Preface

This thesis is written in partial fulfilment of the requirements for th degree in engineering from the Technical University of Denmark The reported research has been carried out at the Department of matical Modelling (IMM) at the same university in the period from S ber, 1995, through September, 1998. Associate professor Niels K Poulsen was the academic advisor on the project.

Uffe Høgsbro Thygesen
Lyngby, September 30, 1998

Acknowledgements

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I also wish to thank students and staff at IMM for what they have taught me and for providing an inspiring and pleasant working environment. The list of people is too long to include here; let me just mention that I shared an office with Dr. Morten B. Lauritsen during the first half of my studies, which led to many interesting discussions on stochastic control theory.

I have spent much time with staff and students at the Department of Mathematics. Professor Jakob Stoustrup has influenced my view on control theory significantly, and has continued to do so after he left the department. I have also enjoyed many long discussions about mathematical control theory with Dr. Eric Beran, Dr. Marc Cromme, Dr. Mikael Larsen and Dr. Mike Lind Rank.

I am grateful to Professor D. Prätzel-Wolters and Dr. Jörg Hoffmann, with whom I had the pleasure to work during and immediately after my M.Sc. studies. I have had great benefit from the control theory and the mathematics which I learned at the University of Kaiserslautern.

I also wish to express my gratitude to Professor Robert Skelton, who convinced me about the importance of convex optimization in control theory. I spent a year in his group, which was at Purdue University at that time, where I worked with him on his Finite Signal-to-Noise Ratio models. These models also appear in this thesis and I gratefully acknowledge the inspiration from him. During this year I also learned much from Professor Martin Corless, in particular Lyapunov theory, and from Professor Mario Rotea, who introduced me to dissipation theory. I also enjoyed many stimulating discussions with my fellow students, in particular Dr. Goujun Shi and Dr. Jianbo Lu.

Last but not least, I thank Dr. Marc Cromme who carefully read through an early version of this thesis and whose comments improved the final result considerably.

Previous publication of the material

Parts of the material in this thesis have previously been published in re-
erences

[113] U.H. Thygesen and N.K. Poulsen. Min-max control of nonlinear sys-
tems with multi-dissipative perturbations. Tech. Rep. 23, Dep-
Math. Modeling, Tech. Uni. Denmark, `http://www.imm.dtu.d`
1997. Pres. at the 6th Viennese WOCDGNLDAS, Vienna, 1997.

[114] U.H. Thygesen and N.K. Poulsen. On multi-dissipative perturbation
in linear systems. Technical Report 1, Dept. Math. Modeling, Tec-
Uni. Denmark, `http://www.imm.dtu.dk`, 1997.

[115] U.H. Thygesen and N.K. Poulsen. Robustness of linear system
with multi-dissipative perturbations. In *Proceedings of The Ame-
ican Control Conference*, pages 3444–3445, 1997.

[116] U.H. Thygesen and N.K. Poulsen. Simultaneous $\mathcal{H}_\infty$ control of
finite number of plants. Technical Report 24, Dept. Math. Modelin-
Tech. Uni. Denmark, `http://www.imm.dtu.dk`, 1997.

[117] U.H. Thygesen and N.K. Poulsen. Simultaneous output feedback $\mathcal{H}$
control of p plants using switching. In *Proceedings of the Europea-
Control Conference*, 1997.

Other parts are included in papers which are in the process of being re-
viewed for publication:

[109] U. H. Thygesen. On dissipation in stochastic systems. Under re-
view for publication in a journal. A short version is submitted to
conference, 1998.

[111] U.H. Thygesen. On multi-dissipative dynamic systems. Submitte-
to a journal. A short version is submitted to a conference, 1998.

Finally, some results are included in manuscripts in preparation:

[112] U.H. Thygesen. On the conditional expectation of first passage time-
Manuscript in preparation, 1998.

The material published in the references [72, 108, 110] is not included i-
this thesis.

Summary

The topic of the present dissertation is robustness and performance issues in nonlinear control systems.

The control systems in our study are described by nominal models consisting of nonlinear deterministic or stochastic differential equations in a Euclidean state space. The nominal models are subject to perturbations which are completely unknown dynamic systems, except that they are known to possess certain properties of dissipation. A dissipation property restricts the dynamic behaviour of the perturbation to conform with a bounded resource; for instance energy. The main contribution of the dissertation is a number of sufficient conditions for robust performance of such systems.

Since the perturbations in these uncertain models possess several dissipation properties simultaneously, we study fundamental properties of such multi-dissipative systems. These properties are related to convexity and topology on the space of supply rates. For instance, we give conditions under which the available storage is a continuous convex function of the supply rate.

Dissipation theory in the existing literature applies only to deterministic systems. This is unfortunate since robust control applications typically also contain uncertainty which is better modelled in a probabilistic framework, such as measurement noise. This motivates an extension of the theory of dissipative dynamic systems to stochastic systems. This dissertation presents such an extension: We propose a definition and generalize fundamental results from deterministic dissipation theory to stochastic systems.

Furthermore, we argue that stochastic dissipation is a natural fundament for a theory of robust performance of stochastic systems. To this end, we present a number of performance requirements to stochastic systems which can be formulated in terms of dissipation, after which we give sufficient conditions for these requirements to be robust towards multi-dissipative perturbations.

A final contribution of the dissertation concerns the problem of simultaneous $\mathcal{H}_\infty$ control of a finite number of linear time invariant plants. This problem is a prototype of robust adaptive control problems. We show that the optimal (minimax) controller for this problem is finite dimensional but not based on certainty equivalence, and we discuss the heuristic certainty equivalence controller.

Resumé (in Danish)

Emnet for denne afhandling er robusthed og ydelse (performance) af ikk
lineære reguleringssystemer.

Reguleringssystemerne er beskrevet af nominelle modeller bestående a
ikke-lineære deterministiske eller stokastiske differentialligninger i et eu
klidisk tilstandsrum. Disse nominelle modeller underkastes perturbatione
som er ukendte dynamiske systemer om hvilke det dog vides at de besidde
visse dissipationsegenskaber. En dissipationsegenskab indskrænker pertur
bationens dynamiske opførsel ved at påtrykke en begrænset ressource, fo
eksempel energi. Hovedbidraget i denne afhandling er et antal tilstrække
lige betingelser for robust ydelse af sådanne systemer.

Eftersom perturbationerne i disse usikre modeller besidder flere dissipa
tionsegenskaber samtidigt, studerer vi fundamentale egenskaber af sådann
multi-dissipative systemer. Disse egenskaber omhandler konveksitet o
topologi på rummet af tilførselsrater (supply rates). For eksempel opstille
vi betingelser under hvilke det tilgængelige lager (available storage) er e
kontinuert konveks funktion af tilførselsraten.

Den eksisterende litteratur beskriver kun dissipationsteori for determin
istiske systemer. Det er uheldigt fordi anvendelser af robust regulerin
typisk også indeholder usikkerhed som bedst modelleres sandsynligheds
teoretisk, såsom målestøj. Det er motivationen for at denne afhandlin
udvider dissipationsteorien til stokastiske systemer: Vi foreslår en defin
tion og generaliserer nogle af de grundliggende resultater fra deterministis
dissipationsteori til stokastiske systemer.

Derefter argumenterer vi for at stokastisk dissipation er et naturligt udgang
punkt for en teori for robust ydelse af stokastiske systemer. Til dette forma
opstiller vi et antal kvalitetskriterier for stokastiske systemer som kan for
muleres som dissipationsegenskaber, og dernæst angiver vi tilstrækkelig
betingelser for at disse kriterier er robuste overfor multi-dissipative pertur
bationer.

Herudover behandler denne afhandling også problemet om simultan $\mathcal{H}$
regulering af et endeligt antal lineære tidsinvariante anlæg. Dette prob
lem fungerer som en prototype på robust adaptiv regulering. Vi viser a
den optimale regulator (d.v.s. minimax-regulatoren) for dette problem e
endelig-dimensional men ikke bygger på certainty equivalence. Derudove
diskuterer vi heuristisk certainty equivalence regulering.

Contents

Chapter 1

Introduction

The subject of this dissertation lies within the field of mathematical contro
theory.

In this chapter we give a broad introduction to the field of mathematica
control theory. Those among the readers who are more interested in th
specific contributions of the dissertation may prefer to jump to section 1.
which outlines the dissertation, and from there to the succeeding chapte
which presents the new material.

1.1 What is control theory?

The subject of control theory is the interconnection of the dynamic system
Σ and K in figure 1.1. Here Σ is a given dynamic plant (a mathematic
model of a physical system) and K is the controller (which is also a math
ematical model of a physical system). The objective is to design the con
troller K, i.e. to find a suitable K, such that the interconnection has som
desirable properties. These properties typically describe how the intercon
nection responds to an exogenous input w and are quantified through th
output z. The controller affects the response by choosing the control signa
u. The controller has at least partial access to information about the stat
of the plant, quantified by the measurement signal y.

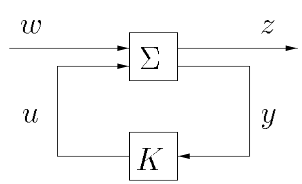

Figure 1.1: A control problem

The questions addressed by control theory are *analysis* questions and *synthesis* questions. Analysis questions investigate properties of the interconnection for a given controller K, while the synthesis question is how to choose the controller K such that the interconnection has certain properties. The motivation for considering analysis questions is twofold: First, analysis questions are most often much easier to answer than synthesis questions, but good answers to analysis questions often enable the control theorist to find answers to the corresponding synthesis questions. Second, the control engineer may have found a candidate controller by solving one specific synthesis question and then wish to know if this controller provides satisfactory answers to other analysis questions.

In order to answer analysis as well as synthesis questions, control theory employs several disciplines from the field of applied mathematics. Qualitative and quantitative theory for deterministic and stochastic dynamic systems is essential as is optimization theory. In addition, statistical inference or deterministic estimation theory is necessary to address problems where the measurements y contain only incomplete information about the state of the plant Σ.

The resulting theory depends greatly on the specifics of the interconnection in figure 1.1: If the systems are linear or nonlinear, deterministic or stochastic, and if the dynamic systems are described in continuous or discrete time. The next section describes these differences in some further detail, as well as clarifies and motivates how the present dissertation is placed in this discourse.

1.2 Paradigm and state of the art

Robustness or performance?

While most control theorists and engineers agree that robustness and per formance are desirable properties of a controlled system, and that as result analysis and synthesis must address these issues, there is much les consensus regarding the exact meaning of these properties and their rela tion.

Performance measures the quality of the controlled system: How fast, ho accurate or how effective is the system. In this work we use the term per formance to describe how a cost, accumulated during the operation of th system, depends on the initial state of the system, or on exogenous dete ministic or stochastic disturbances. The lower cost, the better performanc

The issue of *robustness* arises because the mathematical model, which the object of the mathematical analysis, never fully describes the physica control system. Loosely, robustness means that the mathematical analys predicts the behaviour of the physical system with sufficient accuracy. W assign a much more precise meaning to the word robustness: We model th physical system by a *family* of mathematical models (typically obtained a an interconnection of a nominal model and an unknown perturbation), an say that a property is robust if it holds for any model in this family.

An often heard statement is that one must trade off robustness and per formance: For instance, if one wishes a fast response of a servo system one must accept that the system is sensitive to parasitic dynamics. W do not disagree that such trade-off considerations between sensitivity an nominal performance are helpful. However, we prefer to discuss the issue How fast a response can we obtain in presence of parasitic dynamics? Thu the objective is to guarantee a level of performance which is robust toward a given family of perturbations. In summary, the question *Robustness o performance?* should be answered: *Robust performance!*

Linear or nonlinear theory?

A seemingly never-ending controversy among control theorists concerns lin ear versus nonlinear theory. Advocates of nonlinear theory emphasize tha nonlinear models provide more accurate descriptions of technical system which makes it plausible that better control systems can be obtained with nonlinear theory. On the other hand, nonlinear theory quickly becomes so involved that the designer can be forced to oversimplify the problem, for instance by neglecting certain dynamics, and in these situations it is plausible that linear theory is more effective. Also, in industrial applications it cannot be neglected that nonlinear control theoretic investigations consume great resources which perhaps would be more beneficial if allocated to complete different parts of the design project. A fact that keeps the controversy going is that some fields of applications manage quite well with linear models whereas nonlinearities are essential to the problems of other fields.

Tools for analysis and design of controllers for linear plants are well developed and implemented in commercial software packages such as `MATLAB`. The engineering appeal of frequency domain techniques is an important factor, as is the fact that it is possible to give standardized recipes which work for most linear problems. There is an abundance of methodologies, ranging from parameter tuning in PID-controllers to μ-synthesis [128]. For engineers who wish to pose their own non-standard design criteria, the framework of linear matrix inequalities is an option [20, 19]. Important open problems within the linear paradigm, which are topics of current research, concern mixed and multi-criteria problems, the problem of designing controllers of fixed structure, and interdisciplinary topics such as simultaneous design of system and controller.

Regarding nonlinear control theory significant progress has been made but a fully operational general theory is still far away; indeed it is plausible that such a theory is utopian. The field of Lyapunov stability [74, 59] illustrates the hurdle: The theory is fairly complete from an analytical point of view, but the problem of computing Lyapunov functions for a general system is overwhelming. The same discussion applies to optimal control and differential games where it is known as Bellman's *curse of dimensionality*: The computational complexity grows exponentially with the dimension of the underlying state space. Despite increased computational power and improved numerical methods [65, 11] we cannot expect to be able solve all control problems by direct solution of partial differential equations on state space: It is not unusual for technical control problems to have 75 states as in [18]. One may imagine the effort required to compute, implement and understand a nonlinear controller feeding back a state of this dimension.

As a consequence a myriad of special cases have been investigated and

sometimes the special structure enables progress. For instance, within th last decades the differential geometric framework [51] has evolved. Th associated tools such as feed-back linearization are valuable, although the are prone to robustness problems and require special structure. Backstep ping [63] and other recursive design techniques provide a methodology fo systematic design, but requires considerable computational effort and certain skill of the designer. Inverse optimality [36] is another promisin concept; with this approach one solves the linearized problem at first an then constructs a nonlinear control law such that certain robustness prop erties of the linearized system hold globally for the nonlinear system.

With this state of the art, researchers and engineers must in each projec choose pragmatically between the linear and the nonlinear paradigm. Ther is little doubt that nonlinear theory is becoming increasingly important a models grow in fidelity and complexity, as desired operating regions gro larger, and as better controller hardware allows more complex controlle algorithms to be implemented. What is more, many concepts and idea are clearer for nonlinear systems than for linear systems where matrix ma nipulations tend to obscure the picture; this is perhaps most evident in th field of stability and of optimal and $\mathcal{H}_\infty$ control. Therefore, our ambitio in this dissertation is to develop control theory which is based on principle applicable to general nonlinear systems.

Time domain or frequency domain?

Within the linear paradigm, a great strength of control theory is the abilit to combine considerations in frequency domain and time domain. Unfortu nately, frequency domain tools are less than effective in a general nonlinea context where even the elementary concept of bandwidth is problemati It remains a formidable project to find suitable substitutes.

Therefore, this dissertation considers systems in time domain exclusivel Without making a virtue out of necessity, an advantage of time domai techniques is that they appeal to that intuition for dynamic systems whic engineers develop by studying physical systems. Not only does this facil tate the study and teaching of control theory, but it is also advantageous i industrial environments where a sharp distinction between controller an plant cannot be maintained, and where the control engineer works in a interdisciplinary team. A splendid example where experience from physic is of great value in control theory is the field of Lyapunov stability [74 calculus of variations and dynamic optimization [41, 3], which originately concerned mechanical and in particular astronomical systems, but which in the last decades have been developed by control theorists [12, 21, 122].

Deterministic or stochastic representation of uncertainty?

The explicit consideration of uncertainty lies at the core of control theory. Uncertainty may be represented by unknown parameters, unknown inputs or uncertain dynamical elements, and although much recent work [26, 60, 68, 69, 86, 98, 106, 129, 132] is devoted to mixed problems combining two or more types of uncertainty, there does not yet exist a general operational framework within which all these representations of uncertainty can be embedded.

Models of uncertainty can be divided into two groups: The stochastic ones, i.e. those that build on an underlying probability space, and the deterministic ones, which typically result in worst-case considerations. It is not uncommon for control theorists and engineers to have a very firm preference for one of the two groups, and occasionally this results in attempts to demonstrate that the one group can cover all models of uncertainty.

This dissertation is based on the pragmatic point of view that control theory should, to the widest extent possible, allow for both groups of uncertainty. With such a theory at hand, the control engineer can in each application choose to use deterministic or stochastic models, or both. This becomes increasingly important as control objects grow in complexity, since a complex control problem may contain both elements which require stochastic descriptions and elements which are suited for deterministic worst-case considerations.

1.3 Two recent advances in control theory

In this section we outline two recent developments in the field of control theory which have, too, provided background for the present work: Nonlinear $\mathcal{H}_\infty$ control theory and semidefinite programming. In short, nonlinear $\mathcal{H}_\infty$ theory is an analytical framework for addressing issues of robustness of nonlinear systems towards dynamic uncertainty. Semidefinite programming is a special case of convex optimization which can be used as a computational tool in control problems.

1.3.1 Nonlinear $\mathcal{H}_\infty$ control

One of the important products of control research of the 1980's was the for mulation and solution of the linear $\mathcal{H}_\infty$ control problem. The backgroun for this work was the robustness of LQG (Linear dynamics, Quadratic co functions, Gaussian noise distributions, [66, 2]) controllers - robustness here in the sense of the classical gain and phase margins. It was known tha linear quadratic *state feedback* regulators provide universal robustness gai margins of $(\frac{1}{2}, \infty)$ and phase margins of ± 60 degrees [1, 93]; an impre sive result which can be generalized in several directions, [118, 120]. Th led to the question if similar universal margins existed for LQG controlle where the state is not available for feedback. Unfortunately, this is not th case [27]; optimal controllers are not necessarily robust. This motivated th formulation [127] of the $\mathcal{H}_\infty$ control problem. Here the design objective is t guarantee stability in presence of perturbations with $\mathcal{H}_\infty$ norm less than specified number; this definition of robustness implies gain and phase ma gins, but is more general and more appealing from a mathematical point view. A later generalization was the μ superstructure which allows severa uncertain elements at different places in the closed-loop system; see [12 and the references therein.

Although the $\mathcal{H}_\infty$ framework originated in the frequency domain,[1] th celebrated DGKF solution [29] exploited the fact that the $\mathcal{H}_\infty$ norm of transfer function is also the $\mathcal{L}_2$-gain of the associated input/output operat and thus relied on time-domain techniques, in particular completion of th square under an integral. This solution hinted towards two-player zero-su differential games, thus suggesting that it would be possible to extent th $\mathcal{H}_\infty$ problem to nonlinear systems. An early development in this directio was [6]; the textbook [9] contains a large number of such results, mainl as extensions to the linear theory and focusing on different patterns information available to the players.

Further insight into the nonlinear $\mathcal{H}_\infty$ problem was achieved in [119, 12 by stressing the connection to the theory of dissipative systems [124] an to that of Hamiltonian dynamics [3]. This also brought the field in touc with that of passive systems which has played a central rôle in moder control theory; bounded $\mathcal{L}_2$-gain and passivity constitute by now the mos carefully investigated dissipation properties.

[1] The symbol $\mathcal{H}_\infty$ refers to the Hardy space [50] of transfer functions $G : \mathbb{C} \to \mathbb{C}^{m\times}$ which are analytical in the right half plane, equipped with the supremum norm $||G||_\infty$ $\sup_{\omega \in R} \bar{\sigma} G(i\omega)$.

At the time of writing, there exists a well-established solution to the state feed-back nonlinear $\mathcal{H}_\infty$ control problem in terms of a Hamilton-Jacobi-Isaacs partial differential equation or inequality, which results from applying dynamic programming to the differential game. Nevertheless, issues related to the smoothness and properness of the value functions have yet to be worked out; here the notion of viscosity solutions [23] to partial differential equations has proved effective [7, 105]. Also, the numerical burden of actually computing bounds on value functions is still prohibitive except for problems with very low-dimensional state spaces; up to four, say, depending on the system at hand. Thus Bellman's *curse of dimensionality* also applies to these problems. As the paradigm of robust control includes a use of high-order dynamic weights we conclude that there is a need for heuristics and sub-optimal strategies which can deal with higher-dimensional problems.

Another remaining obstacle for the practical application of nonlinear $\mathcal{H}_\infty$ control theory is the design of state estimators in the situation where the state is not directly measurable, rather the controller is a causal map from a measured signal y to the control signal u. While static or finite-dimensional controllers may be optimal in special situations, [120, 121], it does not in general suffice to make use of a state observer of the same dimension as the control object [120]. In fact, general output feedback problems are very difficult and not fully resolved; not with respect to theoretical analysis and certainly not with respect to practical implementations. The most general framework for approaching these problems is that of the *information state*, see [55] and, in the context of stochastic optimal control problems on finite state spaces, [16]. With this technique the output feedback problem is first reduced to a state feedback problem. The state in this reduced problem is the information state which is a real-valued function on state space (termed the cost-to-go function by other authors, e.g. [25, 120, 9]) and hence the new problem requires infinite-dimensional dynamic programming.

In some situations the information state can be restricted to a finite-dimensional function space which facilitates the problem, see e.g. [56] or chapter 4 in this dissertation. In other situations one can *a priori* guarantee that a certainty equivalence principle holds [54, 14]. This reduces the complexity of the solution so that only two partial differential equations must be solved; one off-line (which governs the original problem with full state information) and one on-line (which governs the state estimation problem). While certainty equivalence principles thus simplifies control problems, it can be argued that certainty equivalence architectures lack the most in-

triguing feature of control with incomplete information: That additiona information can be obtained by proper use of the control signal. This i the effect of probing, or *duality*.

1.3.2 Semidefinite programming and LMIs

A framework which has attracted much attention among control researcher relies on numerical solution of a special type of convex optimization prob lems, namely *semidefinite programs*. These optimization problems consis of optimizing a linear functional

$$\inf_x c'x$$

over all $x \in \mathbb{R}^n$ which satisfy a *linear matrix inequality* (LMI) constraint

$$A(x) \leq 0, \quad (\text{or} \quad A(x) < 0 \ \ )$$

where $c \in \mathbb{R}^n$ is a fixed co-vector and $A : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is an affine functio taking symmetric matrix values. Such semidefinite programs are convex and it is feasible to solve them numerically; powerful polynomial-time algo rithms based on interior-point methods exist [82]. See also [13] for furthe references.

The surprising fact is that a large number of performance requirement in linear control theory can be formulated as linear matrix inequalitie see [19]. Thus semidefinite programming can be used to solve especiall analysis problems but also some of the classic design problems, notabl $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis. The simplest example is the well-known stabilit result [74] that, given a real matrix $A \in \mathbb{R}^{n\times n}$, the existence of a rea symmetric *Lyapunov* matrix $P = P'$ such that

$$\begin{bmatrix} \Leftrightarrow P & 0 \\ 0 & PA + A'P \end{bmatrix} < 0$$

is necessary and sufficient for A to have all eigenvalues in the open le half of the complex plane. In chapters 3 and 6 we demonstrate that linea matrix inequalities provide the natural tool to deal with robustness analy sis in linear systems where uncertainty is represented by *multi-dissipati perturbations* and *finite signal-to-noise ratios.*

[2] Meaning, we minimize a convex functional over a convex set.

The limitation of the LMI approaches to control *design* is that for the majority of controller design problems it is difficult or impossible to make the problem convex by variable substitutions. Rather, the resulting problems are bi-linear matrix inequalities which in general are non-convex. Numerical solution of bi-linear matrix inequalities is a topic of current research; see [13] and the references therein.

Finite-dimensional convex optimization, and in particular linear matrix inequalities, can also be employed as a computational tool for nonlinear control problems. Consider as an example the problem of finding a non-negative function $V : \mathbb{R}^n \to \mathbb{R}$ which satisfies the Hamilton-Jacobi inequality on $\mathbb{R}^n$

$$\sup_{w \in \mathbb{R}^m} \frac{\partial V}{\partial x}(x) f(x) + \frac{\partial V}{\partial x}(x) g(x) w \Leftrightarrow |w|^2 + |h(x)|^2 \leq 0 \quad .$$

The existence of such a *storage function* V implies (and is under certain conditions equivalent to) that the system

$$\dot{x}(t) = f(x(t)) + g(x(t))w \ , \quad z(t) = h(x(t))$$

has $\mathcal{L}_2$-gain less than or equal to 1. Here $f(x) \in \mathbb{R}^n$ and $g(x) \in \mathbb{R}^{n \times m}$. The set of those functions V which satisfies this inequality is convex; in fact the inequality is equivalent to

$$\begin{bmatrix} \frac{\partial V}{\partial x} f + |h|^2 & \frac{1}{2}\frac{\partial V}{\partial x} g \\ \frac{1}{2}(\frac{\partial V}{\partial x} g)' & \Leftrightarrow I \end{bmatrix} \leq 0 \quad . \tag{1.1}$$

A computational strategy for this infinite-dimensional convex feasibility problem is to search for a V of the form

$$V(x) = \sum_{i=1}^{N} \alpha_i V_i(x)$$

where V_i are basis functions, and require that the inequality (1.1) and $V \geq 0$ holds only at a finite set of points x_j, $j = 1, \ldots, M$. Inserting $V = \sum_i \alpha_i V_i$ in (1.1) and evaluating at x_j leads to M linear matrix inequalities in α_i

$$\left.\begin{bmatrix} \sum_{i=1}^N \alpha_i \frac{\partial V_i}{\partial x} f + |h|^2 & \frac{1}{2}\sum_{i=1}^N \alpha_i \frac{\partial V_i}{\partial x} g \\ \frac{1}{2}(\sum_{i=1}^N \alpha_i \frac{\partial V_i}{\partial x} g)' & \Leftrightarrow I \end{bmatrix}\right|_{x=x_j} \leq 0 \ , \quad j = 1, \ldots, M$$

which must holds together with with the M constraints

$$\sum_{i=1}^{N} \alpha_i V_i(x_j) \geq 0 \quad .$$

Thus LMI solvers such as [38, 32] may be used to search for storage func tions V, and hence to compute the $\mathcal{L}_2$ gain of nonlinear systems. It ca be argued that other numerical methods based on partial differential *equa tions* would be at least as effective for this particular problem of $\mathcal{L}_2$ gai analysis, but this dissertation contains numerous examples of nonlinea analysis problems which can be solved by convex optimization but no with equations. Admittedly, realistic control problems quickly lead to s large problems that the existing numerical tools for semidefinite program will be ineffective, but as these tools are improving rapidly we expect tha the approach may have practical applicability in the not so far future.

1.4 Problem formulation

Consider the control problem depicted in figure 1.2. The problem is to fin a controller K in some set which maps measurements y to control signal u such as to achieve some design specifications on the output z. $\Sigma(\theta$ is a plant which may be nonlinear and stochastic. The exogenous inpu w, the parameters θ, the dynamic perturbation Δ and the static nonlinea function ϕ represent uncertainty. All these uncertain elements are unknow but known to belong to some specified set. Additional uncertainty may b introduced by stochastic disturbances internal to $\Sigma(\theta)$.

With the current state of the art, this control problem is much too am bitious. The far more modest objective of this dissertation is simply t develop a framework within which this control problem can be formulate Furthermore, to approach various subproblems, for instance by excludin some of the uncertain elements and considering analysis problems rathe than synthesis problems.

As a starting point the theory of dissipative systems (in the sense o Willems [124] and Hill and Moylan, e.g. [46]) was adopted. See section 2. on page 28 below for an introduction and further references to dissipatio theory. This is a natural choice in that problems of robust performance ar easily formulated in terms of dissipation. Furthermore, dissipation theor

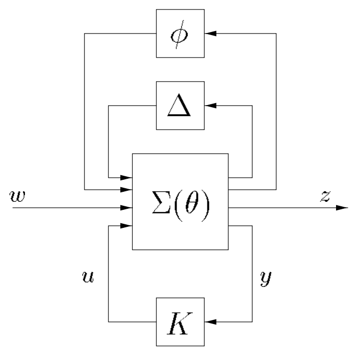

Figure 1.2: The ultimate control problem

assisted major theoretical achievements reached during the first half of the nineties in the field of nonlinear $\mathcal{H}_\infty$ control, which can indeed be seen as one of the subproblems mentioned above (see section 1.3.1 above).

Our research concentrated on three subproblems:

1. The problem arising by only considering a dynamic perturbation Δ which is known to possess several dissipation properties.
2. The problem arising when only considering uncertain parameters and robustness in the $\mathcal{H}_\infty$ sense.
3. The problem of incorporating stochastic noise signals in a dissipation-based framework for robustness.

The first item led to the study of multi-dissipative dynamic systems, see chapter 2, and to the study of robustness towards multi-dissipative perturbations, see chapter 3. The second item is that of adaptive $\mathcal{H}_\infty$ control or robust adaptive control; a topic which has been researched intensively over the past few years by several groups. Chapter 4 presents some new contributions to this topic, especially regarding the rôle of certainty equivalence, which were obtained by making the further simplification that the parameter θ belongs to a known, finite set. Finally, the last item motivated the notion of dissipative stochastic systems, a class of systems which is defined and investigated in chapter 5, and the investigation of a class of robust

stochastic control problems, namely those which involve multi-dissipati
perturbations, see chapter 6.

1.5 Outline of the dissertation

The dissertation is divided into two parts. The first concerns *determin*
istic models. This means that the nominal models are deterministic, i.
ordinary, differential equations, and that uncertainty is represented by pe
turbations which belong to a specified set. The primary novelty of th
part is the study of systems, in particular perturbations, which are diss
pative with respect to *several* supply rates. Although dissipative system
are well-studied objects [124, 53, 122], multi-dissipative systems have n
been discussed previously.

In part I, chapter 2 on *multi-dissipative dynamic systems* is devoted t
fundamental properties of these systems; these properties concern convexit
and continuity associated with the supply rates. Chapter 3 on *robustne*
towards multi-dissipative perturbations develops *sufficient* conditions f
robust stability and performance of systems subject to such perturbation
The conditions involve certain weights, or multipliers, associated with th
dissipation properties and the results are shown using state-space tim
domain techniques in the tradition of Lyapunov [74, 59]. The results hav
some conservativeness inherent which is illustrated by a simple examp
where non-conservative conditions can be obtained using an input-outpu
approach. Chapter 4 on *simultaneous $\mathcal{H}_\infty$ control* assumes that the pla
to be controlled is unknown, but belongs to a given finite collection. This
prototype of an adaptive $\mathcal{H}_\infty$ control problem and contains the problem
duality which remains a hurdle in stochastic adaptive control. We obtain a
implicit solutions in terms of a partial differential equation, and discuss th
structure of its solution as well as heuristic certainty equivalence control.

Part II concerns *stochastic models* where the nominal systems are describe
by stochastic differential equations in the sense of Itô. The aim of this pa
is to develop tools for problems which include both stochastic and dete
ministic representations of uncertainty. To this end, we develop in chapter
a theory of *dissipation in stochastic systems*, generalizing the framework
Willems [124]. We show that dissipative stochastic systems are as we
behaved as their deterministic counterparts; for instance dissipation h

implications for stability and is preserved under interconnections of systems. This is exploited in chapter 6 on *robustness of stochastic systems*, where performance as well as uncertainty is described in terms of stochastic dissipation properties. Examples include $\mathcal{H}_2$ performance as well as disturbances with finite signal-to-noise ratios in the sense of Skelton [103]. The perspective of this framework is that it allows a modular approach to robustness analysis, using convex optimization as a numerical tool.

Concluding remarks and suggestions for future work are given in chapter 7.

Appendix A concerns autonomous stochastic differential equations and derives a formula for the *conditional expectation of first passage times.* The conditioning is here on a specified part of the target set being reached before the remainder. Such conditional expectations are natural performance measures for control systems in certain applications. Nevertheless, the material is somewhat peripheral to the robust performance questions which are the main topic of the dissertation; hence it has been placed in appendix.

Appendix B contains a few *technicalities.* These are long but elementary computations needed in proofs in the body of the dissertation. Appendix C contains tables of *frequently used symbols and acronyms.*

The appendices are followed by a bibliography and an index.

1.6 Prerequisites of the reader

Part I in this dissertation assumes that the reader has had some exposure to system theory, linear $\mathcal{H}_\infty$ control and nonlinear deterministic optimal control, e.g. at the level of [67, 128]. Part II assumes in addition some familiarity with stochastic differential equations, e.g. [83].

Part I

Deterministic models

Chapter 2

Multi-dissipative dynamic systems

We consider deterministic dynamic systems with state space representations which are dissipative in the sense of Willems [124] with respect to several supply rates. This property is of interest in robustness analysis and in multi-objective control. We show that under certain assumptions, the dissipated supply rates form a closed convex cone. Furthermore we show convexity and semi-continuity properties of the available storage and required supply as functions of the supply rate.

2.1 Introduction

Dynamic systems which are dissipative in the sense of Willems [124] appea in several areas of control theory. Roughly speaking, a system is dissipativ if it is unable to produce a specified quantity, such as energy. The frame work is applicable to large-scale systems and robustness problems becaus dissipativity is preserved under interconnections of systems and becaus dissipativity for autonomous systems implies stability. Indeed, the frame work is a natural extension of Lyapunov theory to input/output system Although the notion of dissipativity is a quite general one, most attentio has been given to two special cases: passive systems and systems with bounded $\mathcal{L}_2$ gain.

In this chapter we consider deterministic dynamic systems which are dissipative in the sense of [124] with respect to *several* supply rates. Such multi-dissipative systems are interesting from a control perspective for two reasons: It may be a *design objective* that a system should be multi-dissipative, for instance that the closed loop has small gain and that the controller is passive. Secondly, uncertain dynamic elements in the system may be modeled as *multi-dissipative perturbations*. For instance, consider a mechanical system containing two parasitics, each of which is passive and has small $\mathcal{L}_2$ gain. This results in a total of four dissipation properties which the parasitics satisfy together. Such information can be used to show robust stability and performance of the overall system.

Although much literature has been devoted to the topic of systems which are dissipative w.r.t *one* supply rate, it appears that simultaneous dissipation properties have not been studied. In this contribution we show that convexity properties appear nicely when several supply rates are considered at once; for instance, the set of supply rates w.r.t. which a system is dissipative is a convex cone, and for a fixed initial state, the available storage is a convex lower semi-continuous function of the supply rate (see below for definitions and exact statements). These properties are important both from an analytical and a computational point of view.

The chapter is organized as follows: In section 2.2 we summarize some definitions and properties associated with dissipative systems, mostly following [124]. Section 2.3 presents our new results for systems which are dissipative with respect to several supply rates while section 2.4 offers some conclusions.

2.2 Preliminaries

We consider dynamic systems Σ defined by ordinary differential equations in state-space:

$$\Sigma : \quad \begin{array}{rcl} \dot{x}(t) & = & f(x(t), w(t)) \\ z(t) & = & g(x(t), w(t)) \end{array} \tag{2.1}$$

Here, the system has input $w(t) \in \mathbb{W}$, output $z(t) \in \mathbb{Z}$ and state $x(t) \in \mathbb{X}$, and the spaces $\mathbb{X}, \mathbb{W}$ and $\mathbb{Z}$ are Euclidean. We restrict the input signal

$w(\cdot)$ to a signal space $\mathcal{W}$ which is chosen such that the differential equatio defines a state transition map $\phi(x, t, w(\cdot))$: If $x(\cdot)$ solves the equation, the $x(t) = \phi(x(0), t, w(\cdot))$.

Associated with the system we have a *supply rate* $s : \mathbb{W} \times \mathbb{Z} \to \mathbb{R}$ whic describes a *flow* of some quantity into the system. When the initial stat x_0 and the input $w(\cdot)$ is clear from the context we use the shorthand

$$s(t) := s(w(t), g(\phi(x_0, t, w(\cdot)), w(t))) \quad .$$

We do not wish to dwell on technicalities regarding existence, uniquenes and regularity of state trajectories and supplies. We hence simply as sume that the input space $\mathcal{W}$ is chosen such that $\phi(x_0, t, w(\cdot))$ is a we defined semi-group, continuous in t, consistent with x_0, causal in $w(\cdot)$ an such that all resulting signals are measurable locally bounded functions of time. Furthermore $\mathcal{W}$ must be closed under switching to guarante that the principle of optimality holds. These assumptions are for instanc met if $f : \mathbb{X} \times \mathbb{W} \to T\mathbb{X}$ is Lipschitz continuous, $g : \mathbb{X} \times \mathbb{W} \to \mathbb{Z}$ an $s : \mathbb{W} \times \mathbb{Z} \to \mathbb{R}$ are locally bounded and measurable, and if $\mathcal{W}$ is the set o piecewise continuous locally bounded signals.

We remark that one could avoid the differential equations all together an define the dynamic system by ϕ, see [124]. One can also define dissipatio for input-output systems, see [47].

Our notion of dissipation is the original one of Willems [124]:

Definition 1: A dynamic system Σ is said to be *dissipative* with respec to the supply rate s if there exist a *storage* function $V : \mathbb{X} \to \bar{\mathbb{R}}_+$ suc that for all time intervals $[0, T]$, initial conditions x_0 and inputs $w \in \mathcal{W}$ th *dissipation inequality*

$$V(x(T)) \leq V(x(0)) + \int_0^T s(t)\ dt \tag{2.2}$$

holds.

We use the following formulations interchangeably: Σ is dissipative w.r. s; Σ dissipates s; s is dissipated by Σ.

The reader is encouraged to always keep the energy interpretation in min s denotes an (abstract) energy flow into the system and V denotes th energy stored in the system.

[1] A function is said to be locally bounded if the image of any bounded set is bounde

We remark that James has proposed a slightly different definition in [53] where the storage function is required to be locally bounded. It is then possible to restrict attention to lower semi-continuous storage functions which are shown to be exactly the non-negative viscosity solutions in the sense of [23] to the differential formulation of the dissipation inequality

$$\forall w \in \mathbb{W}: \quad V_x(x) f(x,w) \leq s(w, g(x,w)) \tag{2.3}$$

which must hold for all $x \in \mathbb{X}$. The two definitions coincide when the system is locally controllable; then all storage functions are continuous [47, 7].

In many situation it is possible to use the storage function as a Lyapunov function to show various stability properties [124]. For instance, assume that V attains an isolated local minimum at some point x_0 and is continuous in a neighbourhood of x_0 and that $w(\cdot)$ is chosen such that $s(\cdot) \leq 0$, then $x(\cdot) = x_0$ is a Lyapunov stable solution.

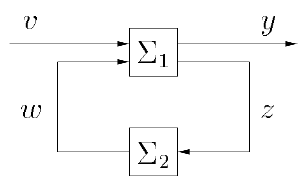

Figure 2.1: Feedback interconnection of dissipative systems

If a collection of dissipative system components are connected in a suitable fashion, then the resulting system will be dissipative as well; as storage function one can use the sum of the storage in each component. This statement seems obvious if one keeps the energy interpretation in mind; however there are a few technical requirements [124]. The simplest such statement is as follows: Assume that the system Σ_1 in figure 2.1 dissipates the supply rate $s_1(v,y) + s_2(w,z)$ and that Σ_2 dissipates $\Leftrightarrow s_2(w,z)$, then the interconnection (Σ_1, Σ_2), which is a system with input v and output y, dissipates $s_1(v,y)$. Here we have assumed that the interconnection is a well defined dynamic system with a state space representation. In fact a repeated use of this simple statement is sufficient for our purposes.

Sometimes it is useful to consider *strict* dissipation inequalities, i.e. to ask if the system is dissipative w.r.t. $s(w,z) \Leftrightarrow \alpha(x)$ for some suitable non-negative function α. This is particularly relevant when one is interested in

time constants associated with the system, in robustness w.r.t. perturba tions in dynamics or in supply rate, or in stronger stability properties tha Lyapunov stability. In particular we follow [99] and say that the system *strictly output dissipative* w.r.t. the supply rate s iff it is dissipative w.r $s \Leftrightarrow \epsilon |z|^2$ for some constant $\epsilon > 0$. This property is of interest in $\mathcal{L}_2$ stabilit and performance analysis.

The available storage and the required supply

A dissipative system will in general have many different storage functio for each supply rate, but two are of special interest. First we follow [12 and define the *available* storage

$$V_a(x) = \sup_{w(\cdot),T} \int_0^T \Leftrightarrow s(t)\ dt$$

where the integral is along the trajectory starting in x and correspondin to $w(\cdot)$. It is easy to see [124] that the available storage is finite everywhe if and only if the system is dissipative, in which case it is in itself a stora function and satisfies $V_a(x) \leq V(x)$ for any other storage function $V(\cdot$ Furthermore, the available storage has infimum 0 (to see this, let V be storage function, then so is $V(x) \Leftrightarrow \inf_\xi V(\xi)$ which implies $V_a(x) \leq V(x)$ $\inf_\xi V(\xi)$ and hence $\inf_x V_a(x) = 0$). On the other hand, the infimum nee not be attained; consider as an example a system of two electrons movi frictionless in space subject to an external input force u. The supply the energy delivered by u, the unique storage function is the energy in th system. Minimum storage is found in the limit as the electrons come rest infinitely far from each other.

Secondly, we define the *required supply* as the least possible supply whi can bring the system from a state of minimal available storage to the desir terminal state. More precisely:

$$V_r(x) = \inf_{x(\cdot),w(\cdot),T} \int_0^T s(t)\ dt$$

where the trajectory $x(\cdot)$ must be consistent with $w(\cdot)$ and furthermo satisfy $V_a(x(0)) = 0$ and $x(T) = x$. When no such trajectory exists w define $V_r(x) = \infty$. The required supply satisfies $V_r(x) \geq V(x)$ for an storage function V which has been normalized so that $V(x) = 0$ whenev

$V_a(x) = 0$. Furthermore, if $V_r(x)$ is finite everywhere (i.e. the system is dissipative, there exists at least one point of minimum available storage and any state is reachable from such a point) then $V_r(x)$ is in itself a storage function.

We remark that it is possible to give a more general definition which does not assume the existence of a point of minimum available storage; we shall not pursue this. Also, the reachability assumption will be used frequently in the following. In some situations where it does not hold it may be advantageous to redefine the state space of the system to contain exactly those states which are reachable.

