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Thermal sine-Gordon system in the presence of different types of dissipation

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The effects of thermal fluctuations on solitons and phonons of the sine-Gordon system are investigated in the presence of a $\alpha\phi_t-\beta\phi_{xxt}$ dissipation. The analysis requires the assumption of a more general autocorrelation function for the noise than the one used in previous works. We verify that this leads to the correct results for the average kinetic energies of solitons and phonons in the system. We also evaluate the linewidth for a Josephson oscillator in the presence of both α and β dissipation, and lastly we briefly discuss the extension of the theory to more general dissipative terms.

I. INTRODUCTION

The effects of thermal fluctuations both on solitons and phonons of the sine-Gordon system are relevant in the description of many physical systems in contact with a heat reservoir.[1–8] In the context of Josephson junctions, for example, it was shown that thermal fluctuations in the fluxon velocity are directly related to the appearance of a very narrow oscillator linewidth.[4,5] The coupling of the sine-Gordon system to the heat reservoir can be schematized as shown in Fig. 1, where A represents an ordered flow of energy from the system to the heat reservoir (due to dissipation) and B represents a disordered flow of energy from the reservoir to the system (thermal fluctuations). This means that the loss term in the sine-Gordon equation is intrinsically connected to a noise term (dependent on temperature) representing the effect of the reservoir on the system. This scheme leads to the following thermal sine-Gordon (TSG) equation:

$$\phi_{xx}-\phi_{tt}-\sin\phi=\eta+\Gamma(\phi)+n(x,t) , \quad (1)$$

where $\Gamma(\phi)$ represents a generic dissipation and $n(x,t)$ is the stochastic force associated with the loss. [In Eq. (1) a bias term η which represents ordered energy input into the system, suitable for many practical applications, is also included.] In recent papers the TSG equation was studied by assuming a loss term proportional to ϕ_t [i.e., $\Gamma(\phi)=\alpha\phi_t$ in Eq. (1)], and an autocorrelation function for the noise given by

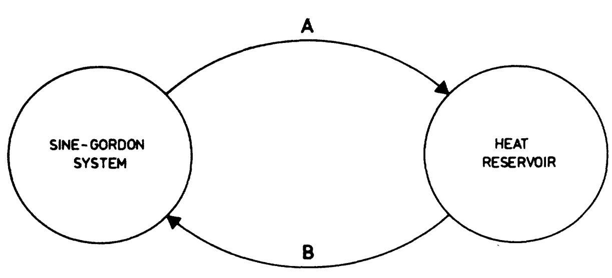

FIG. 1. Schematized representation of the thermal sine-Gordon system. A represents ordered flow of energy from the sine-Gordon system to the heat reservoir and B represents disordered flow from the reservoir to the system.

$$\langle n(x,t)n(x',t')\rangle=16\alpha\frac{k_BT}{E_0}\delta(x-x')\delta(t-t') . \quad (2)$$

In Eq. (2) $\langle \cdots \rangle$ means ensemble average, E_0 is the rest energy of the soliton (used to fix the scale in energy), k_B is the Boltzmann constant, and T is the temperature. The prefactor in Eq. (2) was determined by applying the fluctuation-dissipation theorem to a soliton with small velocity.[6] Among other results, it was shown that as a consequence of the thermal reservoir, solitons have an average energy of $\frac{1}{2}k_BT$ per mode.[6,7] This analysis was also applied to a Josephson junction, leading to an expression for the oscillator linewidth in agreement with experimental measurements.[5] In the context of the Josephson junction however, besides a loss term proportional to ϕ_t, it is of interest to include a loss proportional to ϕ_{xxt} which is due to normal surface currents through the junction. This kind of dissipation is found to be responsible for several interesting phenomena such as bunching of fluxons,[9] and appearance of strong deformations on the fluxon tail.[10] The aim of the present paper is to extend the analysis in Ref. 6 to include this ϕ_{xxt} dissipative term. More precisely we will consider Γ in Eq. (1) to be given by

$$\Gamma(\phi)\equiv\alpha\phi_t-\beta\phi_{xxt} \quad \alpha,\beta\in R^+ \quad (3)$$

and assume for the noise the following autocorrelation function:

$$\langle n(x,t)n(x',t')\rangle=16\frac{k_BT}{E_0}\delta(t-t')\times\left[\alpha-\beta\frac{\partial^2}{\partial x^2}\right]\delta(x-x') . \quad (4)$$

