Prewhitening for Narrow-Band Noise in Subspace Methods for Noise Reduction

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***Abstract*— A fundamental issue in connection with subspace methods for noise reduction is that the covariance matrix for the noise is required to have full rank, in order for the prewhitening step to be defined. However, there are important cases where this requirement is not fulfilled, typically when the noise has narrow-band characteristics, including the case of tonal noise. We extend the concept of prewhitening to include the case when the noise covariance matrix is rank deficient, using a weighted pseudoinverse and the quotient SVD, and we show how to formulate a general rank-reduction algorithm that works also for rank deficient noise. We also demonstrate how to formulate this algorithm by means of a quotient ULV decomposition, which allows for faster computation and updating. Finally we apply our algorithm to a problem involving a speech signal contaminated by narrow-band noise.**

I. Introduction

SUBSPACE methods and rank reduction have emerged as important techniques for noise reduction in many applications, including speech enhancement, see [4], [5], [12], [13], [19]. In all these applications there is a fundamental restriction, namely, that the covariance matrix for the noise must have full rank; this is necessary because the prewhitening step essentially consists of post multiplying the signal-matrix with the inverse of the Cholesky factor of the noise covariance matrix.

However, there also exist important applications where the requirement of full rank is not satisfied, for example, in the case of narrow-band noise. It is therefore preferable to have a general method which is guaranteed to work in all cases, independently of the rank of the noise correlation matrix. Hence it is of interest to extend the concept of prewhitening for subspace methods, such that it can also handle rank-deficient noise covariance matrices – in such a way that the new technique is identical to standard prewhitening in the full-rank case.

The underlying mathematics of a general prewhitener for a rank-deficient noise covariance matrix was recently developed in [10]. In that paper the quotient (or generalized) SVD was used to demonstrate that the prewhitening matrix, to be post-multiplied to the signal matrix, should be a weighted pseudoinverse. An algorithm suited for efficient updating, based on the rank-revealing quotient ULV decomposition, was also outlined in [10].

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The purpose of this paper is to complete the theory of the rank-deficient prewhitening algorithm, to give a filter-bank interpretation of the algorithm (similar to the work in [11]), and to demonstrate the usefulness of the new algorithm in connection with realistic signals. Our work makes use of a weighted pseudoinverse which originates in work by Mitra and Rao [16] and Eldén [7], and which is related to the oblique projection, a tool that is currently receiving attention in the signal processing literature [1], [2].

One of the features of subspace methods is that they do not require detailed models of neither the signal nor the noise. In the present work there are no assumptions about the rank of neither the signal covariance matrix nor the noise covariance matrix.

Our paper is organized as follows. In Section II we discuss full-rank and low-rank prewhitening in terms of the quotient singular value decomposition, and we give a FIR filter interpretation of the algorithm. In Section III we introduce the rank-revealing quotient ULV decomposition, which is a computationally attractive alternative to the quotient SVD, and we demonstrate how to formulate the rank-deficient prewhitening in terms of this decomposition. Section IV discusses some technical issues related to a special case where one of the matrices in the decompositions is singular, and we demonstrate that in practice this case does not lead to difficulties with the algorithms. Finally, in Section V we illustrate the performance of our algorithms by an example involving a speech signal with narrow-band low-rank noise.

II. Prewhitening for Subspace Methods

The fundamental idea in subspace methods is, from a given noisy signal, to determine a (low-dimensional) signal subspace which contains most of the desired signal. This typically involves the formation of a Toeplitz or Hankel matrix X, in such a way that the cross product $X^T X$ is a scaled estimate of the signal's covariance matrix, and such that the desired signal subspace is a proper subspace of the column space (range) of X.

If the noise in the signal is additive and white, and if it is uncorrelated with the pure signal, then the signal subspace simply consists of the principal left singular vectors of the signal matrix X; see, e.g., [3] and [18].

A. Full-Rank and Low-Rank Prewhiteners

If the noise is not white, then it is still possible to use the same basic approach provided that the covariance matrix C for the noise has full rank. The key idea is to use prewhitening; if C has the Cholesky factorization $C = R^T R$ then the matrix

$X\,R^{-1}$ represents a new signal whose noise component is white, and the principal left singular vectors of this matrix form the desired signal subspace. A matrix $\widehat{X}_{\rm filt}$ that represents the filtered signal is then obtained by filtering the singular values of $X\,R^{-1}$ followed by right multiplication with R ("dewhitening").

We emphasize that this approach requires that we are able to estimate C, typically by forming a "noise matrix" E from samples of pure noise (similar to X) such that $E^T E$ is a scaled estimate of C. This is often possible in speech processing applications where the noise can be recorded in speech-less frames.

The prewhitening breaks down when the noise covariance matrix C is rank deficient and the matrix R^{-1} therefore no longer exists. We shall now demonstrate that in connection with rank-deficient noise a weighted pseudoinverse takes the place of R^{-1}.

The basis for our analysis is the *quotient singular value decomposition* (QSVD)[1] of the two matrices X and E. Let $X \in \mathbb{R}^{m_X \times n}$ and $E \in \mathbb{R}^{m_E \times n}$ with $m_X \geq n$ and $m_E \geq n$, and assume that $\mathrm{rank}(E) = p \leq n$. Moreover, assume that the matrix $(\, X^T \,,\, E^T \,)$ has full rank. Then the QSVD takes the form

$$X \;=\; Q_X \begin{pmatrix} \Sigma & 0 \\ 0 & I_{n-p} \end{pmatrix} \Theta^T \qquad (1)$$

$$E \;=\; Q_E \,(\, M \,,\, 0 \,)\, \Theta^T \qquad (2)$$

where $Q_X \in \mathbb{R}^{m_X \times n}$ and $Q_E \in \mathbb{R}^{m_E \times p}$ have orthonormal columns; $\Theta \in \mathbb{R}^{n \times n}$ is a nonsingular matrix; and Σ and M are diagonal $p \times p$ matrices satisfying $\Sigma^2 + M^2 = I_p$. It is convenient to partition the two matrices Q_X and Θ into submatrices

$$Q_X = (\, Q_{X1} \,,\, Q_{X2} \,), \qquad \Theta = (\, \Theta_1 \,,\, \Theta_2 \,) \qquad (3)$$

with p and $n-p$ columns, respectively.

