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Locating a circle on a sphere

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Abstract

We consider the problem of locating a spherical circle with respect to existing facilities on a sphere, such that the sum of weighted distances between the circle and the facilities is minimized, or such that the maximum weighted distance is minimized. The problem properties are analyzed, and we give solution procedures. When the circle to be located is restricted to be a great circle, some simplifications are possible.

1 Introduction

The location of a linear facility in space has many potential applications. For example, the facility may represent a new highway in two-dimensional space or a pipeline. It could be an electrical power line, a string of radio or mobile phone transmission towers, or radar stations, and on a smaller scale, a main electrical conduit on a circuit board.

The problem of locating a linear facility in the plane has been well studied beginning with the work of Wesolowsky [15]. Here the objective is to find a line that minimizes the weighted sum of shortest Euclidean distances from the line to a set of fixed points representing the users or customers. A fundamental property of this problem that leads to an efficient solution procedure is that the 'median' line must intersect at least two of the existing points. Further refinements and extensions to the basic model are investigated by Morris and Norback [8, 9] and

Norback and Morris [10]; meanwhile Schöbel [13] examines general distance measures and other forms of generalizations to the problem. Finding a line that minimizes the maximum distance to a set of users has been studied in Schömer et al. [14], and in the context of determining the width of a set in Houle and Toussaint [4]. For a recent overview of line location in the plane, see Schöbel [13], and [7] for a survey on extensive facility location in general.

It is well recognized that the planar model becomes inaccurate when the users are spread over larger areas of the earth's surface, and that spherical distances should be used to account for the earth's curvature. Early work on locating a point facility on a sphere was done by Drezner and Wesolowsky [2] and Katz and Cooper [5], among others, and summarized by Wesolowsky [16]. As the general single facility minisum problem on a sphere is nonconvex, unlike the planar model, Hansen et al. [3] propose a branch-and-bound algorithm to solve it. The idea is to divide the surface of the sphere into smaller and smaller sections, using alternative bounds provided by the authors to fathom unattractive zones, and to proceed in this fashion until the solution is found within an acceptable accuracy. The problem of locating a point facility on a sphere with the minimax objective is examined more recently by Das et al. [1] and Patel and Chidambaram [11].

The purpose of this paper is to study the problem of locating circles on a sphere, which is a natural (yet new) extension of the line location problem on the plane. In the next section, some basic concepts of spherical distances are reviewed and the notation we will use is specified. Section 3 investigates the problem of locating great circles by the minimax criterion, while section 4 generalizes to any spherical circle. The remaining sections are devoted to finding minisum great circles and general circles.

2 Notation

We use the following notation, based on [6]. The sphere is denoted by S, and without loss of generality we may assume that the radius of the sphere is 1.

A point $x = (x_1, x_2)$ on the sphere is given by its latitude x_1 (angle from the equator) and its longitude x_2 (angle from the Greenwich meridian). We assume $-\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2}$, with a negative latitude denoting a point south of the equator, and $-\pi \leq x_2 \leq \pi$, with a negative longitude denoting a point west of Greenwich. Henceforth, by a point we mean a point on the sphere.

A great circle is the intersection between the sphere and a plane through the center of the sphere. The distance between two points is measured along the great circle containing the points; it is the shorter of the lengths of the two great circle arcs connecting the points, measured in radians. The largest possible distance between two points is π, realized when one point is the antipode of the other point. The distance $d(x, a)$ between the two points $x = (x_1, x_2)$ and

$a = (a_1, a_2)$ may be computed from the relation

$$\cos d(x, a) = \cos x_1 \cos a_1 \cos(x_2 - a_2) + \sin x_1 \sin a_1.$$

A spherical circle is the intersection between the sphere and a plane. Henceforth, by a circle we mean a spherical circle. A circle is the locus of points with a fixed distance r from a given point $c = (c_1, c_2)$. c is called the center and r is called the radius of the circle. Denoting the circle by $C(c, r)$, we have

$$C(c, r) = \{x \in S : d(x, c) = r\}.$$

If a circle $C(c, r)$ has radius $r > \frac{\pi}{2}$, it may be viewed as a circle with center in the antipode of c and radius $\pi - r$. Thus it suffices to consider circles with radii in the interval $0 \leq r \leq \frac{\pi}{2}$, and henceforth we shall do so. A circle with radius 0 is a point, and a circle with radius $\frac{\pi}{2}$ is a great circle.

