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# One-dimensional map lattices: Synchronization, bifurcations, and chaotic structures

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The paper presents a qualitative analysis of coupled map lattices (CMLs) for the case of arbitrary nonlinearity of the local map and with space-shift as well as diffusion coupling. The effect of synchronization where, independently of the initial conditions, all elements of a CML acquire uniform dynamics is investigated and stable chaotic time behaviors, steady structures, and traveling waves are described. Finally, the bifurcations occurring under the transition from spatiotemporal chaos to chaotic synchronization and the peculiarities of CMLs with specific symmetries are discussed. [S1063-651X(96)05309-3]

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## I. INTRODUCTION

As simplified models of spatially extended systems under nonequilibrium conditions, the dynamics of coupled map lattices (CMLs), i.e., of systems with discrete time, discrete space, and a continuous state, has attracted a rapidly growing interest in recent years [1–6]. Computer simulations have revealed a variety of behaviors from the very simple to the very complex and the core problem of the transition from one-dimensional chaos associated with the temporal behavior of the local element to multidimensional spatiotemporal chaos in the coupled map lattice has been elucidated for different maps and different types of coupling.

In particular, Kaneko [1] has investigated the development of spatiotemporal intermittency in the form of a laminar motion interrupted by bursts. This study was performed with a class of coupled map lattices for which the individual map was close to a transition to temporal intermittency and the observed geometric structures in space-time resembled the structures found in cellular automata. As the coupling was increased, the number of positive Lyapunov exponents also increased and a kind of fully developed turbulence appeared. Under certain conditions, localized chaos was observed, i.e., the burst regions were confined to specific areas and could not propagate throughout the whole space. Kaneko [2] has also studied the information flow in coupled map lattices and has introduced the concept of comoving Lyapunov exponents to characterize convective instabilities in open flow systems. More recently, Kaneko [3] has studied the dynamics of globally coupled maps and developed a mean-field description of the fluctuations in such systems.

Willeboordse [4] has studied the problem of pattern selection in chains of diffusively coupled logistic maps. Besides globally incoherent patterns in the form of frozen random lattices or spatiotemporal intermittency, slowly moving coherent structures were observed and Willeboordse proposes a scheme to encode standing as well as traveling waves into the CML dynamics. Xie, Hu, and Qu [5] have investigated so-called on-off intermittency for a coupled map lattice operating near a spatiotemporal period-2 solution. When applying noise at a single site, they have determined the relative

probabilities of different durations of the laminar phases along the chain. Near the forced site, this probability was found to decay exponentially. For sites further away, however, the exponential decay was replaced by a power law with an exponent of  $-3/2$ .

In order to examine their thermodynamic properties, Bourzutschky and Cross [6] have considered the long-wavelength limit of the behavior of simple CMLs. Using a generalization of the fluctuation-dissipation theorem, they have tried to define a temperature that could be useful in the description of spatially extended, nonequilibrium systems. Such a temperature could provide constraints, for example, on the effective noise term in the corresponding Langevin equation.

The purpose of the present paper is to proceed with a more analytic approach to the description of coupled map lattices. Previous contributions in this direction are due, for instance, to Afraimovich and Nekorkin [7], who analyzed the stability of chaotic waves propagating in a discrete chain of diffusively coupled maps, and to Kuznetsov [8], who applied renormalization-group theory to study universality and scaling in CML dynamics. The scaling behavior of coupled map lattices has also been studied by Kook, Ling, and Schmidt [9], by Kaspar and Shuster [10], and by Bohr and Christensen [11].

Amritkar *et al.* [12] have investigated the stability of spatially and temporally periodic orbits in one- and two-dimensional coupled map lattices. Using the fact that the stability matrices for such solutions are block circulant and hence can be brought onto a block diagonal form through a unitary transformation, they derive conditions for the stability of periodic solutions in terms of the criteria for smaller orbits. Druzhinin and Mikhailov [13] have considered a particular type of CML where the coupling affects only the bifurcation parameter of the local maps. In the continuum limit this corresponds to a reaction-diffusion equation for which the diffusion coefficient depends on the state of the system. They show that the formation of stable solitonlike patterns is possible and, assuming the local map to be logistic, they obtain an expression for the solution close to the period-doubling bifurcation where the fixed point turns unstable. In

a detailed numerical study of bounded one- and two-dimensional CML with diffusively coupled logistic maps, Giberti and Vernia [14] have followed the change in the steady states as the coupling parameter is increased. Ascribing to the single map a bifurcation parameter slightly above the Feigenbaum accumulation point, the dynamics of the uncoupled lattice is totally chaotic. As the coupling parameter is increased, more and more periodic orbits of period  $2^k$  start to arise, with the orbits with longer periods appearing first. Above a certain coupling strength, almost all randomly chosen initial conditions lead to a stable period orbit. By means of continuation techniques, Giberti and Vernia [14] also show how the stable periodic orbits develop from the unstable periodic orbits of the uncoupled system.

Giacomelli, Lepri, and Politi [15] have examined the statistical properties of the bidimensional patterns generated from delayed and extended maps and Losson and Mackey [16] have considered coupled map lattices as models of deterministic and stochastic differential-delay equations. By rotating the time-space reference frame, a delayed map can be transformed into a CML with asymmetric coupling. The equivalence between these systems is formal, however, and different causality conditions apply to the two cases. Some future events in the delayed map are past events in the CML representation and vice versa. As a result, certain statistical properties are the same for the two representations, while others are not [15]. In several cases of interest, the Hopf equation describing the evolution of the ensemble density in phase space for a delayed differential equation may be approximated by a Perron-Frobenius equation in  $\mathbb{R}^N$  for a CML system. This can be used to explain the statistical cycles observed numerically in delayed differential equations in terms of stable density limit cycles [16].

