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Sadegh, Payman

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A Maximum Feasible Subset Algorithm with Application to Radiation Therapy

Payman Sadegh

Dept. of Mathematical Modeling, Tech. University of Denmark, DK-2800 Lyngby, Denmark.
ps@imm.dtu.dk

Abstract

Consider a set of linear one sided or two sided inequality constraints on a real vector X. The problem of interest is selection of X so as to maximize the number of constraints that are simultaneously satisfied, or equivalently, combinatorial selection of a maximum cardinality subset of feasible inequalities. Special classes of this problem are of interest in a variety of areas such as pattern recognition, machine learning, operations research, and medical treatment planning. This problem is generally solvable in exponential time. A heuristic polynomial time algorithm is presented in this paper. The algorithm relies on an iterative constraint removal procedure where constraints are eliminated from a set proposed by solutions to min-max linear programs. The method is illustrated by a simulated example of a linear system with double sided bounds and a case from the area of radiation therapy.

1 Introduction

Consider the set $\{X^\top a_i | a_i, X \in \mathcal{R}^p, i = 1, \cdots, N\}$ where typically $N \gg p$. Assume that for each element, $X^\top a_i$, a one sided or two sided bound is given. The problem of interest is selection of X so as to maximize the number of elements that simultaneously lie within their corresponding bounds. Special classes of this problem are of interest in areas such as pattern recognition [1], machine learning [2, 3], and medical treatment planning. The numerical study of the paper considers a problem from the area of medical treatment planning.

The problem may be formulated as a combinatorial optimization with the objective of selecting a maximum cardinality subset of $\{X^\top a_i\}$ that together with their corresponding bounds define a feasible system. The variable X is then found as a point in the selected feasible region. This combinatorial problem is solvable only in exponential time [4]. In [1] an algorithm is proposed

for a particular class of pattern recognition problems which is better than exhaustive search. The problem may alternatively be formulated as a continuous search over X where the objective function is the number of elements that lie within the given bounds. The continuous formulation does not remedy the complexity of the problem since the objective function is discontinuous and typically exhibits numerous local optima. As a result, the usual gradient based algorithms are not appropriate. Techniques such as genetic algorithm may be useful but not very efficient. Other heuristic search procedures have been proposed that apply to particular classes of the above problem [2]. The problem has been studied from a linear programming infeasibility analysis point of view [5]. [6] presents an algorithm based on alternative formulation of the problem as a minimum weight cover for irreducible inconsistent subsystems (an infeasible subsystem with smallest number of elements). The approach of [6] relies on integer programming and suffers from complexity and difficulty of implementation. The approach followed in this paper is in spirit close to the work of [5, 7]. In both approaches a polynomial time algorithm is proposed that at each iteration solves a linear program to determine candidate inequalities for removal from the set. These candidates are removed and the procedure is repeated until a feasible subset is rendered. The main difference between the approach here and the work of [5, 7] is in the constraint removal technique. While [7] determines removal candidates for elimination by solving an elastic linear program [8] obtained by minimizing the sum of positive violations from the bounds, the constraint removal of the present work relies on solutions to minmax linear programs. The advantage is reduced number of variables for the linear programs (solved at each iteration) and much fewer number of candidates for removal. Moreover, should there at each iteration exist a single constraint whose elimination renders a feasible subset, that single constraint will be identified by the elimination process of the algorithm.

The rest of the paper is organized as follows. The algorithm is presented in Section 2 and illustrated by two numerical examples in Section 3 which contains a

simulated linear system with two sided bounds and a case from radiation therapy planning. Section 4 offers concluding remarks.

2 Maximum feasible subset algorithm

In this section, the algorithm for maximum feasible subset selection is presented. As mentioned in the introduction, the algorithm relies on iterative solutions to minmax linear programs.

Let S denote $\{(a_i, L_i, U_i)|i = 1, \cdots, N\}$ and consider double sided bounds of the form $L_i \leq X^\top a_i \leq U_i$ on X. Generalization to the case where some or all constraints are one sided is straightforward. The problem of interest is maximization (with respect to X) of $\sum_i I(L_i \leq X^\top a_i \leq U_i)$ where the indicator function $I(\cdot)$ is equal to 1 if its argument is true and 0 otherwise. The above optimization problem is equivalent to minimization of the sum $\sum_i \alpha_i^q$, $q \to 0^+$, where

$\alpha_i = \max\{0, \max\{X^\top a_i - U_i, L_i - X^\top a_i\}\}$, ie largest positive deviation from the bounds for the i'th constraint (note that $\alpha_i^{0^+}$ is zero if $\alpha_i = 0$ and 1 otherwise). Hence, the problem may be regarded as minimizing the 0^+-norm of the (deviation) sequence $\{\alpha_i\}$ with respect to X. The objective function $\sum_i \alpha_i^q$ is non-convex for $0 < q < 1$ and exhibits an increasing number of local optima as $q \to 0^+$. We present a heuristic polynomial time algorithm in the following.

