Fourier Series, the DFT and Shape Modelling

Karl Skoglund

Informatics and Mathematical Modelling
Technical University of Denmark

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Abstract

This report provides an introduction to Fourier series, the discrete Fourier transform, complex geometry and Fourier descriptors for shape analysis. The content is aimed at undergraduate and graduate students who wish to learn about Fourier analysis in general, as well as its application to shape modelling and analysis. The theory is based on and borrows largely from the excellent book Konkret Analys [4]*, which is a Swedish text on complex analysis.*

1 Fourier Analysis of Periodic Functions

Fourier analysis is due to the French mathematician Jean Baptiste Joseph Fourier who presented his theory in 1807 and again in 1812. His work shows how any function can be formulated as an infinite series of sines and cosines. The original paper did not get a warm welcome, mainly because of Fourier's statement that an *arbitrary* function could be described as a trigonometric series. Also, the proof supplied was vague. Furthermore, other scientists, including Lagrange, Euler and Bernoulli, had contributed largely to the development of the Fourier transform. Nevertheless, Fourier analysis has seen much use and research over the last 200 years. An explanation for this is that sums of sines and cosines are simple to work with and commonly describe separate physical properties of a process.

This text presents how Fourier analysis can be used to describe outlines of shapes in two dimensions. Starting from the definition of Fourier coefficients, the theory of discrete Fourier analysis is explained step by step. This gives the necessary background to properly understand how Fourier analysis can be used in shape analysis.

1.1 Euler's Formula

Trigonometric functions are commonly described by complex exponential functions. This seemingly complicates things, however, exponential functions have a number of advantages. For instance, they are compact and easy to integrate and

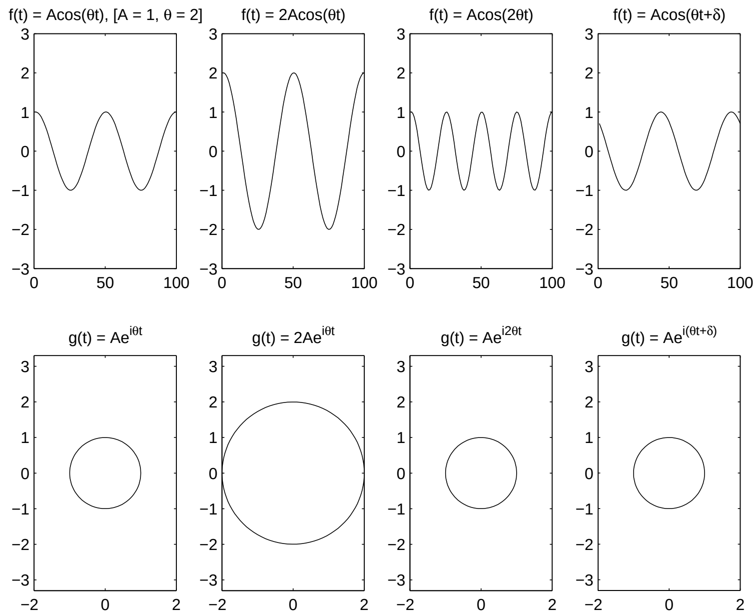

Figure 1: Real and complex oscillations. Note that amplitude is the only parameter that has effect on the shape of complex periodic functions.

derivate. The link between the complex exponential function and trigonometric functions is described by Euler's formula.

$$\left\{ \begin{array}{r} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \end{array} \right. \tag{1}$$

1.2 Complex Trigonometric Functions

The common real harmonic function $f(t) = A\cos\theta t$ can be expressed using Euler's formulas as $f(t) = \mathrm{Re}(Ae^{i\theta t})$. But what kind of function is $g(t) = Ae^{i\theta t}$? Whereas $f(t)$ is real and depends on the single variable t, $g(t)$ is complex valued and the dependent variable t can be interpreted as **arc-length**. Figure 1 shows $f(t)$ versus $g(t)$ for amplitudes A and $2A$, (angular) frequencies θ and 2θ and phase shift δ. As can be seen, altering the angular frequency does not modify the curve, it only changes the velocity with which the curve is drawn.

What about the phase shifts? This can be performed by multiplication with the complex number $\mathcal{A} = Ae^{i\delta}$ since $Ae^{i\delta}e^{i\theta t} = Ae^{i(\theta t+\delta)} = A(\cos(\theta t + \delta) + i\sin(\theta t + \delta))$. This changes the start and end point positions along the curve for $g(t)$. As can be seen in figure 1, the shape of periodic curves are invariant to phase shifts. The value $\mathcal{A} = Ae^{i\delta}$ is therefore denoted **complex amplitude**. Despite this apparent simplification, phase shifts will play an important role later in this text, when complex ellipses are dealt with.

Real harmonic functions have a number of representations. These are depicted in figure 2, and will be useful for deriving the different forms of the Fourier transform.

