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Poulsen, Niels Kjølstad

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Discrete Time Control
of
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Niels Kjølstad Poulsen
Department of Informatics and Mathematical Modelling
The Technical University of Denmark

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Abstract

This report is meant as a supplement or an extension to the material used in connection to or after the courses **Stochastic Adaptive Control (02421)** and **Static and Dynamic Optimization (02711)** given at the department **Department of Informatics and Mathematical Modelling, The Technical University of Denmark.**

The focus in this paper is control of a continuous time system by means of a digital control. In this context the control signal can only change at sample instants and is constant between samples. The cost function do include the variations of output between samples.

1 Introduction

In the standard discrete time LQ (and H_2) control (see Appendix A and B) of dynamic system we only consider the state vector (state variable) and the control actions at the sampling instants. In the continuous time version of the LQ problem the state vector and the control are considered at all times. In this report we will consider the states at all times, but only consider control actions which can change at the sampling instants (and are constant between samples i.e. using a zero order hold network).

The results presented here can to a certain extend be found in [2], but is here presented in the same (standard) settings as in [3] or in [1]. In this paper we will use the following definitions

$$|x|_P^2 = x^T P x \qquad |x|^2 = x^T x$$

where x (here) is any vector.

Consider the problem of controlling a continuous time LTI system

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \qquad\qquad x(0) = \underline{x}(0) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

such that the objective function

$$J = |x(T)|_P^2 + \int_0^T |y(t)|_V^2 \ dt \tag{2}$$

is minimized. That is to determine an input signal such that the system is taken from its initial state and along a trajectory such that the cost function is minimized. This is (a finite horizon formulation of) the H_2 problem. Notice, the cost function in (2) can also be formulated as

$$J = |x(T)|_P^2 + \int_0^T \left| \left[\begin{array}{c} x(t) \\ u(t) \end{array} \right] \right|_W^2 \ dt$$

where

$$W = \left[\begin{array}{cc} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{array} \right] = \left[\begin{array}{c} C^T \\ D^T \end{array} \right] V \left[\begin{array}{cc} C & D \end{array} \right]$$

This is the standard LQ formulation (with cross coupling between state and control actions in the cost function). The control objective is then related to the standard LQ problem dealt with in Appendix A and B. It is often written in the more recognizable way

$$J = x(T)^T P x(T) + \int_0^T x_t^T Q x_t + u_t^T R u_t + 2 x_t^T \mathbb{S} u_t \ dt$$

In discrete time control (and in digital control) the control action is normally assumed to be constant between samples, i.e.

$$u(t) = u_i \qquad \text{for} \qquad ih < t \leq ih + h$$

where h is the (constant) length of the sampling period. We assume for the sake of simplicity that the horizon is a multiple of the sampling period, i.e. $T = Nh$.

For this problem the Bellman equation becomes:

$$V_i(x_i) = \min_{u_i} \left[\int_{ih}^{ih+h} |y(t)|_V^2 \ dt + V_{i+1}(x_{i+1}) \right] \tag{3}$$

$$V_N(x_N) = |x_N|_P^2$$

where the Bellman function, $V_i(x_i)$, is the optimal cost to go. By definition (notation) $x_i = x(ih)$ and $x_N = x(T)$. In a local sampling period, $ih \leq t \leq ih + h$, we can use the local time $s = t - ih$ where $0 \leq s \leq h$.

If the control action is constant between sample instants the solution to (1) is well known and is

$$\begin{aligned} x(t) &= e^{As} x_i + \int_0^s e^{A(s-\tau)} B \ d\tau \ u_i \\ &= \Phi_s x_i + \Gamma_s u_i \end{aligned}$$

where

$$\Phi_s = e^{As} \qquad \Gamma_s = \int_0^s e^{A(s-\tau)} B \; d\tau = \int_0^s e^{A\tau} B \; d\tau$$

Now, it is easy to see that

$$\begin{aligned} |y(t)|_V^2 &= [Cx(t) + Du(t)]^T V [Cx(t) + Du(t)] \\ &= \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} \Phi_s^T C^T V C \Phi_s & \Phi_s^T C^T V (D + C\Gamma_s) \\ (D + C\Gamma_s)^T V C \Phi_s & (D + C\Gamma_s)^T V (D + C\Gamma_s) \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} \end{aligned}$$

