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Comments on `A discrete optimal control problem for descriptor systems'

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process is governed by

$$d\eta^*(t) = -\text{sgn}(\eta^*(t) - y)\,dt + dw(t). \quad (33)$$

The corresponding Fokker–Planck equation can be written, in terms of a generalized function, as

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial z}(\text{sgn}(z-y)p) + \frac{1}{2}\frac{\partial^2 p}{\partial z^2}$$
$$= 2\delta(z-y)p + \text{sgn}(z-y)\frac{\partial p}{\partial z} + \frac{1}{2}\frac{\partial^2 p}{\partial z^2}$$
$$\lim_{t\to 0} p(t, z|x) = \delta(z-x). \quad (34)$$

We apply (27), noticing that $-|z-y|$ is an indefinite integral of $-\text{sgn}(z-y)$, to obtain

$$p(t, z|x) = e^{|x-y|-|z-y|-\frac{t}{2}} G(t, z-x) E^x[e^{2L_t(y)}|w(t) = z]$$
$$= e^{|x-y|-|z-y|-\frac{t}{2}} G(t, z-x)\psi(t; 1; x, y, z)$$
$$= e^{|x-y|-|z-y|-\frac{t}{2}}[G(t, z-x) + H(1, t, |z-y| + |x-y|)] \quad (35)$$

which is obviously positive and continuous on $(0, \infty) \times \mathbb{R}$ and is $C^{1,2}$ on $(0, \infty) \times (\mathbb{R}\backslash\{y\})$.

Using Properties 5 and 6 in Proposition 1, one can easily verify

$$\int_{-\infty}^{\infty} p(t, z|x)\,dz = 1 \qquad \forall (t, x) \in (0, \infty) \times \mathbb{R}.$$

In fact,

$$\int_{-\infty}^{\infty} p(t, z|x)\,dz$$
$$= e^{|x-y|-\frac{t}{2}} \int_{-\infty}^{\infty} e^{-|z-y|}[G(t, z-x) + H(1, t, |z-y| + |x-y|)]\,d$$
$$= e^{|x-y|-\frac{t}{2}} \int_{0}^{\infty} e^{-u}[G(t, u+y-x) + G(t, u+x-y)]\,du$$
$$+ 2\int_{0}^{\infty} e^{-u} H(1, t, u + |x-y|)\,du$$
$$= e^{|x-y|-\frac{t}{2}}[H(-1, t, -|x-y|) + H(1, t, |x-y|)]$$
$$= 1.$$

Next, we compute the mean $u(t, x) \overset{\text{def}}{=} E[\xi(t)|\xi(0) = x]$ by using (35). As we know, $u(t, x)$ solves the backward equation

$$\frac{\partial u}{\partial t} = -\text{sgn}(x-y)\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \qquad x \neq y$$
$$u(0, x) = x.$$

Computation gives

$$u(t, x) = y + E[\xi(t) - y|\xi(0) = x]$$
$$= y + e^{|x-y|-\frac{t}{2}} \int_{-\infty}^{\infty} z e^{-|z|} G(t, z+y-x)\,dz$$
$$= y + e^{|x-y|-\frac{t}{2}}[(x-y+t)H(-1, t, x-y)$$
$$+ (x-y-t)H(-1, t, y-x)].$$

It is easy to see $u(t, x)$ is $C^{1,1}$ on $(0, \infty) \times \mathbb{R}$ and $\partial^2 u(t, x)/\partial x^2$ exits and is continuous for $x \neq y$.

The steady-state density function $p_s(z)$ can be obtained by directly taking the limit

$$p_s(z) = \lim_{t\to\infty} p(t, z|x)$$
$$= \lim_{t\to\infty} e^{|x-y|-|z-y|-\frac{t}{2}} H(1, t, |z-y| + |x-y|)$$
$$= \lim_{t\to\infty} e^{-2|x-y|}\left[1 - \Phi\left(\frac{|z-y| + |x-y|}{\sqrt{t}} - \sqrt{t}\right)\right]$$
$$= e^{-2|x-y|}$$

where we have used the relation (6). And, of course, the invariant measure of (33) is $\mu(dz) = e^{-2|z-y|}\,dz$.