The remainder of this text will look at periodic functions. The period is

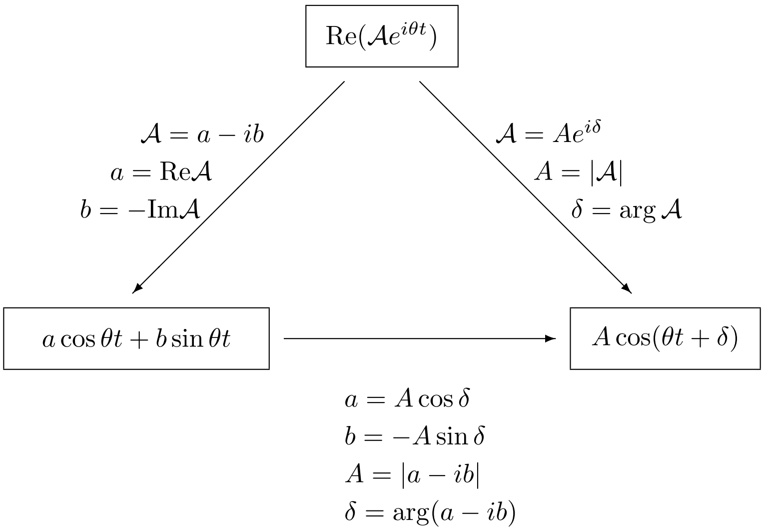

Figure 2: Different forms of a simple, real, trigonometric function

assumed to be $T = 2\pi/\Omega$ where Ω is the **angular frequency**. The angular frequency directly relates to the number of times a periodic function is run over a length of 2π. $\Omega = 4$ means that the function is run 4 times. A function with period T is said to be **T-periodic**.

1.3 Fourier Series

The definition of the discrete Fourier transform used in most practical applications is closely related to Fourier series. This text will therefore begin with a look at Fourier series, state an outline of a proof on the formula for the Fourier coefficients and discuss some of the conditions under which the Fourier series converges.

Theorem 1 Assume that $f(t)$ is a (real or complex) function which can be expanded according to Fourier's principle.

$$f(t) = \sum_{k=-\infty}^{k=\infty} c_k e^{ik\Omega t}$$

The coefficients c_k are then given by

$$c_k = \frac{1}{T} \int_P e^{-ik\Omega t} f(t) dt$$

where P is any interval over one period.

Proof (sketch) Multiply $f(t)$ by $e^{-il\Omega t}$ and integrate

$$\int_P e^{-il\Omega t} f(t) dt = \int_P e^{-il\Omega t} \sum_{k=-\infty}^{k=\infty} c_k e^{ik\Omega t} dt =$$

$$\sum_{k=-\infty}^{k=\infty} c_k \int_P e^{-il\Omega t} e^{ik\Omega t} dt = \sum_{k=-\infty}^{k=\infty} c_k \int_P e^{i(k-l)\Omega t} dt$$

But $g(t) = \int_P e^{i(k-l)\Omega t} dt$ is the integral over one period of pure sine and cosine functions, i.e. $g(t) = 0$ for $k \neq l$. For $k = l$ the integral $g(t)$ is equal to the wave length, T. This means that

$$\int_P e^{-il\Omega t} f(t) dt = \sum_{k=-\infty}^{k=\infty} c_k \int_P e^{i(k-l)\Omega t} dt = c_l T$$

Solving for c_l and switching indices completes the proof.

1.3.1 Convergence

The question is under what conditions a periodic function can be described as in theorem 1. The theory of convergence for Fourier series is still an active area of research with many open questions remaining. The convergence conditions stated here will cover a large class of periodic functions sufficient in most practical applications.

A necessary condition for the Fourier series to converge, is that the coefficients c_k are sufficiently small. This is clear seeing that $|c_k e^{ik\Omega t}| = |c_k|$. But for what functions $f(t)$ does this occur? To answer this we need a couple of definitions.

A function is **piecewise continuous** on the interval P if it is continuous over the whole interval except at a finite number of points where it is either discontinuous or not defined. A function f is **piecewise smooth** if both f and its derivative f' is piecewise continuous over the interval P. Piecewise continuity implies the existence of one-sided limits

$$f(t_i^+) = \lim_{h \to 0^+} f(t_i + h) \qquad \text{and} \qquad f(t_i^-) = \lim_{h \to 0^-} f(t_i + h) \tag{2}$$

Piecewise smoothness also implies the existence of one-sided derivatives

$$f'(t_i^+) = \lim_{h \to 0^+} \frac{f(t_i + h) - f(t_i)}{h} \qquad \text{and} \qquad f'(t_i^-) = \lim_{h \to 0^-} \frac{f(t_i + h) - f(t_i)}{h} \tag{3}$$

With these definitions at hand it is possible to state the following useful sufficient condition for the convergence of Fourier series.

Theorem 2 Let $f(t)$ be a piecewise smooth function defined on an interval P. Then the Fourier series

$$f(t) = \sum_{k=-\infty}^{k=\infty} c_k e^{ik\Omega t} \qquad \text{where} \qquad c_k = \frac{1}{T} \int_P e^{-ik\Omega t} f(t) dt \tag{4}$$

converges for every t to the value

$$\frac{f(t^+) + f(t^-)}{2} \tag{5}$$

For a proof, consult e.g. [1, 4].

As it turns out, most functions possess the necessary properties for Fourier analysis. This is shown by the great span of research areas benefiting from these methods.

1.3.2 Fourier's Formula

We are now ready to formulate the main theorem of this text.