If we furthermore define

$$\begin{aligned} Q_1 &= \int_0^h \Phi_s^T C^T V C \Phi_s \; ds \\ Q_{12} &= \int_0^h \Phi_s^T C^T V (D + C\Gamma_s) \; ds \\ Q_2 &= \int_0^h (D + C\Gamma_s)^T V (D + C\Gamma_s) ds \end{aligned} \tag{4}$$

then

$$\int_{ih}^{ih+h} |y(t)|_V^2 \; dt = \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

Let

$$V_i(x_i) = |x_i|_{S_i}^2 = x_i^T S_i x_i$$

be a candidate function. With the chosen candidate function the inner part of the minimization in (3) can be written as

$$I = \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q_1 + \Phi_h^T S_{i+1} \Phi_h & Q_{12} + \Phi_h^T S_{i+1} \Gamma_h \\ Q_{12}^T + \Gamma_h^T S_{i+1} \Phi_h & Q_2 + \Gamma_h^T S_{i+1} \Gamma_h \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} \tag{5}$$

which (see e.g. Appendix C) has its minimum for

$$\boxed{u_i = -\left[Q_2 + \Gamma_h^T S_{i+1} \Gamma_h\right]^{-1} \left[Q_{12}^T + \Gamma_h^T S_{i+1} \Phi_h\right] x_i}$$

If we use methods in Appendix C, we notice the candidate function indeed is a solution to (1) and S_i is given by the recursion

$$\begin{gathered} S_i = Q_1 + \Phi_h^T S_{i+1} \Phi_h - \left[Q_{12} + \Phi_h^T S_{i+1} \Gamma_h\right] \left[Q_2 + \Gamma_h^T S_{i+1} \Gamma_h\right]^{-1} \left[Q_{12}^T + \Gamma_h^T S_{i+1} \Phi_h\right] \\ S_N = P \end{gathered} \tag{6}$$

We also notice that the problem is closely related to a standard discrete LQ problem, just the weight matrices (as well as the system matrices) are transformed in the sampling process.

If Q_2 is invertible then the recursion in (6) can be reduced significantly if we introduce a new decision variable, v_i, through

$$u_i = v_i - Q_2^{-1} Q_{12}^T x_i$$

In that case the minimization in (5) becomes

$$I = \begin{bmatrix} x_i^T & v_i^T \end{bmatrix} \begin{bmatrix} \bar{Q}_1 + \bar{\Phi}_h^T S_{i+1} \bar{\Phi}_h & \bar{\Phi}_h^T S_{i+1} \Gamma_h \\ \Gamma_h^T S_{i+1} \bar{\Phi}_h & Q_2 + \Gamma_h^T S_{i+1} \Gamma_h \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

where

$$\bar{Q}_1 = Q_1 - Q_{12} Q_2^{-1} Q_{12}^T \qquad \bar{\Phi} = \Phi - \Gamma Q_2^{-1} Q_{12}^T$$

The solution to this problem is the solution to the standard problem (except off course the substitution of variable)

$$v_i = -\bar{L}_i x_i \qquad \bar{L}_i = \left[Q_2 + \Gamma^T S_{i+1} \Gamma\right]^{-1} \Gamma^T S_{i+1} \bar{\Phi}$$

where:

$$S_i = \bar{Q}_1 + \bar{\Phi}_h^T S_{i+1} \bar{\Phi}_h - \bar{\Phi}_h^T S_{i+1} \Gamma_h \left[Q_2 + \Gamma_h^T S_{i+1} \Gamma_h\right]^{-1} \Gamma_h^T S_{i+1} \bar{\Phi}_h \qquad S_N = P$$

The solution (to the original problem) can be written as

$$\boxed{u_i = -\left[\bar{L}_i + Q_2^{-1} Q_{12}^T\right] x_i \qquad \bar{L}_i = \left[Q_2 + \Gamma^T S_{i+1} \Gamma\right]^{-1} \Gamma^T S_{i+1} \bar{\Phi}}$$

where S is given by the Riccati equation

$$S_i = \bar{Q}_1 + \bar{\Phi}_h^T S_{i+1} \bar{\Phi}_h - \bar{\Phi}_h^T S_{i+1} \Gamma_h \left[Q_2 + \Gamma_h^T S_{i+1} \Gamma_h\right]^{-1} \Gamma_h^T S_{i+1} \bar{\Phi}_h \qquad S_N = P$$

or in a shorter form

$$\boxed{S_i = \bar{\Phi}_h^T S_{i+1} (\bar{\Phi} - \Gamma \bar{L}_i) + \bar{Q}_1 \qquad S_N = P}$$

References

[1] Bryson and Ho. *Applied Optimal Control*. Hemisphere, 1975.

