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Abstract (max. 2000 char.):

On the basis of Corrsin's independence hypothesis, in conjunction with specific assumptions about the form of the distance-neighbour function, an equation is derived for twoparticle dispersion in isotropic turbulence with no mean motion. It is formulated in terms of the mean-square difference between the particle positions $\mathbf{r}_1(t_1)$ and $\mathbf{r}_2(t_2)$ at arbitrary times t_1 and t_2 after the release of the particles with a given initial separation. Eddy removal and eddy decay are included with wave-number dependent time scales. The equation, which in general must be solved numerically, has been considered for the scale free $k^{-5/3}$ energy spectrum as well as for the von K´arm´an spectrum. The model implies that only when the outer scale is infinite, i.e. in the limit where the energy spectrum is of the form $k^{-5/3}$, will there be a $C\varepsilon t^3$ range of the mean-square separation between the two particles. In this limiting case it is possible to estimate the dimensionless Richardson- Obukhov constant C as a function of a dimensionless eddy-decay parameter. A reasonable choice of this parameter leads to a C-value of the order 1.

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Foreword

The work described in this form was originally formulated as article and submitted to *Journal of Fluid Mechanics* in 2005. It was reviewed by three referees. Based on their report to the editor the manuscript was rejected. One of the three referees had made a strong effort to understand our ideas and provided constructive comments. In this report we have, where we agree, reformulated and expanded the text in accordance with these comments. We are grateful to this referee for the effort.

1 Introduction

We imagine that we follow neutrally buoyant particles suspended in a fluid in which the turbulence is stationary and isotropic. Our main purpose is to predict the time evolution of the mean-square separation $D^2(t)$ between two particles which are initially separated by the distance $d_\circ$. We consider a fluid in rest. The implication is that the Eulerian velocity field has zero ensemble mean and that, due to stationarity, all the particle-velocities have zero ensemble means.

Real fluids are of course neither isotropic on all length scales nor stationary over infinite long times. In the real world there are only approximately *locally* isotropic and approximately stationary fluids. For these flows it is particularly interesting to study pairs of particles with spatial displacements d which are not large compared to a turbulence spatial scale ℓ defined as

$$\ell = \frac{\langle u^2 \rangle^{3/2}}{\varepsilon}. \tag{1}$$

Here $\langle u^2 \rangle$ is the variance of one velocity component and ε the rate of dissipation of specific kinetic energy. Angle brackets indicate ensemble averaging. We consider ℓ the outer scale of the turbulence in the sense that eddies with linear dimensions larger than ℓ start having decreasing spatial density and start becoming—usually—increasingly anisotropic. The turbulence temporal scale is also defined by means of $\langle u^2 \rangle$ and ε:

$$\mathcal{T} = \frac{\langle u^2 \rangle}{\varepsilon}, \tag{2}$$

It follows that

$$\ell^2 = \varepsilon\, \mathcal{T}^3. \tag{3}$$

In the following we consider initial-separation magnitudes large compared to the Kolmogorov microscale $\eta = (\nu^3/\varepsilon)^{1/4}$, given by ε and the kinematic viscosity ν.

The description of neutrally buoyant particle pairs in a turbulent fluid has a long story going back to Richardson (1926) and one of the conclusions from that time, namely that $D^2(t)$ is proportional to ε and t^3 when $D(t)$ has become large compared to the initial separation $d_\circ$, is still widely accepted. This law,

$$D^2(t) = C\, \varepsilon\, t^3, \tag{4}$$

is called the Richardson-Obukhov law as explained by Ott & Mann (2000) in a discussion of the theoretical and experimental investigations of relative dispersion. They present an account of the various determinations of the dimensionless factor of proportionality C, the Richardson-Obukhov constant, which seem to vary from about 0.1 to about 5.5. A comprehensive review of the theoretical and experimental investigations concerning relative dispersion in turbulent media has been given by Sawford (2001).

More recent investigations are reported by Ishihara & Kaneda (2002), and by Yeung & Borgas (2004) and Borgas & Yeung (2004) in two juxtaposed articles. These investigations are based on numerical simulations of particle trajectories in isotropic and stationary turbulence. The first two are based on direct numerical simulation (DNS) of the Navier-Stokes equations to obtain a box of an isotropic, stationary Eulerian velocity field at Reynolds numbers R_λ, based on $\langle u^2 \rangle^{1/2}$ and the Taylor microscale $\lambda = (15\nu \langle u^2 \rangle/\varepsilon)^{1/2}$, ranging from 90 to 280. Many thousands of particle pairs were released and tracked in this velocity field and the statistics of two-particle separations as functions of time are calculated. In the third investigation (Borgas & Yeung 2004) particle pairs were followed by a Langevin equation in a Lagrangian frame and the results are compared to those obtained by the (Yeung & Borgas 2004) DNS simulation.

Here we suggest, as an alternative, a simple statistical model. We derive equations for second-order ensemble means of particle positions, in single and in pairs. Basically we are therefore dealing with the Lagrangian problem of the statistics of particle trajectories. For a single particle Taylor's (1921) theory provides a very accurate and useful description in the limits of small times and large times. When it comes to the relative dispersion of two particles the situation is much more difficult (Batchelor 1952) because information about the spatial structure of the turbulence is required. In other words, a connection between Lagrangian and Eulerian statistics is must be spedified. The model presented here makes use of the bridge known as 'Corrsin's independence hypothesis' (Shlien & Corrsin 1974). In this way it becomes possible to establish an auxiliary relation between the Lagrangian velocity covariance of two particles and the covariance between their positions. A similar approach was described by Ishihara & Kaneda (2002) and also for the growth of a collection of particles, a puff, by Kristensen & Kirkegaard (1987).

In the following section we will present some preliminary considerations about relative dispersion. In section 3 the model will be described. It will include a presentation of Corrsin's independence hypothesis for one particle and for particle pairs, the time-lag dependent energy spectrum, and the distance-neighbour functions to be used. Sections 4 and 5 will contain results of integrating the model equations and a discussion of the solutions. In the final section we will review the assumptions and the results.

2 Preliminary Considerations

We consider two particles with the initial separation $\boldsymbol{d}_\circ$, where $d_\circ = |\boldsymbol{d}_\circ|$ is assumed small compared to ℓ. Their positions are

$$\boldsymbol{r}_n(t) = \boldsymbol{r}_n^\circ + \int_0^t \boldsymbol{v}_n(t_1)\,\mathrm{d}t_1, \quad n = 1, 2, \tag{5}$$

where

$$\boldsymbol{r}_n^\circ = \boldsymbol{r}_n(0) \tag{6}$$

is the initial position of particle no n and $\boldsymbol{v}_n(t)$ its velocity. The initial positions are considered non-random. It is customary to include the initial position $\boldsymbol{r}_n^\circ$ as a parameter in the velocity. We have decided to omit this extra argument to make the equations easier to read.

The separation vector between the two particles is

$$\boldsymbol{d}(t) = \boldsymbol{r}_2(t) - \boldsymbol{r}_1(t) \tag{7}$$

with the mean square $D^2(t) = \langle \boldsymbol{d}(t)^2 \rangle$.

Obviously,

$$\langle \boldsymbol{d}(t) \rangle = \boldsymbol{r}_2^\circ - \boldsymbol{r}_1^\circ = \boldsymbol{d}_\circ. \tag{8}$$

At this point it is convenient to introduce the *mean-square excess separation* $D_E^2(t)$ (MES) as the mean square of the vector $\boldsymbol{d}(t) - \boldsymbol{d}_\circ$:

$$D_E^2(t) = \left\langle (\boldsymbol{d}(t) - \boldsymbol{d}_\circ)^2 \right\rangle. \tag{9}$$

Clearly we have

$$D^2(t) = d_\circ^2 + D_E^2(t). \tag{10}$$

Initially, i.e. when $D(t) \simeq d_\circ$, the positions of the two particles are strongly correlated so that

$$\boldsymbol{d}(t) - \boldsymbol{d}_\circ = \boldsymbol{r}_2(t) - \boldsymbol{r}_1(t) - \boldsymbol{d}_\circ \simeq (\boldsymbol{v}_2(t) - \boldsymbol{v}_1(t))\, t \simeq (\boldsymbol{u}(\boldsymbol{r}_1^\circ + \boldsymbol{d}_\circ, 0) - \boldsymbol{u}(\boldsymbol{r}_1^\circ, 0))\, t, \tag{11}$$

where $\boldsymbol{v}(t)$ and $\boldsymbol{u}(\boldsymbol{r}, t)$ are Lagrangian and Eulerian velocity vectors, respectively. Hence,

$$D_E^2(t) \simeq \langle (\boldsymbol{u}(\boldsymbol{r}_1^\circ + \boldsymbol{d}_\circ, 0) - \boldsymbol{u}(\boldsymbol{r}_1^\circ, 0))^2 \rangle\, t^2 = D_{ii}(\boldsymbol{d}_\circ)\, t^2, \tag{12}$$

expressed in terms of the trace of the *structure-function tensor*

$$D_{ij}(\boldsymbol{\varrho}) \equiv \left\langle \left(u_i(\boldsymbol{r} + \boldsymbol{\varrho}, 0) - u_i(\boldsymbol{r}, 0)\right) \left(u_j(\boldsymbol{r} + \boldsymbol{\varrho}, 0) - u_j(\boldsymbol{r}, 0)\right) \right\rangle. \tag{13}$$