Our definition of the required supply differs slightly from the one of Willems [124]: In this reference infimum is taken over trajectories which start in a fixed, specified point $x(0) = x^*$ with $V_a(x^*) = 0$. In contrast, we allow $x(0)$ to vary as long as $V_a(x(0))$ holds. We believe our definition is more suitable when multiple points of zero available storage exist, for instance several equilibrium points. A consequence of our definition is that any x which satisfies $V_a(x) = 0$ also satisfies $V_r(x) = 0$.

Sometimes we use the notation $V_a(x, s)$ and $V_r(x, s)$ to stress which supply rate we are referring to. We remark that the availabe storage and the required supply are viscosity solutions to differential dissipation *equalities* corresponding to the inequality (2.3), provided that they are continuous and under certain assumptions [7]. See also the example in section 5.7 on page 116 below.

2.3 Properties of multi-dissipative dynamic systems

In this section we consider a system Σ of the form (2.1) which is dissipative with respect to more than one supply rate. We investigate the set of supply rates which are dissipated by the system and we show several properties which are related to the convexity of this set.

It was noted already in [124] that the storage functions for a dynamical system with respect to a single supply rate form a convex set. For multi-dissipative systems this fact extends easily to the following:

Proposition 2: Let $\mathcal{V}$ be a linear space of functions $\mathbb{X} \to \mathbb{R}$ and let $\mathcal{S}$ be a linear space of supply rates $\mathbb{W} \times \mathbb{Z} \to \mathbb{R}$. Then those pairs (V, s) for

which V is a storage function w.r.t. the supply rate s form a convex con
i.e.

$\{(V, s) \in \mathcal{V} \times \mathcal{S} \mid V \geq 0$ and (V, s) satisfy the dissipation inequality (2.2)

is a convex cone. Furthermore this set is closed with respect to pointwis (in $\mathbb{X}$) convergence of storage functions V and local uniform convergenc (over $\mathbb{W} \times \mathbb{Z}$) of supply rates s.

Regarding the last closedness statement, one could have considered severa different topologies on the space $\mathcal{S}$ of supply rates. Throughout, we sha restrict attention to the topology corresponding to local uniform conver gence over $\mathbb{W} \times \mathbb{Z}$; this mode of convergence appears to be most useful i applications. We recall the standard definition:

Definition 3: We say that $s_i \to s$ locally uniformly if, for every compac subset Ω of $\mathbb{W} \times \mathbb{Z}$ and every $\epsilon > 0$, there exists an $N > 0$ such tha $\sup_{(w,z) \in \Omega} |s_i(w, z) \Leftrightarrow s(w, z)| < \epsilon$ for $i > N$.

We remark that if $s_i \to s$ locally uniformly, then $\sup_{t \in [0,T]} |s_i(t) \Leftrightarrow s(t)| \to$ for any finite T and any trajectory which satify our standing assumptio that all signals are locally bounded functions of time.

The proof of proposition 2 is a quite straightforward exercise of the machi ery of dissipation theory; we include it for the convenience of the reader.

Proof: [of the proposition] Too see that the set is a convex cone, l the system be dissipative w.r.t. the supply rates s_1 and s_2 with storag functions V_1 and V_2, respectively. We must then show that $\lambda_1 V_1 + \lambda_2 V$ is a storage function with respect to $\lambda_1 s_1 + \lambda_2 s_2$ for any $\lambda_1, \lambda_2 \geq 0$. T this end, let the initial state $x(0)$, the input $w(\cdot)$ and the final time T b arbitrary; then the dissipation inequalities

$$V_i(x(T)) \leq V_i(x(0)) + \int_0^T s_i(t)\ dt$$

hold for $i = 1, 2$. Multiply these inequalities with $\lambda_1, \lambda_2 \geq 0$ and add th two to obtain

$$\sum_{i=1}^{2} \lambda_i V_i(x(T)) \leq \sum_{i=1}^{2} \lambda_i V_i(x(0)) + \int_0^T \sum_{i=1}^{2} \lambda_i s_i(t)\ dt$$

which says that $\lambda_1 V_1 + \lambda_2 V_2$ is a storage function with respect to $\lambda_1 s_1 + \lambda_2 s$

To see that the set is closed, let $V_i \to V$ pointwise in $\mathbb{X}$ and let $s_i \to s$ locally uniformly over $\mathbb{W} \times \mathbb{Z}$. Consider an arbitrary trajectory such that

$$V_i(x(T)) \leq V_i(x(0)) + \int_0^T s_i(t)\ dt \quad .$$

Then we have $\int_0^T s_i(t)\ dt \to \int_0^T s(t)\ dt$ due to local uniform convergence of s_i since all signals by assumption are bounded on the bounded interval $[0, T]$. Combining with pointwise convergence of $V_i(x(\cdot))$ we get

$$V(x(T)) \leq V(x(0)) + \int_0^T s(t)\ dt$$

which should be shown. ■

We see from proposition 2 that if the system dissipates any supply rate in a given set $\mathbb{S} \subset \mathcal{S}$, then it is dissipates any supply rate in the convex conic hull of $\mathbb{S}$. This was also noted in [45].

An interesting question is if the set of dissipated supply rates is closed under some given topology on the space $\mathcal{S}$ of supply rates. For instance, in $\mathcal{L}_2$-gain analysis one considers supply rates $s_\gamma(w, z) = \gamma^2 |w|^2 \Leftrightarrow |z|^2$ and define the $\mathcal{L}_2$-gain γ^* as the infimum over all numbers $\gamma > 0$ such that the system is dissipative w.r.t. s_γ. The question if the system is dissipative w.r.t. s_{γ^*} hence arises naturally. In this case it is [120], but the question has not been considered for more general families of supply rates. Notice that the closedness shown in proposition 2 does not answer this question.

A first result in this direction is obtained with the notion of a cyclo-dissipative system:

Definition 4: The system Σ is cyclo-dissipative w.r.t. the supply rate s if

$$\int_0^T s(t)\ dt \geq 0$$

for any T and any pair $w(\cdot), x(\cdot)$ such that $x(0) = x(T)$. □

This definition deviates slightly from the one in [47] where the inequality is required to hold only when $x(0) = x(T) = 0$; here, we have no reason to discriminate the state $x = 0$. A dissipative system is obviously cyclo-dissipative whereas the converse implication does not hold in general [47]. We can now pose the result:

Proposition 5: Assume that Σ is cyclo-dissipative w.r.t. s_i, $i \in \mathbb{N}$, an that $s_i \to s$ locally uniformly as $i \to \infty$. Then Σ is cyclo-dissipative w.r. s.

Proof: Let $w(\cdot), x(\cdot)$ be any trajectory such that $x(0) = x(T)$, let $z($ be the corresponding output. Since all signals by assumption are locall bounded, there exists a bounded set $\Omega \in \mathbb{W} \times \mathbb{Z}$ such that $(w(t), z(t)) \in$ for $t \in [0, T]$. Let $\epsilon > 0$ be arbitrary and let i be sufficiently large such tha $\sup_{(w,z)\in\Omega} |s_i(w, z) \Leftrightarrow s(w, z)| < \epsilon$. Then

$$\int_0^T s(t)\ dt \geq \int_0^T s_i(t)\ dt \Leftrightarrow \epsilon T \geq \Leftrightarrow \epsilon T$$

since the system is cyclo-dissipative w.r.t. s_i. Letting $\epsilon \to 0$ yields th desired conclusion.

It follows that the set of cyclo-dissipated supply rates is a closed conve cone. An appealing conjecture is that the same statement holds if on replaces the word cyclo-dissipated with dissipated. This is not the cas however, as the following example demonstrates.

Example 6: Consider a scalar integrator, i.e. a system with state spac $\mathbb{X} = \mathbb{R}$ and dynamics

$$\dot{x} = w, \quad z = x$$

and let the space $\mathcal{S}$ of supply rates be the span of the two rates wz an $z^3 w$. Consider a sequence of supply rates

$$s_i(w, z) = \Leftrightarrow 2wz + \frac{4}{i} z^3 w \quad .$$

It is then easy to see that the system is dissipative w.r.t. s_i; the availab storage is

$$V_a(x, s_i) = \Leftrightarrow x^2 + \frac{1}{i} x^4 + \frac{i}{4} \quad .$$

In fact the dissipation inequalities always hold with equality (in the term nology of [124] the system is *lossless* w.r.t. s_i). The supply rates s_i converg locally uniformly to $s(w, z) = \Leftrightarrow 2wz$ and it is easy to see that the syster does not dissipate s: For the dissipation inequality to be satisfied the stor age function must necessarily be in the form $V(x) = \Leftrightarrow x^2 + K$ and no exists such that V is non-negative. However, the system is cyclo-dissipativ w.r.t. s in accordance with the previous result.

In order to get the desired result we need an additional assumption on the states of zero available storage:

Proposition 7: Let s_i, $i \in \mathbb{N}$, be a sequence of dissipated supply rates which converges locally uniformly to the supply rate s. Assume that the set of minimal available storage $\{x \mid V_a(x, s_i) = 0\}$ is independent of $i \in \mathbb{N}$ and non-empty, and that the entire state space $\mathbb{X}$ is reachable from this set. Then the system is dissipative w.r.t. s. $\triangle$

Proof: Consider an arbitrary trajectory such that $V_a(x(0), s_i) = 0$ and define

$$J := \int_0^T s(t)\ dt$$

where $T > 0$ is arbitrary. Let $\epsilon > 0$ be arbitrary and choose i sufficiently large such that $|s_i(t) \Leftrightarrow s(t)| < \epsilon$ for $t \in [0, T]$; this is possible since all signals are bounded on $[0, T]$ and $s_i \to s$ locally uniformly on $\mathbb{W} \times \mathbb{Z}$. It follows that

$$0 \leq \int_0^T s_i(t)\ dt \leq J + \epsilon T$$

where the first inequality holds because the trajectory starts in a point of zero available storage w.r.t. s_i. Since $\epsilon > 0$ was arbitrary we conclude that $J \geq 0$.

Now consider any continuation of the trajectory starting at time T in the state $x(T)$ and ending at time $T' > T$. Repeating the above argument we see that

$$\int_0^{T'} s(t)\ dt \geq 0$$

which in turn implies that

$$\int_T^{T'} \Leftrightarrow s(t)\ dt \leq J \quad .$$

We conclude that $V_a(x(T), s) \leq J < \infty$. Now notice that the point $x(T)$ can be chosen arbitrarily since the entire state space is reachable; it follows that the available storage w.r.t. s is finite everywhere. We conclude that the system is dissipative w.r.t. s. ∎

The hypothesis that the set of zero initial storage is independent of i fails in example 6 above. In this example we have $V_a(x, s_i) = 0 \Leftrightarrow |x| = \sqrt{i/2}$.

In many applications there is only one set which can be a set of minim
available storage, for instance a single zero-input equilibrium point. I
these situations we conclude that the dissipated supply rates form a close
convex cone.

One can also derive closedness properties using theory for partial differenti
equations, rather than system theory, for instance following [23, 53]. W
point out that in comparison with this approach, proposition 7 has th
strength of not imposing local boundedness, continuity, or other regularit
requirements on the storage functions.

The previous results clarifies the structure of the set of dissipated suppl
rates. We now turn to the properties of the available storage and require
supply, seen as functions of the supply rate.

Proposition 8: Let $\mathbb{S}$ be a convex set of dissipated supply rates and le
$x \in \mathbb{X}$ be fixed. Then $V_a(x, s)$ is a convex lower semi-continuous functio
of $s \in \mathbb{S}$. If furthermore the set $\{x | V_a(x, s) = 0\}$ is independent of $s \in$
and non-empty, and if the entire state space is reachable from this set, the
$V_r(x, s)$ is a concave upper semi-continuous function of $s \in \mathbb{S}$.

Proof: First we show that $V_a(x, \cdot)$ is convex in the supply rate: Fix th
initial condition $x(0)$ and define the functional J_a on $\mathcal{W} \times \mathbb{R}_+ \times \mathcal{S}$ by

$$J_a(w(\cdot), T, s) = \int_0^T \Leftrightarrow s(t)\ dt$$

where the integrand is evaluated along the trajectory starting in $x(0)$ an
corresponding to $w(\cdot)$. Notice that J_a is convex in s; even linear. Hence

$$V_a(x(0), s) = \sup_{w(\cdot), T} J_a(w(\cdot), T, s)$$

is also convex since the supremeum of any family of convex functionals
convex.

Next we show that $V_a(x, \cdot)$ is lower semi-continuous: Let $s \in \mathbb{S}$ and let
be a sequence in $\mathbb{S}$ which converges locally uniformly to s; we must the
show that

$$\liminf_{i \to \infty} V_a(x, s_i) \geq V_a(x, s) \quad .$$

Choose $\epsilon > 0$ and let $x(\cdot)$ be a trajectory with $x(0) = x$ such that

$$\int_0^T \Leftrightarrow s(t)\ dt \geq V_a(x, s) \Leftrightarrow \epsilon \quad .$$

Now choose i sufficiently large such that $|s_i(t) - s(t)| < \epsilon/T$ on $[0,T]$; then

$$\int_0^T -s_i(t)\ dt \geq \int_0^T -s(t)\ dt - \epsilon \geq V_a(x,s) - 2\epsilon$$

which implies that $V_a(x,s_i) \geq V_a(x,s) - 2\epsilon$ for i sufficiently large. Now let $\epsilon \to 0$ to obtain the desired conclusion.

To show concavity of $V_r(\bar{x},\cdot)$ under the additional assumptions, we follow the argument above: Let Ω denote those $(w(\cdot), x_0, T)$ in $\mathcal{W} \times \mathbb{X} \times \bar{\mathbb{R}}_+$ for which $V_a(x_0,s) = 0$ for $s \in \mathbb{S}$, and for which the trajectory starting in x_0 and corresponding to $w(\cdot)$ satisfies $x(T) = \bar{x}$. Now define the functional J_r on $\Omega \times \mathcal{S}$ by

$$J_r(w(\cdot), x_0, T, s) = \int_0^T s(t)\ dt$$

which is concave, in fact linear, in s. Now notice that $V_r(\bar{x},s)$ is the infimum of J_r over the set Ω and hence concave.

Finally we show upper semi-continuity of $V_r(x,\cdot)$. Choose $\epsilon > 0$ and let $x(\cdot)$ be a trajectory which starts with zero available storage, i.e. $V_a(x(0),s) = 0$, ends in $x(T) = x$, and which satisfies

$$\int_0^T s(t)\ dt \leq V_r(x,s) + \epsilon \quad .$$

Now choose i sufficiently large such that $|s_i(t) - s(t)| < \epsilon/T$ on $[0,T]$, then

$$\int_0^T s_i(t)\ dt \leq \int_0^T s(t)\ dt + \epsilon \leq V_r(x,s) + 2\epsilon$$

which implies that $V_r(x,s_i) \leq V_r(x,s) + 2\epsilon$ for i sufficiently large. Again let $\epsilon \to 0$ to obtain the desired result. ∎

With this result in mind it is natural to ask if the available storage is also an upper semi-continuous function of the supply rate, and thus continuous. In general, the answer to this question is negative:

Example 9: Consider the autonomous system with state space $\mathbb{X} = \mathbb{R}$ and dynamics

$$\dot{x} = -x, \quad z = x.$$

Let a sequence of supply rates s_i be given by

$$s_i(z) = \begin{cases} -\frac{1}{i} & \text{if } 2^{-i(i+1)/2} \leq z \leq 2^{-(i-1)i/2} \\ 0 & \text{else.} \end{cases}$$

The space $\mathcal{S}$ is the linear span of these s_i for $i \in \mathbb{N}$, and we take $\mathbb{S} = \mathcal{S}$. I
is then straightforward to see that

$$V_a(x, s_i) = \log 2$$

for any $x \geq 1$. On the other hand, the supply rates s_i converge uniforml
to the supply rate $s = 0$ for which $V_a(x,0) = 0$. Hence the available storag
$V_a(x,\cdot)$ is not an upper semi-continuous function of the supply rate.

However, an example of particular interest is when the set $\mathbb{S}$ of dissipate
supply rates is a convex polytope, i.e. the convex hull of a finite collection o
supply rates. In this situation upper semi-continuity follows from convexity

Corollary 10: Take the same assumptions as in proposition 8 and assum
in addition that $\mathbb{S}$ is a convex polytope. Then $V_a(x,\cdot)$ and $V_r(x,\cdot)$ ar
continuous functions of $s \in \mathbb{S}$.

Proof: The statement follows from a standard result [92, p. 84] accordin
to which a convex function defined on a convex polytope is upper sem
continuous.

Another situation where continuity follows is when the available storag
and the required supply coincide. This is the case for lossless system
under certain assumptions, see [124].

We summarize and illustrate the discussion with the following simple ex
ample concerning $\mathcal{L}_2$-gain analsysis of a scalar linear system.

Example 11: Consider the system

$$\dot{x} = -x + w, \quad z = x$$

and the two supply rates $s_1 = |w|^2$ and $s_2 = -|z|^2$ corresponding to a
analysis of $\mathcal{L}_2$-gain from w to z. Let the space $\mathcal{S}$ of supply rates be th
span of s_1 and s_2.

Since the system is linear and the supply rates are quadratic we know [12
that if the system is dissipative w.r.t. the rate $\lambda_1 s_1 + \lambda_2 s_2$ then ther
exist a quadratic storage function $V(x) = \alpha x^2$. The differential dissipatio
inequality then reduces to the linear matrix inequality

$$\begin{bmatrix} -2\alpha + \lambda_2 & \alpha \\ \alpha & -\lambda_1 \end{bmatrix} \leq 0 \quad .$$

The set of those $\alpha, \lambda_1, \lambda_2$, for which the linear matrix inequality hold
is a cone. Let us concentrate on the subcone for which $\lambda_2 > 0$. W

may obtain a cross-section of this cone by fixing $\lambda_2 = 1$ and examine which values α and λ_1 result in a supply rate and a storage function which satisfy the dissipation inequality. A little manipulation yields that this set is characterized as

$$\alpha \geq \frac{1}{2}, \quad (2\alpha \Leftrightarrow 1)\lambda_1 \geq \alpha^2 \quad .$$

This set is depicted in figure 2.2. It has the structure which was predicted by the previous results: It is convex and closed as is its projection on the λ_1-axis. Furthermore, the available storage and the required supply are continuous functions of λ_1, convex and concave, respectively. In addition the set has the special feature of being unbounded since s_1 is sign definite. □

2.4 Chapter conclusion

For a dissipative dynamic system, we have asked the question: With respect to which supply rates is the system dissipative? We have shown elementary properties associated with these *dissipated* supply rates: They form a convex cone which is also closed under additional assumptions. Furthermore we have investigated continuity properties of the available storage and the required supply, seen as functions of the supply rate. For the important special case of convex polytopes of supply rates, we have shown that these functions are continuous.

Many of our results have been shown under the assumption that the sets of zero available storage are independent of the supply rate under consideration. An interesting topic of future research would be to relax this assumption.

Our original motivation for this study was the situation where a dynamic system contains perturbations which are known to be multi-dissipative. In this situation the inherent convexity can be employed to obtain quite sharp conditions for robust stability and performance by means of convex optimization: we optimize over the set of supply rates with respect to which the nominal system is dissipative. In the special case of linear systems and quadratic supply rates the dissipation inequalities are linear matrix inequalities and the numerical tool is semidefinite programming. Some results along these lines follow in chapter 3 below.

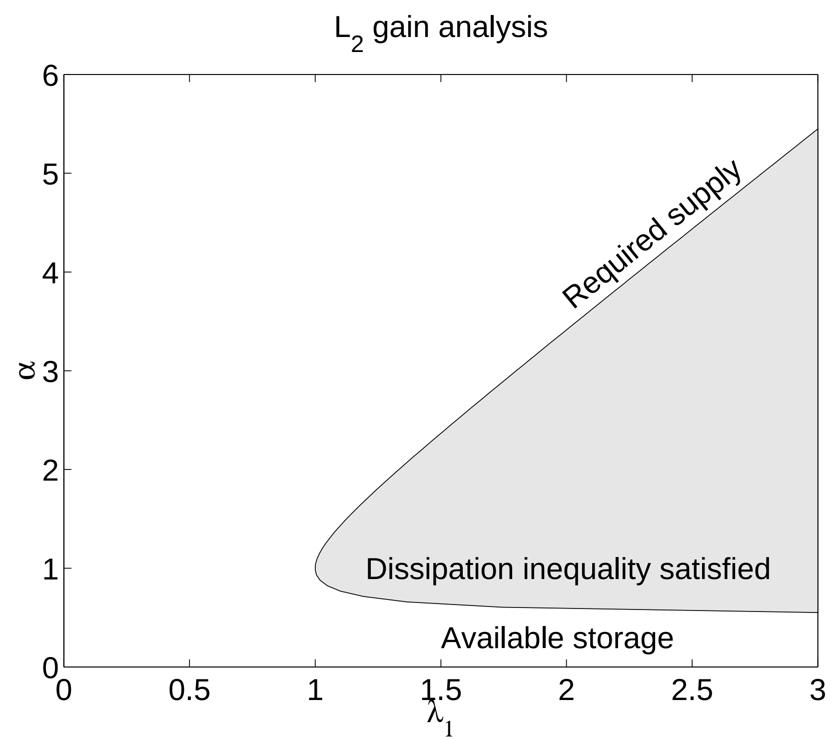

Figure 2.2: A cross-section of the cone of supply rates and storage function which satisfy the dissipation inequality.

Another application of the theory of multi-dissipative systems is the problem of *control for multi-dissipation*, i.e. find a controller in some class such that the resulting closed-loop system is dissipative w.r.t. a family of supply rates. For instance, one may require that the closed loop is small gain and that the controller is passive. In chapter 4 below, which concerns the problem of simultaneous $\mathcal{H}_\infty$ control, such a problem of control for multi-dissipation arises.

Chapter 3

Robustness towards multi-dissipative perturbations

We investigate the robustness of an interconnection of a nominal system, described by nonlinear ordinary differential equations, and an unknown perturbation which is dissipative with respect to several supply rates. We give sufficient conditions for global robust stability and performance in terms of existence of solutions to nonlinear partial differential inequalities of the Hamilton-Jacobi-Bellman type with certain extra degrees of freedom, namely a vector of weights. We then specialize to linear systems with quadratic supply rates where the analysis reduces to linear matrix inequality problems.

It is popular to deal with uncertainty in control problems using the frame work of dissipation (in the sense of Willems [124] and the previous chapter because dissipativity is preserved under interconnections of systems and be cause dissipativity for autonomous systems implies stability. This makes practical to model uncertainty by dissipative perturbations, and to pose a design specification that the overall system is dissipative. A common exam ple of a dissipation property is bounded $\mathcal{L}_2$-gain. This particular propert leads to linear or nonlinear $\mathcal{H}_\infty$ control, where the uncertainty is modelled by perturbations which have bounded $\mathcal{L}_2$-gain, and where performance of the overall system is measured by its $\mathcal{L}_2$-gain as well. Also passive pertubations are common; for instance stability proofs of certain adaptive control systems employ passivity-based arguments.

In this chapter we consider robustness towards deterministic dynamic perturbations which are dissipative with respect to *several* supply rates. Section 3.1 motivates this problem by providing examples of such multi-dissipative perturbations. In this section we also compare the framework with that of integral quadratic constraints. In section 3.2, we demonstrate that information regarding multiple dissipation properties of the perturbations can be included in an robustness analysis in an operational fashion. The resulting conditions on the nominal part of the system are partial differential inequalities of the Hamilton-Jacobi-Bellman type with certain extra degrees of freedom. Section 3.3 specializes the discussion to linear systems and quadratic supply rates; in these situations linear matrix inequalities becomes an efficient numerical tool with which we can also address related problems involving parameter uncertainty, or of robust $\mathcal{H}_2$ performance. Finally, section 3.4 offers some concluding remarks and points out a number of open problems.

3.1 Multi-dissipative perturbations

The aim of this section is to provide a few examples of multiple dissipation properties of perturbations in control systems. The section merely summarizes some ideas - some well known, others seemingly new - and does not present new results.

For a single dynamic perturbation $w(\cdot) = \Delta z(\cdot)$, typical dissipation properties are related to gain and phase properties. For instance, linear positive real perturbations - or more generally nonlinear passive perturbations - are dissipative w.r.t. the supply rate $s(w,z) = \langle w, z\rangle$. Similarly Δ has $\mathcal{L}_2$-gain (or $\mathcal{H}_\infty$ norm) less than or equal to $\gamma > 0$ if and only if Δ is dissipative w.r.t. the supply rate $s(w,z) = \gamma^2|z|^2 \Leftrightarrow |w|^2$ - this can be generalized to any $\mathcal{L}_p$ induced norm for finite p.

When the perturbation represents parasitic dynamics, for instance oscillatory modes in a mechanical or electrical system, the passivity follows from

the fact that such oscillations cannot produce physical energy. More ger
erally, physical conservation laws give rise to dissipation properties. Cor
servation of mass, volume, free thermodynamic energy, or momentum ca
be cast as dissipation properties.

Bounds on static (memoryless) nonlinearities can also be expressed in term
of dissipation properties, although the information that Δ is static is los
Specifically, let $w(t)$ and $z(t)$ be scalar and let Δ be given by

$$w(t) = (\Delta z)(t) = \phi(z(t))$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is known to satisfy the inequality $\psi(z, \phi(z)) \geq 0$, the
obviously Δ is dissipative w.r.t. the supply rate $s(w,z) = \psi(z,w)$.
particular popular class of bounds are the linear *sector bounds* which ar
common in the field of absolute stability, see [59] and the references therei
For instance, if the graph of ϕ lies between the lines $w = az$ and $w = b$
for known real numbers $a < b$ then the corresponding function ψ may b
taken as the quadratic form

$$\psi(z,w) = (w \ z) \begin{bmatrix} \Leftrightarrow 1 & \frac{a+b}{2} \\ \frac{a+b}{2} & \Leftrightarrow ab \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \quad . \qquad (3.1$$

It is important to examine how much conservativeness one introduces b
neglecting that the perturbation is static. When the supply rate s
quadratic, a partial answer to this question is obtained by examining whic
linear time invariant systems dissipate s.

The above examples illustrate how one may establish *single* dissipatio
properties of perturbations. Our prime example of a multi-dissipative dy
namic perturbation concerns parasitic dynamics which are bounded an
passive:

Example 12: [Modelling of multi-dissipative perturbations] Conside
the spring-mass system in figure 3.1, which is a simple model of a on
dimensional position regulator system. The force u is the output of an linea
time invariant controller. We consider the small mass as an unmodelle
parasitic, and the parameters associated with it to be very uncertain.

The overall interconnection of the small mass and the remaining syste
may be written in the form of figure 3.2. The error signal z is then th
velocity $\dot{y}$ of the large mass while the disturbance w is the force acting fro
small mass on the large mass. With this formulation, Δ is given by

$$\Delta(s) = \frac{(k+cs)ms}{ms^2 + cs + k} \quad .$$

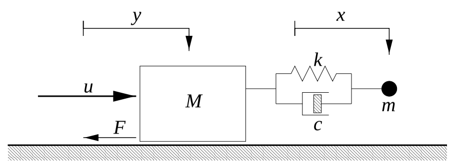

Figure 3.1: A position regulator with parasitic dynamics

The transfer function Δ is positive real, i.e. dissipative w.r.t. the supply rate $\Leftrightarrow s_1(z,w) = zw$, since this supply rate corresponds to the mechanical energy supplied to the parasitic. It is also small gain, i.e. dissipative w.r.t. the supply rate $\Leftrightarrow s_2(z,w) = \gamma^2 z^2 \Leftrightarrow w^2$ for $\gamma \geq ||\Delta||_\infty \approx km/c\sqrt{1+c^2/km}$.

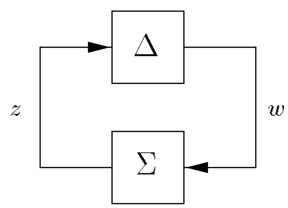

Figure 3.2: System and perturbation in feed-back

One can easily imagine situations where physical considerations or a few simple experiments provide a reasonable bound on γ but where we nevertheless do not wish to estimate m, c and k. Indeed, in many situations we do not wish to specify the order of Δ. In these situations, the only information about Δ we wish to make use of in the subsequent analysis is the two dissipation properties. □

See also [33] for a discussion of this example in the context of integral quadratic constraints.

Another way multiple dissipation properties arise is when the input z and the output w to the perturbation Δ can be partitioned as

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix}$$

and the perturbation Δ is block diagonal, i.e.

$$w = \Delta z \;\Leftrightarrow\; w_i = \Delta_i z_i,\; i = 1, \ldots p \quad .$$

This block structure occurs when each perturbation Δ_i is associated with different physical component in the system (if each block diagonal elemen Δ_i is bounded in $\mathcal{H}_\infty$-norm, then the resulting problem is μ-analysis). I is clear that the total perturbation Δ inherits the dissipation propertie of each perturbation element Δ_i. This may quickly lead to a quite larg number of dissipation properties of Δ as a realistic control problem typicall will contain uncertain elements in many different places in the control loo

We conclude this section with an example demonstrating how one ma establish necessary and sufficient conditions for the robustness of a syster containing a multi-dissipative perturbation:

Example 13: [A non-conservative robustness condition] Continuing exam ple 12 above, the suitable analysis question is: When is Σ robustly stab towards perturbations Δ which are linear time invariant and dissipati with respect to $\Leftrightarrow s_i$ for $i \in \{1,2\}$? Here we construct necessary and su ficient conditions through frequency domain considerations. First, resca the system such that that $\gamma = 1$. The requirement that Δ is linear tim invariant, passive and small gain then is equivalent to the transfer functio $\Delta(s)$ mapping the closed right half of the complex plane into the set A i figure 3.3, i.e.

$$\forall s \in \bar{\mathbb{C}}^+ :\; \Delta(s) \in A = \{s \in \mathbb{C} \mid \mathbf{Re}\, s \geq 0 \wedge |s| \leq 1\} \qquad (3.2$$

The interconnection is unstable if and only if the closed loop has a pole i the closed right half plane, i.e. there exists an $s \in \bar{\mathbb{C}}^+$ such that

$$\Sigma(s)\Delta(s) = 1 \quad .$$

So the interconnection is stable *for all* Δ such that (3.2) holds if and onl if Σ maps the closed right half plane into the region B in figure 3.3, i.e.

$$\forall s \in \bar{\mathbb{C}}^+ :\; \Sigma(s) \in B := \{s \in \mathbb{C} \mid \frac{1}{s} \notin A\} \qquad (3.3$$

$$= \{s \in \mathbb{C} \mid \mathbf{Re} s < 0 \vee |s| < 1\} \quad .$$

An alternative characterization of the set B is

$$B = \{s \in \mathbb{C} \mid \exists \alpha > 0 :\; |s + \alpha| < |i + \alpha|\} \quad .$$

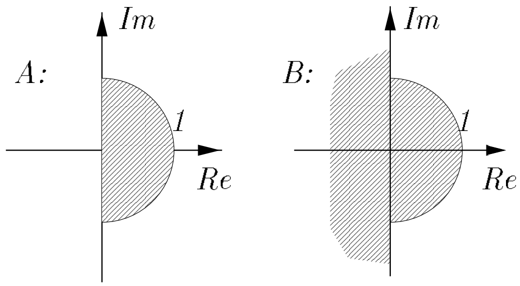

Figure 3.3: The subsets A and B of the complex plane

So the condition (3.3) holds if and only if there exists a map $\alpha : \bar{\mathbb{C}}^+ \to \mathbf{R}^+$ such that

$$\forall s \in \bar{\mathbb{C}}^+ : \ |\Sigma(s) + \alpha(s)| < |i + \alpha(s)| \quad .$$

The inequality in this expression can be restated as

$$(1 \ \bar{\Sigma}(s)) \begin{bmatrix} 1 & \Leftrightarrow\alpha(s) \\ \Leftrightarrow\alpha(s) & \Leftrightarrow 1 \end{bmatrix} \begin{pmatrix} 1 \\ \Sigma(s) \end{pmatrix} > 0 \quad . \tag{3.4}$$

To recapitulate, the feedback system is stable if and only if there exists an $\alpha : \bar{\mathbb{C}}^+ \to \mathbb{R}^+$ such that this holds for all $s \in \bar{\mathbb{C}}^+$. This is exactly the type of stability conditions that appear in [57] (see also the references therein); in the nomenclature used there Δ satisfies two *integral quadratic constraints* (IQC's). □

In more complicated situations, involving several perturbations or non-linear systems, the problem of obtaining non-conservative conditions for robustness is still untractable. In the remainder of this chapter we derive *sufficient* conditions only. We shall later return to this example in order to comment on the conservativeness inherent in our conditions.

3.2 Robustness analysis

We now turn to the interconnection of figure 3.4 where Σ is the nominal system, Δ is a multi-dissipative perturbation and v is an exogenous deterministic signal. Throughout the section, x denotes a state of Σ while ξ denotes a state of Δ.

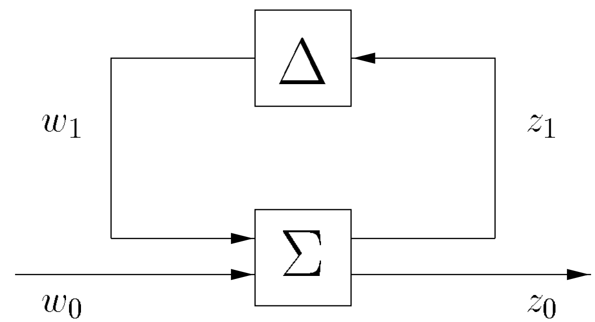

Figure 3.4: Setup for robust performance analysis.

The problems we consider are generalizations of robust non-linear $\mathcal{H}_\infty$ ana ysis problems. Several versions of these problems exist; one is the followin The unknown perturbation Δ is a causal system with $\mathcal{L}_2$-gain less than c equal to 1, i.e. is dissipative w.r.t. the supply rate $|z|^2 \Leftrightarrow |w|^2$. Commonl the interconnection is assumed to be at rest at the initial time $t = 0$. Th aim of the analysis is to establish an upper bound for

$$\int_0^T |y(t)|^2 \ dt$$

which holds for all perturbations Δ, all final times T and all inputs $v($ with $\int_0^T |v|^2 \ dt \leq 1$.

The aim of this section is to consider robustness analysis problems whic generalize this $\mathcal{H}_\infty$ problem above in several directions. The objective to establish a bound on

$$\int_0^T l(y(t)) \ dt$$

where l is a given non-negative *running cost.* To retrieve robust $\mathcal{H}_\infty$ ana ysis, use $l(y) = |y|^2$.

The following list makes precise the assumptions under which we obtai our robustness result:

Assumption 14:

1. The interconnection of Σ and Δ is well posed in the following sens To each initial conditions x_0 and ξ_0 and each input $v(\cdot)$ correspon

unique state trajectories $x(\cdot)$ and $\xi(\cdot)$ which are continuous and depend causally on $v(\cdot)$, and the signals $w(\cdot)$, $z(\cdot)$ are measurable and locally bounded[1] functions of time, at least up to some finite escape time.

2. The perturbation Δ is dissipative w.r.t. to the p measurable and locally bounded supply rates $\Leftrightarrow s_i$, $i = 1, \ldots, p$, with available storage $V_a(\xi_0, \Leftrightarrow s_i) \leq \beta_i$ for a known set of bounds β_i. Here ξ_0 is the initial state of the perturbation Δ.

3. The input $v(\cdot)$ satisfies

$$\int_0^T s_v(t)\ dt \leq \beta_v$$

for a known bound β_v and any time $T > 0$, provided that no finite escape time occurs before T. Here $s_v(v, y)$ is a given measurable and locally bounded supply rate.

□

The motivation behind the first assumption is as follows: We disregard pathological situations where non-unique or discontinuous state trajectories occur, but we do not wish to exclude a finite escape time *a priori*; rather we wish to establish conditions under which a finite escape time cannot occur.

Regarding the second assumption, we retrieve robust $\mathcal{H}_\infty$ analysis by choosing $p = 1$, $s_1 = |w|^2 \Leftrightarrow |z|^2$, and $\beta_1 = 0$. In applications, it is not always reasonable to assume zero initial storage in the perturbations (i.e, $\beta_i = 0$) as is done in the robust $\mathcal{H}_\infty$ problem as outlined here; in order to study robustness of transient behaviour it is essential to allow some bounded amount of initial storage in the perturbation. (If focus is on stability or steady-state behavior assumptions corresponding to zero initial storage may be reasonable and are seen in the IQC literature, e.g. [77, 58, 33]; an exception is [96]). On the other hand, the assumption that the perturbations have bounded initial storage is often quite reasonable - although it may be difficult to establish the exact size of these bounds. A similar discussion applies to the assumption that the input v has a bounded resource given by the rate s_v and the bound β_v.

[1] A function is said to be locally bounded if the image of any bounded set is bounded.

With this problem setup the sufficient condition for our objectives to b met is that the nominal system is dissipative w.r.t. some supply rate whic matches the rates s_i, s_v and the running cost l. More precisely we can stat the following theorem:

Theorem 15: [Robustness implications of dissipativity] Let assumption 1 hold and assume in addition that the nominal system Σ is dissipative w.r. the supply rate $\sum_i d_i s_i + d_v s_v \Leftrightarrow l$ for some non-negative weights d_i, d_v. Le V be a corresponding storage function. Then the following holds:

1. If no finite escape time occurs, then the interconnection is dissipati w.r.t. the supply rate $d_v s_v \Leftrightarrow l$.

2. The state $x(T)$ remains in the set

$$\{x \mid V(x) \leq V(x_0) + \sum_i d_i\beta_i + d_v\beta_v\}$$

for any $T > 0$ such that no finite escape time occurs before T.

3. If $V(\cdot)$ and $\sum_i d_i V_a(\cdot, s_i)$ are proper[2] functions, then no finite escap time occurs.

4. The performance bound

$$\int_0^T l\ dt \leq V(x_0) + \sum_i d_i\beta_i + d_v\beta_v$$

holds for any $T > 0$ such that no finite escape time occurs before T

Proof: Fix the initial states x_0 and ξ_0 and the input v and let $T > 0$ be time such that no finite escape time occurs before T. As candidate storag function for the interconnection w.r.t. the supply rate $d_v s_v \Leftrightarrow l$ we tak $W(x, \xi) = V(x) + \sum_i d_i V_a(\xi, \Leftrightarrow s_i)$. It is then easy to see that W satisfie the dissipation inequality which proves item 1. Using the non-negativenes

[2] A real-valued function is said to be proper iff all preimages $V^{-1}(I)$ of bounde intervals $I \subset \mathbb{R}$ are bounded.

of $V_a(\xi, \Leftrightarrow s_i)$ and of l we get

$$\begin{aligned} V(x(T)) &\leq W(x(T), \xi(T)) \\ &\leq W(x_0, \xi_0) + \int_0^T d_v s_v \Leftrightarrow l \; dt \\ &\leq V(x_0) + \sum_i d_i \beta_i + d_v \beta_v \end{aligned}$$

as claimed in item 2. If furthermore V and $\sum_i d_i V_a(\cdot, \Leftrightarrow s_i)$ are proper then this implies that $x(t)$ and $\xi(t)$ remain in a fixed bounded set which excludes the existence of a finite escape time and hence proves item 3. Finally item 4 is simply a rearrangement of the dissipation inequality of item 1. ■

A key feature of the theorem is that the characterization is *convex*: The set of those storage functions V and weights d_i, d_v which satisfy the dissipation inequality is convex (proposition 2 on page 32 above). Furthermore, if we wish to search for the *best* weights d_i, d_v, i.e. those that lead to smallest available storage in a fixed initial point, then this involves minimizing a convex continuous function (proposition 8 on page 37 above and the subsequent corollary 10).

Another feature of the theorem is that it simultaneously addresses robust stability and performance: Robust performance in the sense of a bound on an integral is given in item 4. To demonstrate that item 2 can be used to show robust stability, we first establish a useful lemma:

Lemma 16: [Bounding the state trajectory] Let $\Omega \subset \mathbb{X}$ be an open set and let $x(t)$, $t \geq 0$, be a state trajectory such that $x(0) \in \Omega$. Let $\gamma > 0$ be such that $V(x(t)) \leq \gamma$ at least until $x(t)$ leaves Ω. Let $\mathbb{A}$ be the largest connected subset of $V^{-1}([0, \gamma]) \cap \Omega$ which contains the initial state $x(0)$. Assume that $\mathbb{A}$ is compact. Then $x(t)$ remains in $\mathbb{A}$ for $t \in [0, \infty)$. □

Proof: Assume that $x(t)$ leaves Ω in finite time. Let t_2 denote the time of first exit from Ω and let t_1 denote the last preceding time of exit from $\mathbb{A}$. Since $\mathbb{A}$ is closed and Ω is open we have $t_1 < t_2$. Let $t \in (t_1, t_2)$ and define $\mathbb{B} = \mathbb{A} \cup \{x(\tau) \; : \; \tau \in [t_1, t]\}$. Then $\mathbb{B}$ contains x_0 and is a connected subset of $V^{-1}([0, \gamma]) \cap \Omega$ of which $\mathbb{A}$ is a strict subset. This is in contradiction with the definition of $\mathbb{A}$. We conclude that $x(t)$ remains in Ω, hence also in $\mathbb{A}$, until a finite escape time. Since $\mathbb{A}$ is bounded this excludes finite escape times; hence $x(t)$ always remains in $\mathbb{A}$. ■

The importance of $\mathbb{A}$ being closed is illustrated in figure 3.5. Here $\mathbb{A}$ is not

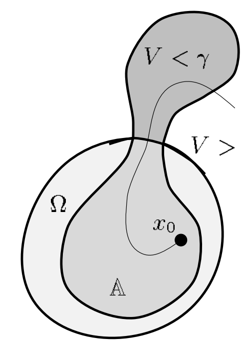

Figure 3.5: A pre-image $\mathbb{A} = V^{-1}([0, \gamma]) \subset \Omega$ which is not closed.

closed and hence the state trajectory can leave $\mathbb{A}$ and Ω simultaneousl
once the state has exited Ω the bound $V(x) \leq \gamma$ needs not hold.