[2] T. Chen and B. Francis. *Optimal Sampled-Data Control Systems*. Communications and Control Engineering Series. Springer-Verlag New York Inc., 1995.

[3] F. L. Lewis. *Applied Optimal Control and Estimation*. Prentice Hall, 1992. ISBN 0-13-040361-X.

[4] N. K. Poulsen. The matrix exponential, dynamic systems and control. Technical report, Informatics and Mathematical Modelling, Technical University of Denmark, DTU, Richard Petersens Plads, Building 321, DK-2800 Kgs. Lyngby, 2004.

A The Standard DLQ Control Problem

In this appendix we will consider the standard discrete time LQ control problem. Consider the problem of controlling a dynamic system in discrete time

$$x_{i+1} = \Phi x_i + \Gamma u_i \qquad x_0 = \underline{x}_0 \tag{7}$$

such that the (standard LQ) cost function

$$J = x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \tag{8}$$

is minimized.

The Bellman equation will in case be

$$V_i(x_i) = \min_{u_i} \left[x_i^T Q x_i + u_i^T R u_i + V_{i+1}(x_{i+1}) \right] \tag{9}$$

with the end point constraints

$$V_N(x_N) = x_N^T P x_N$$

If we test the candidate function

$$V_i(x_i) = x_i^T S_i x_i$$

then the inner part of the minimization in (9) will be

$$I = \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q + \Phi^T S_{i+1} \Phi & \Phi^T S_{i+1} \Gamma \\ \Gamma^T S_{i+1} \Phi & R + \Gamma^T S_{i+1} \Gamma \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

The minimum for the this function is according to Appendix C given by

$$u_i = -L_i x_i \qquad L_i = \left[R + \Gamma^T S_{i+1} \Gamma \right]^{-1} \Gamma^T S_{i+1} \Phi$$

and the candidate function is in fact a solution to the Bellman equation in (9) if

$$S_i = Q + \Phi^T S_{i+1} \Phi - \Phi^T S_{i+1} \Gamma \left[R + \Gamma^T S_{i+1} \Gamma \right]^{-1} \Gamma^T S_{i+1} \Phi \qquad S_N = P$$

If the gain, L_i, is used in the recursion for S_i

$$S_i = \Phi_h^T S_{i+1} (\Phi - \Gamma L_i) \qquad S_N = P$$

As a simple implication from the proof we that

$$V(0(x_0) = J^\star = x_0^T S_0 x_0$$

which among other things is useful in connection to a interpretation of S.

B DLQ and cross terms

In order to connect the (very) related LQ formulation and H_2 formulation we have to augment the standard problem with cross terms in the cost function. Assume a discrete time (LTI) system is given as in (7) and the cost function (instead of (8)) is:

$$J = x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + 2 x_i^T \mathbb{S} u_i$$

or

$$J = x_N^T P x_N + \sum_{i=0}^{N-1} \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

then the situation becomes a bit more complicated. The cross terms especially occurs if the control problem is formulated as a problem in which (the square of) an output signal

$$y_i = Cx_i + Du_i$$

is minimized. In that case

$$\begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}$$

The Bellman equation becomes in the special case

$$V_i(x_i) = \min_{u_i} \left[\begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} + V_{i+1}(x_{i+1}) \right]$$

$$V_N(x_N) = x_N^T P x_N$$

and again we will try the following candidate function

$$V_i(x_i) = x_i^T S_i x_i$$

This can be solved head on or by transforming the problem into the standard one. If R is invertible then we can introduce a new decision variable, v_i, given by:

$$u_i = v_i - R^{-1}\mathbb{S}^T x_i$$

The instaneous loss term (first term in the Bellman equation) can be expressed as:

$$\begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} = x_i^T \bar{Q} x_i + v_i^T R v_i$$

