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A Primal-Dual Interior-Point Linear Programming Algorithm for MPC

Kristian Edlund, Leo Emil Sokoler and John Bagterp Jørgensen

Abstract—Constrained optimal control problems for linear systems with linear constraints and an objective function consisting of linear and l_1 -norm terms can be expressed as linear programs. We develop an efficient primal-dual interior point algorithm for solution of such linear programs. The algorithm is implemented in Matlab and its performance is compared to an active set based LP solver and linprog in Matlab's optimization toolbox. Simulations demonstrate that the new algorithm is more than one magnitude faster than the other LP algorithms applied to this problem.

I. INTRODUCTION

In MPC applications, the performance and reliability of the optimization algorithm solving the constrained optimal control problem is important as the optimization problem is solved repeatedly online. In this paper we develop a primal-dual interior-point algorithm for model predictive control (MPC) with input and input-rate constraints and an objective function consisting of linear stage costs as well as l_1 -norms penalizing deviation from target and movements [1], [2], [3]. The primal-dual interior point algorithm is based on Mehrotra's predictor-corrector algorithm [4], [5], [6], [7], [8]. Linear programs for MPC have previously been considered by [9], [10], [11]. Interior-point algorithms based on Riccati iterations for solution of an l_2 constrained regulation problem [12] and a robust l_1 constrained regulation problem [13] have been reported. In this paper, we use state elimination to construct a structured linear program with upper and lower limits on the decision variables, and highly structured general constraints. The special structure of the constraints in this linear program is utilized by the primal-dual interior-point algorithm.

A. Power Portfolio Control

DONG Energy is the main power generating company in Denmark. It operates a portfolio of power plants and wind turbine farms for electricity and district heating production. The wind turbines constitute a large share: 30% of the installed generation capacity in Western Denmark. The share is expected to increase even further as a new wind turbine park is added to the portfolio at the end of 2009. In addition a large pool of electric cars are added to the power network.

In a liberalized electricity market, such an interconnected power and heating system with significant stochastic generators and consumers needs an agile and robust control system to coordinate the most economic power generation

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respecting constraints, long-term contracts, and short-term demand-fluctuations.

By simulation Model Predictive Control has been demonstrated as a very promising technology for dynamic regulation and coordination of power generation in the DONG Energy portfolio [3]. This controller is called the DONG Energy portfolio balance controller. The controller reduces the deviation between sold and actual production in the most economical way. This is an example of Model Predictive Control with an economical rather than a traditional target deviation objective function [14].

The models used in this paper has been derived in [15]. To test different optimization algorithms, and the possibility to exploit the structure of the problem, we consider a single subsystem of the entire power generation portfolio. The subsystem is a single boiler load effectuator with the simplification that rate-of-movement limits can be specified as parameters [15].

B. Paper Organization

In Section II we state the constrained optimal control problem with a linear cost and l_1 -norm penalties. We derive the LP problem used to compute the solution of the constrained optimal control problem. Section III describes the interior-point algorithm for an inequality constrained linear program. Section IV specializes the operations in this algorithm to the LP problem for the constrained optimal control problem with linear cost and l_1 -penalties. Section V compares the developed interior-point algorithm for the MPC-LP to off-the-shelf LP solvers. Section VI concludes on the results.

II. PROBLEM DEFINITION

We state the control problem that is to be used in the power balance controller in controlling one power generating unit (a power plant). The problem and the models are described in detail in [3], [15]. The power balance controller is a Model Predictive Controller in which a constrained optimal control problem is solved at each sampling instant. Only the input associated to the first time period is implemented and the computations are repeated at the next sampling instant. We consider long horizons to have economic performance as well as stability. This implies that the constrained optimal control problem solved at each sampling instant is relatively large. It is important that this large constrained optimal control problem is solved robustly and fast as it is embedded in a real-time system.

The objective function used to measure the quality of a

power trajectory is

$$\phi = \sum_{k=0}^{N-1} c'_{k+1} z_{k+1} + \|z_{k+1} - r_{k+1}\|_{1,q_{k+1}} + \|\Delta u_k\|_{1,s_k} \quad (1)$$

z_k is the output (power production), r_k is the reference (planned power production), and u_k is the input (modified power production to meet short term fluctuations in demand). k is a time index and we consider these cost for a finite period, $\mathcal{N} = \{0, 1, \dots, N-1\}$, characterized by the control and prediction horizon, N .