The case where E has full rank, i.e., $p = n$, was studied in [13]. Here the three submatrices Q_{X2}, Θ_2 and I_{n-p} vanish, and the QSVD takes the simpler form

$$X = Q_X \,\Sigma\, \Theta^T, \qquad E = Q_E \, M\, \Theta^T.$$

In this case, E has the QR factorization $E = Q\,R$ where R is the above-mentioned Cholesky factor, and the Moore-Penrose pseudoinverse of E is given by

$$E^\dagger = R^{-1} Q^T = \left(\Theta^T\right)^{-1} M^{-1} Q_E^T.$$

Hence, the matrix quotient

$$X\,E^\dagger = X\,R^{-1} Q^T = Q_X \,(\Sigma M^{-1})\, Q_E^T,$$

has the same singular values and left singular vectors as the prewhitened matrix $X\,R^{-1}$ introduced above. Moreover, we see that when using this prewhitener, the QSVD immediately provides the SVD of the matrix quotient $X\,E^\dagger$, and thus the desired signal subspace is immediately available from the QSVD in the form of the vectors of Q_X. Neither R^{-1} nor $E^\dagger$ is needed. To see that the prewhitened noise is indeed white, we note that $E\,E^\dagger = Q_E\,Q_E^T$ represents a signal with covariance matrix I_n.

To reconstruct the filtered matrix $\widehat{X}_{\rm filt}$ via the QSVD, we first filter the singular values of the matrix quotient $X\,E^\dagger$ and then right-multiply with E. Inserting the QSVD it is easy to see that the complete process can be written as

$$\widehat{X}_{\rm filt} = Q_X \,\widehat{\Psi}\, \Sigma\, \Theta^T,$$

where $\widehat{\Psi}$ denotes a diagonal filter matrix [13]. It follows immediately that the covariance matrix for the filtered signal is given by

$$\widehat{X}_{\rm filt}^T \widehat{X}_{\rm filt} = \Theta\, \widehat{\Psi}^2\, \Sigma^2\, \Theta^T$$

As an example, the least squares (LS) estimate of rank k is obtained by choosing $\widehat{\Psi}$ such that the k largest elements of Σ are retained while the rest are discarded.

We now return to the case with a rank-deficient noise matrix E, still assuming that $(\, X^T \,,\, E^T \,)$ has full rank. By means of the QSVD (1)–(2) we can express the scaled covariance matrix of the observed signal as

$$\begin{aligned} X^T X \;&=\; \Theta \begin{pmatrix} \Sigma & 0 \\ 0 & I_{n-p} \end{pmatrix}^2 \Theta^T \\ &=\; \Theta_1\, \Sigma^2\, \Theta_1^T + \Theta_2\, \Theta_2^T. \end{aligned}$$

This expression shows that we can consider the observed signal as a sum of two signal components with covariance matrices $\Theta_1\,\Sigma^2\,\Theta_1^T$ and $\Theta_2\,\Theta_2^T$, respectively. Moreover, since the scaled covariance matrix for the noise is

$$E^T E = \Theta_1\, M^2 \Theta_1^T,$$

we see that the first signal component is associated with the same p-dimensional subspace $\mathcal{R}(\Theta_1)$ as the rank-deficient noise, while the second component is associated with the subspace $\mathcal{R}(\Theta_2)$. These two subspaces are disjoint but not orthogonal.

The key observation is that the second signal component, which lies in $\mathcal{R}(\Theta_2)$, is not influence by the noise. Only the first component, lying in $\mathcal{R}(\Theta_1)$, is affected by the noise, and only this component needs to be filtered. Hence the covariance matrix for the filtered signal must take the form

$$\widehat{X}_{\rm filt}^T \widehat{X}_{\rm filt} = \Theta_1\, \widehat{\Psi}^2\, \Sigma^2\, \Theta_1^T + \Theta_2\, \Theta_2^T,$$

where again $\widehat{\Psi}$ is a diagonal $p \times p$ filter matrix. This, in turn, corresponds to writing the reconstructed matrix as

$$\widehat{X}_{\rm filt} = Q_X \begin{pmatrix} \widehat{\Psi}\Sigma & 0 \\ 0 & I_{n-p} \end{pmatrix} \Theta^T. \qquad (4)$$

Again, the necessary filtering is easily accomplished via the QSVD, and we see that the desired signal subspace is spanned by columns of the matrix Q_X.

The above analysis also sheds light on the existence and form of a matrix which can take the place of the full-rank prewhitening matrices R^{-1} and $E^\dagger$. Inserting the QSVD it follows immediately that if we multiply X from the right with the matrix

$$E_X^\dagger = \left(\Theta^T\right)^{-1} \begin{pmatrix} M^{-1} \\ 0 \end{pmatrix} Q_E^T \qquad (5)$$

[1]The QSVD is also known as the generalized SVD (GSVD); it is computed in Matlab by means of the function `gsvd`.

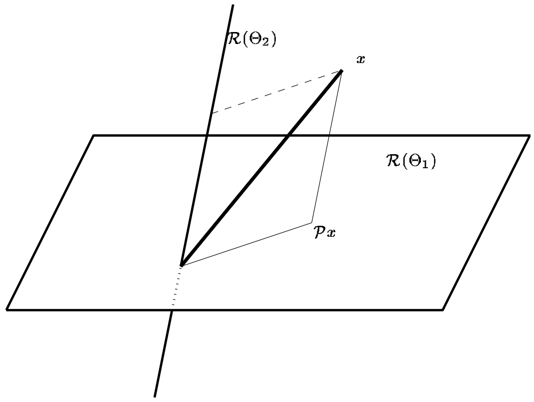

Fig. 1. Illustration of the oblique projection. The subspaces $\mathcal{R}(\Theta_1)$ and $\mathcal{R}(\Theta_2)$ are represented by a plane and a line, intersecting at an angle less than 90°. Any vector x can be written as a sum $x = \mathcal{P}x + (I - \mathcal{P})x$, where $\mathcal{P}x \in \mathcal{R}(\Theta_1)$ is the oblique projection of x on $\mathcal{R}(\Theta_1)$ along $\mathcal{R}(\Theta_2)$.

then we obtain

$$X\,E_X^{\dagger} = Q_X \begin{pmatrix} \Sigma M^{-1} \\ 0 \end{pmatrix} Q_E^T = Q_{X1}\,(\Sigma M^{-1})\,Q_E^T \qquad (6)$$

which is the desired SVD of $X\,E_X^{\dagger}$, from which we construct the filter matrix $\widehat{\Psi}$. The matrix $E_X^{\dagger}$ in (5) is known as the X-*weighted pseudoinverse* of E; see [7] for details about the weighted pseudoinverse and its relation to the QSVD.

