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# THE DIRECT FLOW PARAMETRIC PROOF OF GAUSS' DIVERGENCE THEOREM REVISITED

STEEN MARKVORSEN

ABSTRACT. The standard proof of the divergence theorem in undergraduate calculus courses covers the theorem for static domains between two graph surfaces. We show that within first year undergraduate curriculum, the flow proof of the dynamic version of the divergence theorem - which is usually considered only much later in more advanced math courses - is comprehensible with only a little extension of the first year curriculum. Moreover, it is more intuitive than the static proof. We support this intuition further by unfolding and visualizing a few examples with increasing complexity. In these examples we apply the key instrumental concepts and verify the various steps towards this alternative proof of the divergence theorem.

## 1. INTRODUCTION

With the advent and aid of modern tools for visualizations and calculations, the study of integral curves of vector fields in space has become much more accessible and comprehensible. It is now an easy matter to simulate flows along given vector fields in 3D and thence visualize the corresponding deformation of curves, surfaces and bodies floating along with the flow.

Here we think of the flow deformation of a given geometric object as being organized so that each point of the object follows its own determined flow line so that the full object is pushed forward and at the same time deformed by the flow map in the direction of the given vector field.

Intuition concerning the flow map is thereby firmly developed and supported. Several first rate questions emerge naturally among students when seeing this deformation take place in, say, a 3D animation: How much is this curve elongated during the flow? What is the total deformation of that surface? Why doesn't it self-intersect during the flow? What is the volume of a compact floating body after time  $t$ ?

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The answers to these questions are all contained in (the flow proof of) Gauss' divergence theorem which we now state in its full dynamic generality for compact domains and vector fields in 3D:

**Theorem 1.1** (Gauss' divergence theorem, flow version). *Let  $\Omega_0$  denote a compact domain in  $\mathbb{R}^3$  with piecewise smooth boundary  $\partial\Omega_0$  and outward pointing unit normal vector field  $\mathbf{n}_{\partial\Omega_0}$  on  $\partial\Omega_0$ . Let  $\mathbf{V}$  be a vector field in  $\mathbb{R}^3$  with corresponding flow map  $\mathbf{F}^t$ . Let  $\Omega_t = \mathbf{F}^t(\Omega_0)$ . Then*

$$(1.1) \quad \left( \frac{d}{dt} \right)_{|_{t=0}} \text{Vol}(\Omega_t) = \int_{\Omega_0} \text{div}(\mathbf{V}) d\mu = \int_{\partial\Omega_0} \mathbf{V} \cdot \mathbf{n}_{\partial\Omega_0} d\nu \quad .$$

The flow map  $\mathbf{F}^t$  will be defined in detail via the examples below and in Theorem 2.5.

The right hand side of (1.1) is the outwards directed *flux* of the vector field through the boundary of the domain.

Of course, along the way of proving Theorem 1.1 we must expect and answer yet another pertinent and natural question from the students: What is the precise rôle of the divergence in this statement? How does this emerge from the action of the flow map?

Before entering into the details of answering these questions and proving the theorem, we make a few remarks concerning the development of the theorem, i.e. concerning its origin and history and concerning its momentum within standard first year curriculum.

Gauss' divergence theorem is of the same calibre as Stokes' theorem. They are both members of a family of results which are concerned with 'pushing the integration to the boundary'. The eldest member of this family is the following:

**Theorem 1.2** (Fundamental theorem of calculus). *Let  $f$  be a continuous function on  $\mathbb{R}$ . Then the function*

$$A(x) = \int_0^x f(u) du$$

*is differentiable with*

$$(1.2) \quad A'(x) = f(x) \quad ,$$

*and moreover, if  $F(x)$  is any (other) function satisfying  $F'(x) = f(x)$ , then*

$$(1.3) \quad \int_a^b f(u) du = F(b) - F(a) \quad .$$

The message of this theorem is that two fundamental problems - that of finding a function whose derivative is a given function and that of finding the average of a given function - have a common solution. It is also the first result which displays - in equation (1.3) - the astounding success of 'pushing the integration to the boundary'.

The divergence theorem is not - conceptually speaking - 'far' from the fundamental theorem of calculus. Most textbook proofs of the divergence theorem covers only the special setting of a static domain whose boundary consists of the graphs of two functions, each of two variables. This enables in fact a direct proof in this special case via Theorem 1.2, see [EP] pp. 1058–1059. Stokes' theorem is a little harder to grasp, even locally, but follows also in the corresponding setting (for graph surfaces) from Gauss' theorem for planar domains, see [EP] pp. 1065–1066. Stokes' theorem can alternatively be presented in the same vein as the divergence theorem is presented in this paper. One version of such a proof can be found in [Mar].

The idea of unfolding the details of Gauss' divergence theorem in this generality via a flow parametric proof on the platform of first year undergraduate curriculum has - to the best of the present authors knowledge - not yet been implemented in any calculus book. We must mention, however, that the dynamic version of the divergence theorem in two dimensions has been covered in a similar vein via an application of the co-area formula in the recent beautiful paper [ES] by Eisenberg and Sullivan.

Among the many interesting efforts to improve the teaching of vector calculus in general - and of its applications, in particular to electromagnetism - we also refer to [FLS], [She], [A], [DuB] and [DoB]. These efforts are typically initiated from the community of physics teachers. However, as expressed clearly in [DuB] there is room for - as well as a need for - a fruitful dialogue with the mathematics teachers concerning these issues.

The main purpose of the present work is in this spirit to try and facilitate the first presentation, i.e. the undergraduate teaching, as well as the general intuition of the divergence theorem and thus to prepare it for later use in the applied fields.

The presentation is 'compatible' with the Worldwide calculus curriculum in the sense that throughout this paper we make explicit use of vector parametrizations of surfaces and curves and their corresponding Jacobian functions. This gives maximal flexibility for choosing concrete examples and illustrations, not only for the divergence theorem

itself, but also for each step in the proof as will be indicated by three examples in the present paper.

What may be new in comparison with the typical first year curriculum is the necessary tools and results from systems of differential equations and their solutions. At the Technical University of Denmark we have successfully chosen to introduce  $2 \times 2$  linear ODE systems and their complete solutions (via eigen-solutions of the system matrix) into the first semester curriculum. This gives in particular a wonderful application of linear algebra to establish existence and uniqueness of solutions in these simplest (linear, planar) cases. The step to  $3 \times 3$  linear ODE systems is then not a difficult one. The step to *non-linear*  $3 \times 3$  systems is, of course, the real hurdle but conceptually still within reach - in particular with the advent and aid of modern computer facilities.

Finally, concerning the *history* of the divergence theorem, one interesting account is given by Crowe in [C], from where we quote:

The history of these theorems [Green's, Stokes', and Gauss' theorems] has never (to my knowledge) been written. It essentially lies outside the history of vector analysis, for the theorems were all developed originally for Cartesian analysis, and by people who did not work with vectors. Some comments may however be made. Gauss' Theorem (often called the Divergence Theorem) is attributed (by Hermann Rothe [1931?]) to Gauss [1840].

James Clerk Maxwell [in his *Treatise on Electricity and Magnetism*] stated that Gauss' Theorem "... seems to have been first given by Ostrogradsky in a paper read in 1828, but published in 1831 [...]". This note is not contained in the first edition of his *Treatise* [1891]. This fact is doubly interesting as possibly indicating where Maxwell first found the theorem.

Oliver Dimon Kellogg, *Foundations of Potential Theory* [1953] wrote the following in regard to Gauss' Theorem: "A similar reduction of triple integrals to double integrals was employed by Lagrange [1760]. The double integrals are given in more definite form by Gauss [1813]. A systematic use of integral identities equivalent to the divergence theorem was made by George Green in his *Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism, Nottingham, 1828.*"

M. J. Crowe, [C] p. 146.

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**Outline of paper .** In section 2 we consider and illustrate, mainly by examples, the flow maps associated to given vector fields. The proof of the divergence theorem is orchestrated in sections 3, 4, and 5. The covariant (directional) derivative of a vector field is applied in section 3 to give precise information about the deformation of curves. The  $t$ -derivative of the volume of a floating domain can then be expressed in two ways: An 'intrinsic' calculation gives rise to the divergence integral (section 4), and an 'extrinsic' calculation gives rise to the flux integral (section 5).