In the present paper we consider the  $N$ -dimensional (or infinite-dimensional) map

$$T: x_i \rightarrow f(x_i) + \varepsilon[x_{i+1} - (1 + \gamma)x_i + \gamma x_{i-1}], \quad (1)$$

where  $x_i \triangleq x_i(j) \in \mathbb{R}^1$ .  $j \in \mathbb{Z}^+$  is a discrete time and  $i \in [1, N]$  or  $i \in \mathbb{Z}$ , depending on the boundary conditions, is a discrete space coordinate. Obviously, with  $g(x) = f(x) - x$ , one can derive (1) from the partial differential equation

$$\frac{\partial x}{\partial t} = g(x) + (1 - \gamma) \frac{\partial x}{\partial s} + \frac{\partial^2 x}{\partial s^2} \quad (2)$$

by approximating the time and space derivatives with differences and by rescaling the nonlinearity. This relates our model to the dynamics of extended, nonequilibrium media. Our aim is to study some global aspects of the dynamics of the map  $T$  such as the relation between the dynamics of the single map  $S: x \rightarrow f(x)$  and of the coupled map lattice for different values of the coupling parameter  $\varepsilon$ , and the general features of the bifurcations that take place as  $\varepsilon$  is increased. We first consider the sufficient conditions for the map  $T$  to have an attractive domain (Theorem 1). This allows us to estimate the limits of variation for the state variables. In Theorem 2, one of the main properties of the map  $T$  is stated, namely, its ability to generate simple behavior as a result of synchronization when all elements of the CML, independently of the initial conditions, acquire a uniform behavior determined by the local one-dimensional (1D) map. The

stable chaotic time behaviors, steady structures, and traveling waves are studied by applying the detailed rigorous analysis of an associated two-dimensional map proposed by Belykh *et al.* [17]. In contrast to Afraimovich and Nekorkin [7], we prove that the coordinates of traveling chaotic waves need not be confined to some particular states of the lattice elements. For weak coupling, the ability of the map  $T$  to produce an extremely complex dynamics as a result of the chaotic time behavior of the local maps is discussed in Sec. III. There we also state Theorem 5 on the bifurcation set corresponding to the disappearance of a complex limiting set under the transition to synchronization. Finally, we discuss the symmetric solutions to the map  $T$  in the case of pure diffusive coupling.

## II. SYNCHRONIZATION IN COUPLED ONE-DIMENSIONAL MAPS

Consider the diffusively coupled 1D map array  $T$  given by (1) with  $x_i \triangleq x_i(j)$ . As before,  $j \in \mathbb{Z}^+$  is a discrete time coordinate,  $f(x_i) \in \mathbb{C}^k$  ( $k \geq 1$ ) is a nonlinear mapping function, and  $i \in [1, N]$  or  $i \in \mathbb{Z}$  is a discrete space coordinate.  $\varepsilon$  and  $\gamma$  are non-negative coupling parameters to be referred to as diffusion and shift coefficients. For finite arrays we shall consider *zero-flux*

$$x_0 \equiv x_1, \quad x_N \equiv x_{N+1} \quad (3a)$$

or *periodic*

$$x_0 \equiv x_N, \quad x_{N+1} \equiv x_1 \quad (3b)$$

boundary conditions. The single map  $S: x \rightarrow f(x)$  is assumed to have an attractive interval  $I_0 \subset \mathbb{R}^1$ . The purpose of the present section is to discuss some general properties of the map  $T$  relating to its stable and regular behaviors. Let us first determine the attracting domain of the map  $T$ . Here we may state the following.

*Theorem 1.* (a) Let the map  $S_\alpha: x \rightarrow \alpha + f(x)$  have an attracting domain  $I_\alpha$  for  $|\alpha| \leq \alpha_0$  and let  $I^* = (x_1^* \leq x \leq x_2^*)$  be an interval such that for any  $x \in I^*$  and any  $\alpha \in [-\alpha_0, \alpha_0]$ ,  $S_\alpha I^* \subset I^*$ . (b) If there exists a value  $r$  satisfying the conditions  $-r \leq x_1^* < x_2^* \leq r$ ,  $\varepsilon r \leq \alpha_0/2(1 + \alpha)$ , then the map  $T$  has an attracting domain  $D = \{x_i | x_i < r, i = 1, 2, \dots, N\}$ . The proof of this theorem can be found in the Appendix.

Let us now consider the possibility of synchronization of the individual maps into an overall uniform behavior. The coupled map  $T$  has a one-dimensional invariant manifold (*diagonal*)  $\mathcal{D} = \{x_1 = x_2 = \dots = x_{N-1} = x_N\}$ . Indeed, if  $x_i(0) \in \mathcal{D}$ ,  $i = 1, 2, \dots, N$ , then  $x_i(j) \in \mathcal{D}$  for  $j \in \mathbb{Z}^+$ , such that  $T|_{\mathcal{D}} = S$ . To analyze the stability of  $\mathcal{D}$  let us introduce the variables

$$y_i(j) = x_i(j) - x_{i+1}(j), \quad i = 1, 2, \dots, n, \quad n \triangleq N - 1 \quad (4)$$

and the differences

$$f(x_i(j)) - f(x_{i+1}(j)) = f'(\xi(x_i(j), x_{i+1}(j))) y_i(j), \quad (5)$$

where  $\xi \in (x_i, x_{i+1})$ . According to the mean value theorem,  $f'(\xi)$  is a piecewise smooth function of  $x_i$  and  $x_{i+1}$ . Let us denote  $\beta_i(j) = f'(\xi(x_i(j), x_{i+1}(j))) - \varepsilon \gamma_i$ , where  $\gamma_1 = 1$ ,

$\gamma_n = \gamma$ , and  $\gamma_i = 1 + \gamma$ ,  $i = 2, 3, \dots, n - 1$ , and let us introduce the vector  $Y(j) = \text{column } (y_1(j), y_2(j), \dots, y_n(j))$  and the  $n \times n$  matrix  $Q(\beta_i(j))$ , where

$$Q(\beta_i) \triangleq \begin{pmatrix} \beta_1 & \varepsilon & 0 & \cdots & 0 & 0 & 0 \\ \varepsilon \gamma & \beta_2 & \varepsilon & \cdots & 0 & 0 & 0 \\ 0 & \varepsilon \gamma & \beta_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{n-2} & \varepsilon & 0 \\ 0 & 0 & 0 & \cdots & \varepsilon \gamma & \beta_{n-1} & \varepsilon \\ 0 & 0 & 0 & \cdots & 0 & \varepsilon \gamma & \beta_n \end{pmatrix}. \tag{6}$$

By subtracting from each equation in (1) the subsequent equation, we get the map

$$L(j); \quad Y(j+1) = Q(\beta_i(j))Y(j), \tag{7}$$

where the dependence on  $j$  is determined by the original map (1).