Again we must verify that the prewhitened noise is white, and therefore we consider the matrix $E\,E_X^{\dagger}$. Inserting the QSVD we obtain

$$E\,E_X^{\dagger} = Q_E\,(\,M\,,\,0\,)\,\Theta^T\left(\Theta^T\right)^{-1}\begin{pmatrix} M^{-1} \\ 0 \end{pmatrix} Q_E^T = Q_E\,Q_E^T$$

which represents white noise in a p-dimensional subspace. We have thus shown that the prewhitener $E_X^{\dagger}$ indeed produces a new signal with white noise. Moreover, the filtered component of the reconstructed signal lies in the p-dimensional subspace $\mathcal{R}(\Theta_1)$, while the unfiltered component of the reconstructed signal lies in the subspace $\mathcal{R}(\Theta_2)$ of dimension $n - p$.

The above theory generalizes the existing theory from [13], and in the full-rank case the two methods are identical because $E_X^{\dagger} = E^{\dagger}$ when E has full column rank.

We note that the matrix $Q_{X1}\,\Sigma\,\Theta_1^T$, which represents the signal component in $\mathcal{R}(\Theta_1)$, can be written as

$$Q_{X1}\,\Sigma\,\Theta_1^T = X\,E_X^{\dagger}E$$

and that the matrix

$$\mathcal{P} = \left(E_X^{\dagger}E\right)^T = \Theta\begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}\Theta^{-1}$$

is the projection matrix for the oblique projection onto $\mathcal{R}(\Theta_1)$ along $\mathcal{R}(\Theta_2)$. Figure 1 illustrates an oblique projection. See, e.g., [2] concerning the use of oblique projections in signal processing.

It is the use of this oblique projection that allows us to prewhiten with rank-deficient noise via a splitting of $\mathbb{R}^n$ into the noise and noise-free subspaces, precisely in such a way that the signal component in the noise-free subspace $\mathcal{R}(\Theta_2)$ is left unfiltered, while only the component in the noise subspace $\mathcal{R}(\Theta_1)$ is filtered. The Moore-Penrose pseudoinverse $E^{\dagger}$ (for which the matrix $(E^{\dagger}E)^T = E^{\dagger}E$ is an orthogonal projection) does not provide this subspace splitting.

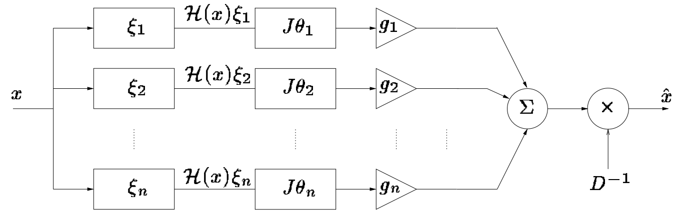

Fig. 2. FIR filter interpretation of the QSVD algorithm. The rightmost block is a time varying amplifier. Replace “D” with “Dˆ{-1}” and swap ξ and θ.

B. FIR Filter Interpretation of the QSVD Algorithm

As shown in [6] and [11], subspace-based noise reduction algorithms can be interpreted by means of FIR filters defined by the right singular vectors of the SVD or QSVD. Here we extend these results to the rank-deficient case considered in this paper.

First we need to introduce some notation from [11]. Given a vector x of length N, we define the Hankel matrix $\mathcal{H}(x) \in \mathbb{R}^{m\times n}$ with $m + n - 1 = N$ by

$$\mathcal{H}(x) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & x_{n+1} \\ \vdots & \vdots & \iddots & \vdots \\ x_m & x_{m+1} & \cdots & x_N \end{pmatrix}.$$

This is precisely the matrix X used throughout our paper. We also need the augmented Hankel matrix

$$\mathcal{H}_{\mathrm{aug}}(x) = \begin{pmatrix} 0 & 0 & \cdots & x_1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & x_1 & \cdots & x_{n-1} \\ & & \mathcal{H}(x) & \\ x_{m+1} & x_{m+2} & \cdots & 0 \\ \vdots & \vdots & \iddots & \vdots \\ x_N & 0 & \cdots & 0 \end{pmatrix}$$

and the $N \times N$ diagonal matrix

$$D = \mathrm{diag}(1, 2, 3, \ldots, n, n, \ldots, n, \ldots, 3, 2, 1).$$

Finally we write

$$\Theta = (\theta_1, \ldots, \theta_n) \quad \text{and} \quad \Theta^{-T} = (\xi_1, \ldots, \xi_n).$$

Then it follows from the theory in Section IV of [11] that if the reconstructed signal $\hat{x}$ is obtained by averaging along the antidiagonals of $\widehat{X}_{\mathrm{filt}}$ then this signal can be expressed as

$$\hat{x} = D^{-1}\sum_{i=1}^{n} g_i\,\mathcal{H}_{\mathrm{aug}}\big(\mathcal{H}(x)\xi_i\big)\,J\,\theta_i \qquad (7)$$

where J is the exchange matrix. We have introduced the gains $g_i = \widehat{\Psi}_{ii}$ for $i \le p$ and $g_i = 1$ otherwise. Referring to Eq. (4), the first p filtered components correspond to $Q_{X1}\widehat{\Psi}\Sigma\Theta_1^T$ while the last $n - p$ unfiltered components correspond to $Q_{X2}\Theta_2^T$.