## 2. THE FLOW OF A VECTOR FIELD

A smooth parametrized curve  $\gamma(t)$  in  $\mathbb{R}^3$  has a tangent vector field  $\gamma'(t)$  along the curve. The curve is an integral curve for a given vector field  $\mathbf{V}$  if the tangent vector at every point of the curve is precisely the given vector field evaluated at that point. The integral curves are therefore solutions to the following differential equation:

$$\gamma'(t) = \mathbf{V}(\gamma(t)) \quad .$$

In cartesian coordinates in  $\mathbb{R}^3$  we have

$$\begin{aligned} \gamma(t) &= (x(t), y(t), z(t)) \quad , \\ \gamma'(t) &= (x'(t), y'(t), z'(t)) \quad \text{and} \\ \mathbf{V}(x, y, z) &= (V_1(x, y, z), V_2(x, y, z), V_3(x, y, z)) \quad . \end{aligned}$$

The system of ordinary differential equations corresponding to the vector equation  $\gamma'(t) = \mathbf{V}(\gamma(t))$  is thus

$$(2.1) \quad \begin{aligned} x'(t) &= V_1(x(t), y(t), z(t)) \\ y'(t) &= V_2(x(t), y(t), z(t)) \\ z'(t) &= V_3(x(t), y(t), z(t)) \quad . \end{aligned}$$

The integral curve through a given point  $(x_0, y_0, z_0)$  is obtained by using this point as the initial condition for the system of differential equations.

We present 3 examples with slightly increasing complexity, but always with emphasis on different elementary aspects of the theory.

**Example 2.1.** The vector field in Figure 1,  $\mathbf{V}(x, y, z) = (-y, x, z/7)$  has integral curves defined by the following system of ordinary first

order differential equations:

$$(2.2) \quad \begin{aligned} x'(t) &= -y(t) \\ y'(t) &= x(t) \\ z'(t) &= z(t)/7 \quad . \end{aligned}$$

Given any point  $p = (x_0, y_0, z_0)$  there is a unique solution to equation (2.2) with this initial condition, i.e. there is an integral curve, a flow curve, through that point:

$$(2.3) \quad \begin{aligned} x(t) &= x_0 \cos(t) - y_0 \sin(t) \\ y(t) &= x_0 \sin(t) + y_0 \cos(t) \quad . \\ z(t) &= z_0 e^{t/7} . \end{aligned}$$

Such a solution is considered as the trajectory of a (test) particle starting at the point  $p = (x_0, y_0, z_0)$  and flowing along the vector field.

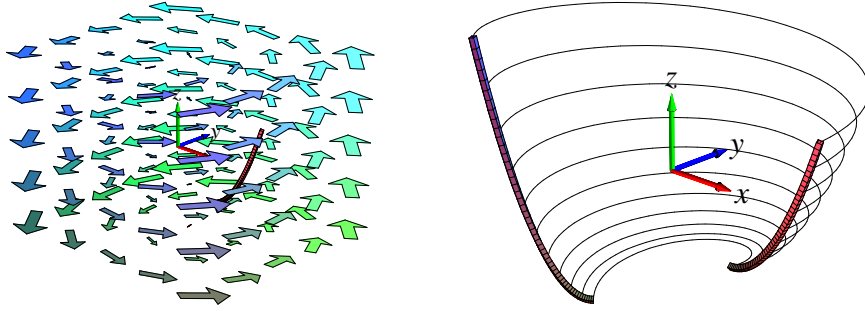


FIGURE 1. A curve flow along the vector field  $\mathbf{V}(x, y, z) = (-y, x, z/7)$ . The base curve, which is flown by this vector field, is:  $\mathbf{r}(u) = (1 + u, 0, u^2 - 1)$  where  $u \in [0, \sqrt{2}]$ . The flow time is  $t \in [0, 5\pi/4]$ . The deformation of the curve is obtained by pushing every point of the curve for time  $t$  along the respective integral curves.

The flow curves are organized by a *flow map*  $\mathbf{F}^t$  as follows. For each  $t$  the particle starting at  $(x_0, y_0, z_0)$  is moved forward along the vector field  $\mathbf{V}$  - or rather along the integral curves of  $\mathbf{V}$  - to the position

$$\begin{aligned} \mathbf{F}^t(x_0, y_0, z_0) &= (x(t), y(t), z(t)) \\ &= (x_0 \cos(t) - y_0 \sin(t), x_0 \sin(t) + y_0 \cos(t), z_0 e^{t/7}) \quad . \end{aligned}$$

From now on we will drop the index 0 in  $(x_0, y_0, z_0)$  and simply consider all possible starting points  $(x, y, z)$  for the flow map  $\mathbf{F}^t$ , to stress the fact that for each fixed  $t$  it is a map  $(x, y, z) \in U \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$  and for each fixed point  $(x, y, z)$  it is a map  $t \in I \subset \mathbb{R} \mapsto \mathbb{R}^3$ . The following

pertinent question emerges naturally: Given a time  $t$ , what is then the maximal set  $U = \mathcal{M}_t$  for which the flow map is defined? And given a point  $(x, y, z)$  what is then the maximal interval  $I = \mathcal{D}_{(x,y,z)}$  for which the flow map is defined? These maximal sets will be considered in the examples and defined precisely below in Definition 2.2.

In the particular example 2.1 considered above the map  $\mathbf{F}^t$  is linear for each fixed  $t$  and may be represented by a  $t$ -dependent matrix as follows. (For a given fixed time  $t$  we use shorthand  $(\hat{x}, \hat{y}, \hat{z})$  for  $(x(t), y(t), z(t))$ .) Then

$$(2.4) \quad \mathbf{F}^t(x, y, z) = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{t/7} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} .$$

This flow map is clearly smooth and regular for every  $t$ . The determinant of the matrix representative is  $e^{t/7} > 0$ . The inverse map is also linear and has determinant  $e^{-t/7} > 0$ . The inverse map is simply obtained by changing sign on time  $t$  throughout the expression for  $\mathbf{F}^t$ . Indeed:

$$(2.5) \quad \begin{aligned} (\mathbf{F}^t)^{-1}(\hat{x}, \hat{y}, \hat{z}) &= \begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{t/7} \end{bmatrix}^{-1} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & e^{-t/7} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \\ &= \begin{bmatrix} \cos(-t) & -\sin(-t) & 0 \\ \sin(-t) & \cos(-t) & 0 \\ 0 & 0 & e^{-t/7} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \\ &= \mathbf{F}^{-t}(\hat{x}, \hat{y}, \hat{z}) . \end{aligned}$$

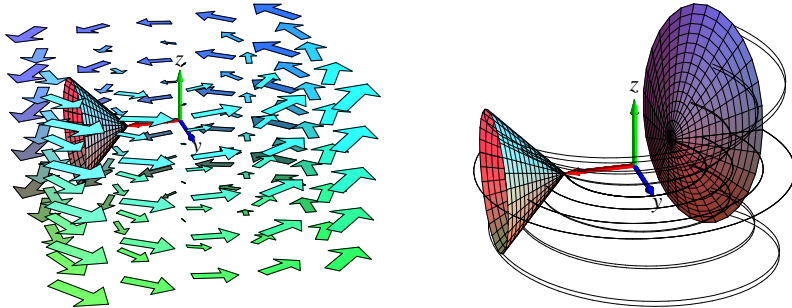


FIGURE 2. A conic surface is deformed via the vector field  $\mathbf{V}(x, y, z) = (-y, x, z/7)$ .