Incompressible, isotropic turbulence has the property that this tensor reduces to the expression

$$D_{ij}(\boldsymbol{\varrho}) = D_T(\varrho)\, \delta_{ij} + (D_L(\varrho) - D_T(\varrho)) \frac{\varrho_i\, \varrho_j}{\varrho^2}, \tag{14}$$

where $D_L(\varrho)$ and $D_T(\varrho)$ are the scalar structure functions of $\varrho = |\boldsymbol{\varrho}|$ for the velocity components along and transverse to the displacement vector $\boldsymbol{\varrho}$, respectively.

For turbulence in the inertial subrange where local isotropy prevails we further have

$$D_T(\varrho) = \frac{4}{3} D_L(\varrho) = \frac{36}{55}\, \Gamma\left(\frac{1}{3}\right)\, \alpha\, (\varepsilon \varrho)^{2/3}, \tag{15}$$

where $\alpha \approx 1.7$ is the empirical Kolmogorov constant for the energy spectrum. This means that

$$D_{ii}(\boldsymbol{d}_\circ) = D_L(d_\circ) + 2\, D_T(d_\circ) = \frac{9}{5}\, \Gamma\left(\frac{1}{3}\right)\, \alpha\, (\varepsilon d_\circ)^{2/3} \tag{16}$$

which, combined with (12) and (10), yields

$$D^2(t) = d_\circ^2 + \frac{9}{5}\, \Gamma\left(\frac{1}{3}\right) \alpha (\varepsilon d_\circ)^{2/3}\, t^2 = d_\circ^2 + 8.25\, (\varepsilon d_\circ)^{2/3}\, t^2. \tag{17}$$

Sawford (2001) stated the same conclusion.

When $D \gg \ell$ the two particles move independent of one another and $D^2(t)$ becomes just twice the mean square excursion (MSE) $\sigma^2(t)$ of the position of a single particle, i.e., $D^2(t)$ is proportional to t. As discussed by Batchelor (1952), there should be three regimes of $D^2(t)$: the initial stage, where $D(t)$ follows (17) ($D_E(t) \propto t$), the intermediate stage, where $D(t)$ according to the Richardson-Obukhov law is proportional to $t^{3/2}$, and the final stage where $D(t) \propto t^{1/2}$. Figure 1 summarizes the situation.

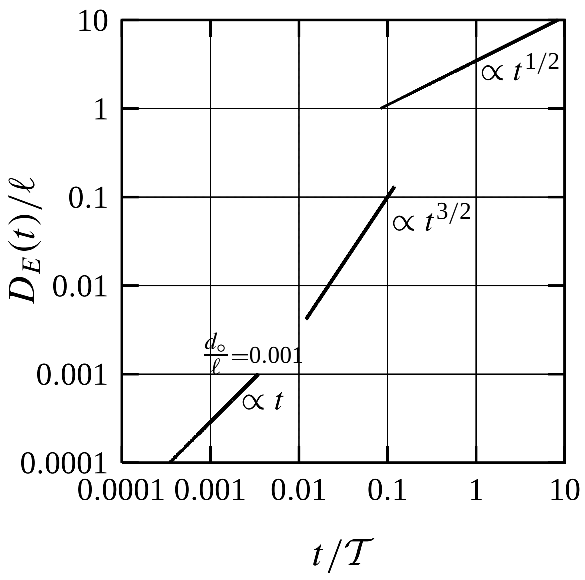

Figure 1. Sketch of the asymptotic behavior by Batchelor (1952) of the root-mean-square excess distance $D_E(t)$ between two particles, initially separated by the distance $d_\circ$. The variables are made dimensionless by means of (1) and (2).

3 Model Formulation

To proceed we introduce the generalized, mean-square excess separation (GMES)

$$\Sigma^2(t_1, t_2) = \left\langle \left(\boldsymbol{r}_2(t_2) - \boldsymbol{r}_1(t_1) - \boldsymbol{d}_\circ \right)^2 \right\rangle, \tag{18}$$

which is a symmetric function of t_1 and t_2*. Comparing with (9), we see that

$$\Sigma^2(t, t) = D_E^2(t) = D^2(t) - d_\circ^2 \tag{19}$$

and that, according to (8),

$$\Sigma^2(t, 0) = \left\langle \left(\boldsymbol{r}_n(t) - \boldsymbol{r}_n^\circ \right)^2 \right\rangle \tag{20}$$

becomes the MSE $\sigma^2(t)$ of the displacement of a single particle from its position at $t = 0$. This quantity can be expressed as (Taylor 1921)

$$\sigma^2(t) = \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 R_L(t_2 - t_1), \tag{21}$$

where

$$R_L(\tau) = \langle \boldsymbol{v}_n(t + \tau) \cdot \boldsymbol{v}_n(t) \rangle \tag{22}$$

is the Lagrangian autocovariance function.

*The turbulence is isotropic, which includes reflection and rotation symmetry, and the vector $\boldsymbol{d}_\circ$ enters only with its magnitude $d_\circ$.

Note that from (21) follows

$$\frac{\mathrm{d}\sigma^2(t)}{\mathrm{d}t} = 2\int_0^t R_L(\tau)\mathrm{d}\tau \tag{23}$$

and

$$\sigma^2(t) = 2\,t\int_0^t \left(1-\frac{\tau}{t}\right)\,R_L(\tau)\,\mathrm{d}\tau. \tag{24}$$

The asymptotes of (21) are:

$t \to\ 0$:

$$\sigma^2(t) \simeq \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\,\underbrace{R_L(0)}_{3\langle u^2\rangle} = 3\langle u^2\rangle\,t^2 \tag{25}$$

$t \to \infty$:

$$\sigma^2(t) \simeq 2\,t\int_0^\infty R_L(\tau)\,\mathrm{d}\tau, \tag{26}$$

where (26) follows from (24).

Then we get

$$\sigma^2(t) = 6\,\langle u^2\rangle\,\mathcal{T}_L\,t, \tag{27}$$

where

$$\mathcal{T}_L = \frac{1}{R_L(0)}\int_0^\infty R_L(\tau)\,\mathrm{d}\tau \tag{28}$$

is the Lagrangian integral time scale.

An equation for the GMES $\Sigma^2(t_1,t_2)$—the counterpart to the MSE $\sigma^2(t)$ of the one-particle distance from the initial position—is obtained by expanding (18):

$$\Sigma^2(t_1,t_2) = \sigma^2(t_1) + \sigma^2(t_2) - 2\int_0^{t_1}\mathrm{d}\tau_1\int_0^{t_2}\mathrm{d}\tau_2 R_{L2}(\tau_1,\tau_2), \tag{29}$$

where we have introduced the symmetrical, two-particle Lagrangian covariance function

$$R_{L2}(t_1,t_2) = \langle \boldsymbol{v}_1(t_1)\cdot\boldsymbol{v}_2(t_2)\rangle\,. \tag{30}$$

$R_{L2}(t_1,t_2)$ depends on the initial separation $\boldsymbol{d}_\circ$, but according to our convention, the initial positions have been suppressed as arguments in the velocities.

3.1 Lagrangian and Eulerian Statistics

The model we propose requires the establishment of a relation between Lagrangian and Eulerian second-order velocity statistics and at this point we introduce Corrsin's independence hypothesis with the aim of reformulating both (21) and (29). We shall follow a step-by-step procedure and consider first a single particle with the velocity $\boldsymbol{v}(t)$ and the position $\boldsymbol{r}(t)$.

The identity

$$\boldsymbol{v}(t) = \boldsymbol{u}(\boldsymbol{r}(t), t) \tag{31}$$

allows us to reformulate $R_L(t_2 - t_1)$ in terms of Eulerian velocities:

$$R_L(t_2 - t_1) = \langle \boldsymbol{u}(\boldsymbol{r}(t_1), t_1) \cdot \boldsymbol{u}(\boldsymbol{r}(t_2), t_2) \rangle. \tag{32}$$

If the difference $\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1)$ were not random, but rather a specified vector $\boldsymbol{\varrho}$, this would have been equal to the trace of the Eulerian autocovariance tensor defined by

$$R_{ij}(\boldsymbol{\varrho}, \tau) = \langle u_i(\boldsymbol{r}, t)\, u_j(\boldsymbol{r} + \boldsymbol{\varrho}, t + \tau) \rangle \tag{33}$$

with the time lag $\tau = t_2 - t_1$.