*Theorem 2.* Let the map  $T$  have an attractive domain  $D$ . If the matrix  $Q(\beta_i(j))$  for  $x_i(j) \in D$ ,  $j \in \mathbb{Z}^+$  has the eigenvalues  $s_k(j)$ ,  $k = 1, 2, \dots, n$ , inside the unit circle in the complex plane [ $|s_k(j)| < 1$ ], then the manifold  $\mathcal{D}$  is absolutely asymptotically stable.

Indeed, as each iterate of the map (7) is contracting, the composition of maps  $L_l \triangleq L(j+l)L(j+l-1) \cdots L(j)$  is contracting as well. The asymptotic stability of the manifold  $\mathcal{D}$  implies the global synchronization of the map  $T$ . Hence, when  $j \rightarrow \infty$ , each cell acquires a uniform behavior with respect to the map  $S$ , independently of the initial conditions [17].

In order to calculate the eigenvalues of  $Q(\beta_i(j))$  one can estimate the trace elements  $\beta_i(j)$  relating the derivatives  $f'(\theta)$  to the attracting domain  $D$  and use the recurrent formula for  $\Delta_n \triangleq \det Q(j)$ ,

$$\Delta_k = \beta_k \Delta_{k-1} - \gamma \varepsilon^2 \Delta_{k-2} \tag{8}$$

with the initial conditions  $\Delta_{-1} = 0$  and  $\Delta_0 = 1$ . A similar approach was used by Afraimovich and Nekorkin [7] to examine the stability of the steady states. If one is to use the norm of a matrix  $P = (p_{kl})$ ,  $\|P\| = \sum_{k,l} |p_{kl}|$ , another criterion of synchronization is  $\|Q\| < 1$ . From this criterion follows the inequality

$$2\varepsilon(1 + \gamma)(N - 2) + (N - 1) [\max_{x \in D} |f'|] < 1. \tag{9}$$

Hence the sufficient condition for the manifold  $\mathcal{D}$  to be absolutely stable is

$$\max_{x \in D} |f'| < \frac{1 - 2\varepsilon(1 + \gamma)(N - 2)}{N - 1}. \tag{10}$$

*Example 1.* Consider  $N = 2$  ( $n = 1$ ). The map  $L(j)$  in (7) takes the form  $y(j+1) = [f' - \varepsilon(1 + \gamma)]y(j)$ , which is stable if  $-1 + \varepsilon(1 + \gamma) < f' < 1 + \varepsilon(1 + \gamma)$ ,  $x \in D$ .

### III. STEADY STATES AND STABLY TRAVELING CHAOTIC WAVES

Let us hereafter study the problem of the existence and stability of the steady states of the map  $T$ . The fixed points of  $T$  are defined by the conditions

$$x_i(j+1) = x_i(j) \triangleq x_i, \quad i = 1, 2, \dots, N \tag{11a}$$

or

$$g(x_i) + \varepsilon[x_{i+1} - (1 + \gamma)x_i + \gamma x_{i-1}] = 0, \tag{11b}$$

where as before  $g(x) \triangleq f(x) - x$ . Equation (11b) may be considered as a spatial map and the steady states for  $T$  arise as the solutions to this map satisfying the boundary conditions (3) or, in the case of unbounded array, without boundary conditions.

Introducing the new variables

$$u_i = \gamma(x_i - x_{i-1}), \quad u_{i+1} = \gamma(x_{i+1} - x_i), \tag{12}$$

we obtain the two-dimensional map  $F$ :

$$\begin{aligned} x_{i+1} &= x_i + u_i - \varepsilon^{-1}g(x_i), \\ u_{i+1} &= \gamma[u_i - \varepsilon^{-1}g(x_i)], \quad i = 1, 2, \dots, N \text{ or } i \in \mathbb{Z}. \end{aligned} \tag{13}$$

This map has the Jacobian  $J = \gamma > 0$  and hence  $F$  is a one-to-one map. Moreover, in the case where  $\gamma = 1$  and the function  $g(x)$  is periodic, the map  $F$  reduces to the standard map [18,19]. Depending on the boundary conditions, the trajectories of  $F$ , representing the steady states of the map  $T$ , are as follows. In the case of zero-flux boundary conditions, each fixed point of  $T$  is a trajectory of  $F$  satisfying the condition

$$u_1 = 0, \quad u_{N+1} = 0. \tag{14}$$

In the case of periodic boundary conditions, each fixed point of  $T$  is a period- $N$  cycle of the map  $F$ , i.e.,

$$u = F^N u, \quad u_1 = u_{N+1}, \quad u_{i_1} \neq u_{i_2}, \quad i_{1,2} \in [1, N+1]. \tag{15}$$

In the limit  $N \rightarrow \infty$ , each bounded trajectory of (13)  $(x_k, u_k) = T^k(x_0, u_0)$ ,  $k \in \mathbb{Z}$ , corresponds to a steady state of the map  $T$ .

The trajectories of the map  $F$  were studied by Belykh [20] for an arbitrary nonlinear function  $g(x)$  having  $l$  zeros and  $l - 1$  extrema alternating between one another. In particular, the case of an odd sinelike periodic function was considered. The fixed points of  $F$  are alternatingly of saddle type and of elliptical (or reverse saddle) type. Applying these results to the present case, we obtain the following.

*Theorem 3.* (a) For any  $N \geq 2$  there exist values of the parameters  $\varepsilon, \gamma$  such that the map  $F$  has a trajectory satisfying the boundary conditions (14) or (15). (b) The map  $F$  displays the bifurcation curve  $\{\gamma = \gamma^*(\varepsilon^{-1})\}$ ,  $\gamma^*(0) = 1$ ,  $\gamma^*(\varepsilon_*^{-1}) = 0$  at which the homoclinic orbit is in the tangency of the stable and unstable manifolds of a saddle point. For the range of parameters  $\mathfrak{H} = \{\gamma \in (\tilde{\gamma}, 1], \tilde{\gamma} = \gamma^*$  for  $\varepsilon > \varepsilon_*$ ,  $\tilde{\gamma} = 0$  for  $\varepsilon \leq \varepsilon_*\}$ , the map  $F$  has a structurally stable ho-

moclinic orbit in the neighborhood of which the trajectories of  $F$  are topologically conjugated to the Bernoulli shift over  $p \geq 2$  symbols.