The distance between a point a and a circle $C = C(c, r)$, defined as

$$D(C, a) = \min_{x \in C} d(x, a)$$

can be calculated as follows: Consider the great circle containing a and c. If this great circle intersects the circle C in the points x and z, the point to circle distance $D(C, a)$ is given by

$$D(C, a) = \min\{d(x, a), d(z, a)\} =: d(y, a).$$

The closer of the two points x and z is called the *footpoint* y of a with respect to C. Furthermore, let P be the shorter part of the great circle connecting a and y. Note that the length of P equals $D(C, a)$.

For the special case $r = 0$, we have $D(C, a) = d(c, a)$, and for the special case $c = a$, we have $D(C, a) = r$.

It is particularly easy to compute the point to circle distance, when the center of the circle is a Pole of the sphere. Suppose for instance that the center of a circle C is the North Pole; then the distance from the point $a = (a_1, a_2)$ to the circle is $r + a_1 - \frac{\pi}{2}$ if a is north of the circle, and $\frac{\pi}{2} - r - a_1$ otherwise, or $D(C, a) = |r + a_1 - \frac{\pi}{2}|$.

In general, we have

$$D(C, a) = |r - d(c, a)|.$$

For the case when C is a great circle, there is a simple relation between the point to great circle distance, $D(C, a)$ and the smallest Euclidean distance from the point to the plane H containing the great circle, $E(H, a)$: $\sin D(C, a) = E(H, a)$.

Let n be the number of existing point facilities, located at $a_j = (a_{j1}, a_{j2}) \in S$ with positive weight w_j, for $j = 1, \ldots, n$. Denote the set of existing facility locations by A.

Then our optimization problem is finding a circle $C = C(c, r)$ with center $c = (c_1, c_2)$ and radius $r \in [0, \frac{\pi}{2}]$ so as to minimize

$$f(C) = f(c, r) = \sum_{j=1}^{n} w_j D(C(c, r), a_j)$$

or so as to minimize

$$g(C) = g(c, r) = \max_{j=1,\dots,n} w_j D(C(c, r), a_j).$$

The first objective refers to the minisum or median problem, whereas the second objective refers to the minimax or center problem.

Any given circle $C = (c, r)$ separates the sphere in two parts, and it is convenient to define the index sets $J_+ = \{j : d(a_j, c) < r\}$, $J_- = \{j : d(a_j, c) > r\}$, and $J_= = \{j : d(a_j, c) = r\}$.

3 Finding minimax great circles

We first consider the equally weighted *Great-Circle-Minimax problem* (GCM) of locating a great circle on the sphere, minimizing the maximum distance to the existing facilities. When all the weights are equal, we may assume without loss of generality that they are all 1. Let us assume that $n \geq 3$.

Lemma 1 *Let C^* be an optimal solution of (GCM) with objective value $g(C)$. Then there exist at least three existing facilities $a \in A$ satisfying*

$$D(C, a) = g(C).$$

<u>Proof:</u> Let C^* be an optimal great circle and assume first that there exist exactly two points $a_i, a_j \in A$ with $g(C^*) = D(C^*, a_i) = D(C^*, a_j)$. Without loss of generality let $i = 1$, $j = 2$ and assume that $g(C^*) > 0$, otherwise all points $a \in A$ satisfy $D(C, a) = g(C)$.