Before we conclude this example, let us make the following observation: let the diffusion $\eta_0(t)$ be governed by

$$d\eta_0(t) = -\text{sgn}(\eta_0(t) - y)\,dt + \text{sgn}[f(\eta_0(t))]\,dw(t) \quad (36)$$

where $f: \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

We claim that $\eta_0(t)$ and $\eta^*(t)$, determined by (33), share the same transition probability density given in (35) and the same Fokker–Planck equation (34).

In fact, to see this, it is sufficient to notice that

$$W(t) \overset{\text{def}}{=} \int_0^t \text{sgn}[f(\eta_0(s))]\,dw(s)$$

is another Brownian motion because $\{W(t), \mathfrak{F}_t, t \geq 0\}$ and $\{W^2(t) - t, \mathfrak{F}_t, t > 0\}$ are both martingales. Therefore, (36) can be rewritten as

$$d\eta_0(t) = -\text{sgn}(\eta_0(t) - y)\,dt + dW(t),$$

i.e., $\eta^*(t)$ of (33) is a weak solution of (36).

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Comments on "A Discrete Optimal Control Problem for Descriptor Systems"

HANS F. RAVN

Abstract—**In a recent paper,[1] necessary and sufficient optimality conditions are derived for a discrete-time optimal control problem, as well as other specific cases of implicit and explicit dynamic systems. We correct a mistake and demonstrate that there is not an "if and only if" correspondence between stationarity conditions and minimization of the Hamiltonian.**

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I. Introduction

In the paper,[1] the following problem was considered:

$$\min G(x_N) + \sum_{k=0}^{N-1} L_k(x_k, u_k) \tag{1a}$$

$$x_{k+1} = f_k(x_k, u_k), \qquad k = 0, \cdots, N-1 \tag{1b}$$

$$q_k(x_k, u_k) \geq 0, \qquad k = 0, \cdots, N-1 \tag{1c}$$

$$x_0 = \underline{x}_0 \qquad \text{(fixed value)} \tag{1d}$$

$x_k \in R^n$, $u_k \in R^m$, $G: R^n \to R$, $L_k: R^{n+m} \to R$, $f_k: R^{n+m} \to R^n$, $q_k: R^{n+m} \to R^p$, as well as other specific cases of implicit and explicit dynamic systems.

In this note, we correct an error in the paper[1] and extend the results by weakening the assumptions on constraint qualifications.

The approach taken in the paper,[1] as well as here, is to derive optimality conditions by considering (1) as a specific case of a nonlinear programming problem. In this approach, a central element is the derivation of the Kuhn–Tucker conditions, and the identification of assumptions under which these conditions are necessary and/or sufficient, respectively, for optimality. This is supplemented with the control approach, where the Kuhn–Tucker stationarity conditions are supplemented with (or partially substituted by) minimization of the Hamiltonian with respect to the control u_k.

II. Main Results

Let us introduce the following assumptions.

Assumption 1: L_k, f_k, q_k, $k = 0, \cdots, N-1$ and G are continuously differentiable with respect to all their variables.

Assumption 2: The Managasarian–Fromowitz constraint qualification holds at the optimal points (x_k^*, u_k^*), x_N^*, i.e., there holds that:

i) the gradients $\nabla F(y)$ are linearly independent; and

ii) there exists a $z \in R^{N(n+m)}$ such that

$$\nabla F(y^*)z = 0$$

$$\nabla q_{ik}(y^*)z > 0 \qquad \text{for all } (i, k) \text{ for which } q_{ik}(y^*) = 0;$$

here

$$y = (x_1, \cdots, x_N, u_0, \cdots, u_{N-1})',$$

$$F = (f_0, \cdots, f_{N-1}), \qquad f_k = (f_{1k}, \cdots, f_{nk})$$

$$q = (q_0, \cdots, q_k, \cdots, q_{N-1}), \qquad q_k = (q_{1k}, \cdots, q_{pk}).$$

Remark 1: This assumption is weaker than the assumption of linear independence of $\nabla F(y^*)$ and those $\nabla q_{ik}(y^*)$, for which $q_{ik}(y) = 0$, which was used in the paper.[1]

Assumption 3: At the optimal solution x_k^*, L_k is convex, f_k is affine, and q_k is quasi-concave with respect to u_k, $k = 0, \cdots, N-1$.