*Example 2.* At  $\gamma=0$  the coupled map lattice (1) is unidirectional and the map  $F$  is reduced to the one-dimensional mapping  $x \rightarrow x - \varepsilon^{-1}g(x)$ , which becomes the short map  $x \rightarrow \mu - x - x^2$  for  $\varepsilon^{-1}g(x) = -(\mu - 2x - x^2)$ . For this function, under the coordinate transformation  $(x, u) \rightarrow [(1 + \mu)^{-1/2}(x+1), (1 + \mu)^{-1/2}u]$ , the map  $F$  for  $\gamma > 0$  takes the form

$$\bar{x} = x + u - \sqrt{1 + \mu}(1 - x^2), \quad \bar{u} = \gamma[u - \sqrt{1 + \mu}(1 - x^2)], \quad (16)$$

i.e., the form (13) with  $\varepsilon^{-1} = \sqrt{1 + \mu}$  and  $g(x) = 1 - x^2$ . Here an overbar is used to denote the next iterate. Figure 1(a) illustrates the homoclinic orbit bifurcation curve  $\gamma = \gamma^*(\sqrt{1 + \mu})$  for the map (16) and Figs. 1(b) and 1(c) show the stable and unstable manifolds of the saddle point for the values  $\mu=0.3, \gamma=0.6$  in the region  $\mathcal{S}$  after tangency and for the values  $\mu=0.5, \gamma=0.1$  before tangency, respectively. Note that the bifurcation curve for  $\mu > 0$  follows the rough approximation  $\gamma^* = (15 - 12\mu)/20$ .

A corollary of Theorem 3 is that the coupled map  $T$  has fixed points in the form of a regular stationary space distribution of coordinates for bounded arrays and in the form of chaotic distributions for unbounded arrays. Note that in the degenerate case  $\gamma=0$  when the map  $T$  becomes unidirectional and the stationary distributions of coordinates (fixed points) are determined by the one-dimensional map  $x \rightarrow x - \varepsilon^{-1}g(x)$ , one also has the possibility of observing complex behavior of  $T$ .

Following Afraimovich and Nekorkin [7], let us now consider solutions to the coupled map lattice (1) in the form of waves traveling at a constant speed and with unchanged shape  $x_i(j) = \Psi(i + j)$ . The equations for such solutions become

$$x_{k+1} = f(x_k) + \varepsilon[x_{k+1} - (1 + \gamma)x_k + \gamma x_{k-1}], \quad (17)$$

where  $k = i + j$  is a *traveling coordinate* and the  $x$  notation  $x_k = \Psi(k)$  has been preserved. Introducing in analogy with (12) the new coordinates

$$u_i = (x_i - x_{i-1}) \frac{\varepsilon \gamma}{1 - \varepsilon}, \quad (18)$$

we obtain a 2D mapping

$$x_{i+1} = x_i + u_i + \frac{g(x_i)}{1 - \varepsilon}, \quad u_{i+1} = \frac{\varepsilon \gamma}{\varepsilon - 1} \left( u_i + \frac{g(x_i)}{1 - \varepsilon} \right) \quad (19)$$

of the same form as (13). By virtue of Theorem 3, the map  $T$  generates traveling waves of chaotic profile for parameters corresponding to the homoclinic orbits and, consequently, Smale horseshoes exist for each previously defined sine-wave type of function  $g(x)$  having not less than two zeros.

The stability analysis of the steady states and traveling waves performed by Afraimovich and Nekorkin [7] for cubic functions  $g(x)$  in the finite as well as the infinite-dimensional case may now be extended to arbitrary nonlin-

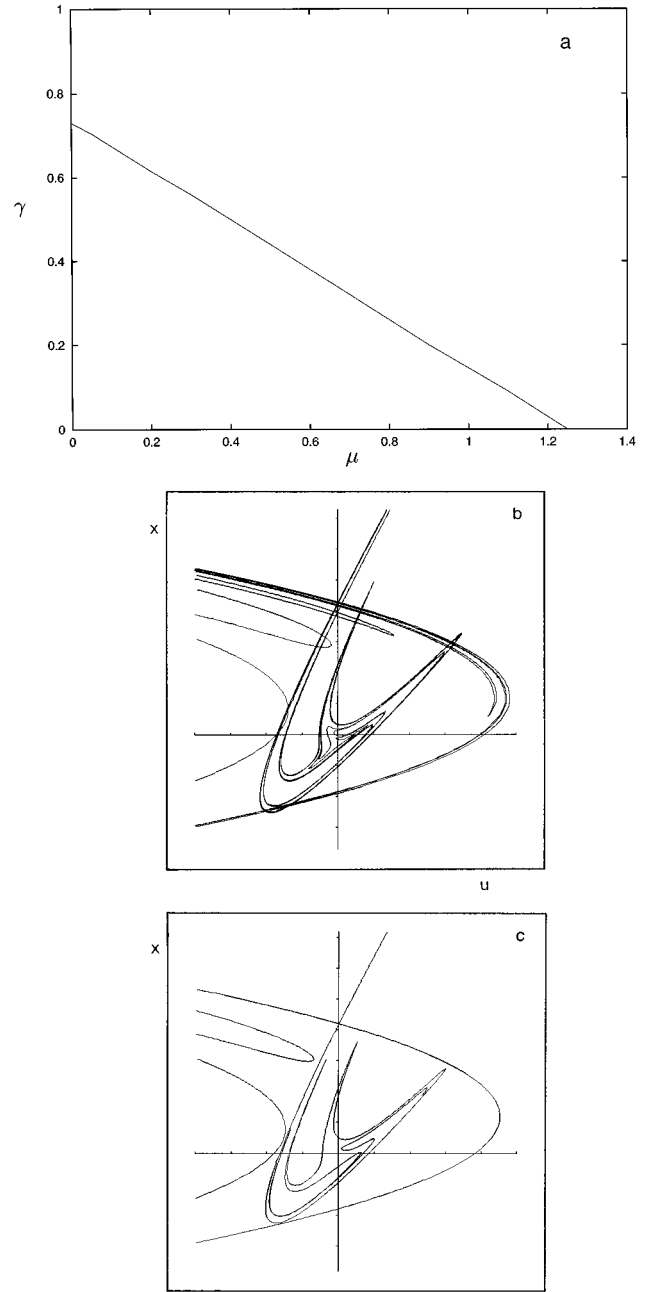


FIG. 1. Homoclinic orbit bifurcation for a two-dimensional map corresponding to two diffusively coupled short maps. (a) Bifurcation diagram. (b) Stable and unstable manifolds after bifurcation ( $\mu=0.3, \gamma=0.6$ ). (c) Phase picture before bifurcation ( $\mu=0.5, \gamma=0.1$ ).

