"DECOMPOSITION OF GENERAL QUEUEING NETWORK MODELS"

An investigation into the implementation of hierarchical decomposition schemes of general closed queueing network models using the principle of minimum relative entropy subject to fully decomposable constraints.

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ABSTRACT

Decomposition methods based on the hierarchical partitioning of the state space of queueing network models offer powerful evaluation tools for the performance analysis of computer systems and communication networks. These methods being conventionally implemented capture the exact solution of separable queueing network models but their credibility differs when applied to general queueing networks. This thesis provides a universal information theoretic framework for the implementation of hierarchical decomposition schemes, based on the principle of minimum relative entropy given fully decomposable subset and aggregate utilization, mean queue length and flow-balance constraints. This principle is used, in conjuction with asymptotic connections to infinite capacity queues, to derive new closed form approximations for the conditional and marginal state probabilities of general queueing network models. The minimum relative entropy solutions are implemented iteratively at each decomposition level involving the generalized exponential (GE) distributional model in approximating the general service and asymptotic flow processes in the network. It is shown that the minimum relative entropy joint state probability, subject to mean queue length and flow-balance constraints, is identical to the exact product-form solution obtained as if the network was separable. An investigation into the effect of different couplings of the resource units on the relative accuracy of the approximation is carried out, based on an extensive experimentation. The credibility of the method is demonstrated with some illustrative examples involving first-come-first-served general queueing networks with single and multiple servers and favourable comparisons against exact solutions and other approximations are made.

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CHAPTER I

INTRODUCTION

Most of the scientific activity nowadays is directed towards understanding and dealing with complexity. To this end the development of modern computing systems has probably played the most important role.

Nature and the art of man have no difficulty whatsoever in inventing systems whose exact analysis defies the most powerful present or prospective computers - and large computers and communication networks themselves constitute one class of such systems. Thus, considering performance analysis, even at the top level of technology, seems to be inevitable.

The average computer user, who views computing power as an investment, naturally requires optimum utilization of his system as well as prediction of its performance subject to future alterations of the workload. Moreover, performance prediction is probably most useful for systems at their designing stages. For in todays competitive market designing errors are too costly to be discovered posterior to the development/installation of a computing system.

The most effective method for conducting performance analysis involves modelling a computer system as a network of queues. This is due to the fact that, in the actual system, processes (jobs), generated by users,. compete for access to the various resources (service stations) of the system. Thus, they make their way through this network of units by queueing and receiving service repeatedly until their task is completed. The purpose of modelling is to examine this contention for resources and quantify it in terms of system's performance measures such as: mean queue lengths, utilizations, throughput rates, mean response times.

Parameters of the network that are assumed to be known are:

i) The network configuration which describes the connections between the units and the type of routing.

ii) The number of servers and the service time characteristics for each unit of the network, usually consisting of the first and second moments of the service time, which can be measured in the actual system using special monitors.

iii) The workload intensity which consists of the information available on the jobs that circulate in the network. Depending on the characterization of the workload we have three types of networks, namely open,closed and mixed.

In an open type of network (fig. 1.1) the population of jobs varies over time. In such models jobs arrive into the network