Determine the corresponding footpoints and great circle segments $y_1 \in P_1$ and $y_2 \in P_2$. Let $\epsilon > 0$ and define two points $y_1' \in P_1$ and $y_2' \in P_2$ such that

$$\begin{aligned} d(a_1, y_1) - d(a_1, y_1') &= \epsilon, \\ d(a_2, y_1) - d(a_2, y_1') &= \epsilon. \end{aligned}$$

Since a great circle is uniquely defined by two points we define C' as the great circle passing through y_1' and y_2'. Note that the function $\mathcal{C}$ mapping two points y_1, y_2 to the great circle defined by these points is well-defined and continuous whenever $y_1 \neq y_2$ and y_1 and y_2 are not antipodes to each other. For this reason we distinguish the following three cases.

- First assume that $y_1 \neq y_2$ and the points are not antipodes to each other. Hence, $y'_1 \neq y'_2$.

  We obtain for $k = 1, 2$:

$$D(C', a_k) = \min_{x \in C'} d(x, a_k) \leq d(y'_i, a_k) < d(y_i, a_k) = g(C^*)$$

  Denote $g' = \max\{D(C', a_1, D(C', a_2)\}$ and note that $g' < g(C^*)$. For all other $a \in A \setminus \{a_1, a_2\}$ the continuity of $\mathcal{C}$ yields

$$g(C^*) > D(C^*, a),$$

  hence $g' \geq D(C', a)$ holds if ϵ is chosen small enough.

  Together,

$$g(C') = \max_{a \in \mathcal{A}} D(C', a) = g' < g(C^*)$$

  contradicting the optimality of C^*.

- In the case that $y_1 = y_2$, the existing facilities a_1 and a_2 must be on opposite sides of C^* (otherwise they would coincide). Then rotate C^* a small amount as in the previous case, but this time about the axis through the common foot point, $y_1 = y_2$. Again we obtain a reduction in distance, $d(a_1, y'_1) = d(a_2, y'_2) < g(C^*)$, leading to a similar contradiction as before.

- If y_1 and y_2 are antipodes of each other, there are two possibilities: either a_1 and a_2 are on the same side of C^*, in which case rotate C^* a small amount about the axis through y_1 and y_2; or a_1 and a_2 are on the opposite sides of C^*, in which case use the line on C^* perpendicular to (y_1, y_2) as the axis of rotation. Again we obtain a similar contradiction as before.

To exclude that there exists only one unique point a on C^* satisfying $g(C) = D(C^*, a)$ we proceed as follows. Let a be such a unique point, $y \in P$ be the corresponding footpoint and P be the great circle segment between a and y. Similar to the first part of the proof, we fix an arbitrary point $x \in C^* \setminus \{y\}$, find $y' \in P$ such that

$$d(a, y) - d(a, y') = \epsilon$$

for some $\epsilon > 0$ and choose a new circle C' as the great circle passing through x and y'. Since $g(C') < g(C^*)$, we again have a contradiction.

QED

Note that a similar proof carries through also for the weighted great circle minimax problem. In this case we allow positive weights for the existing facilities and minimize the maximum weighted distance to the circle. Lemma 1 then states that there exist three existing facilities i, j, k satisfying

$$w_i D(C, a_i) = w_j D(C, a_j) = w_k D(C, a_k) = g(C).$$

Now we turn our attention back to computing an optimal great circle in the unweighted case.

First, we remark that it can happen that all three existing facilities with the maximum distance to the circle may lie on the same side of the circle, as the following example demonstrates.

Consider three existing facilities all on the northern hemisphere, but all three of them close to the equator, e.g.,

$$A = \{(\epsilon, 0), (\epsilon, \frac{2}{3}\pi), (\epsilon, -\frac{2}{3}\pi)\}.$$

Using Lemma 1 and checking all great circles at equal distance to the three existing facilities yields the equator $C((\frac{\pi}{2}, 0), \frac{\pi}{2})$ with optimal distance of ϵ to all three points as optimal great circle.