earities. Let us consider small values of the coupling parameter  $\varepsilon$ . The stability of a fixed point  $(x_1^*, x_2^*, \dots, x_N^*)$  of  $T$  with respect to a perturbation  $Z_i = x_i - x_i^*$  is defined by the linear map

$$Z(j+1) = AZ(j), \quad (20)$$

where  $Z(j)$  is the column matrix  $(Z_1(j), Z_2(j), \dots, Z_n(j))$ .  $A$  is the Jacobian of  $T$  at  $(x_1^*, x_2^*, \dots, x_N^*)$  with the boundary conditions (3). For zero-flux boundary conditions, the Jacobian matrix becomes  $A = Q(a_i)$ , where

$$\begin{aligned} a_i &= f'(x_i^*) - \varepsilon \gamma_i, \quad \gamma_1 = 1, \quad \gamma_N = \gamma, \\ \gamma_i &= 1 + \gamma \quad \text{for } i = 2, 3, \dots, N-1. \end{aligned} \quad (21)$$

For practical calculations of the stability conditions the following formula for  $\Delta_N \triangleq \det A$  turns out to be useful

$$\begin{aligned} \Delta_k &= a_k \Delta_{k-1} - \gamma \varepsilon^2 \Delta_{k-2}, \quad \Delta_{-1} = 0, \quad \Delta_0 = 1, \\ a_k &= f'(x_k^*) - \varepsilon \gamma_k, \\ k &= 1, 2, \dots, N, \quad \gamma_1 = 1, \quad \gamma_N = \gamma, \\ \gamma_k &= 1 + \gamma \quad \text{for } k = 2, \dots, N-1. \end{aligned}$$

In the limit of weak coupling, when terms of  $O(\varepsilon^2)$  may be neglected, we obtain

$$\Delta_1 = a_1, \quad \Delta_2 = a_2 \Delta_1, \dots, \quad \Delta_N = \prod_{k=1}^N a_k. \quad (22)$$

Hence the characteristic equation can be written in the form

$$\prod_{k=1}^N (a_k - s) = 0$$

and the eigenvalues are given by  $s_k = f'(x_k^*) - \varepsilon \gamma_k$ ,  $k = 1, 2, \dots, N$ . The condition of stability  $|s_k| < 1$ , related to the function  $g(x)$ , then attains the form

$$-2 + \varepsilon(1 + \gamma) < g' < \varepsilon \gamma, \quad x \in D, \quad (23)$$

where  $D$  is the attracting domain of (1). Note that no accumulation of terms of  $O(\varepsilon^2)$  occurs when  $N \rightarrow \infty$ . Hence, without performing a detailed analysis of the transition  $N \rightarrow \infty$ , we can state that the condition of stability (23) applies for any  $N \in \mathbb{Z}^+$ . By means of an example we shall show that the map (13) has chaotic trajectories under the condition (23). This proves the existence of stable chaotic stationary distributions of  $T$ .

*Example 3.* For  $\gamma = 1$  and  $g(x) = \varepsilon a \sin x$ , the map (13) reduces to the standard map

$$\bar{x} = x + u - a \sin x, \quad \bar{u} = u - a \sin x. \quad (24)$$

In this case homoclinic orbits exist for any  $a > 0$ . Moreover, for  $a \sim 1$  the map (24) has no closed invariant curves and arbitrary homoclinic connections exist. On the other hand, the stability condition (23) holds for  $g = \varepsilon a \sin x$ .

The stability of  $k$ -cycles of the map is given by (20) where the matrix

$$A = \prod_{j=1}^k Q(f'(x_j^*(j)) - \varepsilon \gamma_j). \quad (25)$$

Here  $(x_i^*(1), x_i^*(2), \dots, x_i^*(k) = x_i^*(1))$ ,  $i = 1, 2, \dots, N$ , are the coordinates of the  $k$ -cycle and  $\{\gamma_i\} = (1, 1 + \gamma, \dots, 1 + \gamma, \gamma)$ . In the particular case of a 2-cycle of the map  $T$  [for small  $\varepsilon$ , where terms of  $O(\varepsilon^2)$  can be neglected except when they occur in the trace terms], the characteristic equation becomes

$$\prod_{i=1}^N (a_i(1)a_i(2) + \varepsilon^2 \gamma^{(i)} - s) = 0, \quad (26)$$

where  $\{\gamma^{(i)}\} = (\gamma, 2\gamma, \dots, 2\gamma, \gamma)$  and  $a_i(j) = f'(x_i^*(j)) - \varepsilon \gamma_j$ .

It is easy to verify that for functions  $g(x)$  that are proportional to  $\varepsilon$ , the condition of stability for (26) may be fulfilled in the same way as in Example 3. Using the property of the matrix (25) that all its elements outside the trace are of  $O(\varepsilon)$ , the stability conditions for any  $k$ -cycle with  $k > 2$  can be obtained in a similar way as for the 2-cycle.