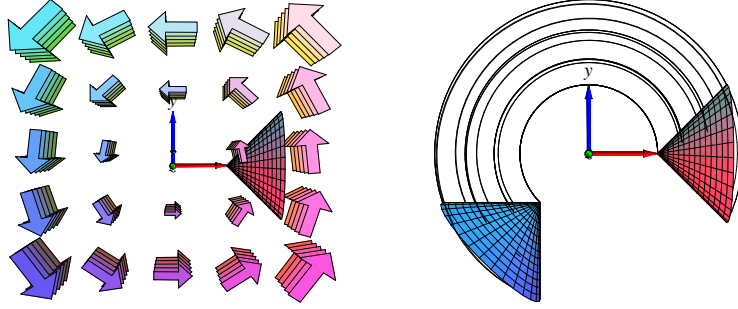


FIGURE 3. The same setting as in figure 2. Seen directly along the  $z$ -axis the action looks like a pure rotation.

The forward flow of duration  $t$  is a smooth regular map whose inverse is the forward flow along  $\mathbf{V}$  of duration  $-t$ . (The latter inverse flow is also the same as the forward flow along  $-\mathbf{V}$  of duration  $t$ .) This is intuitively quite reasonable and should be expected to hold true for every smooth vector field. But there is one problem which has already been alluded to above. For each initial point there is a maximal duration time interval for the flow which may not be the full real line, see example 2.3 below.

The proper definition of the sets  $\mathcal{D}_{(x,y,z)}$  and  $\mathcal{M}_t$  for the flow  $\mathbf{F}^t$  are:

**Definition 2.2.** *Whenever the collection of integral curves of a given vector field  $\mathbf{V}$  is expressed in terms of a flow map  $\mathbf{F}^t(x, y, z)$  as in the previous example, we define the two sets  $\mathcal{D}_{(x,y,z)}$  and  $\mathcal{M}_t$  as follows: For every given point  $(x, y, z) \in \mathbb{R}^3$  and every given time  $t \in \mathbb{R}$ , respectively.  $\mathcal{D}_{(x,y,z)}$  = the maximal  $t$ -interval containing 0 for which  $\mathbf{F}^t(x, y, z)$  exists and  $\mathcal{M}_t$  = the maximal set of points  $(x, y, z)$  in  $\mathbb{R}^3$  for which  $\mathbf{F}^t(x, y, z)$  exists.*

In the above example 2.1 we have  $\mathcal{D}_{(x,y,z)} = \mathbb{R}$  for all  $(x, y, z) \in \mathbb{R}^3$  and  $\mathcal{M}_t = \mathbb{R}^3$  for all  $t \in \mathbb{R}$ .

**Example 2.3.** The following vector field does not have so trivial flow domains  $\mathcal{D}_{(x,y,z)}$  and  $\mathcal{M}_t$ :

$$(2.6) \quad \mathbf{V}(x, y, z) = (-y, x, z^2) \quad .$$

The flow map for this field is

$$(2.7) \quad \mathbf{F}^t(x, y, z) = \begin{pmatrix} x \cos(t) - y \sin(t), \\ x \sin(t) + y \cos(t), \\ z/(1 - tz) \end{pmatrix} \quad .$$

Indeed, the following shows that the integral curves do have the vector field as tangent vectors as demanded by definition in equation 2.1:

$$\begin{aligned}
(2.8) \quad \frac{\partial}{\partial t} \mathbf{F}^t(x, y, z) &= (x'(t), y'(t), z'(t)) \\
&= (-y(t), x(t), z(t)^2) \\
&= \mathbf{V}(x(t), y(t), z(t)) \\
&= \mathbf{V}(\mathbf{F}^t(x, y, z)) \quad .
\end{aligned}$$

If  $z = 0$  the flow map is defined for all values of  $t$  - the maximal flow-time interval is  $\mathbb{R}$  for all the initial points lying in the  $z$ -plane - but for  $z > 0$  the maximal flow-time interval for the flow is  $] -\infty, 1/z[$ , and for  $z < 0$  the maximal flow-time interval is  $]1/z, \infty[$ . In the latter two cases, when  $t$  approaches  $1/z$  the corresponding particle which flows along the flow line, is simply howling towards infinity and is eventually ripped out of space in finite time. Note that this dramatic fate stems directly from the well known fact that the general solution to the equation  $z'(t) = z(t)^2$  is singular (except for  $c = 0$ :  $z(t) = c/(1 - ct)$ ), where  $c \in \mathbb{R}$  is the arbitrary constant of integration.

We note that for the vector field in this example we therefore have:

$$(2.9) \quad \mathcal{D}_{(x,y,z)} = \begin{cases} ] -\infty, 1/z[ & \text{for } z > 0 \\ \mathbb{R} & \text{for } z = 0 \\ ]1/z, +\infty[ & \text{for } z < 0 \end{cases} \quad .$$

and

$$(2.10) \quad \mathcal{M}_t = \{(x, y, z) \in \mathbb{R}^3 \mid z \neq 1/t\} \quad .$$

The inverse flow map is again determined by a sign change on the time parameter wherever time is well defined: As in the previous example (see equation (2.5)) we have:

$$(2.11) \quad (\mathbf{F}^t)^{-1}(\hat{x}, \hat{y}, \hat{z}) = \mathbf{F}^{-t}(\hat{x}, \hat{y}, \hat{z}) \quad .$$

Indeed,

$$\begin{aligned}
(2.12) \quad \mathbf{F}^{-t}(\hat{x}, \hat{y}, \hat{z}) &= (\hat{x} \cos(-t) - \hat{y} \sin(-t), \\
&\quad \hat{x} \sin(-t) + \hat{y} \cos(-t), \\
&\quad \hat{z}/(1 - (-t)\hat{z})) \\
&= (\hat{x} \cos(t) + \hat{y} \sin(t), \\
&\quad -\hat{x} \sin(t) + \hat{y} \cos(t), \\
&\quad \hat{z}/(1 + t\hat{z})) \quad ,
\end{aligned}$$

so that for all  $(x, y, z) \in \mathcal{M}_t$  we have

$$\mathbf{F}^{-t}(\mathbf{F}^t(x, y, z)) = \left( x, y, \frac{(z/(1-tz))}{(1+t(z/(1-tz)))} \right) = (x, y, z)$$

and for all  $(\hat{x}, \hat{y}, \hat{z}) \in \mathcal{M}_{-t}$

$$\mathbf{F}^t(\mathbf{F}^{-t}(\hat{x}, \hat{y}, \hat{z})) = \left( \hat{x}, \hat{y}, \frac{(\hat{z}/(1+t\hat{z}))}{(1-t(\hat{z}/(1+t\hat{z})))} \right) = (\hat{x}, \hat{y}, \hat{z}) \quad .$$

The inverse map is thus well defined and smooth on  $\mathbf{F}^t(\mathcal{M}_t) = \mathcal{M}_{-t}$ .

**Example 2.4** (The pendulum flow map). The pendulum consists of a particle which is free to move on a vertical circle of radius  $l$  without friction and only acted upon by gravity  $\mathbf{g}$ . The position on the circle is given by the oriented angle  $\theta(t)$  between  $\mathbf{g}$  and the radial vector from the origin of the circle to the particle. The well known equation for  $\theta(t)$  is then as follows, see e.g. [MacM] pp. 310 ff.:

$$(2.13) \quad l\theta''(t) = -g \sin(\theta(t)) \quad .$$

If we set  $\theta(t) = x(t)$ ,  $\theta'(t) = y(t)$  then the dynamics of a simple pendulum with  $l = g$  is given by the following planar vector field:

$$(2.14) \quad \mathbf{V}(x, y) = (y, -\sin(x)) \quad .$$

We apply the analysis provided in [Law] pp. 114 ff. and in [MacM] pp. 310 ff. There are essentially two modes of behavior for the pendulum depending on the energy  $E$  of the initial state  $(x, y)$  of the pendulum.

Here  $E = E(x, y) = (1/2)y^2 - \cos(x)$  (once set into swing with this initial energy, the pendulum preserves this energy for all times): The 'back and forth' mode with small energy:  $E = (y^2/2) - \cos(x) < 1$  and the 'revolving' mode with large energy:  $E = (y^2/2) - \cos(x) > 1$ . In 'between' these two modes, the 'separating' mode is determined by  $E = (y^2/2) - \cos(x) = 1$ .