We can now pose a result regarding robust Lyapunov stability of the inter
connection:

Corollary 17: [Dissipativity implies robust Lyapunov stability] Take th
same assumptions as in the theorem. Let $\bar{x}$ be a strict local minimu
point of the storage function V and assume that V is continuous in
neighbourhood Ω of $\bar{x}$. Then there exists another neighbourhood $\Omega' \subset$
of $\bar{x}$ such that the following holds: If the initial state x_0 is in Ω', and if th
positive bounds β_i, β_v are small enough, then the state $x(t)$ remains in
for $t \in [0, \infty)$; in addition the performance bound

$$\int_0^\infty l \; dt \leq V(x_0) \Leftrightarrow V(\bar{x}) + \sum_i d_i \beta_i + d_v \beta_v$$

holds.

The proof of the corollary is conceptually identical to standard Lyapun
stability proofs, e.g. [59], although some extra technicalities are neede
because $V(x(t))$ is not necessarily a non-increasing function of time.

Proof: Set $\alpha = V(\bar{x})$. Assume without loss of generality that Ω
bounded and that $\inf_{x \in \Omega} V(x) = \alpha$: If not, then replace Ω with $\Omega \cap$
where B is a sufficiently small bounded neighbourhood of $\bar{x}$. Let $\gamma >$
and let $\mathbb{A}$ denote the largest connected subset of $V^{-1}([\alpha, \gamma]) \cap \Omega$ whic

contains $\bar{x}$; notice that $\bar{x}$ is an interior point in $\mathbb{A}$. Assume that γ is chosen such that $\mathbb{A}$ is closed; this is possible since $\bar{x}$ is a strict local minimum. Assume that β_i and β_v are small enough, i.e. $\alpha + \sum_i d_i + d_v \beta_v < \gamma$. Let Ω' be any neighbourhood of $\bar{x}$ contained in $V^{-1}([\alpha, \gamma \Leftrightarrow \sum_i d_i \beta_i \Leftrightarrow d_v \beta_v)) \cap \mathbb{A}$; Now assume that $x_0 \in \Omega'$. According to item 2 in theorem 15 $x(t)$ remains in $V^{-1}([0, \gamma])$ at least up to a finite escape time. Now apply lemma 16 to see that $x(t)$ remains in the bounded set $\mathbb{A} \subset \Omega$ for $t \in [0, \infty)$. The performance bound follows from the dissipation inequality since $V(x(t)) \geq V(\bar{x})$. ■

In the proofs above the dissipation inequality does not need to hold *everywhere* but only along the possible trajectories. This is particularly useful when studying *local* behaviour. Developments along these lines are reported in [113].

3.3 Linear systems and quadratic supply rates

In this section we specialize the previous discussion to the case of linear systems Σ defined by ordinary differential equations in state-space:

$$\Sigma : \quad \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bw(t) \\ z(t) & = & Cx(t) + Dw(t) \end{array} \quad . \tag{3.5}$$

For systems consisting of a nominal part in feed-back with a multi-dissipative perturbation, we show that stability and various performance properties can be described by linear matrix inequalities (LMIs) which describes some dissipativity property of the nominal part. Such linear matrix inequalities can be verified directly numerically with commercially available packages such as [32, 38]. The connection between dissipativity for linear-quadratic systems and LMIs was noted already in [124] and has received much interest during the last few years [19] due to efficient numerical algorithms for solving LMI problems [82].

3.3.1 Robust stability

Consider the connection in figure 3.6 (a) where Σ is the nominal system and Δ is a perturbation in a set $\boldsymbol{\Delta}$; both are assumed to be causal, linear, finite dimensional, and time invariant systems. We say that the

configuration $(\Sigma, \boldsymbol{\Delta})$ is robustly stable if for every $\Delta \in \boldsymbol{\Delta}$ the configuration is well posed (i.e, the dynamics of the closed loop can be writte $(\dot{x}(t), \dot{\xi}(t)) = \bar{A}(x(t), \xi(t))$ for some linear $\bar{A}$), and if furthermore $z(\cdot) \in \mathcal{L}$

For a deterministic linear time invariant systems with quadratic suppl rates (i.e, when $s(w,z) = (w' \;\; z')Q(w' \;\; z')'$), there is no loss of genera ity [124] in assuming the storage function V to be quadratic ($V(x) = x'P$ with $P = P' \geq 0$), in which case the differential dissipation inequality (2.3 becomes [124]

$$\forall x \in \mathbb{X},\; w \in \mathbb{W} : (x' \;\; w')\Phi \begin{pmatrix} x \\ w \end{pmatrix} \leq 0$$

where Φ is shorthand for

$$\Phi = \begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q \begin{bmatrix} O & I \\ C & D \end{bmatrix}$$

This is a linear matrix inequality (LMI) in P.

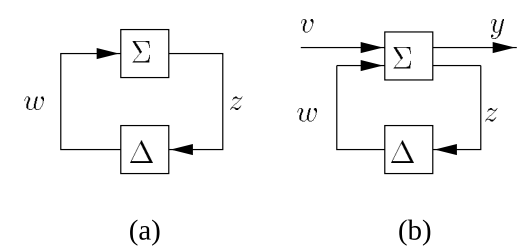

Figure 3.6: The two problems considered: (a) Robust stability. (b) Robus $\mathcal{H}_2$ performance.

Lemma 18: Assume that every $\Delta \in \boldsymbol{\Delta}$ is linear, time invariant, an dissipative w.r.t. $\Leftrightarrow s$, and that Σ is strictly output dissipative with respec to the supply rate s. Then the feed-back configuration $(\Sigma, \boldsymbol{\Delta})$ is robustl stable.

Remark 19: If Σ is dissipative w.r.t. s but not strictly output diss pative, and if the interconnection is well posed (which in this case is no guaranteed by the dissipativity), and if the storage functions are (quadrat and) positive definite, then we have Lyapunov stability [124], but z need not be in $\mathcal{L}_2$. □

Proof: The dissipation inequalities for Σ and Δ are

$$\begin{aligned} V(x(T)) &\leq V(x(0)) + \int_0^T s(w,z) \Leftrightarrow \epsilon ||z||^2 \, dt \quad , \\ W(\xi(T)) &\leq W(\xi(0)) + \int_0^T \Leftrightarrow s(w,z) \, dt \quad . \end{aligned}$$

Since we know that V has a local minimum (possibly non-strict) at $x=0$, dissipativity of Σ implies that $s(w,z) \Leftrightarrow \epsilon||z||^2 \geq 0$ for $x=0$. Repeating the argument for W we get that $s(w,z) \leq 0$ whenever $\xi = 0$. Combining the two we get that $z=0$ whenever $(x,\xi) = (0,0)$. Due to linearity of the output equation of Δ we conclude that $w=0$. This implies (using linearity) that the interconnection is well posed, and hence the solutions exist on $[0,\infty)$.

Adding the two dissipation inequalities and using the non-negativity of the storage functions give

$$\int_0^T ||z||^2 \, dt \leq \frac{1}{\epsilon}\left(V(x(0)) + W(\xi(0))\right) \quad .$$

Since this holds for all T, the configuration is robustly stable. ■

Remark 20: It can be argued that the lemma is of limited interest since it is very restrictive to assume that perturbations are linear time invariant (a similar point was also emphasized in [106]). Notice, however, that these assumptions on Δ are only used to guarantee existence and uniqueness of the state trajectories; the $\mathcal{L}_2$-bound on z follows directly from the storage functions. Luckily, there exist other ways of guaranteeing existence and uniqueness when Δ is nonlinear and/or time varying. For instance, local well-posedness may be established through linearizations, algebraic loops may be avoided by assuming that Δ is strictly causal, and finally including w in z and assuming properness of W guards against finite escape times.

To avoid these details we will in the remainder of the chapter always assume that any feedback connection which we analyse is well-posed in the sense that there exist unique signals which solve the describing equations. We remark that if one is willing to make the assumption that Δ is LTI, then well-posedness is guaranteed by the stability conditions we derive. □

In this remainder of the chapter we consider analysis of systems Σ connecte in feedback as in figure 3.6 with a perturbation Δ, which is dissipative wit respect to the p supply rates

$$\Leftrightarrow s_i = \Leftrightarrow (w' \;\; z') Q_i \begin{pmatrix} w \\ z \end{pmatrix}, \qquad i \in \{1,\ldots,p\} \qquad (3.6$$

Without loss of generality we assume that Q_i are symmetric. We use th symbol $\mathbf{\Delta}$ to denote this particular class of *multi-dissipative* perturbation i.e.

$$\mathbf{\Delta} = \{\Delta : \text{ dissipative w.r.t. } \Leftrightarrow s_i;\ i \in \{1,\ldots,p\}\} \qquad (3.7$$

To each $\Delta \in \mathbf{\Delta}$ and each supply rate $\Leftrightarrow s_i$ corresponds a storage functio $W_i(\xi)$ defined on the state space of Δ.

Specializing theorem 15 to the linear-quadratic case leads the followin result:

Theorem 21: Given the system Σ defined in (3.5) and the class c perturbations $\mathbf{\Delta}$ of (3.7). The configuration $(\Sigma, \mathbf{\Delta})$ is robustly stable if th following linear matrix inequality

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} \Leftrightarrow \sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} + \epsilon \begin{bmatrix} C' \\ D' \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \leq \qquad (3.8$$

is satisfied for some $P = P' \geq 0,\ \epsilon > 0, d_i \geq 0$

We emphasize that the theorem gives a less restrictive condition than e.g small gain criterion or positive real criterion because of the extra freedor associated with d_i.

Remark 22: One will often examine the following linear matrix inequalit problem in $\bar{P}$ and $\bar{d}$ in stead:

$$\begin{bmatrix} \bar{P}A + A'\bar{P} & \bar{P}B \\ B'\bar{P} & 0 \end{bmatrix} \Leftrightarrow \sum_{i=1}^{p} \bar{d}_i \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} < 0 \quad ,$$

$$\bar{P} > 0,\ \bar{d}_i > 0 \quad . \qquad (3.9$$

Feasibility of this LMIP is a sufficient condition for robust stability c $(\Sigma, \mathbf{\Delta})$ and - under weak assumptions on the data - equivalent to feas bility of (3.8).

Proof: Given a solution pair P, d_i, ϵ, the function $x'Px$ acts as a storage function for Σ with respect to the supply rate $\sum_i d_i s_i - \epsilon|z|^2$ and since any $\Delta \in \mathbf{\Delta}$ is dissipative w.r.t. $-\sum_i d_i s_i$ (proposition 2 on page 32 above) we have shown robust stability (lemma 18).

It is also easy to see that feasibility of (3.9) implies feasibility of (3.8): In fact, given solutions $\bar{P}, \bar{d}_i$ to (3.9), one may find sufficiently small $\epsilon > 0$ such that $\bar{P}, \bar{d}_i, \epsilon$ solves (3.8). ■

A similar result was derived independently in the recent contribution [126].

Example 23: [The conservativeness of the sufficient condition] Continuing example 13 above, the two supply rates dissipated by the perturbation Δ are $-s_1 = zw$ and $-s_2 = |z|^2 - |w|^2$ corresponding to

$$Q_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad , \quad Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad .$$

The sufficient condition of remark 22 is that Σ is strictly output dissipative with respect to a combination of s_1 and s_2, i.e. with respect to

$$s_\alpha(w,z) = (w \ z) \begin{bmatrix} 1 & -\alpha \\ -\alpha & -1 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \tag{3.10}$$

for some $\alpha \geq 0$. Here we have taken $d_2 = 1$ which is possible due to the conicity, and $d_1 = \alpha$. This will be the case if and only if

$$(1 \ \bar{\Sigma}(s)) \begin{bmatrix} 1 & -\alpha \\ -\alpha & -1 \end{bmatrix} \begin{pmatrix} 1 \\ \Sigma(s) \end{pmatrix} > 0$$

holds for all s in the closed right half $\bar{\mathbb{C}}^+$ of the complex plane. For $\alpha \to 0$, we retrieve the condition that Σ has $\mathcal{L}_2$-gain less than 1, while for $\alpha \to \infty$ the permittable circle approaches the entire left half plane, and thus we get the condition that Σ is strictly negative real. For high order plants, the latter condition is often difficult or even impossible to obtain, while the former may impose too severe constraints on bandwidth. Also taking $\alpha \in (0, \infty)$ into account obviously increases the possibility of reaching a good design.

In comparison, the sufficient *and* necessary condition of equation (3.4) requires the existence of a function $\alpha : \bar{\mathbb{C}}^+ \to \mathbb{R}_+$ such that the inequality holds. In other words, the sufficient conditions of theorem 21 and remark 22 are conservative in that they do not allow frequency dependent weights d_i.

Notice however that linearity and time invariance of Δ is essential to th derivation of equation (3.4), whereas theorem 21 holds also for nonlinea and time-varying Δ provided that the interconnection is well posed.

Example 24: [A graphic interpretation] Continuing the preceding exam ple, we can also give a graphic interpretation of the sufficient condition tha Σ is strictly output dissipative w.r.t. s_α for some $\alpha \geq 0$: Let $S_{-\alpha}$ denot the circle in the complex plane which is centered in $-\alpha \in \mathbb{R}$ and whos boundary contains the point i - see figure 3.7. Then Σ is strictly outpu dissipative w.r.t. s_α if and only Σ maps the right half of the complex plan into the interior of the circle $S_{-\alpha}$. Combining with the maximum modu lus theorem, the sufficient condition of remark 22 is that Σ is stable an its Nyquist plot is contained in such a circle $S_{-\alpha}$ for some $\alpha \geq 0$. Th graphic criterion is reminiscent of the circle criterion for absolute stabilit see e.g. [59], except that we need only find *one* suitable circle in a certai family.

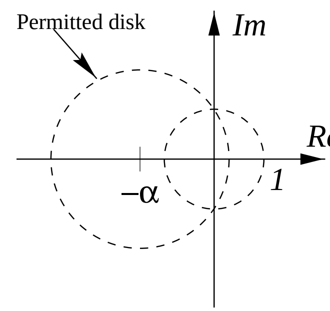

Figure 3.7: Permitted area for $\Sigma(\bar{\mathbb{C}}^+)$ with $\alpha(s)$ independent of s

A further understanding of the conservativeness of theorem 21 and re mark 22 is obtained from the following observation: If Σ dissipates s_α fo some $\alpha \geq 0$, then (Σ, Δ) is stable for any perturbation Δ which maps th right half of the complex plane into the circle S_α. Notice that any suc circle S_α contains the original set A of figure 3.3 on page 48. The conserva tiveness of theorem 21 is thus illustrated by the difference between the se A and the sphere S_α which covers A. This interpretation is not restricte to this particular example, but applies to theorem 21 and remark 22 i general.

3.3.2 Parameter uncertainty

A popular model of parameter uncertainty is that the system matrices A, B, C and D of the system (3.5) are time varying but remain in a given polytope:

$$\begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \mathbf{Co}\left(\left\{\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \mid j = 1, \ldots, m\right\}\right) \quad . \tag{3.11}$$

Here $\mathbf{Co}(\cdot)$ denotes convex hull. This situation covers not only parameter uncertainties but also non-linear systems with sector-bounded nonlinearities. In [19] a model like this is called a *Polytopic Linear Differential Inclusion*, and many properties of such models are reduced to feasibility of LMIs. It is therefore not surprising that also robustness in the presence of multi-dissipative perturbations can be guaranteed with LMIs. For this situation, a sufficient condition for robust stability in the presence of multi-dissipative perturbations is given by the following:

Theorem 25: Let Σ satisfy (3.11). Assume that each of the supply rates s_i is concave in z and that the following LMI problem in P, d_i and ϵ is feasible:

$$\forall j \in \{1, \ldots, m\} :$$

$$\begin{bmatrix} PA_j + A_j'P & PB_j \\ B_j'P & 0 \end{bmatrix} \Leftrightarrow \sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C_j' \\ I & D_j' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C_j & D_j \end{bmatrix}$$

$$+\epsilon \begin{bmatrix} C_j' \\ D_j' \end{bmatrix} \begin{bmatrix} C_j' & D_j' \end{bmatrix} \leq 0 \quad ,$$

$$P \geq 0,\ d_i \geq 0,\ \epsilon \geq 0 \quad .$$

Then the feed-back configuration $(\Sigma, \mathbf{\Delta})$ is robustly stable. □

Remark 26: The assumption that s_i is concave in z may be written

$$Q_i^{zz} \leq 0 \text{ where } Q_i = \begin{bmatrix} Q_i^{ww} & Q_i^{wz} \\ Q_i^{zw} & Q_i^{zz} \end{bmatrix}$$

and essentially says, that when the input w is zero, the flow s_i is zero or directed out of the system Σ. □

Proof: The proof consists of tedious though straightforward manipulations of linear matrix inequalities and can be found in appendix B.1 on page 167. The idea is that the time-varying system is strictly output dissipative with respect to the supply rate $\sum_i d_i s_i$; as storage function we use the time-invariant function $V(x) = x'Px$. ■

3.3.3 Guaranteed $\mathcal{H}_2$ Performance

We now expand the system with an exogenous input $v(t)$ and a performanc output $y(t)$, corresponding to figure 3.6 (b) on page 55:

$$\begin{aligned} \dot{x} &= Ax + Bw + Gv \\ z &= Cx + Dw \\ y &= Hx + Jw \end{aligned} \tag{3.12}$$

As before, we have $w = \Delta z$ where $\Delta \in \mathbf{\Delta}$. We use the symbol (Σ, Δ) t denote the closed-loop system with input v and output y. As a measure c performance for (Σ, Δ) we use its $\mathcal{H}_2$-norm.

When Δ is nonlinear and/or time-varying one needs to specify what i meant by the $\mathcal{H}_2$-norm of the interconnection, since it cannot be represente by a transfer function. Two possibilities exist: One can use the $\mathcal{L}_2$-norn of the impulse response which we will call the deterministic $\mathcal{H}_2$-norm c the interconnection, or one can assume that v is white noise and conside the steady-state variance of y. At this point we discuss the deterministi interpretation while the stochastic approach is taken in the second part c this thesis, in chapter 6 below.

Theorem 27: The $\mathcal{H}_2$ norm of the closed loop system from v to y i bounded above by

$$||(\Sigma, \Delta)||^2_{\mathcal{H}_2} \leq \inf_{P, d_i, \epsilon} \operatorname{tr}(G'PG)$$

where the infimization is subject to

$$P = P' \geq 0,\ d_i \geq 0,\ \epsilon > 0$$

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} \Leftrightarrow \sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C' \\ I & D' \end{bmatrix} Q_i \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}$$

$$+ \begin{bmatrix} H' \\ J' \end{bmatrix} \begin{bmatrix} H & J \end{bmatrix} + \epsilon \begin{bmatrix} C \\ D \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \leq 0$$

Remark 28: Computation of the upper bound on $\mathcal{H}_2$-performance is a LMI problem in P, d_i and ϵ.

Notice that if we remove Δ, the LMI in P reduces to $PA + A'P + H'H \leq 0$, i.e. we retrieve the standard way of computing the $\mathcal{H}_2$-norms of known system using the obervability Gramian [128]. □

Proof: Given P, d_i and ϵ, we know that Σ *with* $v \equiv 0!$ is strictly output dissipative w.r.t. $\sum_i d_i s_i \Leftrightarrow ||y||^2$. This implies strictly output dissipativity w.r.t. $\sum_i d_i s_i$ and hence it is reasonable to assume well-posedness of the interconnection of Σ and Δ, cf. remark 20.

Assume that the interconnection (Σ, Δ) is at rest for $t < 0$ and that we at $t = 0$ excite the interconnection with an impulse at v, i.e. $v(t) = v_0\delta(t)$ where $\delta(\cdot)$ is the Dirac delta. We then have $x(0^+) = Gv_0$. Assume that P, d_i and ϵ solve the LMI problem in the theorem, then the integral dissipation inequality for the interconnection reads

$$\int_0^T ||y||^2 \, dt + \int_0^T \epsilon ||z||^2 \, dt \leq x'(0^+) P x(0^+)$$

(and holds because $v(t) = 0$ for $t > 0$). Hence,

$$\int_0^\infty ||y||^2 \, dt \leq v_0' G' P G v_0 \quad .$$

Now let v_j be the jth unit vector in the input space $\mathbb{V} = \mathbb{R}^{n_v}$ $(v(t) \in \mathbb{V})$ and let $y_j(t)$ be the impulse response of the interconnection (Σ, Δ) to the input $v(t) = v_j\delta(t)$. We then have

$$||(\Sigma, \Delta)||^2_{\mathcal{H}_2} = \sum_{i=1}^{n_v} \int_0^\infty ||y_j(t)||^2 \, dt \leq \sum_{i=1}^{n_v} v_i' G' P G v_i = \text{trace } G'PG \quad .$$

Since this holds for any P, d_i and ϵ that solve the LMI problem the conclusion in the theorem follows. ■

3.4 Chapter conclusion

The concept of dissipation is widely used in the area of robust control and control of large scale systems, but except for the special cases of μ theory [28, 128] or more generally integral quadratic constraints [77, 57, 97], there has been no systematic use of the fact that systems possess several dissipation properties at once.

In this chapter we have reported results on the use of such multiple prop erties of dissipation. Our results essentially follow from the fact that th supply rates dissipated by a given system form a convex cone. We hav derived results corresponding to robust Lyapunov stability as well as robus performance. The framework allows generalization of several other stan dard Lyapunov-type results; of particular practical relevance is ultimat boundedness, slowly varying systems and parametric uncertainty. Man such extensions are straightforward.

The appeal of the framework is that it allows combination of informatic and specifications of different types. Admittedly the resulting condition will be conservative in that only sufficient conditions are given. Compare to common practice, however, where either several dissipation properties c the involved uncertain subsystems are ignored or the uncertainty is simpl left out of the analysis, the framework is an improvement.

It is appealing that the analysis reduces to linear matrix inequalities in th special, but very important, case of linear systems and quadratic suppl rates. For nonlinear systems the issue of numerical methods is more critica see the note below.

3.5 Notes and references

Comparison to the IQC framework

Consider a perturbation Δ which maps $z(\cdot)$ to $w(\cdot)$ and which is dissipativ with respect to a supply rate $s(z, w)$ which is quadratic, i.e. $s(z, w)$ = $(w'\ z')Q(w'\ z')'$. Assume that the available storage is 0 at time 0; then th signals satisfy the *integral quadratic constraint* (IQC)

$$\int_0^T (w'(t)\ z'(t)) Q (w'(t)\ z'(t))' \, dt \geq 0$$

for all times T. The converse also holds: If the IQC holds for all input $z(\cdot)$, and if the state space of the perturbation Δ is reachable, then Δ dissipative w.r.t. s with available storage 0 at time 0.

With this perspective, it is reasonable to compare our framework of mult dissipative perturbations to that of integral quadratic constraints. Clearl the motivation behind the two frameworks are identical as is the idea c *modularity*: reducing a large complex problem to a collection of smaller and more managable subproblems, viz. performing dissipation (or IQC) analysis on components. The techniques used are quite different, though.

The IQC framework, in the sense of [77, 57], makes heavy use of frequency-domain techniques. Although it is feasible to pose IQCs for specific nonlinear perturbations, see for instance [57], the resulting conditions on the nominal system are in frequency domain and no results are given as to how to verify these conditions for nonlinear nominal systems. The sufficient conditions are less conservative than the ones we have obtained in this chapter in that they make use of frequency dependent *multipliers* corresponding to our weights d_i (c.f. examples 13 and 23 above). In order to make use of these extra degrees of freedom in the *linear* case one needs to consider infinite-dimensional convex optimization problems associated with the choice of multipliers; this is the major numerical challenge. In comparison the hurdle in our framework of multi-dissipative perturbations is the computation of storage functions for *nonlinear* nominal systems, which also can be cast as an infinite-dimensional convex optimization problem.

Another approach to integral quadratic constraints is found in [95, 96, 97]. These papers use time-domain techniques and the approach is closer to this chapter than is [77, 57]. Only linear nominal systems are considered as are integral quadratic constraints corresponding to $\mathcal{L}_2$-gain of the perturbations.

Numerical methods for optimal control problems

In order to verify if a given system dissipates a given supply rate one needs to consider the optimal control problem which defines the available storage or the required supply. Except for systems with low dimensional state spaces or a particular structure, this is an overwhelming numerical challenge which is the major obstacle to the practical use of the results in this chapter.

Among the numerical methods for optimal control problems, those based on dynamic programming rather than the maximum principle are most natural: In fact the optimal trajectories are of less interest whereas approximations of the value functions may serve as storage functions.

For a fixed supply rate, storage functions may be approximated by discretization of the differential dissipation inequality, [52, 65] or by a spectral

method where a storage function is sought in a given finite-dimensiona space [11]. An alternative is a recursive scheme due to Lukes [73, 120, 7 for computing the Taylor expansion of the value function around an isolate equilibrium point.

The sufficient conditions for robustness presented in this chapter requi finding a storage function *and a supply rate* simultaneously which adds a extra twist to the optimal control problem. One approach is to restri the storage function to a finite dimensional space and employ convex opt mization techniques, optimizing over the storage function and the d-weigh simultaneously. For input affine-quadratic systems, the LMI based proc dure described on page 20 may easily be modified to search simultaneous for the storage function V and the set of weights d_i, d_l required by the rem 15. Although the convexity makes a convergence analysis feasible, th size of the optimization problems grows exponentially with the number o states; this is Bellman's curse of dimensionality. More heuristic approach may be useful. For instance we presented in [113] an example where th d-weights first where fixed considering only the linearization of the syste afterwards higher order terms were included using a Lukes' scheme.

State feed-back controller design

We briefly comment on the problem of finding a state feedback controll $u(t) = \mu(x(t))$ such that the resulting closed loop system satisfies the su ficient condition derived in this chapter.

For a *fixed* supply rate, the problem of control for dissipation requires pra tically the same tools as the problem of dissipation analysis as is evide in [120]. This reference treats the special problem of $\mathcal{L}_2$-gain analysis a nonlinear $\mathcal{H}_\infty$ control, but the discussion applies to broad classes of supp rates: In stead of optimal control problems we consider differential gam and the differential dissipation inequality is replaced by a Hamilton-Jaco Isaacs equation. In both cases the Hamiltonian dynamics provides inform tion about existence of a value function. The issue of smoothness of stora functions becomes more problematic since control strategies are found fro the partial derivatives of the value functions; see [7, 105]. Local approx mations to value functions may be found by Lukes' scheme, [120, 78].

To employ the sufficient conditions in this chapter, we need to find a co trol law, a storage function, *and a supply rate*. In the reference [113] w suggested to fix the supply rate in a first step (which considered only the linearization of the system) and then apply Lukes' scheme.

An alternative is the following value-policy iteration: In the *value step*, for a *fixed* controller, we find a supply rate and a storage function such that the differential dissipation inequality holds. This analysis problem can for instance be solved with convex optimization as outlined above. Then, in the *policy step* we fix the supply rate and the storage function and compute the *maximum dissipation* controller, i.e. the control law which at each point in state space maximizes the worst-case dissipation. This is a family of static min-max problems. Then the value step and the policy step are iterated. It is easy to show monotonicity of such an algorithm; under suitable hypothesis this implies convergence. We have in [114] given the details in such an algorithm for the case of linear systems and quadratic supply rates and demonstrated it on a numerical example.

For a linear system and a quadratic supply rate it is possible to give a convex parametrization of linear controllers (static state feedback or full order output feedback) which make the system dissipative; this trick appeared first in [15] for the state feedback problem, see also [37, 126]. This motivates a two-step iterative procedure where the first step optimizes the supply rate while the second finds a controller which makes the closed loop system dissipative w.r.t. the current supply rate. A similar procedure is suggested in the recent reference [126]; see also [125].

Regarding output feedback control of nonlinear plants, it is principle possible to combine a search over the d-weights with the information state approach [55] to differential games. The resulting problems are in general deterringly complex and with the present state of the art heuristic approaches should be more fruitful; for instance, first solving the linearized problem and then applying Lukes scheme.

Towards a nonconservative condition

The technique in this chapter is essentially the following: If $V(x)$ is a storage function for Σ with respect to $s+\sum_i d_i s_i$ and $V_a(\xi, \Leftrightarrow s_i)$ are storage functions for Δ w.r.t. $\Leftrightarrow s_i$, then $V(x)+\sum_i d_i V_a(\xi, \Leftrightarrow s_i)$ is a storage function for the interconnection (Σ, Δ) w.r.t. s. One way to generalize this is to find a function $\bar{V}(x, \beta_i)$ such that the available storage of (Σ, Δ) is less than $\bar{V}$ provided that $V_a(\xi, \Leftrightarrow s_i) \leq \beta_i$. This leads to a less conservative

condition since we do not require $\bar{V}$ to be in the form $V(x) + \sum_i d_i\beta_i$. I fact this condition is nonconservative in a certain sense, and can be verifie by performing dissipation analysis of an extended plant. We do not pursu this further at this point but will return to the stochastic analogy of th idea in part II of this dissertation; see page 133.

Chapter 4

Simultaneous $\mathcal{H}_\infty$ Control

We consider the problem of finding one output feedback controller which achieves $\mathcal{H}_\infty$ performance when connected to any one of p linear time invariant plants. This is a prototype of an adaptive $\mathcal{H}_\infty$ control problem. We formulate the problem as a non-linear $\mathcal{H}_\infty$ problem and show that the minimax controller is finite dimensional but not based on certainty equivalence. Synthesis of the minimax controller involves solving a partial differential equation, namely a state feedback Hamilton-Jacobi-Isaacs equation. We investigate the structure of the solution and derive the heuristic certainty equivalence controller which has a switching architecture.

4.1 Introduction

Robustness in presence of both parametric and dynamic perturbations i an important problem which poses great theoretical difficulties. In applica tions, parametric uncertainty is typically effective at low frequencies, an is often highly structured. On the other hand, less structured dynami perturbations always affect high frequency behaviour [128, p. 216].

With a low level of parametric uncertainty and with a $\mathcal{H}_\infty$ bounded dy namic perturbation, *linear* controllers may suffice, which then can be de signed using μ synthesis [128, 5] or quadratic stabilization [130, 39, 15]; se also [34]. For larger levels of parametric uncertainty one would expect that improvement can be achieved by using nonlinear controllers which include an adaptation mechanism. This motivates the field of adaptive $\mathcal{H}_\infty$ control.

A natural approach to adaptive $\mathcal{H}_\infty$ control is to *extend* the state with the unknown parameters, thus obtaining a $\mathcal{H}_\infty$ control problem for a new nonlinear plant. Then, one may apply the differential game techniques [9, 120, 55] to nonlinear $\mathcal{H}_\infty$ control. This approach has been pursued in for instance [22, 25]. In these references uncertainty is restricted to special parts of the system such that the minimax controller is finite dimensional and based on certainty equivalence principles such as the one in [14].

In view of this, an immediate question is: With a dynamic game approach to adaptive $\mathcal{H}_\infty$ control, is certainty equivalence and finite-dimensional minimax controllers the generic situation, or a special case? To study this question we consider the special situation where the unknown parameter *a priori* is restricted to a known, finite set. Such problems of *simultaneous control* can be considered as a prototype of adaptive control problems - see e.g. [44]. Our conclusion is negative: Certainty equivalence can *not* be expected to hold in adaptive $\mathcal{H}_\infty$ control problems. Furthermore, the minimax controller must run a linear $\mathcal{H}_\infty$ filter for each possible value of the parameter. Therefore we expect the minimax controller to be infinite-dimensional when there is a continuum of possible parameter values.

Next, we show that the *heuristic* certainty equivalence controller guarantees $\mathcal{H}_\infty$ performance, provided that the minimax control input is uniquely defined for *almost* all times. This *weak* certainty equivalence principle emphasizes the following point: The important issue is not if the *best* (i.e, minimax) controller is based on certainty equivalence, but if a certainty equivalence based controller is *good enough,* i.e, guarantees that the $\mathcal{H}_\infty$ design objective is met.

Besides being prototypes of adaptive control problems, simultaneous control problems have been the subject of considerable independent research. Linear controllers are investigated in [17, 19]; in general nonlinear control leads to improvement. *Switching control* is studied in [79, 80, 81] and the references therein: These controllers consist of a bank of linear low-level controllers and a high-level logical switch, which connects one of the low-level controllers to the plant. One way of designing the switch is to find that *estimator* which supplies the best fit with observations, and then switch in the corresponding controller. This technique appears in [81, 79, 49].

In this chapter we point out that this technique, when applied to the prob lem of simultaneous $\mathcal{H}_\infty$ control, must be modified so that the switch com pares not just estimation errors but also a *control error* associated wit each controller. The resulting switching controller is exactly the heuristi certainty equivalence controller.

A problem with switching architectures is chattering. Chattering is rapi switching back and forth, or that unique (classical) solutions to the dynami equations seize to exist, depending on ones point of view. Modification o the switch to avoid chattering are suggested in [79]; in this chapter w suggest an alternative based on a smooth approximation of the switch.

The chapter is organized as follows: Section 4.2 formulates the simultaneou $\mathcal{H}_\infty$ control problem. Section 4.3 deals with the extended state feedbac problem while section 4.4 develops the filter for the worst-case extende state estimate. Section 4.5 discusses the the minimax controller. Section 4. concerns the heuristic certainty equivalence controller. Finally section 4. offers some conclusions.

Notation

If P is a two-port plant with disturbance input w, control input u, mea surements y and error signal z, and K is a controller with input y an output u, then (P, K) denotes the closed-loop system with inputs w an outputs z (see figure 4.1 below).

We use the standard notion of $\mathcal{L}_2$-gains, see [119] and page 17 above:

Definition 29: [$\mathcal{L}_2$-gain] The $\mathcal{L}_2$-gain of a state-space system Σ (mappin inputs $w(\cdot)$ to outputs $z(\cdot)$ through states $\zeta(\cdot)$) is denoted $||\Sigma||$ and is th infimum of all numbers $\gamma > 0$ such that

$$\forall \zeta_0 : \ \exists M(\zeta_0) : \ \forall t_f > t_0, \ w \in \mathcal{L}_2([t_0, t_f]) :$$

$$\int_{t_0}^{t_f} |z(t)|^2 \, dt \leq \gamma^2 \int_{t_0}^{t_f} |w(t)|^2 \, dt + M(\zeta_0) \quad .$$

Here $z(\cdot)$ is the output corresponding to the input $w(\cdot)$ and the initial stat $\zeta(t_0) = \zeta_0$. If no such γ exists we write $||\Sigma|| = \infty$.

We consider only measurable locally bounded inputs and assume that a systems map such inputs to measurable locally bounded states and output

4.2 Problem statement

We consider systems of the form

$$\begin{array}{rcl} \dot{x}(t) & = & A_\theta x(t) + B_\theta u(t) + G_\theta w(t) \\ y(t) & = & C_\theta x(t) + v(t) \\ z(t) & = & \begin{pmatrix} H_\theta x(t) \\ u(t) \end{pmatrix} \\ \theta & \in & \{1, \ldots, p\} \stackrel{\triangle}{=} \Theta \end{array} \tag{4.1}$$

Here, x is the state of the system, u is a control input, w is an process disturbance, y is the measured signal, v is the measurement noise, z is the generalized error signal. All signals take values in Euclidean spaces.

The matrices $(A_\theta, B_\theta, G_\theta, C_\theta, H_\theta)$ are known functions of the unknown parameter θ. With P_θ we denote the linear system from (w, v, u) to (z, y) obtained by fixing θ.

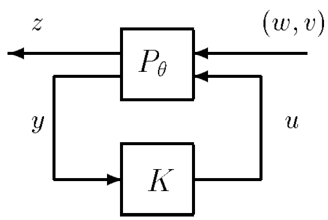

Figure 4.1: Simultaneous Control Problem

Problem of Simultaneous $\mathcal{H}_\infty$ Control with Stability: Given a constant $\gamma > 0$, find a causal control law $K : y(\cdot) \to u(\cdot)$ such that for any parameter $\theta \in \Theta$, the closed-loop system (P_θ, K) from (w, v) to z has $\mathcal{L}_2$-gain less than γ and in addition (P_θ, K) is internally stable in the sense that $w(\cdot) \in \mathcal{L}_2([0,\infty))$, $v \in \mathcal{L}_2([0,\infty))$ implies that $x(t) \to 0$ as $t \to \infty$. $\diamond$

We adopt the following standard assumptions on the system matrices:

Assumption 30:

1. For any $i \in \Theta$, the triple (C_i, A_i, B_i) is detectable and stabilizable.
2. For any $i \in \Theta$, the triple (H_i, A_i, G_i) is observable and controllable.

$\square$

The standard discussion regarding these assumptions applies [128]: Th first part is necessary for the existence of an internally stabilizing con troller. The second part is mainly a technical regularity assumption, whic guarantees that any closed-loop system with finite $\mathcal{L}_2$-gain must be inte nally stable and furthermore that certain loss functions are positive definit As in linear $\mathcal{H}_\infty$ control for a fixed plant, the second assumption can b relaxed quite a bit but to keep the exposition simple, we shall not do thi

Adding the parameter dynamics

$$\dot{\theta}(t) = 0 \tag{4.2}$$

we obtain a non-linear system description in the *extended state* $(x', \theta)'$ b combining (4.2) and (4.1). We then attack the problem of non-linear $\mathcal{H}$ control for this system using the *differential game* techniques for outpu feedback design presented in [9, 120]. To be specific, we consider the di ferential min-max problem

$$\min_K \sup_{w(\cdot), x(\cdot), \theta} \left[\int_0^\infty \Leftrightarrow s_\theta(t)\ dt \Leftrightarrow \frac{1}{2} x(0)' N_\theta x(0) \Leftrightarrow \Lambda_\theta \right] \tag{4.3}$$

where we have used the shorthand

$$s_i = \frac{1}{2}\gamma^2 |w|^2 + \frac{1}{2}\gamma^2 |y \Leftrightarrow C_i x|^2 \Leftrightarrow \frac{1}{2}|u|^2 \Leftrightarrow \frac{1}{2}|H_i x|^2 \quad . \tag{4.4}$$

Here, the supremum is subject to the dynamics (4.2) and (4.1), and th minimization is subject to the causality restriction on the controller K Notice that the initial condition $x(0)$ is chosen by the maximizing playe If this minimax problem has finite upper value for some choice of N_i an Λ_i, then there exists a controller (viz., the minimax controller) which gua antees that the closed loop from $(w(\cdot), v(\cdot))$ to $z(\cdot)$ has $\mathcal{L}_2$-gain less tha or equal to γ. The converse also holds.

In this minimax problem $\Lambda_i \geq 0$ represents prior information about θ; ou prior estimate of θ is $\arg\min_i \Lambda_i$ (assuming a unique minimum point). Fo simplicity and without loss of generality we assume $\min_i \Lambda_i = 0$.

Similarly, $N_i > 0$ represents our confidence in the prior estimate $x(0)$ = 0, given that $\theta = i$. The choice of N_i influences the transients of th state estimator but not steady-state behaviour such as the closed-loop $\mathcal{L}$ gain. The standard discussion from linear $\mathcal{H}_\infty$ theory [42] applies; th situation corresponds to the initialization of variances in Kalman filter

For simplicity, we are going to assume that the estimator starts in steady state; see section 4.4 below.

In the following sections we approach this min-max problems, following the general procedure of [9] closely as far as possible. It is interesting to see that no problems are caused by the fact that our state space $\mathbb{R}^n \times \{1, \ldots, p\}$ is *hybrid*, i.e. has a continuous as well as a discrete part. The main steps in the procedure are:

1. The full information problem where x and θ is available to the controller on-line. This problem reduces to p standard linear $\mathcal{H}_\infty$ problems; section 4.3.
2. The problem of estimating x and θ using the measured signal $y(\cdot)$. The solution is a *bank* of linear state estimators, one for each parameter value, which run in parallel. The final state estimate is found by comparing residuals associated with these estimators; section 4.4.
3. In [9], a certainty equivalence principle [14] is verified at this point. In our case, the hypothesis for this principle is not met. In stead, we reduce the problem to a finite-dimensional full information minimax control problem. Our procedure is similar to the information state machinery [55]. The minimax controller is then characterized by a Hamilton-Jacobi-Isaacs equation. We discuss this equation and the structure of its solution; section 4.5.
4. Finally we investigate the *heuristic* certainty equivalence controller; section 4.6.

4.3 Control with known extended state

We address the subproblem where $y = (x, \theta)$. A trivial but helpful observation is that this *extended state feedback* problem reduces to a standard *linear* $\mathcal{H}_\infty$ problem *for each parameter* θ. Following [128], we consider the p control algebraic Riccati equations

$$A_i' X_i + X_i A_i + X_i \left(\frac{1}{\gamma^2} G_i G_i' \Leftrightarrow B_i B_i' \right) X_i + H_i' H_i = 0 \qquad (4.5)$$

which we explicitly assume have the needed solutions:

Assumption 31: For each $i = 1, \ldots, p$, the algebraic Riccati equa-
tion (4.5) admits a solution X_i such that $A_i \Leftrightarrow B_i B_i' X_i + \frac{1}{\gamma^2} G_i G_i' X_i$
asymptotically stable. In addition X_i is positive semi-definite.

Remark 32: For the relevant theory of Riccati equations as (4.5) we refe
to [128]. We note if such an X_i exists, it must be unique. Furthermore X
must be positive definite since (H_i, A_i) is assumed observable.

Well known results from linear $\mathcal{H}_\infty$ theory thus immediately gives:

Proposition 33: [c.f. [128, theorem 16.9], [9, theorem 4.8]] Let th
plant (4.1), (4.2) satisfy assumption 30. Then there exists a causal contro
law $(\theta(\cdot), x(\cdot)) \to u(\cdot)$ such that the closed-loop system from $w(\cdot)$ to $z(\cdot)$
internally stable and has $\mathcal{L}_2$-gain less than γ, if and only if assumption 3
holds. In this case, one such control law is the minimax control

$$u(t) = \Leftrightarrow B_\theta' X_\theta x(t) \quad . \qquad (4.6$$

The associated cost-to-go is

$$P(x_t, \theta) \triangleq \sup_{w(\cdot)} \int_t^\infty \frac{1}{2}|z(\tau)|^2 \Leftrightarrow \frac{1}{2}\gamma^2 |w(\tau)|^2 \, d\tau = \frac{1}{2} x_t' X_\theta x_t \qquad (4.7$$

where the supremum is subject to the initial condition $x(t) = x_t$ and th
dynamic equations (4.1,4.2,4.6) governing the closed loop.