The first term represent the production costs, i.e. the cost of fuel, emission taxes etc. The second term describes the costs for deviating from the production plan computed by the production planner. The last term is a cost related to plant wear that penalizes excessive movement of the input.

The models describing the dynamics of the system are linear. The inputs have bound and rate-of-movement constraints [15]. Therefore, the constrained optimal control problem solved at each sampling period is

$$\min_{\{u_k\}_{k=0}^{N-1}} \phi = \phi(\{u_k\}_{k=0}^{N-1}; x_0, u_{-1}, \{d_k, r_{k+1}\}_{k=0}^{N-1}) \quad (2a)$$

$$s.t. \quad x_{k+1} = Ax_k + Bu_k + Ed_k \quad k \in \mathcal{N} \quad (2b)$$

$$z_{k+1} = Cx_{k+1} \quad k \in \mathcal{N} \quad (2c)$$

$$u_{\min,k} \leq u_k \leq u_{\max,k} \quad k \in \mathcal{N} \quad (2d)$$

$$\Delta u_{\min,k} \leq \Delta u_k \leq \Delta u_{\max,k} \quad k \in \mathcal{N} \quad (2e)$$

$\mathcal{N} = \{0, 1, \dots, N-1\}$. Note that the input bounds and the rate-of-movement constraints are time varying.

Combination of (2b) and (2c) yields

$$z_k = CA^k x_0 + \sum_{i=0}^{k-1} H_{u,k-i} u_i + \sum_{i=0}^{k-1} H_{d,k-i} d_i \quad (3)$$

with $k = 1, 2, \dots, N$ and the impulse response coefficients defined in the usual way

$$H_{u,i} = CA^{i-1}B \quad i = 1, 2, \dots, N \quad (4a)$$

$$H_{d,i} = CA^{i-1}E \quad i = 1, 2, \dots, N \quad (4b)$$

Define the vectors

$$U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} \quad D = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{N-1} \end{bmatrix} \quad \Delta U = \begin{bmatrix} \Delta u_0 \\ \Delta u_1 \\ \vdots \\ \Delta u_{N-1} \end{bmatrix}$$

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix} \quad R = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \quad W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

and the matrices

$$\Phi = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix} \quad \Gamma_\alpha = \begin{bmatrix} H_{\alpha,1} & 0 & \dots & 0 \\ H_{\alpha,2} & H_{\alpha,1} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ H_{\alpha,N} & H_{\alpha,N-1} & \dots & H_{\alpha,1} \end{bmatrix}$$

with $\alpha \in \{u, d\}$. Using (3) the stacked outputs, Z , may be expressed by the linear relation

$$Z = \Phi x_0 + \Gamma_u U + \Gamma_d D \quad (5)$$

Introduce the matrices (shown for the case $N = 5$)

$$I_0 = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Psi = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ -I & I & 0 & 0 & 0 \\ 0 & -I & I & 0 & 0 \\ 0 & 0 & -I & I & 0 \\ 0 & 0 & 0 & -I & I \end{bmatrix}$$

to have the following expression

$$\Delta U = \Psi U - I_0 u_{-1} \quad (6)$$

Consequently, the constrained optimal control problem (2) may be expressed as

$$\min_U \phi = c'Z + \|Z - R\|_{1,q} + \|\Delta U\|_{1,s} \quad (7a)$$

$$s.t. \quad Z = \Phi x_0 + \Gamma_u U + \Gamma_d D \quad (7b)$$

$$\Delta U = \Psi U - I_0 u_{-1} \quad (7c)$$

$$U_{\min} \leq U \leq U_{\max} \quad (7d)$$

$$\Delta U_{\min} \leq \Delta U \leq \Delta U_{\max} \quad (7e)$$

Theorem 2.1 (Linear Program for l_1 -approximation):
The l_1 -approximation problem

$$\min_{x \in \mathbb{R}^n} \phi = \|Ax - b\|_1 \quad (8)$$

with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ can be represented as the linear program

$$\min_{x,y} \phi = e'y \quad (9a)$$

$$s.t. \quad -y \leq Ax - b \leq y \quad (9b)$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $e = [1 \dots 1]'$.