Theorem 3, Fourier's formula If $f(t)$ is T-periodic and fulfills certain reasonable conditions, and the series c_k is defined as

$$c_k = \frac{1}{T}\int_P e^{-ik\Omega t} f(t)dt \tag{6}$$

then

$$f(t) = \sum_{k=-\infty}^{k=\infty} c_k e^{ik\Omega t} \tag{7}$$

The set $< c_k >$ is called the **Fourier transformation** of f, and the coefficients c_k is called the **Fourier coefficients** of f. The series (7) is called the **Fourier series** of f. Equation 6 is known as **Fourier analysis** and equation 7 is referred to as **Fourier synthesis**.

1.3.3 Relation to the general Fourier transform

The definition of the Fourier series may easily be confused with the general Fourier transformation

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi\omega t}dt \tag{8}$$

and its inverse

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{i2\pi\omega t}d\omega \tag{9}$$

which holds for any absolutely integrable function. In words, this definition states that a function $f(t)$ can be synthesized from combinations of all frequencies ω.

1.4 Examples

In this section, two examples of periodic functions and their fourier coefficients are given.

A simple complex harmonic function

The first example examines the complex function $f(t) = \cos t + i\sin\frac{1}{5}t$ which naturally can be described as a sum of harmonic functions. Using Euler's functions we get

$$f(t) = \frac{1}{2}(e^{it} + e^{-it}) + i\frac{1}{2i}(e^{i\frac{1}{5}t} - e^{-i\frac{1}{5}t}) =$$
$$\frac{1}{2}(e^{i5\frac{1}{5}t} + e^{i(-5)\frac{1}{5}t} + e^{i1\frac{1}{5}t} - e^{i(-1)\frac{1}{5}t})$$

Comparing this expression with equation 7 it is seen that the Fourier coefficients for $f(t)$ are

$$c_{\pm 5} = \frac{1}{2}, \quad c_{-1} = -\frac{1}{2}, \quad c_1 = \frac{1}{2}, \quad c_k = 0, \;\; k \notin \{\pm 5, \pm 1\}$$

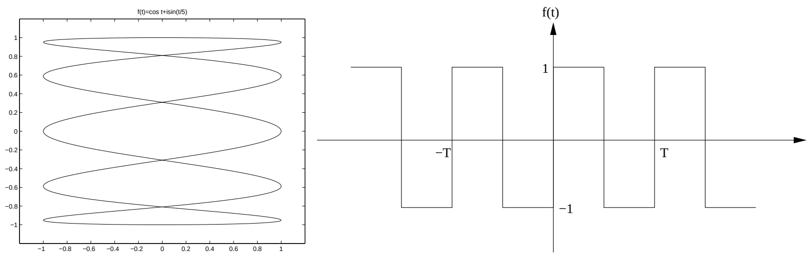

Figure 3: Functions in example one and two

The square waveform

The second example examines the square waveform defined as

$$f(t) = \begin{cases} 1, & 0 < t < \frac{T}{2} \\ -1, & -\frac{T}{2} < t < 0 \end{cases}$$

We calculate the coefficients according to the definition,

$$c_k = \frac{1}{T}\int_P e^{-ik\Omega t} f(t)dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-ik\Omega t} f(t)dt =$$

$$\frac{1}{T}(-\int_{-T/2}^{0} e^{-ik\Omega t}dt + \int_{0}^{T/2} e^{-ik\Omega t}dt) =$$

$$\overset{k\neq 0}{=} \frac{1}{T}(\left[-\frac{e^{-ik\Omega t}}{-ik\Omega}\right]_{-T/2}^{0} + \left[\frac{e^{-ik\Omega t}}{-ik\Omega}\right]_{0}^{T/2}) = \frac{1}{T}\frac{-1 + e^{ik\Omega T/2} + e^{-ik\Omega T/2} - 1}{-ik\Omega} =$$

$$\frac{1}{T}\frac{2(1-\cos(k\Omega T/2))}{ik\Omega} = \frac{1}{T}T\frac{1-\cos k\pi}{ik\pi} = \frac{1-\cos k\pi}{ik\pi} \iff$$

$$c_k = \frac{1-(-1)^k}{\pi ik} = \begin{cases} \frac{2}{\pi ik} & k \text{ odd} \\ 0 & k \text{ even or } k = 0 \end{cases}$$

The coefficient for $k = 0$ can be calculated directly from equation 6.

$$c_0 = \frac{1}{T}\int_P f(t)dt = 0$$

1.5 Trigonometric Fourier series

The complex exponential formulation of Fourier series that we've seen so far is compact and easy to work with. However, on many occasions, real functions are described and it may seem strange to use a complex expression. We will therefore derive an alternative formulation which is quite common in the literature.