where

$$\bar{Q} = Q - \mathbb{S} R^{-1} \mathbb{S}^T$$

In similar way we find for the dynamics

$$\begin{aligned} x_{i+1} &= \Phi x_i + \Gamma u_i \\ &= (\Phi - \Gamma R^{-1}\mathbb{S}) x_i + \Gamma v_i \\ &= \bar{\Phi} x_i + \Gamma v_i \end{aligned}$$

where

$$\bar{\Phi} = \Phi - \Gamma R^{-1} \mathbb{S}$$

For the future cost to go (the second term in the Bellman equation) we have:

$$V_{i+1}(x_{i+1}) = x_{i+1}^T S_{i+1} x_{i+1} = (\bar{\Phi} x_i + \Gamma v_i)^T S_{i+1} (\bar{\Phi} x_i + \Gamma v_i)$$

We have now transformed the problem to the standard form and the inner minimization in the Bellman equation

$$V_i(x_i) = \min_{u_i} \left[x_i^T Q x_i + u_i^T R u_i + V_{i+1}(x_{i+1}) \right]$$

is then simply:

$$I = \begin{bmatrix} x_i^T & v_i^T \end{bmatrix} \begin{bmatrix} \bar{Q} + \bar{\Phi}^T S_{i+1} \bar{\Phi} & \bar{\Phi}^T S_{i+1} \Gamma \\ \Gamma^T S_{i+1} \bar{\Phi} & R + \Gamma^T S_{i+1} \Gamma \end{bmatrix} \begin{bmatrix} x_i \\ v_i \end{bmatrix}$$

with the solution

$$v_i = -\bar{L}_i x_i \qquad \bar{L}_i = \left[R + \Gamma^T S_{i+1}\Gamma\right]^{-1} \Gamma^T S_{i+1}\bar{\Phi}$$

The candidate function is a solution to the Bellman equation if

$$\begin{aligned} S_i &= \bar{Q} + \bar{\Phi}^T S_{i+1}\bar{\Phi} - \bar{\Phi}^T S_{i+1}\Gamma\left[R + \Gamma^T S_{i+1}\Gamma\right]^{-1}\Gamma^T S_{i+1}\bar{\Phi} \\ &= \bar{\Phi}^T S_{i+1}(\bar{\Phi} - \Gamma\bar{L}_i) + \bar{Q} \end{aligned} \tag{10}$$

This means that

$$u_i = -\left[\bar{L}_i + R^{-1}\mathbb{S}\right] x_i \qquad \bar{L}_i = \left[R + \Gamma^T S_{i+1}\Gamma\right]^{-1} \Gamma^T S_{i+1}\bar{\Phi}$$

If R is not invertible then we are forced to use a more direct approach which results in the following inner minimization (minimization of the inner part in the Bellman equation):

$$I = \begin{bmatrix} x_i^T & u_i^T \end{bmatrix} \begin{bmatrix} \bar{Q} + \bar{\Phi}^T S_{i+1}\bar{\Phi} & \mathbb{S} + \bar{\Phi}^T S_{i+1}\Gamma \\ \mathbb{S}^T + \Gamma^T S_{i+1}\bar{\Phi} & R + \Gamma^T S_{i+1}\Gamma \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

with the solution

$$u_i = -L_i x_i \qquad L_i = \left[R + \Gamma^T S_{i+1}\Gamma\right]^{-1}\left[\mathbb{S}^T + \Gamma^T S_{i+1}\Phi\right]$$

and a Riccati equation

$$S_i = Q + \Phi^T S_{i+1}\Phi - \left[\mathbb{S} + \Phi^T S_{i+1}\Gamma\right]\left[R + \Gamma^T S_{i+1}\Gamma\right]^{-1}\left[\mathbb{S}^T + \Gamma^T S_{i+1}\Phi\right] \tag{11}$$

Notice, that (10) is the standard Riccati equation, whereas (11) contains (directly) the cross term $\mathbb{S}$. The transformation method do require that R is invertible.