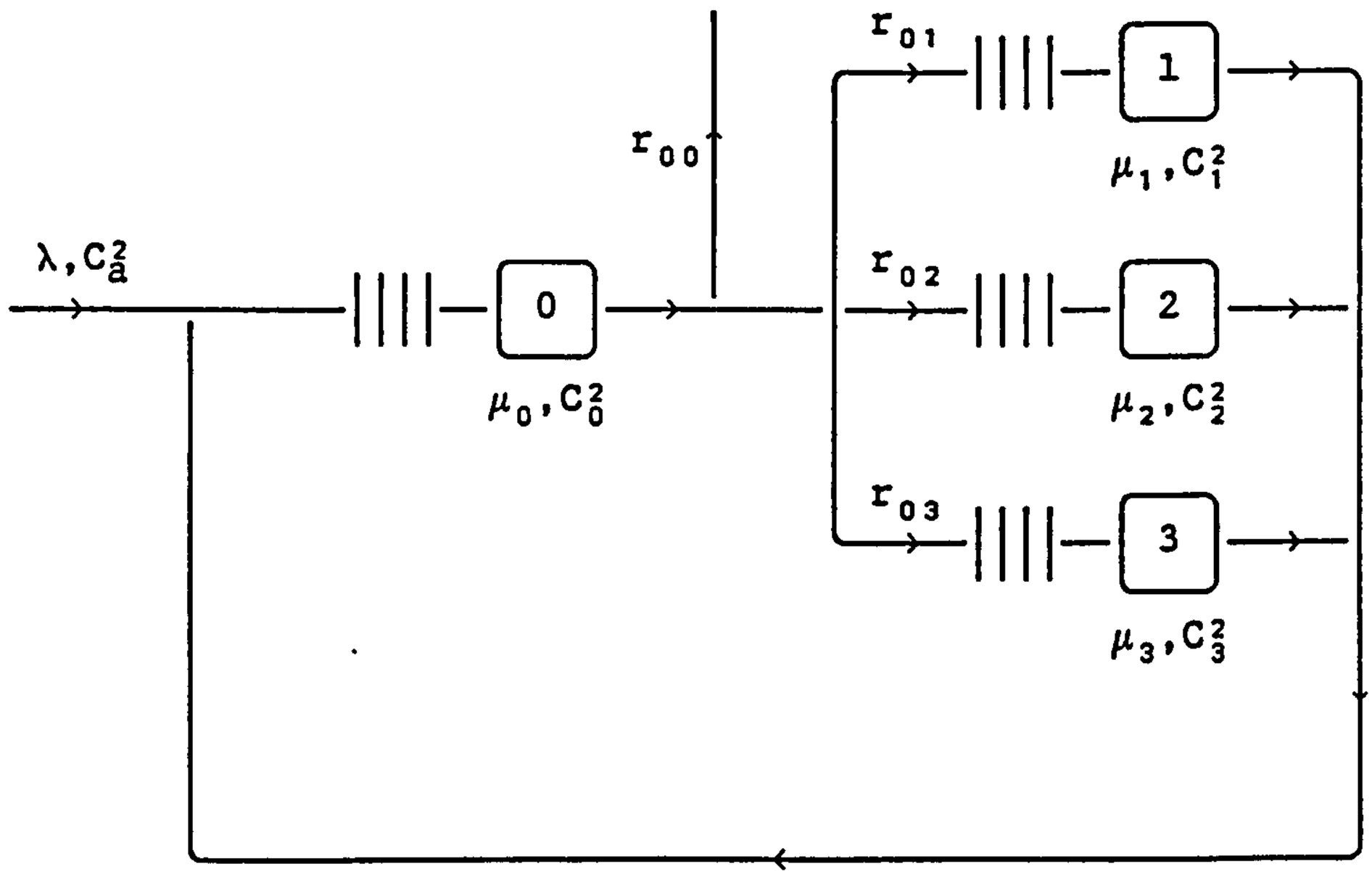

Figure 1.1. Open central server model.

from an external source and having completed their service they leave the network. Open networks are used to model computer communication networks e.t.c. In figure 1.1 we see a small open network of central server type. It consists of a single CPU (unit 0) and three I/O units. External arrivals join the CPU. A job that completes service

at the CPU, joins one of the I/O queues with probability r_{0i}, i=1,2,3, while it departs with probability r_{00}. For each unit i (i=0,1,2,3), μ_i and C_i^2 are the mean service rate and squared coefficient of variation of the service time. The squared coefficient of variation of a random variable X is defined as the ratio of the variance and the square of the mean, $C_x^2=VAR[X]/E[X]^2$. The information about the interarrival times also consists of the arrival rate λ (workload intensity), and the squared coefficient of variation C_a^2.

The closed type involves a fixed number of jobs N circulating in the network. In this case there is no external arrivals into the