Assumption 4: L_k is pseudoconvex, f_k is quasi-linear (i.e., quasi-convex and quasi-concave) and q_k is quasi-concave at (x_k, u_k), $k = 0, \cdots, N-1$; and G_N is pseudoconvex at x_N.

Theorem 1: If $(x, u)^* = (x_1^*, \cdots, x_N^*, u_0^*, \cdots, u_{N-1}^*)$ is optimal in (1), then under Assumptions 1 and 2, there exist vectors $\lambda_k \in R^n$ and $\mu_k \in R^p$ such that at $(x, u)^*$ there hold

$$\lambda_k = \frac{\partial H_k}{\partial x_k} - \frac{\partial q_k^T}{\partial x_k}\mu_k \qquad k = 1, \cdots, N-1 \tag{2a}$$

$$\lambda_N = \frac{\partial G}{\partial x_N} \tag{2b}$$

$$x_{k+1} = \frac{\partial H_k}{\partial \lambda_{k+1}} \qquad k = 0, \cdots, N-1 \tag{2c}$$

$$x_0 = \underline{x}_0 \tag{2d}$$

$$\frac{\partial H_k}{\partial u_k} - \frac{\partial q_k^T}{\partial u_k}\mu_k = 0 \qquad k = 0, \cdots, N-1 \tag{2e}$$

$$q_k(x_k u_k) \geq 0, \; \mu_k^T q_k(x_k u_k) = 0, \qquad \begin{array}{l} \mu_k \geq 0 \\ k = 0, \cdots, N-1 \end{array} \tag{2f}$$

where

$$H_k(x_k, u_k, \lambda_{k+1}) = L_k(x_k, u_k) + \lambda_{k+1}^T f_k(x_k, u_k) \qquad k = 0, \cdots, N-1. \tag{2g}$$

If Assumption 3 holds also, then for $k = 0, \cdots, N-1$, u_k^* is a solution to

$$\min_v H_k(x_k^*, v, \lambda_{k+1}) \tag{3a}$$

$$q_k(x_k^*, v) \geq 0. \tag{3b}$$

Proof: The first result is proved in [5]. For the second result observe, that for $x_k = x_k^*$, the Hamiltonian H_k is convex (L_k convex, f_k affine, and hence, also $\lambda_{k+1}^T f_k$ affine) with respect to u_k, and therefore, also pseudoconvex [1, p. 108] and q_k quasi-concave with respect to u_k. Therefore, the conditions (2e)–(2g) are sufficient for optimality of u_k^* in (3) [1, pp. 147–148]. □

Remark 2: The assumption of convexity of L_k in the last part of Theorem 1 cannot be substituted by an assumption of pseudoconvexity of L_k. To see this, consider the following example. Let $n = 1$, $m = 1$, $N = 1$, $p = 0$, $L_0(x_0, u_0) = -e^{-u_0^2}$, $f_0(x_0, u_0) = x_0 + u_0$, $G_1(x_1) = (x_1 - c)^2$ with $c = \sqrt{2}(1+e^{-1/2})/2$, $\underline{x}_0 = 0$. We find the following unique solution to (2): $u_0^* = \sqrt{2}/2$, $x_1^* = \sqrt{2}/2$, $\lambda_1 = -\sqrt{2}e^{-1/2}$. Since Assumption 4 is fulfilled, this is the optimal solution (cf. Theorem 2). However, u_0^* is not minimizing $H_0(x_0, v, \lambda_1)$; in fact, $\inf H_0(\underline{x}_0, v, \lambda_1) = -\infty$.