Algorithm 1 Initialize $S_0 = S$ and $k = 0$.

Step 1: Solve the linear program

$$\begin{aligned}(X^{(k)}, \beta^{(k)}) &= \arg\min_{X, \beta} \beta, \\ L_i - \beta &\leq a_i^\top X \leq U_i + \beta, \\ \beta &\geq 0, \end{aligned} \quad (2.1)$$

where $(a_i, L_i, U_i) \in S_k$.

Step 2: If $\beta^{(k)} = 0$ (or smaller than a predetermined positive threshold) then propose $X^{(k)}$ and S_k as the solution and terminate. Otherwise continue.

Step 3: Determine all the elements of S_k that are involved in at least one of the $p + 1$ active constraints of the linear program (2.1). These elements constitute candidates for removal from the set (ie there is a maximum number of $p + 1$ elements in the candidate set at each iteration).

Step 4: Select one element of the candidate set for removal from S_k. Update k to $k + 1$ and go to Step 1.

The algorithm may be explained as follows. At each iteration, say iteration k, the linear program (2.1) minimizes the maximum positive deviation from the bounds for the inequalities induced by S_k, or more precisely, minimizes $\max\{0, \max_i\{X^\top a_i - U_i, L_i - X^\top a_i\}\}$ for $(a_i, L_i, U_i) \in S_k$. The constraints of the linear program (2.1) are obviously feasible (since β should merely be selected large enough for the inequalities to hold) and the feasibility of the subsystem $L_i \leq X^\top a_i \leq U_i$, $(a_i, L_i, U_i) \in S_k$, holds iff $\beta^{(k)} = 0$ (the termination criterion of Step 2). The algorithm then proceeds by removing an element from S_k that results in a small value for $\beta^{(k+1)}$. Removal of an element that is *not* a candidate proposed by Step 3 will obviously leave the optimal value of $\beta^{(k+1)}$ unchanged relative to $\beta^{(k)}$. Hence, we only consider removal of S_k's elements that are contained in the candidate set (Step 4). One way of selecting among candidate elements is removal of each candidate at a time, solving the linear program followed by the removal, and selecting the element whose removal results in the smallest value for $\beta^{(k+1)}$. Note that this approach is optimal in a one step ahead sense meaning that if removal of only one element renders a feasible subset then that element will be identified by the procedure. Note that this technique requires solutions to (at most) $p + 1$ linear programs at each iteration. Other techniques might be proposed to speed up calculations, eg removal of the element involved in an equality with largest Lagrange multiplier value at the solution to (2.1).

3 Numerical Examples

In this section, we present two numerical examples to illustrate the performance of the algorithm. The first one is a simulated case where it is of interest to maximize the number of feasible two sided bounds. The second example is an application from the area of radiotherapy planning.

A two sided case: Consider the system of two sided inequalities $19.5 \leq a_i^\top X \leq 20.5$, $i = 1, \cdots, 100$, where the elements of $a_i \in \mathcal{R}^{20}$ are randomly sampled from a uniform distribution between 0.75 and 1.25. Denoting $\alpha_i = \max\{0, \max\{X^\top a_i - U_i, L_i - X^\top a_i\}\}$, we apply Algorithm 1 to minimize $\sum_i \alpha_i^{0^+}$ where the removal is optimal in a one step ahead sense. The percentage of satisfied constraints upon termination of the algorithm is 88% and the computations take about $3\frac{3}{4}$ minutes on a HP-PA8000 work station using MATLAB 5.2 software. Further, we apply linear programming to minimize $\sum_i \alpha_i$ as a convex approximation to the original problem. The percentage of satisfied constraints for

the solution of the linear program is 74% which is noticeably lower than the result obtained by Algorithm 1.

Radiotherapy planning: Coronary angioplasty is a minimally invasive technique for treatment of atherosclerosis, the principle process of heart disease. Despite its wide acceptance, coronary angioplasty is limited by rates of restenosis (renarrowing of the vessel) of 30-60% [9]. A major component of the restenosis process is intimal hyperplasia which refers to the proliferative response to vascular injury. While coronary stents virtually remove recoil and remodeling (other components of the restenosis process) they do not decrease the proliferative response caused by angioplasty. Recent studies show that ionizing radiation, administered during or after angioplasty, can inhibit the proliferative component of restenosis [10]. Possible radiation techniques are temporary or high dose rate catheter-based brachytherapy using radioactive seeds, wire, or liquid filled balloons, permanent or low dose rate brachytherapy using radioactive stents, and external beam radiation. Treatment planning is essential since low doses of radiation stimulate neointimal proliferation and high doses may cause vascular complications [11].

