Technical University of Denmark

Reorienting a Quasi-Rigid Body Using Shape Changes

Duun, Marie Bro

Published in:
Proceedings of RoboMat 2007

Publication date:
2008

Reorienting a Quasi-Rigid Body Using Shape Changes

Marie Bro* Maria Mose†

Abstract

It is well-known that a falling cat can reorient itself merely by changing its shape. We present here a ball and stick model which captures the features necessary for a cat to turn without external torque. We calculate the reorientation of this model resulting from a cyclic sequence of shape changes, or in terms of differential geometry the geometric phase obtained by following a given closed path in shape space. We further determine via numerical implementation of this model the most efficient series of shape changes leading to a given reorientation. In other words we solve numerically a version of the isoholonomic problem.

1 The Isoholonomic Problem

The concept of geometric phase have been studied by a number of authors [1], [6], [7], [9]. We give here a brief outline of the concept, and how it should be understood in the context of control theory.

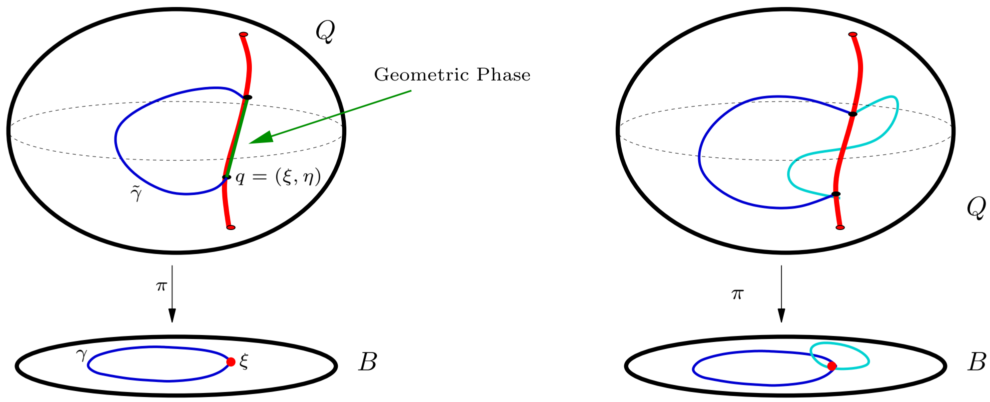
The geometric phase of a dynamical system is a certain type of phase shift. Any continuous change in the variables of the system corresponds to a curve its configuration space. If the system is a control system the variables can be split into control variables and state variables, and the configuration space can be given the structure of a fiber bundle. The set of control variables will constitute the base space and the state variables form the fibers.

One may then ask the question: 'Given a curve in base space, which curve in the configuration space will result from the changes in the control variables along this curve?'. In terms of differential geometry this question adresses the problem of lifting a curve from base space to the fiber bundle in a unique way. Lifts in general are not unique, but the so-called horisontal lifts are. They occur in the presence of a connection since a connection define the horisontal spaces. Connections often emanate from conservation laws or other constraints on the system. The phase shift in the state variables obtained from lifting a closed curve

*Department of Mathematics, Technical University of Denmark. E-mail:M.Bro@mat.dtu.dk.
†Department of Mathematics, Technical University of Denmark. E-mail:mariamose@hotmail.com.

in the base space horisontally is the geometric phase (in references [7] and [8] refered to as holonomy).

When falling, the cat controls its orientation by changing its shape. Hence, the control variables are those that describe the shape, and the state of the system is the orientation. The falling cat's problem is to obtain a certain reorientation (i.e geometric phase) as efficiently as possible. It is therefore a special case of The Isoholonomic Problem [8]: *Among all curves with a fixed geometric phase, find the loop of minimum length.*



(a) The geometric phase is the phase shift in the state variables (i.e. the fiber variables) obtained when the control variables follow a closed curve in base space. The geometric phase is determined by performing a horisontal lift of the curve to the configuration space

(b) Sketch of the isoholonomic problem. Several closed curves in the base space with the same starting point might lead to the same geometric phase. The isoholonomic problem is to determine the shortest of these.