The error in the proof of Theorem 2 in the paper[1] is the conclusion that the sum (viz. the Hamiltonian) of a pseudoconvex function (viz. L_k) and an affine function (viz. $\lambda_{k+1}^T f_k$, where f_k in the paper[1] was assumed affine) is pseudoconvex.

Theorem 2: Assume that Assumption 1 holds, and that there exist λ, μ such that (2) holds at $(x, u)^*$. If Assumption 4 holds also, then $(x, u)^*$ is optimal in (1).

Proof: We first show that the criterion function (1a) is pseudoconvex. The key observation is that (1a) is additive (viz. the sum of L_k, $k = 0, \cdots, N-1$, and G_N). Since all terms in (1a) are continuously differentiable (1a) is continuously differentiable; therefore, the gradient is zero, if and only if any partial derivative is zero. If the partial derivative with respect to (x_k, u_k) is zero, then L_k attains a minimum since L_k is pseudoconvex, and similarly holds for G_N. Since (1a) is additive, the attainment of a minimum in each term implies that (1a) attains a minimum. Therefore, (1a) is pseudoconvex. Now, the result is proved as in [1, pp. 147–148] by observing that (2c), (2f), and (2g) imply that $(x, u)^*$ is feasible in (1). □

Remark 3: This result can also be obtained under the following weaker assumption on f_k: f_{ik} is continuously differentiable, f_{ik} is quasi-convex at (x_k^*, u_k^*) if $\lambda_{ik+1} > 0$, f_{ik} is quasi-concave at (x_k^*, u_k^*) if $\lambda_{ik+1} > 0$ [1, pp. 147–148].

Remark 4: In Theorem 2, the stationarity condition (2e) cannot be substituted by the condition that u_k^* is optimal in (3). However, (2e) may be substituted by the condition that u_k^* is an optimal solution to

$$\min_v H_k(x_k^*, v, \lambda_{k+1}) - \mu_k^T q_k(x_k^*, v). \tag{4}$$

But this condition is actually stronger than (2e); since (4) is an unconstrained problem with a continuously differentiable criterion function, the optimal point in (4) is a stationary point [1, p. 125] and this implies that (2e) holds.

III. Discussion

We have given necessary and sufficient optimality conditions for a discrete-time optimal control problem.

The conditions are derived from similar stationary conditions in nonlinear programming, and supplemented by conditions from the control approach, in which the Hamiltonian is minimized. It is shown that the distinction between convexity and pseudoconvexity is essential, and that the results from the two approaches thus differ, implying that there is not an "if and only if" correspondence between stationarity conditions and minimization of the Hamiltonian.

The discussion about the equivalence or nonequivalence between various versions of optimality conditions in connection with discrete-time optimal control is old (see [7]). The mathematical programming approach has maybe been most extensively treated in [2]. Derivation of optimality conditions from the saddle-point theorem of mathematical programming was done in [8]. A discussion of the connection between mathematical programming and discrete-time optimal control was performed in [4].

In all the aforementioned references, the Hamiltonian was defined as in (2g). By a suitable generalization of the Hamiltonian it is possible to specify weaker assumptions under which the Hamiltonian is minimized (see, e.g., [3], [6], or [7]).

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Authors' Reply[2]

JING-YUE LIN AND ZI-HOU YANG

The authors would like to thank Prof. Ravn for his comments on the paper.[1]

While we appreciated the comments, we wish to give a revised version of Theorem 2 in the paper[1] in the context of the rest of this response, to achieve a balance of emphasis on the control problem for descriptor systems which has not been adequately explored in the literature.

The revised version of Theorem 2 in the paper[1] is given by the following theorems without proofs which can be given by a slight modification of those in the paper,[1] according to the correction given by Prof. Ravn.

Theorem 2.1: Consider the control problem (19). Let L_k be convex, and q_k be quasi-concave in x_k and u_k, $k = 0, 1, \cdots, N-1$. If the sequence $\{(x_k, u_k),\ k = 1, \cdots, N\}$ is an optimal solution to the problem, then there exist vectors $\lambda_1, \cdots, \lambda_N$, $\mu_0, \cdots, \mu_{N-1}$ such that (20a)–(20f) and (21)–(23) hold.