TABLE I

AMPLITUDES a_j OF THE PURE SIGNAL AND THE PURE NOISE, AND AMPLITUDES $\hat{a}_j^{(k)}$ OF THE RECONSTRUCTION.

j	1	2	3	4	5	6
f_j	25	50	21	30	45	57
a_j	3.00	5.00	0.80	0.80	0.80	0.80
$\hat{a}_j^{(0)}$	1.58	2.51	0.04	0.11	0.12	0.04
$\hat{a}_j^{(1)}$	1.82	4.36	0.01	0.09	0.23	0.18
$\hat{a}_j^{(2)}$	3.02	5.03	0.09	0.24	0.41	0.25
$\hat{a}_j^{(3)}$	3.01	4.97	0.23	0.30	0.44	0.31
$\hat{a}_j^{(4)}$	2.99	4.99	0.48	0.58	0.52	0.40

The interpretation of the result in Eq. (7) is that the reconstructed signal consists of a weighted sum (with weights g_i) of n filtered signals. Each of these filtered signals is obtained by first filtering the original signal x with a FIR filter whose coefficients are the elements of ξ_i, and then filtering this intermediate signal with another FIR filter whose coefficients are the elements of the vector θ_i in reverse order. The complete process is illustrated in Fig. 2.

C. Examples of Low-Rank Prewhitening

Before turning to computationally efficient methods for working with full-rank and low-rank prewhitening, it is worthwhile to illustrate the ideas of the low-rank prewhitening approach described above. We do this by two simple examples.

Sinusoids in Low-Rank Noise. The pure signal $\bar{x}$ has length $N = 128$ and consists of a sum of two sampled sinusoids

$$\bar{x}_i = a_1 \sin\!\left(i \tfrac{2\pi}{N} f_1\right) + a_2 \sin\!\left(i \tfrac{2\pi}{N} f_2\right), \quad i = 1, \ldots, N$$

whose amplitudes and frequencies are given in Table I. The noise is an interfering signal consisting of a sum of four sinusoids

$$e_i = \sum_{j=3}^{6} a_j \sin\!\left(i \tfrac{2\pi}{N} f_j + \phi_j\right), \quad i = 1, \ldots, N$$

whose amplitudes and frequencies are also given in Table I while their phases are chosen randomly. The observed signal is $x = \bar{x} + e$. The signal matrix X is the 119×10 Hankel matrix defined by the vector x, and this matrix has full rank, i.e., $\mathrm{rank}(X) = n = 10$. The matrix corresponding to the pure signal $\bar{x}$ has rank 4. Finally, the matrix E for the pure noise has $\mathrm{rank}(E) = p = 8$, i.e., E is rank deficient.

Since the pure signal matrix has rank 4, we expect that a signal subspace of dimension 4 will lead to the best reconstruction. In our experiments, we choose the LS filter matrix $\widehat{\Psi}$ that selects the k largest values of Σ, thus ensuring that $\mathrm{rank}(\widehat{X}_{\mathrm{filt}}) = n - p + k = 2 + k$. Finally the reconstructed signal is obtained by averaging along the antidiagonals of $\widehat{X}_{\mathrm{filt}}$.

The amplitudes $\hat{a}_j^{(k)}$ of the reconstructed signal at the six relevant frequencies are listed in the bottom rows of Table I. For $k = 0$ and $k = 1$ the dimension of the signal subspace is not large enough to capture the desired signal, while it is well reconstructed for $k = 2$, 3 and 4. As k increases the signal subspace captures an increasing amount of the low-rank noise. The optimal reconstruction is obtained for $k = 2$, as expected, where the errors in $a_1^{(2)}$ and $a_2^{(2)}$ are less than 1%, while all four noise amplitudes $\hat{a}_j^{(2)}$ for $j = 3, 4, 5, 6$ are damped.

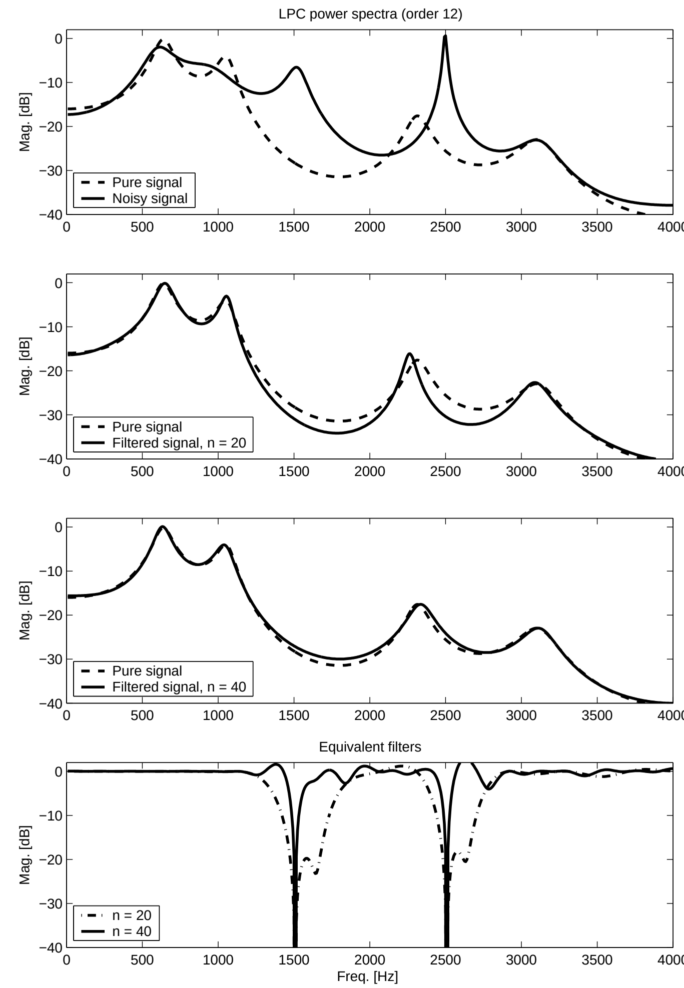

Fig. 3. LPC power spectra of order 12 and equivalent filters. Top: pure and noisy speech signals, the noise consisting of two sinusoids at 1.5 kHz and 2.5 kHz. Middle: pure and filtered signals using $k = 0$, for $n = 20$ and $n = 40$. Bottom: equivalent filters for $n = 20$ and $n = 40$; both are notch filters located at the frequencies of the interfering noise signal.

For $k = 2$ the dimensions of the estimated and the pure signal subspaces are both equal to 4, and we can compute the angle between these two subspaces. The angle is 0.24 radians (about $14°$) which is quite small compared to the large SNR in the noisy signal.