The first step in the design of an optimal treatment plan is outlining points on tomographic sections of the vessel (obtained by eg ultrasonography) that should receive proper amounts of radioactive dose. These points are typically selected on an inner and an outer surface that circumscribe a smooth muscle cell layer of the vessel [9]. The next step is determination of the treatment variables so as to deliver desirable dose values to the outlined points. In this example, we consider the common technique of high dose rate radiation delivery where an afterloader is used to step a train of seeds along the vessel path. The train of seeds dwells at predetermined positions (dwell positions) along the path for predetermined durations of time (dwell times). The treatment variable consists of delivery configuration parameters (relative positioning of the seeds within a train, initial and final positions of the trains on the paths, stepping lengths, seed geometry, etc.) and the sequence of dwell times. Here we consider dwell time optimization for a fixed delivery configuration. With the assumption that the total irradiation times are much smaller than seeds half lives, the dose delivery rates may be assumed constant within the time intervals of radiation. Then for a train of seeds dwelling at a certain position along its path, the amount of delivered dose to a point is equal to the dwell time of the train multiplied by the dose delivery rate to the point of interest for that dwell position. Moreover, the dose delivered during transition between consequent dwell

positions may be ignored due to fast afterloading (negligible transition times). Hence, the dose delivered to a point, say v, is given by $\sum_{i=1}^{p} \phi_i(v)T_i$ where $\phi_i(v)$ is dose delivery rate at point v from the dwell position i, T_i is the dwell time i, and p is the total number of dwell positions. Since the dose delivery rate to a point is only affected by the delivery configuration (which is assumed to be fixed), the problem of treatment planning concerns optimal selection of $T = (T_1, \cdots, T_p)^\top$. The desirable dose values are most commonly given as bounds on delivered radioactive dose to the outlined points. The ideal dose bounds for treatment of intimal hyperplasia are usually given as maximum $D_{\max} = 30$ Gy (radiation unit) for the points on the inner surface and minimum $D_{\min} = 8$ Gy for the points on the outer surface [9]. These objectives may be stated as

$$\sum_{i=1}^{p} \phi_i(v)T_i \leq D_{\max} \; \forall v \in \text{inner surface,}$$
$$\sum_{i=1}^{p} \phi_i(v)T_i \geq D_{\min} \; \forall v \in \text{outer surface.} \quad (3.2)$$

Under infeasibility of the above prescribed bounds, an alternative will be maximization of the fraction of outlined points that receive dose values within the prescribed bounds, ie maximum feasible subset problem for the system (3.2). Indeed, numbers such as fraction of a volume or surface receiving desirable dose values are commonly used in clinical practice for assessment of a treatment plan and are closely related to the so called dose-volume histograms (DVH) or dose-surface histograms (DSH). DVH's and DSH's are cumulative versions of histogram over delivered dose to the volume or surface of interest and are widely accepted figures of merit for evaluation of a treatment plan [12].

Optimization results are presented for a vessel with tubular inner and outer surfaces where the inner surface narrows in the middle of the vessel (see Figure 1) and the delivery configuration and point sampling are identical to those of the numerical study of [13]. The point sampling and delivery configuration used here result in a system of 288 inequalities and 6 variables (ie $N = 288$ and $p = 6$). Similar to [13], calculation of dose rates are based on line source approximation following the American Association of Physicists in Medicine (AAPM) task group 43 (TG43) formalism [14] as recommended by the AAPM-TG60 [15] for photon emitting sources.

Denoting $\alpha_i = \max\left\{0, \max\{X^\top a_i - U_i, L_i - X^\top a_i\}\right\}$, we apply Algorithm 1 to minimize $\sum_i \alpha_i^{0^+}$ where the removal is optimal in a one step ahead sense. The percentage of satisfied constraints upon termination of the
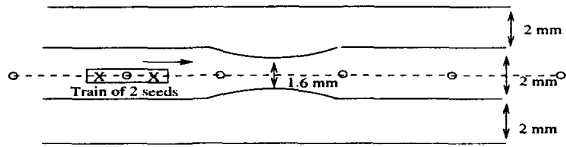
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Figure 1: Longitudinal cross section of the vessel under radiation delivery. Seeds (marked by X) travel in a train of two (represented by the box containing the X marks). The train of seeds travels along the center line of the vessel (represented by the dashed line) and dwells at the positions marked by o along its path.

algorithm is 70% and the computations take about $7\frac{2}{3}$ minutes on a HP-PA8000 work station using MATLAB 5.2 software. Further, we apply linear programming to minimize $\sum_i \alpha_i$ as a convex approximation to the original problem. The percentage of satisfied constraints for the solution of the linear program is 62% which is again noticeably lower than the result obtained by Algorithm 1.

Finally, it should be noted that the planning problems arising in connection with radiotherapy of cancer are very similar to the planning problem considered here. The approach is therefore expected to be of enormous importance in radiotherapy of cancer.

4 Conclusion

We have presented a heuristic polynomial time algorithm for maximum feasible subset selection. The performance of the algorithm has been illustrated on a simulated linear system with double sided bounds and a case from radiation therapy planning for treatment of intimal hyperplasia. It will be of interest to test the performance of the algorithm versus existing other techniques and apply the maximum feasible subset approach to various treatment planning problems, eg radiotherapy of cancer.

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