Proof: The l_1 -approximation problem (8) is equivalent to $\min_{x,y} \{\phi = e'y : y \geq |Ax - b|\}$. The constraint $y \geq |Ax - b|$ may be written as the linear constraints $-y \leq Ax - b \leq y$. ■

Corollary 2.2 (l_1 -approximation as LPs in standard form):
The l_1 -approximation problem (8) may be expressed as the linear program in the form

$$\min_{x,y} \phi = \begin{bmatrix} 0 \\ e \end{bmatrix}' \begin{bmatrix} x \\ y \end{bmatrix} \quad (10a)$$

$$s.t. \quad \begin{bmatrix} A & I \\ -A & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} b \\ -b \end{bmatrix} \quad (10b)$$

(8) may also be expressed as the linear program in the form

$$\min_{x,y} \phi = \begin{bmatrix} 0 \\ e \end{bmatrix}' \begin{bmatrix} x \\ y \end{bmatrix} \quad (11a)$$

$$s.t. \quad \begin{bmatrix} b \\ -\infty \end{bmatrix} \leq \begin{bmatrix} A & I \\ A & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} \infty \\ b \end{bmatrix} \quad (11b)$$

Proof: Follows by rearrangement of (9). ■

Using Theorem 2.1 we may express (7) as

$$\min_{U,V,W} \phi = c'Z + s'V + q'W \quad (12a)$$

$$s.t. \quad Z = \Phi x_0 + \Gamma_u U + \Gamma_d D \quad (12b)$$

$$\Delta U = \Psi U - I_0 u_{-1} \quad (12c)$$

$$U_{\min} \leq U \leq U_{\max} \quad (12d)$$

$$\Delta U_{\min} \leq \Delta U \leq \Delta U_{\max} \quad (12e)$$

$$-V \leq \Delta U \leq V \quad (12f)$$

$$-W \leq Z - R \leq W \quad (12g)$$

which by elimination of Z and ΔU is equivalent to the inequality constrained linear program

$$\min_{U,V,W} \phi = c'(\Phi x_0 + \Gamma_u U + \Gamma_d D) + s'V + q'W \quad (13a)$$

$$s.t. \quad U_{\min} \leq U \leq U_{\max} \quad (13b)$$

$$\Delta U_{\min} \leq \Psi U - I_0 u_{-1} \leq \Delta U_{\max} \quad (13c)$$

$$-V \leq \Psi U - I_0 u_{-1} \leq V \quad (13d)$$

$$-W \leq \Phi x_0 + \Gamma_u U + \Gamma_d D - R \leq W \quad (13e)$$

This linear program along with Corollary 2.2 may be used to arrive at the following linear program

$$\min_x \psi = g'x \quad (14a)$$

$$s.t. \quad x_l \leq x \leq x_u \quad (14b)$$

$$b_l \leq Ax \leq b_u \quad (14c)$$

with the variables and data defined as

$$x = \begin{bmatrix} U \\ V \\ W \end{bmatrix} \quad x_l = \begin{bmatrix} U_{\min} \\ 0 \\ 0 \end{bmatrix} \quad x_u = \begin{bmatrix} U_{\max} \\ \infty \\ \infty \end{bmatrix} \quad g = \begin{bmatrix} g_u \\ s \\ q \end{bmatrix} \quad (15a)$$

$$A = \begin{bmatrix} \Psi & 0 & 0 \\ \Psi & I & 0 \\ \Psi & -I & 0 \\ \Gamma_u & 0 & I \\ \Gamma_u & 0 & -I \end{bmatrix} \quad (15b)$$

$$b_l = \begin{bmatrix} \Delta U_{\min} + I_0 u_{-1} \\ I_0 u_{-1} \\ -\infty \\ b \\ -\infty \end{bmatrix} \quad b_u = \begin{bmatrix} \Delta U_{\max} + I_0 u_{-1} \\ \infty \\ I_0 u_{-1} \\ \infty \\ b \end{bmatrix} \quad (15c)$$

$$g_u = \Gamma_u' c \quad (15d)$$

$$b = R - (\Phi x_0 + \Gamma_d D) \quad (15e)$$

The original objective function is $\phi = \psi - c'b$ where $c'b$ is a constant.