Voiced Speech in Low-Rank Noise. The pure signal is a voiced speech signal of length $N = 160$ and sampled at 8 kHz, while the low-rank noise is an interfering signal consisting of a sum of two sinusoids

$$e_i = \sin(i\, 2\pi f_1\, \Delta t) + \sin(i\, 2\pi f_2\, \Delta t), \quad i = 1, \ldots, N.$$

The two frequencies $f_1 = 1.5$ kHz and $f_2 = 2.5$ kHz of the noise are selected such that the former is between the second and the third formant, while the latter is close to the fourth formant. The signal-to-noise ratio is 5 dB.

The data matrix X and the noise matrix E are again Hankel matrices with n columns. The noise matrix has rank $p = 4$ while the data matrix has full rank. In order to suppress the interference as much as possible, we choose $k = 0$, i.e., our reconstructed signal lies solely in the noise-free subspace $\mathcal{R}(\Theta_2)$ of dimension $n-4$. Moreover, we use $n = 20$ and $n = 40$ to illustrate the relation between matrix dimensions and noise-reduction performance.

The 12th-order LPC spectra of the pure signal, the observed noisy signal and the two reconstructed signals are shown in Fig. 3, together with the equivalent filters obtained from Eq. (7) with weights

$$g_i = 0,\ i = 1,2,3,4, \qquad g_i = 1,\ i = 5,\ldots,n.$$

Clearly we are able to suppress the noise by the QSVD approach, which effectively puts two notch filters at the frequencies f_1 and f_2 of the interfering noise signal. The width of these filters decreases as n increases.

III. Implementation by The Rank-Revealing Quotient ULV Decomposition

Although the QSVD is ideal for defining the weighted pseudoinverse and the low-rank prewhitening algorithm, the QSVD algorithm may be too computationally demanding for realtime applications. Hence we need an alternative decomposition which is easier to compute and update, and which yields good approximations to the quantities in the QSVD. The rank-revealing quotient ULV (QULV) decomposition, also referred to as the ULLV decomposition, is such a tool.

A. The ULV and QULV Decompositions

Before introducing the QULV, we first briefly describe the ULV decomposition [17] which was introduced as a computationally attractive alternative to the SVD. The ULV decomposition of X takes the form

$$X = U\,L\,V^T = (\,U_r\,,\,U_o\,)\begin{pmatrix} L_r & 0 \\ F & G \end{pmatrix}(\,V_r\,,\,V_o\,)^T$$

where U and V have orthonormal columns, L is lower triangular, and the numerical rank of X is revealed in L in the sense that both norms $\|F\|_2$ and $\|G\|_2$ are small. Hence the matrix $U_r\,L_r\,V_r^T$ is a low-rank approximation to X, and the range of U_r is an approximation to the desired signal subspace. The ULV decomposition can therefore replace the SVD in subspace algorithms for the white-noise case.

The QULV decomposition[2] [10], [15] factors the two matrices X and E as products of a left orthogonal matrix, one or two lower triangular matrices, and a common right orthogonal matrix. Specifically, the QULV decomposition takes the form

$$X = U_X\,L\begin{pmatrix} \bar{L} & 0 \\ 0 & I_{n-p} \end{pmatrix}V^T \tag{8}$$

$$E = U_E\,(\,\bar{L}\,,\,0\,)\,V^T \tag{9}$$

in which $U_X \in \mathbb{R}^{m_X\times n}$, $U_E \in \mathbb{R}^{m_E\times p}$ and $V \in \mathbb{R}^{n\times n}$ have orthonormal columns, while $L \in \mathbb{R}^{n\times n}$ and $\bar{L} \in \mathbb{R}^{p\times p}$ are lower triangular.[3] Similar to the QSVD, it is convenient to work with the partitionings

$$U_X = (\,U_{X1}\,,\,U_{X2}\,), \quad V = (\,V_1\,,\,V_2\,)$$

and

$$L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}$$

where U_{X1} and V_1 have p columns, and L_{11} is $p\times p$.

The similarity between the QULV and the QSVD is perhaps better revealed by rewriting the QULV decomposition in the form

$$X = U_X\begin{pmatrix} L_{11} & 0 \\ 0 & I_{n-p} \end{pmatrix}(\,\bar{\Theta}_1\,,\,\bar{\Theta}_2\,)^T$$

$$E = U_E\,(\,I_p\,,\,0\,)\,(\,\bar{\Theta}_1\,,\,\bar{\Theta}_2\,)^T$$

where we have defined the matrix $\bar{\Theta} = (\,\bar{\Theta}_1\,,\,\bar{\Theta}_2\,)$ with

$$\bar{\Theta}_1 = V_1\bar{L}^T$$

and

$$\bar{\Theta}_2 = V_1\bar{L}^T L_{21}^T + V_2 L_{22}^T = \bar{\Theta}_1 L_{21}^T + V_2 L_{22}^T.$$

The column spaces of the QULV matrices U_{X1}, U_{X2}, U_E, $\bar{\Theta}_1$ and $\bar{\Theta}_2$ are approximations to the column spaces of the corresponding QSVD matrices Q_{X1}, Q_{X2}, Q_E, Θ_1 and Θ_2, respectively.

When E has full rank, the matrices U_{X2}, I_{n-p} and V_2 vanish and the QULV takes the simpler form $X = U_X\,L\,\bar{L}\,V^T$ and $E = U_E\,\bar{L}\,V^T$. This is the original version of the QULV from [14].

The QULV decomposition is rank-revealing in the following sense. As shown in [10] the X-weighted pseudoinverse of E can be written in terms of the QULV decomposition as

$$E_X^\dagger = \left(\bar{\Theta}^T\right)^{-1}\begin{pmatrix} I_p \\ 0 \end{pmatrix}U_E^T = V\begin{pmatrix} \bar{L}^{-1} \\ -L_{22}^{-1}L_{21} \end{pmatrix}U_E^T.$$

Consequently the matrix quotient $X\,E_X^\dagger$ can be expressed in terms of the QULV factors simply as

$$X\,E_X^\dagger = U_{X1}\,L_{11}\,U_E^T,$$

which is a rank-revealing ULV decomposition of $X\,E_X^\dagger$. Hence the numerical rank of $X\,E_X^\dagger$ is immediately revealed in the $p\times p$ triangular submatrix L_{11} in the QULV. To incorporate filtering we must filter or truncate this submatrix. Hence, the QULV-based reconstruction takes the form

$$\widehat{X}_{\rm filt} = U_X\begin{pmatrix} \widehat{\Psi}\,L_{11} & 0 \\ 0 & I_{n-p} \end{pmatrix}(\,\bar{\Theta}_1\,,\,\bar{\Theta}_2\,)^T = U_{X1}\,\widehat{\Psi}\,L_{11}\,\bar{\Theta}_1^T + U_{X2}\,\bar{\Theta}_2^T,$$

where $\widehat{\Psi}$ is the diagonal filter matrix.

