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Stability and response bounds of non-conservative linear systems

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Abstract

For a linear system of second order differential equations the stability is studied by Lyapunov's direct method. The Lyapunov matrix equation is solved and a sufficient condition for stability is expressed by the system matrices. For a system which satisfies the condition for stability the Lyapunov function is used to derive amplitude bounds of displacement and velocity in the homogeneous as well as in the inhomogeneous case. The developed results are illustrated by examples.

1 Introduction

Given the linear system of differential equations of the form

$$M\ddot{x}(t) + (D+G)\dot{x}(t) + (K+N)x(t) = f(t) \quad (1)$$

where M, D and K are Hermitian (in particular real symmetric) matrices. The mass matrix M, the damping matrix D and the stiffness matrix K are all positive definite ($M > 0$, $D > 0$, $K > 0$), while the matrix G of the gyroscopic forces and the matrix N of the circulatory forces are skew-Hermitian (in particular real skew-symmetric). If $G = 0$ and $N = 0$ we deal with a classical damped system which of course is stable. Recently response bounds of such systems are given by Schiehlen, Hu and Eberhard[1, 2, 3, 4]. If both circulatory forces N and gyroscopic forces G are present, the stability of the system depends on the relation between the stabilizing forces characterized by D and G and the destabilizing forces characterized by N. If the system is stable, we can ask for bounds of the response of the system. To examine whether the system is stable, we find a Lyapunov function by solving the Lyapunov matrix equation. Then a sufficient condition for stability is expressed in terms of the properties of the system matrices. This stability condition includes for a certain choice of the involved parameters the more restrictive stability criterion earlier found by Frik[5]. We then achieve bounds of the responses by using the Lyapunov function associated with the stable system. Finally we give examples which demonstrate the usefulness of the results.

2 Lyapunov's direct method

The homogeneous linear system obtained from (1)

$$M\,\ddot{x}(t) + (D+G)\,\dot{x}(t) + (K+N)\,x(t) = 0 \quad (2)$$

can be rewritten as a first order system $\dot{z} = Az$ where

$$A = \begin{pmatrix} 0 & I \\ -K-N & -D-G \end{pmatrix}, \; z = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \quad (3)$$

and I is the identity matrix. As a Lyapunov function $V(z)$ for system (3) we take

$$V = z^*(t)\,P\,z(t) \quad , \quad (4)$$

where $P = P^*$ is a Hermitian matrix, which satisfies the Lyapunov matrix equation

$$PA + A^*P = -Q \quad , \quad (5)$$

and $Q = Q^* \geq 0$. The system (3) is asymptotically stable, if there exist Hermitian matrices $P > 0$ and $Q > 0$ which satisfy the Lyapunov matrix equation (5).

3 Stability

The crucial point here is to find a matrix P satisfying (5) . Starting with a first integral of the of the equation of motion we find a proper solution P . We then formulate the following

Theorem: *If $b^2 - 4ac > 0$ and $b > 0$, then system (2) is asymptotically stable.*

The scalars a, b and c are defined by

$$a = \lambda_{max}(M + G^* K^{-1} G)\,,$$
$$b = \lambda_{min}\big(2D + \frac{1}{2}(G^* K^{-1} N + N^* K^{-1} G)\big)\,,$$
$$c = \lambda_{max}(N^* K^{-1} N)\,, \qquad (6)$$

where λ_{min} and λ_{max} denote the smallest and the largest eigenvalues of the respective matrices. A more restrictive stability condition we achieve by making a rough estimate of the scalars a and b

$$a \le m_{max} + \frac{1}{4}\, g_{max}^2 \,/\, k_{min} \quad ,$$
$$b \ge 2\, d_{min} - g_{max}\, n_{max}/k_{min} \quad . \qquad (7)$$

Here m_{max} is the largest eigenvalue of M , g_{max} and n_{max} are largest values of the moduli of the eigenvalues of G and N , respectively, and k_{min} denotes the smallest eigenvalue of K. Then the stability conditions of the theorem become

$$(\,2d_{min} - g_{max} n_{max} \,/\, k_{min})^2$$
$$- 4\,(m_{max} + \frac{1}{4}\, g_{max}^2 \,/\, k_{min})\, n_{max}^2 \,/\, k_{min} \,>\, 0\,,$$
$$2d_{min} - g_{max} n_{max}/k_{min} \,>\, 0\,. \qquad (8)$$

Both inequalities in (8) are satisfied if

$$d_{min}^2\, k_{min} - d_{min}\, g_{max}\, n_{max}$$
$$- m_{max}\, n_{max}^2 > 0\,. \qquad (9)$$

This inequality is known as a sufficient condition for asymptotic stability, se Frik[5] and Kliem and Seyranian[6] .