4.4 The estimation problem

In this section we define the problem of estimating the extended state an
derive the dynamic filter of the estimator. As in [9, 25, 120], we defin
the *cost-to-come* function (termed the *information state* by other author
e.g. [55])

$$R(x_t, i, t) = \inf_{w(\cdot), x(\cdot)} \left(\int_0^t s_i(\tau) \, d\tau + \frac{1}{2} x'(0) N_i x(0) \right) + \Lambda_i \qquad (4.8$$

where s_i is as in (4.4). The infimization in (4.8) is subject to the constraint

$$\begin{aligned} x(t) &= x_t \quad , \\ \dot{x}(\tau) &= A_i x(\tau) + B_i u(\tau) + G_i w(\tau) \ , \quad 0 \le \tau \le t \ . \end{aligned}$$

The cost-to-go is the worst-case loss over the time interval $[0,t]$, given $y(\cdot)$ and $u(\cdot)$ and assuming that $x(t) = x_t$ and that $\theta = i$.

Following the notation in [120], we denote by $S(x,i,t)$ the worst-case total cost over the time interval $[0,\infty)$ consistent with the observations of $u(\tau), y(\tau)$ for $\tau \in [0,t]$ and such that $x(t) = x$, $\theta = i$, and subject to full information control for $\tau \geq t$. Hence

$$S(x,i,t) = R(x,i,t) - P(x,i) \quad .$$

We can now define the worst-case extended state estimate:

$$\begin{pmatrix} \hat{x}(t) \\ \hat{\theta}(t) \end{pmatrix} = \arg \min_{x,i} S(x,i,t) \quad . \tag{4.9}$$

The extended state estimate is instrumental to the minimax controller: A certainty equivalence controller [14, 9, 120] applies the full information control law (4.6) with the state x, θ substituted with $\hat{x}, \hat{\theta}$. Without certainty equivalence, we demonstrate in the following section that the problem can be transformed into one where the extended state estimate $\hat{x}, \hat{\theta}$ is the controlled variable.

In order to derive the dynamics of the extended state estimate we split the estimation into two parts: First a conditional state estimate which estimates x conditioned on assumptions on θ, and second the (unconditional) parameter estimate. To be specific, the conditional state estimate is

$$\xi(i,t) = \arg \min_{x} S(x,i,t) \tag{4.10}$$

and is the worst-case state estimate based on the assumption that the true parameter equals i. Correspondingly the worst case parameter estimate is

$$\hat{\theta}(t) = \arg \min_{i} S(\xi(i,t),i,t) \quad . \tag{4.11}$$

With this formulation the state estimate is $\hat{x}(t) = \xi(\hat{\theta}(t),t)$. Determining $\xi(i,t)$ for fixed i is a purely linear problem which can be solved as in [9, 128]:

Assumption 34: For each $i = 1,\ldots,p$, the filter algebraic Riccati equation

$$Y_i A_i' + A_i Y_i + G_i G_i' + Y_i (\frac{1}{\gamma^2} H_i' H_i - C_i' C_i) Y_i = 0 \tag{4.12}$$

admits a positive semi-definite solution Y_i such that $A_i' + (\frac{1}{\gamma^2} H_i' H_i - C_i' C_i) Y_i$ is asymptotically stable. □

By duality of remark 32, such a Y_i will be unique and positive definit
Define $Q_i := \gamma^2 Y_i^{-1}$. then Q_i satisfies

$$A_i' Q_i + Q_i A_i + \frac{1}{\gamma^2} Q_i G_i G_i' Q_i + H_i' H_i - \gamma^2 C_i' C_i = 0 \quad .$$

For ease of notation we assume that the game (4.3) has been chosen suc that $Q_i = N_i$ for all $i = 1,\ldots,p$; thus the filters start in steady-state. Se the discussion on page 74 above, and appendix B.2.

The implication of assumption 34 is that the cost-to-go is always well de fined and for each i has a minimum over x which is attained at a uniqu point. For the same to hold for $S(x,i,t)$ we need $S(x,i,t)$ to be strictl convex in x, i.e:

Assumption 35: For each $i = 1,\ldots,p$, the coupling condition

$$Q_i - X_i > 0$$

holds.

Summarizing, linear $\mathcal{H}_\infty$ theory gives us the following proposition:

Proposition 36: Let the plant (4.1), (4.2) satisfy assumption 30. Ther exist causal controllers $K_i : y(\cdot) \to u(\cdot)$ such that (P_i, K_i) are internall stable and have $\mathcal{L}_2$-gain less than γ if and only if assumptions 31, 34 an 35 hold. Assume in addition that $N_i = Q_i$, then $\xi(i,t)$ is well defined fo all t and all $u \in \mathcal{L}_2([0,t])$, $y \in \mathcal{L}_2([0,t])$ and can be computed on-line a the solution to the ODE

$$\begin{aligned} \dot{\xi}(i,t) = & \\ & \left(A_i + \gamma^{-2} G_i G_i' X_i - B_i B_i' X_i\right) \cdot \xi(i,t) \\ & + \gamma^2 (Q_i - X_i)^{-1} C_i' \cdot (y(t) - C_i \xi(i,t)) \\ & + (Q_i - X_i)^{-1} Q_i B_i \cdot (u(t) + B_i' X_i \xi(i,t)) \end{aligned} \tag{4.13}$$

with initial condition $\xi(i,0) = 0$. Furthermore the conditional worst-cas loss $S(\xi(i,t),i,t)$ is computed on-line as the solution to the ODE

$$\frac{d}{dt} S(\xi(i,t),i,t) = \frac{1}{2}\gamma^2 |y(t) - C_i \xi(i,t)|^2 - \frac{1}{2} |u(t) + B_i' X_i \xi(i,t)|^2 \tag{4.14}$$

with the initial condition $S(\xi(i,0),i,0) = \Lambda_i$.

All statements in the proposition can be found in [9, theorem 5.5] (see also [128, theorem 16.4]) *except* the dynamic equations for ξ and S. Nevertheless, these equations can easily be derived using the method of [9, 120]; the calculations can be found in appendix B.2.

The structure of the single estimator $\xi(i,\cdot)$ is illustrated in figure 4.2 where we have omitted the subscript i and used the notation

$$\begin{aligned} E &:= \gamma^{-2}G'X \\ F &:= \Leftrightarrow B'X \\ K &:= (Q \Leftrightarrow X)^{-1}QB \\ L &:= \gamma^{2}(Q \Leftrightarrow X)^{-1}C' \end{aligned}$$

The block diagram (and the ODEs) for the conditioned state estimate $\xi(i,t)$ is identical to estimator in the standard central $\mathcal{H}_\infty$ controller [128, p. 435], *except* for the last term $(Q_i \Leftrightarrow X_i)^{-1}Q_iB_i\,(u(t) + B_i'X_i\xi(i,t))$ (the block K in the block diagram). This term vanishes when the control signal is conditionally minimax (i.e, $\tilde{u} = 0$ as will happen when $\hat{\theta}(t) = i$ and certainty equivalence control is used; see below) and is therefore not present in the central $\mathcal{H}_\infty$ controller for a single linear plant. The way $\tilde{u}$ affects the dynamics of the conditional state estimate corresponds to a parametrization of all $\mathcal{H}_\infty$ suboptimal controllers [128, p. 420] (we will elaborate further on this connection in remark 38 below).

We see from equation (4.14) that $S(\xi(i,t),i,t)$ is an integrated *residual* associated with the model P_i. The *estimation error* $y(t) \Leftrightarrow C_i\xi(i,t)$ appears also in residuals of stochastic system identification, but the subtraction of the *control error* $u(t) + B_i'X_i\xi(i,t)$ is a new feature due to the minimax setting. Notice that $\Leftrightarrow B_iX_i\xi(i,t)$ is an *estimate* of the full information minimax control (4.6).

In the remainder of the chapter we will use the shorthands

$$\xi_i(t) := \xi(i,t) \quad \text{and} \quad S_i(t) := S(\xi(i,t),i,t) \quad .$$

The total cost function $S(x,i,t)$ can be computed as

$$S(x,i,t) = \frac{1}{2}(x \Leftrightarrow \xi_i(t))'(Q_i \Leftrightarrow X_i)(x \Leftrightarrow \xi_i(t)) + S_i(t)$$

after which the cost-to-come function can be computed as

$$R(x,i,t) = S(x,i,t) + P(x,i) \quad .$$

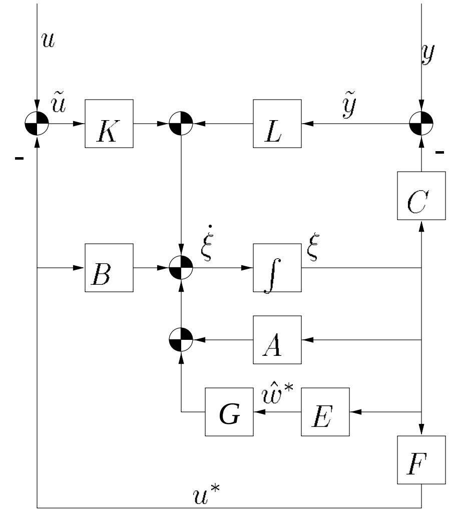

Figure 4.2: Block diagram of each conditional worst-case state estimato
The subscripts i are omitted.

4.5 The minimax controller

Having derived the minimax estimator in the previous section the first thin
to verify is if the certainty equivalence (CE) principle of [14] can be applie
as in [9, 120]. This principle states that *if* the worst-case extended stat
estimate $(\hat{x}(t), \hat{\theta}(t))$ is always well defined by equation (4.9) on page 76 i
the sense that the minimum exists and is attained at a unique point, the
the minimax control strategy associated with the game (4.3) on page 73

$$u(t) = \Leftrightarrow B'_{\hat{\theta}(t)}X_{\hat{\theta}(t)}\hat{x}(t) \quad . \qquad (4.15$$

This is a *certainty equivalence controller* since it applies the state feedbac
law (4.6) to the estimates $\hat{x}, \hat{\theta}$. In general, a CE principle is one whic
states that a CE controller is optimal (in this case minimax). If a C
controller is applied without a justifying CE principle, then we emphasiz
this by calling it a *heuristic* certainty equivalence controller.

We know from proposition 36 that the conditional state estimates $\xi(i,$
are always well defined by equation (4.10), which implies that the minima

controller in the case of a *single* plant is based on certainty equivalence [9, theorem 5.3]. However, the parameter estimate $\hat{\theta}(t)$ needs not always be well defined by equation (4.11) since the minimum of $S_i(t)$ over i may be attained for two or more values of i. In fact, if $\hat{\theta}(t)$ is always well defined then $\hat{\theta}(t)$ is a constant function of time t; clearly such an assumption would be rather detrimental to the whole idea of adaptation. We conclude that certainty equivalence does not necessarily hold.

Despite this it is possible to characterize the minimax controller implicitly in terms of a Hamilton-Jacobi-Isaacs equation; this is the subject of the remainder of this section. First we reduce the problem to a dynamic game with full information. Next we derive the Hamilton-Jacobi-Isaacs equation associated with this full information game. Finally we state a theorem and pose a conjecture about the structure of the value function.

Reduction to a full information game

At this point we adopt a technique similar to the information state machinery in order to reduce the output feedback problem to a full information game. See [55] for the information state machinery in the context of nonlinear $\mathcal{H}_\infty$ control. The corresponding approach to optimal control of Markov chains is described in [16, ch. 4].

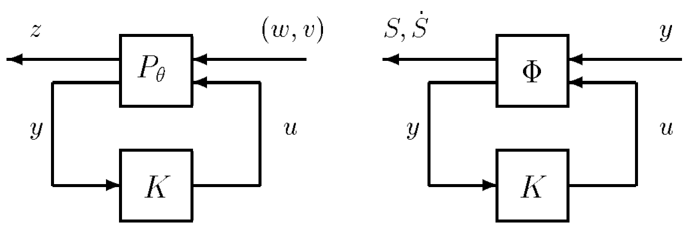

Figure 4.3: The original output feedback problem and the equivalent full information problem of controlling the total cost.

Use the symbol Φ to denote the filter with states

$$(\xi_1(t), \ldots, \xi_p(t), S_1(t), \ldots, S_p(t)) \quad ,$$

inputs u and y, outputs y, S_i and $\dot{S}_i$, and dynamics given by equations (4.1 and (4.14) above. Let $\xi = (\xi_1, \ldots, \xi_p)$ and $S = (S_1, \ldots, S_p)$. Use the sym bol (Φ, K) to denote the interconnection of the filter Φ and a controll $K : y(\cdot) \to u(\cdot)$:

$$(\Phi, K)(y(\cdot)) = \Phi(K(y(\cdot)), y(\cdot))$$

The interconnection (Φ, K) thus has input $y(\cdot)$ and outputs $S(\cdot)$ and $\dot{S}(\cdot$ See figure 4.3.

The important step in the reduction of the problem to one of full informa tion is the following proposition, which says that as control object we ma take the filter Φ rather than the plant P_θ:

Proposition 37: Let K be a causal controller $y(\cdot) \to u(\cdot)$ with a stat space representation. Then the closed loop (P_i, K) has $\mathcal{L}_2$-gain less than o equal to γ *if and only if* the interconnection (Φ, K) dissipates the suppl rate $\dot{S}_i$.

The proposition follows directly from the definition of the worst-case los S_i: (Φ, K) dissipates $\dot{S}_i$ iff $S_i(\cdot)$ can be bounded below in terms of th initial condition, and such a bound is exactly what is needed according t the definition of the $\mathcal{L}_2$ gain.

The problem of controlling Φ is essentially a *full information* problem sinc the initial conditions in Φ are known and all inputs to Φ are available o line. So also the states of Φ can be considered known to the controller.

Remark 38: Loosely said, (Φ, K) dissipates $\dot{S}_i$ if and only if $u + B_i' X_i$ is smaller than $\gamma(y \Leftrightarrow C_i\xi_i)$ in $\mathcal{L}_2$ norm. Therefore, we can construct such controller K in the following way: Take a system $\tilde{Q}$ with $\mathcal{L}_2$-gain less tha or equal to γ. Let the input to $\tilde{Q}$ be $\tilde{y} = y \Leftrightarrow C_i\xi$ and denote the outpu $\tilde{u}$. Now choose the control signal u such that $\tilde{u} = u + B_i' X_i \xi_i$. Thus w have established the connection to the parametrization of $\mathcal{H}_\infty$ suboptima controllers [128, theorem 16.5], see also [9, corollary 5.2].

Recall that a simultaneous $\mathcal{H}_\infty$ controller was required also to be stab lizing. However, under the observability assumption 30 on page 72, an γ-suboptimal $\mathcal{H}_\infty$ controller is internally stabilizing:

Proposition 39: If (P_θ, K) has $\mathcal{L}_2$-gain less than or equal to $\gamma > 0$ an $w(\cdot) \in \mathcal{L}_2([0, \infty))$, $v(\cdot) \in \mathcal{L}_2([0, \infty))$, then $x(t) \to 0$ as $t \to \infty$.

If only *linear* controllers K are considered, then the proposition is containe in corollary 16.3 in [128, p. 418]. A statement which allows smooth stati nonlinear control is [120, prop. 3.4]; however smoothness of the storage function is required there. Given these results, there is little novelty in the proposition. For the sake of completeness we give an elementary proof.

Proof: Let L be such that $A_\theta + LH_\theta$ is asymptotically stable; such an L exists since (H_i, A_i) is assumed observable. Now write the state dynamics $\dot{x} = (A_\theta + LH_\theta)x \Leftrightarrow LH_\theta x + B_\theta u + G_\theta w$. The hypothesis implies $H_\theta x(\cdot) \in \mathcal{L}_2$ and $u(\cdot) \in \mathcal{L}_2$, so x is the state of a stable system with $\mathcal{L}_2$ inputs; thus $x(\cdot) \in \mathcal{L}_2$. Now $\dot{x}$ is a linear combination of $\mathcal{L}_2$ signals; hence also $\dot{x}(\cdot) \in \mathcal{L}_2$. Finally $x(\cdot) \in \mathcal{L}_2$ and $\dot{x}(\cdot) \in \mathcal{L}_2$ implies that $x(t) \to 0$ as $t \to \infty$. ■

Notice that the proof merely uses that $\mathcal{L}_2$ inputs w, v, are mapped to an $\mathcal{L}_2$ output z (which is somewhat weaker than finite $\mathcal{L}_2$-gain), and that the causality of K is not used.

If we ignore the slight difference between obtaining an $\mathcal{L}_2$-gain from w, v to z less than γ, and less than or equal to γ, the propositions allow us to consider a problem of *control for multi-dissipation* rather than the original problem of simultaneous $\mathcal{H}_\infty$ control. The control object in this new problem is the worst case filter Φ with states ξ, S, the control objective is to make the interconnection (Φ, K) dissipate S_i for $i = 1, \ldots, p$, and the controller has access to both the state ξ, S and the disturbance y. In summary, we have reduced the problem of simultaneous $\mathcal{H}_\infty$ control to a multi-objective min-max control problem with full information.

This multi-objective problem may be reduced to a single-objective one; in fact the interconnection (Φ, K) dissipates $\dot{S}_i$ for all $i = 1, \ldots, p$ if and only if it dissipates the supply rate

$$\frac{d}{dt} \min_i S_i(t) \quad .$$

This supply rate is regular (i.e, for any ξ and u there exists a y such that $d/dt \ \min_i S_i(t) \leq 0$) and hence the problem of control for dissipation is equivalent to a differential game on infinite horizon:

$$U(\xi(0), S(0)) = \sup_{u(\cdot)} \inf_{y(\cdot)} \min_i S_i(\infty) \tag{4.16}$$

for which we can *a priori* pose the bound

$$U(\xi(0), S(0)) \in [\Leftrightarrow\infty, \min_i S_i(0)] \quad .$$

Note that the value $\Leftrightarrow\infty$ is included; the value functions in this section ar in general extended real-valued. The players are allowed to play closed loo strategies which result in locally bounded $\mathcal{L}_2$ signals $u(\cdot)$, $y(\cdot)$. Note als that the limits $S_i(\infty) = \lim_{t\to\infty} S_i(t)$ are well defined for all such strategie since the filters (4.13) are stable.

The following theorem, which follows immediately from the discussio makes precise the statement that the problem of simultaneous $\mathcal{H}_\infty$ con trol is equivalent to a full information game:

Theorem 40: The following are equivalent:

1. There exists a controller K such that (P_θ, K) has $\mathcal{L}_2$-gain less tha or equal to γ for any θ, and such that $x(\cdot) \to 0$ for $w(\cdot)$, $v(\cdot) \in \mathcal{L}_2$.
2. There exists a controller K such that (Φ, K) dissipates $d/dt \ \min_i S_i(t$
3. For each pair of initial conditions ξ, S, the lower value $U(\xi, S)$ a defined above is finite.

The Hamilton-Jacobi-Isaacs equation

To study the game associated with U we follow the terminology of [12 and define the *pre-Hamiltonian*

$$K(\xi, S, \lambda, \mu, u, y) = \lambda\dot{\xi} + \mu\dot{S} \tag{4.17}$$

where $\dot{\xi}$ and $\dot{S}$ are given by the filter dynamics (4.13) and (4.14) of Φ. He λ and μ are co-states to ξ and S, respectively, i.e. λ is a row vector in $\mathbb{R}^{p\times}$ while μ is a row vector in $\mathbb{R}^p$.

We also define the *Hamiltonian* H

$$H(\xi, S, \lambda, \mu) = \sup_u \inf_y K(\xi, S, \lambda, \mu, u, y) \quad . \tag{4.18}$$

We restrict attention to co-states for which $\sum_i \mu_i > 0$. Thus the Hamilto nian is finite, smooth, independent of S and quadratic in ξ, λ for fixed μ Furthermore, the static game in (4.18) can be solved by completion of th squares:

$$K(\xi, S, \lambda, \mu, u, y) = H(\xi, S, \lambda, \mu) \Leftrightarrow \frac{1}{2}\sum_i \mu_i |u \Leftrightarrow u^*|^2 + \frac{1}{2}\sum_i \mu_i |y \Leftrightarrow y^*|^2$$

so Isaacs' condition[1] holds. Here y^* and u^* are smooth functions of ξ, λ and μ:

$$u^*(\xi,\lambda,\mu) = \arg\max_u \min_y K(\xi,S,\lambda,\mu,u,y) \quad ,$$

$$y^*(\xi,\lambda,\mu) = \arg\min_y \max_u K(\xi,S,\lambda,\mu,u,y) \quad .$$

The functions u^* and y^* are linear in ξ, λ for fixed μ. It is straightforward but unnecessary to give explicit expressions for these functions.

It is well known [10] that value functions of differential games, such as U, are related to Hamilton-Jacobi-Isaacs equations, in this case

$$H(\xi,S,\psi_\xi(\xi,S),\psi_S(\xi,S)) = 0 \quad . \tag{4.19}$$

The results of [10] does not cover the particular games in our study, but with analogous arguments we may obtain similar results. First we show that if (4.19) admits a *subsolution*, then it provides a *guaranteed cost strategy* for u:

Proposition 41: Let $\psi(\xi,S)$ be C^1 and satisfy $H(\xi,S,\psi_\xi,\psi_S) \geq 0$ as well as $\psi(\xi,S) \leq \min_i S_i$. Let the *maximum dissipation* controller K_ψ be specified by the state feedback law

$$u^\psi(\xi,S) = u^*(\xi,\psi_\xi,\psi_S) \quad ,$$

then (Φ, K_ψ) dissipates $d/dt\ \min_i S_i(t)$. Furthermore ψ is a lower bound on the lower value function: $\psi \leq U$. △

Proof: We claim that $\min_i S_i \Leftrightarrow \psi(\xi,S)$ is a storage function, i.e. that the dissipation inequality

$$\min_i S_i(T) \Leftrightarrow \psi(\xi(T),S(T)) \leq \min_i S_i(0) \Leftrightarrow \psi(\xi(0),S(0)) + \int_0^T \frac{d}{dt}\min_i S_i(t)\, dt$$

holds. This is equivalent to $\psi(\xi(T),S(T)) \geq \psi(\xi(0),S(0))$ which follows from

$$\dot{\psi} = K(\xi,S,\psi_\xi,\psi_S,u^\psi,y) \geq H(\xi,S,\psi_\xi,\psi_S) \geq 0 \quad .$$

Thus dissipation is established. Furthermore, we have

$$\min_i S_i(T) \geq \psi(\xi(T),S(T)) \geq \psi(\xi(0),S(0))$$

[1] I.e. the game in (4.18) has saddle point u^*, y^* for each ξ, S, λ, μ; see [10, p. 349].

which holds for all inputs $y(\cdot)$ and all T - hence in particular in the lim $T \to \infty$ - and thus implies that $U(\xi(0),S(0)) \geq \psi(\xi(0),S(0))$.

Remark 42: In the case of a single plant, $p = 1$, we may take ψ = $\min_i S_i = S_1$. The corresponding storage function is identically 0, and th resulting maximum dissipation controller is $u = \Leftrightarrow B_i' X_i \xi_i$. Thus we recove the central controller from linear $\mathcal{H}_\infty$ theory [128, p. 419].

The next question is if the lower value function U, or cost bounding func tions corresponding to guaranteed cost strategies, must necessarily satisf the Hamilton-Jacobi-Isaacs equation (4.19), or the related inequality. Her matters are complicated by the observation that it is not reasonable to ex pect U to be differentiable everywhere. Within the last decade, the notio of *viscosity* solutions [23, 35] to equations such as (4.19) has become th standard tool with which to approach these issues of non-differentiabilit The following definition is taken from [23] and specialized[2] to the case o first order partial differential equations:

Definition 43: We say that $\kappa(\xi,S)$ is a *viscosity supersolution* to th Hamilton-Jacobi-Isaacs equation $H(\xi,S,\kappa_\xi,\kappa_S) = 0$ if κ is lower sem continuous and $H(\bar{\xi},\bar{S},\phi_\xi,\phi_S) \leq 0$ holds for every $\bar{\xi}$, $\bar{S}$ and every $\phi(\xi,S$ which is C^∞ and satisfies $\phi \leq \kappa$, $\phi(\bar{\xi},\bar{S}) = \kappa(\bar{\xi},\bar{S})$.

We say that κ is a *viscosity subsolution* if κ is upper semi-continuous an $H(\bar{\xi},\bar{S},\phi_\xi,\phi_S) \geq 0$ holds for every $\bar{\xi}$, $\bar{S}$ and every $\phi(\xi,S)$ which is C^∞ an satisfies $\phi \geq \kappa$, $\phi(\bar{\xi},\bar{S}) = \kappa(\bar{\xi},\bar{S})$.

We say that κ is a *viscosity solution* if it is both a subsolution and supersolution.

If κ is a viscosity supersolution, then we also say that κ solves the inequalit $H(\xi,S,\kappa_\xi,\kappa_S) \leq 0$ in the viscosity sense. Notice that viscosity solutions ar by definition continuous, and that a differentiable function κ is a viscosit solution if and only if it is a classical solution. We refer to [23] for furthe discussion of viscosity solutions.

It is by now a standard exercise to show that value functions satisfy Hamilt Jacob-Isaacs equations in the viscosity sense. It complicates matters, how ever, that the inputs u, y are not restricted to bounded sets. See page 11 for an example where the value function does not solve the PDE since near optimal controls are unbounded. Most contributions, e.g. [70], consider onl

[2] To see that our definition coincides with that in [23], substitute $F = -H$.

problems where the controls are restricted to compact sets. The recent reference [7] explicitly assumes that near-optimal controls are bounded before proving that value functions are viscosity solutions, but does not discuss how to verify the assumption for a given system.

For our system, we are able to show that the value function is indeed a viscosity solution. The key element is, roughly speaking, that controls leading to fast trajectories also lead to large running costs, as will be made precise in the proof:

Proposition 44: Assume that U is finite everywhere and continuous. Then U solves the Hamilton-Jacobi-Isaacs equation

$$H(\xi, S, \phi_\xi, \phi_S) = 0$$

in the viscosity sense. △

Proof: We show that U is a subsolution only; the other statement follows similarly. Let $\bar{\xi}$ and $\bar{S}$ be a fixed initial condition and let ϕ be a C^∞ function such that $\phi(\bar{\xi}, \bar{S}) = U(\bar{\xi}, \bar{S})$ and $\phi \geq U$. Notice that this implies that $\sum_i \phi_{S_i}(\bar{\xi}, \bar{S}) = 1$. Hence $H(\xi, S, \phi_\xi, \phi_S)$ is finite and smooth on a neighbourhood of $(\bar{\xi}, \bar{S})$.

Our proof is by contradiction: Assume that $H(\bar{\xi}, \bar{S}, \phi_\xi, \phi_S) < 0$. Then there exists a neighbourhood Ω of $\bar{\xi}$, $\bar{S}$ and $\delta, \epsilon > 0$ such that $\sum_i \phi_{S_i} > 2\delta$ and $H(\xi, S, \phi_\xi, \phi_S) < \Leftrightarrow\epsilon$ on Ω.

Now let $T > 0$ be arbitrary, let $\bar{T}_\Omega$ the time of first exit time from Ω, and let $T_\Omega = \min\{T, \bar{T}_\Omega\}$.

Let the minimizing player use the smooth feedback strategy $y = y^*(\xi, \phi_\xi, \phi_S)$. Let $\rho > 0$ be such that the ρ-ball around $\bar{\xi}$, $\bar{S}$ is contained in Ω. Let $c > 0$ be such that

$$\epsilon + \delta|u \Leftrightarrow u^*|^2 > c|(\dot{\xi}, \dot{S})|$$

holds for all ξ, S in Ω and all u. Such a c exists since Ω is bounded and since $\dot{\xi}$, $\dot{S}$, are affine-quadratic in u. This inequality makes precise the statement that controls leading to fast trajectories also lead to large running costs.

Thus, for any strategy for the maximizing player, we have

$$\begin{aligned} \phi(\xi(T_\Omega), S(T_\Omega)) \Leftrightarrow \phi(\bar{\xi}, \bar{S}) &= \int_0^{T_\Omega} K(\xi, S, \phi_\xi, \phi_S, u, y^*)\ dt \\ &\leq \Leftrightarrow \int_0^{T_\Omega} \epsilon + \delta|u \Leftrightarrow u^*(\xi, \phi_\xi, \phi_S)|^2\ dt \\ &\leq \Leftrightarrow \int_0^{T_\Omega} c|(\dot{\xi}, \dot{S})|\ dt \\ &\leq \Leftrightarrow c \cdot \rho < 0 \end{aligned}$$

which holds for any policy for the maximizing player. This implies that

$$\sup_{u(\cdot)} \inf_{y(\cdot)} \phi(\xi(T_\Omega), S(T_\Omega)) < \phi(\bar{\xi}, \bar{S}) \quad .$$

Combining this with $\phi \geq U$ we obtain

$$\sup_{u(\cdot)} \inf_{y(\cdot)} U(\xi(T_\Omega), S(T_\Omega)) < U(\bar{\xi}, \bar{S})$$

which contradicts the dynamic programming principle. We conclude tha the hypothesis $H(\bar{\xi}, \bar{S}, \phi_\xi, \phi_S) < 0$ cannot hold; in other words, U is subsolution in the viscosity sense.

Remark 45: In the light of remark 42, it is instructive to consider $\min_i S$ as a candidate solution to to the Hamilton-Jacobi-Isaacs equation (4.19 First, $\min_i S_i$ is a viscosity *supersolution* as can readily be verified. Henc we can deduce a guaranteed cost strategy for y: at each instant y is chose such that $\min_i S_i$ is non-increasing.

Second, $\min_i S_i$ is not *in general* a viscosity subsolution and therefore doe not in general help us derive guaranteed cost strategies for u.

Third, $\min_i S_i$ is a *generalized* solution to (4.19) in the sense that th equation holds for almost all ξ, S (viz. whenever $\hat{\theta} = \arg\min_i S_i$ is we defined). This property is important in the following section where w discuss a *weak* certainty equivalence principle concerning the *heuristic* cer tainty equivalence controller.

A theorem and a conjecture on the structure of U

Consider the canonical equations governing the *Hamiltonian* dynamics as sociated with U:

$$\dot{\xi}_i = \frac{\partial H}{\partial \lambda_i}, \quad \dot{\lambda}_i = \Leftrightarrow \frac{\partial H}{\partial \xi_i}, \quad \dot{S}_i = \frac{\partial H}{\partial \mu_i}, \quad \dot{\mu}_i = \Leftrightarrow \frac{\partial H}{\partial S_i} = 0 \quad . \qquad (4.20$$

It is well known (see e.g. [120]) that *if* the lower value function U is C^1, the the trajectories (ξ, S, λ, μ) corresponding to the saddle point strategies u

y^*, solve the canonical equations. Hence the co-state μ is *constant* along the saddle point trajectories. Now $u^*(\xi, \lambda, \mu)$ and $y^*(\xi, \lambda, \mu)$ are linear in (ξ, λ) for fixed μ which implies that the trajectories also solve a *linear* system. Furthermore this linear system is the canonical equations associated with the *weighted* linear-quadratic game

$$Z(\xi(0), S(0); \alpha) = \sup_{u(\cdot)} \inf_{y(\cdot)} \sum_i \alpha_i S_i(\infty) \tag{4.21}$$

where $\alpha = \mu$.

This fits with the following observation: If a controller K is such that (Φ, K) dissipates $\dot{S}_i$ for $i = 1, \ldots, p$, then (Φ, K) also dissipates $\sum_i \alpha_i \dot{S}_i$ for any non-negative weights α_i with $\sum_i \alpha_i = 1$ (proposition 2 on page 32).

This leads us to believe that the minimax controller at each instant chooses an *equivalent* linear-quadratic game, given by $\alpha = \mu$, and plays the minimax control of that game. In fact we have the following theorem:

Theorem 46: Assume that Z is finite everywhere and C^1, and that a differentiable function $\alpha^*(\xi, S)$ exists such that

$$Z(\xi, S; \alpha^*(\xi, S)) = \min_\alpha Z(\xi, S; \alpha) \quad .$$

Here minimization is over $\alpha_i \geq 0$ with $\sum_i \alpha_i = 1$. Then $U(\xi, S) = \min_\alpha Z(\xi, S; \alpha)$. Furthermore, the control law

$$u^U(\xi, S) = u^*(\xi, U_\xi(\xi, S), U_S(\xi, S))$$

guarantees that (P_θ, K) is has $\mathcal{L}_2$-gain less than or equal to γ for all θ, and that $x(t) \to 0$ as $t \to \infty$ for any $\mathcal{L}_2$ disturbances $w(\cdot)$, $v(\cdot)$. □

Proof: First, note that the one half of the statement $U = \min_\alpha Z$ is trivial:

$$U(\xi, S) = \sup_u \inf_y \min_\alpha \sum_i \alpha_i S_i(\infty) \leq \min_\alpha Z(\xi, S; \alpha) \quad .$$

To show that also the other inequality holds, we denote

$$\psi(\xi, S) = \min_\alpha Z(\xi, S; \alpha) = Z(\xi, S; \alpha^*(\xi, S))$$

and aim to show $\psi \leq U$ using proposition 41. First, take $\alpha = (1, 0, \ldots, 0)$; then $\psi(\xi, S) \leq Z(\xi, S; \alpha) = S_1$. Thus $\psi \leq \min_i S_i$.

Second, we must show that $H(\xi, S, \psi_\xi, \psi_S) \geq 0$. Here α^* being a minimize implies that

$$\begin{aligned}
\psi_\xi(\xi, S) &= Z_\xi(\xi, S; \alpha^*(\xi, S)) + Z_\alpha(\xi, S; \alpha^*(\xi, S)) \frac{\partial \alpha^*}{\partial \xi}(\xi, S) \\
&= Z_\xi(\xi, S; \alpha^*(\xi, S)) \quad , \\
\psi_S(\xi, S) &= Z_S(\xi, S; \alpha^*(\xi, S)) + Z_\alpha(\xi, S; \alpha^*(\xi, S)) \frac{\partial \alpha^*}{\partial S}(\xi, S) \\
&= Z_\xi(\xi, S; \alpha^*(\xi, S)) \quad .
\end{aligned}$$

Since Z solves the Hamilton-Jacobi-Isaacs equation $H(\xi, S, Z_\xi, Z_S) = 0$ fo each α, these expressions imply that also $H(\xi, S, \psi_\xi, \psi_S) = 0$. Thus w can apply proposition 41 to show that $\psi \leq U$, hence $\psi = U$, and that th control law u^U guarantees $\min_i S_i(T) \geq U(\xi(0), S(0))$ for all T and input $y(\cdot)$. Finally combine with theorem 40 to see that this control applied to P guarantees an $\mathcal{L}_2$ gain less than or equal to γ as well as internal stabilit

The theorem provides the following solution to the simultaneous $\mathcal{H}_\infty$ con trol problem: First, construct the filter bank (4.13), (4.14) which generate the estimates ξ_i, S_i. Second, determine off-line the quadratic value func tions $Z(\xi, S; \alpha)$ by finding the stabilizing solutions to a family of Ricca equation; one for each α. This yields the corresponding feedback controls

$$u^Z(\xi, S; \alpha) = u^*(\xi, Z_\xi(\xi, S; \alpha), Z_S(\xi, S; \alpha))$$

which are *linear* in ξ. Then, on-line, determine the minimizing argumen α^* and apply the control $u^U(\xi, S) = u^Z(\xi, S; \alpha^*(\xi, S))$.

One could argue that this solution is only partial since differentiability o Z and α^* is *sufficient* but not *necessary* for the existence of a simultaneou $\mathcal{H}_\infty$ controller. Indeed, Z may take the value $+\infty$ for some values of ξ an α, and - more importantly - α^* may be discontinuous, when more tha one minimizing argument of $\min_\alpha Z(\xi, S; \alpha)$ exist. At this point it is no clear how profound these difficulties are, and this topic deserves furthe attention. To this end, a good working hypothesis is the following:

Conjecture 47: The lower value function $U(\xi, S)$ is finite for all ξ, S and only if $Z(\xi, S; \alpha) > \Leftrightarrow\infty$ for all ξ, S, α. In this case

$$U(\xi, S) = \min_\alpha Z(\xi, S; \alpha)$$

where minimization is over weights $\alpha_i \geq 0$ such that $\sum_i \alpha_i = 1$. □

A result corresponding to the conjecture was stated recently in [90] for a finite horizon problem in discrete time; the proof has not been published. A complication related to the continuous-time setting is that discontinuities in the control law for u may lead to a closed loop system which is not well-posed.

In general, it adds some credibility to the conjecture is that many multi-objective optimization problems have been shown to be equivalent to weighted problems. A recent contribution concerning minimax control of a discrete system on a finite horizon is found in [89], which also contains further references. In order to prove the conjecture, or similar results, one probably needs to make use of viscosity solutions as well as minimax theorems [101], and investigate in further detail how the games associated with Z depend on α.

Summary of the discussion of the minimax controller

Let us briefly recapitulate our results concerning the minimax controller:

- The minimax controller is not based on certainty equivalence.
- The output feedback minimax control problem can be formulated as a full information problem of control for multi-dissipation (proposition 40). The control object in this equivalent problem is the filter bank (4.13), (4.14) which generates the minimax estimates of x, θ.
- The lower value function U of the corresponding game is not necessarily C^1, but solves a certain Hamilton-Jacobi-Isaacs equation in the viscosity sense (proposition 44). In addition, C^1 subsolutions of this equation generate guaranteed cost controllers, which solve the problem of simultaneous $\mathcal{H}_\infty$ control (proposition 41).
- These controllers can be implemented with the $p \times (n+1)$ states ξ, S.
- Theorem 46 reduces the task of solving the Hamilton-Jacobi-Isaacs equation by determining the structure of U: U can be derived from a study of the weighted optimization problems, providing that additional assumptions hold. Finally conjecture 47 suggests that these additional assumptions can be removed.

4.6 Heuristic certainty equivalence control

Even if the assumptions of theorem 46 are met, the resulting minimax cor troller is quite complex and requires substantial computation, both off-lin and on-line. From a practical point of view it is therefore of great interes to investigate what can be obtained with simpler controller architectures

In this section we consider the *heuristic certainty equivalence* controller

$$u(t) = u^{\hat\theta(t)}(t) = \Leftrightarrow B_{\hat\theta(t)'} X_{\hat\theta(t)} \xi(\hat\theta(t), t) \qquad (4.22$$

where

$$\hat\theta(t) = \arg\ \min_i S_i(t) \qquad (4.23$$

as before. Notice that the control law has yet to be defined at points wher the minimum $\min_i S_i$ is attained at more than one i. Controllers which ar based on certainty equivalence, but without justifying certainty equivalenc principles, are common in adaptive control [4].

The state of the controller is $(\xi(t), S(t))$. Recalling that the control ob jective is that the closed loop dissipates the supply rate s_θ as defined i equation (4.4), consider as a candidate control storage function

$$V(\theta, x(t), \xi(t), S(t)) = R(x(t), \theta, t) \Leftrightarrow \min_i S_i(t) \qquad (4.24$$

which may be computed as

$$V(\theta, x, \xi, S) =$$

$$\frac{1}{2}(x \Leftrightarrow \xi_\theta)'(Q_\theta \Leftrightarrow X_\theta)(x \Leftrightarrow \xi_\theta) + S_\theta + \frac{1}{2}x' X_\theta x \Leftrightarrow \min_i S_i \quad .$$

The candidate control storage function V is locally Lipschitz and henc differentiable almost everywhere, viz. wherever $\hat\theta = \arg\ \min_i S_i$ is we defined. Here the differential dissipation inequality holds, i.e.

$$\frac{dV}{dt} = \frac{1}{2}\gamma^2|v|^2 + \frac{1}{2}\gamma^2|w|^2 \Leftrightarrow \frac{1}{2}|H_\theta x|^2 \Leftrightarrow \frac{1}{2}|u|^2$$

$$\Leftrightarrow \frac{1}{2}|\gamma w \Leftrightarrow \frac{1}{\gamma}G'_\theta(X_\theta x + (Q_\theta \Leftrightarrow X_\theta)(x \Leftrightarrow \xi_\theta))|^2$$

$$\Leftrightarrow \frac{1}{2}\gamma^2|v + C_\theta x \Leftrightarrow C_{\hat\theta}\xi_{\hat\theta}|^2 + \frac{1}{2}|u + B'_{\hat\theta} X_{\hat\theta}\xi_{\hat\theta}|^2 \quad .$$

We see that whenever $\hat{\theta}(t)$ is well defined, the heuristic CE control $u(t) = u^{\hat{\theta}(t)}(t)$ is the *maximum dissipation* control law with which V indeed satisfies the differential dissipation inequality $\dot{V} \leq s$. Thus V is a *generalized solution*[3] to the differential dissipation inequality, in the sense of [35, p. 20].

It is now straightforward to pose the following result:

Proposition 48: Let assumptions 30, 31, 34 and 35 hold, let the heuristic CE control law (4.22) be used and assume that $\hat{\theta}(t)$ is well defined by (4.23) almost everywhere on $[0,T]$. Then the $\mathcal{L}_2$ gain objective is met, i.e.

$$\frac{1}{2}\int_0^T |z|^2\,dt \leq \frac{1}{2}\gamma^2\int_0^T |w|^2 + |v|^2\,dt + \frac{1}{2}x_0' Q_\theta x_0 + \Lambda_\theta \quad .$$

If furthermore $w \in \mathcal{L}_2([0,\infty))$, $v \in \mathcal{L}_2([0,\infty))$ and $\hat{\theta}(t)$ is well defined almost everywhere on $[0,\infty)$ then $x(t) \to 0$ as $t \to \infty$. $\triangle$

Proof: We have

$$\frac{1}{2}\int_0^T \gamma^2|w|^2 + \gamma^2|v|^2 \Leftrightarrow |z|^2\,dt + \frac{1}{2}x_0' Q_\theta x_0 + \Lambda_\theta$$
$$\geq R(x(T),\theta,T) \geq S(x(T),\theta,T) \geq \min_i S_i(T) \quad .$$

Since $\hat{\theta}(t)$ was assumed to be well defined for almost all $t \in [0,T]$ we have

$$\frac{d}{dt}\min_i S_i(t) = \frac{d}{dt}S_{\hat{\theta}(t)}(t) = \frac{1}{2}\gamma^2|y(t) \Leftrightarrow C_{\hat{\theta}(t)}\xi_{\hat{\theta}(t)}|^2 \geq 0$$

for almost all $t \in [0,T]$ due to the control law (4.22) and hence

$$\min_i S_i(T) \geq \min_i S_i(0) \geq 0$$

from which the result follows.