C Quadratic optimization

Consider the problem of minimizing a quadratic cost function

$$\begin{aligned} J &= \frac{1}{2}\begin{bmatrix} x^T & u^T \end{bmatrix}\begin{bmatrix} h_{11} & h_{12} \\ h_{12}^T & h_{22} \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \\ &= \frac{1}{2}x^T h_{11}x + x^T h_{12}u + \frac{1}{2}u^T h_{22}u \end{aligned}$$

It is quite elementary to find the derivative of the cost function

$$\frac{\partial}{\partial u}J = x^T h_{12} + u^T h_{22}$$

and the stationary point must fulfill

$$h_{12}^T x + h_{22}u = 0$$

The stationary point

$$\boxed{u = -h_{22}^{-1}h_{12}^T x}$$

is a minimum to the cost function if h_{22} is positive definite. Furthermore, the minimum of the cost function is quadratic in x:

$$\begin{aligned} J &= \frac{1}{2}x^T h_{11} x - x^T h_{12} h_{22}^{-1} h_{12}^T x + \frac{1}{2} x^T h_{12} h_{22}^{-1} h_{22} h_{22}^{-1} h_{12}^T x \\ &= \frac{1}{2} x^T \left(h_{11} - h_{12} h_{22}^{-1} h_{12}^T \right) x \\ &= \frac{1}{2} x^T S x \end{aligned}$$

where

$$\boxed{S = h_{11} - h_{12} h_{22}^{-1} h_{12}^T}$$

D Numerical methods

The numerically determination of the matrices in (4) is described in more details in [4].

The following Lemma can be found in e.g. [2] (page 235). Consider matrices A_{11}, A_{12} and A_{22} with adequate dimensions. Let

$$\begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} = exp\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} h \right) \tag{12}$$

Then

$$F_{11} = e^{A_{11} h} \qquad F_{22} = e^{A_{22} h}$$

and

$$\boxed{F_{12} = \int_0^h e^{A_{11}(h-s)} A_{12} e^{A_{22} s} \; ds}$$

Since the matrices are block upper triangular, we easily get

$$F_{11} = e^{A_{11} h} \quad \text{and} \quad F_{22} = e^{A_{22} h}$$

If we differentiate (12) we get

$$\frac{d}{dt} \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}$$

and

$$\frac{d}{dt} F_{12} = A_{11} F_{12} + A_{12} F_{22}$$

Using the solution for F_{22} and $F_{12}(0) = 0$ we have

$$F_{12} = \int_0^h e^{A_{11}(h-s)} A_{12} e^{A_{22} s} ds$$

As stated in he lemma.

Now focus on the determination of the matrices in (4). Let

$$\Sigma = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \quad \text{and} \quad \mathbf{Q}_c = \begin{bmatrix} C^T \\ D^T \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix}$$

Define the square matrix

$$\mathbf{A} = \left[\begin{array}{cc} A & B \\ 0 & 0 \end{array} \right]$$

Then by the Lemma

$$e^{\mathbf{A}t} = \left[\begin{array}{cc} e^{At} & \int_0^h e^{A(t-s)}ds \\ 0 & I \end{array} \right] = \left[\begin{array}{cc} e^{At} & \int_0^t e^{As}ds \\ 0 & I \end{array} \right]$$

It is straight forward to check that

$$\Sigma = \int_0^h e^{\mathbf{A}^T s} \left[\begin{array}{c} C^T \\ D^T \end{array} \right] V \left[\begin{array}{cc} C & D \end{array} \right] e^{\mathbf{A}s} \; ds = \int_0^h e^{\mathbf{A}^T s} \mathbf{Q}_c e^{\mathbf{A}s} \; ds$$

Compute the matrix

$$\left[\begin{array}{cc} F_{11} & F_{12} \\ 0 & F_{22} \end{array} \right] = exp\left(\left[\begin{array}{cc} -\mathbf{A}^T & \mathbf{Q} \\ 0 & \mathbf{A} \end{array} \right] h \right)$$

Then

$$\boxed{\Sigma = F_{22}^T F_{12}}$$

and

$$F_{22} = \left[\begin{array}{cc} \Phi & \Gamma \\ 0 & I \end{array} \right]$$

The Matlab implementation of this algorithm is listed in Appendix F as `smplq.m`.

E The Continuous Time LQ Control

In this section we will review the results given in the previous sections but in continuous time. We will start with the standard LQ problem and then in order to connect with the H_2 formulation review the LQ problem with a cross term in the cost function.

