Gaussian Integrals

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1 Product of Gaussians

Suppose that $\boldsymbol{x} \in \mathcal{R}^d$ and

$$p_i(\boldsymbol{x}) \sim \mathcal{N}(\boldsymbol{a}_i, \boldsymbol{A}_i) = \frac{1}{|2\pi \boldsymbol{A}_i|^{\frac{1}{2}}} e^{-\frac{1}{2}\boldsymbol{x}_i^\top \boldsymbol{A}_i^{-1}\boldsymbol{x}_i} \tag{1}$$

where $\boldsymbol{x}_i = \boldsymbol{x} - \boldsymbol{a}_i$, $i = 1, 2, \cdots, M$.

Consider the evaluation of integrals like

$$\int |\boldsymbol{x}|^q \boldsymbol{x} \prod_i p_i(\boldsymbol{x})\, d\boldsymbol{x}, \qquad \int |\boldsymbol{x}|^q \prod_i p_i(\boldsymbol{x})\, d\boldsymbol{x} \tag{2}$$

The solutions can be obtained by using the fundamental proposition Eq. (6).

2 Fundamental Proposition

Define a common Gaussian density $p_c(\boldsymbol{x}) \sim \mathcal{N}(\boldsymbol{c}, \boldsymbol{C})$ with

$$\boldsymbol{x}_c = \boldsymbol{x} - \boldsymbol{c} \tag{3}$$

$$\boldsymbol{x}_c = \boldsymbol{C}\left(\sum_i \boldsymbol{A}_i^{-1}\boldsymbol{x}_i\right) \tag{4}$$

$$\boldsymbol{C} = \left(\sum_i \boldsymbol{A}_i^{-1}\right)^{-1} \tag{5}$$

then

$$p_c(\boldsymbol{x}) = \frac{\prod_\ell |2\pi \boldsymbol{A}_\ell|^{\frac{1}{2}}}{|2\pi \boldsymbol{C}|^{\frac{1}{2}}} \cdot \prod_k p_k(\boldsymbol{x}) \cdot \prod_{i>j} \exp\left[\frac{1}{2}(\boldsymbol{a}_i - \boldsymbol{a}_j)^\top \boldsymbol{B}_{ij}(\boldsymbol{a}_i - \boldsymbol{a}_j)\right] \tag{6}$$

where

$$\boldsymbol{B}_{ij} = \boldsymbol{B}_{ji} = \boldsymbol{A}_i^{-1}\left(\sum_k \boldsymbol{A}_k^{-1}\right)^{-1}\boldsymbol{A}_j^{-1} \tag{7}$$

Proof Consider writing

$$\boldsymbol{x}_c^\top \boldsymbol{C}^{-1}\boldsymbol{x}_c = \sum_i \boldsymbol{x}_i^\top \boldsymbol{A}_i^{-1}\boldsymbol{x}_i - \sum_{i>j}(\boldsymbol{x}_i - \boldsymbol{x}_j)^\top \boldsymbol{B}_{ij}(\boldsymbol{x}_i - \boldsymbol{x}_j) \tag{8}$$

The left hand side of (8) can cf. (4) and (5) be expressed as:

$$\boldsymbol{x}_c^\top \boldsymbol{C}^{-1}\boldsymbol{x}_c = \sum_{ij} \boldsymbol{x}_i^\top \boldsymbol{A}_i^{-1}\left(\sum_k \boldsymbol{A}_k^{-1}\right)^{-1}\boldsymbol{A}_j^{-1}\boldsymbol{x}_j \tag{9}$$

The right hand side of (8) can be rewritten as

$$\sum_i \boldsymbol{x}_i^\top\left(\boldsymbol{A}_i^{-1} - \sum_{j\neq i}\boldsymbol{B}_{ij}\right)\boldsymbol{x}_i + \sum_{j\neq i}\boldsymbol{x}_i^\top \boldsymbol{B}_{ij}\boldsymbol{x}_j \tag{10}$$

Comparison of (9) and (10) provides the relations:

$$\boldsymbol{B}_{ij} \quad = \quad \boldsymbol{A}_i^{-1}\left(\sum_k \boldsymbol{A}_k^{-1}\right)^{-1}\boldsymbol{A}_j^{-1} \tag{11}$$

$$\boldsymbol{A}_i^{-1} - \sum_{j\neq i}\boldsymbol{B}_{ij} \quad = \quad \boldsymbol{A}_i^{-1}\left(\sum_k \boldsymbol{A}_k^{-1}\right)^{-1}\boldsymbol{A}_i^{-1} \tag{12}$$

Equation (12) is true since

$$\boldsymbol{A}_i^{-1} - \sum_{j\neq i}\boldsymbol{A}_i^{-1}\left(\sum_k \boldsymbol{A}_k^{-1}\right)^{-1}\boldsymbol{A}_j^{-1} = \boldsymbol{A}_i^{-1}\left(\sum_k \boldsymbol{A}_k^{-1}\right)^{-1}\boldsymbol{A}_i^{-1} \tag{13}$$

$\Updownarrow$

$$\boldsymbol{A}_i^{-1} - \sum_j \boldsymbol{A}_i^{-1}\left(\sum_k \boldsymbol{A}_k^{-1}\right)^{-1}\boldsymbol{A}_j^{-1} = \boldsymbol{0} \tag{14}$$

$\Updownarrow$

$$\sum_j \boldsymbol{A}_i^{-1}\left(\sum_k \boldsymbol{A}_k^{-1}\right)^{-1}\boldsymbol{A}_j^{-1} = \boldsymbol{A}_i^{-1} \tag{15}$$

