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Potential dominance of oscillating crescent waves in finite width tanks

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Recently, it has been proposed that the emergence of previously observed oscillating crescent water wave patterns, created by class II (three-dimensional) instabilities which are in principle not dominant, could in fact be explained as an artifact of a finite width tank, combined with a suppression of the class I (Benjamin–Feir) instability. Within this context, we investigate quantitatively the dominance of class II deep water wave instabilities for particular transversal wavenumbers, and it is shown that the regions where non-phase-locked (oscillating) crescent wave patterns are locally dominant is surprisingly large, particularly for low to moderate wave steepness. This is an important realization for both experimentalists and numerical modelers currently studying these phenomena. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1852291]

The observation by Collard and Caulliez[1] of oscillating crescent waves in their physical experiments has puzzled scientists, as these are due to what is, in principle, a nondominant class II (Ref. 2) wave instability (the dominant instability corresponds to the phase-locked crescents observed, e.g., by Su *et al.*[3]). Indeed, Collard and Caulliez state that the emergence of their oscillating crescents could not be explained using existing theories. Recently, however, Fuhrman *et al.*[4] have demonstrated that the emergence of oscillating crescent wave patterns, very similar to those observed by Collard and Caulliez, could in fact be explained directly from the stability analysis of McLean,[2] when effects from a finite width tank were taken into account, combined with a suppression of the class I instability. They discuss a single example, fitting precisely within the actual tank width used by Collard and Caulliez, and having perturbation frequencies exactly matching those from the experiments. This finding raises the previously unanswered question: How prevalent are such regions in the instability plane where, for a given transversal wavenumber, non-phase-locked (i.e., oscillating) crescent wave patterns are in fact locally dominant? Hence, there is an apparent need for a reinvestigation of the instability regions to examine this issue, which is the aim of the present work.

In what follows, we work under the assumption that the two-dimensional class I (Benjamin–Feir[5]) instability is suppressed, in some way, so that it is weaker than the class II instability for any given wave steepness. This is justified within the current context, as Collard and Caulliez report observing crescent waves for (waveheight divided by wavelength) $H/L \approx 0.051$, i.e., well below $H/L \approx 0.10$ where they theoretically become dominant. This is most likely explained in their case by the use of a plastic film, though wind[6] and randomness[7] have also been reported to have a suppressive effect on the class I instability, while their relative effects on the class II instability remain open. Furthermore, we neglect any potential effects of extraneous parameters (e.g., the plas-

tic film, wind, etc.) on the instability regions themselves, as there is currently no realistic way of incorporating them. In other words, we limit our consideration to class II instabilities, and we assume that the basic assumptions of potential flow are valid.

To investigate quantitatively the dominance of various class II instabilities, we use an analysis based on the seminal work of McLean[2] (see also the relevant work of Kharif and Ramamonjiarisoa[8,9]), which considers the stability of finite amplitude plane deep water waves having wavenumber vector $\mathbf{k} = (k, 0)$ (computed here using the stream function solution of Fenton[10]), subject to infinitesimal three-dimensional perturbations having wavenumber vectors $\mathbf{k}_1 = (p+1, q)k$ and $\mathbf{k}_2 = (2-p, -q)k$. These wavenumbers satisfy the quintet resonance condition

$$3\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2, \qquad (1)$$

which is the physical mechanism responsible for the formation of such crescent patterns, as noted, e.g., by Shrira *et al.*[11] As shown by McLean,[2] the analysis leads to an eigenvalue problem for the complex frequency σ, where $\mathrm{Re}(\sigma)$ governs the perturbation frequencies (in a frame of reference moving with the unperturbed wave), while $\mathrm{Im}(\sigma)$ determines the relative strength of the instability. Throughout this work, we pay sole attention to the imaginary part, as this governs the relative dominance of various unstable modes. Following McLean,[2] we here take $k = g = 1$, where g is the gravitational acceleration. As this analysis has been described numerous times in the literature, we do not provide additional details here. Note that the analysis used here has been previously validated against McLean's results in Ref. 4, see their Table III, where very minor differences can be seen. These differences have since been found to be from McLean apparently using *exactly* the steepnesses, e.g., $ak = 0.1$, 0.2, 0.3, 0.35 (with $a = H/2$ the wave amplitude) in his analysis, while curiously reporting rounded values in terms of the steepness $H/L = ak/\pi$ (when we use the exact values for ak we arrive precisely at his values for σ).