Start with the exponential Fourier series.

$$f(t) = \sum_{k=-\infty}^{k=\infty} c_k e^{ik\Omega t}$$

Combine each pair of terms having the same angular frequency $k\Omega$.

$$c_k e^{ik\Omega t} + c_{-k} e^{-ik\Omega t} = (c_k + c_{-k}) \cos k\Omega t + i * (c_k + c_{-k}) \sin k\Omega t =$$

$$a_k \cos k\Omega t + b_k \sin k\Omega t$$

where

$$\left\{ \begin{array}{lcl} a_k & = & c_k + c_{-k} \\ b_k & = & i(c_k - c_{-k}) \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{lcl} c_k & = & \frac{1}{2}(a_k - ib_k) \\ c_{-k} & = & \frac{1}{2}(a_k + ib_k) \end{array} \right. \tag{10}$$

Inserting the equations for c_k and simplifying, we get

$$\left\{ \begin{array}{lcll} a_k & = & \frac{2}{T} \int_P \cos k\Omega t f(t) dt & k \geq 0 \\ b_k & = & \frac{2}{T} \int_P \sin k\Omega t f(t) dt & k \geq 1 \end{array} \right. \tag{11}$$

It is seen that $a_0 = 2c_0$. The resulting **trigonometric Fourier series** is

$$\frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos k\Omega t + b_k \sin k\Omega t \tag{12}$$

Using this formulation, it is clear that if $f(t)$ is a real function, the coefficients a_k and b_k are also real. Using the equations of section 1.2, we get a third representation,

$$A_0 + \sum_{k=1}^{\infty} A_k \cos(k\Omega t + \delta_k) \tag{13}$$

where $A_0 = \frac{1}{2} a_0$, $A_k = |a_k - ib_k|$ and $\delta_k = \arg(a_k - ib_k)$, $k > 0$.

1.6 Discrete Fourier Transforms

After this rather long introduction to Fourier series we are well prepared to tackle the discrete Fourier transform. The DFT is *an approximation to the Fourier series coefficients* c_k, a statement we will prove shortly.

In practical applications $f(t)$ is seldom a continuous functions, most of the times $f(t)$ is represented by samples from some process we wish to model. To emphasize this, f sampled at time k is denoted $f_k = f(t_k)$. Assume f is a table of N samples gathered over the interval length T at equally spaced time points $t_k = k\Delta t$, $k = 0 \ldots N-1$ and $\Delta t = T/N$. At intervals outside of the chosen one, the function is assumed to repeat, something called the **periodic extension** of f. According to theorem 2, the Fourier series converges to the average of the endpoints at any discontinuity. In the discrete setting, $f(0^+) = f_0$ and $f(T^-) = f_N$. Bearing this and the periodic extension in mind, it is seen that the first and last value of f_k must be averaged, $f_0 = f_N = (f_0 + f_N)/2$.

The integrand in equation 6 can be approximated using the **trapezoid rule**. The area of a trapezoid consists of the sum of the area of a rectangle and a triangle. Assume $f_k > f_{k+1}$. The total area of the trapezoid is then $f_{k+1}\Delta t + (f_k - f_{k+1})\Delta t/2 = \Delta t(f_k + f_{k+1})/2$. The same result is reached when $f_k \leq f_{k+1}$.

The trapezoid approximation works as follows

$$c_j = \frac{1}{T} \int_P e^{-ij\Omega t} f(t) dt \approx \left[g_k = e^{-ij\Omega t_k} f_k \right] \approx \frac{1}{T} \sum_{k=0}^{N-1} \frac{\Delta t}{2} (g_k + g_{k+1}) =$$

$$\frac{1}{T}\frac{\Delta t}{2}\left(\sum_{k=0}^{N-1} g_k + \sum_{k=1}^{N} g_k\right) = \frac{1}{T}\frac{\Delta t}{2}\left(g_0 + 2\sum_{k=1}^{N-1} g_k + g_N\right) =$$

$$\frac{1}{T}\Delta t \sum_{k=0}^{N-1} g_k = \frac{1}{N}\sum_{k=0}^{N-1} g_k = \frac{1}{N}\sum_{k=0}^{N-1} e^{-ij\Omega t_k} f_k = \frac{1}{N}\sum_{k=0}^{N-1} e^{-ij(2\pi/T)(kT/N)} f_k$$

We have arrived at the discrete Fourier transform.

Theorem 4, the DFT The discrete Fourier transform is an approximation of the Fourier series coefficient and is given by

$$F_j = \frac{1}{N}\sum_{k=0}^{N-1} e^{-2\pi ijk/N} f_k \tag{14}$$

and its inverse is

$$f_k = \sum_{j=0}^{N-1} e^{2\pi ijk/N} F_j \tag{15}$$

1.7 The Reciprocity Relations

To deepen the understanding of the discrete Fourier transform, something ought to be mentioned about the *reciprocity relations*[1]. These shed some light on how interval lengths, grid spacing and number of samples in spatial and frequency space relate.

We have just seen how the spatial domain is sampled at N regular intervals Δt over the whole interval T, that is, $\Delta t = T/N$. For each time point t_k there will be a specific frequency γ_k in the frequency domain. According to the basic Fourier assumption, the function to be analyzed is a sum of sines and cosines in the spatial domain. The lowest possible frequency that still is periodic has one whole period on the interval T. The so-called **fundamental frequency** is therefore $1/T$. Since this is the smallest unit of frequency, $\Delta\gamma = 1/T$. This is the grid spacing in frequency space. Since there are N samples, $N\Delta\gamma = N/T = \Gamma$, where Γ denotes the entire frequency interval. Similarly, $N\Delta t = T$. This gives the following.