The constant energy is composed of potential and kinetic energy. The integral curves are (non-parametrized) level curves for this energy function, see Figure 4. The hard part of this example (as in general) is to actually parametrize these integral curves by time.

We let  $\text{am}$ ,  $\text{cn}$ ,  $\text{sn}$ , and  $\text{dn}$  denote Jacobi's elliptic 'trigonometric' functions and denote by  $k$  and  $q$  the following values depending on the

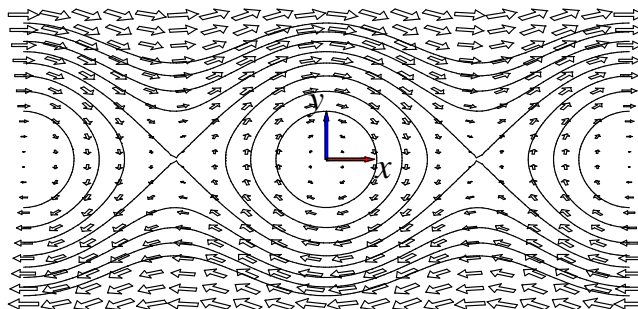


FIGURE 4. The pendulum vector field and level curves for the energy function  $E(x, y)$ . The level curves are clearly the (non-parametrized) integral curves for the vector field.

initial state  $(x, y)$ :

$$k = k(x, y) = \begin{cases} \sin((1/2) \arccos(\cos(x) - (1/2)y^2)) & \text{for } E < 1 \\ 1 & \text{for } E = 1 \\ \sqrt{2/((y^2/2) - \cos(x) + 1)} & \text{for } E > 1 \end{cases},$$

and

$$q = q(x) = \begin{cases} \operatorname{sn}^{-1}(\sin(x/2)/k, k) & \text{for } E < 1 \\ \operatorname{sn}^{-1}(\sin(x/2), 1) = \tanh^{-1}(\sin(x/2)) & \text{for } E = 1 \\ k \operatorname{sn}^{-1}(\sin(x/2), k) & \text{for } E > 1 \end{cases}.$$

The flow map is then explicitly:

$$\mathbf{F}^t(x, y) = \begin{cases} (2 \arcsin(k \operatorname{sn}(t + q, k)), 2k \operatorname{cn}(t + q, k)) & \text{for } E < 1 \\ (4 \arctan(\exp(t + q)) - \pi, 2 \operatorname{sech}(t + q)) & \text{for } E = 1 \\ (2 \operatorname{am}((t + q)/k, k), (2/k) \operatorname{dn}((t + q)/k, k)) & \text{for } E > 1 \end{cases}.$$

We note that the solution  $\theta(t)$  to the original 2.nd order differential equation 2.13 with given initial conditions  $\theta(0) = a$  and  $\theta'(0) = b$  is  $\theta(t) = x(t) =$  the 1.st coordinate function of  $\mathbf{F}^t(a, b)$ .

In spite of this complicated solution, the divergence theorem for the underlying vector field is quite simple. Since the divergence of the vector field  $\mathbf{V}$  is 0, the flow map preserves the area of every domain in the flow domain  $\mathbb{R}^2$  for all times  $t$ . This is indicated by the flow deformation of a rectangle in Figure 5.

The flow map is smooth with a well defined and easily constructed inverse flow map. According to Theorem 2.5 below the inverse map is also smooth and is obtained by changing sign on the time parameter  $t$  in the expression for  $\mathbf{F}^t(x, y)$  above. As in the case of the previously considered linear vector field (example 2.1) we also have in the present case:  $\mathcal{M}_t = \mathbb{R}^2$  for all  $t \in \mathbb{R}$  and  $\mathcal{D}_{(x,y)} = \mathbb{R}$  for all  $(x, y) \in \mathbb{R}^2$ .

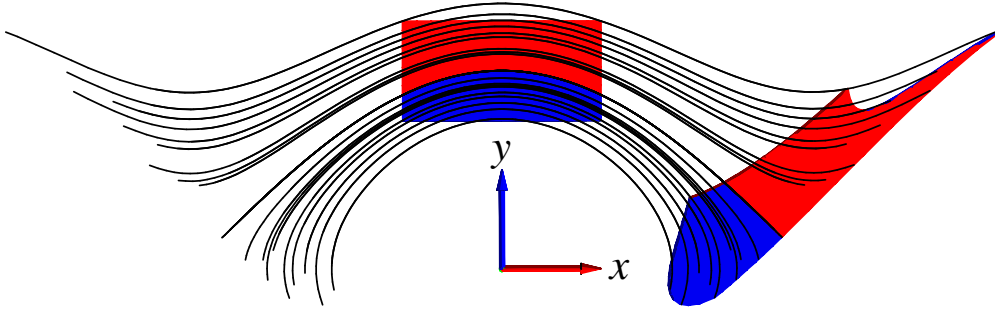


FIGURE 5. The pendulum flow of a rectangle. The lower part of the rectangle is trapped in a cell and oscillates around the value  $x = 0$ . The upper part is revolving and increases the  $x$ -value for all time. All the flow lines shown correspond to the time interval  $t \in [-2, 2]$ .

The properties observed so far, concerning the existence and uniqueness of the (suitably restricted) flow maps and the openness of the maximal sets  $\mathcal{M}_t$  and  $\mathcal{D}_{(x,y,z)}$ , are generally true as proved in the study of ordinary differential equations, see e.g. [H] or [Lee] Chapter 17 p. 442:

**Theorem 2.5** (Fundamental Theorem on Flows). *Let  $\mathbf{V}$  denote a smooth vector field in  $\mathbb{R}^3$ . Then there is a unique maximal flow map  $\mathbf{F}^t(p)$  such that:*

- a) *For each point  $p$  the maximal integral curve for  $\mathbf{V}$  through  $p$  is  $\mathbf{F}^t(p)$ ,  $t \in \mathcal{D}_p$ , i.e.*

$$\frac{\partial}{\partial t} \mathbf{F}^t(p) = \mathbf{V}(\mathbf{F}^t(p)) \quad .$$

- b) *For each  $t \in \mathbb{R}$  the set  $\mathcal{M}_t$  is an open set in  $\mathbb{R}^3$  and  $\mathbf{F}^t : \mathcal{M}_t \mapsto \mathcal{M}_{-t}$  is a smooth map with smooth inverse  $\mathbf{F}^{-t}$ .*

## 3. THE CURVE FLOW AND COVARIANT DERIVATIVES

Note that at points where  $\mathbf{V}(x, y, z) = (0, 0, 0)$  the flow map of the vector field  $\mathbf{V}$  is the identity for all  $t$ . There is no flow, no deformation of such points or point sets. When  $\mathbf{V} \neq (0, 0, 0)$ , the flow map  $\mathbf{F}^t$  moves - and in general deforms - any given smooth curve, surface or domain into a new smooth curve, surface and domain - see Figures 1 and 2. The main idea in the present paper is to understand Gauss' divergence theorem in terms of this deformation. In fact we only need to understand the  $t$ -derivative of the deformation at  $t = 0$  in order to extract the divergence theorem from such an analysis.

We begin by studying in detail what happens to a given parametrized curve  $\mathbf{r}(u)$ ,  $u \in [a, b]$ , when it is floating along a given vector field  $\mathbf{V}$ . The first obvious question concerns the length of the deformed curve  $\mathbf{F}^t(\mathbf{r}(u))$ . To find the length we need to find the tangent vector field along the curve, i.e. the vector field  $\frac{\partial}{\partial u} \mathbf{F}^t(\mathbf{r}(u))$  for every  $u \in [a, b]$ . For this purpose we need to introduce the so-called covariant derivative of the vector field.