This is obviously not the case, and to remedy this problem we rewrite (32) in the equivalent form

$$R_L(t_2 - t_1) = \int \mathrm{d}^3\boldsymbol{x}_1 \int \mathrm{d}^3\boldsymbol{x}_2 \left\langle (\boldsymbol{u}(\boldsymbol{x}_1, t_1) \cdot \boldsymbol{u}(\boldsymbol{x}_2, t_2))\ \delta(\boldsymbol{x}_1 - \boldsymbol{r}(t_1))\, \delta(\boldsymbol{x}_2 - \boldsymbol{r}(t_2)) \right\rangle. \tag{34}$$

One way of formulating Corrsin's independence hypothesis is to assume that $\boldsymbol{r}(t)$ and $\boldsymbol{u}(\boldsymbol{x}, \tau)$ are statistically independent for all $(\boldsymbol{x}, t, \tau)$. Then we may write (McComb 1990, Cambon et al. 2004)

$$R_L(t_2 - t_1) = \int \mathrm{d}^3\boldsymbol{x}_1 \int \mathrm{d}^3\boldsymbol{x}_2\, R_{ii}(\boldsymbol{x}_2 - \boldsymbol{x}_1, t_2 - t_1) \left\langle \delta(\boldsymbol{x}_1 - \boldsymbol{r}(t_1))\, \delta(\boldsymbol{x}_2 - \boldsymbol{r}(t_2)) \right\rangle, \tag{35}$$

where we have used (33) and where summation over double indices is assumed. Introducing new variables

$$\left\{ \begin{array}{c} \boldsymbol{x} \\ \boldsymbol{y} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{x}_2 - \boldsymbol{x}_1 \\ (\boldsymbol{x}_2 + \boldsymbol{x}_1)/2 \end{array} \right\} \tag{36}$$

(35) reduces to

$$R_L(t_2 - t_1) = \int R_{ii}(\boldsymbol{x}, t_2 - t_1) \left\langle \delta(\boldsymbol{x} - \{\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1)\}) \right\rangle \mathrm{d}^3\boldsymbol{x}. \tag{37}$$

The quantity $\langle \delta(\boldsymbol{x} - \{\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1)\}) \rangle\, \mathrm{d}^3\boldsymbol{x}$ is the relative average number of cases (out of infinitely many trials) where the difference $\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1)$ falls in the volume element $\mathrm{d}^3\boldsymbol{x}$ around $\boldsymbol{x}$. Hence

$$\langle \delta(\boldsymbol{x} - \{\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1)\}) \rangle = p_1(\boldsymbol{x}; t_1, t_2) \tag{38}$$

can be interpreted as the probability density function for $\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1) = \boldsymbol{x}$ (McComb 1990). The isotropy implies that it depends on $\boldsymbol{x}$ only through the magnitude x. The parametric time dependence of p_1, which enters through the difference $t_2 - t_1$, is conveniently expressed in terms of the MSE $\sigma^2(t_2 - t_1) = \langle \{\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1)\}^2 \rangle$ in the period of time from t_1 to t_2:

$$p_1(\boldsymbol{x}; t_1, t_2) = \varphi_1\Big(x; \sigma^2(t_2 - t_1)\Big) . \tag{39}$$

Inserting (39), we get

$$R_L(t_2 - t_1) = \int R_{ii}(\boldsymbol{x}, t_2 - t_1)\, \varphi_1\Big(x; \sigma^2(t_2 - t_1)\Big)\ \mathrm{d}^3\boldsymbol{x}. \tag{40}$$

The next step is to consider the two-particle case. We go back to (30), and by an analogous procedure we obtain an equation similar to (37) for $R_{L2}(t_1, t_2) = \langle \boldsymbol{u}(\boldsymbol{r}_1(t_1), t_1) \cdot \boldsymbol{u}(\boldsymbol{r}_2(t_2), t_2) \rangle$:

$$R_{L2}(t_1, t_2) = \int R_{ii}(\boldsymbol{x}, t_2 - t_1) \Big\langle \delta(\boldsymbol{x} - \{\boldsymbol{r}_2(t_2) - \boldsymbol{r}_1(t_1)\}) \Big\rangle \mathrm{d}^3\boldsymbol{x}. \tag{41}$$

We also interpret $\langle \delta(\boldsymbol{x} - \{\boldsymbol{r}_2(t_2) - \boldsymbol{r}_1(t_1)\}) \rangle$ as a probability density function $p_2(\boldsymbol{x}; t_1, t_2)$, now for two particles with $\boldsymbol{r}_2(t_2) - \boldsymbol{r}_1(t_1) = \boldsymbol{x}$. This generalization of Corrsin's independence hypothesis was also applied by Ishihara & Kaneda (2002). We assume that p_2 depends on $\boldsymbol{x}$ through $|\boldsymbol{x} - \boldsymbol{d}_\circ|$. Such a dependence is clearly justified at the initial stage $\boldsymbol{x} \simeq \boldsymbol{d}_\circ$, and when $|\boldsymbol{x}|$ is large the influence of $\boldsymbol{d}_\circ$ is weak in any case. We shall make the extra assumption that the dependence on t_1 and t_2 can be parameterized in terms of the quantity $\Sigma^2(t_1, t_2)$ introduced in (18). Thus we assume the form

$$p_2(\boldsymbol{x}; t_1, t_2) = \varphi_2\Big(|\boldsymbol{x} - \boldsymbol{d}_\circ|; \Sigma^2(t_1, t_2)\Big) . \tag{42}$$

With this assumption (41) becomes

$$R_{L2}(t_1, t_2) = \int R_{ii}(\boldsymbol{x}, t_2 - t_1)\, \varphi_2\Big(|\boldsymbol{x} - \boldsymbol{d}_\circ|; \Sigma^2(t_1, t_2)\Big)\ \mathrm{d}^3\boldsymbol{x}. \tag{43}$$

We now reformulate (40) and (43) by means of the spectral tensor $\Phi_{ij}(\boldsymbol{k}, \tau)$ which, in turn, can be expressed in terms of the energy spectrum $E(k, \tau)$ by (Batchelor 1953)

$$\Phi_{ij}(\boldsymbol{k}, \tau) = \frac{E(k, \tau)}{4\pi k^2} \left\{ \delta_{ij} - \frac{k_i k_j}{k^2} \right\}, \qquad k = |\boldsymbol{k}|. \tag{44}$$

This means that

$$R_{ii}(\boldsymbol{x}, \tau) = \int \Phi_{ii}(\boldsymbol{k}, \tau) \exp(\mathrm{i}\, \boldsymbol{k} \cdot \boldsymbol{x})\ \mathrm{d}^3\boldsymbol{k} = \int \frac{E(k, \tau)}{2\pi k^2} \exp(\mathrm{i}\, \boldsymbol{k} \cdot \boldsymbol{x})\ \mathrm{d}^3\boldsymbol{k}, \tag{45}$$

so that (40) and (43) become

$$\begin{aligned} R_L(t_2 - t_1) &= \int \frac{E(k, t_2 - t_1)}{2\pi k^2}\, \mathrm{d}^3\boldsymbol{k} \int \exp(\mathrm{i}\, \boldsymbol{k} \cdot \boldsymbol{x})\ \varphi_1\Big(x; \sigma^2(t_2 - t_1)\Big)\ \mathrm{d}^3\boldsymbol{x} \\ &= \frac{1}{2\pi} \int \frac{E(k, t_2 - t_1)}{k^2}\, \widehat{\varphi}_1(k; \sigma^2(t_2 - t_1))\, \mathrm{d}^3\boldsymbol{k}. \end{aligned} \tag{46}$$

and

$$R_{L2}(t_1,t_2) = \int \frac{E(k,t_2-t_1)}{2\pi k^2}\,\mathrm{d}^3\boldsymbol{k}\int \exp(\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x})\;\varphi_2\Big(|\boldsymbol{x}-\boldsymbol{d}_\circ|;\,\Sigma^2(t_1,t_2)\Big)\;\mathrm{d}^3\boldsymbol{x}$$

$$= \frac{1}{2\pi}\int \frac{E(k,t_2-t_1)}{k^2}\,\widehat{\varphi}_2(k;\,\Sigma^2(t_1,t_2))\exp\left(\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{d}_\circ\right)\;\mathrm{d}^3\boldsymbol{k}. \qquad (47)$$