E.1 The Standard CLQ Control problem

Consider the problem of controlling a continuous time LTI system

$$\frac{d}{dt} x_t = Ax_t + Bu_t \qquad\qquad x_0 = \underline{x}_0 \tag{13}$$

such that the performance index

$$J = x_T^T P x_T + \int_0^T x_t^T Q x_t + x_t^T Q x_t \; dt$$

is minimized. The Bellman equation is for this situation

$$-\frac{\partial}{\partial t} V_t(x_t) = \min_{u_t} \left[x_t^T Q x_t + u_t^T R u_t + \frac{\partial}{\partial x} V_t(x_t) \left(Ax_t + Bu_t \right) \right]$$

with

$$V_T = x_T^T P x_T$$

as boundary condition. For the candidate function

$$V_t(x_t) = x_t^T S_t x_t$$

this (Bellman) equation becomes

$$-x_t^T \dot{S}_t x_t = \min_{u_t} \left[x_t^T Q x_t + u_t^T R u_t + 2\ x_t^T S_t A x_t + 2\ x_t^T S_t B u_t \right]$$

This is fulfilled for

$$\boxed{u_t = -R^{-1} B^T S_t x_t}$$

The candidate function is indeed a Bellman function if S_t is the solution to the Riccati equation

$$-\dot{S}_t = S_t A + A^T S_t + Q - S_t B R^{-1} B^T S_t \qquad S_T = P$$

In terms of the gain

$$L_t = R^{-1} B^T S_t$$

the Riccati equation can also be expressed as

$$\begin{aligned}
-\dot{S}_t &= S_t A + A^T S_t + Q - L_t^T R L_t \\
-\dot{S}_t &= S_t(A - BL_t) + A^T S_t + Q = (A - BL_t)^T S_t + S_t A + Q \\
-\dot{S}_t &= S_t(A - BL_t) + (A - BL_t)^T S_t + Q + L_t^T R L_t
\end{aligned}$$

It can be shown that

$$J = x_0^T S_0 x_0$$

E.2 CLQ and cross terms

Let us now focus on the problem where performance index has a cross term, i.e. where

$$J = x_T^T P x_T + \int_0^T x_t^T Q x_t + x_t^T Q x_t + 2 x_t^T \mathbb{S} u_t \ dt$$

As in the discrete time case this will typically be the case if the problem arise from a minimization of the weighted (V) square of the output

$$y_t = C x_t + D u_t$$

i.e. the H_2 problem. In that case

$$\begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} = \begin{bmatrix} C^T \\ D^T \end{bmatrix} V \begin{bmatrix} C & D \end{bmatrix}$$

The Bellman equation is now for this situation

$$-\frac{\partial}{\partial t} V_t(x_t) = \min_{u_t} \left[\begin{bmatrix} x_t^T & u_t^T \end{bmatrix} \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} + \frac{\partial}{\partial x} V_t(x_t) \left(A x_t + B u_t \right) \right] \quad (14)$$

with

$$V_T = x_T^T P x_T$$

as boundary condition. Again we can go directly for a solution, but if R is invertible, we can transform the problem to the standard form. If we use the same method as in the discrete time and introduce a new decision variable, v_t through

$$u_t = v_t - R^{-1} \mathbb{S}^T x_t$$

then instantaneous loss term is rewritten to

$$\begin{bmatrix} x_t^T & u_t^T \end{bmatrix} \begin{bmatrix} Q & \mathbb{S} \\ \mathbb{S}^T & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} = x_t^T \bar{Q} x_t + v_t^T R v_t$$

where

$$\bar{Q} = Q - \mathbb{S} R^{-1} \mathbb{S}^T$$

Furthermore the dynamics is transformed to

$$Ax_t + Bu_t = (A - BR^{-1}\mathbb{S}^T)x_t + Bv_t = \bar{A}x_t + Bv_t$$

where

$$\bar{A} = (A - BR^{-1}\mathbb{S}^T)$$

The Bellman equation is now in the newly introduced variable

$$-\frac{\partial}{\partial t} V_t(x_t) = \min_{v_t} \left[x_t^T \bar{Q} x_t + v_t^T R v_t + \frac{\partial}{\partial x} V_t(x_t) \left(\bar{A} x_t + B v_t \right) \right]$$

with

$$V_T = x_T^T P x_T$$

as boundary condition. For the candidate function

$$V_t(x_t) = x_t^T S_t x_t$$

this Bellman equations becomes

$$-x_t^T \dot{S}_t x_t = \min_{v_t} \left[x_t^T \bar{Q} x_t + v_t^T R v_t + 2\ x_t^T S_t \bar{A} x_t + 2\ x_t^T S_t B v_t \right]$$