#### IV. NONWANDERING SET OF THE WEAKLY COUPLED MAP

Consider now the dynamics of the map  $T$  for small values of  $\varepsilon$ . Let  $\Omega_s$  be a limiting set of a single map  $S$  and let us associate this set with each cell of the uncoupled map lattice  $T$  such that  $\Omega_i = \Omega_s$ ,  $i = 1, 2, \dots, N$ . Obviously, for the coupled map lattice, the limiting set in  $\mathbb{R}^N$  has a topological limit

$$\lim_{\varepsilon \rightarrow 0} \Omega(\varepsilon) = \Omega(0) = \prod_{i=1}^N \Omega_i, \quad (27)$$

which is the topological product of the limiting sets  $\Omega_i$  for each cell in the array. Recall that  $\Omega_s = \bigcup_{x(0) \in \mathbb{R}^1} \omega_{x(0)}$ , where  $\omega_{x(0)} = \lim_{k \rightarrow \infty} f^k(x(0))$  depends on  $x(0)$ , unless  $\Omega_s$  is a unique fixed point of  $f$ . Hence, allowing for the dependence on the initial conditions, the limiting set of the uncoupled map can be expressed as

$$\Omega(0) = \bigcup_{X(0) \in \mathbb{R}^N} [\lim_{j \rightarrow \infty} X(j)] = \prod_{i=1}^N \Omega_i(x_i(0)). \quad (28)$$

This implies that any combination of  $N$  initial conditions of the single map interpreted as an initial condition  $X(0)$  of the map  $T$  leads to a certain component of the limiting set  $\Omega(0)$ . For example, if  $\omega_{x_i(0)}$  is the  $k_i$ -cycle of the map  $S$ , then the component of  $\Omega(0)$  corresponding to that initial condition is the  $k$ -cycle with  $k$  being the lowest common multiple of  $k_1, k_2, \dots, k_N$ . Another important feature of the limiting set  $\Omega(0)$  of the uncoupled maps is that each of the  $k$ -cycles of  $T$  has real multipliers.

In terms of the structural stability theory we may define the hyperbolic subset of  $\Omega(0), \Omega_h(0)$ , including both the trivial unity of nondegenerate  $k$ -cycles (with multipliers  $|s_i| \neq 1$ ,  $i = 1, 2, \dots, N$ ) and a nontrivial set with a Bernoulli shift over some symbols. In this sense the general theory of hyperbolic systems is applied for the map  $T$  in the neighborhood of  $\Omega_h$  and the following assertion is valid.

*Theorem 4.* There exists such a small value of  $\varepsilon_0$  that, for  $\varepsilon \in (0, \varepsilon_0)$ , the coupled map  $T$  has a structurally stable hyperbolic component  $\Omega_h(\varepsilon)$  of a limiting set  $\Omega(\varepsilon)$  with the topological limit

$$\lim_{\varepsilon \rightarrow 0} \Omega_h(\varepsilon) = \Omega_h(0).$$

The immediate corollary of this assertion is that all the nondegenerate  $k$ -cycles of the uncoupled map  $T$  are preserved under a small increase of  $\varepsilon$  from zero. Another conclusion is

that, magnified by multiplying the dimensions, the most significant features of the 1D map (such as its chaotic dynamics and the complex bifurcation structure) are preserved for a small range of the coupling parameter  $\varepsilon$ .

Next we compare the case of large  $\varepsilon$ , where the coupled map  $T$  is synchronized and the limiting set  $\Omega(\varepsilon)$  lies on the one-dimensional manifold  $\mathfrak{D}$  (Theorem 2), and the case of small  $\varepsilon$ , where the component of  $\Omega(\varepsilon)$  outside  $\mathfrak{D}$  is nontrivial (Theorem 4). Observing that the map  $T$  depends continuously on the parameter  $\varepsilon$  and on the parameters of the single map, we obtain the following.

*Theorem 5.* Let the single map have a parameter  $a$  as a multiplier  $S: x \rightarrow af(x)$ , such that for  $a > a_1$  it has a nontrivial limiting set  $\Omega_s$  and let the invariant manifold  $\mathfrak{D}$  of the coupled map  $T(\varepsilon, a)$  be absolutely stable for some region of parameters ( $a < a_2$ ,  $\varepsilon > \varepsilon_1$ ). Then the map  $T$  has an infinite number of bifurcations corresponding to the disappearance of the set  $\Omega(\varepsilon) \setminus [\Omega(\varepsilon) \cap \mathfrak{D}]$  under a transition in parameter space from the region ( $\varepsilon \leq 1$ ,  $a > a_1$ ) to the region ( $\varepsilon > \varepsilon_1$ ,  $a < a_2$ ).

In the general case, the only knowledge we have about these bifurcations is that they are bifurcations of periodic orbits and of homo- or heteroclinic orbits. As previously noted, Giberti and Vernia [14] have conducted a numerical study of the mechanisms by which stable periodic orbits arise in CMLs as the coupling between the local maps is increased. Considering a lattice of nine diffusively coupled logistic maps with periodic boundary conditions and with the local map operating slightly above the Feigenbaum accumulation point, they find that the stable periodic orbits typically emerge in inverse period-doubling bifurcations to subsequently disappear in saddle-node bifurcations as they collide with unstable orbits. Secondary bifurcations in which periodic orbits are stabilized or destabilized while a pair of eigenvalues simultaneously pass the unit circle through  $+1$  or  $-1$  may also occur. Giberti and Vernia [14] also point to the role played by the formation of normally attracting, one-dimensional manifolds connecting  $2N$  weakly hyperbolic orbits,  $N$  being the number of lattice sites. Half of these periodic orbits are stable and the other half are unstable. Relaxation of the trajectory towards such an invariant manifold is usually found to occur relatively fast. Once on the manifold, however, the dynamics becomes very slow, characterized, as it may be, by eigenvalues that deviate from 1 by as little as  $10^{-12}$ . These one-dimensional manifolds may be involved in various global bifurcations in which, at the same time, the stability of the manifold and of the periodic orbits are affected. In particular, Giberti and Vernia describe a special type of bifurcation that may occur in CMLs and in which a one-dimensional manifold of the type described above collapses with an unstable cycle of half the period.