The covariance matrix for $\widehat{X}_{\rm filt}$ thus takes the form

$$\widehat{X}_{\rm filt}^T\widehat{X}_{\rm filt} = \bar{\Theta}_1 L_{11}^T\widehat{\Psi}^T\widehat{\Psi}L_{11}\bar{\Theta}_1^T + \bar{\Theta}_2\bar{\Theta}_2^T.$$

This expression shows that, again, the reconstructed signal has a filtered component lying in the p-dimensional subspace

[2] Matlab software for computing the rank-revealing QULV decomposition in the full-rank and rank-deficient cases is available in [9] and [8], respectively.

[3] We emphasize that L and V in the ULV and QULV decompositions are not the same matrices.

TABLE II

AMPLITUDES a_j OF THE PURE SIGNAL AND THE PURE NOISE, AND AMPLITUDES $\hat{a}_j^{(k)}$ OF THE QULV-BASED RECONSTRUCTION.

j	1	2	3	4	5	6
f_j	25	50	21	30	45	57
a_j	3.00	5.00	0.80	0.80	0.80	0.80
$\hat{a}_j^{(0)}$	1.58	2.51	0.04	0.11	0.13	0.04
$\hat{a}_j^{(1)}$	2.45	3.59	0.07	0.22	0.19	0.10
$\hat{a}_j^{(2)}$	2.94	4.94	0.07	0.24	0.38	0.20
$\hat{a}_j^{(3)}$	3.00	5.01	0.26	0.37	0.50	0.34
$\hat{a}_j^{(4)}$	2.95	4.98	0.36	0.41	0.58	0.42

$\mathcal{R}(\bar{\Theta}_1)$, and an unfiltered component in the subspace $\mathcal{R}(\bar{\Theta}_2)$ of dimension $n-p$. The two subspaces are disjoint but not orthogonal, and in the Appendix we prove that if $\angle(\bar{\Theta}_1, \bar{\Theta}_2)$ denotes the subspace angle between the column spaces of $\bar{\Theta}_1$ and $\bar{\Theta}_2$ then

$$\cos\angle(\bar{\Theta}_1, \bar{\Theta}_2) \leq \|L_{22}^{-1} L_{21}\, \bar{L}\|_2 \tag{10}$$

showing that the smaller the norm of L_{21} the larger the angle between the two subspaces.

We note in passing that there is also an equivalent quotient URV decomposition with upper triangular matrices. However, the analysis in [10] shows that this decomposition is impractical in connection with the applications that we have in mind.

B. Examples of the QULV-Based Algorithm

We illustrate the use of the QULV-based algorithm by means of the two examples from the previous section.

Sinusoids in Low-Rank Noise. We applied the QULV-based algorithm to the first test problem, and computed a least squares estimate by keeping the leading $k \times k$ block of L_{11} and setting the remaining elements of L_{11} to zero. The reconstructions are of essentially the same quality as those computed by means of the QSVD, cf. Table II. The subspace angle (for $k = 2$) between the exact and estimated signal subspaces is again 0.24 radians. This illustrates that the QULV decomposition is indeed able to yield good approximations to the quantities defined by the QSVD.

Voiced Speech in Low-Rank Noise. We also applied the QULV algorithm to the second test problem, using only the component of the solution in $\mathcal{R}(\bar{\Theta}_2)$. We obtained reconstructed signals whose LPC power spectra are very similar to those obtained by means of the QSVD; the spectral distance between the QSVD and QULV spectra is of the order 1 dB for $n = 20$ and less than one for $n = 40$.

IV. THE RANK DEFICIENT CASE

So far we have assumed that the matrix $(\, X^T \,,\, E^T \,)$ has full rank. A rank deficient $(\, X^T \,,\, E^T \,)$ implies that the order n of the model used for describing the system (n is the size of the covariance matrices of X and E) is larger than necessary. Therefore one cure for rank deficiency is to reduce the order n.

However, for generality of our algorithms it is important to be able to treat the rank-deficient case, because this allows an implementation with a fixed n. We shall now demonstrate that the QSVD and QULV decompositions described above can also be used to handle this case.

From the condition $\Sigma^2 + M^2 = I_p$ it follows that the middle matrix in the expression

$$\begin{pmatrix} X \\ E \end{pmatrix} = \begin{pmatrix} Q_X & 0 \\ 0 & Q_E \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & I_{n-p} \\ M & 0 \end{pmatrix} \Theta^T$$

has full rank. The leftmost matrix has orthonormal columns and therefore also full rank. Hence $\mathrm{rank}\big((\, X^T \,,\, E^T \,)\big) = \mathrm{rank}(\Theta)$, i.e., any rank deficiency must manifest itself in the matrix Θ. Consequently, when $(\, X^T \,,\, E^T \,)$ is rank deficient we cannot infer about the ranks of X and E merely from inspection of Σ and M.

The situation is the same in the QULV setting, in which

$$\begin{pmatrix} X \\ E \end{pmatrix} = \begin{pmatrix} U_X & 0 \\ 0 & U_E \end{pmatrix} \begin{pmatrix} L_{11} & 0 \\ 0 & I_{n-p} \\ I_p & 0 \end{pmatrix} \bar{\Theta}^T$$

showing that $\mathrm{rank}\big((\, X^T \,,\, E^T \,)\big) = \mathrm{rank}(\bar{\Theta})$. A closer look at

$$\bar{\Theta}^T = \begin{pmatrix} I_p & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \bar{L} & 0 \\ 0 & I_{n-p} \end{pmatrix} V^T$$

reveals that any rank deficiency in $\mathrm{rank}\big((\, X^T \,,\, E^T \,)\big)$ manifests itself in L_{22} being singular, because $\bar{L}$ has full rank.