The effects of the noise term (4) in Eq. (1), will be then studied in the cases of pure soliton and pure fluxon motion, respectively, in Secs. II and III. As a result we find that the "thermal" solitons and phonons will still have an average energy of, respectively, $\frac{1}{2}k_BT$ and k_BT per mode; however, the presence of the β term in (3) will decrease the diffusion constant of a soliton by a factor $\alpha/(\alpha+\beta/3)$. In Sec. III we also relate the above results to the Josephson junctions by showing that there will be no change in the linewidth expression given in Ref. 5 due to the presence of the β dissipation. Finally in Sec. IV we give a short summary of the main results, including a brief discussion on the generalization of the above-mentioned results to higher-order dissipative terms of the type $\sum_{i=1}^{m}\alpha_i D_x^{2i}\phi_t$ with $\alpha_i\in R$, $D_x\equiv\partial/\partial x$, and $m\in N$.

II. THERMAL PHONONS

In this section we consider the TSG equation

$$\phi_{xx}-\phi_{tt}-\sin\phi=\eta+\alpha\phi_t-\beta\phi_{xxt}+n(x,t) \qquad (5)$$

in the small-amplitude limit and with no solitons in the system. Phonon modes ψ are seen as small oscillations around the ground state $\phi_0=-\sin^{-1}\eta$ satisfying the boundary conditions

$$\psi_x(0,t)=\psi_x(L,t)=0\ . \qquad (6)$$

The field $\phi(x,t)$ in the small-amplitude limit can be written as

$$\phi(x,t)=-\sin^{-1}(\eta)+\psi(x,t) \text{ with } \|\psi\|\ll 1\ . \qquad (7)$$

By substituting (7) in (5) we get the following stochastic equation for thermal phonons:

$$\psi_{xx}=\psi_{tt}-(1-\eta^2)^{1/2}\psi+\alpha\psi_t-\beta\psi_{xxt}+n(x,t)\ . \qquad (8)$$

When $\alpha=0$, $\beta=0$, and $n(x,t)\equiv 0$, these phonons are just classical Klein-Gordon modes with energy given by

$$H_{\text{ph}}=\frac{E_0}{16}\int_0^L dx[\psi_x^2+\psi_t^2+\psi^2(1-\eta^2)^{1/2}] \qquad (9)$$

(here L is the length of the system). The general solution of Eq. (8) can then be expressed in terms of the complete set ϕ_n of orthonormal Klein-Gordon modes as

$$\psi(x,t)=\sum_n A_n(t)\phi_n(x)=\sqrt{2/L}\sum_n A_n(t)\cos(k_n x) \qquad (10)$$

with $k_n=n\pi/L$ and $\sqrt{2/L}$ just a normalization factor. Inserting (10) in (8) and projecting the resulting equation along the unperturbed eigenstate we get

$$A_{n,tt}+(\alpha+\beta k_n^2)A_{n,t}+\omega_n^2 A_n=\varepsilon_n(t)\ , \qquad (11)$$

where

$$\varepsilon_n(t)=\sqrt{2/L}\int_0^L n(x,t)\cos(k_n x)dx \qquad (12)$$

and

$$\omega_n^2=(1-\eta^2)^{1/2}+k_n^2\ . \qquad (13)$$

By using (4) we obtain for the autocorrelation function and for the power spectrum of the normal process $\varepsilon_n(t)$ the following expressions:

$$R_{\varepsilon_n}(t-t')\equiv\langle\varepsilon_n(t')\varepsilon_n(t)\rangle=16(\alpha+\beta k_n^2)\frac{k_BT}{E_0}\delta(t-t')\ , \qquad (14)$$

$$S_{\varepsilon_n}(\omega)=16(\alpha+\beta k_n^2)\frac{k_BT}{E_0}\ . \qquad (15)$$

By identifying $\alpha+\beta k_n^2$ with α we see that Eqs. (11), (14), and (15) coincide, respectively, with Eqs. (3.9), (2.12), and (2.13) of Ref. 6. One can follow the same analysis of Ref. 6 to show that the average energy per phonon mode is

$$\langle H_n\rangle=k_BT\ . \qquad (16)$$

[This easily follows by solving by harmonic analysis Eq. (11) and by using Eq. (9).] It is worth remarking that this result does not depend on the particular boundary conditions used, nor on the smallness requirements of α, β, and η.