Definition 1, The Reciprocity Relations

$$T\Gamma = N \tag{16}$$

$$\Delta x \Delta\gamma = \frac{1}{N} \tag{17}$$

These relations show for example that if the spatial interval T is made longer, the frequency interval Γ will be shorter. If the number of samples N is doubled and the spatial interval is fixed, the frequency interval will also be doubled.

1.8 The Fast Fourier Transform

The fast Fourier transform (FFT) is an *implementation* of the DFT. It is not a separate transform and the results of an FFT are exactly the same as those of a DFT on the same data. However, it reduces the computational effort from

$\mathcal{O}(N^2)$ (direct implementation from the definition of the DFT) to $\mathcal{O}(\mathcal{N}\log\mathcal{N})$. It is based on maxim of Machiavelli, "divide and conquer". This algorithm is implemented in MATLAB, MAPLE and many numerical analysis packages. It is therefore neither of practical nor theoretical interest in this context. There are many good books and web sites for the interested reader.

2 Shape Modelling

2.1 Complex Ellipses

To understand how shape outlines in two dimensions can be described in frequency space using Fourier analysis it is necessary to understand how ellipses may be represented using complex exponential functions. In section 1.2 we saw how an exponential function can be used to describe a circle in the complex plane. The function is

$$ae^{i\theta t} = a\left(\cos\theta t + i\sin\theta t\right)$$

that is, a circle with radius a. If a is complex, the circle will have radius $|a|$ and a rotation of $\arg(a)$. Also in section 1.2, we proved that multiplication by a complex number amounts to a scaling and a phase shift. In other words, phase shifts and rotations are synonymous for complex exponential functions.

An ellipse is spanned by its major and minor axis, the major axis being the longer of the two. These are usually orthogonal, but as we shall see, this is not necessary. In the space of all possible ellipses, the circle is the subset with equal length axes. To turn a circle into an ellipse, we simple change the length of one axis.

$$\begin{aligned} a\cos\theta t + c(ai\sin\theta t),\ \ c\neq 0 \Leftrightarrow \left[\begin{array}{c} r_1 = (a+ca)/2 \\ r_2 = (a-ca)/2 \end{array}\right] \Leftrightarrow \\ (r_1+r_2)\cos\theta t + (r_1-r_2)i\sin\theta t = \\ r_1(\cos\theta t + i\sin\theta t) + r_2(\cos\theta t - i\sin\theta t) = \\ r_1e^{i\theta t} + r_2e^{-i\theta t} \end{aligned} \tag{18}$$

This shows how an ellipse can be written using two exponentials of the same frequency but with differing frequency signs.

What constrains apply to r_1 and r_2 for equation 18 to describe a proper ellipse? Obviously, if r_1+r_2 is purely real and r_1-r_2 is purely imaginary, the imaginary part will disappear and the ellipse will collapse into a line segment along the real line. It is realized that the same phenomenon always occur if r_1+r_2 and $i(r_1-r_2)$ have the same argument. Let's further study the general case where r_1 and r_2 are arbitrary complex numbers.

$$r_1e^{i\theta t} + r_2e^{-i\theta t} \equiv r_1\cos\theta t + ir_2\sin\theta t$$

When r_1 and r_2 are complex numbers, the axes of the ellipse are rotated independently of each other. The right hand side of figure 4 shows the function $f(t) = (1+3i)\cos\theta t + i(2+i)\sin\theta t$. When the axes are no longer orthogonal with respect to each other, will the resulting shape still be a perfect ellipse? In that case, there should be a new set of axes that are orthogonal and that spans the same ellipse. We have already seen that the shape of complex periodic functions are invariant to phase shifts. The shift will rotate both axes along the ellipse curve. The angle between the axes will be different for different phase shifts since an axis will move slowly past the first principal direction of the ellipse and fast past the second principal direction. Hopefully this will change the angle between the axes such that they are orthogonal precisely when the axes point in the principal directions.

Let's start by adding a phase shift to the equation.

$$r_1 \cos(\theta t + \delta) + i r_2 \sin(\theta t + \delta) \tag{19}$$

Using the trigonometric equalities

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \tag{20}$$

$$\sin(x + y) = \cos x \sin y + \sin x \cos y \tag{21}$$

the equation can be expanded to

$$\begin{gathered} r_1 \left(\cos\delta \cos\theta t - \sin\delta \sin\theta t\right) + i r_2 \left(\sin\delta \cos\theta t + \cos\delta \sin\theta t\right) = \\ \left(r_1 \cos\delta + i r_2 \sin\delta\right) \cos\theta t + i \left(r_2 \cos\delta + i r_1 \sin\delta\right) \sin\theta t = \\ c_1(\delta) \cos\theta t + i c_2(\delta) \sin\theta t \end{gathered}$$

As can be seen, the equation reduces to what we started with, a cosine and a sine multiplied by complex numbers. The difference is that we now have a parameter δ which can be chosen arbitrarily. We wish to find the δ for which the angle between $c_1(\delta)$ and $ic_2(\delta)$ is $\pi/2$. This is true when the dot product between $c_1(\delta)$ and $c_2(\delta)$ is zero. Denote the dot product function $\Psi(\delta)$ and set $r_1 = a_1 + ib_1$ and $r_2 = a_2 + ib_2$ where $a_{1,2}, b_{1,2} \in \Re$.