A. The QSVD Algorithm

To extend the QSVD algorithm from Section II-A to the case where Θ is rank deficient, we seek a prewhitening matrix $E_X^{\oplus}$ of the form

$$E_X^{\oplus} = Z\, M^{-1} U_E^T$$

where $Z \in \mathbb{R}^{n\times p}$ is a matrix to be determined. There are two requirements to $E_X^{\oplus}$, namely, that $E\, E_X^{\oplus}$ must represent white noise, and that $X\, E_X^{\oplus}$ must represent a prewhitened signal with no component in the noise-free subspace. From the expressions

$$\begin{aligned} E\, E_X^{\oplus} &= Q_E\, M\, \Theta_1^T Z\, M^{-1} Q_E^T \\ X\, E_X^{\oplus} &= Q_X \begin{pmatrix} \Sigma\, \Theta_1^T Z \\ \Theta_2^T Z \end{pmatrix} M^{-1} Q_E^T \end{aligned}$$

we see that the two requirements are achieved if we choose Z such that $M\, \Theta_1^T Z\, M^{-1}$ is an orthogonal projection matrix, and such that $\Theta_2^T Z = 0$.

It is straightforward to show that if W is a matrix whose columns span the null space of Θ_2^T then the choice

$$Z = W\, \big(M\, \Theta_1^T W\big)^{\dagger}\, M$$

satisfies both requirements. Specifically, we obtain

$$\begin{aligned} E\, E_X^{\oplus} &= Q_E\, P\, Q_E^T \\ X\, E_X^{\oplus} &= Q_{X1}\, \Sigma\, M^{-1} P\, Q_E^T, \end{aligned}$$

with the orthogonal projection matrix P given by

$$P = M\, \Theta_1^T W\, \big(M\, \Theta_1^T W\big)^{\dagger}.$$

We remark that if Θ has full rank then W consists of the first p columns of $(\Theta^T)^{-1}$ and consequently $Z = W$, $\Theta_1^T Z =$

I_p, $P = I_p$ and $E_X^{\oplus} = E_X^{\dagger}$. Therefore our choice of the prewhitening matrix $E_X^{\oplus}$ is a natural extension of the weighted pseudoinverse $E_X^{\dagger}$.

The QSVD algorithm from Section II-A never forms the matrices $E_X^{\dagger}$ and $X\,E_X^{\dagger}$ explicitly, it only needs the diagonal matrix ΣM^{-1} to reveal the rank of $X\,E_X^{\dagger}$ in Eq. (6). The desired signal is then reconstructed from $\widehat{X}_{\rm filt}$ in Eq. (4).

When Θ is rank deficient we should ideally work with the prewhitened matrix $X\,E_X^{\oplus}$. However it is not practical to compute the matrix P, and instead we prefer to use the original QSVD algorithm and ignore P. To understand the consequence of this we need to examine $X\,E_X^{\oplus}$ closer. Assume that $\mathrm{rank}(\Theta_1) = q < p$ and write $P = Q\,Q^T$ with $Q \in \mathbb{R}^{p\times q}$. Then $X\,E_X^{\oplus}$ takes the form

$$X\,E_X^{\oplus} = Q_{X1}\,(\Sigma M^{-1} Q)\,(U_E Q)^T.$$

If we ignore P (and thus Q) then the decision about which columns of Q_{X1} to include in the signal subspace is based solely on the elements of the diagonal matrix ΣM^{-1}. Ideally, however, the decision should be based on the matrix $\Sigma M^{-1} Q$.

Hence, if the number of columns of $\Sigma M^{-1} Q$ with large norm is smaller than the number of large elements in ΣM^{-1}, then the signal subspace based on ΣM^{-1} may be too large, i.e., it may include noise components. The opposite situation where the dimension is chosen too small, such that genuine signal components are ignored, cannot happen. For this reason we believe that it is safe to use the original QSVD algorithm, independent of the rank of $(\,X^T\,,\,E^T\,)$ and thus avoiding the rank check and avoiding to work with the projection matrix P.

B. The QULV Algorithm

We now repeat the above analysis for the QULV algorithm from Section III-A. When $\bar\Theta$ is rank deficient, we seek a matrix $\bar Z$ such that $E_X^{\oplus} = V\,\bar Z\,U_E^T$ and such that the two previous requirements on

$$E\,E_X^{\oplus} = U_E\,(\,\bar L\,,\,0\,)\,\bar Z\,U_E^T$$

and

$$X\,e_X^{\oplus} = U_X \begin{pmatrix} (\,L_{11}\bar L\,,\,0\,)\,\bar Z \\ (\,L_{21}\bar L\,,\,L_{22}\,)\,\bar Z \end{pmatrix} U_E^T$$

are again satisfied, i.e., such that $(\,L_{11}\bar L\,,\,0\,)\,\bar Z$ is an orthogonal projection matrix and $(\,L_{21}\bar L\,,\,L_{22}\,)\,\bar Z = 0$. If $\overline{W}$ is a matrix whose columns span the null space of $(\,L_{21}\bar L\,,\,L_{22}\,)$, then

$$\bar Z = \overline{W}\left((\,L_{11}\bar L\,,\,0\,)\,\overline{W}\right)^{\dagger},$$

and it follows that

$$\begin{aligned} E\,E_X^{\oplus} &= U_E\,\bar P\,U_E^T \\ X\,E_X^{\oplus} &= U_{X1}\,L_{11}\,\bar P\,U_E^T \end{aligned}$$

with the orthogonal projection matrix $\bar P$ given by

$$\bar P = (\,L_{11}\bar L\,,\,0\,)\,\overline{W}\left((\,L_{11}\bar L\,,\,0\,)\,\overline{W}\right)^{\dagger}.$$

We note that when $\bar\Theta$ has full rank, then $\overline{W} = \begin{pmatrix} I_p \\ -L_{22}^{-1} L_{21} \end{pmatrix}$ and we obtain the results from Section III-A, showing that the above approach is a natural extension of the original QULV algorithm.