**Definition 3.1.** *Let  $\mathbf{V}$  denote a smooth vector field with coordinate functions  $\mathbf{V}(x, y, z) = (V_1(x, y, z), V_2(x, y, z), V_3(x, y, z))$  and let  $\mathbf{X} = (X_1, X_2, X_3)$  be any (other) smooth vector field in  $\mathbb{R}^3$ . The covariant derivative of  $\mathbf{V}$  with respect to  $\mathbf{X}$  is then the following vector field:*

$$(3.1) \quad \nabla_{\mathbf{X}} \mathbf{V} = [\mathbf{D} \mathbf{V}] \mathbf{X} \quad ,$$

where  $[\mathbf{D} \mathbf{V}]$  denotes the matrix (operator):

$$(3.2) \quad [\mathbf{D} \mathbf{V}] = \begin{bmatrix} \frac{\partial V_1}{\partial x} & \frac{\partial V_1}{\partial y} & \frac{\partial V_1}{\partial z} \\ \frac{\partial V_2}{\partial x} & \frac{\partial V_2}{\partial y} & \frac{\partial V_2}{\partial z} \\ \frac{\partial V_3}{\partial x} & \frac{\partial V_3}{\partial y} & \frac{\partial V_3}{\partial z} \end{bmatrix} \quad .$$

When evaluating the right hand side of (3.1) we get

$$(3.3) \quad \begin{aligned} \nabla_{\mathbf{X}} \mathbf{V} &= \left( X_1 \frac{\partial V_1}{\partial x} + X_2 \frac{\partial V_1}{\partial y} + X_3 \frac{\partial V_1}{\partial z}, \quad * \quad , \quad * \right) \\ &= (\mathbf{grad}(V_1) \cdot \mathbf{X}, \quad \mathbf{grad}(V_2) \cdot \mathbf{X}, \quad \mathbf{grad}(V_3) \cdot \mathbf{X}) \quad , \end{aligned}$$

so that the covariant derivative is - in this precise sense - the *directional derivative* of  $\mathbf{V}$  with respect to  $\mathbf{X}$ .

We note that the divergence of  $\mathbf{V}$  is precisely the trace of the matrix  $[\mathbf{D} \mathbf{V}]$ . This trace will appear below when we express the local volume deformation (via the Jacobian) induced by the flow map on a given domain.

The covariant derivative appears naturally from an application of the chain rule as follows: We evaluate the vector field  $\mathbf{V}$  along a given

parametrized curve  $\mathbf{r}(u)$  and consider the  $u$ -derivative of the coordinate functions  $V_i(\mathbf{r}(u))$ ,  $i = 1, 2, 3$ , i.e.

$$(3.4) \quad \frac{\partial}{\partial u} V_i(\mathbf{r}(u)) = \mathbf{grad}(V_i) \cdot \mathbf{r}'(u) \quad .$$

In view of equation (3.3) we therefore have:

**Lemma 3.2.**

$$(3.5) \quad \frac{\partial}{\partial u} \mathbf{V}(\mathbf{r}(u)) = [\mathbf{D} \mathbf{V}]_{|\mathbf{r}(u)} \mathbf{r}'(u) \quad .$$

We are now ready to set free the curve and let it flow along the integral curves of the vector field.

**Proposition 3.3.** *Given a smooth parametrized curve  $\mathbf{r}(u)$ ,  $u \in [a, b]$ . When the points of this curve flow along their respective integral curves of  $\mathbf{V}$ , the deformed curve  $\mathbf{F}^t(\mathbf{r}(u))$  has the following tangent vector field at time  $t$ :*

$$(3.6) \quad \frac{\partial}{\partial u} \mathbf{F}^t(\mathbf{r}(u)) = \left( \mathbf{I} + t \left( [\mathbf{D} \mathbf{V}]_{|\mathbf{r}(u)} + [\mathbf{D} \boldsymbol{\varepsilon}^t]_{|\mathbf{r}(u)} \right) \right) \mathbf{r}'(u) \quad ,$$

where  $\mathbf{I}$  denotes the identity matrix and  $[\mathbf{D} \boldsymbol{\varepsilon}^t]_{|(t,x,y,z)}$  is a matrix function all of whose elements go to 0 when  $t \rightarrow 0$ .

*Proof.* By definition we know the  $t$ -derivative of  $\mathbf{F}^t$  in terms of the vector field  $\mathbf{V}$ , see Theorem 2.5:

$$(3.7) \quad \frac{\partial}{\partial t} \mathbf{F}^t(x, y, z) = \mathbf{V}(\mathbf{F}^t(x, y, z)) \quad .$$

The Taylor series expansion of  $\mathbf{F}^t(x, y, z)$  with respect to  $t$  at  $t = 0$  for fixed  $(x, y, z)$  is therefore:

$$(3.8) \quad \begin{aligned} \mathbf{F}^t(x, y, z) &= \mathbf{F}^0(x, y, z) + t \left( \frac{\partial}{\partial t} \right)_{|t=0} \mathbf{F}^t(x, y, z) + t \boldsymbol{\varepsilon}^t(x, y, z) \\ &= (x, y, z) + t \mathbf{V}(\mathbf{F}^0(x, y, z)) + t \boldsymbol{\varepsilon}^t(x, y, z) \\ &= (x, y, z) + t \mathbf{V}(x, y, z) + t \boldsymbol{\varepsilon}^t(x, y, z) \quad , \end{aligned}$$

where  $\boldsymbol{\varepsilon}^t$  has the property that  $\boldsymbol{\varepsilon}^t(x, y, z) \rightarrow \mathbf{0}$  for  $t \rightarrow 0$  and for all  $(x, y, z)$ . Moreover it follows from equation (3.8) that for every fixed value of  $t$  in  $\mathcal{D}_{(x,y,z)}$  the function  $\boldsymbol{\varepsilon}^t(x, y, z)$  is a smooth function of the three space variables  $x$ ,  $y$ , and  $z$ . In coordinates, let

$$\boldsymbol{\varepsilon}^t(x, y, z) = (\varepsilon_1^t(x, y, z), \varepsilon_2^t(x, y, z), \varepsilon_3^t(x, y, z)) \quad .$$

The covariant derivative matrix for  $\boldsymbol{\varepsilon}^t$  (with respect to the space variables) is then:

$$(3.9) \quad [\mathbf{D} \boldsymbol{\varepsilon}^t]_{|(x,y,z)} = \begin{bmatrix} \frac{\partial \varepsilon_1^t}{\partial x} & \frac{\partial \varepsilon_1^t}{\partial y} & \frac{\partial \varepsilon_1^t}{\partial z} \\ \frac{\partial \varepsilon_2^t}{\partial x} & \frac{\partial \varepsilon_2^t}{\partial y} & \frac{\partial \varepsilon_2^t}{\partial z} \\ \frac{\partial \varepsilon_3^t}{\partial x} & \frac{\partial \varepsilon_3^t}{\partial y} & \frac{\partial \varepsilon_3^t}{\partial z} \end{bmatrix} \quad .$$

Since the derivatives entering this matrix are derivatives with respect to the space variables only, the matrix  $[\mathbf{D}\boldsymbol{\varepsilon}^t]$  retains the property that all the elements of  $[\mathbf{D}\boldsymbol{\varepsilon}^t]_{|(t,x,y,z)}$  go to 0 for  $t \rightarrow 0$ .