Consequently, we may solve the constrained optimal control problem (2) by solution of the linear program (14). The coefficient matrix (15b) is highly structured. It is composed of the matrices Ψ and Γ_u which themselves are structured matrices. We develop a primal-dual interior-point algorithm that exploits this structure to efficiently solve the constrained optimal control problem (2) in the MPC.

III. INTERIOR-POINT METHODS

Before proceeding to a description of the interior-point algorithm applied to (14), we describe the interior-point algorithm for the structural simpler linear program

$$\min_{x \in \mathbb{R}^n} \phi = g'x \quad (16a)$$

$$s.t. \quad Ax \geq b \quad (16b)$$

The algorithm and its principles are discussed in [8].

A. Optimality Conditions

The Lagrangian of (16) is

$$\mathcal{L}(x, \lambda) = g'x - \lambda'(Ax - b) \quad (17)$$

and a stationary point of the Lagrangian satisfies

$$\nabla_x \mathcal{L}(x, \lambda) = g - A'\lambda = 0 \quad (18)$$

Consequently, the first order necessary and sufficient optimality conditions may be stated as

$$g - A'\lambda = 0 \quad (19a)$$

$$Ax - b \geq 0 \quad \perp \quad \lambda \geq 0 \quad (19b)$$

in which \perp is used to denote complementarity. Introduce slack variables defined as

$$s = Ax - b \geq 0 \quad (20)$$

and let

$$S = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_m \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix} \quad (21)$$

such that the complementarity conditions $s_i \lambda_i$ for $i = 1, 2, \dots, m$ may be stated compactly as $S\Lambda e = 0$ with $e = [1 \dots 1]'$. Consequently, the optimality conditions (19) may be stated as the systems of equations and inequalities

$$r_L = g - A'\lambda = 0 \quad (22a)$$

$$r_s = s - Ax + b = 0 \quad (22b)$$

$$r_{s\lambda} = S\Lambda e = 0 \quad (22c)$$

$$(s, \lambda) \geq 0 \quad (22d)$$

B. Newton Step

Given an iterate (x, λ, s) satisfying $(s, \lambda) > 0$, (22) may be solved by a sequence of Newton steps with modified search directions and step lengths.

The Newton direction is computed as the solution of

$$\begin{bmatrix} 0 & -A' & 0 \\ -A & 0 & I \\ 0 & S & \Lambda \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{bmatrix} = - \begin{bmatrix} r_L \\ r_s \\ r_{s\lambda} \end{bmatrix} \quad (23)$$

The structure of this linear system of equations may be utilized to solve it efficiently. Note that the second block row of (23) yields

$$\Delta s = -r_s + A\Delta x \quad (24)$$

Using that $S > 0$ and easily invertible as it is a diagonal matrix with positive entries, the third block row of (23) along with (24) yield

$$\begin{aligned}\Delta\lambda &= -S^{-1}(r_{s\lambda} + \Lambda\Delta s) \\ &= S^{-1}(-r_{s\lambda} + \Lambda r_s) - S^{-1}\Lambda A\Delta x\end{aligned}\quad (25)$$

Finally, the first block row of (23) along with (25) yield

$$\begin{aligned}-r_L &= -A'\Delta\lambda \\ &= (A'S^{-1}\Lambda A)\Delta x - A'S^{-1}(-r_{s\lambda} + \Lambda r_s) \\ &= \bar{H}\Delta x + \bar{r}\end{aligned}\quad (26)$$

in which

$$\bar{H} = A'(S^{-1}\Lambda)A \quad (27a)$$

$$\bar{r} = A'[S^{-1}(r_{s\lambda} - \Lambda r_s)] \quad (27b)$$

Consequently, (23) may be solved by solution of

$$\bar{H}\Delta x = -\bar{g} = -(r_L + \bar{r}) \quad (28)$$

for Δx and subsequent computation of Δs by (24) and $\Delta\lambda$ by (25). The next iterate in the Newton iteration is computed as

$$\begin{bmatrix} x \\ \lambda \\ s \end{bmatrix} \leftarrow \begin{bmatrix} x \\ \lambda \\ s \end{bmatrix} + \alpha \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} \quad (29)$$

with the step length $\alpha \in (0, \alpha_{\max}) \cap (0, 1]$ selected such that $(\lambda, s) > 0$, i.e. with the maximum step length computed as

$$s + \alpha_{\max}\Delta s \geq (1 - \tau)s \quad (30a)$$

$$\lambda + \alpha_{\max}\Delta\lambda \geq (1 - \tau)\lambda \quad (30b)$$

with $\tau \rightarrow 1$ as the iterate approaches the solution.