Figure 1 shows four computed regions of class II instability for waves having steepnesses $H/L = 0.05$, 0.064,

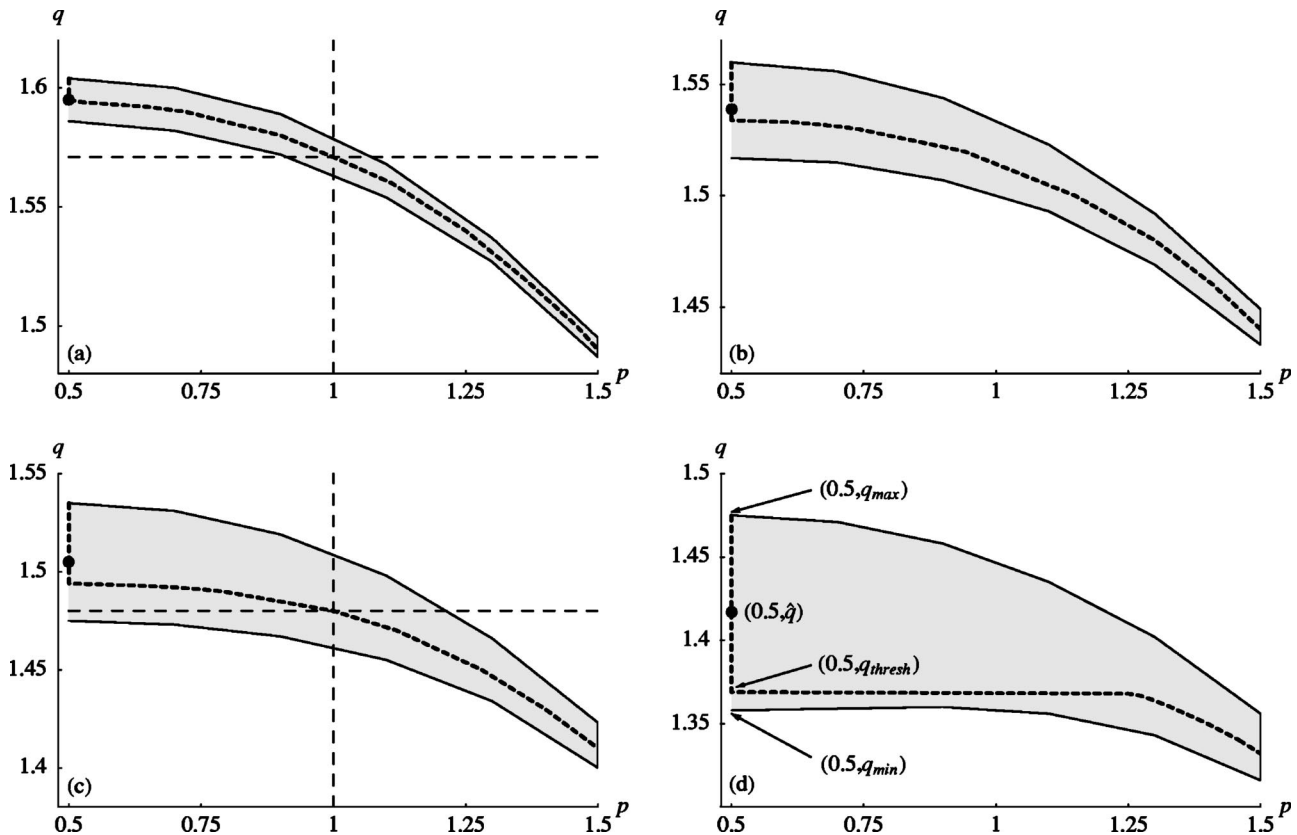[a]Electronic mail: drf@mek.dtu.dk
[b]Electronic mail: prm@mek.dtu.dk