## V. COUPLED MAPS WITH SYMMETRY

Consider the diffusively coupled maps (1) for  $\gamma=1$  and  $N=2m$  and assume that we have periodic boundary conditions ( $m \rightarrow \infty$  when the array is unbounded). Using a new notation  $y$  for the variables  $x_i$  with odd (or even)  $i$  and a new numeration, we obtain an alternative form of Eq. (1) for the map  $T$ :

$$\begin{aligned}\bar{x}_l &= f(x_l) + \varepsilon(y_{l+1} - 2x_l + y_{l-1}), \\ \bar{y}_l &= f(y_l) + \varepsilon(x_{l+1} - 2y_l + x_{l-1}), \\ l &= 1, 2, \dots, m, \quad x_0 \equiv x_1, \quad y_0 \equiv y_1.\end{aligned}\tag{29}$$

If  $N=2m+1$ ,  $T$  may be written in a similar form only with the additional equation  $\bar{y}_0 = f(y_0) + \varepsilon(x_2 - 2y_0 + x_1)$  and without the restriction  $x_0 \equiv x_1$ ,  $y_0 \equiv y_1$ .

Let us denote  $x$  as the column matrix  $(x_1, x_2, \dots, x_m)$ ,  $y$  as the column matrix  $(y_1, y_2, \dots, y_m)$ ,  $h(x, y)$  as the column matrix  $(y_2 - 2x_1 + y_0, y_3 - 2x_2 + y_1, \dots, y_{m+1} - 2x_m + y_{m-1})$ , and  $F(x)$  as the column matrix  $(f(x_1), f(x_2), \dots, f(x_m))$ . The system (29) then takes the form

$$\bar{x} = F(x) + \varepsilon h(x, y), \quad \bar{y} = F(y) + \varepsilon h(y, x).\tag{30}$$

Systems of this type, exhibiting the symmetry  $(x, y) \leftrightarrow (y, x)$ , were considered in a recent paper by Reick and Mosekilde [21]. In this section we present some different properties of such symmetrically coupled systems with reference to maps of the form (29). We first observe that the general system (30) has an  $m$ -dimensional invariant manifold  $\mathfrak{D}^{(m)} \equiv \{x = y\}$  with the map on it,

$$\bar{x} = F(x) + \varepsilon h(x, x), \quad x \in \mathbb{R}^m.\tag{31}$$

In the case of (29),  $h(x, x) \equiv 0$  and the map (31) splits into  $m$  single maps  $S: x_l = f(x_l)$ .

*Theorem 6.* The map (29) has a two-dimensional manifold

$$\mathfrak{D}^{(2)} = \{x_1 = x_2 = \dots = x_m, \quad y_1 = y_2 = \dots = y_m\}$$

with the map on it,

$$\bar{x} = f(x) + 2\varepsilon(y - x), \quad \bar{y} = f(y) + 2\varepsilon(x - y), \quad x, y \in \mathbb{R}^1.\tag{32}$$

This assertion follows in a straightforward manner from Eq. (29) by subtracting  $x_l(0) = x$ ,  $y_l(0) = y$ ,  $l = 1, 2, \dots, m$ . Note that the map (32) is the same as the initial map  $T$  for two coupled maps with the substitution  $2\varepsilon \rightarrow \varepsilon$  and with zero-flux boundary conditions. We also note that a symmetric period-2 orbit of the map (32)  $(x^{(1)}, y^{(1)}) \rightarrow (y^{(1)}, x^{(1)})$  represents a traveling wave of wavelength 2 moving along the array  $T$  at unit velocity. In general, each period- $k$  orbit may be interpreted as a stationary wave and as a generalization of Theorem 6 we state the following.

*Theorem 7.* For  $N$  being a multiple of  $p$  (for any  $p$  when  $N \rightarrow \infty$ ) each space periodic solution of the system (1)  $x_i(j) = y(i, j)$ ,  $y(i + p, j) = y(i, j)$ ,  $y(i, j) = y(-i, j)$  with even period  $p$  lies on a  $p$ -dimensional manifold  $\mathfrak{D}^{(p)}$  with a dynamics on it of the form (29) with  $m = p$ .

Indeed, denoting  $\varphi(2l-1, j) = x_l(j)$  and  $\varphi(2l, j) = y_l(j)$ , instead of (1) we obtain Eqs. (32) with  $m = p$ . Here  $(x_l, y_l)$ ,  $l = 1, 2, \dots, p$  are the coordinates of  $\mathfrak{D}^{(p)}$  such that from  $(x_l(0), y_l(0)) \in \mathfrak{D}^{(p)}$  it follows that  $(x_l(j), y_l(j)) \in \mathfrak{D}^{(p)}$ ,  $j \in \mathbb{Z}^+$ .

Next, under the linear transformation

$$x = u + v, \quad y = u - v\tag{33}$$

the map (30) is transformed into the map  $G$ :

$$\begin{aligned}\bar{u} &= \frac{1}{2}[F(u+v) + F(u-v) + \varepsilon(h(u+v, u-v) \\ &\quad + h(u-v, u+v))] \triangleq U(u, v), \\ \bar{v} &= \frac{1}{2}[F(u+v) - F(u-v) + \varepsilon(h(u+v, u-v) \\ &\quad - h(u-v, u+v))] \triangleq V(u, v).\end{aligned}\quad (34)$$

Since  $V(u, 0) = 0$ ,  $\{v = 0\}$  is an invariant manifold of (34). Moreover, the map  $G$  is invariant with respect to involution  $(u, v) \leftrightarrow (u, -v)$  due to the obvious equalities

$$U(u, -v) = U(u, v), \quad V(u, -v) = -V(u, v).$$

This implies that the limiting set  $\Omega^s$  of the trajectories of (34) is symmetric with respect to the manifold  $\{v = 0\}$ , i.e., if a point  $(u^*, v^*) \in \Omega^s$  then  $(u^*, -v^*) \in \Omega^s$ . Note that each symmetric trajectory  $\Gamma_s \in \Omega^s$  is mapped onto itself by an involution transformation  $\Gamma_s = \mathfrak{J}\Gamma_s$ , where the matrix

$$\mathfrak{J} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad (35)$$

$E$  being a unit matrix. Similarly, each asymmetric trajectory  $\Gamma_a^+ \in \Omega^s$  is mapped onto its reflection twin  $\Gamma_a^-$ , i.e.,  $\Gamma_a^{+(-)} = \mathfrak{J}\Gamma_a^{-(+)}$ .