From Lemma 1 we know that all optimal circles of (GCM) have the same positive distance to at least three points $a_j, a_i, a_k \in A$. Note that in the case that not all points are contained in one common great circle, no pair of these points a_j, a_i, and a_k can be antipodes to each other, since they all have the same positive distance to an optimal great circle. Furthermore, at least two of these points lie on the same side of C, without loss of generality let us assume that $i, k \in J_+$. Since $D(C, a) = |r - d(a, c)|$ and $i, k \in J_+$ we obtain $d(c, a_i) = \frac{\pi}{2} - D(C, a_i) = \frac{\pi}{2} - D(C, a_k) = d(c, a_k)$, i.e., the distance from both points a_i and a_k to the center c of the circle is the same. In other words, c lies in the set $B_{ik} = \{x \in S : d(x, a_i) = d(x, a_k)\}$, which is the bisector of a_i and a_k. Note that bisectors on the sphere are great circles. Hence, to find the center point c^* of an optimal circle C^* only points on one of the bisectors $B_{ik}, i \neq k, i, k \in \{1, \ldots, n\}$ need to be investigated. Finding the best great circle with center c on some bisector (great circle) B_{ik} hence reduces to a one-dimensional optimization problem.

To tackle this problem we can furthermore use that the distance of an optimal great circle $D(C^*, a_j)$ to a third point a_j is the same as to the points a_i and a_k defining the bisector B_{ik}.

We hence only have to investigate the points a_j satisfying that $D(C, a_j) = D(C, a_i)$ which can be reformulated as $|\frac{\pi}{2} - d(c, a_j)| = |\frac{\pi}{2} - d(c, a_i)|$.

We need to consider two cases.

Case 1: Assume that a_j is on the same side of the optimal circle C^* as a_i and a_k. In this case, a_j satisfies

$$d(c, a_j) = D(C, a_j) - \frac{\pi}{2} = D(C, a_i) - \frac{\pi}{2} = d(c, a_i),$$

hence the set of candidates to be investigated can be determined by intersecting B_{ik} with all bisectors B_{ij}, $j = 1, \ldots, n, j \neq i, j \neq k$.

Case 2: Assume that a_j lies on the opposite side of C^* as a_i and a_k do. Consequently, a_j satisfies $\frac{\pi}{2} - d(c, a_j) = -\frac{\pi}{2} + d(c, a_i)$, or, equivalently,

$$\pi = d(c, a_i) + d(c, a_j).$$

As candidates for the optimal center c we hence have to consider all points $c \in C_{ij} = \{x \in S : d(x, a_i) + d(x, a_j) = \pi\}$

For Case 1 it is well known that B_{ij} is a great circle. In the following we show that also C_{ij} is a great circle and how the centers of B_{ij} and C_{ij} can be constructed. To this end, let C denote the (unique) great circle passing through a_i and a_j. Let c_{ij} be the midpoint on the great circle segment of C joining a_i and a_j and choose $b_{ij} \in C$ such that $d(b_{ij}, c_{ij}) = \frac{\pi}{2}$. Note that by construction we have $d(c_{ij}, a_i) = d(c_{ij}, a_j)$ and $d(b_{ij}, a_i) + d(b_{ij}, a_j) = \pi$. Then the following holds.

Lemma 2

1. $B_{ij} = C(b_{ij}, \frac{\pi}{2})$

2. $C_{ij} = C(c_{ij}, \frac{\pi}{2})$

<u>Proof:</u>

1. Consider any point $x \in S$ such that $d(x, a_1) = d(x, a_2)$. Since $d(c_{ij}, a_i) = d(c_{ij}, a_j)$ it follows by congruence that the great circle segment joining x and c_{ij} belongs to B_{ij}, or $d(b_{ij}, x) = \frac{\pi}{2}$. The reverse obviously holds: If $x \in B_{ij}$, then $d(x, a_i) = d(x, a_j)$.

2. Suppose $x \in S$ satisfies $d(x, a_i) + d(x, a_j) = \pi$, and without loss of generality, $d(x, a_1) \leq d(x, a_2)$. Consider the plane containing $\{a_1, a_2, x\}$. We may use symmetry of triangles and projections back on the sphere to conclude that

$$d(x, c_{ij}) = \frac{1}{2}\left(d(x, a_1) + d(x, a_2)\right) = \frac{\pi}{2},$$

hence $x \in C_{ij}$. The argument also applies in reverse such that from $x \in C_{ij}$ we conclude that $d(x, a_i) + d(x, a_j) = \pi$.