Here we have introduced the three-dimensional Fourier transform of the function $\varphi_n(r;\,\Lambda^2)$ by

$$\widehat{\varphi}_n(k;\,\Lambda^2) = \int \varphi_n\Big(r;\,\Lambda^2\Big)\;\exp(\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r})\;\mathrm{d}^3\boldsymbol{r}, \quad n=1,2. \qquad (48)$$

We note that the triple integral (48) can be reduced to a single integral over $r = |\boldsymbol{r}|$:

$$\widehat{\varphi}_n(k;\,\Lambda^2) = 4\pi\int_0^\infty \varphi_n(r;\,\Lambda^2)\,\mathrm{sinc}(kr)\,r^2\,\mathrm{d}r, \qquad (49)$$

where $\mathrm{sinc}(x) = \sin x/x$, and by a similar reduction over $k = |\boldsymbol{k}|$ both (46) and (47) can be expressed as single integrals over k as follows:

$$R_L(t_2-t_1) = 2\int_0^\infty E(k,t_2-t_1)\,\widehat{\varphi}_1(k;\sigma^2(t_2-t_1))\,\mathrm{d}k, \qquad (50)$$

and

$$R_{L2}(t_1,t_2) = 2\int_0^\infty E(k,t_2-t_1)\,\mathrm{sinc}(kd_\circ)\,\widehat{\varphi}_2(k;\,\Sigma^2(t_1,t_2))\,\mathrm{d}k, \quad d_\circ = |\boldsymbol{d}_\circ|. \qquad (51)$$

Combining (29) and (51) we obtain the following hyperbolic partial differential equation for $\Sigma^2(t_1,t_2)$

$$\frac{\partial^2\Sigma^2}{\partial t_1\partial t_2} = -4\int_0^\infty E(k,t_2-t_1)\,\mathrm{sinc}(kd_\circ)\,\widehat{\varphi}_2(k;\,\Sigma^2)\,\mathrm{d}k \qquad (52)$$

with the boundary conditions $\Sigma^2(0,t) = \Sigma^2(t,0) = \sigma^2(t)$, which follow from (21) and (29).

At this point we must specify the Eulerian energy spectrum $E(k,\tau)$ for isotropic turbulence as well as the functions $\varphi_n(r;\,\Lambda^2)$.

3.2 The Energy Spectrum

We assume that the temporal part of the energy spectrum is an exponential decay factor and write

$$E(k,\tau) = E_\circ(k)\;\exp(-|\tau|/\mathcal{T}_\circ(k)), \qquad (53)$$

where the mean presence time $\mathcal{T}_\circ(k)$ of an eddy of the size k^{-1} will in general be a function of k.

A Eulerian eddy can in general disappear in two ways. It may be advected away by larger eddies or decay by the action of smaller eddies. Since we must assume that more intense turbulence causes shorter presence times, both of these times will be inversely proportional to characteristic turbulence velocities: $v_a(k)$ for advection and $v_d(k)$ for decay. We assume the forms

$$\mathcal{T}_a^{-1}(k) = \mu_a^\circ \, k \, v_a(k) \tag{54}$$

for presence time under the advective action of the larger eddies and

$$\mathcal{T}_d^{-1}(k) = \mu_d^\circ \, k \, v_d(k) \tag{55}$$

for the mean lifetime of an eddy which decays due to the action of the smaller eddies. μ_a° and μ_d° are two dimensionless constants, which must be determined by means of auxiliary information. The total mean presence time is then given by

$$\mathcal{T}_\circ(k)^{-1} = \mathcal{T}_a(k)^{-1} + \mathcal{T}_d(k)^{-1}. \tag{56}$$

We assume

$$v_a^2(k) = \int_0^k E_\circ(k')\,\mathrm{d}k' \tag{57}$$

and

$$v_d^2(k) = \int_k^\infty E_\circ(k')\,\mathrm{d}k'. \tag{58}$$

At this point we specify the wave-number part of the energy spectrum. Often the scale free form

$$E_\circ(k) = \alpha\,\varepsilon^{2/3}\,k^{-5/3}, \qquad 0 < k < \infty, \tag{59}$$

is applied to obtain general results. Therefore we assume (59) to see if we may obtain such general results.

We first note that the integrals in neither (50)–(52) nor (57) are convergent if $E_\circ(k)$ is given by (59). However, when this form is used it is required that $k \gg \ell^{-1}$ so that (57) is approximated by

$$v_a^2(k) \simeq \int_0^\infty E_\circ(k')\,\mathrm{d}k' = \frac{3}{2}\,\langle u^2 \rangle. \tag{60}$$

To obtain an equation which can replace (52), we recast the problem by including single-particle dispersion. From (50) and (23) we deduce

$$\frac{\partial^2 \sigma^2(t_2 - t_1)}{\partial t_1 \partial t_2} = -4 \int_0^\infty E(k, t_2 - t_1)\,\widehat{\varphi}_1(k; \sigma^2(t_2 - t_1))\,\mathrm{d}k \tag{61}$$

Introducing

$$\widetilde{\Sigma}^2(t_1, t_2) = \Sigma^2(t_1, t_2) - \sigma^2(t_2 - t_1), \tag{62}$$

we obtain the hyperbolic, partial differential equation

$$\frac{\partial^2 \widetilde{\Sigma}^2}{\partial t_1 \partial t_2} = 4\int_0^\infty E(k, t_2 - t_1) \left\{\widehat{\varphi}_1(k; \sigma^2(t_2 - t_1)) - \widehat{\varphi}_2\big(k; \widetilde{\Sigma}^2 + \sigma^2(t_2 - t_1)\big)\mathrm{sinc}(kd_\circ)\right\} \mathrm{d}k \quad (63)$$

with the boundary conditions $\widetilde{\Sigma}^2(t, 0) = \widetilde{\Sigma}^2(0, t) = 0$. We note that

$$\widetilde{\Sigma}^2(t, t) = \Sigma^2(t, t) = D_E^2(t). \quad (64)$$

Applying (60) and carrying out the integration (58), (54) and (55) become, in view of the relation (3),

$$\mathcal{T}_a^{-1}(k) = \mu_a \frac{k\ell}{\mathcal{T}} \quad (65)$$

and

$$\mathcal{T}_d^{-1}(k) = \mu_d \frac{(k\ell)^{2/3}}{\mathcal{T}}, \quad (66)$$

where we for convenience have introduced

$$\mu_a = \sqrt{\frac{3}{2}}\mu_a^\circ \quad (67)$$

and

$$\mu_d = \sqrt{\frac{3\alpha}{2}}\mu_d^\circ. \quad (68)$$

Thus the energy spectrum in (63) can be written

$$E(k, \tau) = \alpha\, \varepsilon^{2/3}\, k^{-5/3} \exp\!\left(-\left\{\mu_a\, k\ell + \mu_d\, (k\ell)^{2/3}\right\} \frac{|\tau|}{\mathcal{T}}\right). \quad (69)$$

and (63) becomes

$$\begin{aligned} \frac{\partial^2 \widetilde{\Sigma}^2}{\partial t_1 \partial t_2} &= 4\,\alpha\, \varepsilon^{2/3} \int_0^\infty \exp\!\left(-\left\{\mu_a\, k\ell + \mu_d\, (k\ell)^{2/3}\right\} \frac{|t_2 - t_1|}{\mathcal{T}}\right) \times \\ &\qquad \left\{\widehat{\varphi}_1(k; \sigma^2(t_2 - t_1)) - \widehat{\varphi}_2(k;\, \widetilde{\Sigma}^2 + \sigma^2(t_2 - t_1))\, \mathrm{sinc}(kd_\circ)\right\} \frac{\mathrm{d}k}{k^{5/3}}. \quad (70) \end{aligned}$$

We simplify this equation by noting that we are here considering dispersion for which $t \ll \mathcal{T}_L$. This means that $\sigma^2(t)$ can be approximated by (25) and, consequently, also, as (3) shows, that $\sigma^2(t) \ll \ell^2$. Now (70) becomes

$$\frac{\partial^2 \widetilde{\Sigma}^2}{\partial t_1 \partial t_2} = 4\,\alpha\,\varepsilon^{2/3} \int_0^\infty \exp\!\left(-\left\{\mu_a\, k\ell + \mu_d\,(k\ell)^{2/3}\right\}\frac{|t_2-t_1|}{\mathcal{T}}\right) \times$$
$$\left\{\widehat{\varphi}_1(k;\,3\langle u^2\rangle\{t_2-t_1\}^2) - \widehat{\varphi}_2(k;\,\widetilde{\Sigma}^2 + 3\langle u^2\rangle\{t_2-t_1\}^2)\,\mathrm{sinc}(kd_\circ)\right\}\frac{\mathrm{d}k}{k^{5/3}}. \quad (71)$$

In the limit $(t_1, t_2) \to (0, 0)$, both (39) and (42) approach delta functions. The corresponding Fourier transforms (48) are therefore both equal to unity. In this case (71) reduces to

$$\frac{\partial^2 \widetilde{\Sigma}^2}{\partial t_1 \partial t_2} = 4\,\alpha\,\varepsilon^{2/3} \int_0^\infty \{1 - \mathrm{sinc}(kd_\circ)\}\frac{\mathrm{d}k}{k^{5/3}} \quad (72)$$

with the solution

$$\widetilde{\Sigma}^2(t_1, t_2) = \frac{9}{5}\,\Gamma\!\left(\frac{1}{3}\right)\,\alpha\,(\varepsilon d_\circ)^{2/3}\,t_1 t_2. \quad (73)$$

Applying (10) and (64), we retrieve for $t_1 = t_2 = t$ the solution (17) for the initial stage.

