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Potential Theory and the Lorenz Condition

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Abstract - When potentials are used to calculate the electromagnetic field intensities choices may be made. The present study is carried out in order to outline the various possibilities. It looks like the types of choices and their number depend on the actual situation and the preferences of the researcher. Also the explanations as to why choices can be made vary. Since potentials probably are more basic than field intensities, it is of some interest to understand whether choices can be related to physical interpretations or computational advantages in specific situations. In particular, it is discussed whether the Lorenz condition is a choice or a consequence. It turns out that facts in relation to potentials depend on the postulates made.

INTRODUCTION

The structure of Maxwell's equations is such that their solution may be obtained by following different paths. When potentials are used choices are sometimes possible. The present study was initiated in part by reading [Datta, p. 230] stating that in free space $\nabla \cdot \overline{A} = 0$ is a fact for an electromagnetic field and in part by reading [Aharonov and Bohm] stating that the scalar potential V and the vector potential $\overline{A}$ may have their own meaning. Since these two statements seem to contradict the possibility of making choices, it was decided to study the facts in electromagnetism as related to potentials. It may be said that it is not necessary. The argument may be that, when $\overline{A}$ and V are calculated as so-called retarded potentials, they automatically satisfy the Lorenz condition. This is a consequence of the continuity equation as shown already by Lorenz, see also [Stratton, p.429] who did not refer to Lorenz. Sometimes, it is even stated that in real problems the Lorenz condition must always be satisfied [Jordan and Balmain, 1950,p.301]. However, due to some freedom (gauge invariance), it is not necessary.

THE LORENZ CONDITION

Usually, the name of the condition is given after H. A. Lorentz and not after L. V. Lorenz, who was the first to get the condition [Van Bladel, April 1991]. In [Van Bladel, p.77], the condition is named after Lorenz. This notation is also used in the present work. However, the notation can be discussed. In order to do so, the work by the L. V. Lorenz, Danish, in relation to those of J. C. Maxwell, English, and H. A. Lorentz, Dutch, will be outlined. The emphasis is on potentials.

Detailed historical reviews of the development of electromagnetic theory can be found in the references listed in [Appel-Hansen and Jiaxiang, 1993]. In order to give a simple introduction, the development may be described by the following comments which are based on the form Maxwell's equations have today. Due to the structure of the electromagnetic field it is possible to choose among several relationships or conditions. As a result, the expressions for the potentials depend on the condition chosen. In particular, two choices are popular. The first choice is to use a vector potential which is divergence free. If this choice is made in the presence of sources, the scalar potential will be found by using the same expression as in the electrostatic case. Therefore, this choice is often called the Coulomb condition. The second choice involves the vector potential as well as the scalar potential. The possibility of making choices is due to the freedom to choose the divergence of the vector potential.

However, it took a long time to realize that these choices were possible. It was questioned whether the various equations were correct and it was not seen that, in fact, the same phenomena were treated and that the final results were the same. It turns out that Maxwell postulated a vector potential in accordance with the first choice. For several years this choice was subjected to interpretations and discussions. In 1867, Lorenz, about two years after Maxwell, postulated retarded potentials and showed, by using the continuity equation, that they were related. Lorenz made no comments to the relation. It turns out that this relation corresponds to the second

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choice. It should be noted that Maxwell associated a physical meaning with the vector potential, but he considered the scalar potential as an auxiliary magnitude. According to the expressions Lorenz suggested, one may, if desired, associate a physical meaning to both potentials. In connection with the discrepancy between Maxwell and Lorenz, it should be noted that new investigations seem to show that it is possible to set up regions in space where the field intensities are equal to zero and it is nevertheless possible to measure phenomena because the potentials are different from zero. Thus, both potentials seem to be more than auxiliary magnitudes and may be said to be more basic than field intensities.

From the exposition given above, it is seen that Maxwell used the electrostatic potential and postulated a divergence free vector potential, whereas Lorenz postulated a scalar potential and a vector potential with retardation and showed a relation (the condition) between the potentials. This is a difference between Maxwell and Lorenz. Another difference is that Maxwell considered wires as guides for the fields in the surrounding media, but Lorenz considered fields propagating entirely in conducting material. However, Lorentz was the first to use the retarded potentials in nondissipative media.

Whereas the first choice of vector potential condition is often called the Coulomb condition, the second choice has two designations, viz., the Lorenz condition or the Lorentz condition. Arguments can be given in favour of both designations. At the outset Lorenz postulated the potentials to be retarded. Therefore, Lorenz lost some freedom to choose the divergence of the vector potential, i.e., Lorenz had (without noting it) a restricted gauge invariance. But, using the continuity equation, Lorenz was the first to get the relation corresponding to the second choice. On the other hand, Lorentz had the Lorenz condition as a Lorentz-invariant equation. Thus, Lorenz postulated potentials, which may be more fundamental than field intensities, and which automatically satisfy a much more fundamental relation than Lorenz could have imagined in 1867 considering a restricted class of problems.

Procedures In Dealing With Potentials

In [Appel-Hansen, 1993] the following four procedures in dealing with potentials are compared. In the first procedure relations between field intensities and potentials are considered. Due to the possibility of choosing the divergence of the vector potential, the Lorenz condition is a choice. In the second procedure relations between retarded potentials, currents and charges are considered. In the third procedure the potentials are introduced in an action integral. In the fourth procedure potentials are introduced in the Schrödinger equation used in quantum mechanics. In the first procedure the potentials are introduced as auxiliary functions whereas in the remaining procedures they are introduced as postulates. Thus, the facts relating potentials depend on the postulates made. In the previous section we have seen that the Lorenz condition itself is a postulate or a consequence of postulates. In free space, $V = 0$ is a natural choice and therefore according to the Lorenz condition $\nabla \cdot \overline{A} = 0$. But, due to gauge invariance other values of V and $\overline{A}$ may give the same field intensities. The present study has found no examples in which potentials not satisfying the Lorenz condition are superior with respect to physical interpretations to potentials satisfying the condition.

A special discussion is given in relation to the problem of potentials having their own physical meaning.

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