K. E. Lonngren, Phys. Fluids **26**, 2393 (1983).
[8] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars—The Physics of Compact Objects* (John Wiley & Sons, New York, 1983); M. Houbiers and H. T. Stoof, Phys. Rev. A **54**, 5055 (1996).
[9] C. Sulem and P. Sulem, *The Nonlinear Schrödinger Equation* (Springer-Verlag, Berlin, 1999).
[10] E. A. Kuznetsov, J. J. Rasmussen, K. Rypdal, and S. K. Turitsyn, Physica D **87**, 273 (1995).
[11] V. E. Zakharov, in *Handbook of Plasma Physics Vol. 2. Basic Plasma Physics II*, edited by A. A. Galeev and R. N. Sudan (North-Holland, Amsterdam, 1984).
[12] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[13] O. Bang, J. J. Rasmussen, and P. L. Christiansen, Nonlinearity **7**, 205 (1994); Physica D **68**, 169 (1993).
[14] M. D. Feit and J. A. Fleck, J. Opt. Soc. Am. B **5**, 623 (1988).
[15] O. Bang *et al.*, Phys. Rev. E **49**, 4627 (1994); Appl. Anal. **57**, 3 (1994).
[16] O. Bang, D. Edmundson, and W. Krolikowski, Phys. Rev. Lett. **83**, 5479 (1999).
[17] A. G. Litvak, JETP Lett. **4**, 230 (1966).
[18] D. Suter and T. Blasberg, Phys. Rev. A **48**, 4583 (1993).
[19] A. G. Litvak, V. A. Mironov, G. M. Fraiman, and A. D. Yunakovskii, Sov. J. Plasma Phys. **1**, 31 (1975).
[20] T. A. Davydova and A. I. Fishchuk, Ukr. J. Phys. **40**, 487 (1995).
[21] H. L. Pécseli and J. J. Rasmussen, Plasma Phys. **22**, 421 (1980).
[22] L. Bergé and A. Couairon, Phys. Plasmas **7**, 210 (2000).
[23] K. Goral, K. Rzazewski, and T. Pfau, Phys. Rev. A **61**, 051601R (2000).
[24] V. M. Pérez-García, V. V. Konotop, and J. J. García-Ripoll, Phys. Rev. E **62**, 4300 (2000).
[25] A. Parola, L. Salasnich, and L. Reatto, Phys. Rev. A **57**, R3180 (1998).
[26] S. K. Turitsyn, Teor. Mat. Fiz. **64**, 226 (1985).
[27] D. V. Fedorov and A. S. Jensen, Phys. Rev. A **63**, 063608 (2001).
[28] G. P. Agrawal, *Nonlinear Fiber Optics*, 2nd ed. (Academic Press, San Diego, 1995); J. Wyller, Physica D **157**, 9 (2001).
[29] W. Królikowski and O. Bang, Phys. Rev. E **63**, 016610 (2001).
[30] X. Wang, D. W. Brown, K. Lindenberg, and B. J. West, Phys. Rev. A **37**, 3557 (1988).
[31] A. W. Snyder and D. J. Mitchell, Science **276**, 1538 (1997).
[32] J. La Salle and S. Lefschetz, *Stability by Liapunov's Direct Method* (Academic Press, New York, 1961).
[33] G. Rosen, J. Math. Phys. **6**, 1269 (1965).
[34] N. G. Vakhitov and A. A. Kolokolov, Radiophys. Quantum Electron. **16**, 783 (1975).
[35] W. Krolikowski, O. Bang, J. J. Rasmussen, and J. Wyller, Phys. Rev. E **64**, 016612 (2001).
[36] N. Akhmediev, A. Ankiewicz, and R. Grimshaw, Phys. Rev. E **59**, 6088 (1999).