QED

This means, in both cases we have to find the intersection of two great circles. This gives two points $x_1, x_2 \in S$. But note that x_1 and x_2 are antipodes to each other and hence define the same great circle. This means that only one of these points needs to be further investigated.

Algorithm 1 for (GCM)

Step 1: Let the candidate set $K = \emptyset$
Step 2: For all triples $\{i, j, k\} \in \{1, \ldots, n\}$, determine an intersection point h^1_{ijk} of the bisectors B_{ik} and B_{ij}, and h^2_{ijk} of the bisector B_{ik} and the great circle C_{ij}. let $K = K \cup \{h^1_{ijk}, h^2_{ijk}\}$
Step 3: Evaluate all candidates $c \in K$ by calculating
$f(C(c, \frac{\pi}{2})) = \max_{j=1,\ldots,n} |d(a_j, c) - \frac{\pi}{2}|$ and take the one with the best objective value.

Consider an arbitrary great circle C and the plane containing it, H. For all existing facilities, we have $E(H, a_j) = \sin(D(C, a_j))$. Since $\sin(v)$ is increasing on the relevant interval, $0 \leq v \leq \frac{\pi}{2}$, the existing facility, j', that is furthest from H (measured by Euclidean distance) is also furthest from C (measured by angle). This observation means that $E(H, a_{j'}) = \max_{j=1,\ldots,n} E(H, a_j)$ and $D(C, a_{j'}) = \max_{j=1,\ldots,n} D(C, a_j)$, and allows us to characterize the relationship between the two problems.

Lemma 3 *The problems (GCM) and (REM) are equivalent: If a plane H^* solves (REM), then the great circle contained in H^* solves (GCM), and if a great circle C^* solves (GCM), then the plane containing C^* solves (REM).*

Proof:
Let H^* solve (REM), and let j^* be the furthest existing facility. Optimality means that $E(H^*, a_{j^*}) \leq E(H, a_{j'})$ for an arbitrary plane H with furthest existing facility j'. Since arcsin is increasing, we have $\arcsin(E(H^*, a_{j^*})) \leq \arcsin(E(H, a_{j'}))$. Let C^* and C be the great circles contained in H^* and H, respectively. Then we obtain

$$\max_{j=1,\ldots,n} D(C^*, a_j) = D(C^*, a_{j^*}) = \arcsin(E(H^*, a_{j^*}) \leq$$
$$\arcsin(E(H, a_{j'}) = D(C, a_{j'}) = \max_{j=1,\ldots,n} D(C, a_j),$$

or $\max_{j=1,\ldots,n} D(C^*, a_j) \leq \max_{j=1,\ldots,n} D(C, a_j)$. Since C is arbitrary, this establishes the optimality of C^*.
The converse is shown similarly.

QED

A similar proof establishes the equivalence for the weighted case. This gives an alternative proof of Lemma 1 by using Theorem 3 of [12] which states that all optimal hyperplanes of (REM), i.e., all hyperplanes in $\mathbb{R}^3$ through one specified point that minimize the maximum distance to a given set of points $a_1, \ldots, a_n$ pass through at least three affinely independent points of this set.

The algorithmic implication is clear: To solve the great circle minimax problem, we just need to solve the restricted Euclidean minimax problem.

4 Finding minimax circles

Now we pass our attention to the unweighted *Circle-Minimax problem* (CM) in which we relax the restriction that the optimal circle must be a great circle, but allow any circle on the sphere. Again our goal is to minimize the maximum distance to the existing facilities. Although many more circles are allowed as feasible solutions the problem is easier to solve. First of all we prove the following result which is stronger than the result of Lemma 1. Here we need to assume that $n > 3$.