3.3 The Functions $\varphi_n(r;\,\Lambda^2)$

These functions satisfy the norm and second-order moment constrains

$$\int \varphi_n(r;\,\Lambda^2)\,\mathrm{d}^3\boldsymbol{r} = 4\pi \int_0^\infty \varphi_n(r;\,\Lambda^2)\,r^2\,\mathrm{d}r = 1 \quad (74)$$

$$\int \boldsymbol{r}^2 \varphi_n(r;\,\Lambda^2)\,\mathrm{d}^3\boldsymbol{r} = 4\pi \int_0^\infty \varphi_n(r;\,\Lambda^2)\,r^4\,\mathrm{d}r = \Lambda^2. \quad (75)$$

Considering first the dispersion of a single particle, Batchelor argued (1949, 1953) that there are good reasons to believe that $p_1(\boldsymbol{r};\,t_1, t_2)$ is a three-dimensional Gaussian in $\boldsymbol{r}$ with the one-component variance $\sigma^2(t_2 - t_1)/3 = \Lambda^2/3$, i.e.

$$\varphi_1(r;\,\Lambda^2) = \left(\frac{3}{2\pi\,\Lambda^2}\right)^{3/2} \exp\!\left(-\frac{3r^2}{2\Lambda^2}\right). \quad (76)$$

For two particles the function $\varphi_2(r;\,\Lambda^2)$ is equal to the *distance-neighbour function*, introduced by Richardson (1926) in the special case when $\Lambda^2 = \Sigma^2(t, t) = D_E^2(t)$. Batchelor (1952) subsequently discussed the form of this function. Richardson's (1926) original form can, if we neglect the initial displacement $\boldsymbol{d}_\circ$, be written

$$\varphi_2(r;\,D_E^2(t)) = \frac{1}{315}\left(\frac{1287}{2\pi\,D_E^2(t)}\right)^{3/2} \exp\!\left(-\left\{\frac{1287}{8}\,\frac{r^2}{D_E^2(t)}\right\}^{1/3}\right). \quad (77)$$

Batchelor (1952) found it more reasonable to assume that $\varphi_2(r;\,D_E^2(t))$ has a Gaussian form, i.e. $\varphi_2(r;\,D_E^2(t)) = \varphi_1(r;\,D_E^2(t))$, given by (76). Later experiments, e.g., Ott &

Mann (2000), seem to support the assumption (77) by Richardson (1926). Taking this into account, $\varphi_2(r;\, \Sigma^2(0, t_2))$ (particle one fixed at its initial position) must be expected to be Gaussian, whereas $\varphi_2(r;\, \Sigma^2(t_1, t_2))$ with $t_1 \simeq t_2$ is of the form (77). Instead of devising a gradual transformation from the Richardson (1926) form where $t_1 = t_2$ to the Gaussian form where $t_1 = 0$—an interpolation which, in any case, will be arbitrary and probably very complicated—we have decided to consider the two possible cases where $\varphi_1(r;\, \Lambda^2)$ and $\varphi_2(r;\, \Lambda^2)$ are given by the same form. The two cases are

$$\varphi_1(r;\, \Lambda^2) = \varphi_2(r;\, \Lambda^2) = \left(\frac{3}{2\pi\Lambda^2}\right)^{3/2} \exp\left(-\frac{3r^2}{2\Lambda^2}\right) \equiv \varphi_B(r;\, \Lambda^2) \tag{78}$$

and

$$\varphi_1(r;\, \Lambda^2) = \varphi_2(r;\, \Lambda^2) = \frac{1}{315}\left(\frac{1287}{2\pi\,\Lambda^2}\right)^{3/2} \exp\left(-\left\{\frac{1287}{8}\,\frac{r^2}{\Lambda^2}\right\}^{1/3}\right) \equiv \varphi_R(r;\, \Lambda^2). \tag{79}$$

The corresponding Fourier transforms (49) are

$$\widehat{\varphi}_B(k;\, \Lambda^2) = \exp\left(-\frac{k^2\Lambda^2}{6}\right) \tag{80}$$

and

$$\widehat{\varphi}_R(k;\, \Lambda^2) = W(k^2\Lambda^2), \tag{81}$$

where we have introduced the function

$$W(x) = \left(\frac{143}{6\,x}\right)^{11/6} U\left(\frac{11}{6}, \frac{2}{3}, \frac{143}{6\,x}\right), \tag{82}$$

expressed in terms of the confluent hypergeometric function $U(a, b, x)$ (Wolfram 1999).

3.4 Dimensionless Formulation

For convenience, we restate the basic equation (71) in a dimensionless form by using ℓ and $\mathcal{T}$ and their interrelations (3) as scaling parameters. The new variables are defined by

$$\left\{\begin{array}{l} s \\ \theta \\ \Psi(\theta_1, \theta_2) \\ \chi(\theta) \\ \Delta_\circ \end{array}\right\} = \left\{\begin{array}{l} k\ell \\ t/\mathcal{T} \\ \widetilde{\Sigma}^2(t_1, t_2)/\ell^2 \\ D_E^2(t)/\ell^2 \\ d_\circ/\ell \end{array}\right\}. \tag{83}$$

We have in the purely Gaussian case, suggested by Batchelor (1952),

$$\begin{aligned}\frac{\partial^2\Psi}{\partial\theta_1\partial\theta_2} &= 4\,\alpha\int_0^\infty \exp\Big(-\Big\{\mu_a\,s+\mu_d\,s^{2/3}\Big\}\;|\theta_2-\theta_1|-(\theta_2-\theta_1)^2 s^2/2\Big)\times\\ &\qquad\Big\{1-\exp\Big(-\frac{s^2\Psi}{6}\Big)\;\mathrm{sinc}(s\,\Delta_\circ)\Big\}\;\frac{\mathrm{d}s}{s^{5/3}},\end{aligned} \tag{84}$$

and in the "Richardson case"

$$\begin{aligned}\frac{\partial^2\Psi}{\partial\theta_1\partial\theta_2} &= 4\,\alpha\int_0^\infty \exp\Big(-\Big\{\mu_a\,s+\mu_d\,s^{2/3}\Big\}\;|\theta_2-\theta_1|\Big)\times\\ &\qquad\Big\{W(3s^2\{\theta_2-\theta_1\}^2)-W(s^2[\Psi+3\{\theta_2-\theta_1\}^2])\;\mathrm{sinc}(s\,\Delta_\circ)\Big\}\;\frac{\mathrm{d}s}{s^{5/3}}.\end{aligned} \tag{85}$$

The boundary conditions are in both cases $\Psi(\theta,0)=\Psi(0,\theta)=0$.

4 Results

The two equations (84), the "Batchelor case", and (85), the "Richardson case", are solved numerically in the (θ_1,θ_2)-plane. The integration is initiated at the point $(\theta_1,\theta_2)=(0,0)$ and the reflection symmetry with respect to the diagonal line $\theta_2=\theta_1$ is utilized. Ultimately, the solution $\chi(\theta)=\Psi(\theta,\theta)$ on the diagonal is obtained by calculating $\Psi(\theta_1,\theta_2)$ on a progressing front, characterized by θ_2.

We note that (73), which in dimensionless form can be written

$$\Psi(\theta_1,\theta_2)=\frac{9}{5}\,\Gamma\Big(\frac{1}{3}\Big)\;\alpha\,\Delta_\circ^{2/3}\,\theta_1\,\theta_2 \tag{86}$$

is an exact, asymptotic solution for the initial stage in both the Batchelor and the Richardson case.