To show internal stability we follow the proof of proposition 39. ■

This result can be termed a *weak* certainty equivalence principle: Whereas the CE principle in [14] requires that the extended state estimate is *always* unique and concludes that the minimax controller is based on certainty

[3] V is also viscosity subsolution but in general not a supersolution which would imply dissipation [53]. Compare also with remark 45 above.

equivalence, the present result states that if the estimate is *almost alwa* unique, then the *heuristic* CE controller solves the original control problen although it may not be minimax.

The condition is not completely satisfying since it imposes a restriction o the disturbances $w(\cdot)$ and $v(\cdot)$. One can draw a parallel to the assumptio of persistent excitation in stochastic adaptive control: This condition is als not verifiable *a priori*, and a safety system must be added to the controlle so that proper action can be taken if the condition fails to hold. Howeve in contrast to the assumption of persistent excitation, it is difficult to se exactly which disturbances $w(\cdot)$, $v(\cdot)$ yield $\theta(t)$ being well defined almos everywhere, and hence it is difficult for the practicing engineer to judge the restriction is reasonable. Further work on this issue is needed.

A smooth approximation of the controller

Since the control law is discontinuous at points where the minimum $\min_i S$ is attained for more than one i (indeed, the control law has yet not bee defined at the points of discontinuity), some modification is needed t avoid chattering. Dwell-time switching or hysteresis switching are sug gested in [79]. Here we consider as an alternative to approximate th control law with a smooth one. This will ease the load on the actuato hardware and prevent excitation of unmodeled fast dynamics. To this end let us modify the candidate control storage function (4.24) to

$$\tilde{V}(\theta,x,\xi,S) = R(x,\theta,t) \Leftrightarrow f(S)$$

where the function f is the approximation of $\min_i S_i$ given by

$$f(S) = \Leftrightarrow\frac{1}{\eta}\log\left(\sum_{j=1}^{p} e^{-\eta S_j}\right) \quad . \qquad (4.25$$

Here $\eta > 0$ is a fixed parameter which determines the accuracy of th approximation. The function $f(S)$ enjoys the following properties whic make it a suitable approximation of $\min_i S_i$: 1) f is C^∞, 2) f satisfie $\partial f/\partial S_i \geq 0$ and $\sum_i \partial f/\partial S_i = 1$, and finally 3) $f(S) < \min_i S_i < f(S)$ + $\eta^{-1}\log p$.

The maximum dissipation controller corresponding to $\tilde{V}$ is

$$u^{\tilde{V}}(\xi,S) = \sum_{i=1}^{p}\frac{\partial f}{\partial S_i}(\Leftrightarrow B_i X_i \xi_i) \ ,$$

i.e., it is a weighted sum of the conditional minimax control suggested by each estimator. The derivation of this expression, as well as some further comments on this control law, can be found in [116].

Supervision of the controller

As mentioned above, the heuristic certainty equivalence controller should be supervised since we cannot prove that it guarantees satisfying operation. The dissipation analysis suggests that such a supervisory system should monitor the signals $S_i(\cdot)$. In particular, a decrease in $\min_i S_i$ or $f(S_i)$ indicates that the controller has not identified the plant and is uncertain about which control signal to actuate. On the other hand, a sudden increase in this signal should also attract attention as it indicates that the disturbances behave unexpected - a possible cause could be that a change in system parameters has occured.

We conclude the discussion of heuristic certainty equivalence control with a brief description of a simulation study:

Example 49: In [116] we discussed a case study regarding position control of an inverted pendulum, see figure 4.4. Here we briefly recapitulate the discussion; see [116] for further details.

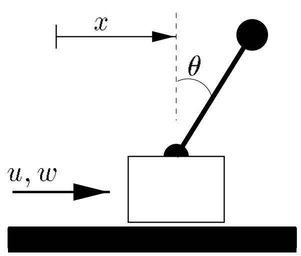

Figure 4.4: An inverted pendulum with force control and disturbance

The inverted pendulum is popular in benchmark problems because it is nonlinear, unstable, and minimum-phase (from the control force to the cart position), and yet relatively simple.

In our study, the plant is equipped with three sensors: One measuring the position of the cart, one measuring the angular position of the rod, and one

measuring the angular velocity of the rod. The latter is subject to fault s
we use two models to represent the control object; a nominal model an
one corresponding to the sensor fault.

A simultaneous controller for the two *linearized* plant models is constructec
using the heuristic certainty equivalence architecture developed above.

The residuals S_i are pre-filtered with a first-order low-pass filter before th
plant estimate $\hat{\theta}(t)$ is generated. This corresponds to exponential forgettin
in adaptive control, [85, 4].

Simulation results with the nonlinear plant and the switching controller ar
shown in figures 4.5 and 4.6. Here a sensor fault occurs at time 7.4 seconds
which is at a critical stage after a step in the position reference. The faul
is detected within approximately 0.2 seconds (figure 4.6). Some oscillation
result from the fault but the system is rapidly stabilized (figure 4.5). Afte
the fault has been detected system performance is worse since the one les
sensor implies worse state estimates.

The residuals S_i seem to be quite well suited as indicators of model fit, an
the heuristic certainty equivalence controller works nicely in this exampl
Although further work is needed with respect to forgetting schemes an
modifications of the switching mechanism, the controller architecture seem
to be reasonable and holds some promise.

4.7 Conclusion

In this chapter we have applied nonlinear $\mathcal{H}_\infty$ theory to the problem c
simultaneous $\mathcal{H}_\infty$ control of a finite number of linear plants. Our mot
vation for investigating this problem is that it appears to be the simples
problem of adaptive $\mathcal{H}_\infty$ control, if one excludes problems where paramete
uncertainty is restricted to special system parameters.

We have shown that simultaneous $\mathcal{H}_\infty$ control involves a nonlinear $\mathcal{H}_c$
problem which possesses a number of simplifying features: The full info
mation subproblem can be solved using linear theory. The cost-to-go, c
the information state, is a quadratic function on state space which als
can be found using linear theory. Although certainty equivalence does nc
apply, the simultaneous $\mathcal{H}_\infty$ control problem can be reduced to a stat
feedback problem on the worst-case filter, and hence be solved with finite
dimensional dynamic programming. However, these worst-case filters wi

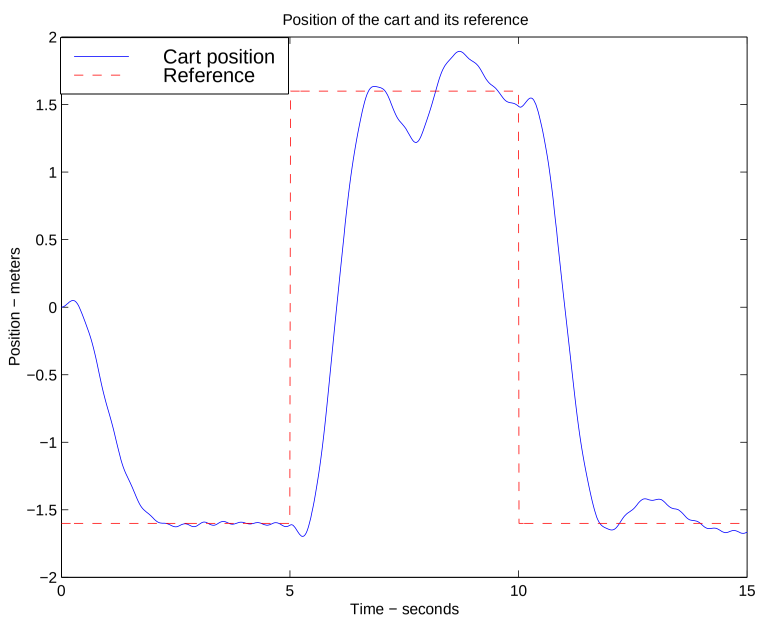

Figure 4.5: The position of the cart and its reference. Sensor fault in the angular velocity sensor at time 7.4 seconds.

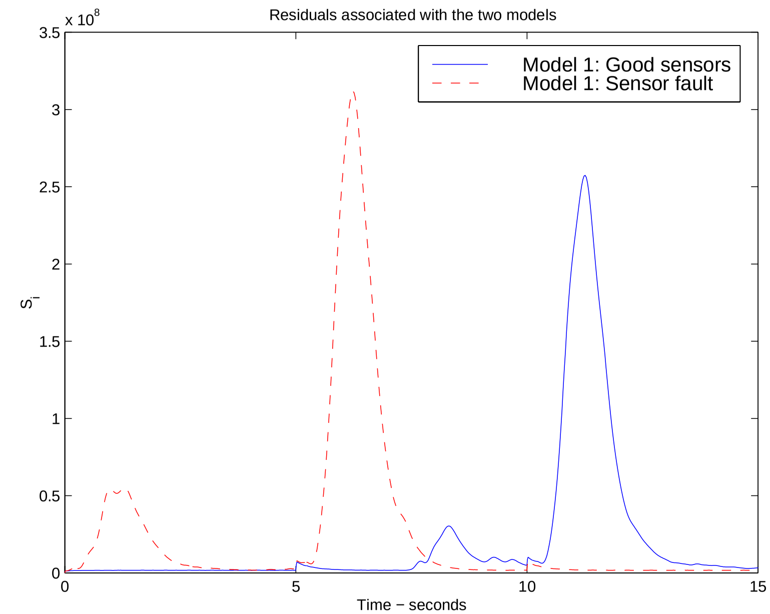

Figure 4.6: The residuals associated with the two models. The controlle corresponding to the lower residual is connected to the plant. The fault a time 7.4 seconds is detected when the lines cross; approximately at tim 7.6 seconds.

always be high-dimensional (example 49 leads to a filter with 14 states despite being somewhat academic) which makes direct numerical solution impossible and *a priori* insight into the structure of the solution necessary. We have made considerable progress in this direction, although a complete solution requires conjecture 47 to be verified a falsified. Furthermore we have investigated the heuristic certainty equivalence controller, and although the assumptions under which we can guarantee its performance are very restrictive, our simulations study suggests that its architecture is quite reasonable.

The work reported in this chapter may be continued in several directions: Further theoretical study of the problem may lead to conjecture 47 being resolved, or less restrictive conditions under which the heuristic certainty equivalence controller is sufficient. Also approximations of the heuristic certainty equivalence controller and investigation of various forgetting schemes is a subject which deserves more attention. Some hints towards other subjects are given in the succeeding notes.

Acknowledgements

The author wishes to thank Dr. A. Rapaport for stimulating discussions, and Prof. G. Vinnicombe for pointing out the work of S. Rangan and K. Poolla [90], which is described below.

4.8 Notes and references

Related recent literature

A study of simultaneous output feedback $\mathcal{H}_\infty$ control using digital controllers is presented by Savkin in [94]. The approach in this reference is reminiscent of the information state machinery, although it is not explicitly used since the problem is not formulated as a nonlinear $\mathcal{H}_\infty$ problem. The main result of the paper is that a feasible controller exists if and only if *a)* the filter algebraic Riccati equation (4.12) admits a suitable solution (although a small perturbation of the equation is necessary due to the use of digital controllers), *and b)* a full information minimax control problem, which except for the use of piecewise constant control signals is similar to

the problem discussed in section 4.5, admits a solution. As a consequenc
we may restrict attention to finite dimensional controllers. The problem o
explicitly characterizing the solution of this state feedback minimax contro
problem is not addressed. It is interesting to notice that the use of digita
controllers leads to a certain amount of technical simplification.

In [90] Rangan and Poolla consider a problem of simultaneous $\mathcal{H}_\infty$ contro
which is similar to the one studied here, but is formulated in discrete tim
and on finite horizon. The approach is based on the information state ma
chinery. Difficulties regarding regularity of the value function are avoide
due to the finite horizon discrete time setting. Very interestingly, a resul
which resembles our conjecture 47 is stated. The proof of the result ha
not been published (the result is not included in [88]), and therefore it i
not clear if it can be modified to assist in the verification of our conjec
ture 47. Other problems related to identification and control of a plan
with multiple models are investigated in [88].

Jumping parameters

In applications one must usually expect that the parameter θ is not constan
for all time, but will occasionally jump. This holds whether the problem i
one of fault handling or an approximation to an adaptive control problen
where the continuous parameter space has been discretized.

As we mentioned in example 49 in section 4.6, one may add a exponentia
forgetting scheme to the heuristic certainty equivalence controller in orde
to make it handle parameter jumps. This forgetting scheme and others ar
popular in adaptive control [4, 85], and although it is most often difficul
to carry through a rigorous analysis of the resulting system, experienc
indicates that they work quite well.

A rigorous approach to the problem with jumping parameters is to mode
the parameter variations with a Markov chain, which then leads to
stochastic dynamic game. The full information, finite time version of th
game is treated in [8], where the solution is shown to be governed by
p coupled differential Riccati equations. The corresponding output feed
back problem is open and involves several new difficulties, regarding th
characterization and rôle of the information state.

Relaxing the simplifying assumptions

We have assumed the *simple case* [128] of the p linear $\mathcal{H}_\infty$ control problems, i.e. observability of (H_i, A_i), controllability of (A_i, G_i), decoupled process and measurement noise w and v, and a decoupled error signal $z = (\,(Hx)', u'\,)'$. Relaxing these assumptions involves mainly algebraic manipulations [128], although certain details require attention.

Another assumption which can easily be removed is that the p plants have state space representations with the same dimension.

Part II

Stochastic models

Chapter 5

Dissipation in stochastic systems

We define the property of dissipativity for controlled Itô diffusions, and we investigate elementary properties, such as differential dissipation inequalities, convexity, and the connection to stability.

5.1 Introduction

Dissipative systems play a central rôle in the deterministic theory of robus stability, as evident from the works of numerous authors and also from th first part of this thesis. The deterministic theory also enables performanc analysis, where performance is measured by the response to initial cond tions, or by the worst-case response to an input in some set. The resultin framework has much appeal from a theoretical as well as from an eng neering point of view, and is in accordance with the currently dominatin paradigm for robust control.

A drawback of this framework is that it does not allow for stochastic repre sentations of uncertainty, such as white noise disturbances, or for stochasti performance measures, such as risk of failure. On the other hand, the li erature on stochastic systems does little to address the issues of robustnes towards dynamic perturbations which motivated for instance the development of $\mathcal{H}_\infty$ control.

This suggests that it may be fruitful to extend the theory of dissipation to stochastic systems, and apply it to robustness analysis of stochastic systems. In this chapter we report results which indicate that the concept of dissipation is indeed meaningful in a stochastic context, and that much of the deterministic theory applies more or less directly.

Dissipation-like properties of stochastic systems do appear in the literature. For instance [31] uses stochastic Lyapunov functions to achieve bounds on the $\mathcal{L}_2$-gain of a wide sense linear system with deterministic inputs and stochastic outputs. Another example is the stochastic small gain theorem in [30] which connects input-output properties to Riccati equations, the solutions of which are subsequently used to obtain a stochastic stability result.

5.2 Preliminaries

We consider a controlled process x_t in a Euclidean state space $\mathbb{X} = \mathbb{R}^n$ given by an Itô stochastic differential equation evolving on the time interval $\mathbb{T} = [0, \infty)$

$$dx_t = f(x_t, w_t)\ dt + g(x_t, w_t)\ dB_t, \quad x_0 = x \in \mathbb{X} \tag{5.1}$$

where B_t is standard m-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with respect a given filtration $\mathcal{F}_t$. The initial condition x is deterministic. The input w_t is an $\mathcal{F}_t$-adapted process taking values in Euclidean space $\mathbb{W} = \mathbb{R}^p$. See [83] for the necessary background material.

The system exchanges some quantity with its environment, specified by a supply rate $r : \mathbb{X} \times \mathbb{W} \to \mathbb{R}$. The accumulated flow from environment into the system during the time interval $[0, t]$ is R_t where

$$dR_t = r(x_t, w_t)\ dt, \quad R_0 = 0 \quad . \tag{5.2}$$

Notice that we here consider the supply to be a function of state and input, rather than a function of input and output. The motivation for this is simply to achieve a shorter notation, and the reader may substitute $r(x, w)$ with $s(z, w)$ if he so pleases, where $z = h(x, w)$ is the output.

We do not wish to dwell on technicalities regarding existence and unique-ness of solutions. Hence we simply restrict the input w_t to a set $\mathcal{W}$ o
$\mathcal{F}_t$-adapted inputs for which there exists a unique t-continuous solution x
and assume that $\mathcal{W}$ is sufficiently large and closed under switching so tha
the principle of optimality holds.

Associated with the equation (5.1) we define for each $w \in \mathbb{W}$ the differentia
operator $L^w : C^2(\mathbb{X}, \mathbb{R}) \to C^0(\mathbb{X}, \mathbb{R})$ given by $L^w V(x) = V_x f + \frac{1}{2}\mathrm{tr} g' V_{xx}$
where the right hand side is evaluated at (x, w).

If J is a functional on sample paths of the processes x_t, w_t, then $E^x J$
expectation w.r.t. the probability measure generated by x_t, w_t with initia
condition $x_0 = x$. In this notation the dependence of $E^x J$ on the input w
is suppressed.

5.3 Definition of dissipativeness and elementary properties

Recall that the fundamental element in the deterministic theory of dissipa-
tion [124] is the storage function $V : \mathbb{X} \to \mathbb{R}$ which satisfies the dissipatio
inequality

$$V(x_t) \le V(x_0) + \int_0^t r(x_s, w_s)\ ds$$

along every trajectory of the system. This inequality can be generalize
to a stochastic setting in several ways, but it appears that the most usef
framework is achieved by requiring the inequality to hold in expectation:

Definition 50: We say that the system (5.1) is *dissipative* w.r.t. th
supply rate r, if there exists a non-negative *storage function* $V : \mathbb{X} \to$
such that the *integral dissipation inequality*

$$E^x\{V(x_\tau) \Leftrightarrow \int_0^\tau r(x_s, w_s)\ ds\} \le V(x) \tag{5.3}$$

holds for all bounded stopping times τ and all solutions x_t, w_t of the syste
with $x_0 = x \in \mathbb{X}$.

We emphasize that the dissipation inequality is only required to hold fo
bounded stopping times τ; see p. 114 below for a comment.

Using the results in e.g. [83], it is easy to see that storage functions are related to a differential version of the dissipation inequality:

Proposition 51: A nonnegative C^2 function $V : \mathbb{X} \to \mathbb{R}$ is a storage function if and only if it satisfies the differential dissipation inequality

$$\sup_{w \in \mathbb{W}} L^w V(x) \Leftrightarrow r(x,w) \le 0 \tag{5.4}$$

on $\mathbb{X}$. △

Proof: Sufficiency: Let V be C^2 and satisfy the inequality (5.4). Let x_t, w_t be a solution with $x_0 = x \in \mathbb{X}$ and let τ be a bounded stopping time, i.e. $\tau < T$. Let $(x_{t\wedge\tau}, R_{t\wedge\tau})$ be the process (x_r, R_t) stopped at τ, i.e.

$$dx_{t\wedge\tau} = dx_t \cdot \chi_{t\le\tau}, \quad dR_{t\wedge\tau} = dR_t \cdot \chi_{t\le\tau}$$

(Here $\chi_{t\le\tau}$ is the indicator function, i.e. $\chi_{t\le\tau} = 1$ if and only if $t \le \tau$ and 0 otherwise). Now consider $V(x_{t\wedge\tau}) \Leftrightarrow R_{t\wedge\tau}$. By Itô's lemma this process is again an Itô process and the differential dissipation inequality implies $E^x(V(x_{t\wedge\tau}) \Leftrightarrow R_{t\wedge\tau}) \le V(x)$. Now notice that $x_{T\wedge\tau} = x_\tau$ and $R_{T\wedge\tau} = R_\tau$; we have thus shown that the inequality (5.3) holds.

Necessity: Let V be a C^2 storage function and consider a solution (x_t, w_t) for which the input is constant and deterministic, $w_t \equiv w \in \mathbb{W}$, and the initial condition $x_0 = x \in \mathbb{X}$ is deterministic. Then V is in the domain of the characteristic operator $\mathcal{A}^w$ (see [83, p. 116]) and the dissipation inequality (5.3) implies that $L^w V(x) = \mathcal{A}^w V(x) \le r(x,w)$. Since x and w were arbitrary the conclusion follows. ■

We define the available storage of the system (5.1) w.r.t. the supply rate r in a manner analogous to [124], namely by

$$V_a(x) = \sup_{w_t,\tau} E^x \int_0^\tau \Leftrightarrow r \; ds \tag{5.5}$$

where the supremum is over all bounded stopping times τ and all solutions x_t, w_t with $x_0 = x$. With this definition we immediately have a result analogous to theorem 1 in [124, p. 328]:

Proposition 52: The available storage is finite for all $x \in \mathbb{X}$ if and only if the system is dissipative. Furthermore, in this case the available storage is in itself a storage function and any other storage function V satisfies

$$V(x) \ge V_a(x), \quad \forall x \in \mathbb{X} \quad .$$

Finally $\inf\{V_a(x) : x \in \mathbb{X}\} = 0$.

Proof: First we show that if the available storage is finite, then it is storage function. It is immediate that $V_a \ge 0$ (if necessary, this is obtaine by letting $\tau \to 0$). The dissipation inequality then reads

$$E^x\{V_a(x_\tau) \Leftrightarrow \int_0^\tau r \; ds\} \le V_a(x)$$

which follows from the principle of optimality. Hence the system is diss pative.

Second we show that if the system is dissipative with storage function V then we have $V_a \le V$; in particular the available storage is finite. To se this we rewrite the dissipation inequality as

$$E^x \int_0^\tau \Leftrightarrow r \; ds \le V(x) \Leftrightarrow E^x V(x_\tau) \le V(x)$$

where the second inequality follows from V being non-negative. Since th inequality holds for all bounded stopping times τ and all solutions x_t, w which satisfy $x_0 = x$ we have $V_a(x) \le V(x) < \infty$. The conclusion follows

Finally we show the last claim: Let $V(x)$ be a storage function, then it easy to see that so is $V(x) \Leftrightarrow \inf_\xi V(\xi)$, hence $V_a(x) \le V(x) \Leftrightarrow \inf_\xi V(\xi)$. follows that $\inf_x V_a(x) \le 0$.

The available storage is related to a differential dissipation *equality*; see th note on page 116 below.

In chapter 2 on deterministic systems, we stated that storage functions an supply rates satisfy a joint convexity property (proposition 2 on page 32 This generalized a statement of Willems [124, theorem 3, p. 331] an was the key to the chapter 3, which reduced robustness analysis to conve optimization. This approach to robustness analysis is also fruitful in stochastic context, which is the subject of the succeeding chapter. At th point we state a result similar to the deterministic proposition 2:

Proposition 53: Given a diffusion (5.1), a linear space $\mathcal{V}$ of candidat storage functions $V : \mathbb{X} \to \mathbb{R}$ and a linear space $\mathcal{R}$ of supply rates. The the subset

$$\{(V,r) \subset \mathcal{V} \times \mathcal{R} \mid V \ge 0 \text{ and } (V,r) \text{ satisfy } (5.3)\}$$

is a convex cone.

Proof: Let $r_i \in \mathcal{R}$ for $i = 1, 2$ be chosen such that the system is dissipative w.r.t. r_i and let two corresponding storage functions be $V(x; r_i)$. Let x_t, w_t be a solution with $x_0 = x \in \mathbb{X}$ and let τ be a bounded stopping time; we then know that

$$E^x\{V(x_\tau; r_i) \Leftrightarrow \int_0^\tau r_i \, dt\} \leq V(\bar{x}; r_i) \quad .$$

By multiplying these two inequalities with positive constants α_i and adding the results we see that $\alpha_1 V(x; r_1) + \alpha_2 V(x; r_2)$ is a storage function for the system w.r.t. the supply rate $\alpha_1 r_1 + \alpha_2 r_2$. ■

In particular, the set of dissipated supply rates in $\mathcal{R}$ is a convex cone, as is the case for deterministic systems (see chapter 2 or [45]). A related fact is the following:

Proposition 54: Let $V_a(x; r) \in [0, \infty]$ be the available storage of the system (5.1) with respect to the rate $r \in \mathcal{R}$, then for each x the function $V_a(x; r)$ is convex in r. △

Proof: The available storage is for each x defined as the supremum of a family of functionals which are convex in r; the same holds therefore for $V_a(x, \cdot)$. ■

5.4 Linear systems and quadratic supply rates

Consider a homogeneous wide sense linear system

$$dx_t = [Ax_t + Bw_t] \, dt + \sum_{i=i}^{m} [F_i x_t + G_i w_t] \, dB_t^i \tag{5.6}$$

with a quadratic supply rate $r(x, w) = (x' \; w')Q(x' \; w')'$. We assume that r is concave-convex in (x, w) which implies that r is regular in the sense $r(x, 0) \leq 0$. This system is linear in the sense that the set of solutions (x_t, w_t) is a linear space; in other words, the map from input process w_t and initial condition x to the state process x_t is linear. It can be shown that if such a system is dissipative then the available storage is a quadratic function of the initial state x, i.e. may be written as

$$V_a(x) = x' P_a x$$

where $P_a = P_a' \geq 0$. Furthermore, the quadratic storage functions $V(x) =$ $x' P x$ with $P = P'$ are exactly those that satisfy $P \geq 0$ and the differentia dissipation inequality (5.4) which can be rewritten as the linear matri inequality

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} + \sum_{i=1}^{m} [F_i, G_i]' P [F_i, G_i] \leq Q \quad . \qquad (5.7$$

It is thus possible to use LMI solvers as [38, 32] to answer the analys questions: Is the system dissipative? If yes, what is the available storage?

Remark 55: It is well known (see e.g. [76] and the references therein) tha multiplicative noise terms F_i, G_i can be advantageous for a linear syster from the point of view of stability in probability. But such a noise tern will always contribute positively to the left hand side of the inequality (5.7 which shows that multiplicative noise terms are always disadvantageous i analysis of dissipation w.r.t. a quadratic supply rate.

Supply rates of special interest are those corresponding to passivity an small gain. Stochastic $\mathcal{L}_2$ gains have recently received some attention an stochastic bounded real lemmas as well as other results can be found in [3 31, 48]. Stochastic passivity has, to our knowledge, not been considered i the literature, probably because stochastic passivity of an isolated syster is of no particular interest. However, if a nominal stochastic system connected to an unknown passive perturbation, then it is of great relevanc if the nominal system is stochastically passive. We will later return to suc robustness issues; at this point we state a stochastic positive real lemma

Proposition 56: For the system (5.6), let the supply rate be $r(x, w) =$ $2\langle w, z \rangle$ with $z = Cx + Dw$. Then the following are equivalent:

1. The system is stochastically strictly input passive, i.e. stochasticall dissipative w.r.t. $r \Leftrightarrow \epsilon |w|^2$ for some $\epsilon > 0$, and the autonomous syster obtained with $w = 0$ is exponentially mean square stable.

2. There exists a $P = P' > 0$ such that

$$\begin{bmatrix} PA + A'P & PB \\ B'P & 0 \end{bmatrix} + \sum_{i=1}^{m} [F_i, G_i]' P [F_i, G_i] < \begin{bmatrix} 0 & C' \\ C & D' + D \end{bmatrix} \quad . \qquad (5.8$$

Before we prove the proposition it is convenient to state an elementary matrix lemma:

Lemma 57: Let $Q = Q' > 0$, $R = R' > 0$, S and $T = T'$ be of compatible dimensions, then for $\alpha > 0$ sufficiently small the matrix inequality

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & Q \end{array}\right] > \alpha \left[\begin{array}{cc} \Leftrightarrow R & S \\ S' & T \end{array}\right]$$

holds. □

Proof: [of the lemma] By Schur complement, the inequality holds if and only if $\alpha R > 0$ and $Q \Leftrightarrow \alpha T \Leftrightarrow \alpha S' R^{-1} S > 0$. These conditions are satisfied for $\alpha > 0$ small enough since $Q > 0$ and $R > 0$. ■

Proof: [of the proposition] To see that the linear matrix inequality condition is sufficient one needs only verify that $V(x) = x'Px$ is a storage function function w.r.t. $r \Leftrightarrow \epsilon |w|^2$ for sufficiently small ϵ, and that, with $w = 0$, V serves as a stochastic Lyapunov function to show exponential mean square stability using a standard sufficient condition [43, p. 200].

To show necessity we use that exponential mean square stability implies [43, p. 201] the existence of a $Z = Z' > 0$ such that

$$ZA + A'Z + \sum_{i=1}^{m} F^{i'} Z F^i < \Leftrightarrow \delta\ I$$

for some $\delta > 0$. Now let $\epsilon > 0$ and let $V(x) = x'Xx$ be a storage function for the system with respect to the supply rate $r \Leftrightarrow \epsilon |w|^2$, i.e. $X = X' \geq 0$ and

$$\left[\begin{array}{cc} XA + A'X & XB \\ B'X & 0 \end{array}\right] + \sum_{i=1}^{m} [F_i, G_i]' X [F_i, G_i] \leq \left[\begin{array}{cc} 0 & C' \\ C & D' + D \Leftrightarrow \epsilon\ I \end{array}\right] \quad .$$

We claim that $P = X + \alpha Z$ solves the linear matrix inequality (5.8) for $\alpha > 0$ sufficiently small. To see this, insert $P = X + \alpha Z$ in (5.8) and reduce terms using the LMIs which X and Z satisfy, thus obtaining

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & \epsilon I \end{array}\right] > \alpha \left[\begin{array}{cc} \Leftrightarrow \delta I & ZB + \sum F_i' Z G_i \\ B'Z + \sum G_i' Z F_i & \sum G_i' Z G_i \end{array}\right] \quad .$$

This inequality holds for $\alpha > 0$ small enough according to lemma 57 which completes the proof. ■

Apart from strict input passivity, one could imagine several other definitions of strict positive realness of a stochastic system, just as in the deterministic case [123].

5.5 Stability and interconnections of dissipa tive systems

As in proposition 56 and in deterministic theory [124, 45] a storage functio often serves as a Lyapunov function to show that the isolated system stable. Indeed, this is one of the properties which make dissipative system interesting from a control point of view.

In order to investigate stability of the autonomous system

$$dx_t = f(x_t, 0)\ dt + g(x_t, 0)\ dB_t \qquad (5.9$$

we use the terminology of Has'minskiĭ [43]:

Definition 58: A constant solution $x_t \equiv \bar{x}$ of the autonomous equa tion (5.9) is stable in probability if for any $\epsilon > 0$

$$\lim_{x \to \bar{x}} P^x \{ \sup_{t \geq 0} |x_t \Leftrightarrow \bar{x}| > \epsilon \} = 0$$

where the diffusion x_t solves (5.9) with $x_0 = x$.

Using the existing Lyapunov-type criterion for stochastic stability [43] w immediately get the following:

Theorem 59: Let the supply rate r be regular in the sense that $r(x, 0) \leq$ for all x. Let the system (5.1) be dissipative with respect to r and let be a continuous storage function which attains an isolated local minimu at $\bar{x} \in \mathbb{X}$. Then the process $x_t \equiv \bar{x}$ is a solution of the autonomou equation (5.9) and is stable in probability.

Proof: The proof runs along the same lines as theorem 3.1 in [43, 164], the only deviation being that V is not required to be C^2 around Let $a = V(\bar{x})$, let Ω be a neighbourhood of $\bar{x}$ such that $a < V(x)$ fo $x \in \bar{\Omega} \backslash \{\bar{x}\}$. Let τ be the stopping time $\tau = \inf\{t : x_t \notin \Omega\}$. It follows fro the dissipation inequality that $V(x_{t \wedge \tau}) \Leftrightarrow a$ is a supermartingale for an initial condition $x \in \Omega$. In particular if $x = \bar{x}$ then $x_t = \bar{x}$ w.p. 1 for a t which proves the first claim. Furthermore for $x \neq \bar{x}$ the supermartinga inequality of Doob (see e.g. [83, p. 28]) yields

$$P^x \{ \sup_{t \geq 0} V(x_{t \wedge \tau}) \Leftrightarrow a \geq \epsilon \} \leq \frac{V(x) \Leftrightarrow a}{\epsilon}$$

which holds for all ϵ. Now pick arbitrarily small $\epsilon, \epsilon' > 0$ such the ϵ-ball around $\bar{x}$ is contained in Ω. We must show that there exists $\delta > 0$ such that $|x \Leftrightarrow \bar{x}| < \delta$ implies $P^x\{\sup_{t\geq 0} |x_t \Leftrightarrow \bar{x}| > \epsilon\} \leq \epsilon'$. To this end choose $V_2 > a$ such that $\xi \in \Omega$ and $V(\xi) < V_2$ together imply $|\xi \Leftrightarrow \bar{x}| < \epsilon$. Then choose $V_1 > a$ such that $(V_1 \Leftrightarrow a)/(V_2 \Leftrightarrow a) \leq \epsilon'$. Finally choose $\delta \in (0, \epsilon)$ such that $|\xi \Leftrightarrow \bar{x}| < \delta$ implies that $V(\xi) < V_1$. We then have the implications

$$
\begin{aligned}
|x \Leftrightarrow \bar{x}| < \delta \quad &\Rightarrow \quad V(x) \Leftrightarrow a < V_1 \Leftrightarrow a \\
&\Rightarrow \quad P^x\{\sup_{t\geq 0} V(x_{t\wedge\tau}) \Leftrightarrow a > V_2 \Leftrightarrow a\} \leq \frac{V_1 \Leftrightarrow a}{V_2 \Leftrightarrow a} \\
&\Rightarrow \quad P^x\{\sup_{t\geq 0} |x_t \Leftrightarrow \bar{x}| > \epsilon\} \leq \epsilon'
\end{aligned}
$$

as desired. ■

Remark 60: We say that the system (5.1) is *locally* dissipative around $\bar{x}$ w.r.t. the supply rate r if there exists a non-negative V and a bounded neighbourhood Ω of $\bar{x}$ such that the dissipation inequality holds provided $x_t \in \Omega$ for $0 \leq t < \tau$. In this case we say that V is a local storage function. A necessary and sufficient condition for a non-negative C^2 function V to be a local storage function is that it satisfies the differential dissipation inequality (5.4) on Ω. It is easy to see that the above theorem holds if the storage function V is replaced with a local storage function. □

One may show other stability properties such as stochastic sample path boundedness or exponential p-stability by imposing additional constraints on the storage function and the supply rate and using the corresponding Lyapunov-type theorems in [43].

As in the deterministic case, the stability implications of dissipativity is important in robustness analysis since systems consisting of dissipative components are themselves dissipative. Consider the simple case of two systems

$$\Sigma_i \ : \ dx^i = f^i(x^i, w^i)\ dt + g^i(x^i, w^i)\ dB^i$$

connected in feedback through the equations

$$w^1 = h^2(x^2, w^2) + v^1 \ \text{ and } \ w^2 = h^1(x^1, w^1) + v^2 \quad .$$

Here h^i are output functions and v^i are exogenous inputs. Assume that each system is dissipative w.r.t. the rate $r^i(x^i, w^i)$. In addition, assume that the interconnecting equations have unique solutions $w^i = \bar{w}^i$ for all

x^i and v^i (for instance, if one of the two h^i is independent of w^i) and tha
the resulting system satisfies the well-posedness assumptions of section 5.
(in particular, (B^1, B^2) is standard Brownian motion w.r.t. the filtratio
$\mathcal{F}_t$). It is now easy to verify that the interconnection is dissipative w.r.
the supply rate $r(x^1, x^2, v^1, v^2) = r^1(x^1, \bar{w}^1) + r^2(x^2, \bar{w}^2)$. Combining wit
the stability result of theorem 59 we get:

Proposition 61: Assume that the each of the storage functions $V^i(x$
is continuous and attains an isolated local minimum at $x^i = 0$. Assum
in addition that the supply rates satisfy $r(x^1, x^2, 0, 0) \leq 0$ for all x^1,
Then $x^i_t \equiv 0$ is a solution of the interconnected system with $v^i_t \equiv 0$ an
this solution is stable in probability.

The main application of this result is to give a sufficient condition for *r*
bust stability of a stochastic system subject to a deterministic dissipati
perturbation, for instance combining with the positive real lemma of propo
sition 56:

Corollary 62: Let a system Σ be given by the dynamics (5.6) an
the output equation $z = Cx + Dw$, and let Σ be connected in feedbac
with a perturbation $\Delta : z \to w$ which is dissipative w.r.t. $\Leftrightarrow 2\langle w, z\rangle$. Le
the interconnection be well posed and let Δ possess a continuous storag
function of which some point ξ is an isolated minimum point. Assume tha
there exists a $P = P' > 0$ such that the linear matrix inequality (5.8) hold
Then the constant process $(0, \xi)$ is a solution of the interconnection an
this solution is stable in probability.

The corollary demonstrates that, as in the deterministic theory, robustnes
questions can be resolved by computing storage functions; in the case
linear systems this reduces to linear matrix inequalities.

5.6 Chapter conclusion

It can be argued that the concept of dissipation in dynamical systems
the unifying factor behind a broad range of results in deterministic contr
theory, in particular within robust control. We believe that the appeal
the framework is not lost in the transfer to a stochastic context.

Although this chapter demonstrates that several key features of the dete
ministic theory generalizes to the stochastic setting, the stochastic theor

is far from complete. Some comments on remaining problems are discussed in the following.

5.7 Notes and references

Unbounded stopping times

In our definition of a dissipative stochastic system, the integral dissipation inequality (5.3) was required to hold for bounded stopping times only. This leads to the question: If V is a storage function for the diffusion (5.1) and τ is an unbounded stopping time, does the dissipation inequality (5.3) hold?

The short answer to this question is: Not necessarily. Let $\epsilon > 0$, then a trivial counterexample is the stopping time

$$\tau = \inf\{t > 0 \ : \ V(x_t) \Leftrightarrow \int_0^t r \ dt > V(x) + \epsilon\}$$

for which

$$E^x V(x_\tau) \Leftrightarrow E^x \int_0^\tau r \ dt = V(x) + \epsilon$$

provided that V is continuous, implying that the dissipation inequality does not hold. It is possible to construct examples where this stopping time is finite almost surely - the interested reader is encouraged to consider the diffusion $dx_t = \Leftrightarrow x_t \ dt + w_t \ dB_t$ with the supply rate $r = \Leftrightarrow 2x^2 + w^2$ and take the input w_t to be a non-zero constant.

A first step towards a more complete answer is that a sufficient condition for the dissipation inequality (5.3) to hold is that V is a storage function and that the family

$$\{V(x_{t\wedge\tau}) \Leftrightarrow \int_0^{t\wedge\tau} r \ dt\}_{t>0}$$

of random variables is uniformly integrable. This follows from a convergence result for uniformly integrable random variables, [83, p. 41] - we refer to the same reference for the definition of uniform integrability. We expect that more explicit results can be obtained for special classes of unbounded stopping times, such as the first exit time of x_t from a given domain.

Non-smooth storage functions and viscosity solutions

In deterministic theory of dissipation, it has been shown by James [53] tha locally bounded storage functions can without loss of generality be take to be lower semi-continuous (l.s.c.), and that l.s.c. storage functions ar exactly the viscosity solutions to the differential dissipation inequality (5.4

The question is if the analogous statements hold in the stochastic setting It is easy to show that l.s.c. storage functions are indeed viscosity solution to (5.4). We conjecture that also l.s.c. viscosity solutions to (5.4) are stor age functions. Existing stochastic verification theorems in the framewor of viscosity solutions [62, 131] are based on uniqueness results for viscos ity solutions and are therefore not applicable to dissipation inequalities (c even the corresponding equalities) which have many solutions. The de terministic technique in [53] could probably be modified and applied; th additional complication that the dissipation inequality must hold for an *random* bounded stopping time τ could be addressed with the results o optimal stopping in [84].

Further questions are if locally bounded storage functions can taken to b l.s.c. and under what conditions they can be taken to be continuous o even C^2. These issues are left for future research.

The required supply

Recall that we in chapter 2 defined the required supply of a dissipativ deterministic system as

$$V_r(x) = \inf_{x(\cdot),w(\cdot),T} \int_0^T r(t) \ dt$$

where the infimum is subject to the system dynamics and the condition $V_a(x(0)) = 0$ and $x(T) = x$. We see that this definition does not exten directly to stochastic systems, because the presence of noise may make i impossible to reach a specified terminal state in finite time.

An alternative starting point for a definition is

$$V_r(x) = \sup_V V(x)$$

where the supremum is over all l.s.c. storage functions V for which $V(\xi) = 0$ whenever $V_a(\xi) = 0$. Assume that the required supply defined in this fashion is finite; then it is l.s.c. and satisfies

$$E^x\{V_r(x_\tau) \Leftrightarrow \int_0^\tau r\ ds\} \leq 0 \quad \text{if} \quad V_a(x) = 0 \quad .$$

Does the available storage satisfy a PDE?