III. THERMAL SOLITONS

In this section we concentrate on the effect of the α, β, η, and $n(x,t)$ terms in Eq. (5) on an unperturbed sine-Gordon soliton

$$\phi=4\tan^{-1}\exp[\gamma(u)(x-ut)]\ ,\quad \gamma(u)=(1-u^2)^{-1/2}\ . \qquad (17)$$

Note that the η in (5) shifts the ground state from 0 to $-\sin^{-1}\eta$; therefore a soliton in our system should be seen as a 2π kink from $-\sin^{-1}\eta$ to $2\pi-\sin^{-1}\eta$.

An equation of motion for the perturbed soliton can be easily obtained by defining the momentum

$$P=-\tfrac{1}{8}\int_{-\infty}^{+\infty}\phi_x\phi_t dx \qquad (18)$$

and differentiating it with respect to time, this giving

$$\frac{dP}{dt}=\frac{\pi\eta}{4}+\frac{\alpha}{8}\int\phi_x\phi_t dx+\frac{\beta}{8}\int\phi_{xx}\phi_{xt}dx+\varepsilon(t)\ , \qquad (19)$$

where

$$\varepsilon(t)=-\tfrac{1}{8}\int_{-\infty}^{+\infty}\phi_x n(x,t)dx\ . \qquad (20)$$

With neglect of thermal noise, Eq. (19) has stationary 2π-kink ϕ^u solutions moving with the power-balance velocity u_0, for small perturbations, satisfying[11]

$$\frac{\pi\eta}{4}(1-u_0^2)^{3/2}-\alpha u_0(1-u_0^2)-\frac{\beta}{3}u_0=0 \qquad (21)$$

and with momentum

$$P_0=u_0\gamma(u_0)\ . \qquad (22)$$

In the stationary case, the integrals in Eq. (19) can be written as

$$-u\left[\frac{\alpha}{8}\int(\phi_x^u)^2dx+\frac{\beta}{8}\int(\phi_{xx}^u)^2dx\right]=-u\left[\alpha\gamma_1(u)+\frac{\beta}{3}\gamma_2^3(u)\right]\equiv-\frac{\pi\eta(u)}{4}\ , \qquad (23)$$

where we used Eq. (18) together with $\phi_t^u = -u\phi_x^u$. Equation (22) defines the functions $\gamma_1(u)$ and $\gamma_2(u)$ which for small perturbations (or small velocities) reduce to the usual Lorentz factor in (17).

By inserting Eq. (23) into Eq. (19) we get for the momentum the following Langevin equation for P:

$$\frac{dP}{dt} = \frac{\pi\eta}{4} - \pi\frac{\eta(u)}{4} + \varepsilon(t) . \tag{24}$$

The noise term $\varepsilon(t)$ in Eq. (24) introduces fluctuations in the momentum and, from (22), in the velocity of the kink according to

$$\frac{\partial p}{\partial u}\frac{\partial \Delta u}{\partial t} = -\frac{\pi}{4}\frac{\partial \eta}{\partial u}\Delta u + \varepsilon(t) , \tag{25}$$

where Δu measures the deviation of the 2π-kink velocity from the power-balance value (21). The autocorrelation function and the power spectrum of the process $\varepsilon(t)$ in (25) are then easily evaluated by means of Eqs. (4) and (21): we write

$$R_\varepsilon(t-t') = \frac{\pi}{2u}\frac{k_B T}{E_0}\eta(u)\delta(t-t') ,$$
$$S_\varepsilon(\omega) = \frac{\pi}{2u}\frac{k_B T}{E_0}\eta(u) . \tag{26}$$

For small velocities (i.e., $\eta \approx 0$) we have from (23),

$$\eta(u) \simeq \frac{4u}{\pi}\left[\alpha + \frac{\beta}{3}\right] . \tag{27}$$

Then Eq. (25) reduces to a Langevin equation for u,

$$\frac{du}{dt} = -\left[\alpha + \frac{\beta}{3}\right]u + \varepsilon(t) . \tag{28}$$