$$\left\{ \begin{array}{l} c_1(\delta) = r_1 \cos\delta + i r_2 \sin\delta = a_1 \cos\delta + i b_1 \cos\delta + i a_2 \sin\delta - b_2 \sin\delta \\ i c_2(\delta) = i(r_2 \cos\delta + i r_1 \sin\delta) = i a_2 \cos\delta - b_2 \cos\delta - a_1 \sin\delta - i b_1 \sin\delta \end{array} \right. \Longrightarrow$$

$$\begin{gathered} \Psi(\delta) = c_1(\delta).ic_2(\delta) = \left(a_1 \cos\delta - b_2 \sin\delta,\, b_1 \cos\delta + a_2 \sin\delta\right). \\ \left(-b_2 \cos\delta - a_1 \sin\delta,\, a_2 \cos\delta - b_1 \sin\delta\right) = \\ (a_2^2 + b_2^2 - a_1^2 - b_1^2) \cos\delta \sin\delta - (a_1 b_2 - a_2 b_1)(\cos^2\delta - \sin^2\delta) = \\ \frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2) \sin 2\delta - (a_1 b_2 - a_2 b_1) \cos 2\delta = 0 \Leftrightarrow \\ \frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2) \tan 2\delta - (a_1 b_2 - a_2 b_1) = 0 \Leftrightarrow \end{gathered}$$

$$\delta = \frac{1}{2} \arctan \frac{2(a_1 b_2 - a_2 b_1)}{a_2^2 + b_2^2 - a_1^2 - b_1^2} \tag{22}$$

We have shown that by choosing the correct phase shift, it is possible to rectify the axes of an ellipse, thus transforming it to a form without phase shifts and where r_1 and r_2 differ only by magnitude.

How does this relate to Fourier analysis and shape modelling? Let's recapitulate on the equation of Fourier analysis.

$$f(t) = \sum_{k=-\infty}^{k=\infty} c_k e^{ik\Omega t}$$

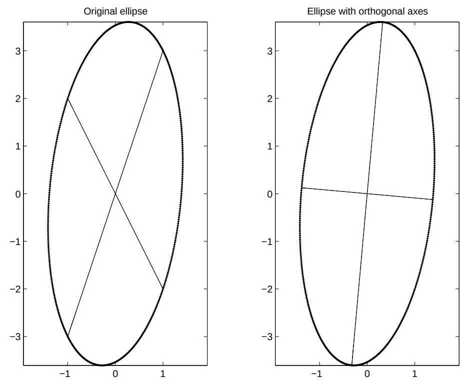

Figure 4: Two representations of the same ellipse. The right one has orthogonal axes

or the equivalent trigonometric form

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos k\Omega t + b_k \sin k\Omega t$$

With the newly acquired knowledge on complex ellipses, we see that Fourier analysis of a periodic complex function results in a sum of ellipses of increasing frequency. This helps a lot in understanding how shapes are formed.

Example

Consider the equation $y = r_1 \cos\theta t + ir_2 \sin\theta t$ with $r_1 = 1 + 3i$ and $r_2 = 2 + i$. Therefore, we have $a_1 = 1$, $b_1 = 3$, $a_2 = 2$ and $b_2 = 1$. We now wish to calculate the phase shift δ in the equation $y = r_1 \cos(\theta t + \delta) + ir_2 \sin(\theta t + \delta)$ such that we get a standard ellipse with orthogonal axes. From the previous section we have

$$\delta = \frac{1}{2} \arctan\left(\frac{2(1 \cdot 1 - 2 \cdot 3)}{2^2 + 1^2 - 1^2 - 3^2}\right) = \frac{1}{2} \arctan 2 \approx 0.55$$

We are now able to calculate the new complex numbers c_1 and c_2,

$$c_1 = r_1 \cos p + ir_2 \sin p \approx 0.32 + 3.6i$$

$$c_2 = r_2 \cos p + ir_1 \sin p \approx 0.12 + 1.4i \approx 0.38c_1$$

The new equation is

$$y = (0.32 + 3.6i) \cos\theta t + i(0.12 + 1.4i) \sin\theta t$$

Figure 4 shows the original ellipse next to its rectified counterpart. Note that the ellipses have equal shape.

2.2 Fourier Representation of Shapes

Most two-dimensional shapes consist of outlines from projections of three-dimensional objects. Many of these consist of a single closed curve. If not, the object

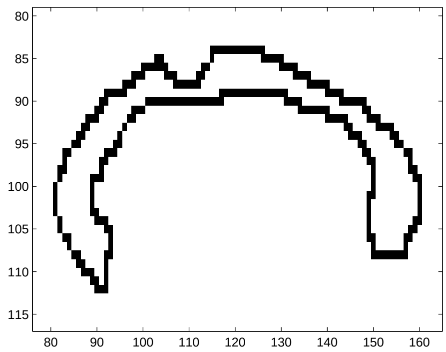

Figure 5: A delineated corpus callosum

may be divided into several closed curves, and open curves can be closed. The remainder of this text will focus on single closed curves. As we have seen, these can be described as a complex periodic function. Usually, the function is not known, instead we have the coordinates of discrete points along the shape outline. Figure 5 depicts a closed curve marking the outline of the *corpus callosum*. This outline may be treated as a discretized periodic complex function. It is therefore possible to perform Fourier analysis on the data using the DFT [5]. Some straight-forward MATLAB-code that follows equation 14 closely is

```
f = complex(xCoordinates, yCoordinates);
N = length(f);
k = 0:N - 1;
j = 0:N - 1;
F = 1/N*exp(-2*pi*i*j*k'/N)*f;
```