It is possible also to obtain an exact, asymptotic solution far from the origin $(\theta_1,\theta_2)=(0,0)$ in the Batchelor case, for "frozen turbulence", where $\mu_a=\mu_d=0$. Near the diagonal $\theta_2=\theta_1$ (84) becomes

$$\begin{aligned}\frac{\partial^2\Psi}{\partial\theta_1\partial\theta_2} &= 4\alpha\int_0^\infty\Big\{1-\mathrm{e}^{-\Psi s^2/6}\mathrm{sinc}(\Delta_\circ s)\Big\}\;\frac{\mathrm{d}s}{s^{5/3}}\\ &= 6^{2/3}\alpha\Gamma\Big(\frac{2}{3}\Big)\;\Psi^{1/3}{}_1F_1\Big(-\frac{1}{3};\frac{3}{2};-\frac{3}{2}\frac{\Delta_\circ^2}{\Psi}\Big),\end{aligned} \tag{87}$$

where ${}_1F_1(a;b;x)$ is the Kummer confluent hypergeometric function (Wolfram 1999).

When θ_1 and θ_2 are so large that $\Psi \gg \Delta_\circ^2$, (87) reduces to

$$\frac{\partial^2 \Psi}{\partial\theta_1 \partial\theta_2} = 6^{2/3} \alpha \, \Gamma\left(\frac{2}{3}\right) \, \Psi^{1/3}. \tag{88}$$

The solution is in this case

$$\Psi(\theta_1, \theta_2) = 6 \left\{ \frac{2}{3} \Gamma\left(\frac{5}{3}\right) \alpha \right\}^{3/2} \theta_1^{3/2} \theta_2^{3/2}. \tag{89}$$

For $\theta_1 = \theta_2 = \theta$ this becomes the Richardson-Obukhov law (4) in dimensionless form with $C = 6\{(2/3)\Gamma(5/3)\alpha\}^{3/2} \simeq 6.3$.

Figures 2, 3, and 4 show the results of integrating (84) and (85). The solutions $\chi(\theta) = \Psi(\theta, \theta)$ are in all cases divided by θ^3 in order to demonstrate that the curves for large values of θ approach the Richardson-Obukhov constant C.

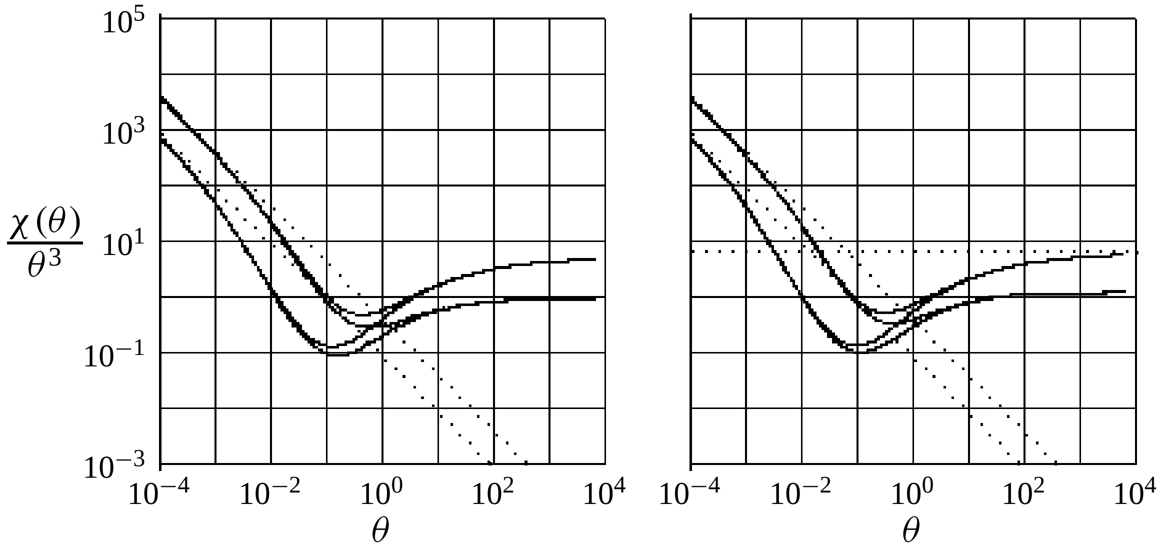

Figure 2. Integration results for $(\mu_a, \mu_d) = (0, 0)$ (thin lines), $(\mu_a, \mu_d) = (0, 1)$ (thick lines). The initial separations are $\Delta_\circ = 0.001$ (lower thick lines) and $\Delta_\circ = 0.01$ (upper thick lines). The left frame is the Richardson case and the right frame the Batchelor case. The dotted lines correspond to the asymptotic solutions (86) for the initial stage and the Batchelor-case solution (89) for frozen turbulence ($(\mu_a, \mu_d) = (0, 0)$) on the diagonal $\theta = \theta_1 = \theta_2$.

Figures 2 and 3 demonstrate that for small values of θ the Richardson and the Batchelor cases become identical and that at large values of θ the dimensionless, mean-square excess separation $\chi(\theta)$ for the Richardson case falls below that of the Batchelor case. Figure 2 also demonstrates C is a decreasing function of μ_d.

According to Fig. 4 the advection parameter μ_a does not influence the solution for neither $\theta \to 0$ nor $\theta \to \infty$. This means that the Richardson-Obukhov constant C in both the Richardson case and the Batchelor case is a function only of the decay parameter μ_d. Figure 5 shows how C depends on μ_d in both cases. We see that the Richardson curve falls below the Batchelor curve for all values of μ_d. The curves in Fig. 5 are obtained by analyzing (84) and (85) in the far field on lines perpendicular to the diagonal $\theta_2 = \theta_1$. Here it is possible to reduce the equations (84) and (85) to second-order, ordinary differential equations from which values $C(\mu_d)$ are obtained.

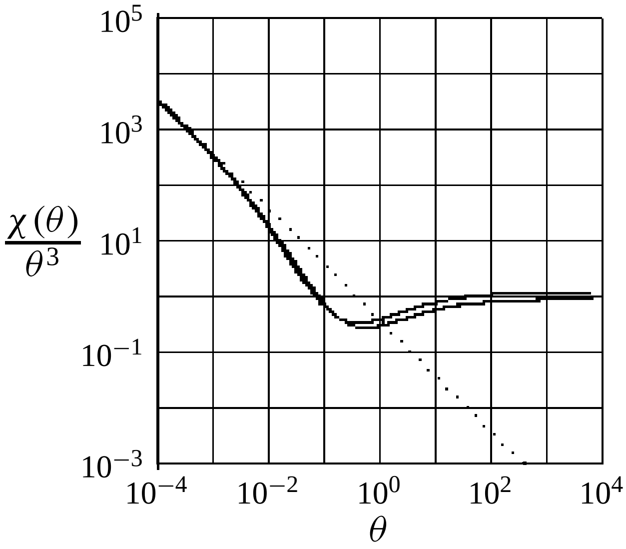

Figure 3. Results for $(\mu_a, \mu_d) = (0, 1)$. *The initial separation is* $\Delta_\circ = 0.01$. *Thick line: Richardson case, thin line: Batchelor case.*

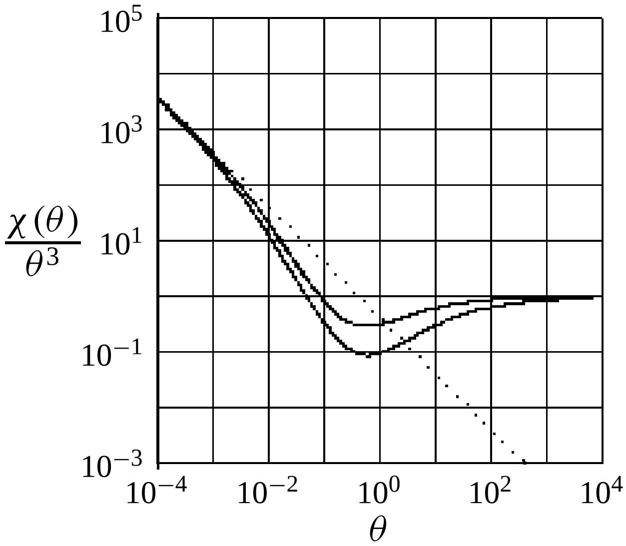

Figure 4. Results in the Richardson case for $(\mu_a, \mu_d) = (0, 1)$ *(thick line) and* $(\mu_a, \mu_d) = (1, 1)$ *(thin line). The initial separation is* $\Delta_\circ = 0.01$.