C. Predictor-Corrector Interior-Point Algorithm

To avoid being restricted to small step lengths as is often the case when (22) is solved directly, the complementarity conditions are modified such that the pairs $s_i\lambda_i$ decrease at the same rate for all i . Instead of solving (22c), we solve

$$r_{s\lambda} = S\Lambda e - \sigma\mu e = 0 \quad \mu = \frac{s'\lambda}{m} = \frac{\sum_{i=1}^m s_i\lambda_i}{m} \quad (31)$$

for some value of $\sigma \in (0, 1]$. In Mehrotra's predictor-corrector algorithm, σ is selected adaptively based on the duality gap reduction for an affine step ($\sigma = 0$). This affine step may also be used to predict $r_{s\lambda}$ and introduce a correction such that the step direction is computed by solution of (23) with

$$r_{s\lambda} = S\Lambda e + \Delta S\Delta\Lambda e - \sigma\mu e \quad (32)$$

ΔS and $\Delta\Lambda$ are the step directions computed in the affine step ($\sigma = 0$).

D. Primal-Dual Interior-Point Algorithm

Algorithm 1 specifies the steps in this procedure for solution of (16).

The main computational efforts in Algorithm 1 are 1) formation of the matrix $\bar{H} = A'DA$ with $D = S^{-1}\Lambda$ being a diagonal matrix with positive entries on the diagonal and 2) Cholesky factorization of \bar{H} .

Algorithm 1 Interior-point algorithm for (16).

Require: ($g \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$)

Residuals and Duality Gap:

$$r_L = g - A'\lambda, r_s = s - Ax + b, r_{s\lambda} = S\Lambda e$$

$$\text{Duality gap: } \mu = \frac{s'\lambda}{m}$$

while *Not_Converged* **do**

$$\text{Compute } \bar{H} = A'(S^{-1}\Lambda)A$$

$$\text{Cholesky factorization: } \bar{H} = \bar{L}\bar{L}'$$

Affine Predictor Step:

$$\text{Compute } \bar{r} = A'(S^{-1}(r_{s\lambda} - \Lambda r_s)), -\bar{g} = -(r_L + \bar{r})$$

$$\text{Solve: } \bar{L}\bar{L}'\Delta x = -\bar{g}$$

$$\Delta s = -r_s + A\Delta x$$

$$\Delta\lambda = -S^{-1}(r_{s\lambda} + \Lambda\Delta s)$$

Determine the maximum affine step length

$$\lambda + \alpha_{\max}\Delta\lambda \geq 0 \quad s + \alpha_{\max}\Delta s \geq 0$$

Select affine step length: $\alpha \in (0, \alpha_{\max}]$

$$\text{Compute affine duality gap: } \mu_a = \frac{(\lambda + \alpha\Delta\lambda)'(s + \alpha\Delta s)}{m}$$

Centering parameter: $\sigma = \left(\frac{\mu_a}{\mu}\right)^3$

Center Corrector Step:

Modified complementarity:

$$r_{s\lambda} \leftarrow r_{s\lambda} + \Delta S\Delta\Lambda e - \sigma\mu e$$

$$\text{Compute } \bar{r} = A'(S^{-1}(r_{s\lambda} - \Lambda r_s)), -\bar{g} = -(r_L + \bar{r})$$

$$\text{Solve: } \bar{L}\bar{L}'\Delta x = -\bar{g}$$

$$\Delta s = -r_s + A\Delta x$$

$$\Delta\lambda = -S^{-1}(r_{s\lambda} + \Lambda\Delta s)$$

Determine the maximum affine step length

$$\lambda + \alpha_{\max}\Delta\lambda \geq 0 \quad s + \alpha_{\max}\Delta s \geq 0$$

Select affine step length: $\alpha \in (0, \alpha_{\max}]$

$$\text{Step: } x \leftarrow x + \alpha\Delta x, \lambda \leftarrow \lambda + \alpha\Delta\lambda, s \leftarrow s + \alpha\Delta s$$