The solution to this problem is

$$v_t = -\bar{L}_t x_t \qquad \bar{L}_t = R^{-1} B^T S_t$$

where

$$\boxed{-\dot{S}_t = S_t \bar{A} + \bar{A}^T S_t + \bar{Q} - S_t B R^{-1} B^T S_t \qquad S_T = P} \tag{15}$$

The last equation ensures that the candidate function indeed is a solution. The total solution is consequently given as

$$\boxed{u_t = -(\bar{L}_t + R^{-1}\mathbb{S}^T)x_t \qquad \bar{L}_t = R^{-1} B^T S_t}$$

or simply as

$$u_t = -R^{-1}(B^T S_t + \mathbb{S}^T)x_t$$

Notice, that (15) is the same Riccati equation that arise from the standard problem except for the transformation of A and Q. Furthermore $\bar{L}$ is the same as arise from the standard problem.

If R is not invertible then (14) must be solved directly. For the candidate function

$$V_t(x_t) = x_t^T S_t x_t$$

the Bellman equation, (14), becomes

$$-x_t^T \dot{S}_t x_t = \min_{u_t} \left[x_t^T Q x_t + u_t^T R u_t + 2 x_t^T \mathbb{S} u_t + 2\ x_t^T S_t A x_t + 2\ x_t^T S_t B u_t \right]$$

which is minimized for

$$\boxed{u_t = -R^{-1}(B^T S_t + \mathbb{S}^T)x_t}$$

where

$$\boxed{-\dot{S}_t = S_t A + A^T S_t + Q - (S_t B + \mathbb{S})R^{-1}(B^T S_t + \mathbb{S}^T) \qquad S_T = P} \tag{16}$$

It is quite easy to check that (for R being invertible) the solutions to (16) and (15) are identical.

F Code

```
function [Ad,Bd,Qd,Rd,Nd]=smplq(varargin)
% Usage: [Ad,Bd,Qd,Rd,Nd]=smplq(A,B,Q,R,N,h)
%        [Ad,Bd,Qd,Rd,Nd]=smplq(A,B,Q,R,h)

if nargin==6,
 A=varargin{1}; B=varargin{2};
  [n,m]=size(B);
 Q=varargin{3}; R=varargin{4};
 N=varargin{5}; h=varargin{6};
elseif nargin==5,
 A=varargin{1}; B=varargin{2};
  [n,m]=size(B);
 Q=varargin{3}; R=varargin{4};
 N=zeros(n,m);  h=varargin{5};
else
  disp('Wrong argument list in smplq');
  return
end

Qc=[Q N; N' R];
Ac=[A B; zeros(m,n+m)];
F=expm([-Ac' Qc; zeros(n+m,n+m) Ac]*h);
F22=F(n+m+1:end,n+m+1:end);
F12=F(1:n+m,n+m+1:end);
Q=F22'*F12;
Ad=F22(1:n,1:n);
Bd=F22(1:n,n+1:end);
Qd=Q(1:n,1:n);
Rd=Q(n+1:end, n+1:end);
Nd=Q(1:n,n+1:end);
```

```
function [Ad,Bd,Cd,Dd]=smph2(A,B,C,D,h)
% Usage: [Ad,Bd,Cd,Dd]=smph2(A,B,C,D,h)
% or                  sysd=smph2(sysc,h)

if nargin==5,
  typ=1;
elseif nargin==2,
  typ=2,
```

```
 [A,B,C,D]=sysenc(sysc);
else
 disp('Wrong argument list');
 return
end

[n,m]=size(B);
Qc=[C';D']*[C D];
Ac=[A B; zeros(m,n+m)];
F=exp([-Ac' Qc; zeros(n+m,n+m) Ac]*h);
F22=F(n+m+1:end,n+m+1:end);
F12=F(1:n+m,n+m+1:end);
Q=F22'*F12;
Ad=F22(1:n,1:n);
Bd=F22(1:n,n+1:end);
[U,S]=svd(Q);
H=sqrt(S)*U';
Cd=H(:,1:n);
Dd=H(:,n+1:end);

if typ==2,
 Ad=ss(Ad,Bd,Cd,Dd,h),
end
```