5 Discussion

Since the present model is based Corrsin's independence hypothesis as presented in (37) and, in a slightly generalized form for two particles, in (41), it is appropriate at this point first to illustrate that this hypothesis can only in a limited sense be consistent with the generally accepted Lagrangian description of the random motion of marked particles. Considering just the one-particle case, we note that—with reference to (31), (32), and (40)—it is a condition for the last equation that $\boldsymbol{r}(t)$ and $\boldsymbol{v}(t) = \boldsymbol{u}(\boldsymbol{r}(t), t)$ are statistically independent. To test this independence we consider the correlation coefficient $\rho_{\mathrm{vr}}(t)$ be-

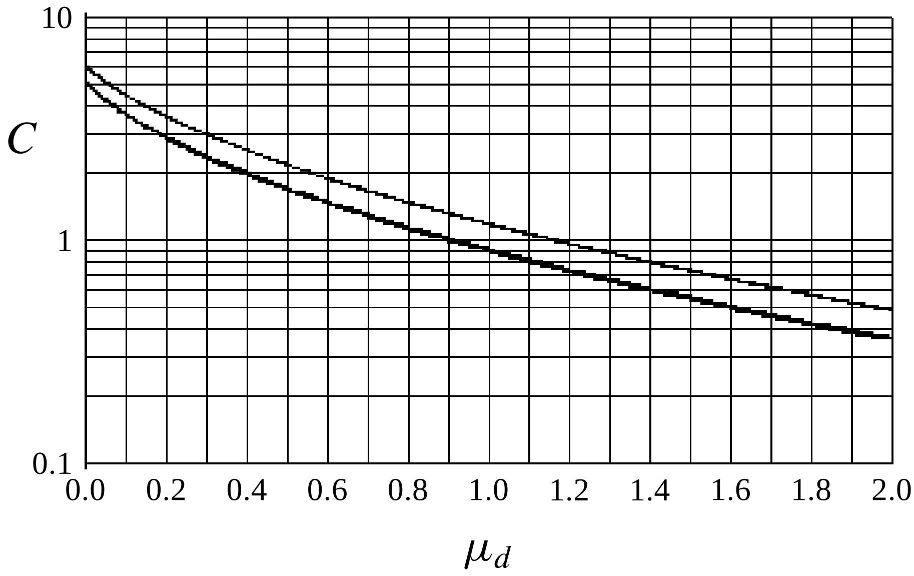

Figure 5. The Richardson-Obukhov constant with distance-neighbour functions by Richardson (thick line) and by Batchelor (thin line).

tween the particle displacement $\boldsymbol{r}_1(t) - \boldsymbol{r}_1^\circ$ and its velocity $\boldsymbol{v}_1(t)$. By definition,

$$\rho_{\mathrm{vr}}(t) = \frac{\langle \boldsymbol{v}_1(t)\cdot(\boldsymbol{r}_1(t)-\boldsymbol{r}_1^\circ)\rangle}{\left\{\langle \boldsymbol{v}_1^2\rangle\,\langle(\boldsymbol{r}_1(t)-\boldsymbol{r}_1^\circ)^2\rangle\right\}^{1/2}} = \frac{\langle \boldsymbol{v}_1(t)\cdot(\boldsymbol{r}_1(t)-\boldsymbol{r}_1^\circ)\rangle}{\langle \boldsymbol{v}_1^2\rangle^{1/2}\,\sigma(t)} \tag{90}$$

Noting that

$$\begin{aligned}\langle \boldsymbol{v}_1(t)\cdot(\boldsymbol{r}_1(t)-\boldsymbol{r}_1^\circ)\rangle &= \left\langle \boldsymbol{v}_1(t)\cdot\int_0^t \boldsymbol{v}_1(t')\,\mathrm{d}t'\right\rangle \\ &= \int_0^t R_L(t')\,\mathrm{d}t' \simeq \begin{cases} 3\langle u^2\rangle t, & t \ll \mathcal{T}_L \\ 3\langle u^2\rangle \mathcal{T}_L, & t \gg \mathcal{T}_L \end{cases},\end{aligned} \tag{91}$$

and that, according to (25) and (27),

$$\sigma^2(t) \simeq \begin{cases} 3\langle u^2\rangle t^2, & t \ll \mathcal{T}_L \\ 6\langle u^2\rangle \mathcal{T}_L t, & t \gg \mathcal{T}_L \end{cases}, \tag{92}$$

the correlation coefficient becomes in the limits of small and large times

$$\rho_{\mathrm{vr}}(t) = \begin{cases} 1 & t \ll \mathcal{T}_L \\ \sqrt{\dfrac{\mathcal{T}_L}{2t}} & t \gg \mathcal{T}_L \end{cases}. \tag{93}$$

We see that for large times the position and the velocity of a Lagrangian particle become uncorrelated. Thus, in this limit a necessary condition for Corrsin's independence hypothesis is fulfilled. At small times the two quantities are highly correlated. However, Corrsin's hypothesis is obviously exactly true in this limit. It is in all likelihood not true in general, but perhaps defendable as a tool for our purpose.

Inspecting Figs. 2, 3, and 4, we note that the asymptotes of constant value of $\chi(\theta)/\theta^3$ are reached at values of θ much larger than one, with corresponding values of $\chi(\theta)$ also much larger than one. These asymptotes are supposedly the dimensionless forms of the intermediate stage of $C = D_E^2(t)/(\varepsilon t^3)$, i.e. the Richardson-Obukhov law with the constant C. The problem is of course that the corresponding dispersion time t is much larger than the temporal scale $\mathcal{T}$ of the turbulence and that the square root of the mean-square excess separation $D_E(t)$ is much larger than the scale of the turbulence. For so large times and separations the assumption of isotropy is not warranted. Rather, we would expect the two marked particles to move independent of one another with $D_E^2(t)$ being proportional to t. This problem is related to the assumption (59) for the wave-number part of the spectrum $E_\circ(k)$. This limiting form of the spectrum implies that the velocity variance $\langle u^2\rangle$ is infinite. According to (1) and (2) both the outer scale ℓ and the time scale $\mathcal{T}$ are then also infinity. It also means that the Reynolds number R_λ is infinitely large. The only dimensional quantity with a physical interpretation is consequently the dissipation ε. From this point of view we have the freedom to reinterpret ℓ and $\mathcal{T}$ as finite scales interconnected by (3) and disregard (1) and (2) entirely. In all the following equations $\langle u^2\rangle$ is replaced by $\ell^2/\mathcal{T}^2$. If we accept this view, we have obtained a method of obtaining the Richardson-Obukhov constant C for two different assumptions about probability density functions entering (61) and (63). Since we cannot specify just one of the parameters ℓ and $\mathcal{T}$, we cannot use the present model to predict where the initial range (17) of $D^2(t)$ goes over into the intermediate range (4). Further, the final range, where the two particles move independent of one another, does not exist in this picture.

Instead of assuming a spectrum $E_\circ(k)$ of the scale free form (59) we could have assumed the less general von Kármán spectrum

$$E_\circ(k) = \frac{\alpha(\varepsilon L)^{2/3}(kL)^4 L}{(1+(kL)^2)^{17/6}} \tag{94}$$

with the finite length scale L. With the original definition (1) of the outer scale, L and ℓ are proportional. Their ratio is

$$\frac{L}{\ell} = \left(\frac{1}{3}\,B\left(\frac{5}{2},\frac{1}{3}\right)\alpha\right)^{-3/2} \approx 0.78, \tag{95}$$

where $B(a,b)$ is the beta function (Wolfram 1999).

Solving (52), we need the MSE $\sigma^2(t)$ for the boundary conditions. The MSE is obtained by solving the differential equation

$$\frac{\mathrm{d}^2\sigma^2}{\mathrm{d}t^2} = 4\int_0^\infty E(k,t)\,\widehat{\varphi}_1(k;\sigma^2)\,\mathrm{d}k, \tag{96}$$

equivalent to (61), with the initial conditions $(\sigma^2(t), \mathrm{d}\sigma^2/\mathrm{d}t)|_{t=0} = (0,0)$. We have used the Batchelor distance-neighbour function (78) and the more complete equations (57) and (58) for determining the eddy presence time $\mathcal{T}_\circ(k)$. Figure 6 shows results with two different initial separations.

We note that the solution apparently does not go through the εt^3 regime before it enters the final range, where it becomes proportional to t.