4 Response bounds

First we consider the homogeneous system (2) which we assume to be stable according to the stability theorem. The Lyapunov function can be used to estimate the 2-norm $\|x(t)\| = \sqrt{x^*(t)x(t)}$ as follows

$$\|x(t)\| \le \sqrt{\frac{V_0}{\lambda_{min}(K + \frac{\gamma}{2}\, D - \frac{\gamma^2}{4}\, M)}} \quad , \qquad (10)$$

where $\gamma = b/(2a)$ and V_0 is the value of the Lyapunov function for $t = 0$ which sounds

$$V_0 = x^*(0)\,(K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M\,)\, x(0) +$$
$$(\dot{x}(0) + \frac{\gamma}{2}\, x(0))^*\, M\, (\dot{x}(0) + \frac{\gamma}{2}\, x(0))\,. \qquad (11)$$

In a similar way we obtain the estimate for the time derivative

$$\|\dot{x}(t)\| \le \frac{\gamma}{2}\, \|x(t)\| + \sqrt{\frac{V_0}{\lambda_{min}(M)}}\,. \qquad (12)$$

We now return to the inhomogeneous system (1) which we again assume to be stable according to the theorem. If we assume $f(t)$ to be a non-transient excitation, it is normally easy to find a particular solution $x_{part}(t)$ e.g. by making a suitable guess. Since every solution $x(t)$ to (1) can be expressed as a sum of a solution $x_{hom}(t)$ to the homogeneous equation (2) and a particular solution $x_{part}(t)$ to the inhomogeneous equation (1) we achieve the bound as follows

$$\|x(t)\| \le \sqrt{\frac{V_{0,h}}{\lambda_{min}(K + \frac{\gamma}{2}\, D - \frac{\gamma^2}{4}\, M)}}$$
$$+ \|x_{part}(t)\| \quad , \qquad (13)$$

where $V_{0,h}$ is given by (11) if we in on the right side substitute $x(t)$ by the $x_{hom}(t)$. For a transient excitation $f(t)$ we find a solution to (1) with the given initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ by calculating the convolution of the impulse response matrix $\Phi(t)$ and $f(t)$

$$x(t) = \int_0^t \Phi(t - \tau)\, f(\tau)\, d\tau \quad . \qquad (14)$$

Taking $f(t) = u\psi(t)$, where u is a constant vector and $\psi(t)$ is a scalar function subjected to the condition

$$p = \int_0^\infty \|\psi(t)\|\, dt < \infty \quad , \qquad (15)$$

we can deduce

$$\|x(t)\| \le \sqrt{\frac{u^*\, M^{-1}\, u}{\lambda_{min}(K + \frac{\gamma}{2}\, D - \frac{\gamma^2}{4}\, M)}}\; p \quad . \qquad (16)$$

5 Examples

5.1 Example 1

To illustrate the formulas for response bounds of the homogeneous system (2) let us consider the 3×3 system described by

$$M = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix},$$
$$D = \begin{pmatrix} 8 & -2 & 2 \\ -2 & 8 & -2 \\ 2 & -2 & 8 \end{pmatrix},$$
$$G = \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix},$$
$$K = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 4 \end{pmatrix},$$
$$N = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}. \qquad (17)$$

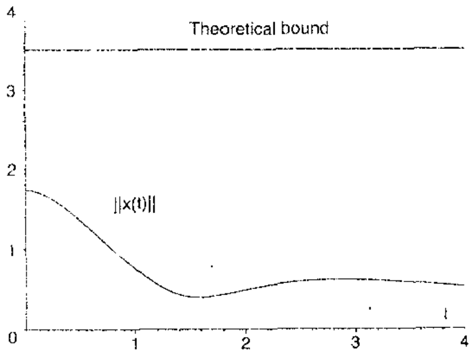

Figure 1: Theoretical bound and $\|x(t)\|$.

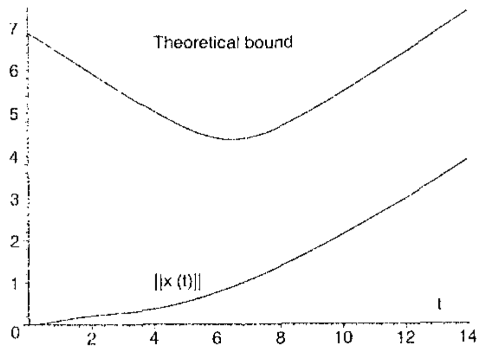

Figure 2: Theoretical bound and $\|\dot{x}(t)\|$

Given the initial conditions $x(0) = [1,1,1]^T$ and $\dot{x}(0) = [0,0,0]^T$, we calculate the constants defined in (6) as