Theorem 1 *Let C^* be an optimal solution of (CM) with objective value $g(C^*)$. Then there exist at least four existing facilities $a \in A$ satisfying*

$$D(C, a) = g(C^*).$$

<u>Proof:</u> Let C^* be an optimal circle which is at maximum distance from exactly $m \in \{1, 2, 3\}$ existing facilities. Without loss of generality assume that these facilities are $a_1, \dots, a_m$ and that $g(C^*) > 0$, i.e.,

$$\begin{aligned} g(C^*) &= D(C^*, a_1) > 0 \\ &\dots \\ g(C^*) &= D(C^*, a_m) > 0 \end{aligned}$$

The goal is to define a circle C' with better objective value. This is done as follows: For $k = 1, \dots, m$ consider the footpoint $y_j \in P_j$ of a_j with respect to C^* and the great circle segment P_j between y_j and a_j. As in the proof of Lemma 1 we assume that no two footpoints coincide, otherwise we disturbe the footpoints slightly. Choose $\epsilon > 0$ and define a new point y'_j by moving y_j along P_j ϵ closer to a_j, i.e., $y'_j \in P_j$ and $d(y_j, a_j) - d(y'_j, a_j) = \epsilon$. Furthermore, choose $3 - m$ arbitrary points in $C \setminus \{a_1, \dots, a_m\}$. This defines $m + 3 - m = 3$ points which uniquely define a new circle C', and if the footpoints are different the function C' mapping these 3 points to a circle C' is well-defined and continuous. Hence, we can choose $\epsilon > 0$ in such a way that

$$|D(C^*, a_j) - D(C', a_j)| \le \delta \text{ for all } j = 1, \dots, n.$$

To calculate the objective value of C' we first consider $j = 1, \dots, m$ and obtain

$$\begin{aligned} D(C', a_j) &= \min_{x \in C'} d(x, a_j) \\ &\le d(y'_j, a_j) \\ &= d(y_j, a_j) - \epsilon \\ &= g(C^*) - \epsilon. \end{aligned}$$

Defining $g' := \max_{j=1,\dots,m} D(C', a_j)$ this yields

$$0 < \epsilon \leq g(C^*) - g' \leq \delta.$$

On the other hand, for all $j = m+1, \dots, n$ we know that

$$D(C^*, a_j) < g(C^*),$$

hence, choosing $\delta \leq \frac{1}{2}\left(g(C^*) - \max_{j=m+1,\dots,n} D(C^*, a_j)\right)$ implies that

$$D(C', a_j) \leq D(C^*, a_j) + \delta \leq g(C^*) - \delta \leq g'.$$

Together, $D(C', a_j) \leq g'$ for all $j = 1, \dots, n$ and hence $g(C') = g' < g(C^*)$ proving the result.

QED

Again, a similar proof can be made for the weighted circle problem, in which positive weights for the existing facilities are allowed. In this case, Theorem 1 can be extended to four existing facilities at the same weighted distance to the optimal circle. Unfortunately that does not help much for finding the optimal circle in the weighted case, since "weighted" bisectors are hard to compute. For the unweighted case, however, the following rather simple procedure can be used to determine an optimal circle with respect to the minimax objective function.

Algorithm 2 for (CM)

Step 1: Let $K = \emptyset$
Step 2: For all pairs $\{i, j\} \in \{1, \dots, n\}$ and all distinct pairs $\{k, l\} \in \{1, \dots, n\}$, determine an intersection point h_{ijkl} of the bisectors B_{ij} and B_{kl}, and let $K = K \cup \{h_{ijkl}\}$
Step 3: Evaluate all candidates $c \in K$ by calculating $f(C(c, \frac{\pi}{2})) = \max_{j=1,\dots,n} |d(a_j, c) - \frac{\pi}{2}|$ and take the one with the best objective value.

5 Finding minisum great circles

Here we consider the problem of finding a great circle minimizing the sum of distances to the given facilities. This problem will be called (GCS).