F now holds the Fourier coefficients F_j of the shape f. Now let's try to relate these to the Fourier coefficients c_k of equations 6 and 7. The first coefficient, F_0, is the result of setting $j = 0$ which reduces the DFT equation to $F_0 = 1/N \sum_{k=0}^{N-1} f_k = \bar{f}$. The first component is clearly the centroid of the points in f. We also notice that $F_0 = c_0$. The following table shows the other correspondences.

F_0	F_1	F_2	F_3	$\ldots$	F_k	F_{k+1}	$\ldots$	F_{N-3}	F_{N-2}	F_{N-1}
c_0	c_1	c_2	c_3	$\ldots$	c_k	c_{-k}	$\ldots$	c_{-3}	c_{-2}	c_{-1}

The seemingly strange ordering is a result of the DFT summation going from 0 to $N - 1$. This is equivalent to the more logical summation range $-N/2 + 1 : N/2$ (N even). The out-of-range frequencies are folded and comes out as the negative frequencies. This effect is called **aliasing** [1]. The $0 : N - 1$ range is preferred since this handles an odd or even number of points similarly, and since he first coefficient F_0 will always be a pure translation. If another form of the DFT is preferred, the following table (from [1]) gives an overview over possible choices.

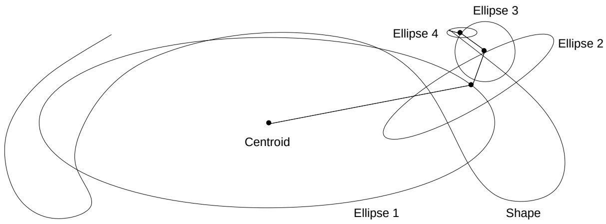

Figure 6: A shape being drawn using a centroid and four ellipses, a total of 9 Fourier coefficients

Type	Comments	Summation range	Highest frequency index
centered	N even, N points	$-N/2+1 : N/2$	$N/2$
centered	N odd, $N-1$ points	$-(N-1)/2+1 : (N-1)/2$	$(N-1)/2$
non-centered	N even, N points	$0 : N-1$	$N/2$
centered	N even, $N+1$ points	$-N/2 : N/2$	$\pm N/2$
centered	N odd, N points	$-(N-1)/2 : (N-1)/2$	$\pm(N-1)/2$
non-centered	N odd, N points	$0 : N-1$	$(N \pm 1)/2$

Apparently, the Fourier transform of f consists of the centroid of f together with coefficients $[c_k,\, c_{-k}]$ of a series of ellipses of increasing frequency. Figure 6 shows a graphical interpretation of this property.

Shape Components of the Fourier Transform

The great advantage of the Fourier representation is the level-of-detail interpretation of the frequency range. Low frequencies represents the coarse structure of an object, while higher frequencies add the details. A certain frequency is always dependent on all frequencies below to approximate a shape. Looking at figure 6, this property is apparent. Figure 7 shows the outline of a corpus callosum represented by one, two, five and 15 components (ellipses). The center of gravity, or origin, of the shape is shown as a point.

2.3 Variance of Fourier Coefficients

In shape analysis, we deal mostly with a set of objects. If any statistical analysis is to be meaningful, all objects must have a common representation, where the only inter-object differences are derived from the actual shape. Differences due to location, scale and rotation need to be filtered out, among other things. This section investigates these differences, how they can be removed, and the effect they have on the Fourier coefficients. The following table (from [2]) summarizes how the Fourier coefficients are altered under certain transformations.

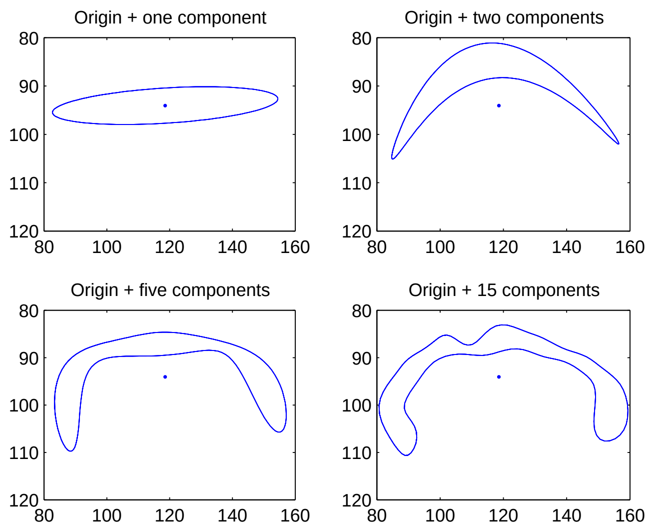

Figure 7: Shape components of a corpus callosum outline

Transformation	Boundary	Fourier Descriptor
Identity	$f(t)$	c_k
Rotation	$e^{i\theta}f(t)$	$e^{i\theta}c_k$
Translation	$f(t)+T$	c_0+T
Scaling	$\alpha f(t)$	αc_k
Starting point	$f(t+\delta)$	$e^{ik\Omega\delta}c_k = e^{2\pi ik\delta/N}c_k$