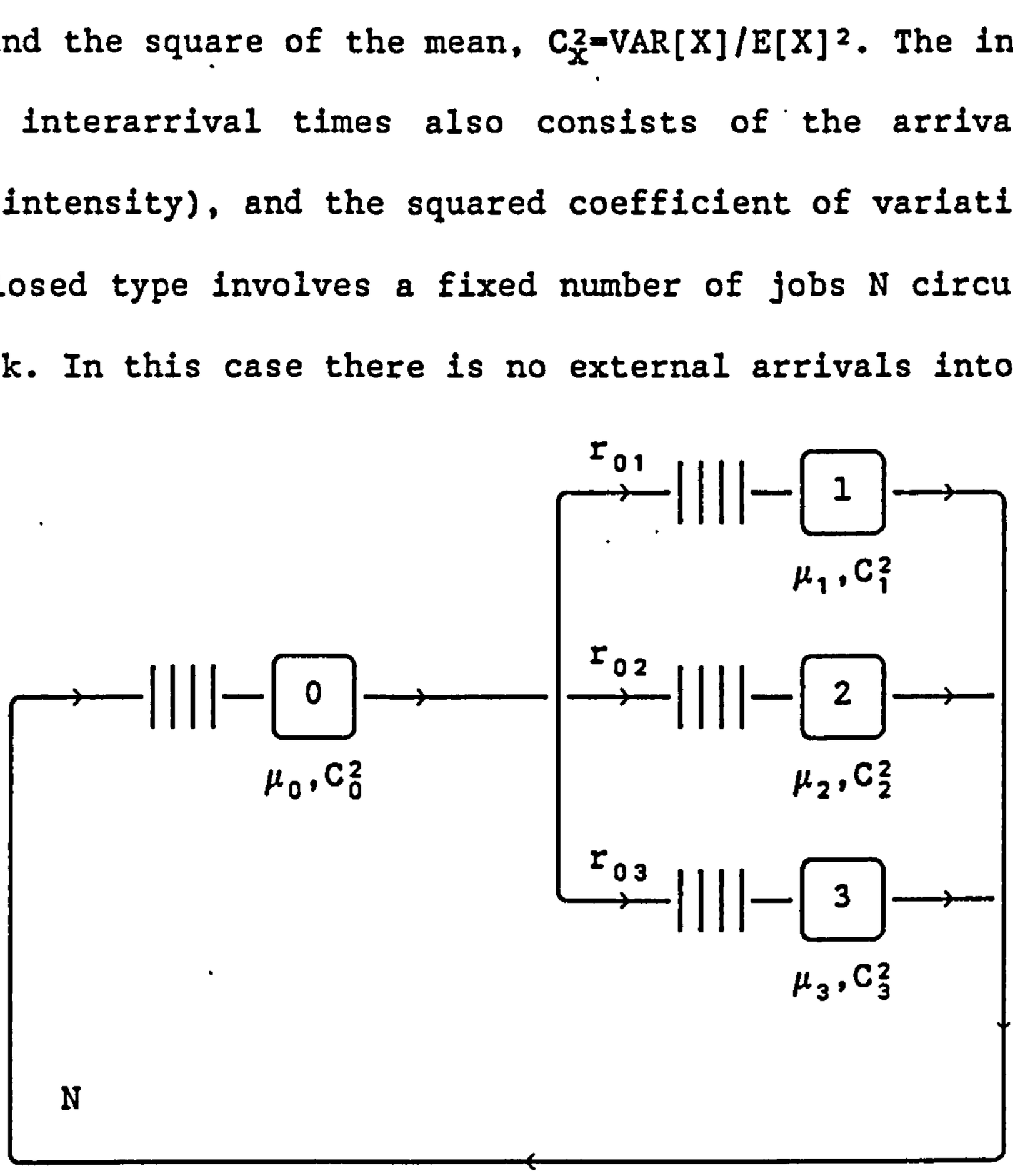

Figure 1.2. Closed central server model.

network and there is also no way out of it. Under a different interpretation, jobs that have completed service can be thought of as leaving the model and being replaced instantaneously from a backlog of waiting jobs accumulated during heavy traffic period. Computing systems are usually modelled as closed type queueing networks. In figure 1.2., we see a closed central server model. Parameters are the same as in figure 1.1 except the workload intensity, which here is described by the fixed number of jobs N.

Mixed networks is a combination of open and closed networks with multiple job classes.

Of course, several other aspects of a system can be parameterized like different classes of customers - first come first served (FCFS) or priority ones - different types of routing like fixed, random or dynamic e.t.c. This project is concerned with the analysis of FCFS, general closed queueing networks with a single class of jobs, fixed or random routing and single or multiple servers.

There are two major types of techniques for the evaluation of a system's performance measures, namely:

i) *Simulation techniques.*

ii) *Analytic techniques.*

The simulation approach to the problem is computationally expensive since one has to model all aspects of the system behaviour, at a level of detail that tends to approach the complexity of the real system. Furthermore there is a difficulty in interpreting the results with a reasonable degree of confidence and thus does not decisevely offer much insight into the problem. Finally such models are usually difficult to modify and expensive to adjust to different design alternatives.

Despite their disadvantages, simulation techniques are quite favourable, especially in cases where analytic methods are not available, usually due to the size and/or complexity of the actual system, and in cases where simplifying assumptions made in an analytic approach critically affect the accuracy of the results due to missing information.

The analytic or mathematical approach to the problem firstly involves the definition of the state space of the network. Each state is a vector whose elements are the states of the individual queues. We are usually interested in calculating the equilibrium values of

the joint state probability distribution.