Along the curve  $\mathbf{r}(u)$  we then have directly from (3.8)

$$(3.10) \quad \mathbf{F}^t(\mathbf{r}(u)) = \mathbf{r}(u) + t \left( \mathbf{V}(\mathbf{r}(u)) + \boldsymbol{\varepsilon}^t(\mathbf{r}(u)) \right) ,$$

where  $\boldsymbol{\varepsilon}^t(\mathbf{r}(u))$  is now a smooth function of  $u$  with:

$$(3.11) \quad \frac{\partial}{\partial u} \boldsymbol{\varepsilon}^t(\mathbf{r}(u)) = [\mathbf{D}\boldsymbol{\varepsilon}^t]_{|\mathbf{r}(u)} \mathbf{r}'(u) .$$

In consequence, upon differentiation of equation (3.10) with respect to  $u$ , we arrive at the desired relation:

$$(3.12) \quad \begin{aligned} \frac{\partial}{\partial u} \mathbf{F}^t(\mathbf{r}(u)) &= \mathbf{r}'(u) + t \left( [\mathbf{D}\mathbf{V}]_{|\mathbf{r}(u)} + [\mathbf{D}\boldsymbol{\varepsilon}^t]_{|\mathbf{r}(u)} \right) \mathbf{r}'(u) \\ &= \left( \mathbf{I} + t \left( [\mathbf{D}\mathbf{V}]_{|\mathbf{r}(u)} + [\mathbf{D}\boldsymbol{\varepsilon}^t]_{|\mathbf{r}(u)} \right) \right) \mathbf{r}'(u) . \end{aligned}$$

□

**Definition 3.4.** *The map of the tangent vector  $\mathbf{r}'(u)$  to the corresponding tangent vector of the deformed curve found at the right hand side of (3.12) is called the vectorial push forward map associated with the vector field  $\mathbf{V}$ . This map only depends on time  $t$  and position  $p = \mathbf{r}(u)$  and is given explicitly by (3.6). We denote it by:*

$$(3.13) \quad \mathbf{F}_{*|p}^t = \mathbf{I} + t \left( [\mathbf{D}\mathbf{V}]_{|p} + [\mathbf{D}\boldsymbol{\varepsilon}^t]_{|p} \right) ,$$

so that (3.12) now reads

$$(3.14) \quad \frac{\partial}{\partial u} \mathbf{F}^t(\mathbf{r}(u)) = \mathbf{F}_{*|\mathbf{r}(u)}^t \mathbf{r}'(u) = \mathbf{F}_*^t \mathbf{r}'(u) .$$

For fixed values of  $t$  and  $p$  the map  $\mathbf{F}_{*|p}^t$  is a linear (matrix valued) map or operator which maps (tangent) vectors at  $p$  to (tangent) vectors at  $\mathbf{F}^t(p)$ . The dependence of  $\mathbf{F}_*^t$  on position - here  $\mathbf{r}(u)$  - will often (as on the rightmost side of (3.14)) be suppressed from the notation, whenever there is no danger of confusion.

For later use (in section 5) we note here the following properties of push forward maps:

**Lemma 3.5.** *If we apply the push forward map associated with a given vector field  $\mathbf{V}$  to the vector field itself we get*

$$(3.15) \quad \mathbf{F}_{*|p}^t \mathbf{V}(p) = \mathbf{V}(\mathbf{F}^t(p)) .$$

*In other words, the vectorial push forward map associated with a given vector field  $\mathbf{V}$  preserves this vector field.*



*Proof.* Let  $\gamma(u)$  denote the maximal integral curve for  $\mathbf{V}$  through the point  $p$  so that  $\gamma'(u) = \mathbf{V}(\gamma(u))$  and  $\gamma(0) = p$ . By uniqueness of integral curves,  $\mathbf{F}^t$  maps the integral curve  $\gamma(u)$  into itself. The corresponding vectorial push forward map  $\mathbf{F}_*^t$  maps tangent vectors of a curve to tangent vectors of the image curve. In particular, the tangent vector  $\gamma'(0)$  at  $\gamma(0)$  is therefore mapped into the tangent vector  $\gamma'(u_t)$  at  $\gamma(u_t)$ , where  $u_t$  is that parameter value which corresponds to the point  $\mathbf{F}^t(\gamma(u_0))$ , i.e.  $\gamma(u_t) = \mathbf{F}^t(\gamma(u_0))$ . (Actually  $u_t = u_0 + t$ , but we shall not need this fact.) Thus we have

$$(3.16) \quad \mathbf{F}_*^t \gamma'(u_0) = \gamma'(u_t) \quad ,$$

so that

$$(3.17) \quad \begin{aligned} \mathbf{F}_*^t \mathbf{V}(p) &= \mathbf{V}(\gamma(u_t)) \\ &= \mathbf{V}(\mathbf{F}^t(\gamma(u_0))) \\ &= \mathbf{V}(\mathbf{F}^t(p)) \quad . \end{aligned}$$

□

**Proposition 3.6.** *The push forward map enjoys - and is in fact determined by - the following matrix differential equation along any given integral curve  $\mathbf{F}^t(p)$ ,  $t \in \mathcal{D}_p$ , for the vector field  $\mathbf{V}$ :*

$$(3.18) \quad \frac{\partial}{\partial t} \mathbf{F}_{*|p}^t = [\mathbf{D}\mathbf{V}]_{|\mathbf{F}^t(p)} \mathbf{F}_{*|p}^t \quad , \quad \mathbf{F}_{*|p}^0 = \mathbf{I} \quad .$$

*Proof.* Let  $t_0 \in \mathcal{D}_p$  be a fixed parameter value and assume  $s$  sufficiently small, so that  $t_0 + s \in \mathcal{D}_p$ . Then by construction

$$(3.19) \quad \mathbf{F}_{*|p}^{t_0+s} = \mathbf{F}_{*|\mathbf{F}^{t_0}(p)}^s \mathbf{F}_{*|p}^{t_0} \quad ,$$

where the right hand side is the matrix product of the two matrices corresponding to the two-step vectorial push forward - first from  $p$  to  $\mathbf{F}^{t_0}(p)$  and then from  $\mathbf{F}^{t_0}(p)$  to  $\mathbf{F}^{t_0+s}(p)$  along the same integral curve. It follows that

$$(3.20) \quad \begin{aligned} \left( \frac{\partial}{\partial t} \right)_{|t=t_0} \mathbf{F}_{*|p}^t &= \left( \frac{\partial}{\partial s} \right)_{|s=0} \mathbf{F}_{*|p}^{t_0+s} \\ &= \left( \left( \frac{\partial}{\partial s} \right)_{|s=0} \mathbf{F}_{*|\mathbf{F}^{t_0}(p)}^s \right) \mathbf{F}_{*|p}^{t_0} \\ &= [\mathbf{D}\mathbf{V}]_{|\mathbf{F}^{t_0}(p)} \mathbf{F}_{*|p}^{t_0} \quad . \end{aligned}$$

The latter identity - as well as the initial condition in (3.18) - follows directly from (3.13). □

The following important identity relates the determinant and the trace of time dependent linear maps (matrices) like  $\mathbf{F}_*^t(p)$ :

**Theorem 3.7** (Liouville, see e.g. [H] p. 46). *Let  $\mathbf{A}(t)$  denote a given square matrix-valued smooth function of  $t$ . Let  $\mathbf{Y} = \mathbf{Y}(t)$  be a matrix solution to the first order linear matrix differential equation:*

$$(3.21) \quad \frac{d}{dt} \mathbf{Y}(t) = \mathbf{A}(t) \mathbf{Y}(t) \quad .$$

Then

$$(3.22) \quad \det(\mathbf{Y}(t)) = \det(\mathbf{Y}(0)) \exp \left( \int_0^t \text{trace}(\mathbf{A}(s)) ds \right) \quad ,$$

so that

$$(3.23) \quad \frac{d}{dt} \det(\mathbf{Y}(t)) = \text{trace}(\mathbf{A}(t)) \det(\mathbf{Y}(t)) \quad .$$

In our case we thus have along every integral curve  $\mathbf{F}^t(p)$  for  $\mathbf{V}$ :

**Corollary 3.8** (See [Ar] p. 112 for no less than two elementary proofs).

$$(3.24) \quad \begin{aligned} \frac{d}{dt} \det(\mathbf{F}_{*|p}^t) &= \text{trace}([\mathbf{D} \mathbf{V}]_{|\mathbf{F}^t(p)}) \det(\mathbf{F}_{*|p}^t) \\ &= \text{div}(\mathbf{V})_{|\mathbf{F}^t(p)} \det(\mathbf{F}_{*|p}^t) \quad , \end{aligned}$$

so that in particular, at  $t = 0$ , where  $\det(\mathbf{F}_{*|p}^0) = \det(\mathbf{I}) = 1$  we get:

$$(3.25) \quad \begin{aligned} \left( \frac{d}{dt} \right)_{|t=0} \det(\mathbf{F}_{*|p}^t) &= \text{trace}([\mathbf{D} \mathbf{V}]_{|p}) \\ &= \text{div}(\mathbf{V})(p) \quad . \end{aligned}$$