Figure 1:

This is an optimal control problem, and for the model we propose in the next section it has the formulation:

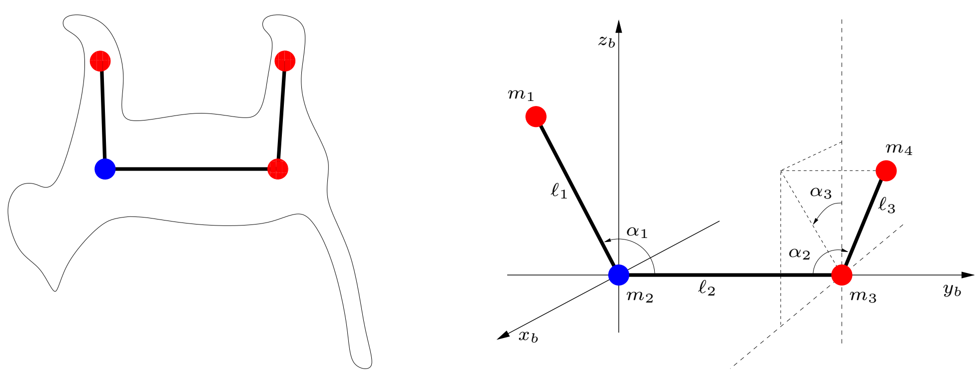
$$
\begin{aligned}
\underset{\xi(t)\in B}{\text{minimize}} \qquad & f(\xi) = \int_{t_0}^{t_1} \|\dot{\xi}(t)\| \mathrm{d}t \\
\text{such that} \qquad & \dot{\eta}(t) = \mathbf{g}\Big(\xi(t), \eta(t)\Big)\dot{\xi}(t), \\
& \xi(t_0) = \xi(t_1) = \xi_0, \\
& \eta(t_0) = \eta_0, \\
& \eta(t_1) = \eta_{end}, \qquad\qquad t \in [t_0, t_1]
\end{aligned}
\tag{1}
$$

Here, the objective funktion $f(\xi)$ is the length of the curve in base space; the first constraint ensures that the lift is horisontal; the second constraint guarantees a closed curve and that the starting point is kept fixed; and finally the third and the fourth constraint gives the correct initial and final orientations, hence the correct geometric phase.

2 The Mechanical Model

Different models of the falling cat have been proposed most of which model the cat as two coupled rigid bodies [3], [4], [5]. These models differ in the nautre of the coupling of the two bodies. Here, we propose a ball and stick model of the falling cat.

Translation plays no part in reorientation through shape changes [6], [8], and therefore we consider a quasi-rigid body, modelling the cat in a frame where origo is at the center of mass. Hence each configuration comprise a shape, described by the internal variables, and an orientation, described by the external variables. As we saw in the previous section the shape variables are the control variables and the orientation variables are the state variables. This means that when viewing the configuration space as a fiber bundle, the shape space constitutes the base space and the group of rotations in $\mathbb{R}^3$ forms the fibers. Since we are dealing with reorientation of the quasi-rigid body, it makes sense to assign $SO(3)$ as the structure group of the fiber bundle thus making it a principal bundle.



(a) A schematic drawing of a cat in relation to the proposed model.

(b) The body frame of the model. The shape is given by the three angles α_1, α_2 and α_3.

Figure 2:

The model consists of four point masses m_1, m_2, m_3, and m_4 which are connected by three massless rods ℓ_1, ℓ_2 and ℓ_3 (see Figures 2(a) and 2(b)). The rods ℓ_1 and ℓ_3 represent the forelegs and the hind legs, respectively, while ℓ_2 respresents the spine of the cat. The shape space is $S^1 \times S^2$, as indicated by figure 2(b).

Elements in shape space are denoted ξ, and elements in $SO(3)$ are denoted η. Hence, the position $\mathbf{s}_i$ of the mass m_i in the laboratory frame is given by:

$$\mathbf{s}_i(\xi, \eta) = \mathbf{R}(\eta)\big(\mathbf{b}_i(\xi) - \mathbf{b}_{CM}(\xi)\big), \quad i = 1, \dots, 4$$

where $\mathbf{R}(\eta)$ is a rotation matrix, and $\mathbf{b}_i(\xi)$ and $\mathbf{b}_{CM}(\xi)$ are the position vectors in the body frame for m_i and the center of mass respectively.