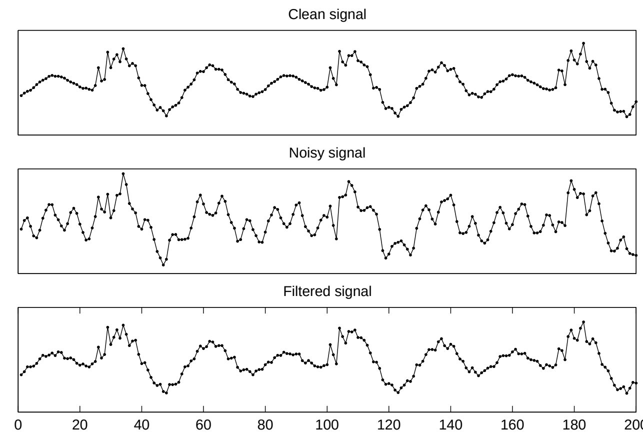

Fig. 4. An example of the signals in the numerical example. Top: pure voiced signal. Middle: noisy signal (SNR = 5 dB). Bottom: filtered signal (SNR = 18.6 dB).

Let us now examine the influence of neglecting the matrix $\bar P$ in the QULV algorithm. We write $\bar P = \bar Q \bar Q^T$ such that

$$X\,E_X^{\oplus} = U_{X1}\,(L_{11} Q)\,(U_E \bar Q)^T,$$

showing that the decision about the signal subspace should ideally be based on the matrix $L_{11}\bar Q$. Hence, if the number of columns in $L_{11}\bar Q$ with large norm is smaller than the number of large-norm columns L_{11} then we might include noise components in the signal subspace. As before, the opposition situation cannot happen, i.e., there is no danger that we omit important signal components.

In conclusion, we find also in the QULV setting that it is safe to ignore $\bar P$ (and thus $\bar Q$) and use the original QULV algorithm, independent of the rank of $(\,X^T\,,\,E^T\,)$.

V. Numerical Example

We illustrate the use of our algorithm with samples of a male voice signal contaminated by noise originating from a buzz saw, with a signal-to-noise ratio of 5 dB. We process the signal by splitting the full time signal into frames of length 200 samples each, and applying the QSVD algorithm in each time frame.

The noise signal from the buzz saw is dominated by a few harmonics whose frequency vary with time. Hence, the noise matrix E changes in each time frame; it is always rank deficient, and its rank changes between time frames. The noise reduction is achieved by maintaining the largest k values of Σ in (4), and discarding the rest. We use a different value of k in each time frame.

Figure 4 shows an example of the signals involved in this example: the clean signal, the noisy signal and the filtered signal. In this frame, the SNR has been improved by about 13 dB.

Figure 5 shows LPC spectra for the signals in four representative time frames. We used an LPC order of 20 in order to captures the spikes in the noise spectra. Above each plot,

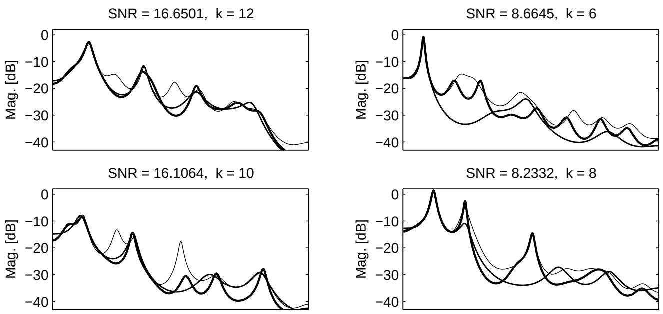

Fig. 5. LPC spectra (order 20) for four representative time frames. Thin line: noisy signal. Medium line: clean signal. Thick line: filtered signal. For each time frame we give the achieved SNR and the number QSVD components k that was used.

we give the SNR that was obtained in the corresponding time frame, together with the value of k that was used.

Clearly, the QSVD algorithm is able to suppress the harmonics of the rank-deficient noise signal in an adaptive fashion.

VI. Conclusion

We have used the results from [10] to extend the QSVD algorithm [13] to the case with narrow-band noise, where the covariance matrix of the noise is rank deficient. In particular we demonstrated that the QSVD of the signal and noise matrices produces all the quantities necessary to perform the rank reduction and construct the signal subspace for the reconstruction. We also demonstrated how the algorithm can be formulated in terms of the rank-revealing QULV algorithm, which has lower computational complexity and is better suited for updating. Finally we demonstrated the efficiency of the QSVD and QULV algorithms with numerical examples involving speech signals and rank-deficient noise.

Appendix

To prove Eq. (10) we first note that a left orthogonal transformation does not change the subspace angle, and therefore the angle between $\bar{\Theta}_1$ and $\bar{\Theta}_2$ is identical to the angle between the ranges of

$$\begin{pmatrix} \bar{L}^T \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{L}^T L_{21}^T \\ L_{22}^T \end{pmatrix}.$$

From the assumption that $(\, X^T \,,\, E^T \,)$ has full rank, it follows that both $\bar{L}$ and L_{22} have full rank. The range of a matrix is not altered by right-multiplication with a full-rank matrix, and hence we seek the subspace angle between the ranges of

$$\begin{pmatrix} I_p \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{L}^T L_{21}^T L_{22}^{-T} \\ I_{n-p} \end{pmatrix}. \tag{11}$$

Let $Q\,R$ denote the "skinny" QR factorization of the latter matrix. Then

$$\begin{aligned} \cos\angle(\bar{\Theta}_1, \bar{\Theta}_2) &= \left\| \begin{pmatrix} I_p \\ 0 \end{pmatrix} \begin{pmatrix} I_p \\ 0 \end{pmatrix}^T Q \right\|_2 \\ &= \left\| \begin{pmatrix} \bar{L}^T L_{21}^T L_{22}^{-T} \\ 0 \end{pmatrix} R^{-1} \right\|_2 \\ &\leq \|\bar{L}^T L_{21}^T L_{22}^{-T}\|_2 \, \|R^{-1}\|_2. \end{aligned}$$

Since the singular values of the second matrix in (11) are greater than or equal to one, it follows that $\|R^{-1}\|_2 \leq 1$ and thus $\cos\angle(\bar{\Theta}_1, \bar{\Theta}_2) \leq \|L_{22}^{-1} L_{21}\, \bar{L}\|_2$.

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