FIG. 1. Selected class II instability regions for plane deep water wave trains having steepness (a) $H/L=0.05$, (b) 0.064, (c) 0.0708, and (d) 0.085. On each plot the dot marks the location of maximum instability, occurring at $(p,q)=(0.5,\hat{q})$, whereas the dotted line indicates the variation of the locally dominant instability for various constant values of q. In (c) the crossing of the dashed lines at $(p,q)=(1,1.480)$ corresponds to the locally dominant point found by Fuhrman et al. (Ref. 4) to match closely the oscillating crescent waves observed by Collard and Caulliez (Ref. 1).

0.0708, and 0.085 (corresponding to $ak\approx0.157$, 0.201, 0.222, and 0.267). These regions are computed by finding upper and lower bounds for q (within ±0.001) for selected discrete values of p. As these regions are symmetric about $p=0.5$, only $p>0.5$ is shown here for simplicity. Within each instability region the dot corresponds to the location of global maximum instability $(p,q)=(0.5,\hat{q})$, whereas the dotted line follows the location of the locally dominant class II instability for various constant values of q. The threshold where this dotted line leaves $p=0.5$ will be denoted q_{thresh}, as indicated in Fig. 1(d). In constructing these dotted lines, for each q considered the locally dominant p is determined within ±0.01. In each case q_{thresh} is likewise determined within ±0.001.

Consideration of $q\neq\hat{q}$ is particularly relevant when finite width tanks are considered (as was used, e.g., by Collard and Caulliez), where the transversal components are then limited to the discrete possibilities,

$$q=\left|\frac{k_y}{k}\right|=\frac{\pi n}{bk}, \qquad (2)$$

where n is a positive integer specifying the number of half transverse wavelengths spanning the width of the tank b, as described, e.g., in Refs. 1 and 4. (Photographs of crescent patterns in, e.g., Su et al.[3] as well as Melville,[12] support the validity of this assertion, which basically implies symmetry about the tank side walls.) Hence, generally speaking, the

more narrow a tank is, the less likely one of the possible transverse modes is to land at (or near) the globally dominant value $\hat{q}$, thus making the entire instability region of practical interest.

Of particular interest in this plot is Fig. 1(c) with H/L $=0.0708$, which corresponds precisely to the case discussed by Fuhrman et al.[4] They showed that by taking into account amplitude dispersion in a primary wave of this steepness, the $q=1.480$ transversal mode fits precisely within the tank used in the experiments, while also having a dominant class II instability at $p=1$ (it should be noted, however, than Collard and Caulliez actually report $q=1.32$). This is marked by the crossing of the dashed lines at $(p,q)=(1,1.480)$ in Fig. 1(c). This local dominance is shown quantitatively in Fig. 2, where $\mathrm{Im}(\sigma)$ is shown along cross sections having constant q values of 1.480, 1.491, and $\hat{q}=1.505$. Clearly, from the full line in this figure the strength of the class II instability at the local maximum $(p,q)=(1,1.480)$ is significantly stronger than at (0.5, 1.480). The other two lines are also rather interesting. The trace with $q=1.491$ is just slightly below q_{thresh} $=1.494$, and it can be seen that for a rather wide range of p the relative strength of the class II instability is nearly constant. Moving slightly upward in the p-q plane to $\hat{q}=1.505$, the picture changes significantly, and the strength of the instability now decreases as p deviates from 0.5. Figure 2 is a good example of what generally happens as one varies q within a given instability region.
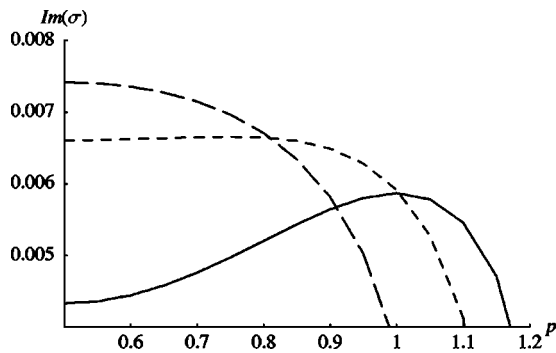
FIG. 2. Strength of instability with $H/L = 0.0708$ corresponding to (full line) $q = 1.480$, (dotted line) $q = 1.491$, and (dashed line) $\hat{q} = 1.505$.