In a first result we relate the median great circle problem to that of locating a plane H through the center of the sphere, such that the sum of the Euclidean distances to the points $a_1, \dots, a_n$ is minimized. We denote this problem as restricted Euclidean minisum problem (RES). It can be stated as the problem of minimizing

$$F(H) = \sum_{j=1,\dots,n} w_j E(H, a_j).$$

Recall that the Euclidean distance $E(H, a) = \sin(D(C, a_j))$, if H is the hyperplane containing the great circle C. Unfortunately, we cannot show that (RES) and (GCS) are equivalent as it is true in the minimax case, but we can at least use the hyperplane location problem for getting an upper and a lower bound.

Lemma 4 *Let H^* be an optimal hyperplane for (RES), and let C^* be an optimal great circle for (GCS). Furthermore, let $C(H^*) = H^* \cap S$ be the great circle contained in H^* and $H(C^*)$ be the hyperplane passing through C^*. Then*

$$F(H^*) \leq F(H(C^*)) \leq f(C^*) \leq f(C(H^*)).$$

Proof:

$$\begin{aligned}
F(H^*) &\leq F(H(C^*)) \\
&= \sum_{j=1,\dots,n} w_j E(H(C^*), a_j) \\
&= \sum_{j=1,\dots,n} w_j \sin D(C^*, a_j) \\
&\leq \sum_{j=1,\dots,n} w_j D(C^*, a_j) \\
&= f(C^*) \\
&\leq f(C(H^*))
\end{aligned}$$

QED

Note that for (RES) it is known that all optimal hyperplanes pass through at least two of the existing facilities, see Theorem 3 of [12], i.e., $C(H^*)$ always contains two of the points a_i, a_j.

Locating a great circle may be viewed as the spherical equivalent of locating a line on the plane; this motivates the following result.

Lemma 5 *An optimal solution C^* of (GCS) may be found that intersects at least two of the existing points.*

Proof: Consider first the trivial case where all existing points are contained on some great circle, C. Obviously, $C^* = C$ is the optimal solution, with $f(C^*) = 0$.

Now assume that all existing points are not contained on the same great circle. The problem is to minimize
$f(C) = f(c, \frac{\pi}{2}) = \sum_{j \in J_+} w_j(\frac{\pi}{2} - d(c, a_j)) + \sum_{j \in J_-} w_j(\frac{\pi}{2} - d(c', a_j))$,
where the index sets J_+ and J_- contain the existing points on each side of the great circle, and c' is the antipode of c. Suppose we have an optimal solution $C^* = (c^*, \frac{\pi}{2})$ that does not contain any existing points. It is known that the

distance from a given point a to a point on the sphere is a convex function within a circle of radius $\frac{\pi}{2}$ and center a (e.g., see [3]). Hence $d(c, a_j)$ is convex in a local neighborhood of c^* for $j \in J_+$, and $d(c', a_j)$ is convex in a local neighborhood of $c^{*\prime}$ for $j \in J_-$. Furthermore, since $c' = c + (\pi, 0)$ the convexity extends to c for $j \in J_-$. We conclude that $f(C)$ is locally concave at c^*, and hence, C^* may be rotated a small amount in any direction without increasing the objective function. (Otherwise, rotating in the opposite direction would decrease f, leading to a contradiction). The argument extends until the rotated great circle intersects one of the existing points, say a_r. Now use the line through a_r and the center of the sphere as the axis of rotation, to conclude in similar fashion (for adjusted J_+, J_-) that C^* may be rotated further until it intersects a second existing point.

QED

This result permits a finite solution method for (GCS): Compute the objective function value for the great circle through each pair of existing points; the optimal solution is the great circle with lowest value.

6 Finding minisum circles

The problem we discuss here, denoted by (CS), is to find a circle minimizing the sum of (weighted) distances to the existing facilities.

Recall that for this purpose we identify a circle $C = C(c, r) \subset S$ by its center point $c \in S$ and its radius r.

The objective function of (CS) may be written as
$f(c, r) = \sum_{j=1}^{n} w_j D(C, a_j) = \sum_{j=1}^{n} w_j |r - d(c, a_j)|$.
The first observation is the following.