Residuals and Duality Gap:

$$r_L = g - A'\lambda, r_s = s - Ax + b, r_{s\lambda} = S\Lambda e$$

$$\text{Duality gap: } \mu = \frac{s'\lambda}{m}$$

end while

Return: (x, λ)

IV. INTERIOR-POINT ALGORITHM FOR MPC-LP

The constrained optimal control problem (2) (which is equivalent with (14)) gives the following A-matrix and b-vector in the standard LP formulation (16)

$$A = \begin{bmatrix} I & 0 & 0 \\ -I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \Psi & 0 & 0 \\ -\Psi & 0 & 0 \\ \Psi & I & 0 \\ -\Psi & I & 0 \\ \Gamma_u & 0 & I \\ -\Gamma_u & 0 & I \end{bmatrix} \quad b = \begin{bmatrix} U_{\min} \\ -U_{\max} \\ 0 \\ 0 \\ \Delta U_{\min} + I_0 u_{-1} \\ -(\Delta U_{\max} + I_0 u_{-1}) \\ I_0 u_{-1} \\ -I_0 u_{-1} \\ b \\ -b \end{bmatrix} \quad (33)$$

This A-matrix is highly structured. Therefore, we may specialize the steps in Algorithm 1 that involves operations with the A-matrix. The following theorems state the computational simplifications used in Algorithm 1 when A has the structure in (33). For notational convenience we use Matlab like notation in some of the theorems.

Lemma 4.1 (Hessian matrix, \bar{H} , in MPC-LP):

Let A have the structure in (33). Let $D = \text{diag}([d_1; d_2; \dots; d_{10}]) = \Lambda^{-1}S$ be a diagonal matrix with positive entries and let $D_i = \text{diag}(d_i)$ for $i = 1, 2, \dots, 10$ be sub-matrices of D corresponding to the division of A in (33).

Then

$$\bar{H} = A'DA = \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{21} & \bar{D} \end{bmatrix} \quad (34)$$

with the sub-matrices

$$\bar{H}_{11} = \bar{D}_1 + \Psi'\bar{D}_2\Psi + \Gamma'_u\bar{D}_3\Gamma_u \quad (35a)$$

$$\bar{H}_{12} = \bar{H}'_{21} = [\Psi'\bar{D}_4 \quad \Gamma'_u\bar{D}_5] \quad (35b)$$

$$\bar{D} = \begin{bmatrix} \bar{D}_6 & \\ & \bar{D}_7 \end{bmatrix} \quad (35c)$$

and

$$\bar{D}_1 = D_1 + D_2 \quad \bar{D}_2 = D_5 + D_6 + D_7 + D_8$$

$$\bar{D}_3 = D_9 + D_{10} \quad \bar{D}_4 = D_7 - D_8$$

$$\bar{D}_5 = D_9 - D_{10} \quad \bar{D}_6 = D_3 + D_7 + D_8$$

$$\bar{D}_7 = D_4 + D_9 + D_{10}$$

Proof: Follows by straightforward matrix multiplications using A in (35). ■

Theorem 4.2 (Cholesky Factorization in MPC-LP):

Solution of $\bar{H}x = b$ corresponds to solution of the system

$$\begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{21} & \bar{D} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad (36)$$

This system may be factorized by

- 1) Compute $\hat{D}_2 = \bar{D}_2 - \bar{D}_4\bar{D}_6^{-1}\bar{D}_4$
- 2) Compute $\hat{D}_3 = \bar{D}_3 - \bar{D}_5\bar{D}_7^{-1}\bar{D}_5$
- 3) Compute $\hat{H}_{11} = \bar{D}_1 + \Psi'\hat{D}_2\Psi + \Gamma'_u\hat{D}_3\Gamma_u$
- 4) Cholesky factorize \hat{H}_{11} : $\hat{H}_{11} = \hat{L}\hat{L}'$

and solved by

- 1) Solve $\hat{L}\hat{L}'x_1 = b_1 - \bar{D}^{-1}b_2$ for x_1 by back substitutions
- 2) Compute $x_2 = \bar{D}^{-1} \left(\bar{b}_2 - \begin{bmatrix} \bar{D}_4(\Psi x_1) \\ \bar{D}_5(\Gamma_u x_1) \end{bmatrix} \right)$