FIG. 3. Transition of (top full line) q_{max}, (bottom full line) q_{min}, (dotted line) $\hat{q}$, and (dashed line) q_{thresh} vs wave steepness.

By further examination of Fig. 1 some rather interesting trends can be seen. In each of the plots, it is first seen that for $q \geqslant \hat{q}$ the dominant class II instability unsurprisingly occurs at $p = 0.5$, corresponding to symmetric perturbations. These lead to the well-known phase-locked L_2 crescent wave patterns, which are always dominant when the full continuous range of q is allowed (e.g., in very wide wave tanks). Alternatively, for $q < \hat{q}$ this is not necessarily the case. In Fig. 1(a) with $H/L = 0.05$ it is seen, quite remarkably, that for $q < \hat{q}$ the dominant class II instability is *never* with $p = 0.5$! Rather, it deviates from this axis immediately below the location of global maximum instability, that is $q_{thresh} = \hat{q}$. As discussed, e.g., by Fuhrman *et al.*,[4] such class II instabilities with $p \neq 0.5$ will in fact result quite generally in oscillating crescent waves, with cases having $p = 1$ closely resembling those observed by Collard and Caulliez[1] (i.e., having crescents arranged in straight rows). We find that $q_{thresh} = \hat{q}$ is generally the case for $H/L \leqslant 0.51$. It is interesting to mention that in this particular case, Fig. 1(a), where the nonlinearity is relatively weak (and the instability region relatively thin), the q value where the $p = 1$ mode is locally dominant is actually stable at $p = 0.5$ [see the dashed horizontal line in Fig. 1(a)], which is also typical for weakly nonlinear cases. This demonstrates that there are physical circumstances where (dominant) oscillating crescents similar to those observed by Collard and Caulliez exist, which are entirely free of competition from any phase-locked modes, which may help explain the described sharp selection in the experiments.

As the steepness is gradually increased, as in Figs. 1(b)–1(d), q_{thresh} gradually moves downward in the p-q plane (i.e., below $\hat{q}$), indicating a relative increase in the fraction of existing phase-locked instabilities (with $p = 0.5$) that will be dominant. This can be more clearly seen in Fig. 3, which shows the variation of the upper and lower bounds of the class II instability region (at $p = 0.5$), denoted here respectively as q_{max} and q_{min}, as well as the locations of $\hat{q}$ and q_{thresh} versus wave steepness H/L. This figure clearly demonstrates the manner in which q_{thresh} gradually shifts from $\hat{q}$ to q_{min} as the steepness is increased.

We will now attempt to quantify the relative dominance of the symmetric class II instability as a function of wave steepness. We here define by γ the fraction of all possible symmetric class II instabilities (with $p = 0.5$) that are locally dominant, i.e., having the maximum value of $Im(\sigma)$ for their