Lemma 6 *There exists an optimal solution C^* to problem (CS) passing through at least one of the existing facilities.*

Proof: Let $C = C(c, r)$ be an optimal solution of (CS) with objective value $f = f(c, r) = \sum_{j=1}^{n} w_j |r - d(c, a_j)|$. Fix the center c and consider the problem of finding the optimal radius r^*, i.e.,
$\min_r \sum_{j=1}^{n} w_j |r - d(c, a_j)|$.
This problem is a one-dimensional (point) location problem for which it is well known that there exists an optimal solution r^* satisfying $r^* = d(c, a_{j^*})$ for some $j^* \in \{1, \ldots, n\}$. Consequently, $C^* = C(c, r^*)$ contains a_{j^*} and its objective value f^* satisfies
$f^* = f(c, r^*) = \sum_{j=1}^{n} w_j |r^* - d(c, a_j)| \leq \sum_{j=1}^{n} w_j |r - d(c, a_j)| = f$.

QED

(CS) is a non-convex problem with many local minima, so one solution method is finding a fair number of local minima and choosing the best one.

For a given C, the circle separates the existing facilities in two sets: the ones on one side of the circle, and the ones on the other side. Using the index sets J_+ and J_- the objective function may be written without the absolute value,

$$f(c,r) = \sum_{j\in J_+} w_j(r - d(c,a_j)) + \sum_{j\in J_-} w_j(d(c,a_j) - r).$$

Ignoring that J_+ and J_- depend on c, and ignoring the terms for which $d(c,a_j) = r$, we obtain these approximate expressions for the partial derivatives of f,

$\partial f/\partial c_1 \approx \sum_{j\in J_-} w_j(\sin c_1 \cos a_{j1} \cos(c_2 - a_{j2}) - \cos c_1 \sin a_{j1})/B_j$
$\quad - \sum_{j\in J_+} w_j(\sin c_1 \cos a_{j1} \cos(c_2 - a_{j2}) - \cos c_1 \sin a_{j1})/B_j$,
$\partial f/\partial c_2 \approx \sum_{j\in J_-} w_j \cos c_1 \cos a_{j1} \sin(c_2 - a_{j2})/B_j$
$\quad - \sum_{j\in J_+} w_j \cos c_1 \cos a_{j1} \sin(c_2 - a_{j2})/B_j$, where
$B_j = \sin(\arccos(\cos c_1 \cos a_{j1} \cos(c_2 - a_{j2}) + \sin c_1 \sin a_{j1}))$.

Setting the two approximations equal to zero and simplifying considerably yields

$$\tan c_2 = \frac{\sum_{j\in J_+} w_j \cos a_{j1}(\sin a_{j2})/B_j - \sum_{j\in J_-} w_j \cos a_{j1}(\sin a_{j2})/B_j}{\sum_{j\in J_+} w_j \cos a_{j1}(\cos a_{j2})/B_j - \sum_{j\in J_-} w_j \cos a_{j1}(\cos a_{j2})/B_j},$$

$$\frac{\tan c_1}{\sin c_2} = \frac{\sum_{j\in J_+} w_j(\sin a_{j1})/B_j - \sum_{j\in J_-} w_j(\sin a_{j1})/B_j}{\sum_{j\in J_+} w_j \cos a_{j1}(\sin a_{j2})/B_j - \sum_{j\in J_-} w_j \cos a_{j1}(\sin a_{j2})/B_j}$$

Now a procedure for finding a local minimum may be outlined. We start by choosing an arbitrary point c on the sphere and use this as the center of a circle. Given this center, the optimal radius, r, is easily found by solving the median problem of locating a point facility on a line. Given these three numbers, we find the index sets, J_+ and J_-. Now the expression for $\tan c_2$ is used for finding a better value for c_2, and the expression for $\tan c_1/\sin c_2$ is then used to find a better value for c_1. The procedure is continued iteratively until significant changes in the three decision variables no longer occur.

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