Theorem 2.2: Consider the problem (19). Suppose the necessary conditions in Theorem 2.1 hold. If G is pseudoconvex in x_N, L_k is pseudoconvex and q_k is quasi-concave in x_k and u_k, $k = 0, 1, \cdots, N-1$, then the sequence $\{(x_k, u_k),\ k = 1, \cdots, N\}$ is an optimal solution to the problem (19).

The distinction between conventional systems and descriptor systems is essential since E_k, $k = 1, \cdots, N$, in (19), may be singular matrices. Thus, our results differ from those for conventional systems and are generalizations of them.

We conclude by noting that there is a typographical error on the first line of p. 178 of the paper[1]: "where g_k is a p-dimensional vector function$\cdots$" should be "where q_k is a p-dimensional vector function$\cdots$".

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Comments on "A Generalization of Kharitonov's Concept for Robust Stability Problems with Linearly Dependent Coefficient Perturbations"

YAU-TARNG JUANG

***Abstract*—It is shown by a counterexample that the main theorem in the above paper[1] may lead to an erroneous D-stability conclusion for certain polynomials among the considered ones. Suggestions are presented and discussed.**

I. Introduction

The Kharitonov stability theorem [1] has attracted much attention to the robust stability problem in the recent literature. Based on Kharitonov's four-polynomial concept, a generalization theorem for robust D-stability assurance of polynomials with linearly dependent coefficient perturbations is presented in the paper.[1]

In this note, we give a counterexample to show that the main theorem in the paper[1] may have a misleading result. Subsequently, suggestions and discussions are made.

Consider a linear system whose characteristic polynomial depends on p physical parameters q_i with $q_i \in [q_i^-, q_i^+]$, $i = 1, 2, \cdots, p$. Suppose that the characteristic polynomial is of the form

$$p(s, q) = \sum_{j=0}^{n} a_j(q) s^j$$

where $q = [q_1\ q_2 \cdots q_p]^T$ and the coefficient perturbations are polytopic. Then the family of polynomials P

$$P = \{p(s, q) : q \in Q \subset R^p\}$$

can be expressed as the convex hull of finitely many generating polynomials $p_1(s), p_2(s), \cdots, p_k(s)$, i.e.,

$$P = \text{conv}\,\{p_1(s), p_2(s), \cdots, p_k(s)\}$$

where

$$p_m(s) = \sum_{i=0}^{n} a_j(q^m) s^j \qquad m = 1, 2, \cdots, k,\ k \le 2^p$$

and q^m denotes the mth extreme point in the bounding set Q

$$Q = \{q : q_i^- \le q_i \le q_i^+, \qquad i = 1, 2, \cdots, p\}.$$

Let D be the union of a finite number (≥ 1) of pathwise connected regions in the complex plane. Define the notation $\phi_D(\delta)$ as a continuous mapping of the scalar variable $\delta \in R$ onto the boundary of D. One choice of $\phi_D(\delta)$ proposed in the paper[1] is

$$\phi_D(\delta) = -\sigma + j\delta \qquad \sigma \ge 0.$$

This implies that D is the half plane described by $\text{Re}(s) < -\sigma$. There is another function used in the paper,[1] namely $\phi_\Gamma(\rho)$, and a simple choice for this function is[1]

$$\phi_\Gamma(\rho) = \cos 2\pi\rho + j \sin 2\pi\rho \qquad \rho \in [0, 1].$$

Then the paper[1] presents the following result.

Theorem: [1]

Assume that the polytope of polynomials P contains at least one D-stable polynomial. Then P is D-stability if and only if for each $\delta \in R$

$$H(\delta) = \max_{\rho \in [0,1]} \min_{m \le k} [\text{Re}(\phi_\Gamma(\rho)) \text{Re}(p_m(\phi_D(\delta))) + \text{Im}(\phi_\Gamma(\rho)) \text{Im}(p_m(\phi_D(\delta)))] > 0.$$

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