A set of ODE's for calculating the geometric phase for this model along a path in shape space can now be derived using the conservation of angular momentum. We assume that the cat is initially at rest, and hence we impose the constraint 'angular momentum = 0' on the system. This constraint is equivalent to the mechanical connection [1], [8]. The angular momentum is calculated from the position vectors, and it turns out that it can be written as the sum of a matrix $\boldsymbol{\mathcal{K}}_\eta(\xi,\eta)$ times the time derivative of the orientation, $\dot{\eta}$ and a matrix $\boldsymbol{\mathcal{K}}_\xi(\xi,\eta)$ times the time derivative of the shape $\dot{\xi}$.

$$\mathbf{L}_{CM} \;=\; \sum_{i=1}^{3}\mathbf{s}_i \times m_i\dot{\mathbf{s}}_i \;=\; \boldsymbol{\mathcal{K}}_\eta(\xi,\eta)\dot{\eta} + \boldsymbol{\mathcal{K}}_\xi(\xi,\eta)\dot{\xi} \tag{2}$$

Given the constraint on the angular momentum, and assuming that $\boldsymbol{\mathcal{K}}_\eta(\xi,\eta)$ is regular, we obtain the following relation between small changes in the orientation and small changes in the shape:

$$\mathrm{d}\eta = -\big(\boldsymbol{\mathcal{K}}_\eta(\xi,\eta)\big)^{-1}\boldsymbol{\mathcal{K}}_\xi(\xi,\eta)\ \mathrm{d}\xi \tag{3}$$

The reorientation obtained by following a given closed curve in shape space (i.e. going through a series of shape changes beginnig and ending with the same shape) can be calculated by integrating the above expression. On top of this, we want to determine an optimal series of shape changes which the (model) cat can perform in order to reorient itself as to land on its feet. This problem is adressed in the next section.

3 Implementation

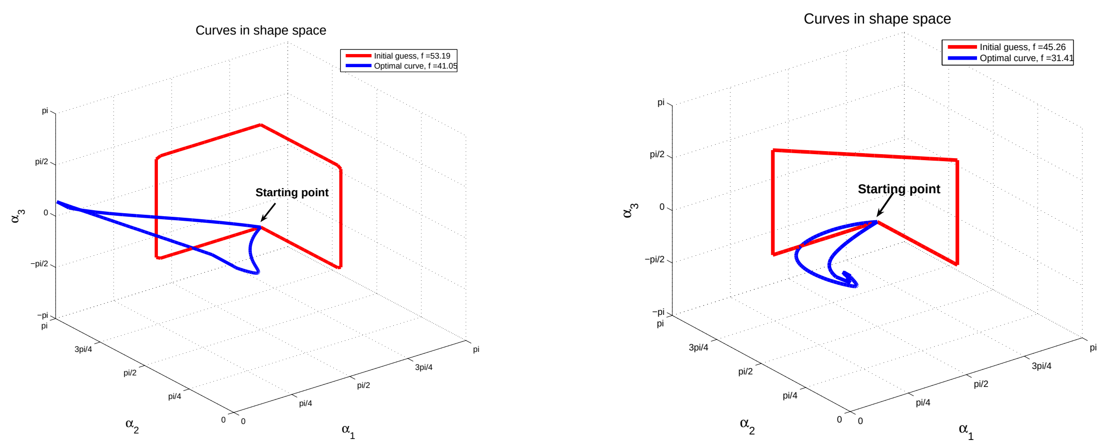
The expression relating small changes in shape to small changes in orientation, (3), can easily be integrated using e.g. a standard ODE solver i MATLAB. However, the software used in solving the isoholonomic problem numerically requires the gradients of the constraints, hence also of the ODE-solver, as input. It is, therefore, necessary to use a known and simple numerical method. To keep things relatively simple the time derivatives of ξ are discretized using simple forward differences, while the ODEs (i.e. the time derivatives of η) are discretized using Eulers Method.