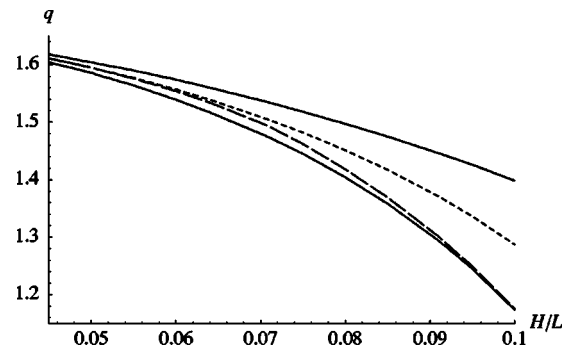respective q. This is simply computed using the previously defined variables by

$$\gamma = \frac{q_{max} - q_{thresh}}{q_{max} - q_{min}}. \tag{3}$$

Under the assumption that a finite width tank is such that exactly one of the discrete q values from (2) lies within the range $q_{min} \leqslant q \leqslant q_{max}$ (with equal likelihood given to the entire range), this defines the probability of having theoretically dominant phase-locked crescent waves at a given steepness. The variation of γ versus the wave steepness is shown in Fig. 4. Here, as expected, it is seen that for $H/L \leqslant 0.051$ only half of all existing symmetric class II instabilities are in fact dominant for their respective q values, consistent with Fig. 1(a). This fraction steadily increases until $H/L \approx 0.09$, after which γ is seen to gradually approach unity. Figure 4 should serve as a useful guide as to (roughly) how likely it might be to observe dominant class II modes that are non-phase-locked (at a given incident steepness) in an arbitrary tank of finite width, provided that the before-mentioned assumptions are satisfied.

This work demonstrates that there is a surprisingly large number of possible symmetric class II deep water wave instabilities (i.e., with $p = 0.5$, corresponding to the classical phase-locked L_2 patterns of Su *et al.*[3]) which are in fact not dominant for a given transversal mode. This is an important consideration when finite width tanks are used. The details presented here also more clearly demonstrate the specific
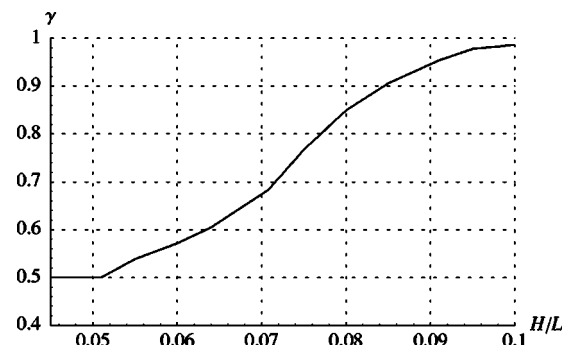

FIG. 4. Fraction γ of existing symmetric class II instabilities (i.e., with $p = 0.5$) that are locally dominant for their respective q values vs wave steepness.

case originally discussed by Fuhrman *et al.*,[4] which could explain the emergence of oscillating crescent waves closely resembling those observed by Collard and Caulliez in their actual wavetank. From this analysis it is also clear that numerous other, similar, examples could be found by combining tanks of varying width with waves of variable steepness. We reiterate, however, that these only appear likely at low to moderate wave steepness, hence this explanation inherently relies on a suppression of the class I instability.

A remaining issue is the question of why Collard and Caulliez observed the specific case with exactly $p=1$, again corresponding to the arrangement of the oscillating crescent waves in straight rows. Based on the current analysis, this specific example would be no more likely to appear than other oscillating cases with $p \neq 0.5$. Those instabilities with $p=1$ would certainly be the most visually striking, however, and it is easily conceivable that the precise order of those crescent waves corresponding to instabilities with neither $p=0.5$ nor $p=1$ might be difficult to distinguish in real time physical experiments. Thus, it is possible that this arrangement in particular "caught the eye" of the experimenters. It is also worth emphasizing that in weakly nonlinear cases, e.g., as demonstrated in Fig. 1(a), it is possible for all existing instabilities for a given q to indeed have $p \approx 1$, which would seem to be an equally valid explanation. Alternatively, perhaps the presence of the plastic film (combined with the wind) somehow favors the growth of modes having even multiples of the carrier wavenumber. Finally, it is also possible that some other (as yet unexplained) external excitation could also have played a role. More knowledge of the effects of external parameters (e.g., those of the plastic film and wind on the instability regions) is apparently necessary before this issue can be resolved.

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