It is well known that there is an intimate connection between the Hamilton-Jacobi-Bellman *equations* and the available storage, the required supply and other value functions, [83, 7]. Nevertheless, the exact nature of this connection is often misquoted, in that situations where the value function does not satisfy the Hamilton-Jacobi-Bellman equation are regarded as pathological. Consider as an example passivity analysis of a scalar wide-sense linear diffusion:

$$dx_t = (\Leftrightarrow x_t + w_t)\ dt + \sigma x_t\ dB_t\ , \quad r(x,w) = xw\ ,$$

where $\sigma \geq 0$ is a parameter. Since the system is linear and the supply rate is quadratic we know that the available storage is a quadratic function of the state. It is easy to verify that a quadratic storage function $V(x) = \alpha x^2$ must satisfy

$$\forall x, w:\ 2\alpha x(\Leftrightarrow x + w) + \alpha\sigma^2 x^2 \leq xw$$

which implies

$$\alpha = \frac{1}{2}, \quad \sigma^2 \leq 2 \quad .$$

We see that the system is dissipative if and only if $\sigma^2 \leq 2$. In this case the available storage satisfies the Hamilton-Jacobi-Bellman *inequality*

$$\sup_w \{L^w V_a(x) \Leftrightarrow r(x,w)\} = (\Leftrightarrow 1 + \frac{1}{2}\sigma^2)x^2 \leq 0 \quad .$$

Only when $\sigma^2 = 2$ does the available storage satisfy the Hamilton-Jacobi-Bellman *equation*. The available storage solves a *strict* HJB-inequality when $\sigma^2 < 2$, for instance in the deterministic situation $\sigma = 0$.

The reason why the value function does not satisfy the HJB-equation is that no optimal solution exists. For the optimal control problem associated with the available storage, almost-optimal Markov controls are $w = \Leftrightarrow f x$ where

$f \to +\infty$. Theorems which state that the value function satisfies a PD (as for instance theorem 11.1 in [83]) need the existence of an optimal pa (x_t^*, w_t^*) (either explicitly assumed or implied by other assumptions) sinc their proofs involve differentiating the value function along x_t^*.

Another situation where the available storage does not satisfy the Hamilto Jacobi-Bellman equation is when the supply rates are not regular, i.e. i some region of state space the input is forced to deliver a positive sup ply to the system. A trivial example is the system above with the suppl rate 1. In general, non-regular supply rates lead to many contra-intuitiv phenomena and should be treated with care or avoided.

Computation of storage functions with convex optimization

Consider the input-affine controlled diffusion on $\mathbb{X} = \mathbb{R}^n$

$$dx_t = (f(x) + \phi(x_t)\ w_t)\ dt + (g(x_t) + \gamma(x_t)\ w_t)\ dB_t$$

with the input-quadratic supply rate

$$r(x,w) = h(x) + 2k(x)\ w + w'\ l(x)\ w \quad .$$

We assume that both B_t and w_t are scalar processes. The case of vecto processes is conceptually the same but the notation is more involved. Th backwards operator is

$$L^w V(x) = V_x f + V_x \phi\ w + \frac{1}{2}(g + \gamma\ w)' V_{xx} (g + \gamma\ w)$$

for $V \in C^2(\mathbb{X}, \mathbb{R})$; we have omitted the argument x on the right han side. The differential dissipation inequality (5.4) can then be written mor explicitly as

$$P(V,x) := \begin{bmatrix} V_x f + \frac{1}{2} g' V_{xx} g \Leftrightarrow h & \frac{1}{2} V_x \phi + \frac{1}{2} g' V_{xx} \gamma \Leftrightarrow k \\ \frac{1}{2}(V_x\phi)' + \frac{1}{2}\gamma' V_{xx} g \Leftrightarrow k' & \gamma' V_{xx} \gamma \Leftrightarrow l \end{bmatrix} \leq 0 \quad . \qquad (5.10$$

Here $P : C^2(\mathbb{X}, \mathbb{R}) \times \mathbb{X} \to \mathbb{R}^{2\times 2}$. A non-negative C^2 function is a storag function if and only if this matrix inequality holds at each point x in stat space (proposition 51 on page 106).

We now suggest the following numerical strategy for computing storage functions: Choose a set of basis functions $V^i \in C^2(\mathbb{X}, \mathbb{R})$ and search for a storage function of the form

$$V(x) = \sum_{i=1}^{N} \alpha_i V^i(x) \quad .$$

The basis functions V^i could for instance be polynomials, trigonometric functions, or wavelets. In order to verify if V is a storage function, we test for dissipation and non-negativity at a set of points x^j, $j = 1, \ldots, M$. This leads to the LMI problem

$$\text{Find } \alpha_1, \ldots, \alpha_N \text{ such that}$$

$$\sum_{i=1}^{N} \alpha_i P(V^i, x_j) \leq 0 \ , \quad \sum_{i=1}^{N} \alpha_i V^i(x_j) \geq 0 \ \text{ for } j = 1, \ldots, M \tag{5.11}$$

for which software such as [38, 32] can find a solution or determine that no solution exists.

The LMI problem has N scalar variables, M scalar constraints and M 2-by-2 matrix constraints. If w is a m-vector rather than a scalar, then the matrix constraints will be $(m+1)$-by-$(m+1)$. Notice that the dimension of the state space does not affect the size of the matrices; however high-dimensional state spaces need a large number of basis functions V^i and a large number of evaluation points x_j in accordance with the curse of dimensionality.

If the differential dissipation inequality (5.10) is merely satisfied at points x_j, it is quite likely that it fails near x_j. Therefore, one may wish to consider strict inequalities in (5.11) and attempt to solve

$$\text{Find } \alpha_1, \ldots, \alpha_N, \beta_1, \beta_2 \text{ such that}$$

$$\sum_{i=1}^{N} \alpha_i P(V^i, x_j) \leq \Leftrightarrow\beta_1 \ \kappa(x) \ , \quad \text{for } j = 1, \ldots, M \quad ,$$

$$\sum_{i=1}^{N} \alpha_i V^i(x_j) \geq \beta_2 \ \lambda(x) \ , \quad \text{for } j = 1, \ldots, M \quad ,$$

$$\beta_1 > 0 \ , \quad \beta_2 > 0$$

where κ and λ are given functions.

Computing storage functions with LMI software is a relatively flexible prin ciple which may be modified in several ways, depending on the specific ap plication. For instance, one may search simultaneously for a supply rate i some convex polytope, add constraints on the storage function, its gradie or curvature, or one may include a linear functional of storage function an supply rate to be minimized.

If one goes beyond input-affine systems with input-quadratic supply rate then storage functions may still be found with convex optimization bu with much greater difficulty since the differential dissipation inequality doe not reduce to LMIs in state space. Further complications arise when th supremum over w in the differential dissipation inequality (5.4) cannot b evaluated explicitly.

While the above discussion may be sufficient for illustrative academic cas studies, it would be necessary for real-world applications to consider th numerics in greater detail. A specific question which deserves attentio concerns the dual to (5.10). As emphasized in [19], when convex opt mization is used as computational tool in control theory, the dual proble often have interesting control theoretic interpretations. See [57] for an ex ample where dualism is utilized in discretized infinite-dimensional conve optimization problems.

Another strategy for numerical computation of storage functions is to solv a partial differential equation corresponding to the differential dissipatio inequality (5.4) using a finite difference scheme [65]. The two approache convex optimization and numerical solution of PDEs, may also be com bined.

Simplifying computations with modularity

For a realistic problem involving more than a couple of states and withou specific simplifying structure, the approaches outlined above become unre alistic as the numerical burden becomes overwhelming. In this case it ma be feasible to decompose the system into a number of sub-systems. Thes sub-systems need not correspond to physical units but could for instanc be dynamic modes which are known to interact weakly. Then one may pe form dissipation analysis on each of the subsystems and after this conclud on the dissipation of the overall system using the results on interconnections of dissipative components. In effect this corresponds to imposing a specific structure on the storage function of the overall system. Needless to say, the effectivity of this approach relies heavily on the physical and mathematical insight into the system.

For deterministic systems with (single) supply rates corresponding to passivity, this approach to analysis goes back to Popov's work on hyperstability [87]. An interesting topic of future research would be systematic modularization. Backstepping and other recursive design techniques [63, 24] can be seen as extreme examples of systematic modularization.

Stratonovich equations

In this dissertation we work exclusively with the Itô interpretation of stochastic differential equations. In some applications it is more natural to model uncertainty with stochastic differential equations in the Stratonovich interpretation. The difference between the interpretations is mainly one of modelling, though; in fact a stochastic process x_t solves the Stratonovich equation

$$dx_t = f(x_t)\ dt + g(x_t) \circ dB_t$$

where B_t is scalar Brownian motion if and only if it solves the *equivalent* Itô equation

$$dx_t = f(x_t)\ dt + \frac{1}{2} g_x(x_t) g(x_t)\ dt + g(x_t)\ dB_t \quad .$$

See [83, p. 75, p. 32 f.] - a similar formula holds for the case of multidimensional Brownian motion. Therefore, if one has modelled a system with a Stratonovich equation, then one may afterwards do the analysis for the equivalent Itô equation.

Chapter 6

Robust performance of stochastic systems

We demonstrate that a number of performance objectives for stochastic systems correspond to stochastic dissipation requirements: stochastic $\mathcal{L}_2$ gain, $\mathcal{H}_2$ gain, probability of failure, and expected time to complete a mission. Then we consider stochastic systems subject to dissipative perturbations and show that a stochastic dissipation analysis of the nominal system can provide sufficient conditions for robust performance of the perturbed system.

6.1 Introduction

The previous chapters in this dissertation have demonstrated that diss pation theory is a very useful tool in addressing deterministic problem of robustness analysis, and that dissipation theory can be generalized t a stochastic setting. The objective of this chapter is to combine thes two statements: Robust performance analysis of stochastic systems can b based on stochastic dissipation.

What motivated us to consider robust performance of stochastic system was the specific problem of robust $\mathcal{H}_2$ performance in presence of $\mathcal{H}$

bounded perturbations. The reader may recall that we in chapter 3 gave a sufficient analysis condition for this problem, for linear systems; there we employed the deterministic interpretation of $\mathcal{H}_2$ performance which is in terms of the response to impulse inputs. This lead to the question if the same condition would also bound the $\mathcal{H}_2$ performance in the stochastic interpretation, i.e. the response to white noise. This chapter employs our notion of dissipative stochastic systems to answer this question affirmatively: For linear systems, one LMI condition on the nominal system implies robust $\mathcal{H}_2$ performance, whether the deterministic or stochastic interpretation of $\mathcal{H}_2$ performance is used. (See the note on page 142 for references to the literature on mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems.)

While our original objective was to bound the variance of an error signal in presence of deterministic and stochastic uncertainty, it soon became clear that many other performance objectives could be addressed with the same framework. Essentially, if performance analysis for the unperturbed stochastic system can be cast in terms of a Lyapunov-type function on state space, then robust performance can be guaranteed by dissipation-type arguments. Particular examples of such performance measures are the risk of failure, as well as expected time to complete a mission. In this chapter we provide bounds for the risk of failure of a stochastic system in presence of deterministic dissipative perturbations; this demonstrates that it is indeed possible to merge stochastic and robust control.

The chapter is organized as follows: First, in section 6.2, we discuss performance measures for autonomous stochastic system which can be formulated in terms of dissipation. Then, in section 6.3 we add an exogenous disturbance and discuss $\mathcal{L}_2$ gain and $\mathcal{H}_2$ performance of the disturbed stochastic system. In section 6.4 we consider *finite signal-to-noise ratio* systems in the sense of Skelton and embed the associated problems in our dissipation-based approach to robust performance.

In section 6.5 we demonstrate that *robust performance* of stochastic systems subject to multi-dissipative perturbations can be guaranteed by performing dissipation analysis on the nominal system. After this general statement we present two examples: robust $\mathcal{H}_2$ performance, and robust bounds on the probability of failure. Finally section 6.6 contains a few concluding remarks.

6.2 Performance of autonomous systems

In this section we consider the autonomous stochastic system

$$\Sigma \;:\; dx_t = f(x_t)\;dt + g(x_t)\;dB_t \qquad (6.1$$

where B_t is standard Brownian motion with respect to a filtration $\mathcal{F}_t$ o
a probability space $(\Omega, \mathcal{F}, P)$. For this system we discuss two propertie
which may be design objectives and give sufficient conditions in terms o
dissipation properties. We do not claim novelty of the conditions. Indeed
they can be found in classical literature, for instance [64]. Our contributio
is simply to point out that these properties can be cast in our framewor
of stochastic dissipation; in particular the characterization is convex. W
will employ this in a later section concerning robustness of the propertie
towards dissipative perturbations, thus obtaining new results.

For the convenience of the reader we include the proofs, which are all quit
straightforward.

Expected time to complete a mission

Assume that the state x_t of system Σ evolves in a domain $D \subset \mathbb{X}$ and tha
the control mission is completed upon first exit from D. We then have th
following bound on the expected time to complete the mission:

Proposition 63: Let $V : \bar{D} \to \mathbb{R}$ be a continuous storage function for
w.r.t. the supply rate ⇔1; then the bound

$$E^x \tau_D \leq V(x)$$

holds for $x \in D$.

Proof: By hypothesis the process $V(x_{t\wedge\tau_D}) + t \wedge \tau_D$ is a supermartinga
and hence

$$E^x\{t \wedge \tau_D\} \leq E^x\{V(x_{t\wedge\tau_D}) + t \wedge \tau_D\} \leq V(x) \quad .$$

Since this holds for all $t \geq 0$ we conclude that τ_D is finite P^x-almost surel
and that $E^x \tau_D \leq V(x)$.

Risk of failure

Assume now that the boundary of D is divided into two components A and B. As before, the process is stopped upon first exit from D and the mission is denoted a *success* if A is reached, whereas exiting through B is a failure. We then have the following bound on the risk of failure:

Proposition 64: Let $V : \bar{D} \to \mathbb{R}$ be a continuous storage function for Σ w.r.t. the supply rate 0 which satisfies the additional constraint $V|_B \geq 1$; then the bound

$$P^x\{\tau_D = \tau_B\} \leq V(x)$$

holds for $x \in D$. $\triangle$

Proof: By hypothesis the process $V(x_{t\wedge\tau_D})$ is a supermartingale and hence

$$P^x\{\tau_D = \tau_B\} \leq P\{\sup_{t\geq 0} V(x_{t\wedge\tau_D}) \geq 1\} \leq V(x) \quad .$$

Here the first inequality holds because $\tau_D = \tau_B$ implies $V(x_{\tau_D}) \geq 1$ and hence $\sup_{0\leq t} V(x_{t\wedge\tau_D}) \geq 1$. The last inequality is the supermartingale inequality. $\blacksquare$

Notice that the proposition does not claim that the probability of success is no smaller than $1 \Leftrightarrow V(x)$; this would in addition require that D is reached in finite time, P^x-almost surely. Propositions 63 and 64 may be combined to yield such a result. A related question is what the expected time to complete the mission is, conditioned on the mission being completed successfully. This is the subject of appendix A (page 151 ff.) where a new formula for this conditional expectation is derived.

6.3 Performance of disturbed systems

In this section we consider a disturbed stochastic system

$$\Sigma : \quad dx_t = f(x_t, v_t)\, dt + g(x_t, v_t)\, dB_t \ , \quad y_t = c(x_t, v_t) \qquad (6.2)$$

where v_t is the disturbance input and y_t is an output which is used in evaluation of the performance of the system. As before, B_t is Brownian motion w.r.t. a filtration $\mathcal{F}_t$. The input v_t is restricted to a set $\mathcal{V}$ of $\mathcal{F}_t$-adapted signals for which there exists a unique t-continuous solution to the

dynamic equation. We assume that $\mathcal{V}$ is sufficiently large and closed unde switching so that the principle of optimality holds.

Following standard notation [83], we define the backward operator assoc ated with the controlled diffusion (6.2):

$$L^v V = V_x f + \frac{1}{2}\mathrm{tr}(g' V_{xx} g) \quad .$$

6.3.1 Stochastic $\mathcal{L}_2$ gain

The $\mathcal{L}_2$ gain is one way of measuring the amplification, or gain, of a deter ministic or stochastic system, and is a good performance measure when w adopt a worst-case view on the inputs and wish to bound their effect o the r.m.s. value of the output. A reasonable question is what makes th $\mathcal{L}_2$ norm (or the r.m.s. value) suitable as a signal norm. Here we adop the pragmatic point of view that in many applications it is not at all clea what signal norm is suitable, and that in these situations it may be mos fruitful to use the $\mathcal{L}_2$ norm since it leads to technical simplicity.

Definition 65: The stochastic $\mathcal{L}_2$ gain of the system (6.2) is denote $||\Sigma||_\infty$ and equals the infimum of all $\gamma > 0$ such that the system is stochas tically dissipative with respect to $\gamma^2|v|^2 \Leftrightarrow |y|^2$.

Thus, we have $||\Sigma||_\infty < \gamma$ if and only if

$$E^x \int_0^\tau |y_t|^2\, dt \leq \gamma^2 E^x \int_0^\tau |v_t|^2\, dt + K(x)$$

holds for some $K : \mathbb{X} \to \mathbb{R}$ and all bounded stopping times τ and all input v_t in $\mathcal{V}$. In this case K must be nonnegative and may be taken to hav infimum 0.

Our choice of notation suggests that $||\Sigma||_\infty$ is a norm. Indeed, it is possibl to organize systems of the form (6.2) in a linear space: Fix the probabilit space $(\Omega, \mathcal{F}, P)$, the filtration $\mathcal{F}_t$ and the input space $\mathcal{V}$. We then view th system as a family of operators from input v_t to output z_t, parametrize by the initial condition x, and define addition and scalar multiplicatio of systems in the obvious way. Then the stochastic $\mathcal{L}_2$ gain $||\Sigma||_\infty$ is semi-norm on the subspace of those systems for which it is finite.

The stochastic $\mathcal{L}_2$ gain is the one property of stochastic dissipation whic has received considerable attention in the literature [30, 31].

6.3.2 $\mathcal{H}_2$ performance

Whereas it is largely agreed that the $\mathcal{L}_2$ gain is a suitable generalization of the $\mathcal{H}_\infty$ norm to nonlinear systems, it is less clear how to define the $\mathcal{H}_2$ norm of a nonlinear system defined by the stochastic differential equation (6.2). Here we suggest a new definition which is based on stochastic $\mathcal{L}_2$ gains and therefore fits into our framework of stochastic dissipation.

Since the $\mathcal{H}_2$ norm of a linear time invariant system concerns the response to a white noise input v_t, we need to modify the model (6.2) to allow for such inputs. Since we have restricted ourselves to Itô diffusions, which only allow a white noise term $d\tilde{B}/dt$ to enter *affinely* in the dynamic equation, we must assume that (6.2) is affine in v_t. Furthermore, recall [128] that a stable rational transfer function has finite $\mathcal{H}_2$ norm if and only if it is strictly causal, so we can assume that the output equation $y = c(x, v)$ is independent of v. Hence we assume that the system (6.2) has the following special form:

$$\Sigma : \quad dx_t = f(x_t)\ dt + g(x_t)\ dB_t + b(x_t) v_t\ dt, \quad y_t = c(x_t) \quad . \tag{6.3}$$

In order to define $\mathcal{H}_2$ performance of such a system Σ, we formally replace the input v_t with a white noise term $\nu_t\ d\tilde{B}/dt$. Here ν_t is a scalar *noise intensity* while $\tilde{B}_t$ is standard Brownian motion with respect to $\mathcal{F}_t$ and independent of B_t. Thus we obtain a new system, mapping the noise intensity ν_t to the output y_t:

$$\tilde{\Sigma} : \quad dx_t = f(x_t)\ dt + g(x_t)\ dB_t + b(x_t)\ \nu_t\ d\tilde{B}_t, \quad y_t = c(x_t) \tag{6.4}$$

Definition 66: The *strong $\mathcal{H}_2$ performance index* of the system (6.3) is denoted $||\Sigma||_2$ and equals the stochastic $\mathcal{L}_2$ gain of the system (6.4). □

The strong $\mathcal{H}_2$ performance index is the worst-case ratio between the variance of the output y_t and the intensity of the white noise input $v_t = \nu_t\ d\tilde{B}_t/dt$. The affix *strong* is due to the feature that the intensity of the white noise input is allowed to vary, for instance as a function of the state.

Implicit in the definition is that the filtration $\mathcal{F}_t$ must be 'large enough' to allow two independent $\mathcal{F}_t$-Brownian motion processes B_t and $\tilde{B}_t$. This mathematical twist will probably cause little concern in engineering applications where we start with statistical properties of noise signals and then, usually implicitly, define the probability space accordingly.

As was the case for stochastic $\mathcal{L}_2$ gains, it is possible to organize system Σ of the form (6.3) in a linear space such that $||\Sigma||_2$ is a seminorm on th subspace where it is finite.

We have the following partial differential inequality condition for $\mathcal{H}_2$ per formance:

Proposition 67: For the system Σ defined by equation (6.3), let ther exist a real number $\gamma \geq 0$ and a C^2 function $V \geq 0$ on $\mathbb{X}$ such that

$$\forall x \in \mathbb{X} : \quad V_x f + \frac{1}{2}\mathrm{tr}(g' V_{xx} g) + \frac{1}{2}|c|^2 \leq 0 \ , \quad \gamma^2 \geq \mathrm{tr}(b' V_{xx} b) \quad .$$

Then $||\Sigma||_2 \leq \gamma$.

Proof: We claim that V is a storage function for the system (6.4) with re spect to the supply rate $\frac{1}{2}\gamma^2\nu^2 - \frac{1}{2}y^2$. The differential dissipation inequalit is

$$V_x f + \frac{1}{2}\mathrm{tr}(g' V_{xx} g) + \frac{1}{2}\nu^2\ \mathrm{tr}(b' V_{xx} b) \leq \frac{1}{2}\gamma^2\nu^2 - \frac{1}{2}|c|^2 \quad ,$$

which is seen to hold for all x and all ν if V and γ are as in the propositio

The condition is only sufficient since storage functions need not in genera be C^2. Notice that the characterization is convex in γ^2 and V. In th narrow sense linear case, i.e.

$$f(x) = Ax \ , \quad g(x) = 0 \ , \quad b(x) = B \ , \quad c(x) = Cx \quad ,$$

we know from chapter 5 that we can restrict attention to quadratic storag functions, i.e. $V(x) = \frac{1}{2}x'Px$, and we recover the Lyapunov-type linea matrix inequality problem

$$P \geq 0 \ , \quad PA + A'P + C'C \leq 0 \ , \quad \gamma^2 \geq \mathrm{tr}(B'PB) \quad .$$

Feasibility of this problem is sufficient *and* necessary for $||\Sigma||_2 \leq \gamma$ sinc linear dissipative systems possess a quadratic storage function. In othe words, the strong $\mathcal{H}_2$ performance index equals the standard $\mathcal{H}_2$ norm o the transfer function $C(sI - A)^{-1}B$.

It is well known [128] that for linear systems the $\mathcal{H}_2$ norm of a transfe function equals the steady-state variance of the output, when the input i white noise with unit intensity. This generalizes to nonlinear systems a follows:

Proposition 68: Let $\gamma = ||\Sigma||_2 < \infty$ and assume that the intensity ν_t in (6.4) is constant and equal to some number $\sigma > 0$, then

$$\limsup_{T\to\infty} \frac{1}{T} E^x \int_0^T |y_t|^2 \, dt \le \sigma^2 \cdot \gamma^2 \quad .$$

If furthermore a stationary solution x_t exists such that $E\ V(x_t) < \infty$, then

$$E\ |c(x_t)|^2 \le \sigma^2 \cdot \gamma^2 \quad .$$

$\triangle$

Proof: The assumption implies that the system $\tilde{\Sigma}$ is dissipative w.r.t. $\gamma^2 \nu^2 \Leftrightarrow |y|^2$. Let V be a storage function; then the dissipation inequality

$$0 \le E^x V(x_T) \le V(x) + E^x \int_0^T \gamma^2 \cdot \sigma^2 \Leftrightarrow |y_t|^2 \, dt$$

holds. This can rewritten as

$$\frac{1}{T} \int_0^T E^x |y_t|^2 \, dt \le \frac{1}{T} V(x) + \gamma^2 \cdot \sigma^2 \quad ,$$

which holds for all $T \ge 0$. Now take lim sup on both sides and notice that $\limsup_{T\to\infty} V(x)/T = 0$. The second claim follows directly from the dissipation inequality

$$E\ V(x_T) \le E\ V(x_0) + E \int_0^T \sigma^2 \cdot \gamma^2 \Leftrightarrow |c(x_t)|^2 \, dt$$

combined with the stationarity property $E\ V(x_T) = E\ V(x_0)$. ■

For general nonlinear systems the bounds in the proposition may be somewhat conservative since we have restricted the noise intensity ν_t to be constant.

In the deterministic case $g = 0$ the condition of proposition 67 is the existence of a Lyapunov function V for the autonomous system $\dot{x} = f(x)$ such that

$$\frac{d}{dt} V(x(t)) \le \Leftrightarrow \frac{1}{2} |c(x(t))|^2$$

and which in addition has small curvature, i.e. $\mathrm{tr}(b' V_{xx} b) \le \gamma^2$. A classical question is to what extent "nice" input-output behaviour (e.g. finite gain)

implies "nice" internal behaviour (e.g. stability). In the deterministic cas $g = 0$ it is possible to employ La Salle's theorem [59, p. 115, p. 440]. Henc finite strong $\mathcal{H}_2$ performance index implies asymptotic stability of the zer solution to the undisturbed system $\dot{x} = f(x)$ if: 1) the autonomous syster $\dot{x} = f(x)$, $y = c(x)$ is zero-state detectable, and 2) the storage function in proposition 67 is proper and satisfies $V^{-1}(\{0\}) = \{0\}$.

A concluding remark concerns $\mathcal{H}_2$ performance of systems (6.2) which d not have the input-affine form (6.3). In this case one needs a more genera framework for stochastic differential equations than Itô diffusions, whic allows a stochastic integral to be a nonlinear function of the driving ma tingale. Such a framework can be found in [75] but is beyond the scope c this dissertation.

6.4 FSN models

In a sequence of papers [102, 104, 100, 110, 72, 71, 103], R.E. Skelton an co-workers have introduced disturbances with *finite signal-to-noise rat* (in short, FSN disturbances) and discussed their use for representation c uncertainty. In this section we demonstrate that FSN disturbances car too, be represented in the framework of stochastic dissipation.

FSN disturbances are white noise signals with intensities which are nc fixed *a priori* but grow with the variance of specified signals in the close loop. As argued in [102], this is a reasonable model of round-off errors i finite word-length computations with floating point, as well as of turbulenc forces around air foils.

To be more specific, consider the linear system

$$\dot{x} = Ax + Gw \ , \quad y = Cx \qquad (6.5$$

where w is an FSN disturbance: i.e., a scalar white noise signal with ir tensity $\sigma_0^2 + \sigma_1^2 E(y'y)$. Here σ_0 and σ_1 are specified constants; σ_1 is calle the *noise-to-signal ratio*. Also other terms such as controls may appea in the expressions for $\dot{x}$ and y but are irrelevant to the present discussior The model can be generalized to allow for vector disturbances w in severa ways.

The model (6.5) is well suited for steady-state analysis: A unique invariar distribution for x exists if and only if there exists a unique non-negativ solution P to the generalized Lyapunov equation

$$PA' + AP + GG'(\sigma_0^2 + \sigma_1^2 \cdot \mathrm{tr}(CPC')) = 0 \quad . \tag{6.6}$$

In this case, this unique invariant distribution for x is $N(0, P)$, i.e. in steady state x is zero mean, has covariance P and is Gaussian. However, the model (6.5) does not fully describe the process x. For instance, assume that we have observed x up to some time t, what is then the conditional distribution of w? Such questions are important if one wishes to study transient behaviour.

The objective of this section is twofold: First, we wish to generalize FSN models to nonlinear and non-stationary systems. Second, we wish to formulate FSN models such that they can be combined with our dissipation-based framework for robustness. We believe that the following model fulfills both objectives:

$$dx_t = f(x_t)\ dt + g(x_t)\left(\sigma_0\ \ dB_t + \sigma_1 \zeta_t\ d\tilde{B}_t\right) \ , \quad y_t = c(x_t) \quad . \tag{6.7}$$

Here ζ_t is the scalar output of an unknown deterministic system Δ which has $\mathcal{L}_2$ gain less than or equal to 1, and the input of which is y_t. Furthermore B_t and $\tilde{B}_t$ are independent standard Brownian motion.

In order to see that this is indeed a generalization of the model (6.5), set $f(x) = Ax$, $g(x) = G$, $c(x) = Cx$, and assume that steady-state has been reached. Assume furthermore that Δ is a *worst-case* perturbation so that the root mean square (r.m.s.) of ζ_t equals that of y_t. Then it is fairly easy to see that x in steady state must have zero mean and variance P where P solves the generalized Lyapunov equation (6.6), which implies that the models (6.5) and (6.7) lead to the same steady-state mean and variance. However, our suggested model (6.7) need not lead to Gaussian distributions in steady state - this will depend on the particular system Δ.

If one wishes to simulate an FSN system, one will obviously have to choose a particular perturbation Δ. Two systems with $\mathcal{L}_2$ gain equal to one are of special interest:

$$\Delta : \quad \zeta(t) = \quad y(t) \qquad \text{(for } y \text{ scalar)}$$

$$\Delta : \quad \zeta(t) = \quad \sqrt{\int_0^\infty \omega \exp(\Leftrightarrow\omega\tau)|y(t \Leftrightarrow \tau)|^2\ d\tau}$$

The first of these two specific perturbations Δ is that of *multiplicative nois* c.f. e.g. [31] and the references therein. In certain analysis problems fo linear FSN systems this perturbation is *worst case*. The second form of illustrates that ζ may be thought of as an r.m.s. estimator for y.

When we analyse FSN models, we take the the perturbation Δ to be a unknown state-space system with $\mathcal{L}_2$ gain less than or equal to one, an we adopt a worst-case view on this class of perturbations. In particular ap plications, one may possess additional knowledge regarding Δ, for instanc concerning time constants.

We have thus demonstrated that FSN models can be embedded in ou general framework for uncertain systems; viz. a nominal system describe by a stochastic differential equation, subject to an unknown perturbatio which possesses a number of specified dissipation properties.

6.5 Performance of perturbed systems

In this section we consider the interconnection of a nominal stochastic sy tem Σ and a multi-dissipative deterministic perturbation Δ; see figure 6. Our objective is to provide conditions on the nominal system Σ unde which the interconnection dissipates a given supply rate r for any mult dissipative perturbation Δ. This is a fairly general problem formulatio later we consider specific applications such as robust $\mathcal{H}_2$ performance i presence of $\mathcal{H}_\infty$ bounded perturbations.

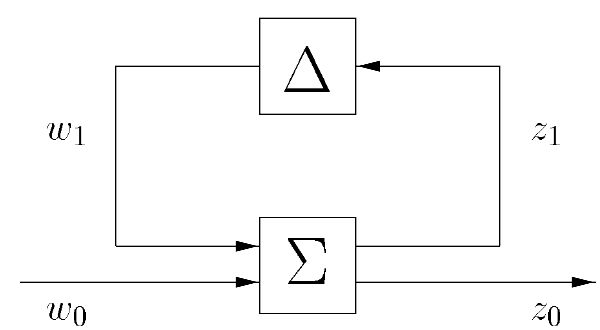

Figure 6.1: Setup for robust performance analysis.

The nominal system Σ is described by a stochastic differential equation

$$\Sigma : \begin{array}{rcl} dx_t &=& f(x_t, w_t, v_t)\ dt + g(x_t, w_t, v_t)\ dB_t\ , \\ y_t &=& c(x_t, w_t, v_t)\ , \\ z_t &=& h(x_t, w_t, v_t)\quad , \end{array} \tag{6.8}$$

whereas the unknown perturbation Δ, mapping z to w, is known to dissipate p given supply rates $-r_i$, $i = 1, \ldots, p$. We let ξ denote the state of the perturbation Δ and let $W(\xi, -r_i)$ denote a storage function for Δ w.r.t. $-r_i$. The backwards operator corresponding to (6.8) is

$$L^{w,v}V(x) = V_x f + \frac{1}{2}\mathrm{tr}(g' V_{xx} g)$$

for $V \in C^2(\mathbb{X}, \mathbb{R})$; here the right hand side is evaluated at x, w, v.

We omit details concerning well-posedness of the interconnection; i.e. we assume that unique t-continuous solutions x_t, ξ_t exist for any $\mathcal{F}_t$-adapted input v_t in a sufficiently large class of inputs.

The vehicle of our analysis of the interconnection (Σ, Δ) is an extended system derived from the nominal system Σ and independent of Δ: Define the system $\bar{\Sigma}$ by appending to (6.8) the dynamic equation

$$d\beta_t^i = -r_i\ dt \quad . \tag{6.9}$$

Thus $\bar{\Sigma}$ has states x_t and $\beta_t = (\beta_t^1, \ldots, \beta_t^p)$, inputs w_t and v_t, and outputs z_t and y_t. The backwards operator associated with $\bar{\Sigma}$ is

$$M^{w,v}U(x,\beta) = U_x f - \sum_{i=1}^{p} U_{\beta_i} r_i + \frac{1}{2}\mathrm{tr}(g' U_{xx} g)$$

for $U \in C^2(\mathbb{X} \times \mathbb{R}^p, \mathbb{R})$; here the right hand side is evaluated at x, β, w, v.

Definition 69: We say that $\bar{\Sigma}$ is *regionally dissipative* on $D \subset \mathbb{X} \times \mathbb{R}^p$ w.r.t. the supply rate r if there exists a function $U(x, \beta)$ which is non-negative on D and such that the dissipation inequality

$$E^{x,\beta} U(x_\tau, \beta_\tau) \le U(x, \beta) + E^{x,\beta} \int_0^\tau r\ dt \tag{6.10}$$

holds for all $(x, \beta) \in D$, all $\mathcal{F}_t$-adapted inputs v_t, w_t and all bounded stopping times τ such that

$$\tau \le \tau_D := \inf\{t \ge 0\ :\ (x_t, \beta_t) \notin D\} \tag{6.11}$$

holds.

Regionally dissipative systems are not directly covered by our definition o stochastic dissipation (page 105); nevertheless it is straightforward to verif that they possess many properties similar to those of dissipative system Let us only state the partial differential inequality condition:

Proposition 70: Let $D \subset \mathbb{X} \times \mathbb{R}_+^p$ be open and let $U \in C^2(D, \bar{\mathbb{R}}_+)$. Th following are equivalent:

1. U is a regional storage function for $\bar{\Sigma}$ on D w.r.t. the supply rate r
2. U satisfies the partial differential inequality

$$\sup_{w,v} M^{w,v} U(x, \beta) - r \le 0 \tag{6.12}$$

on D.

Proof: The proof is merely a repetition of the proof of proposition 51 o page 106 and omitted.

We can now state our main result which is a sufficient condition for th interconnection (Σ, Δ) to dissipate r.

Theorem 71: Assume that $\bar{\Sigma}$ is regionally dissipative on $\mathbb{X} \times \mathbb{R}_+^p$ w.r. r with $U(x, \beta)$ a corresponding regional storage function. Then the inter connection (Σ, Δ) dissipates r; an upper bound on the available storag is

$$U(x, \beta)$$

provided that $\beta^i > W(\xi, -r_i)$.

The idea behind the theorem is that the appended states β_t^i of $\bar{\Sigma}$ boun the storage $W(\xi_t, -r_i)$ in the perturbation. This technique has, to ou knowledge, not been used before in the literature; even in a deterministi context.

Proof: Consider the response x_t, ξ_t of the interconnection (Σ, Δ) t an $\mathcal{F}_t$-adapted input v_t under the initial conditions x and ξ. Let β^i $W(\xi, -r_i)$. We aim to show that

$$E^{x,\xi} \int_0^\tau -r\ dt \le U(x, \beta) \tag{6.13}$$

holds for any bounded stopping time τ.

First, notice that the output w_t of Δ is $\mathcal{F}_t$-adapted since Δ is deterministic. Let $\bar{x}_t$, β_t be the response of $\bar{\Sigma}$ to the inputs v_t and w_t and the initial conditions x and β. Then clearly $x_t = \bar{x}_t$ by uniqueness; the processes x_t and $\bar{x}_t$ solve the same stochastic differential equation (6.8) with the same initial condition.

Next, the dissipation inequalities for Δ are

$$0 \leq W(\xi_t, -r_i) \leq \int_0^t -r_i \; ds + W(\xi, -r_i)$$

and hold for *any* sample trajectory and any $t \geq 0$. This implies that

$$0 \leq \delta_t^i \leq \beta_t^i - \beta^i + W(\xi, -r_i) < \beta_t^i \quad .$$

Finally, let τ be a bounded stopping time. Since $\beta_t^i > 0$ for any $t \geq 0$, the regional dissipativity of $\bar{\Sigma}$ implies that

$$E^{x,\beta} \int_0^\tau -r \; dt \leq U(x,\beta) - E^{x,\beta} U(x_\tau, \beta_\tau) \leq U(x,\beta)$$

which completes the proof. ■

Linear combinations of supply rates

Theorem 71 generalizes the conditions of chapter 3 where we required the nominal system Σ to dissipate a linear combination of the supply rates r, r_i. We may recover this type of results (in a stochastic context) by imposing a specific structure on U:

Corollary 72: Assume that there exists non-negative weights d_i such that Σ dissipates the supply rate

$$r + \sum_{i=0}^{p} d_i r_i$$

then the interconnection (Σ, Δ) dissipates r. □

Proof: In the theorem, take $U(x,\beta) = V(x) + \sum_i d_i \beta_i$ where V is a storage function of Σ w.r.t. the supply rate $r + \sum_{i=0}^{p} d_i r_i$. Let w_t, v_t be

$\mathcal{F}_t$-adapted inputs to Σ and $\bar{\Sigma}$ and let τ be bounded; we then have

$$\begin{aligned} E^{x,\beta} U(x_\tau, \beta_\tau) &= E^x V(x_\tau) + E^{x,\beta} \sum_{i=1}^{p} d_i \beta_\tau^i \\ &\leq V(x) + E^x \int_0^\tau r + \sum_{i=1}^{p} d_i r_i \; dt \\ &\quad + E^{x,\beta} \sum_{i=1}^{p} d_i (\beta^i - \int_0^\tau r_i \; dt) \\ &= E^x \int_0^\tau r \; dt + U(x,\beta) \end{aligned}$$

which implies that the sufficient condition of theorem 71 is satisfied. No
tice that $U(x, W(\xi, -r_i))$ is in this case in fact a storage function for th
interconnection (Σ, Δ).

Conservatism of the condition

Since theorem 71 provides a sufficient condition, but not a necessary one
the question is how conservative the condition is. Before we discuss th
issue we emphasize that the condition is less conservative than those o
chapter 3; this is demonstrated by corollary 72. In fact the condition o
theorem 71 is not very conservative.

First, the theorem does not only guarantee that the interconnection $(\Sigma, \Delta$
dissipates r but also that there exists a bound on the available storage whic
depends only on $W(\xi, -r_i)$, and not on the actual perturbation Δ and it
initial condition ξ. This may be conservative if all we care about is tha
the interconnection is dissipative. On the other hand, in most application
it does not suffice to know that a bound exists for the available storage o
(Σ, Δ); we also want to know what this bound is. Since the initial storage i
Δ may very well be the one quantity we can bound reliably, it is appealin
that this is exactly what we need to bound the available storage of $(\Sigma, \Delta$

Another way conservatism is introduced in the theorem is that the diss
pation inequality (6.10) holds for all $\mathcal{F}_t$-adapted inputs w_t. Notice that
deterministic perturbation Δ must necessarily produce an output w_t whic
is adapted to the sub-filtration generated by z_t. In other words, the theorem
is conservative in that the bound (6.13) holds also for perturbations whic

have access to complete information about the system Σ. This conservatism may even be desirable in applications where Δ is physically integrated in the total control system; for instance if Δ represents parasitic high-frequency dynamics. Then it would be hazardous to let a design depend on Δ not exchanging information with its environment.

A similar discussion concerns the situation where the perturbation Δ is composed of a large number of independent blocks in parallel, i.e. $w^i = \Delta^i z^i$. It appears to be difficult to make use of the fact that multiple perturbations really must solve decentralized control problems in order to make the dissipation inequality fail. In short, we restrict the energy and other resources available to Δ; not the information.

Refining the storage bounds $\beta^i_t > W(\xi_t, \Leftrightarrow r_i)$

The idea in theorem 71 is that we keep track of how much storage is present in the perturbation Δ through the bounds

$$\beta^i_t > W(\xi_t, \Leftrightarrow r_i) \quad .$$

The dynamic equation $d\beta^i_t = \Leftrightarrow r_i \ dt$ simply states that if we supply a quantity to Δ, then the storage in Δ may increase with this quantity but no more.

In some applications it may be essential to incorporate additional knowledge about Δ such as time constants. For instance, consider a welding robot which first moves the arm into correct position with large and fast movements after which the welding process begins and the welding seam is to be followed slowly and accurately. The perturbation Δ is parasitic high-frequency dynamics in the robot arm; the storage in Δ is mechanical energy. During the initial rough placement of the robot arm it is likely that large amounts of energy is supplied to the perturbation. It is then important for the analysis that this energy cannot be hidden in Δ and then released much later, during the fine movements of the actual welding process. In such a situation one may replace the dynamic equation for β^i_t with

$$d\beta^i_t = (\Leftrightarrow \frac{1}{T_\Delta}\beta^i_t \Leftrightarrow r_i) \ dt$$

where T_Δ is the time constant of the perturbation. Of course, also other forms of decay can be used, for instance if physical reasoning gives bounds

to the storage which Δ is capable to keep. In general, these issues are im portant if some phases of the system operation are more critical or sensitiv than others.

The idea of bounding the storage in the perturbation has applications fa beyond the robustness analysis which we concentrate on here. For instanc a supervisory system may keep track on-line of the storage in the perturba tion using the dynamic equation of β^i_t as well as on-line measurements fro the system. A large storage may provoke an alarm, or pause the contr mission until the storage in the perturbation decreases to an acceptab level. For the welding robot above, this means to stop welding until w are confident that parasitic oscillations in the arm have died out. On th other hand, if the bound β^i_t ever goes negative then it can be conclude that the model is inconsistent with the measurements which may trigg a change of control strategy. The reference [88] describes an approach t adaptive $\mathcal{H}_\infty$ control based on a finite number of models and this type o model validation.