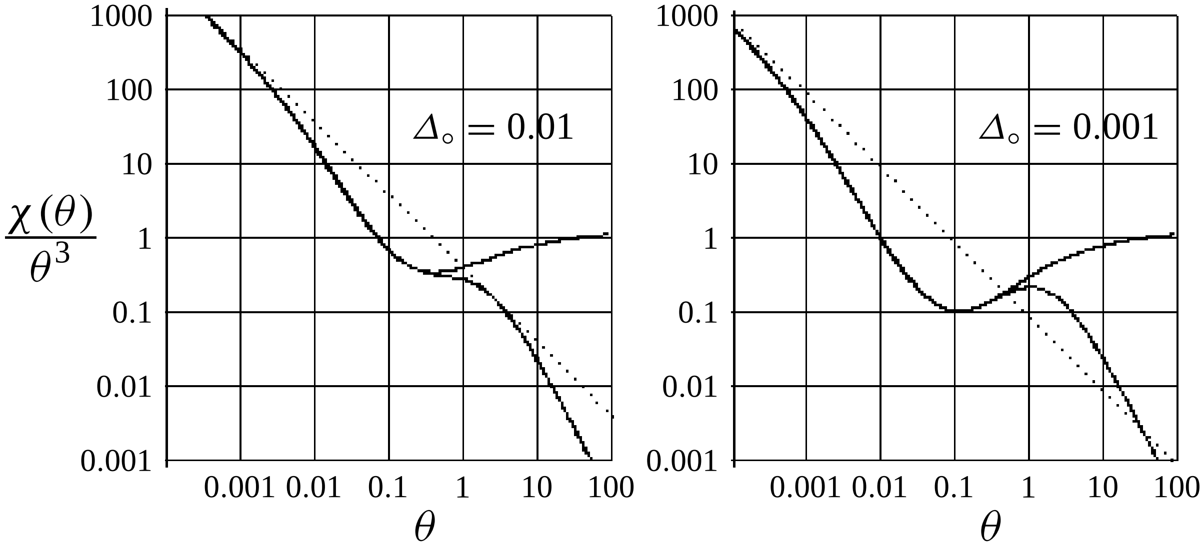

Figure 6. Two solutions to (52) with the Batchelor distance-neighbour function and with $(\mu_a, \mu_d) = (0, 1)$. The normalized $\chi(\theta)$ divided by θ^3 is shown as a function of the normalized time. The two initial separations are $\Delta_\circ = 0.01$ (left frame) and $\Delta_\circ = 0.001$ (right frame). In each frame the thick line is the solution to (52) and the thin line the corresponding scale free solution. The dotted lines correspond to the initial stage of $D^2(t) - d_\circ^2$, given by (17).

6 Conclusions

We have predicted the mean-square separation between two particles as a function of time in isotropic, stationary turbulence. The model assumptions are:

1. Corrsin's independence hypothesis (Shlien & Corrsin 1974, McComb 1990).
2. The probability density function for the one-particle excursions and the (generalized) distance-neighbour function for the two-particle (excess) separation are assumed to be of the same form, either that of Richardson or that of Batchelor (Richardson 1926, Batchelor 1952).
3. The time-lag dependence of the energy spectrum $E(k, \tau)$ is assumed to be exponential, just as spontaneous radioactive decay.
4. The calculations are taken to the limit where the outer scale ℓ is infinitely large.

There is a general agreement that the first assumption is at best a useful approximation for connecting Lagrangian and Eulerian statistics. However, we suggest that it is in itself interesting to study its application to two-particle dispersion.

The second assumption about the probability density functions represents two limits, the Richardson case and the Batchelor case. As discussed by Batchelor (1953), the excursion of a single particle is from an experimental and a theoretical point of view most certainly Gaussian. Batchelor (1952) argued that for two particles the (excess) separation $|\boldsymbol{d}(t) - \boldsymbol{d}_\circ|$ should also be Gaussian. It seems, however, that this is not the case when the two particles do not move independently of one another. Rather, $|\boldsymbol{d}(t) - \boldsymbol{d}_\circ|$ has a probability density function which, compared to a Gaussian, is more pointed at small values and has longer tails for large values. This was predicted by Richardson (1926) and confirmed experimentally by Ott & Mann (2000). On the other hand, when the positions of the two particles are taken at two different times t_1 and t_2, as our model requires, their asynchronous distance $|\boldsymbol{r}_2(t_2) - \boldsymbol{r}_1(t_1) - \boldsymbol{d}_\circ|$ will have a probability density approaching a Gaussian as $|t_2 - t_1|$ increases. This will also be the case if one of the times t_1 or t_2 is kept fixed. Instead of assuming a form which gradually changes from the Richardson form, when t_1 and t_2 are almost equal, to a Gaussian form, when t_1 and t_2 are far from that, we have carried out the calculations in two limiting cases where the probability density functions for single-particle dispersion as well as for $|\boldsymbol{r}_2(t_2) - \boldsymbol{r}_1(t_1) - \boldsymbol{d}_\circ|$ are either of the Richardson form or of the Batchelor form. As it turns out, the predicted difference between the corresponding two values of the Richardson-Obukhov constant C is, according to Fig. 5, about 30%.

Concerning the third assumption, there is no a priori reason forcing us to assume exponential decay of eddies, but it is justified by noting that in a turbulent fluid it just means that the number of eddies of a given size, which are being removed in a short interval of time, is proportional with this number, with a factor of proportionality that is a function of the eddy size only. The removal process can be either destruction by smaller eddies or advection by larger eddies. This analysis shows that the destruction is more important than the advection which, in the limit of large times, becomes insignificant. The explanation is that advection by eddies larger than the distance between the two particles will in most cases sweep them away together.

The fourth assumption shows that in this limit there is no final stage where the two particles move independent of each other. The initial stage goes over into the intermediate stage with a constant value of $C = D_E^2(t)/(\varepsilon t^3)$ and stays there. Checking the calculations with a wave-number spectrum with a finite spatial length scale, we used the von Kármán spectrum and discovered that the intermediate range with $C = D_E^2(t)/(\varepsilon t^3)$ does not seem to exist. The initial stage, where $D_E^2(t)$ is proportional to t^2, goes over into the final stage, where the two particles move independently of each other and where $D_E^2(t)$ is proportional to t, without going through the intermediate t^3-range. This result is contradicted by the experiments by Ott & Mann (2000) where such a range has been identified with $C = 0.5 \pm 0.2$. A similar result was obtained by Ishihara & Kaneda (2002) who, applying direct numerical simulation (DNS), found that C is about 0.7. As mentioned earlier, Ishihara & Kaneda (2002) also applied a closure similar to that presented here. Their equation is equivalent to (84) (Gaussian distance-neighbour function) except that it is presented on integral form and without eddy removal by decay and advection ($\mu_a = \mu_d = 0$). Solving their equation by means of Taylor expansion in what here corresponds to $(\theta_2 - \theta_1)\mathcal{T}$ as described by Kaneda et al. (1999), they found a value of C very close to the value they obtained by DNS. They also solved their equation by using the Lagrangian renormalized approximation (LRA) (Kaneda 1981) and found the slightly larger C-value 1.3. The results by Yeung & Borgas (2004) are not inconsistent with these findings, but as pointed out by Borgas & Yeung (2004), not even at the largest value of $R_\lambda = 230$ there seems to be a significant range with Richardson-Obukhov similarity. In their model for two-particle dispersion in kinematic simulations of turbulentlike, two-dimensional flows Fung & Vassilicos (1998) consider two forms of *unsteadiness*, algebraic and geometric. In this interesting study they identify regions where the two particles stay close to one another (eddy regions and streaming regions) and regions with high separation rates (straining regions). They obtain a C-value of about 0.01. Their model in the algebraic mode has some similarity to the present model in that the unsteadiness "thaws" the turbulence so that eddy decay in terms of a wave-number eddy turnover-time is included. Their dimensionless unsteadiness parameter λ is proportional to our eddy-decay parameter μ_d. It seem reasonable to assume that the ratio is $\mu_d/\lambda = \sqrt{\alpha}/(2\pi) \approx 0.2$. Fung & Vassilicos (1998) find that $\lambda \simeq 0.5$ most convincingly produces a four-decade t^3 regime for $D_E^2(t)$. However, this value will, according to Fig. 5, in our model result in $C \simeq 3$, in striking disagreement with the result by Fung & Vassilicos (1998).

We conclude that Corrsin's independence hypothesis in this model with $R_\lambda = \infty$ leads to

- the observation that the choice of distance neighbour function significantly influences the prediction of the Richardson-Obukhov constant C as a function of the dimensionless decay parameter μ_d for eddy destruction. This constant is about 30% larger in the Batchelor case than than in the Richardson case.
- a maximum value of C equal to 6.3, corresponding to frozen turbulence ($\mu_d = 0$), and
- the observation that the intermediate t^3-range, predicted by Richardson (1926) and observed by Ott & Mann (2000), apparently does not exist for turbulence with a finite length scale. The causes of this discrepancy call for further investigations. One cause could be related to the application of Corrsin's independence hypothesis in the present model.

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