Proof: The results are obtained by application of the Schur complement to (36) and the matrix definitions (35). ■

Theorem 4.3 (Matrix-vector operations in MPC-LP):

- 1) Let A have the structure in (33). Let $x = [U; V; W]$. Then $Ax = [U; -U; V; W; z_1; -z_1; z_3; z_4; z_5; z_6]$ with $z_1 = \Psi U$, $z_2 = \Gamma_u U$, $z_3 = z_1 + V$, $z_4 = -z_1 + V$, $z_5 = z_2 + W$, and $z_6 = -z_2 + W$.

- 2) Let A have the structure in (33).

Let $v = [v_1; v_2; \dots; v_{10}]$. Then

$$A'v = \begin{bmatrix} \bar{v}_1 + \Psi'\bar{v}_2 + \Gamma'_u\bar{v}_3 \\ v_3 + v_7 + v_8 \\ v_4 + v_9 + v_{10} \end{bmatrix} \quad (37)$$

with $\bar{v}_1 = v_1 - v_2$, $\bar{v}_2 = v_5 - v_6 + v_7 - v_8$, $\bar{v}_3 = v_9 - v_{10}$

Proof: Follows by straightforward matrix-vector manipulations. ■

Theorem 4.4 (Operations with Ψ): For illustration consider Ψ for $N = 4$ and let $D = \text{diag}([d_1; d_2; d_3; d_4])$ be a diagonal matrix with $D_i = \text{diag}(d_i)$ for $i \in \{1, 2, 3, 4\}$ also being diagonal matrices. Then

$$\Psi'D\Psi = \begin{bmatrix} D_1 + D_2 & -D_2 & 0 & 0 \\ -D_2 & D_2 + D_3 & -D_3 & 0 \\ 0 & -D_3 & D_3 + D_4 & -D_4 \\ 0 & 0 & -D_4 & D_4 \end{bmatrix} \quad (38)$$

Let $x = [x_1; x_2; x_3; x_4]$ then

$$\Psi x = [x_1; x_2 - x_1; x_3 - x_2; x_4 - x_3] \quad (39a)$$

$$\Psi'x = [x_1 - x_2; x_2 - x_3; x_3 - x_4; x_4] \quad (39b)$$

Proof: Straightforward matrix-matrix and matrix-vector operations with Ψ . ■

The operations $\Gamma'_u D \Gamma_u$, $\Gamma_u U$, and $\Gamma'_u Z$ for some diagonal matrix D, some vector U, and some vector Z are implemented using straightforward matrix operations even though Γ_u is structured. In the current Matlab implementation $\Gamma'_u D \Gamma_u$ is the computational bottleneck. $\Gamma'_u D \Gamma_u$ is implemented using that D is a diagonal matrix but without using the structure of Γ_u .

Remark 4.5 (Operations with Γ_u): Γ_u is a matrix of the impulse response parameters, $\{H_{u,k} = CA^{k-1}B\}_{k=1}^N$. $Z = \Gamma_u U$ transfers a set of inputs $\{u_k\}_{k=0}^{N-1}$ to a set of outputs $\{z_k\}_{k=1}^N$ for the system ($k = 0, 1, \dots, N-1$)

$$x_{k+1} = Ax_k + Bu_k \quad x_0 = 0 \quad (40a)$$

$$z_{k+1} = Cx_{k+1} \quad (40b)$$

Similarly, $U = \Gamma'_u Z$ corresponds to

$$\bar{x}_{k-1} = A'\bar{x}_k + C'z_k \quad \bar{x}_N = 0 \quad (41a)$$

$$u_{k-1} = B'\bar{x}_{k-1} \quad (41b)$$

going backwards with $k = N, N-1, \dots, 1$.

V. RESULTS

Using the boiler load effectuator of a power plant [15], we test the developed interior-point MPC-LP algorithm (Algorithm 1) utilizing the structure of A in (33) for solution of the constrained optimal control problem (2). We compare our MPC-LP algorithm to the solution of (2) using `linprog` in Matlab's optimization toolbox and an active set LP solver applied to (14).