Looking at this table, we realize that the inverse DFT under a general transformation can be formulated as

$$f_k = F_0 + T + \sum_{j=1}^{N-1} \alpha e^{i\theta} e^{2\pi ik\delta/N} e^{2\pi ijk/N} F_j =$$

$$F_0 + T + \alpha e^{i\theta} e^{2\pi ik\delta/N} \sum_{j=1}^{N-1} e^{2\pi ijk/N} F_j$$

We will now go on to discuss and/or prove the results from the table above.

Sensitivity to Point Ordering and Starting Point

From our discussion above, we remember that a typical object outline marked by a number of points is a discretization of a complex, periodic function. It is therefore clear that the points must be ordered by arc-length. If the points are scattered randomly along the outline, the resulting curve will visit these points in turn, and the shape will make no sense.

The remaining free parameter is where on the outline the curve is drawn from. This amounts to a phase shift of the curve function. Equation 6 becomes

$$c_k = \frac{1}{T}\int_P e^{-ik\Omega t} f(t+\delta)dt \Leftrightarrow [\hat{t} = t+\delta] \Leftrightarrow$$

$$c_k = \frac{1}{T}\int_P e^{-ik\Omega(\hat{t}-\delta)} f(\hat{t})dt \Leftrightarrow c_k = e^{ik\Omega\delta}\frac{1}{T}\int_P e^{-ik\Omega(\hat{t})} f(\hat{t})dt$$

This shows that Fourier coefficients of the same frequency from the same contour drawn with different starting points only differ by a factor $e^{ik\Omega\delta}$.

Sensitivity to Number of Points

If the points along the boundary are subsampled, the Fourier coefficients will change, but only slightly. As long as the remaining data points describe the outline reasonably well, the origin coefficient c_0 and the ellipse coefficients $[c_k, c_{-k}]$ will be more or less unaffected.

Sensitivity to Translation

As we already have seen, the first Fourier coefficient c_0 contains the origin of a shape. If a shape $f(t)$ is translated by $f(t) + T$ where $T \in \Im$, this will transfer directly to c_0 as $c_0 + T$. All other coefficients will remain unchanged.

Sensitivity to Scaling

A scaling of a shape $f(t)$ is performed by multiplying f by a single real number α. The scaled function is $\alpha f(t)$. Inserting this number into equation 6 we get

$$\hat{c}_k = \frac{1}{T}\int_P e^{-ik\Omega t}\alpha f(t)dt \Longleftrightarrow \hat{c}_k = \alpha c_k$$

This shows that a scaling α transfers directly onto the Fourier coefficients.

Sensitivity to Rotation

A rotation of $f(t)$ can be carried out by multiplication by a complex number $e^{i\theta}$. This assumes that $f(t)$ is centered around the origin. As before, we insert this into equation 6.

$$\hat{c}_k = \frac{1}{T}\int_P e^{-ik\Omega t}e^{i\theta} f(t)dt \Longleftrightarrow \hat{c}_k = e^{i\theta}c_k$$

As shown, rotation also transfers directly onto the Fourier coefficients.

2.4 Invariance Methods

The Fourier coefficients may be used directly for e.g. classification purposes, but we wish to rid ourselves of the hassle of removing the transformations discussed above. One simple measure which is invariant to all four transformation is

$$\frac{\hat{c}_{k+1}\hat{c}_{-(k-1)}}{\hat{c}_1^2} \tag{23}$$

where $\hat{c}_k = \alpha e^{i\theta}e^{2\pi ik\delta/N}c_k$. Inserting this into the equation above, we get

$$\frac{\hat{c}_{1+k}\hat{c}_{1-k}}{\hat{c}_1^2} = \frac{\alpha e^{i\theta}e^{2\pi i(1+k)\delta/N}c_{k+1}\alpha e^{i\theta}e^{2\pi i(1-k)\delta/N}c_{1-k}}{(\alpha e^{i\theta}e^{2\pi i\delta/N}c_1)^2} =$$

$$\frac{e^{2\pi i\delta/N}e^{2\pi ik\delta/N}c_{k+1}e^{2\pi i\delta/N}e^{-2\pi ik\delta/N}c_{1-k}}{(e^{2\pi i\delta/N}c_1)^2} = \frac{c_{1+k}c_{1-k}}{c_1^2}$$

This was used in [3] for hand printed character recognition.

References

[1] William L. Briggs and Van Emden Henson. *The DFT: An Owner's Manual for the Discrete Fourier Transform*. SIAM, 1995.

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[3] G.H. Granlund. Fourier preprocessing for hand print character recognition. *IEEE Transactions on Computers*, C-21(2):195–201, 1972.

[4] Sven Spanne. *Konkret Analys*. KFS AB, 3rd edition, 1997.

[5] C.T. Zahn and R.Z.L. Roskies. Fourier descriptors for plane closed curves. *IEEE Transactions on Computers*, c-21(3):269–81, 1972.