$\Updownarrow$

$$\sum_j \left(\sum_k \boldsymbol{A}_k^{-1} \right)^{-1} \boldsymbol{A}_j^{-1} = \boldsymbol{I} \tag{16}$$

$\Updownarrow$

$$\sum_j \boldsymbol{A}_j^{-1} = \sum_k \boldsymbol{A}_k^{-1} \tag{17}$$

3 General Cases

3.1 Integral of products

Since $\int p_c(\boldsymbol{x})\, d\boldsymbol{x} = 1$ then by using (6)

$$I = \int \prod_k p_k(\boldsymbol{x})\, d\boldsymbol{x} = \frac{|2\pi \boldsymbol{C}|^{\frac{1}{2}}}{\prod_\ell |2\pi \boldsymbol{A}_\ell|^{\frac{1}{2}}} \cdot \prod_{i>j} \exp\left[-\frac{1}{2} (\boldsymbol{a}_i - \boldsymbol{a}_j)^\top \boldsymbol{B}_{ij} (\boldsymbol{a}_i - \boldsymbol{a}_j) \right] \tag{18}$$

4 Special Cases

4.1 Propagation of mean value

Consider the evaluation of $\int \phi_j(\boldsymbol{x}) p(\boldsymbol{x})\, d\boldsymbol{x}$ where $p(\boldsymbol{x}) = \mathcal{N}(\boldsymbol{u}, \boldsymbol{S})$ and

$$\phi_j(\boldsymbol{x}) = \exp[-(\boldsymbol{x} - \boldsymbol{x}_j)^\top \boldsymbol{\Lambda}^{-1} (\boldsymbol{x} - \boldsymbol{x}_j)/2] = |2\pi \boldsymbol{\Lambda}|^{\frac{1}{2}} \cdot \mathcal{N}(\boldsymbol{x}_j, \boldsymbol{\Lambda}) \tag{19}$$

In the case $M = 2$, cf. (5) and (7),

$$\begin{aligned} \boldsymbol{B}_{12} &= \boldsymbol{A}_1^{-1} \left(\boldsymbol{A}_1^{-1} + \boldsymbol{A}_2^{-1} \right)^{-1} \boldsymbol{A}_2^{-1} = \left(\boldsymbol{A}_1 (\boldsymbol{A}_1^{-1} + \boldsymbol{A}_2^{-1}) \boldsymbol{A}_2 \right)^{-1} = (\boldsymbol{A}_1 + \boldsymbol{A}_2)^{-1} & (20) \\ \boldsymbol{C} &= \left(\boldsymbol{A}_1^{-1} + \boldsymbol{A}_2^{-1} \right)^{-1} & (21) \end{aligned}$$

According to (18) with $\boldsymbol{a}_1 = \boldsymbol{x}_j$, $\boldsymbol{A}_1 = \boldsymbol{\Lambda}$, $\boldsymbol{a}_2 = \boldsymbol{u}$, and $\boldsymbol{A}_2 = \boldsymbol{S}$.

$$\begin{aligned} I &= \frac{|\boldsymbol{C}|^{\frac{1}{2}}}{|2\pi \boldsymbol{A}_1|^{\frac{1}{2}} |\boldsymbol{A}_2|^{\frac{1}{2}}} \cdot \exp\left[-\frac{1}{2} (\boldsymbol{a}_1 - \boldsymbol{a}_2)^\top (\boldsymbol{A}_1 + \boldsymbol{A}_2)^{-1} (\boldsymbol{a}_1 - \boldsymbol{a}_2) \right] \\ &= \frac{1}{|2\pi \boldsymbol{A}_1|^{\frac{1}{2}} \sqrt{|\boldsymbol{C}^{-1} \boldsymbol{A}_2|}} \cdot \exp\left[-\frac{1}{2} (\boldsymbol{a}_1 - \boldsymbol{a}_2)^\top (\boldsymbol{A}_1 + \boldsymbol{A}_2)^{-1} (\boldsymbol{a}_1 - \boldsymbol{a}_2) \right] \\ &= \frac{1}{|2\pi \boldsymbol{A}_1|^{\frac{1}{2}} |\boldsymbol{A}_1^{-1} \boldsymbol{A}_2 + \boldsymbol{I}|^{\frac{1}{2}}} \cdot \exp\left[-\frac{1}{2} (\boldsymbol{a}_1 - \boldsymbol{a}_2)^\top (\boldsymbol{A}_1 + \boldsymbol{A}_2)^{-1} (\boldsymbol{a}_1 - \boldsymbol{a}_2) \right] \qquad (22) \end{aligned}$$

That is,

$$\begin{aligned} I_j &= \int \phi_j(\boldsymbol{x}) p(\boldsymbol{x})\, d\boldsymbol{x} = |2\pi \boldsymbol{\Lambda}|^{\frac{1}{2}} \int \mathcal{N}(\boldsymbol{x}_j, \boldsymbol{\Lambda}) \mathcal{N}(\boldsymbol{u}, \boldsymbol{S})\, d\boldsymbol{x} \\ &= |\boldsymbol{\Lambda}^{-1} \boldsymbol{S} + \boldsymbol{I}|^{-\frac{1}{2}} \cdot \exp\left[-\frac{1}{2} (\boldsymbol{x}_j - \boldsymbol{u})^\top (\boldsymbol{\Lambda} + \boldsymbol{S})^{-1} (\boldsymbol{x}_j - \boldsymbol{u}) \right] \qquad (23) \end{aligned}$$