The boiler effectuator is a SISO system and we use a control horizon of $N = 50$. The number of decision variables (U, V, W) in the LP to be solved is $3N = 150$. The sampling time is $T_s = 5s$ and we run the test problem in closed-loop for 2000 samples. Figure 1 illustrates the benchmark case for which we have compared the tree different LP solvers. All three LP solvers give the same result indicating that our solver is implemented correct. The case study and controller tuning is chosen such that some of the constraints are usually active as indicated to the right in Figure 1.

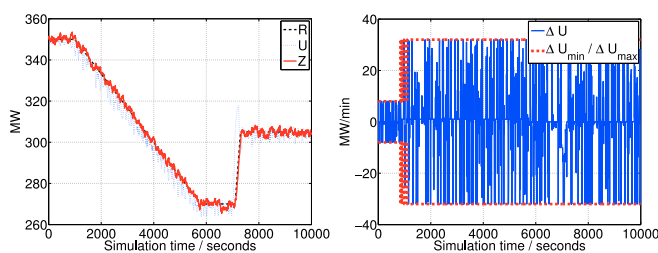


Fig. 1. Input and output for the benchmark case.

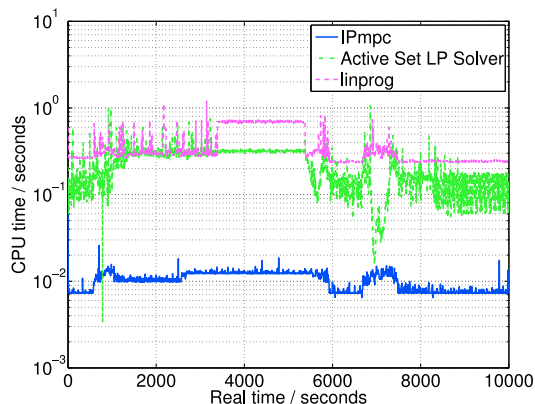


Fig. 2. CPU times for the different LP algorithms solving (2).

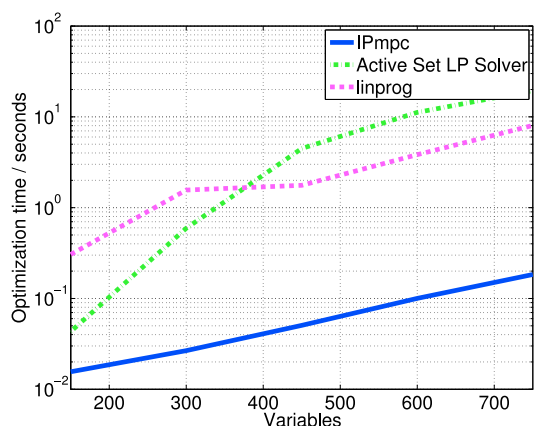


Fig. 3. CPU-time as function of the number of decision variables in the LP corresponding to (2).

As can be read from Figure 2, the runtime of our MPC-LP (IPmpc) solver is about one order of magnitude faster than both the active set LP solver and linprog. Furthermore, the variance of the CPU-time is much smaller for MPC-LP than for both linprog and the active set LP solver. In real-time applications it is desirable to have a predictable computing time. MPC-LP and linprog are implemented in Matlab. The active set LP solver is a highly efficient LP solver for general LPs in the form (14) that is implemented in Fortran and equipped with a mex-interface.

Figure 3 illustrates the CPU-time for the three different LP solvers for (2) as function of the number of decision variables in the LP (14) and (16), respectively. The interior-point

MPC-LP algorithm (IPmpc) is significantly faster than the other algorithms, typically more than one order of magnitude faster.

VI. CONCLUSION

We have developed computationally efficient primal-dual interior point algorithms for constrained optimal control problems that have linear dynamics, input constraints, rate-of-movement constraints, and objective functions containing linear stage costs and l_1 -norm deviation penalties on the set-point and the input movement. MPC for dynamic regulation, coordination and optimization of power generation solves such problems in real-time repeatedly. Fast and robust optimization algorithms are important in these applications. The new primal-dual interior point algorithm is implemented in Matlab and its performance is compared to an active set based LP solver and linprog in Matlab's optimization toolbox. Simulations demonstrate that the new algorithm is more than one magnitude faster than the other LP algorithms.

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