The software used for solving the optimal control problem was SNOPT which is a Fortran based package for MATLAB. SNOPT solves large non-linear optimization problems using an SQP-method [2]. SQP-methods are iterative, and hence an initial guess is required. An initial guess is some closed curve in shape space which represents a series of shape changes that leads to the desired reorientation. We have constructed three, from careful studies of photos of real cats while falling. Two of these are shown as red curves in figures 3(a) and 3(b).

4 Results

We have obtained variuos results from optimizing the initial guesses. The main conclusion to be drawn is that there are several local minima, and that initial guesses that are quite similar can lead to very different minima when optimized. This is not very surprising from a mathematical point of view, and when studying photographs of cats flipping in the air one realizes that different cats use different methods.

Figure 3 shows two different curves in shape space each resulting from optimizing an initial curves with respect to the isoholonomic problem. The two initial curves (the red curves) are quite similar, but the results (the blue curves) are considerably different. One result (see figure 3(a)) is in good agreement with some of the motions we have inferred from photographs of cats. The other 3(b) is quite unphysical. The principal cause for this is the crudity of the model. It has no 'body', no extension. Another reason might be the low order of the discretization.



(a) This result (the blue curve) resembles fairly well some of the motions we have observed in photographs of falling cats. It corresponds to a series of motions where the cat draws its legs towards the body and at the same time twist its spine, followed by stretching out the hind legs, and finally moving both hind- and forelegs to their initial position while twisting the spine in the oppposite direction.

(b) This result is quite unphysical. It corresponds to a series of motions where the cat pull in its legs and 'work a bit' with them in various directions and the unfold them. The reorientation is obtained by the small movements of the legs.

Figure 3: *Initial curves (red) and the resulting optimized curves (blue) in shape space*

Our model of the cat can give an idea about how a quasi-rigid-body can be reoriented. Furthermore we have discovered several curves (the initial guesses) that gives the desired reorientation, and represents a movement similar to the one performed by the cat. However, the optimization results suggest that the presented model does not capture the features of the cat to an extent where it is adequate as a basis for an optimization where the result should resemble the actual movement of a cat.

The two models presented in [4] both take into account the body of the cat, but disregard the effect of the limbs. Our model is opposite of that in the sense that it disregards the bulk of the body and focuses on the limbs. Neither of the models fully captures all the features of the cat involved in the reorientation. Therefore, one should be careful when making biologic conclusions about the cat on the basis of either of the models, and it seems infeasible to make a refined model of that take both the body an the limps into account, given the complexity of these models. On the other hand, if the goal is to discover new principles for efficient reorientation, all three models will be relevant to study.

References

[1] A. M. Bloch. *Nonholonomic Mechanics and Control*. Interdisciplinary Applied Mathematics. Springer, 1. edition, 2003.

[2] Christof Büskens and Helmut Maurer. Sqp-methods for solving optimal control prroblems with control and state constraints: adjoint variables, sensitivity analysis and real-time control. *Journal of Computational and Applied Mathematics*, 120:85–108, 2000.

[3] C. K. Chen and N. Sreenath. Control of coupled spatial two-body systems with nonholonomic constraints. *Proceedings of the 32nd IEEE Conference on Decision and Control*, 2:949–954, december 1993.

[4] Michael J. Enos. On an optimal control problem on $so(3) \times so(3)$ and the falling cat. *Fields Institute Communications: Dynamics and Control of Mechanical systems, The Falling Cat and Related Problems*, 1993.

[5] T. R. Kane and M. P. Scher. A dynamical explanation of the falling cat phenomenon. *International Journal of Solids and Structures*, 5(7):663–670, 1969.

[6] Robert G. Littlejohn and Matthias Reinsch. Gauge fields in the separation of rotations and internal motions in the n-body problem. *Reviews of Modern Physics*, 69(1):213–274, januar 1997.

[7] Jerrold E. Marsden and Tudor S. Ratiu. *Introduction to Mechanics and Symmetry*. Texts in Applied Mathematics. Springer-Verlag, 1994.

[8] R. Montgomery. Isoholonomic problems and some applications. *Communications in Mathematical Physics*, 128:565–592, 1990.

[9] Alfred Shapere and Frank Wilczek. Guage kinematics of deformable bodies. *American Journal of Physics*, 57(6):2051–2054, june 1989.