By using (26) and (27), Eq. (28) is easily integrated by harmonic analysis, this giving

$$S_u(\omega) = \frac{S_\varepsilon(\omega)}{\omega^2 + \left[\alpha + \frac{\beta}{3}\right]^2} , \tag{29}$$

from which it follows that

$$\langle u^2 \rangle = R_u(0) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S_u(\omega) = \frac{k_B T}{E_0} . \tag{30}$$

The time average of the kinetic energy in the Brownian motion of the 2π kink is then evaluated as

$$\langle E_{\text{sol}} \rangle = \tfrac{1}{2}E_0 \langle u^2 \rangle = \tfrac{1}{2}k_B T . \tag{31}$$

From Eq. (29) a diffusion constant D for the 2π-kink motion can also be derived as

$$D = \frac{1}{E_0}\frac{k_B T}{\left[\alpha + \frac{\beta}{3}\right]} , \tag{32}$$

which is just the usual relation reported in Ref. 6 with α identified with $\alpha + \beta/3$. The effect of the β dissipation on the 2π-kink motion is then to decrease its diffusion constant as one would have expected. We finally close this section by showing that the linewidth of a Josephson oscillator with $\beta\phi_{xxt}$ damping will still be given by the same expression reported in Ref. 6. To this end we return to Eq. (24) (which is valid for all u) and rewrite it as

$$\frac{d}{dt}\Delta u + \frac{\pi}{4}\frac{\partial \eta}{\partial p}\Delta u = \frac{\partial u}{\partial p}\varepsilon(t) , \tag{33}$$

this leading to the following expression for the power spectrum of Δu:

$$S_{\Delta u}(\omega) = \left[\frac{\partial u}{\partial \eta}\right]^2 \left[\frac{\partial \eta}{\partial p}\right]^2 \frac{S_\varepsilon(\omega)}{\omega^2 + \left[\frac{\pi}{4}\frac{\partial \eta}{\partial p}\right]^2} , \tag{34}$$

and by performing the same analysis of Ref. 5, one gets the following linewidth expression:

$$\Delta\nu = \frac{\pi k_B T}{\phi_0^2}\frac{R_D^2}{R_S} , \tag{35}$$

where $R_D \propto \partial u/\partial p$, $R_S \propto u/p$, and ϕ_0 is the flux quantum $h/2e$ (for details we refer to Ref. 5).

IV. CONCLUSIONS

It has been shown that the effect of a thermal reservoir on solitons and phonons in the sine-Gordon system in the presence of $\alpha\phi_t - \beta\phi_{xxt}$ dissipations gives an average kinetic energy of, respectively, $\frac{1}{2}k_B T$ and by $k_B T$ per mode. The presence of the β term on the soliton is to decrease its diffusion constant. Furthermore, we showed that the above analysis in the case of the Josephson oscillators leads to the same linewidth expression as obtained in Ref. 5.

Finally, in closing the paper, we wish to point out that the above analysis can be generalized to dissipations of type $\hat{\alpha}\phi_t$ with $\hat{\alpha}$ given by the following differential operator:

$$\hat{\alpha} = \sum_{n=0}^{N} (-1)^n \alpha_n D_x^{2n} \quad \text{where } D_x \equiv \frac{\partial}{\partial x} . \tag{36}$$

In this case we need to replace the autocorrelation function (4) for the noise $n(x,t)$ by the following expression:

$$\langle n(x,t)n(x',t') \rangle = 16\frac{k_B T}{E_0}\delta(t-t')\hat{\alpha}\delta(x-x') \tag{37}$$

in order to get the correct results. Indeed one easily sees that (36) and (37) will give little changes in the above results except for the substitutions of

$$(\alpha + \beta k_n^2) \rightarrow \alpha_0 + \sum_i \alpha_i k_n^{2i} \tag{38}$$

in the phonon case and

$$\left[\alpha\gamma_1(u)+\frac{\beta}{3}\gamma_2^3(u)\right]\rightarrow\alpha_0\gamma_1(u)+\frac{\alpha_1}{3}\gamma_2^3(u)+\tfrac{7}{15}\alpha_2\gamma_3^5(u)+\tfrac{31}{21}\alpha_3\gamma_4^7(u)+\cdots$$

in the soliton case. The linewidth expression Eq. (35) remains the same.

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