**Example 3.9** (Example 2.3 continued). In order to illustrate the inner workings of these findings we show a few explicit calculations concerning the vector field in Example 2.3. So let  $\mathbf{r}(u)$  denote a smooth curve and let  $\mathbf{F}^t$  denote the flow map for the vector field  $\mathbf{V}(x, y, z) = (-y, x, z^2)$ . Then we have from (2.7) that

$$\begin{aligned} \mathbf{F}^t(\mathbf{r}(u)) &= \mathbf{F}^t(x(u), y(u), z(u)) \\ &= (x(u) \cos(t) - y(u) \sin(t), \\ &\quad x(u) \sin(t) + y(u) \cos(t), \\ &\quad z(u)/(1 - tz(u))) \quad . \end{aligned}$$

In consequence

$$\begin{aligned} \frac{\partial}{\partial u} \mathbf{F}^t(\mathbf{r}(u)) &= (x'(u) \cos(t) - y'(u) \sin(t), \\ &\quad x'(u) \sin(t) + y'(u) \cos(t), \\ &\quad z'(u)/(1 - tz(u))^2) \\ &= \begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1/(1 - tz(u))^2 \end{bmatrix} \begin{bmatrix} x'(u) \\ y'(u) \\ z'(u) \end{bmatrix} \end{aligned}$$

At the point  $(x, y, z) = p$  we thus get the vectorial push forward map associated with  $\mathbf{V}$ :

$$\begin{aligned}
 (3.26) \quad \mathbf{F}_{*|p}^t &= \begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1/(1-tz)^2 \end{bmatrix} \\
 &= \begin{bmatrix} 1-t\varepsilon(t) & -t+t\varepsilon(t) & 0 \\ t-t\varepsilon(t) & 1-t\varepsilon(t) & 0 \\ 0 & 0 & 1+2tz+t\varepsilon(t,z) \end{bmatrix} \\
 &= \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2z \end{bmatrix} + t \begin{bmatrix} -\varepsilon(t) & \varepsilon(t) & 0 \\ -\varepsilon(t) & -\varepsilon(t) & 0 \\ 0 & 0 & \varepsilon(t,z) \end{bmatrix} \right)
 \end{aligned}$$

for suitable  $\varepsilon$ -functions. Since the covariant derivative of  $\mathbf{V}$  in this example is given by

$$(3.27) \quad [\mathbf{D}\mathbf{V}]_{|(x,y,z)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2z \end{bmatrix},$$

we have thereby verified Proposition 3.3 in this particular case. Moreover, using the exact expressions for  $\mathbf{F}^t(p)$  from equation (2.7) and  $\mathbf{F}_{*|p}^t$  from equation (3.26) it is a direct matter to verify Proposition 3.6 and Corollary 3.8 as well. In particular we note that along the integral curve starting at  $p = (x, y, z)$  the determinant of the push forward map is

$$(3.28) \quad \det(\mathbf{F}_{*|p}^t) = \frac{1}{(1-tz)^2}.$$

This is in accordance with the fact, that  $t$  must belong to  $\mathcal{D}_p$ : If, say,  $z > 0$  or  $z < 0$ , then neither the flow nor the push forward map is defined past the time value  $t = 1/z$ . On the other hand, if  $z = 0$ , then the flow and the push forward is defined for all values of  $t$  and  $\det(\mathbf{F}_{*|p}^t) = 1$  for all  $t \in \mathbb{R}$ . This is due to the fact, that all points in the  $(x, y)$ -plane are just rotated in that plane by the flow map. A smooth curve  $\mathbf{r}(u)$ ,  $u \in [a, b]$ , which crosses once through the  $(x, y)$ -plane is certainly deformed by the flow: as time goes by all the points of the curve - except the point of crossing - will race towards infinity. Nevertheless, the tangent vector to the curve at the cross point will keep its length and will just rotate around the  $z$ -axis (with a fixed 3.rd coordinate) along with the point of crossing.

#### 4. THE VOLUME FLOW AND THE FIRST HALF OF THE THEOREM

We now consider a 3D domain  $\Omega_0$  in space and assume without lack of generality that  $\Omega_0$  is represented by a piecewise smooth and regular

parametrization as follows:

$$(4.1) \quad \Omega_0 : \quad \mathbf{R}(u, v, w) = (x, y, z) \quad , \quad (u, v, w) \in P \subset \mathbb{R}^3 \quad ,$$

where  $x$ ,  $y$ , and  $z$  are given smooth functions of  $u$ ,  $v$ , and  $w$ . The Jacobian function for this parametrization is then

$$(4.2) \quad \text{Jacobi}_{\mathbf{R}}(u, v, w) = |(\mathbf{R}'_u \times \mathbf{R}'_v) \cdot \mathbf{R}'_w| \quad ,$$

so that the volume of  $\Omega_0$  is

$$(4.3) \quad \text{Vol}(\Omega_0) = \int_{\Omega_0} 1 \, d\mu = \int_P \text{Jacobi}_{\mathbf{R}}(u, v, w) \, du \, dv \, dw \quad .$$

Now consider the flow-deformed domain  $\Omega_t = \mathbf{F}^t(\Omega_0)$ :

$$(4.4) \quad \Omega_t : \quad \mathbf{F}^t(\mathbf{R}(u, v, w)) \quad , \quad (u, v, w) \in P \quad .$$

According to Proposition 3.3 and Definition 3.4 the induced parameter curves of the deformed domain has tangential vector fields as follows:

$$(4.5) \quad \frac{\partial}{\partial u} \mathbf{F}^t(\mathbf{R}(u, v, w)) = \mathbf{F}^t_{*|\mathbf{R}(u, v, w)} \mathbf{R}'_u(u, v, w) = \mathbf{F}^t_* \mathbf{R}'_u \quad ,$$

and similarly for the two other coordinate curves. The Jacobian of the induced parametrization of  $\mathbf{F}^t(\Omega_0)$  is then

$$(4.6) \quad \begin{aligned} \text{Jacobi}_{\mathbf{F}^t(\mathbf{R})}(u, v, w) &= |(\mathbf{F}^t_* \mathbf{R}'_u \times \mathbf{F}^t_* \mathbf{R}'_v) \cdot \mathbf{F}^t_* \mathbf{R}'_w| \\ &= \det(\mathbf{F}^t_*) |(\mathbf{R}'_u \times \mathbf{R}'_v) \cdot \mathbf{R}'_w| \\ &= \det(\mathbf{F}^t_*) \text{Jacobi}_{\mathbf{R}}(u, v, w) \quad . \end{aligned}$$

The volume of  $\Omega_t$  is correspondingly

$$(4.7) \quad \begin{aligned} \text{Vol}(\Omega_t) &= \int_{\Omega_t} 1 \, d\mu \\ &= \int_P \text{Jacobi}_{\mathbf{F}^t(\mathbf{R})}(u, v, w) \, du \, dv \, dw \\ &= \int_P \det(\mathbf{F}^t_*) \text{Jacobi}_{\mathbf{R}}(u, v, w) \, du \, dv \, dw \\ &= \int_{\Omega_0} \det(\mathbf{F}^t_*) \, d\mu \quad . \end{aligned}$$

The derivative of this volume function is therefore - via equation (3.25):

$$(4.8) \quad \begin{aligned} \left( \frac{d}{dt} \right)_{|t=0} \text{Vol}(\Omega_t) &= \int_{\Omega_0} \left( \frac{d}{dt} \right)_{|t=0} \det(\mathbf{F}^t_*) \, d\mu \\ &= \int_{\Omega_0} \text{trace} \left( \left( \frac{d}{dt} \right)_{|t=0} \mathbf{F}^t_* \right) \, d\mu \\ &= \int_{\Omega_0} \text{div}(\mathbf{V}) \, d\mu \quad , \end{aligned}$$

which shows the first half of the divergence theorem, Theorem 1.1.  $\square$

## 5. SURFACE FLOW AND THE SECOND HALF OF THE THEOREM

The volume deformation of a given domain may alternatively be considered as traced out by the surface of the domain. If the surface is pushed outwards in the direction of the outward pointing normal vector  $\mathbf{n}_{\partial\Omega_0}$  of  $\partial\Omega_0$ , then the volume is locally increased. If the surface is pushed inwards, i.e. in the direction of  $-\mathbf{n}_{\partial\Omega_0}$ , then the volume is locally decreased.