$$a = \lambda_{max}(M + G^* K^{-1} G) = \frac{37 + \sqrt{809}}{8},$$

$$b = \lambda_{min}\left(2D + \frac{1}{2}(G^* K^{-1} N + N^* K^{-1} G) \right) = 13,$$

$$c = \lambda_{max}(N^* K^{-1} N) = 3.$$

Because $b > 0$ and $b^2 - 4ac = 70.8 > 0$ the system is stable according to the stability theorem, while the Frik criterion given by (9) is not satisfied in this case. To get the bound of the 2-norm given by (10) we first calculate the values $\gamma = b/(2a) = 0.794$, $V_0 = 33.94$ and $\lambda_{min}(K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M) = 2.752$. The bound is then found to be $\|x(t)\| < 3.51$ and this value is compared to the exact value of $\|x(t)\|$ as shown in figure 1 . Using $\lambda_{min}(M) = 1$ we can calculate the bound of $\|\dot{x}(t)\|$ according to equation (12). Figure 2 shows the bound of $\|\dot{x}(t)\|$ compared to the exact value of $\|\dot{x}(t)\|$.

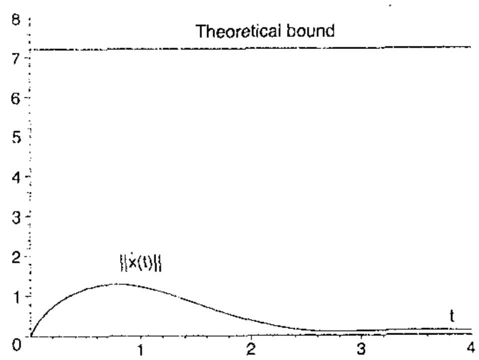

Figure 3: Theoretical bound and $\|\dot{x}(t)\|$

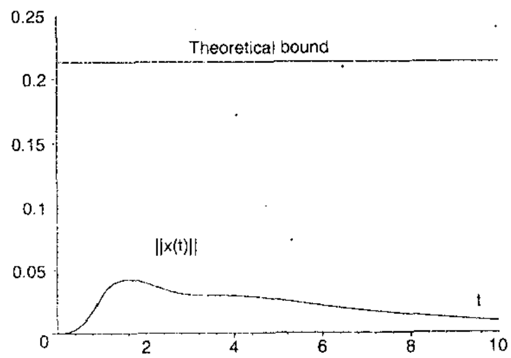

Figure 4: Theoretical bound and $\|\dot{x}(t)\|$

5.2 Example 2

We now look at the inhomogeneous system (1) with the same system matrices (17) as given in example 1 . We assume that the system is excited by a transient force given by $f(t) = [t, 1, 1]^T$. A particular solution to the inhomogeneous equation is

$$x_{part}(t) = t \begin{pmatrix} 13/30 \\ 1/15 \\ -7/30 \end{pmatrix} + \begin{pmatrix} -1087/450 \\ -233/450 \\ 1003/450 \end{pmatrix}. \quad (18)$$

With the initial conditions $x(0) = [0,0,0]^T$ and $\dot{x}(0) = [0,0,0]^T$ we have for the initial conditions of $x_{hom}(t)$

$$x_{hom}(0) = -\begin{pmatrix} -1087/450 \\ -233/450 \\ 1003/450 \end{pmatrix}, \quad (19)$$

$$\dot{x}_{hom}(0) = -\begin{pmatrix} 13/30 \\ 1/15 \\ -7/30 \end{pmatrix}$$

Substituting $x_{hom}(0)$ and $\dot{x}_{hom}(0)$ into (11) we obtain $V_{0,h} = 34.69$. The bound of the 2-norm given by (13) is then found to be $\|x(t)\| < 3.55 + \|x_{part}(t)\|$. This value is compared to the exact norm $\|x(t)\|$ as shown in figure 3 .
If we instead excite the system with a transient force, say $f(t) = u\psi(t)$, where $u = [1,0,0]^T$ and $\psi(t) = t$ for $0 < t < 1$ and otherwise $\psi(t) = 0$, we have $p = 1/2$ and $u^* M^{-1} u = 1/2$. Then the equation (16) gives the bound of the 2-norm $\|x(t)\| < 0.213$. This value is be compared to the exact value of the norm $\|x(t)\|$ as shown in figure 4 .

6 Conclusions

Using the Lyapunov's direct method we have formulated a sufficient condition for stability of a certain class of non-conservative systems. The condition is expressed by the largest and smallest eigenvalues of combinations of the system matrices. The constructed Lyapunov function is used to obtain bounds for the norms of the displacement and the velocity. There exist stable non-conservative systems, which do not satisfy the deduced condition for stability. For such systems, no response bounds are available by this method.

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