Consider now the twin map  $G_t$  defined by

$$\bar{u} = U(u, v), \quad \bar{v} = -V(u, v) \quad (36)$$

and also displaying the symmetry with respect to (35). In this case we have the following.

**Theorem 8.** The limiting set  $\Omega^a$  of the twin map  $G_t$  as a topological set coincides with the limiting set  $\Omega^s$  of the map  $G$ .

Indeed,  $\mathfrak{J}\Omega^s = \Omega^s$  and  $G\Omega^s = \Omega^s$ . Then  $\mathfrak{J}G\Omega^s = \Omega^s$ . But as  $\mathfrak{J}G = G_t$  so  $G_t\Omega^s = \Omega^s$ .

**Corollary.** (a) Let the map  $G$  have a symmetric period  $k = 2k_1$  orbit  $\Gamma_{sk}$ . Then, if  $k_1 = 2k_2 + 1$  the map  $G_t$  has two asymmetric period  $k_1$  orbits  $\Gamma_{ak_1}^{(1)}$  and  $\Gamma_{ak_2}^{(2)}$ , consisting of the points of  $\Gamma_{sk}$ . If  $k_1 = 2k_2$  the map  $G_t$  has a symmetric period  $k = 2k_1$  orbit  $\Gamma_{sk}^t$  consisting of the points of  $\Gamma_{sk}$ , but with another route between them. (b) Let the map  $G$  have an asymmetric period- $k$  orbit  $\Gamma_{ak}^+$  (and its twin  $\Gamma_{ak}^-$ ); then the map  $G_t$  has a period- $2k$  symmetric orbit  $\Gamma_s^{2k}$  consisting of the points of  $\Gamma_{ak}^\pm$ . The fixed point of  $G$ , for example, corresponds to a period-2 orbit of  $G_t$ , the period-2 orbit of  $G$  to two fixed points of  $G_t$ , the period-3 orbit of  $G$  to a period-6 orbit of  $G_t$ , the period-4 orbit of  $G$  to a period-4 orbit of  $G_t$ , etc.

**Example 4.** To illustrate the above results let us consider two diffusively coupled short maps

$$\bar{x} = a(\mu - x - x^2) + \varepsilon(y - x), \quad \bar{y} = a(\mu - y - y^2) + \varepsilon(x - y) \quad (37)$$

where an amplitude  $a > 0$  is introduced as a multiplier. The self-similar box-within-a-box structure of the bifurcations for the single map is described, for instance, by Mira [23]. This structure is preserved for the coupled map (37) with an induced structure of the trajectories in the phase space  $(x, y)$  as described in Sec. II. The conditions for (37) to have an attracting domain for  $a \leq 1$  are (Theorem 1)

$$D = \{|x| < \frac{3}{2}, |y| < \frac{3}{2}\}, \quad \varepsilon < \begin{cases} (\mu + 1)/6 & \text{for } \mu \leq \frac{1}{8} \\ (5 - 4\mu)/24 & \text{for } \mu \in (\frac{1}{8}, \frac{5}{4}) \end{cases} \quad (38)$$

The condition of absolute stability of the invariant manifold ( $x = y$ ) as determined by (10) and (38) has the form

$$0 < \varepsilon < (1 - 3a)/2. \quad (39)$$

Hence, according to Theorem 4, the transition of the parameters  $(\varepsilon, a)$  from a region of small  $\varepsilon$  and  $a$  close to 1 (parameter  $\mu$  is supposed to match Theorem 4) to the region (39) produces an infinite number of bifurcations.

For  $a = 1$ , the map (37) has a period-2 symmetric orbit  $(x^{(2)}, y^{(2)}) \rightarrow (y^{(2)}, x^{(2)})$ , where

$$x^{(2)} = -\varepsilon - \sqrt{\mu + 2\varepsilon - \varepsilon^2}, \quad y^{(2)} = -\varepsilon + \sqrt{\mu + 2\varepsilon - \varepsilon^2}. \quad (40)$$

This orbit appears at  $\mu = \mu_2(\varepsilon) = -2\varepsilon + \varepsilon^2$  and remains stable for  $\mu \in (\mu_2, \mu_4)$ , where  $\mu_4 = (1 - 5\varepsilon)/2 + \varepsilon^2$  is a torus bifurcation point. This agrees with the result obtained by Reick and Mosekilde [21] that the second period-doubling bifurcation for  $\varepsilon = 0$  (here at  $\mu = \frac{1}{2}$ ) is changed into a torus bifurcation for  $\varepsilon > 0$ . A similar phenomenon was observed in the paper by Biragov, Ovsyannikov, and Turaev [22].

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## APPENDIX

Consider the discontinuous map

$$x_i(j+1) = f(x_i(j)) + \alpha_i(j), \quad j \in \mathbb{Z}^+ \quad (A1)$$

where

$$\alpha_i(j) = \begin{cases} \varepsilon[x_{i+1}(j) - (1 + \gamma)x_i(j) + \gamma x_{i-1}(j)] & \text{for } |x_k(j)| < r \\ \alpha_0 & \text{for } |x_k(j)| \geq r, \quad k = i + 1, i, i - 1. \end{cases} \quad (A2)$$

The function  $\alpha_i(j)$  for  $|x_k(j)| < r$  satisfies the inequality

$$|\varepsilon(x_{i+1}(j) - (1 + \gamma)x_i(j) + \gamma x_{i-1}(j))| < 2\varepsilon r(1 + \gamma)$$

and by virtue of the condition  $\varepsilon r < \alpha_0/2(1 + \alpha)$  we have

$$|\alpha_i(j)| \leq \alpha_0, \quad j \in \mathbb{Z}^+, \quad i = 1, 2, \dots, N. \quad (A3)$$

Then, from the first condition of the theorem and from the condition  $-r \leq x^* < x_2^* \leq r$  it follows that for any trajectory of (A1), if  $x_i(0) \in I^*$ , then  $x_i(j) \in I^*$ ,  $i = 1, 2, 3, \dots, N$ ,  $j \in \mathbb{Z}^+$ . But, on the other hand, the initial map  $T_{I^*}$  coincides with (A1). Hence the map  $T$  has an attractive domain  $D$ . In this proof we did not use the boundary conditions and the theorem holds in the case of unbounded array  $i \in \mathbb{Z}$ .



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