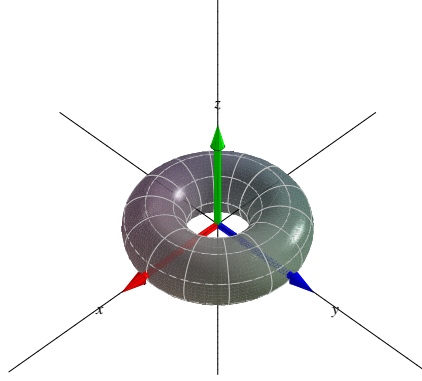


FIGURE 6. A toroidal (solid) domain in space.

We assume without lack of generality, that the boundary  $\partial\Omega_0$  of the domain is a piecewise smooth compact orientable surface which we may, and do, parametrize as follows:

$$(5.1) \quad \partial\Omega_0 : \quad \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \quad , \quad (u, v) \in D \subset \mathbb{R}^2 \quad .$$

The flow map  $\mathbf{F}^t$  deforms the boundary surface  $\partial\Omega_0$  into the boundary surface  $\partial\Omega_t = \mathbf{F}^t(\partial\Omega_0)$ . We now parametrize the 3D shell domain which is traversed by these surfaces as time goes by:

**Lemma 5.1.** *The shell domain  $S^t$  traced out by the surfaces  $\partial\Omega_t$  has the following parametrization:*

$$(5.2) \quad S^t : \quad \mathbf{S}(u, v, w) = \mathbf{F}^w(\mathbf{r}(u, v)) \quad , \quad (u, v) \in D \quad , \quad w \in [0, t] \quad .$$

*The Jacobian function for this parametrization is*

$$(5.3) \quad \text{Jacobi}_{\mathbf{S}}(u, v, w) = \det(\mathbf{F}_*^w) \text{Jacobi}_{\mathbf{r}}(u, v) |\mathbf{n}_{\partial\Omega_0} \cdot \mathbf{V}| \quad .$$

*Proof.* Using Proposition 3.3 and Lemma 3.5 we obtain the respective derivatives:

$$\begin{aligned}
\mathbf{S}'_u(u, v, w) &= \frac{\partial}{\partial u} \mathbf{F}^w(\mathbf{r}(u, v)) \\
&= \mathbf{F}_*^w \mathbf{r}'_u(u, v) , \\
\mathbf{S}'_v(u, v, w) &= \frac{\partial}{\partial v} \mathbf{F}^w(\mathbf{r}(u, v)) \\
&= \mathbf{F}_*^w \mathbf{r}'_v(u, v) , \text{ and} \\
\mathbf{S}'_w(u, v, w) &= \frac{\partial}{\partial w} \mathbf{F}^w(\mathbf{r}(u, v)) \\
&= \mathbf{V}(\mathbf{F}^w(\mathbf{r}(u, v))) \\
&= \mathbf{V}(\mathbf{S}(u, v, w)) \\
&= \mathbf{F}_*^w \mathbf{V}(\mathbf{S}(u, v, 0)) \\
&= \mathbf{F}_*^w \mathbf{V}(\mathbf{r}(u, v)) .
\end{aligned}
\tag{5.4}$$

The Jacobian function of the shell parametrization is thence:

$$\begin{aligned}
\text{Jacobi}_{\mathbf{S}}(u, v, w) &= |(\mathbf{S}'_u \times \mathbf{S}'_v) \cdot \mathbf{S}'_w| \\
&= |(\mathbf{F}_*^w \mathbf{r}'_u \times \mathbf{F}_*^w \mathbf{r}'_v) \cdot \mathbf{F}_*^w \mathbf{V}| \\
&= \det(\mathbf{F}_*^w) |(\mathbf{r}'_u \times \mathbf{r}'_v) \cdot \mathbf{V}| .
\end{aligned}
\tag{5.5}$$

Since we also have by definition that

$$\mathbf{r}'_u \times \mathbf{r}'_v = \text{Jacobi}_{\mathbf{r}}(u, v) \mathbf{n}_{\partial\Omega_0} ,$$

we therefore get as claimed:

$$\text{Jacobi}_{\mathbf{S}}(u, v, w) = \det(\mathbf{F}_*^w) \text{Jacobi}_{\mathbf{r}}(u, v) |\mathbf{n}_{\partial\Omega_0} \cdot \mathbf{V}| .$$

□

The final key point is now to observe, that the previously considered volume  $\text{Vol}(\Omega_t)$  is precisely the volume of  $\Omega_0$  plus the *signed* volume of the shell  $S^t$ , the sign being determined by  $\mathbf{n}_{\partial\Omega_0} \cdot \mathbf{V}$  as alluded to above:

$$\begin{aligned}
\text{Vol}(\Omega_t) &= \text{Vol}(\Omega_0) + \int_{S^t} \text{sign}(\mathbf{n}_{\partial\Omega_0} \cdot \mathbf{V}) d\mu \\
&= \text{Vol}(\Omega_0) + \int_0^t \left( \int_D \text{sign}(\mathbf{n}_{\partial\Omega_0} \cdot \mathbf{V}) \text{Jacobi}_{\mathbf{S}}(u, v, w) du dv \right) dw \\
&= \text{Vol}(\Omega_0) + \int_0^t \left( \int_D (\mathbf{n}_{\partial\Omega_0} \cdot \mathbf{V}) \det(\mathbf{F}_*^w) \text{Jacobi}_{\mathbf{r}}(u, v) du dv \right) dw \\
&= \text{Vol}(\Omega_0) + \int_0^t \left( \int_{\partial\Omega_0} \det(\mathbf{F}_*^w) (\mathbf{n}_{\partial\Omega_0} \cdot \mathbf{V}) d\nu \right) dw ,
\end{aligned}$$

so that, using  $\det(\mathbf{F}_*^0) = \det(\mathbf{I}) = 1$  we finally get from an application of the Fundamental Theorem of Calculus, Theorem 1.2:

$$\begin{aligned} \left(\frac{d}{dt}\right)_{|t=0} \text{Vol}(\Omega_t) &= \left(\frac{d}{dt}\right)_{|t=0} \int_0^t \left( \int_{\partial\Omega_0} \det(\mathbf{F}_*^w) (\mathbf{n}_{\partial\Omega_0} \cdot \mathbf{V}) d\nu \right) dw \\ &= \int_{\partial\Omega_0} \det(\mathbf{F}_*^0) (\mathbf{n}_{\partial\Omega_0} \cdot \mathbf{V}) d\nu \\ &= \int_{\partial\Omega_0} \mathbf{n}_{\partial\Omega_0} \cdot \mathbf{V} d\nu \quad , \end{aligned}$$

which then proves 'the second half' of Theorem 1.1. □

In closing we note that the proof presented here is easily lifted almost verbatim to vector fields and domains in  $\mathbb{R}^n$  for  $n > 3$ .

For planar vector fields and planar domains, i.e. for  $n = 2$ , the divergence theorem follows directly from the 3D version presented above. Indeed, the vector field should just be extended to have 0 as a constant third coordinate and the domain be extended to a finite height 3D cylindrical domain with the given 2D domain as cross section. Then Theorem 1.1 gives the 2D statement except for a constant factor, namely the height of the chosen cylinder.

For  $n = 1$ , the divergence theorem is, of course, nothing but the Fundamental theorem of Calculus, Theorem 1.2.

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