6.5.1 Guaranteed $\mathcal{H}_2$ performance

Consider now the block diagram in figure 6.2 where the system Σ has input w_t, σ_t and v_t and is given by the model

$$\Sigma: \quad dx_t = f(x_t, w_t)\ dt + \sigma_t\ g(x_t)\ dB_t + v_t\ b(x_t)\ dt \qquad (6.14$$

with outputs $y_t = c(x_t)$, $\zeta_t = \eta(x_t)$, and $z_t = h(x_t)$. We make the followin assumptions about the perturbations Δ and Δ_F:

Δ is passive and small $\mathcal{L}_2$-gain, i.e. dissipative w.r.t. $\Leftrightarrow r_1 = \langle w, z\rangle$ an $\Leftrightarrow r_2 = |z|^2 \Leftrightarrow |w|^2$. This could for instance represent unmodelled parasit dynamics.

Δ_F is small $\mathcal{L}_2$-gain, i.e. dissipative w.r.t. $\Leftrightarrow r_3 = |\zeta|^2 \Leftrightarrow |\sigma|^2$. This impli that $\sigma_t\ dB_t/dt$ is a white noise signal which grows in intensity with th variance of ζ_t, i.e. a *finite signal-to-noise ratio disturbance*.

To evaluate the strong $\mathcal{H}_2$ performance index of the total system, we follo our definition 66 and replace the input v_t in (6.14) with a white noise ter $\nu_t\ dW_t/dt$, thus obtaining

$$\tilde{\Sigma}: \quad dx_t = f(x_t, w_t)\ dt + \sigma_t\ g(x_t)\ dB_t + \nu_t\ b(x_t)\ dW_t \quad . \qquad (6.15$$

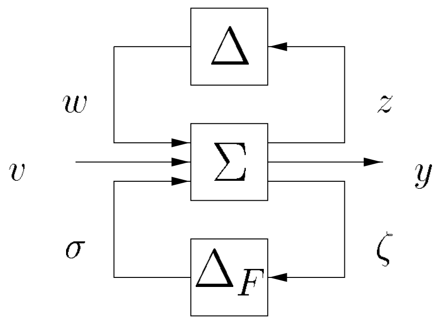

Figure 6.2: Nominal system and perturbations

Assume now that $\tilde{\Sigma}$ is stochastically dissipative w.r.t. $\gamma^2|\nu|^2 \Leftrightarrow |y|^2 + \sum_{i=1}^{3} d_i r_i$ for some $\gamma \geq 0$, $d_i \geq 0$ and that V is a corresponding storage function, then it follows from corollary 72 that the overall interconnection is dissipative w.r.t. $\gamma^2|\nu|^2 \Leftrightarrow |y|^2$; a storage function is $V + \sum_i d_i W_i$. Hence, an upper bound on the square of the strong $\mathcal{H}_2$ performance index is

$$\min_{\gamma^2, d_i, V} \gamma^2 \ \text{ s.t. } \ V \text{ a storage function for (6.15) w.r.t. } \gamma^2|\nu|^2 \Leftrightarrow |y|^2 + \sum_{i=1}^{3} d_i r_i$$

where $\gamma \geq 0$ and $d_i \geq 0$. This infinite-dimensional optimization problem is convex according to proposition 53; if the state x has low dimension it may be solved be restricting V to a finite-dimensional subspace as outlined on page 117.

If the right hand side of the governing equation (6.14) is linear in (x, w, v, σ) then V can be taken to be quadratic and the optimization problem reduces to a linear matrix inequality problem:

Theorem 73: Let the system Σ be given by the linear SDE

$$\Sigma \ : \ dx_t = (Ax_t + \Phi w_t + Bv_t)\ dt + \sigma_t\ G\ dB_t$$

and the output equations $z_t = Hx_t$, $y_t = Cx_t$, $\zeta_t = Jx_t$, and let $w = \Delta z$ and $\nu = \Delta\zeta$ where Δ and Δ_F are as above. Then an upper bound on the square of the strong $\mathcal{H}_2$ performance index of the interconnection is

$$\min_{P, d_1, d_2} \text{tr } B'PB \ \text{ s.t. } \ P \geq 0,\ d_i \geq 0,\ \begin{bmatrix} Y & P\Phi + d_1 H' \\ \Phi' P + d_1 H & \Leftrightarrow d_2 I \end{bmatrix} \leq 0$$

where Y is shorthand for $Y = PA + A'P + C'C + d_2 H'H + J'J$ tr $G'PC$

Proof: The proof is merely a verification that, given feasible P and d_1, d the quadratic form $V(x) = x'Px$ is a storage function of system

$$dx_t = (Ax_t + \Phi w_t)\ dt + \nu_t\ B\ dW_t + \sigma_t\ G\ dB_t$$

with respect to the supply rate $\gamma^2|\nu|^2 \Leftrightarrow |y|^2 + \sum_{i=1}^{3} d_i r_i$ with $\gamma^2 =$ tr $B'P$ and $d_3 =$ tr $G'PG$.

This upper bound can be computed with standard software for linear matri inequalities such as [38, 32]. Notice that if one removes the FSN disturbanc $\sigma_t\ g(x_t)\ dB_t/dt$ in (6.14) and applies the condition in theorem 27 on page 6 for robust $\mathcal{H}_2$ performance in the deterministic sense, then one recovers th condition of theorem 73. On other words, if one is after sufficient condition for robust $\mathcal{H}_2$ performance of linear systems, then it is inessential if on uses the stochastic or the deterministic interpretation of $\mathcal{H}_2$ performance

6.5.2 Robust estimates on the risk of failure

Consider a system

$$\Sigma \ : \ dx_t = f(x_t, w_t)\ dt + g(x_t, w_t)\ dB_t\ , \quad z_t = h(x_t) \qquad (6.16$$

connected in feedback with a deterministic perturbation $\Delta : z \to w$ whic dissipates the p supply rates $\Leftrightarrow r_1, \ldots, \Leftrightarrow r_p$. Let the initial condition x be i an open domain $D \subset \mathbb{X}$, let the boundary ∂D be divided into two disjoi sets A and B; corresponding to success and failure, respectively.

As before, we let $\bar{\Sigma}$ denote the system Σ appended with the states β_t^i wit $d\beta_t^i = \Leftrightarrow r_i\ dt$.

Theorem 74: Assume that $\bar{\Sigma}$ is regionally dissipative on $D \times \mathbb{R}_+^p$ w.r.t. th supply rate 0 with a regional storage function $U(x, \beta)$ which is continuou on $\bar{D} \times \bar{\mathbb{R}}_+^p$ and such that $U(x, \beta) \geq 1$ whenever $x \in B$ and $\beta^i \geq 0$. The we have the following bound on the risk of failure

$$P^{x,\xi}\{x_{\tau_D} \in B\} \leq U(x, \beta)$$

where $\beta^i = W(\xi, \Leftrightarrow r_i)$.

Proof: Let $\beta^i > W(\xi, \Leftrightarrow r_i)$ and let x_t, β_t be the trajectories of $\bar{\Sigma}$ when connected in feed-back with Δ, corresponding to the initial conditions x, ξ and β. We claim that the process $U(x_{t\wedge\tau_D}, \beta_{t\wedge\tau_D})$ is a continuous supermartingale. Continuity is clear since x_t and β_t are continuous processes and U is a continuous function. To see that the process is a supermartingale, notice that $\beta^i_t > W(\xi_t, \Leftrightarrow r_i) \geq 0$, and hence regional dissipativity w.r.t. the supply rate 0 yields

$$E^{x,\xi} U(x_{t\wedge\tau_D}, \beta_{t\wedge\tau_D}) \leq U(x, \beta) \quad .$$

This allows us to pose the probability bound

$$P^{x,\xi}\{x_{\tau_D} \in B\} \leq P^{x,\beta}\{\sup_{0\leq t} U(x_{t\wedge\tau_D}, \beta_{t\wedge\tau_D}) \geq 1\} \leq U(x, \beta) \quad .$$

Here, the first inequality holds because $x_{\tau_D} \in B$ implies that $U(x_{\tau_D}, \beta_{\tau_D}) \geq 1$ and hence $\sup_{0\leq t} U(x_{t\wedge\tau_D}, \beta_{t\wedge\tau_D}) \geq 1$. The second inequality is the supermartingale inequality.

We have thus shown that $P^{x,\xi}\{x_{\tau_D} \in B\} \leq U(x, \beta)$ for any β such that $\beta^i > W(\xi, \Leftrightarrow r_i)$. Now let $\beta^i \to W(\xi, \Leftrightarrow r_i)$ from above and use continuity of U to see that the same bound holds with $\beta^i = W(\xi, \Leftrightarrow r_i)$. ∎

A similar conclusion is obtained if we follow corollary 72 and replace the hypothesis with Σ dissipating $\sum_i d_i r_i$ for non-negative weights d_i, with a continuous storage function V such that $V|_B \geq 1$. However, in this case the resulting bound is

$$P^{x,\xi}\{x_{\tau_D} \in B\} \leq V(x) + \sum_i d_i W(\xi, \Leftrightarrow r_i)$$

which is seen to be quite conservative for large amounts of initial storage in the perturbation Δ; in fact the upper bound may then become $P^{x,\xi}\{x_{\tau_D} \in B\} \leq 1$ which is not very informative. In this situation theorem 71 is of much more use; at least for large amounts of initial storage in the perturbation. In other words, it may well be very conservative to consider only regional storage functions $U(x, \beta)$ which are affine in β.

6.6 Conclusion

This chapter has demonstrated that problems of robust performance of stochastic systems can be addressed with the notion of stochastic dissipation. The three steps in this procedure are:

1. Model the physical system as an interconnection of a nominal syster Σ and a perturbation Δ, where Δ is dissipative w.r.t. the suppl rates $\Leftrightarrow r_i$, $i = 1, \ldots, p$.

2. Formulate the performance property as one of stochastic dissipatior i.e. find the supply rate r such that the overall system has satisfactor performance iff it dissipates r.

3. Perform dissipation analysis on Σ using theorem 71 or corollary 7 i.e. investigate if $\bar{\Sigma}$ dissipates r regionally, or if Σ dissipates $r + \sum_i d_i$ for non-negative weights d_i.

Regarding the first item, the dissipation properties of Δ will typically b the same as in a deterministic analysis, such as passivity or small gai We have also demonstrated that Skelton's finite signal-to-nose ratio (FSN models can be incorporated in this framework.

Regarding the second item, we have shown that stochastic $\mathcal{L}_2$ gain, $\mathcal{H}$ performance, risk of failure and expected time to complete a mission a examples of performance objectives which can be stated in terms of stocha tic dissipation. While it is hardly surprising that the stochastic $\mathcal{L}_2$ gain related to dissipation, it is an innovation that $\mathcal{H}_2$ performance is expresse in this framework. We believe that nonlinear $\mathcal{H}_2$ control, both nomin and robust, is a fruitful field of future research. The two last performan measures, risk of failure and expected time to complete a mission, are we studied in the classical literature on stochastic analysis and control, but is a novelty that they can be embedded in the framework of dissipation an thus subjected to a robustness analysis.

Regarding the last item, the idea of searching through convex conic com binations of supply rates was also employed (in a deterministic context) i chapter 3 and in the recent reference [126], but it is a new observation th this idea is a special case of regional dissipation analysis of the extend system $\bar{\Sigma}$; i.e. that corollary 72 follows from theorem 71.

The practical applicability of our suggested framework depends on tw factors: First, we need numerical methods for performing (regional) diss pation analysis on general nonlinear systems - here it would be interestin to develop the LMI based procedure suggested on page 117 and apply it t some benchmark problems. Second, recognizing that these numerical met ods will not be applicable to systems with high-dimensional state spac due to the curse of dimensionality, we need analytical procedures for simplifying the dissipation analysis using information about the structure of the system. Modularity is one such procedure; time-scale separation would be another interesting issue to investigate.

6.7 Notes and references

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems

The literature contains several different statements of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ analysis and control problems, [26, 60, 68, 69, 86, 98, 106, 129, 132]. Much of this work concerns posing an $\mathcal{H}_2$ bound on one closed loop transfer function and an $\mathcal{H}_\infty$ bound on another. Problems of robust $\mathcal{H}_2$ performance of a linear system in presence of one $\mathcal{H}_\infty$ bounded perturbation are treated in [106, 86]. The setting there is much alike the one used in section 6.5.1; however the object of analysis in these references is a family of Riccati equations rather than a linear matrix inequality. The parameter in this family corresponds to our weight d_1. The final numerical strategy is then to search over this weight, solving a Riccati equation for each d_1. This approach is difficult with more than one dissipation property of the perturbation, since it is not clear how the solution of the Riccati equation depends on the d-weights. We have in [113] presented a numerical example with two dissipation properties; for this example a convexity property makes numerical optimization over the d-weights feasible.

Stability of FSN systems

The simplest FSN model, according to our suggested definition, is

$$dx_t = f(x_t)\ dt + g(x_t)\zeta_t\ dB_t\ , \quad y_t = c(x_t)$$

where $\zeta_t = \Delta y_t$; here Δ is a deterministic system with $\mathcal{L}_2$ gain less than or equal to one. This corresponds to (6.7) where the signal-to-noise ratio σ_1 is 1, and $\sigma_0 = 0$. A sufficient condition for this system to be stable is that the stochastic $\mathcal{L}_2$ gain from ζ_t to y_t is less than one; this is equivalent to the system mapping v_t to y_t given by

$$dx_t = f(x_t)\ dt + g(x_t)v_t\ dt\ , \quad y_t = c(x_t)$$

having strong $\mathcal{H}_2$ performance index less than 1. This is a *small gain* typ
result for nonlinear FSN systems.

Earlier joint work with R.E. Skelton [110], for linear FSN systems, cor
cluded that this condition was sufficient *and* necessary. Furthermore, fo
the situation with *several* FSN disturbances, a necessary and sufficien
condition was given in terms of the spectral radius of a certain matrix, th
elements of which were obtained by $\mathcal{H}_2$ analysis on the nominal system. I
is in fact possible to give a similar *sufficient* condition for stability of nor
linear FSN systems with several FSN disturbances, employing corollary 7
This result will be reported elsewhere.

Chapter 7

Conclusion

We have in this dissertation contributed to the mathematical theory of ro bust performance of control systems in presence of parametric uncertaint dynamic perturbations, and deterministic or stochastic exogenous distur bances.

There are four threads in our work. The first is the opinion that contro theory should employ notions which have some general validity and no only, for instance, make sense in a deterministic linear setting. We believ that our dissipation based framework for robust performance of stochasti systems fulfills this requirement.

The second thread is the opinion that control theory should maintain close connection to physics. This is partly because many techniques fron physics, such as Lyapunov stability, has proven to be valuable to contro theorists, but also because a sound knowledge of the physics in a contro system will assist the control engineer in posing the right mathematica problems.

Thirdly, we consider the uncertainty associated with a nominal mathema ical model to be equally important as the nominal model itself. The rep resentation of uncertainty determines the strategy for analysis and design and the more detailed the information about the uncertainty, the sharpe conclusions. Both the simultaneous $\mathcal{H}_\infty$ controller of chapter 4 and th robust performance analysis of chapter 6 uses explicit quantitative evalu ation of the uncertainty, in terms of the residuals and the storage of the perturbations.

Lastly, we believe that tools for analysis of control systems are as important as tools for synthesis. Good analysis tools, which for instance could be based on dissipation analysis on a closed loop system, can be of great practical value, not only for the theorist but also for the practicing engineer. For instance an inspection of storage functions may conclude that a heuristic controller, although not optimal, solves the control job nicely, or it may identify a weakness in the design of the *plant*.

In the remainder of this chapter we briefly summarize the precise nature of our contributions, and point out a number of issues which deserve further attention.

7.1 Summary of contributions

The purpose of this section is to provide a concentrated overview of the results which were obtained during the Ph.D. study and reported in this thesis.

The introductory chapter 1 does not present new results, although the observation that LMIs can be used to compute storage functions for nonlinear but input affine-quadratic systems seems to be new.

Chapter 2 presents fundamental properties of deterministic systems which are dissipative w.r.t. several supply rates. The convex conicity of the set of dissipated supply rates is mentioned in passing in [45]; this simple property is what enables the robustness analysis of the succeeding chapter. New results are that the set is also closed and that the available storage is a continuous function on this set. These properties are important for a numerical analysis and contribute to the general understanding of multi-dissipative systems.

The contribution of chapter 3 is to demonstrate that analysis of control systems can be done by explicit consideration of the multiple dissipation properties of unknown system components. It is fair to say that this idea is also present in approach of Integral Quadratic Constraints, but several differences exist between this framework and the one of multi-dissipation as explained in section 3.1. The chapter also contains several more technical

contributions which can be seen as exercises in Lyapunov techniques - i
this type of work, the devil is in the details. The results for linear-quadrati
systems are obtained using standard methods for linear matrix inequalitie
The importance of these results is to demonstrate that problems with suc
mixed uncertainty models lead to convex optimization problems, namel
LMIs.

Chapter 4 contributes to the theory of adaptive $\mathcal{H}_\infty$ control by pointin
out that certainty equivalence based minimax controllers for this proble
is not the generic situation. Although the characterization of the minima
controller is done with existing ideas, viz. the information state machiner
the literature contains few applications of this machinery, and the detai
are by no means trivial. One such detail is the characterization of the valu
function as the viscosity solution to the HJI-PDE. In a given applicatio
it will be a cumbersome affair to construct the minimax controller, bu
it is quite straightforward to synthesize the heuristic certainty equivalenc
controller, and this design may have direct practical applicability.

Chapter 5 contains a generalization of dissipation theory to stochastic sy
tems. In the existing literature, dissipation techniques have only been use
to perform analysis of stochastic systems in special cases; it appears to b
a new observation that the framework is applicable and operational in ge
eral. The results of the chapter essentially say that many of the attractiv
features of deterministic dissipative systems apply to stochastic dissipativ
systems as well; these are the inherent convexity, the rôle of the availab
storage, the closedness under interconnections, and the implications f
stability. The strictly positive real lemma for wide-sense linear stochast
systems is new; passivity of stochastic systems has to our knowledge n
been investigated previously.

Chapter 6 constructs a framework for robustness of stochastic system
based on the theory of stochastic dissipation. A minor contribution is th
observation that stochastic performance measures such as the risk of failu
can be formulated in terms of dissipation. It is more innovative that th
same applies to $\mathcal{H}_2$ performance and finite signal-to-noise ratio (FSN) mo
els. The idea of expanding the system with extra states, which keep tra
of the storage in the perturbation, is new. This idea leads to quite shar
sufficient conditions for robust performance; for general nonlinear system
these conditions are more natural than the multiplier-based approach
chapter 3. The idea may also have further applicability in other fields
control theory such as supervision and model validation.

7.2 Perspectives and future works

As is so often the case, each of the answers in this dissertation leads to several new questions. Many of the results could be refined or generalized; the *notes and references* ending each chapter contains such detailed suggestions for future works. At this point we take a step back and outline some fields of research which we believe to be fertile.

The problem of adaptive $\mathcal{H}_\infty$ control remains largely open. As stated in chapter 4, we cannot expect the minimax controller to be based on certainty equivalence or finite dimensional (when there is more than a finite number of possible parameter values). In this situation there is a great need for clever heuristics and sub-optimal strategies as well as for studies of special situations, and although much work has been done in this direction, there are many questions that remain unaddressed. A fundamental question is if the problem formulation itself is a sign of prudence or paranoia. In other words, should we impose some further constraints on those disturbances for which the dissipation inequality must hold, or is it reasonable to anticipate disturbances which in some clever way attempt to confuse the control system?

We have, in the notes at the end of chapter 5, mentioned the possibility of extended the theory of stochastic dissipation to a more general class of stochastic differential equations than Itô diffusions. A related interesting project would be to extend the theory of stochastic dissipation to infinite-dimensional systems, i.e. systems given by stochastic partial differential equations. Initial results in this direction are probably obtained quite easily, following [124] where many results hold for infinite dimensional systems, but we expect it to be quite complicated to obtain more explicit results. A good starting point for such a project would be the corresponding deterministic problem, see [61] and the references therein.

As we have already mentioned on several occasions, numerical methods for analysis and control of nonlinear systems remains the hurdle for the practical applicability of the theory, and is a natural subject of future investigations.

After the robustness analysis results of chapter 6, an obvious next step is to develop a theory of control for stochastic dissipation. The objective of such a theory is to provide techniques for finding a control law, a storage function, and possibly also a supply rate in a given set, which together satisfy

the dissipation inequality. In principle, this can be done by value-polic
iteration but we expect that much more explicit results can be obtainec
at least if some generality is sacrificed.

A special case of such a theory is *nonlinear $\mathcal{H}_2$ control* building on th
definition of strong $\mathcal{H}_2$ performance index of chapter 6. The term nonlinea
$\mathcal{H}_2$ control is most often used in the deterministic meaning, where th
cost is evaluated from the response to initial conditions, and is therefor
unable to conclude on the response to white noise. Similarly, stochasti
nonlinear optimal control is most often used with *fixed* noise intensities, an
does therefore not provide information about the response to other nois
intensities. In some applications it is quite sensible to take a worst-cas
view on the noise intensity (as in our definition of strong $\mathcal{H}_2$ performance
It also embeds nicely in a dissipation-based robustness framework - notic
that fixing the noise intensity (or just bounding it way from zero) lead
to supply rates which are not regular and thus weakens the dissipatio
theory. In short, we believe dissipation-based nonlinear $\mathcal{H}_2$ control to be
promising field.

Appendix A

Conditional Expectations of First Passage Times

We consider an Itô diffusion evolving on a domain in Euclidean space, the boundary of which is divided into two components, A and B. We then ask the question: What is the expected time to pass before the set A is reached, conditioned on A being reached before B?

We derive a partial differential equation which governs this conditionally expected first passage time, seen as a function of the initial state. We also provide a generalization which involves other functionals than the first time of exit, and we show how a partial differential inequality can be useful for establishing bounds.

A classical question concerning Itô diffusions evolving in Euclidean spac is: If the diffusion starts at a point x in some open set Ω, what is th expected time $E^x \tau_{\partial\Omega}$ to pass before it reaches the boundary $\partial\Omega$? It i well known that under suitable technical assumptions this expected firs passage time, seen as a function of the initial state x, is the unique solutio to the second order semi-elliptic partial differential equation

$$L\phi = -1, \quad \phi|_{\partial\Omega} = 0$$

Here, L is the backward differential operator associated with the diffusion - see below for precise definitions and statements.

A related question is: If we divide the boundary $\partial\Omega$ into two disjoint components A and $B = \partial\Omega \setminus A$, what is the probability $P^x\{\tau_{\partial\Omega} = \tau_A\}$ that the process hits A before B? This probability is - again, under suitable technical assumptions - the unique solution to the equation

$$L\psi = 0, \quad \psi|_A = 1, \quad \psi|_B = 0 \quad .$$

One application of these results is performance analysis of a stochastic control system: The control mission is completed upon passage of the boundary $\partial\Omega$; successfully if the boundary is reached at a point in A whereas reaching B before A would be a failure. For instance, the mission could be docking of a ship or a spacecraft. The primary performance measure for this application may be the probability of success, i.e. the function ψ, whereas a secondary performance measure may be the time it takes to complete the mission, averaged only over those missions which are completed successfully. In other words, the question arises: If we condition that A is reached before B, what is then the expected time to reach A?

Although this question seems almost as basic as the two previous ones, we have not been able to find it answered explicitly in the literature. In this note we show that - still, under suitable technical assumption - this conditional expectation of the first passage time can be computed as

$$E^x\{\tau_A \mid \tau_A = \tau_{\partial\Omega}\} = \frac{\kappa(x)}{\psi(x)}$$

where ψ is the probability that A is reached before B, as above, and where κ is the unique solution to the equation

$$L\kappa = \Leftrightarrow\psi, \quad \kappa|_{\partial\Omega} = 0$$

This is our main result which is stated precisely and proved in section A.1 below. In section A.2 we state a rather straightforward generalization where a reward is released upon first passage; making this reward equal to the time of first passage recovers the result of section A.1. In section A.3 we show how one may obtain upper bounds if given solutions to the corresponding partial differential *inequalities*. This is especially useful in situations where the partial differential *equations* have no (classical) solutions which will be the case in many applications.

Notation

Our notation is fairly standard and follows [83]. In particular, the diffusion
we consider in this note are Itô diffusions evolving in Euclidean space $\mathbb{X}$ =
$\mathbb{R}^n$ according to the stochastic differential equation

$$dx_t = f(x_t)\ dt + g(x_t)\ dB_t \qquad \text{(A.1}$$

which we interpret in the Itô sense. Of course, we assume an underlyin
filtered probability space which we however do not refer to explicitly.

We define the (backward) differential operator L associated with the diffu
sion x in the usual way: If $V : \mathbb{X} \to \mathbb{R}$ is C^2, then

$$LV(x) = V_x f + \frac{1}{2}\mathrm{tr}(g' V_{xx} g)$$

where the right hand side is evaluated at x.

If $D \subset \mathbb{X}$ is Borel then we use τ_D to denote the stopping time inf$\{t$
$0 :\ x_t \in D\}$. P^x is the probability law of x_t starting at $x_0 = x$ and E
denotes expectation w.r.t. P^x.

For a set A, $\bar{A}$ denotes the closure of A.

If A is an event such that $P^x A > 0$ and y is a stochastic variable for whic
$E^x|y| < \infty$, then $E^x\{y \mid A\}$ denotes the conditional expectation $E^x\{y \mid \mathcal{A}$
evaluated at some $\omega \in A$; here $\mathcal{A}$ denotes the σ-algebra generated by A.

A.1 The main result

We make the following assumptions on the geometry and the dynamics:

Assumption 75:

i The initial condition x of the stochastic differential equation is i
a domain $\Omega \subset \mathbb{X}$ which is open and bounded and has a smoot
boundary $\partial\Omega$.

ii The drift coefficient f and diffusion coefficient g are Lipschitz contin
uous on the closure $\bar{\Omega}$ of the domain.

iii The diffusion g satisfies the non-degeneracy condition that $gg' > 0$ on $\bar{\Omega}$.

iv The boundary $\partial\Omega$ is divided into two disjoint components A and B which have no common limit points, i.e. $A \cup B = \partial\Omega$ and $\bar{A} \cap \bar{B} = \emptyset$.

□

These assumptions are standard and natural from a classical point of view: The Lipschitz continuity and the boundedness of Ω assure that there exists a unique solution of the stochastic differential equation at least up to the first time the boundary $\partial\Omega$ is reached, see [83]. The non-degeneracy condition on g ensures that the first passage time $\tau_{\partial\Omega}$ is finite w.p. 1 and has finite expectation. It also implies that L is uniformly elliptic which gives us existence and uniqueness of solutions in the classical sense to the partial differential equations we consider. The condition that $\bar{A}$ and $\bar{B}$ are disjoint implies that the probability $\psi(x) = P^x\{\tau_A = \tau_{\partial\Omega}\}$ is Lipschitz continuous on $\partial\Omega$ and hence C^2 on Ω.

Later, we relax some of the assumptions somewhat.

Our main result is the following:

Theorem 76: For the diffusion (A.1) under assumption 75, we have the following formula for the conditional expectation of the first passage time

$$E^x\{\tau_A \mid \tau_A = \tau_{\partial\Omega}\} = \frac{\kappa(x)}{\psi(x)}$$

for any point $x \in \Omega$ such that $\psi(x) > 0$. Here $\psi(x)$ equals $P^x\{\tau_A = \tau_{\partial\Omega}\}$ and is the unique solution to the equation

$$L\psi = 0, \quad \psi|_A = 1, \quad \psi|_B = 0 \tag{A.2}$$

while $\kappa : \mathbb{X} \to \mathbb{R}$ is the unique solution to the equation

$$L\kappa = \Leftrightarrow\psi, \quad \kappa|_{\partial\Omega} = 0 \quad . \tag{A.3}$$

□

Proof: It is well known [107, chpt. 3] that the assumptions imply that $\psi(x) = P^x\{\tau_A = \tau_{\partial\Omega}\}$ is C^2 and the unique solution to (A.2); a compact exposition of the necessary results can be found in [40, sec. 3.5]. This in

turn implies that κ is well defined as the unique solution to (A.3). Further-more, $\tau_{\partial\Omega}$ is finite w.p. 1 and has finite expectation which implies that th conditional expectation is well defined [83, p. 239].

For $s \in \mathbb{R}$, define the process $s_t = s + t$. Let $y_t = (x_t, s_t)$; then y_t is th unique (up to $\tau_{\partial\Omega}$) solution to the stochastic differential equation

$$dy_t = \begin{pmatrix} f(y_t)\ dt + g(y_t)\ dB_t \\ 1\ dt \end{pmatrix} \quad .$$

We stop the process y_t when it hits $\partial\Omega \times \mathbb{R}$ (i.e, at $t = \tau_{\partial\Omega}$) and define th reward function for $y = (x, s)$

$$\lambda(y) = s \cdot \chi_A(x) = \begin{cases} s & \text{if } x \in A \\ 0 & \text{else.} \end{cases}$$

Define the expected reward

$$\nu(y) = E^y \lambda\left(y(\tau_{\partial\Omega})\right)$$

and let $y = (x, s)$ with $x \in \Omega$, then

$$\begin{aligned} \nu(y) &= E^y\left\{\lambda\left(y(\tau_{\partial\Omega})\right) \mid \tau_A = \tau_{\partial\Omega}\right\} \cdot P^y\{\tau_A = \tau_{\partial\Omega}\} \\ &+ E^y\left\{\lambda\left(y(\tau_{\partial\Omega})\right) \mid \tau_B = \tau_{\partial\Omega}\right\} \cdot P^y\{\tau_B = \tau_{\partial\Omega}\} \\ &= E^x\{s + \tau_A \mid \tau_A = \tau_{\partial\Omega}\} \cdot P^x\{\tau_A = \tau_{\partial\Omega}\} \\ &= \left(s + E^x\{\tau_A \mid \tau_A = \tau_{\partial\Omega}\}\right) \cdot \psi(x) \end{aligned}$$

Define the (backward) differential operator M associated with the diffusio y in the usual way: If $W : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ is $C^{2,1}$ then

$$MW(y) = W_x f + W_t + \frac{1}{2}\mathrm{tr}(g' W_{xx} g)$$

where the right hand side is evaluated at $y = (x, s)$. Then $M\nu = 0$ o $\Omega \times \mathbb{R}$. Furthermore, $\nu(x, s) = s$ for $x \in A$ and $\nu(x, s) = 0$ for $x \in B$.

We claim that $\nu(x, s) = \kappa(x) + s \cdot \psi(x)$. To see this notice that $\nu_t(x, s) =$ $\psi(x)$. Together with $M\nu = 0$ and the boundary conditions this implie that $\nu(x, 0) = \kappa(x)$ on $\bar{\Omega}$ from which the conclusion follows.

Combining the above expressions yields

$$E^x\{\tau_A \mid \tau_A = \tau_{\partial\Omega}\} \cdot \psi(x) = \nu(x, s) \Leftrightarrow s \cdot \psi(x) = \kappa(x)$$

which completes the proof. ■

Example 77: [Brownian motion] Consider the case of scalar Brownian motion, i.e. $\mathbb{X} = \mathbb{R}$ and $f = 0$, $g = 1$. Let Ω be the open interval $(0, 1)$ and let $A = \{1\}$, $B = \{0\}$. Then $\psi(x) = x$ and $\kappa(x) = \frac{1}{3}x(1 \Leftrightarrow x^2)$, i.e.

$$E^x\{\tau_A \mid \tau_{\partial\Omega} = \tau_A\} = \frac{1}{3}(1 \Leftrightarrow x^2) \text{ for } 0 < x \leq 1 \quad . \tag{A.4}$$

For comparison we have the unconditional expectation

$$E^x \tau_{\partial\Omega} = x \Leftrightarrow x^2 \text{ for } 0 \leq x \leq 1 \quad .$$

Notice that the conditional expectation and the unconditional expectation coincide for $x = 1/2$ as symmetry predicts.

Figure A.1 shows numerical results which are obtained in the following way: For each initial condition in $\{0.05,\ 0.10,\ \ldots\ , 0.95\}$, we perform a number of simulations until we obtain 100 simulations which exit to the right. Simulations are done with a sample time of $\Delta t = 0.0001$. For these 100 simulations we compute and plot the average first exit time (marked with $\times$ in the figure). The sample means are slightly larger than the conditional expectation as computed by the expression (A.4) (the solid line in the figure). The difference decreases with the sample time Δt (although the plot shows results for only one sample time). This is to be expected: When we only observe the diffusion at discrete points of time we only get an upper bound on the first exit time, and sample paths starting near $x = 0$ are prone to misclassification. □

A.2 A generalization

A way to generalize the result from the previous section is to see that the first passage time is a functional on the set of trajectories and then consider more general functionals. The functionals we consider in this section consist of two components: A cumulative term, i.e. an integral along the trajectory, and a terminal term depending on where the the trajectory hits the boundary. More specifically, we obtain a formula for

$$E^x \left\{ k(x(\tau_{\partial\Omega})) + \int_0^{\tau_{\partial\Omega}} l(x_t)\ dt \;\middle|\; \tau_A = \tau_{\partial\Omega} \right\} \quad .$$

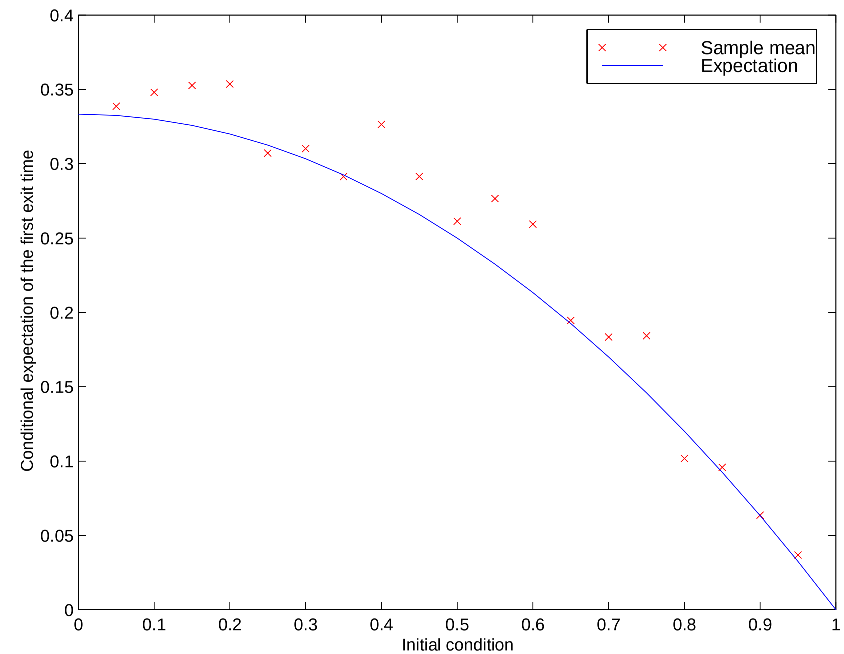

Figure A.1: Numerical results obtained for scalar Brownian motion

Assumption 78:

i: The function $l : \bar{\Omega} \to \mathbb{R}$ is Lipschitz continuous.

ii: The function $k : \partial\Omega \to \mathbb{R}$ is Lipschitz continuous.

□

Theorem 79: For the diffusion (A.1) with the reward functions k, l under assumptions 75, 78, we have the following formula

$$E^x \left\{ k(x(\tau_{\partial\Omega})) + \int_0^{\tau_{\partial\Omega}} l(x_t)\ dt \;\middle|\; \tau_A = \tau_{\partial\Omega} \right\} = \frac{\kappa(x)}{\psi(x)}$$

for any point $x \in \bar{\Omega}$ such that $\psi(x) > 0$. Here $\psi(x)$ is as before and κ is the unique solution to the partial differential equation

$$L\kappa = -l \cdot \psi, \quad \kappa|_A = k, \quad \kappa|_B = 0 \quad .$$

□

Proof: As in the proof for the previous theorem existence and uniqueness of a solution κ to the partial differential equation is guaranteed; notice that the boundary condition $k \cdot \chi_A$ is Lipschitz continuous since k is and since $\bar{A}$ and $\bar{B}$ are disjoint. Also, the conditional expectation is well defined.

Define $y_t = (x_t, z_t)$ where z_t solves the stochastic differential equation

$$dz_t = l(x_t)\ dt \quad .$$

Existence and uniqueness of a solution to this equation is guaranteed since l is Lipschitz continuous. Let $z_0 = z$ be the corresponding initial condition and define the reward

$$\lambda(x, z) = (k(x) + z)\chi_A(x)$$

and the expected reward

$$\nu(x, z) = E^{x,z}\lambda(x(\tau_{\partial\Omega}), z(\tau_{\partial\Omega}))$$

Again, we define the backward differential operator M associated with $y = (x, z)$ in the usual way: If $W : \mathbb{X} \times \mathbb{Z} \to \mathbb{R}$ is $C^{2,1}$, then

$$MW(y) = W_x f + W_z l + \frac{1}{2}\mathrm{tr}(g' W_{xx} g)$$

where the right hand side is evaluated at $y = (x, z)$. Then $M\nu = 0$ on and $\nu = \lambda$ on $\partial\Omega$. Following the proof of theorem 76, we see that $\nu_z =$ on Ω and hence that $\nu(x, z) = \kappa(x) + z \cdot \psi(x)$. Finally we notice that

$$\nu(x, z) = E^{x,z}\{\lambda(x(\tau_{\partial\Omega}), z(\tau_{\partial\Omega})) \mid \tau_{\partial\Omega} = \tau_A\} \cdot \psi(x)$$

which completes the proof.

A.3 An upper bound under weak assump tions

A weakness of the previous results is that the assumptions are rather re strictive. In particular, we would like to allow for non-smooth boundarie degenerate diffusion coefficients and situations where $\bar{A}$ and $\bar{B}$ are not dis joint (although A and B are). This means that we must obtain the desire results without having guaranteed existence and uniqueness of solutions t the involved partial differential equations. For instance, if $\bar{A}$ and $\bar{B}$ are no disjoint then ψ cannot be continuous on $\bar{\Omega}$. This motivates us to establis results which guarantees *bounds* through partial differential inequalities.

In this section we use $\bar{x}_t$ to denote the process x_t stopped at $\partial\Omega$, i. $\bar{x}(t) = x(t \wedge \tau_{\partial\Omega})$.

Assumption 80:

i : The domain Ω is open and bounded.

ii : The drift coefficient f and the diffusion coefficient g are Lipschit continuous on $\bar{\Omega}$.

iii : The boundary $\partial\Omega$ is reached in finite time, almost surely, and fur thermore $E^x\tau_{\partial\Omega} < \infty$ for all $x \in \Omega$.

iv : The boundary $\partial\Omega$ of the domain is divided into two disjoint Bor sets A and B, i.e. $A \cup B = \partial\Omega$ and $A \cap B = \emptyset$.

v : $k : \partial\Omega \to \mathbb{R}$ and $l : \bar{\Omega} \to \mathbb{R}$ are Lipschitz continuous and non-negativ

The assumption that $E^x\tau_{\partial\Omega} < \infty$ is not always immediate; in these situations one can use the sufficient condition that there exists a C^2 function $\phi : \bar{\Omega} \to \mathbb{R}$ such that $L\phi < 0$ on $\bar{\Omega}$.

We start off with an elementary lemma; many similar statements can be found in the literature.

Lemma 81: Let assumption 80 hold and let $\bar{\psi} : \bar{\Omega} \to \mathbb{R}$ be C^2 and satisfy

$$L\bar{\psi} \leq 0, \quad \bar{\psi} \geq 0, \quad \bar{\psi}|_A \geq 1 \quad .$$

Then the bound

$$\psi(x) \leq \bar{\psi}(x)$$

holds. Conversely, let $\underline{\psi} : \bar{\Omega} \to \mathbb{R}$ be C^2 and satisfy

$$L\underline{\psi} \geq 0, \quad \underline{\psi} \leq 1, \quad \underline{\psi}|_B \leq 0 \quad .$$

Then the bound

$$\psi(x) \geq \underline{\psi}(x)$$

holds. □

Proof: The assumptions imply that $\bar{\psi}(\bar{x}_t)$ is an almost surely continuous non-negative supermartingale. We then have the inequalities

$$P^x\{\tau_A = \tau_{\partial\Omega}\} \leq P^x\{\sup_{t\geq 0} \bar{\psi}(\bar{x}_t) \geq 1\} \leq \bar{\psi}(x)$$

using Doob's martingale inequality, see e.g. [83, p. 28]. The converse statement follows similarly after noting that $1 \Leftrightarrow \underline{\psi}(\bar{x}_t)$ is a non-negative supermartingale; here we must use that the process exits Ω in finite time, almost surely. ■

Theorem 82: Let assumptions 80 hold and let $\bar{\kappa}$ be a non-negative C^2 function $\bar{\Omega} \to \mathbb{R}$ which satisfies

$$L\bar{\kappa} \leq \Leftrightarrow l \cdot \bar{\psi}, \quad \bar{\kappa}|_{\partial\Omega} \geq k \cdot \chi_A$$

where $\bar{\psi}$ is as in lemma 81. Let $\underline{\psi}$ satisfy $\underline{\psi} \leq \psi$ on $\bar{\Omega}$. Then the bound

$$E^x\left\{k(x(\tau_{\partial\Omega})) + \int_0^{\tau_{\partial\Omega}} l(x_t)\ dt \;\middle|\; \tau_A = \tau_{\partial\Omega}\right\} \leq \frac{\bar{\kappa}(x)}{\underline{\psi}(x)}$$

holds at any point $x \in \bar{\Omega}$ for which $\underline{\psi} > 0$. □

Proof: As in the previous existence of the conditional expectation i
guaranteed. Let z_t be the unique solution to the stochastic differenti
equation

$$dz_t = l(x_t)\ dt$$

with initial condition $z_0 = z \in \mathbb{R}$. We let $\bar{z}(t)$ denote the stopped proces
$\bar{z}_t = z(t \wedge \tau_{\partial\Omega})$.

Define

$$\bar{\nu}(x,z) = \bar{\kappa}(x) + z \cdot \bar{\psi}(x)$$

Then we have

$$M\bar{\nu} = L\bar{\kappa} + z \cdot L\bar{\psi} + l \cdot \bar{\psi} \leq 0$$

for any x, z with $x \in \Omega$ and $z \geq 0$. Notice that if the initial condition z i
non-negative, then so is $\bar{z}_t$ for $t \geq 0$ since $l \geq 0$. This implies that $\bar{\nu}(\bar{x}_t, \bar{z}_t$
is a non-negative supermartingale with continuous sample paths, almo
surely, which in turn implies that the inequality

$$E^{x,z}\{\bar{\nu}(\bar{x}(\tau_{\partial\Omega}), \bar{z}(\tau_{\partial\Omega})) \mid \tau_A = \tau_{\partial\Omega}\} \cdot P^x\{\tau_A = \tau_{\partial\Omega}\} \leq \bar{\nu}(x,z)$$

holds. Manipulating the left hand side we obtain

$$\begin{aligned}
& E^{x,z}\{\bar{\nu}(\bar{x}(\tau_{\partial\Omega}), \bar{z}(\tau_{\partial\Omega})) \mid \tau_A = \tau_{\partial\Omega}\} \\
\geq\ & E^{x,z}\{z(\tau_{\partial\Omega}) + k(x(\tau_{\partial\Omega})) \mid \tau_A = \tau_{\partial\Omega}\} \\
=\ & z + E^{x,0}\{z(\tau_{\partial\Omega}) + k(x(\tau_{\partial\Omega})) \mid \tau_A = \tau_{\partial\Omega}\} \\
=\ & z + E^x\{\int_0^{\tau_{\partial\Omega}} l(x_s)\ ds + k(x(\tau_{\partial\Omega})) \mid \tau_A = \tau_{\partial\Omega}\} \quad .
\end{aligned}$$

We have thus shown that

$$E^x\left\{\int_0^{\tau_{\partial\Omega}} l(x_s)\ ds + k(x(\tau_{\partial\Omega})) \;\middle|\; \tau_A = \tau_{\partial\Omega}\right\} \cdot \psi(x) \leq \bar{\nu}(x,0) = \bar{\kappa}(x)$$

holds. The result follows.

A.4 Numerical issues

Under the assumptions 75 and 78 there exists smooth solutions to the in
volved partial differential equations and standard methods for their solutio
can be employed.

Example 83: [Two-dimensional Brownian motion] Consider the case $n = 2$, $dx_t = dB_t$ and let the domain Ω be

$$\Omega = \{x \in \mathbb{R}^2 \mid \|x\|_\infty < 1 \ \wedge \ \|x\|_\infty > \frac{1}{4}\}$$

Let $A \subset \partial\Omega$ be the outer boundary, i.e. $A = \{x \mid \|x\|_\infty = 1\}$. The operator L is then $\Delta/2$. Using a quadratic grid with a step length of 0.05, we have discretized the partial differential equations using a finite difference method. The solutions are seen in figures A.2 through A.4. □

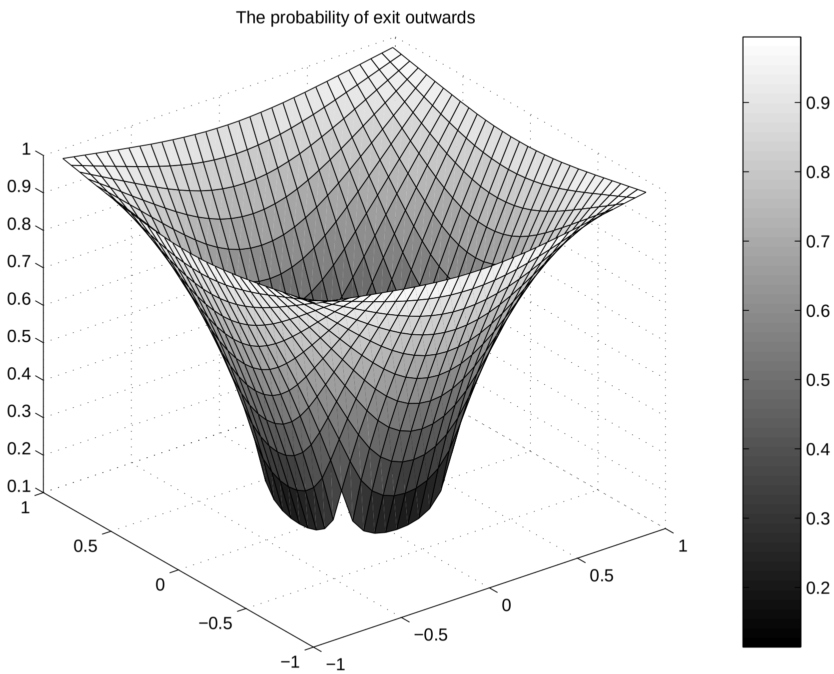

Figure A.2: 2-D Brownian motion: Probability of exit outwards

Under the weaker assumptions 80 one has to consider carefully if the partial differential equations have solutions in the classical sense. One option is to approximate the problem with one which satisfies the assumptions 75. For instance when $\bar{A} \cap \bar{B} \neq \emptyset$ one may choose to approximate χ_A with a function

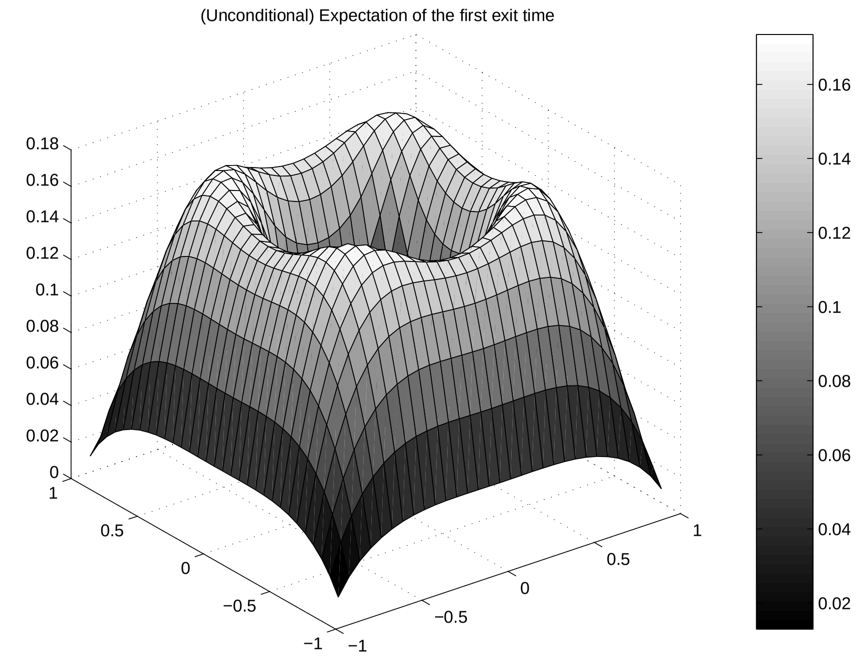

Figure A.3: 2-D Brownian motion: Unconditional expectation of the firs exit time

which is Lipschitz continuous on $\partial\Omega$. The weak maximum principle, se e.g. [91, p. 106], is useful for establishing relations between approximate solutions obtained in this fashion.

An alternative is to search for solutions to the partial differential inequa ities of section A.3 in some finite dimensional subspace, for instance spu by trigonometric functions or polynomials. If one only requires that th inequalities are satisfied at some finite set of points in $\bar{\Omega}$, then the probler of finding the best bounding functions $\bar{\psi}$, $\underline{\psi}$ and $\bar{\kappa}$ becomes one of linea programming.

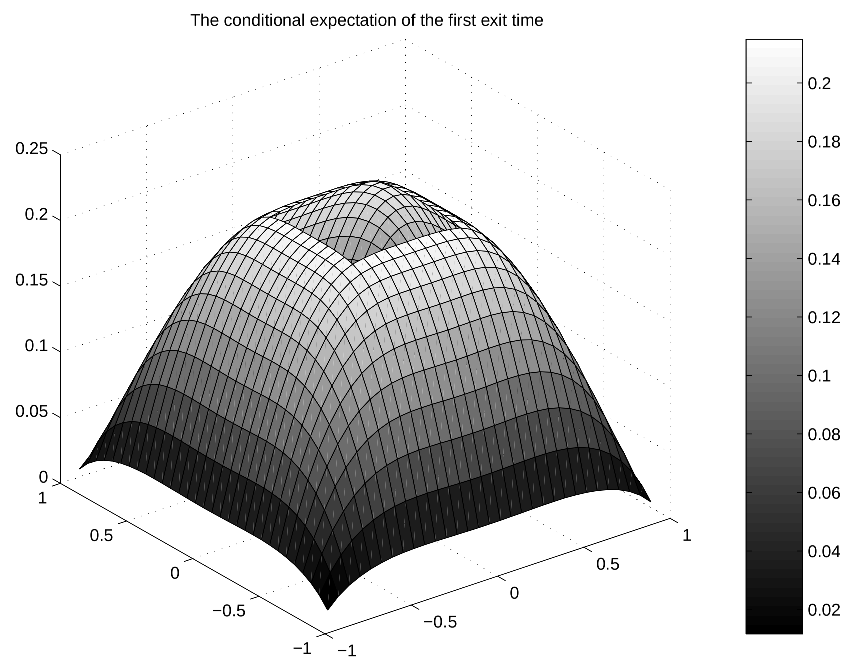

Figure A.4: 2-D Brownian motion: Conditional expectation of the first exit time

A.5 Summary

For solutions to a stochastic differential equation starting is some bounded domain, we have derived a formula for the conditional expectation of the time of first exit from the domain. The conditioning is with respect to the event that a given part of the boundary is reached first. The formula requires the solution of two elliptic partial differential equations. We have also provided a generalization to other functionals than the first time of exit, and we have established bounds which are expressed in terms of partial differential inequalities.

We have concentrated on classical (i.e, C^2) solutions to the involved partial differential equations and inequalities as well as classical conditions for existence and uniqueness of solutions to the equations. Similar results can

be obtained under weaker hypothesis if one employs the notion of viscosit solutions and uses the results of [84]. This is a topic of current research the results will appear in [112].

Appendix B

Various technicalities

This appendix contains various proofs and calculations which are not es
sential for the understanding of the results in this thesis.

B.1 Proof of theorem 25 on page 60

Due to the condition (3.11) we know that there exists parameters $\lambda_j(t) \geq ($
$j = 1, \ldots, m$, such that

$$[A(t), B(t), C(t), D(t)] = \sum_{j=1}^{m} \lambda_j(t)[A_j, B_j, C_j, D_j], \qquad \sum_{j=1}^{m} \lambda_j(t) = 1$$

We omit the time argument after signals and use the notation

$$z_j = C_j x + D_j w$$

Our candidate storage function for Σ is $x'Px$. We then get

$$\frac{d}{dt} x'Px = (x' \quad w') \begin{bmatrix} PA(t) + A'(t)P & PB(t) \\ B'(t)P & 0 \end{bmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

$$= (x' \quad w') \left(\sum_{j=1}^{m} \lambda_j \begin{bmatrix} PA_j + A_j'P & PB_j \\ B_j'P & 0 \end{bmatrix} \right) \begin{pmatrix} x \\ w \end{pmatrix}$$

$$\leq \sum_{j=1}^{m} \lambda_j (x' \;\; w') \left(\sum_{i=1}^{p} d_i \begin{bmatrix} 0 & C_j' \\ I & D_j' \end{bmatrix} Q_i \begin{bmatrix} 0 & I \\ C_j & D_j \end{bmatrix} \Leftrightarrow \epsilon \begin{bmatrix} C_j' \\ D_j' \end{bmatrix} [C_j \;\; D_j] \right) \begin{pmatrix} x \\ w \end{pmatrix}$$

$$= \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{p} d_i (w' \;\; z_j') Q_i \begin{pmatrix} w \\ z_j \end{pmatrix} \Leftrightarrow \epsilon z_j' z_j \right)$$

$$= \sum_{i=1}^{p} d_i \left(\sum_{j=1}^{m} \lambda_j (w' \;\; z_j') Q_i \begin{pmatrix} w \\ z_j \end{pmatrix} \right) \Leftrightarrow \epsilon \sum_{j=1}^{m} \lambda_j z_j' z_j$$

$$\leq \sum_{i=1}^{p} d_i \left((w' \;\; z') Q_i \begin{pmatrix} w \\ z \end{pmatrix} \right) \Leftrightarrow \epsilon z' z$$

We have thus show that the time-invariant function $x'Px$ is a strong storage function for the time-varying system Σ w.r.t. the supply rate $\sum_i d_i s_i$ and hence we may conclude robust stability of the interconnection (Σ, Δ).

B.2 The filter ODE for the conditional state estimate

In this appendix we derive the filter ODEs (4.13) and (4.14) for the conditional worst case state estimate $\xi(i,t)$ and the associated loss $S(\xi(i,t),i,t)$. The derivation follows the general procedure of [120].

The loss function $S(x,i,t)$ is quadratic in x. This means, that the characterization of the worst-case conditional state estimate $\xi(i,t)$ is

$$\frac{\partial}{\partial x} S(\xi(i),i,t) = 0 \tag{B.1}$$

and

$$\frac{\partial^2}{\partial x^2} S(\xi(i),i,t) > 0 \quad .$$

At this point we omit the x and t arguments and adopt the simplified notation S_x for $\frac{\partial}{\partial x} S(x,i,t)$ and so forth.

The cost-to-go and cost-to-come satisfy the PDEs [9, 120]:

$$P_x A_i x + \frac{1}{2} P_x \left(\frac{1}{\gamma^2} G_i G_i' \Leftrightarrow B_i B_i' \right) P_x' + \frac{1}{2} x' H_i' H_i x = 0 \tag{B.2}$$

and

$$R_t + R_x (A_i x + B_i u) + \frac{1}{2\gamma^2} R_x G_i G_i' R_x' \qquad \text{(B.3}$$

$$+ \frac{1}{2} x' H_i' H_i x \Leftrightarrow \frac{1}{2} \gamma^2 \|y \Leftrightarrow C_i x\|^2 + \frac{1}{2} \|u\|^2 = 0 \quad .$$

The PDE (B.2) reduces to the control algebraic Riccati equation (4.5) afte guessing P to be quadratic in x for fixed i. Likewise, the PDE (B.3) is re lated to the filter algebraic Riccati quation (4.12): Guess R to be quadrati in x for fixed i, assume stationarity in the sense $R_{xxt} = 0$ and consider onl second order terms in (B.3). For each i, define $Q_i := R_{xx}(x,i)$, then Q must satisfy the ARE (4.12).

Using $S = R \Leftrightarrow P$ we get by subtracting (B.2) from (B.3) and rearring term

$$S_t + \frac{1}{2} \|u + B_i' P_x'\|^2 \Leftrightarrow \frac{1}{2} \gamma^2 \|y \Leftrightarrow C_i x\|^2 + S_x (A_i x + B_i u) \qquad \text{(B.4}$$

$$+ \frac{1}{2\gamma^2} (S_x + 2P_x) G_i G_i' S_x' = 0 \quad .$$

This must in particular hold for $x = \xi(i,t)$. Using the stationarity cond tion (B.1) we then find the ODE for the conditional loss (4.14)

$$\frac{d}{dt} S(\xi(i,t),i,t) = \frac{1}{2} \gamma^2 \|y \Leftrightarrow C_i \xi(i,t)\|^2 \Leftrightarrow \frac{1}{2} \|u + B_i' P_x'\|^2 \quad .$$

If the parameter estimate $\hat\theta(t) = \arg\min_i S_i(t)$ is well defined, then we ma use the the certainty equivalence control $u(t) = \Leftrightarrow B'_{\hat\theta(t)} P_x'(\xi(\hat\theta(t),t), \hat\theta(t)$

to obtain that the *unconditional* worst-case loss (for $i = \hat\theta(t)$) satisfies

$$\frac{d}{dt} S(\xi(\hat\theta(t),t), \hat\theta(t), t) = \frac{1}{2} \gamma^2 \|y \Leftrightarrow C_{\hat\theta(t)} \xi(\hat\theta(t),t)\|^2$$

and hence is non-decreasing.

To obtain the observer equation (4.13), we again follow [120] and differen tiate the stationarity condition with respect to t to get

$$\frac{d}{dt} S_x(\xi(i,t),t) = 0 \;\Leftrightarrow\; \dot\xi(i,t) = \Leftrightarrow S_{xx}^{-1} S_{tx}' \big|_{x=\xi(i,t)} \quad .$$

The expression for S_{tx} is found by viewing (B.4) as a relation between t, u and y and differentiating with respect to x. Using the stationarit condition to eliminate the terms including S_x we get:

$$S_{tx} + (u + B_i' P_x')' (B_i' X_i) + \gamma^2 (y \Leftrightarrow C_i \xi)' C_i$$

$$+\left(\xi' A_i' + u' B_i' \frac{1}{\gamma^2} P_x G_i G_i\right)' S_{xx} = 0$$

which must hold for all t, u, y at $x = \xi(i,t)$. Combining, we obtain

$$\dot{\xi}(i,t) = A_i\xi + B_i u + \frac{1}{\gamma^2} G_i G_i' P_x' + S_{xx}^{-1}\gamma^2 C_i'(y \Leftrightarrow C_i\xi) + S_{xx}^{-1} X_i B_i (u + B_i' P_x') \quad .$$

This may also be written as (4.13):

$$\dot{\xi}(i,t) = A_i\xi \Leftrightarrow B_i B_i' P_x' + \frac{1}{\gamma^2} G_i G_i' P_x' + S_{xx}^{-1}\gamma^2 C_i'(y \Leftrightarrow C_i\xi) + S_{xx}^{-1} R_{xx} B_i (u + B_i' P_x')$$

Appendix C

Frequently used symbols and acronyms

Miscellaneous

A'	Complex conjugate transpose of matrix A
$\boldsymbol{\Delta}$	A set of dynamic state space systems (perturbations)
Σ	Nominal system
Δ	Perturbation
(Σ, Δ)	Perturbed system; interconnection of Σ and Δ
τ_D	Stopping time; first exit from domain D

Functions and operators

$\operatorname{Arg\,min}_x f(x)$	The set $\{x \mid f(x) = \inf_\xi f(\xi)\}$ where $f(x) \in \mathbb{R}$
$\arg\min_x f(x)$	The unique element of $\operatorname{Arg\,min}_x f(x)$
$o(\delta)$	A function for which $\|\|o(\delta)\|\|/\|\|\delta\|\| \to 0$ as $\|\|\delta\|\| \to 0$
V_x	Gradient of C^1 function V, $\partial V/\partial x$
V_{xx}	Hessian of C^2 function V
$V^{-1}(A)$	Preimage of A under V, i.e. $\{x \mid V(x) \in A\}$
$LV(x)$	Backwards operator of an autonomous diffusion
$L^uV(x)$	Backwards operator of a controlled diffusion

Sets and spaces

$[a,b]$, (a,b), $[a,b)$, $(a,b]$	Closed, open, and half-open real intervals
$\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$	Real, natural, integer numbers
$\mathbb{R}_+, \mathbb{R}_-$	Positive, negative real numbers
$\bar{A}$, $\bar{\mathbb{R}}_+, \bar{\mathbb{R}}_-$	Closure of sets
A^o, ∂A	Interior, boundary of set A
$\mathbf{Co}(A)$	Convex hull of a set A in a linear space
$\mathbb{X}$	State space, typically $\mathbb{R}^n$
$T\mathbb{X}, T^*\mathbb{X}$	Tangent and cotangent bundle of $\mathbb{X}$
$\mathcal{L}_2(\mathbb{X}, \mathbb{Y})$	Lebesque space of square integrable functions from $\mathbb{X}$ to $\mathbb{Y}$
$\mathcal{H}_\infty$	Hardy space of complex functions, analytical in the closed right half plane

Acronyms

CE	Certainty equivalence
FSN	Finite signal-to-noise ratio
HJ	Hamilton-Jacobi
HJB	Hamilton-Jacobi-Bellman
HJI	Hamilton-Jacobi-Isaacs
LMI	Linear matrix inequality
ODE	Ordinary differential equation
PDE	Partial differential equation
PDI	Partial differential inequality
SDE	Stochastic differential equation

Bibliography

[1] B.D.O. Anderson and J.B. Moore. *Linear Optimal Control*. Prentice Hall, Englewood Cliffs, N.J., 1971.

[2] B.D.O. Anderson and J.B. Moore. *Optimal Control: Linea Quadratic Methods*. Prentice-Hall, Englewood Cliffs, N.J., 1989.

[3] V.I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer Verlag, New York, second edition, 1988. Translated into English fro the Russian (1974) edition.

[4] K.J. Åström and B. Wittenmark. *Adaptive Control*. Addison-Wesle second edition, 1995.

[5] G.J. Balas, J.C. Doyle, K. Glover, A. Packard, and R. Smith. μ *Analysis and Synthesis Toolbox, User's Guide*. Musyn, Inc., Mi neapolis, 1991.

[6] J.A. Ball and J.W. Helton. $\mathcal{H}_\infty$ control for nonlinear plants: Co nections with differential games. In *Proceedings of the 28th IEE Conference on Decision and Control*, pages 956–962, 1989.

[7] J.A. Ball and J.W. Helton. Viscosity solutions of Hamilton-Jaco equations arising in nonlinear $\mathcal{H}_\infty$-control. *Journal of Mathemati Systems, Estimation, and Control*, 6(1):1–22, 1996.

[8] T. Başar. Minimax control of switching systems under sampling. I *Proceedings of the 33rd IEEE Conference on Decision and Contr* pages 716–721. IEEE, December 1994.

[9] T. Başar and P. Bernhard. *$\mathcal{H}_\infty$-Optimal Control and Related Minimax Design Problems*. Systems & Control: Foundations & Applications. Birkhäuser, Boston, 2nd edition, 1995.

[10] T. Başar and G.J. Olsder. *Dynamic Noncooperative Game Theory*, volume 160 of *Mathematics in Science and Engineering*. Academic Press, 1981.

[11] R.W. Beard, G.N. Saridis, and J.T. Wen. Approximate solutions to the time-invariant Hamilton-Jacobi-Bellman equation. *Journal of Optimization Theory and its Applications*, 96(3):589–626, 1998.

[12] S. Bennett. A brief history of automatic control. *IEEE Control Systems*, 16(3):17–25, 1996.

[13] E.B. Beran. *Methods for Optimization-based Fixed-Order Control Design*. PhD thesis, Technical University of Denmark, `http://www.iau.dtu.dk/phds.html`, 1997.

[14] P. Bernhard and A. Rapaport. Min-max certainty equivalence principle and differential games. *Int. J. Robust and Nonlinear Control*, 6(8):825–842, 1996.

[15] J. Bernoussou, P.L.D. Peres, and J.C. Geromel. A linear programming oriented procedure for quadratic stabilization of uncertain systems. *Systems and Control Letters*, 13:65–72, 1989.

[16] D.P. Bertsekas. *Dynamic Programming and Stochastic Control*. Academic Press, 1976.

[17] V. Blondel. *Simultaneous Stabilization of Linear Systems*. Number 191 in Lecture Notes in Control and Information Sciences. Springer-Verlag, 1993.

[18] B. Boulet, B.A. Francis, P.C. Hughes, and T. Hong. Uncertainty modeling and experiments in $\mathcal{H}_\infty$ control of large flexible space structures. *IEEE Transactions on Control Systems Technology*, 5(5):504–519, 1997.

[19] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Studies in Applied Mathematics. Siam, Philadelphia, 1994.

[20] S. P. Boyd and C. H. Barratt. *Linear Controller Design - Limits* ... *Performance*. Prentice-Hall, 1991.

[21] A.E. Bryson, Jr. Optimal control – 1950 to 1985. *IEEE Contr* ... *Systems*, 16(3):26–33, 1996.

[22] D.F. Chichka and J.L Speyer. An adaptive controller based on dis... turbance attenuation. *IEEE Transactions on Automatic Contro*... 40(7):1220–1233, July 1995.

[23] M.G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosit... solutions of second order partial differential equations. *Bulletin* ... *the American Mathematical Society*, 27:1–67, 1992.

[24] H. Deng and M. Krstić. Stochastic nonlinear stabilization - I: ... backstepping design. *Systems and Control Letters*, 32:143–150, 199...

[25] G. Didinsky. *Design of Minimax Controllers for Nonlinear Sys... tems using Cost-To-Go Methods*. PhD thesis, U. Illinois at Urbana... Champaign, 1994.

[26] J. Doyle, K. Zhou, K. Glover, and B. Bodenheimer. Mixed $\mathcal{H}_2$ an... $\mathcal{H}_\infty$ performance objectives II: Optimal control. *IEEE Transactio... on Automatic Control*, 39(8):1575–1587, August 1994.

[27] J.C. Doyle. Guaranteed margins of LQG regulators. *IEEE Transac... tions on Automatic Control*, 23(4):756–757, 1979.

[28] J.C. Doyle. Synthesis of robust controllers and filters. In *Proceeding... of the 22nd IEEE Conference on Decision and Control*, pages 109... 114, 1983.

[29] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis. State... space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. *IEE... Transactions on Automatic Control*, 34:831–847, 1989.

[30] V. Dragan, A. Halanay, and A. Stoica. A small gain theorem fo... linear stochastic systems. *Systems and Control Letters*, 30:243–25... 1997.

[31] L. El Ghaoui. State-feedback control of systems with multiplicativ... noise via linear matrix inequalities. *Systems and Control Letter*... 24:223–228, 1995.

[32] L. El Ghaoui, F. Delebecque, and R. Nikoukhah. `LMItool`*: a User--Friendly Interface for LMI Optimization.* Available by FTP at `ftp.ensta.fr/pub/elghaoui`, 1997.

[33] E.G. Eszter and C.V. Hollot. An IQC for uncertainty satisfying both norm-bounded and passivity constraints. *Automatica*, 33(8):1545–1548, 1997.

[34] E. Feron, P. Apkarian, and P. Gahinet. Analysis and synthesis of robust control systems using parameter dependent Lyapunov function. *IEEE Transactions on Automatic Control*, 41(7):1041–1046, July 1996.

[35] W.H. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer, New York, 1993.

[36] R.A. Freeman and P.V. Kokotovic. Inverse optimality in robust stabilization. *SIAM Journal on Control and Optimization*, 34(4):1365–1391, 1996.

[37] P. Gahinet and P. Apkarian. A linear matrix inequality approach to $\mathcal{H}_\infty$ control. *Int. J. Robust and Nonlinear Control*, 4:421–428, 1994.

[38] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali. *LMI Control Toolbox*. MATLAB, 1995.

[39] G. Garcia, J. Bernussou, and D. Arzelier. Stabilization of an uncertain linear dynamic system, by state and output feedback: A quadratic stabilizability approach. *International Journal of Control*, 64(5):839–858, 1996.

[40] T.C. Gard. *Introduction to Stochastic Differential Equations*, volume 114 of *Monographs and textbooks in pure and applied mathematics*. Marcel Dekker, 1988.

[41] H. Goldstein. *Classical Mechanics*. Addison-Wesley, Reading, Mass., 1980.

[42] M. Green and D.J.N. Limebeer. *Linear Robust Control*. Prentice Hall, 1995.

[43] R.Z. Has'minskǐ. *Stochastic Stability of Differential Equations*. Sijthoff & Noordhoff, 1980.

[44] O.B. Hijab. *Stabilization of Control Systems*. Applications of Math ematics. Springer-Verlag, 1987.

[45] D. Hill and P. Moylan. The stability of nonlinear dissipative system *IEEE Transactions on Automatic Control*, 21:708–711, 1976.

[46] D.J. Hill and P.J. Moylan. Stability results for nonlinear feedbac systems. *Automatica*, 13:377–382, 1977.

[47] D.J. Hill and P.J. Moylan. Dissipative dynamical systems: Ba sic input-output and state properties. *J. of the Franklin Insitut* 309:327–357, 1980.

[48] D. Hinrichsen and A.J. Pritchard. Stochastic $\mathcal{H}_\infty$. *SIAM Journal o Control and Optimization*, 36(5):1504–1538, 1998.

[49] J. Hocherman-Frommer, S.R. Kulkarni, and P.J. Ramadge. Con troller switching based on output prediction errors. *IEEE Transa tions on Automatic Control*, 43(5):596–607, 1998.

[50] K. Hoffman. *Banach Spaces of Analytic Functions*. Prentice-Hal 1962. Reprinted by Dover Publications, Inc, in 1988.

[51] A. Isidori. *Nonlinear Control Systems*. Springer-Verlag, third editio 1995.

[52] M.R. James. Computing the $\mathcal{H}_\infty$ norm for nonlinear systems. I *Proc. of the 12th IFAC World Congress*, pages 31–34, 1993.

[53] M.R. James. A partial differential inequality for dissipative nonlinea systems. *Systems and Control Letters*, 21(4):315–320, 1993.

[54] M.R. James. On the certainty equivalence principle and the optima control of partially observed dynamic games. *IEEE Transactions o Automatic Control*, 39(11):2321–2324, 1994.

[55] M.R. James and J.S. Baras. Partially observed differential game infinite-dimensional Hamilton-Jacobi-Isaacs equations, and nonlinea $\mathcal{H}_\infty$ control. *SIAM Journal on Control and Optimization*, 34(4):1342 1364, 1996.

[56] M.R. James and S. Yuliar. A nonlinear partially observed differer tial game with a finite-dimensional information state. *Systems an Control Letters*, 26:137–145, 1995.

[57] U. Jönsson. *Robustness Analysis of Uncertain and Nonlinear Systems*. PhD thesis, Dept. Automatic Control, Lund Inst. Tech., 1996.

[58] U. Jönsson. Stability analysis with Popov multipliers and integral quadratic constraints. *Systems and Control Letters*, 31:85–92, 1997.

[59] K.H. Khalil. *Nonlinear Systems*. McMillan, second edition, 1996.

[60] P.P. Khargonekar and M.A. Rotea. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control: A convex optimization approach. *IEEE Transactions on Automatic Control*, 36(7):824–837, July 1991.

[61] M. Kocan and P. Sorovia. Nonlinear, dissipative, infinite dimensional systems. Technical Report MMR 014-98, Centre for Mahtematics and its Applications, Australian National University, Canberra, 1998.

[62] M. Kohlmann and P. Renner. Optimal control of diffusions: A verification theorem for viscosity solutions. *Systems and Control Letters*, 28:247–253, 1996.

[63] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design*. Adaptive and Learning Systems for Signal Processing. Wiley, 1995.

[64] H.J. Kushner. *Stochastic Stability and Control*, volume 33 of *Mathematics in Science and Engineering*. Academic Press, 1967.

[65] H.J. Kushner and P.G. Dupuis. *Numerical Methods for Stochastic Control Problems in Continuous Time*, volume 24 of *Applications of Mathematics*. Springer-Verlag, 1992.

[66] H. Kwakernaak and R. Sivan. *Linear Optimal Control Systems*. Wiley Interscience, New York, 1972.

[67] F.L. Lewis. *Optimal Control*. Wiley, New York, 1986.

[68] D.J.N. Limebeer, B.D.O. Anderson, and B. Hendel. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering by the theory of Nash games. In B.A. Francis and A.R. Tannenbaum, editors, *Springer Lect. Notes in Ctrl. and Inf. Sci.*, volume 183, pages 9–15. Springer, 1992.

[69] W. Lin. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control via state feedback for nonlinear systems. *International Journal of Control*, 64(5):899–922, 1996.

[70] P.L. Lions and P.E. Sougandis. Differential games, optimal cor trol and directional derivatives of viscosity solutions of Bellman and Isaac's equations. *SIAM Journal on Control and Optimizatio* 23(4):566–583, 1985.

[71] J. Lu and R.E. Skelton. Robust $\mathcal{H}_2$/LQG control for systems wit Finite-signal-to-noise uncertainty: A convergent agorithm. In *Pro ceedings of The American Control Conference*, pages 3505–350 1997.

[72] J. Lu, R.E. Skelton, and U.H. Thygesen. Robust control with var ance finite-signal-to-noise models. In *Proceedings of the 35th IEE Conference on Decision and Control*, pages 3435–3440. IEEE, 1996

[73] D.L. Lukes. Optimal regulation of nonlinear dynamical system *SIAM J. Control*, 7:75–100, 1969.

[74] A.M. Lyapunov. The general problem of stability of motion. *Inte national Journal of Control*, 55(3):531–773, 1992. Translation of th 1892 memoir; from Russian to French by É. Davaux (1907) and fro French to English by A.T. Fuller.

[75] X. Mao. *Exponential Stability of Stochastic Differential Equation* Marcel Dekker, 1994.

[76] X. Mao. Stochastic stabilization and destabilization. *Systems an Control Letters*, 23:279–290, 1994.

[77] A. Megretski and A. Rantzer. System analysis via integtral quadrat constraints. *IEEE Transactions on Automatic Control*, 42(6):81 830, 1997.

[78] J. Møller-Pedersen and M.P. Petersen. Control of nonlinear plant Master's thesis, Department of Mathematics, Technical University Denmark, 1995.

[79] A.S. Morse. Control using logic-based switching. In A. Isidori, edito *Trends in Control - A European Perspective*, pages 69–114. Springe Verlag, 1995.

[80] A.S. Morse. Supervisory control of families of linear set-point co trollers - part 1: Exact matching. *IEEE Transactions on Automat Control*, 41(10):1413–1431, 1996.

[81] A.S. Morse. Supervisory control of families of linear set-point controllers - part 2: Robustness. *IEEE Transactions on Automatic Control*, 42(11):1500–1515, 1997.

[82] Y. Nestorov and A. Nemirovski. *Interior Point Polynomial Methods in Convex Programming*, volume 13 of *Studies Appl. Math.* SIAM, 1993.

[83] B. Øksendal. *Stochastic Differential Equations - An Introduction with Applications*. Springer-Verlag, 1995.

[84] B. Øksendal and K. Reikvam. Viscosity solutions of optimal stopping problems. *Stochastics Stochastics Rep.*, 62(3-4):285–301, 1998.

[85] J.E. Parkum. *Recursive Identification of Time-Varying Systems*. PhD thesis, Dept. of Math. Modeling, Tech. Uni. of Denmark, 1992.

[86] M.A. Peters and A.A. Stoorvogel. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control in a stochastic framework. *Linear Algebra and its Applications*, 205–206:971–996, 1994.

[87] V.M. Popov. *Hyperstability of Control Systems*. Springer, 1973.

[88] S. Rangan. *Validation, Identification and Control of Robust Control Uncertainty Models*. PhD thesis, University of California at Berkeley, 1997.

[89] S. Rangan. Weighted optimization for multiobjective full-information control problems. *Systems and Control Letters*, 31:207–213, 1997.

[90] S. Rangan and K. Poolla. Multimodel adaptive $\mathcal{H}_\infty$ control. In *Proceedings of the 35th IEEE Conference on Decision and Control*, pages 1928–1933, 1996.

[91] M. Renardy and R.C. Rogers. *An Introduction to Partial Differential Equations*, volume 13 of *Texts in Applied Mathematics*. Springer, 1992.

[92] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, 1972.

[93] M.G. Safonov and M. Athans. Gain and phase margins for multiloop LQG regulators. *IEEE Transactions on Automatic Control*, 22(2):173–179, April 1977.

[94] A.V. Savkin. Simultaneous $\mathcal{H}_\infty$ control of a finite collection of linea plants with a single nonlinear digital controller. *Systems and Contr Letters*, 33:281–289, 1998.

[95] A.V. Savkin and I.R. Petersen. Robust $\mathcal{H}_\infty$ control of uncertain sys tems with structured uncertainty. *Journal of Mathematical System Estimation, and Control*, 6(3):1–14, 1996.

[96] A.V. Savkin and I.R. Petersen. Robust state estimation for uncertai systems with averaged integral quadratic constrains. *Internation Journal of Control*, 64(5):923–939, 1996.

[97] A.V. Savkin and I.R. Petersen. Output feedback guaranteed cost con trol of uncertain systems on an infinite time interval. *Int. J. Robu and Nonlinear Control*, 7:43–58, 1997.

[98] C. Scherer. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control. In A. Isidori, editor, *Trends i Control - A European Perspective*, pages 173–217. Springer-Verlag 1995.

[99] R. Sepulchre, M. Janković, and P. Kokotović. *Constructive Nonline Control*. Springer, 1997.

[100] G. Shi and R.E. Skelton. State feedback covariance control for linea finite signal-to-noise ratio models. In *Proceedings of the 34th IEE Conference on Decision and Control*, pages 3423–3428. IEEE, De cember 1995.

[101] S. Simons. Minimax theorems and their proofs. In D.-Z. Du and P.M Pardalos, editors, *Minimax and Applications*, pages 1–23. Kluwe Academic Press, 1995.

[102] R.E. Skelton. Robust control of aerospace systems. In *Symposiu on Robust Control Design*, pages 24–31. IFAC, September 1994.

[103] R.E. Skelton and J. Lu. Iterative identification and control de sign using finite-signal-to-noise models. *Math. Modeling of System* 3(1):102–135, 1997.

[104] R.E. Skelton and G. Shi. Finite signal-to-noise ratio models: Co variance control by state feedback. In *Proceedings of the Europea Control Conference*, pages 77–82, 1995.

[105] P. Sorovia. Equivalence between nonlinear $\mathcal{H}_\infty$ control problems and existence of viscosity solutions of Hamilton-Jacobi-Isaacs equations. *Applied Mathematics & Optimization*, 00:1–16, 1998.

[106] A.A. Stoorvogel. The robust $\mathcal{H}_2$ control problem: A worst-case design. *IEEE Transactions on Automatic Control*, 38(9):1358–1370, 1993.

[107] D. W. Stroock and S.R.S. Varadhan. *Multidimensional Diffusion Processes*. Springer, 1979.

[108] U. H. Thygesen. A survey of Lyapunov techniques for stochastic differential equations. Technical Report 18, Dept. Math. Modeling, Tech. Uni. Denmark, `http://www.imm.dtu.dk`, 1997.

[109] U. H. Thygesen. On dissipation in stochastic systems. Under review for publication in a journal. A short version is submitted to a conference, 1998.

[110] U. H. Thygesen and R. E. Skelton. Linear systems with Finite Signal-to-Noise ratios: A robustness approach. In *Proceedings of the 34th IEEE Conference on Decision and Control*, pages 4157–4162. IEEE, December 1995.

[111] U.H. Thygesen. On multi-dissipative dynamic systems. Submitted to a journal. A short version is submitted to a conference, 1998.

[112] U.H. Thygesen. On the conditional expectation of first passage times. Manuscript in preparation, 1998.

[113] U.H. Thygesen and N.K. Poulsen. Min-max control of non-linear systems with multi-dissipative perturbations. Technical Report 23, Dept. Math. Modeling, Tech. Uni. Denmark, `http://www.imm.dtu.dk`, 1997. Presented at the 6th Viennese Workshop on Optimal Control, Dynamic Games, Nonlinear Dynamics and Adaptive Systems, Vienna, 1997.

[114] U.H. Thygesen and N.K. Poulsen. On multi-dissipative perturbations in linear systems. Technical Report 1, Dept. Math. Modeling, Tech. Uni. Denmark, `http://www.imm.dtu.dk`, 1997.

[115] U.H. Thygesen and N.K. Poulsen. Robustness of linear systems with multi-dissipative perturbations. In *Proceedings of The American Control Conference*, pages 3444–3445, 1997.

[116] U.H. Thygesen and N.K. Poulsen. Simultaneous $\mathcal{H}_\infty$ control of finite number of plants. Technical Report 24, Dept. Math. Modelin Tech. Uni. Denmark, `http://www.imm.dtu.dk`, 1997.

[117] U.H. Thygesen and N.K. Poulsen. Simultaneous output feedback $\mathcal{H}$ control of p plants using switching. In *Proceedings of the Europea Control Conference*, 1997.

[118] J.N. Tsitsiklis and M. Athans. Guranteed robustness properties multivariable nonlinear stochastic optimal regulators. *IEEE Tran actions on Automatic Control*, 29(8):690–696, 1984.

[119] A.J. van der Schaft. $\mathcal{L}_2$-gain analysis of nonlinear systems and non linear state feedback $\mathcal{H}_\infty$ control. *IEEE Transactions on Automat Control*, 37(6):770–784, June 1992.

[120] A.J. van der Schaft. Nonlinear state space $\mathcal{H}_\infty$ control theory. In H.I Trentelman and J.C. Willems, editors, *Essays on Control: Perspe tives in the Theory and its Applications*, pages 153–190. Birkhäuse Basel, 1993.

[121] A.J. van der Schaft. Nonlinear systems which have finite-dimensiona $\mathcal{H}_\infty$ suboptimal central controllers. In *Proceedings of the 32nd IEE Conference on Decision and Control*, pages 202–203. IEEE, Decembe 1993.

[122] A.J. van der Schaft. *$\mathcal{L}_2$-Gain and Passivity Techniques in Nonlinea Control*, volume 218 of *Lecture Notes in Control and Informatio Sciences*. Springer, 1996.

[123] J.T. Wen. Time domain and frequency domain conditions fo strict positive realness. *IEEE Transactions on Automatic Contro* 33(10):988–992, 1988.

[124] J.C. Willems. Dissipative dynamical systems, part i and ii. *Arch Rat. Mech. Analysis*, 45:321–393, 1972.

[125] S. Xie and L. Xie. Robust dissipative control for linear system with dissipative uncertainty and nonlinear perturbation. *System and Control Letters*, 29:255–268, 1997.

[126] S. Xie, L. Xie, and C. E. De Souza. Robust dissipative control fo linear systems with dissipative uncertainty. *International Journal Control*, 70(2):169–191, 1998.

[127] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control*, 26(2):301–320, April 1981.

[128] K. Zhou, J. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, 1996.

[129] K. Zhou, K. Glover, B. Bodenheimer, and J. Doyle. Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives I: Robust performance analysis. *IEEE Transactions on Automatic Control*, 39(8):1564–1574, August 1994.

[130] K. Zhou, P.P. Khargonekar, J. Stoustrup, and H.H. Niemann. Robust performance of systems with structured uncertainties in state space. *Automatica*, 31(2):249–256, February 1995.

[131] X.Y. Zhou, J. Yong, and X. Li. Stochastic verification theorems within the framework of viscosity solutions. *SIAM Journal on Control and Optimization*, 35(1):243–253, January 1997.

[132] G. Zhu and R.E. Skelton. Mixed $\mathcal{L}_2$ and $\mathcal{L}_\infty$ problems by weight selection in quadratic optimal control. *International Journal of Control*, 53(5):1161–1176, 1991.

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