These equilibrium values express the statistical equilibrium of the state process which is reached after an appropriate amount of time which allows us to disregard any time-dependent behaviour. Having solved for this distribution we may obtain the marginal queue length distribution for each service center as well as the performance measures of interest.

The equilibrium probabilities satisfy a set of equations called '*global balance equations*'. Thus the numerical solution of this system of equations is an alternative way of 'solving' the queueing network model (QNM). Unfortunately the size of such systems grows unmanageable as the number of units and/or the population of customers increases. So this approach is restricted to small networks of reduced complexity.

Since classical queueing theory was involved in modelling of computing systems, a vast amount of scientific effort has been directed towards networks with *exponentially* distributed service times. That is because, i) exact solutions for certain types of systems already existed [JACK 57, JACK 63] and ii) solutions of more complex networks were accomodated by the unique properties of exponentiallity [BASK 75].

Hence this effort has been concentrated in tackling more systems that involve exponential assumptions and producing efficient algorithms for the implementation of this type of solutions [BUZE 73, BRUE 80], as well as defining conditions under which a general network may be considered as if it has exponentially distributed service times. Such networks are called '*separable*' [BASK 75], or under the latest extensions '*quasi-reversible*' in time [KELL 79].

The price to be paid for tractability at the exponential level is unfortunately significant. This should be expected, since the

exponential distribution is completely defined by its first moment and thus such analysis utilizes the minimum possible information about the service times involved in a network. Thus, even though some actual systems have been evaluated successfully using such models, cases where this analysis is inadequate occur very often. This fact provides the motivation for the use of more general distributions, which take into account higher moments of the network's service times, aiming at a more realistic approach.

Moving away from exponential assumptions and into general queueing networks, only approximation techniques are available. These kind of techniques could be viewed as tools that can cope with complex networks, the analysis of which lies beyond the limits of the exact models.

Following the argument that it is better to have an approximate treatment of an accurate model than an exact treatment of an inaccurate model, several approximate methods have been introduced [COUR 77, CHAN 75b, MARI 77, KOUV 86c], which are dealing with QNMs that involve general service-time distributions.

One of the most successful methods, that is also of particular interest to this thesis is the maximum entropy (ME) based approximation, developed by Kouvatsos [KOUV 86c], which belongs to a class of techniques that approximate the solution of the joint probability distribution of the network.

The ME principle is a well known method of inference. Using this principle we may uniquely characterize a probability distribution as satisfying certain mean value type constraints, while being maximally non-commital with regard to any other information.

The ME solution for closed QNMs, proposed by Kouvatsos [KOUV 86c], is an approximate product form solution for the joint probability distribution and may be derived by maximizing the entropy

functional subject to mean value constraints concerning the marginal distribution of the network.

The so called universal maximum entropy (UME) algorithm [KOUV 86c], provides an efficient implementation of the ME solution and uses one-dimensional iterative and convolution techniques. This algorithm proves to be powerful at the level of the *generalized exponential* (GE) distribution, while it is exact - as required - at the exponential level.

This thesis will be mainly concerned with another class of methods, namely *hierarchical decomposition schemes*, for closed QNMs. This type of techniques involve a multi-level partition of the state space in order to decompose the system into constituents that can be understood and analyzed separately and furthermore they provide a model of *macrorelations* among these constituents so that the results of the isolated analyses can be combined to give an evaluation of the whole system behaviour.

There are two major decomposition schemes. The first was introduced by Courtois [COUR 77], and is based on the concepts of *near-complete decomposability* and *variable aggregation*, both borrowed from econometrics. The second is the *flow-equivalent* aggregation method introduced by Chandy, Herzog and Woo [CHAN 75a], and is based on the application of *Norton's theorem* - borrowed from the electrical circuit theory - to queueing networks.

This thesis investigates the conventional implementation of decomposition schemes and proposes a new universal information theoretic framework for the development of powerful decomposition algorithms. The work is motivated by the successful application of ME formalism into the analysis of QNMs and is based on the work by Shore [SHOR 82b] on the principle of *minimum relative entropy* (MRE) - a generalization of the ME principle - under the assumption of *fully*

decomposable constraints. More precisely, the thesis is organized as follows:

The *variable aggregation* and *flow-equivalence* decomposition schemes are introduced in chapter 2. Chapter 3 presents the principle of maximum entropy together with some of its applications in the analysis of queueing network models. Furthermore, the generalized exponential (GE) distributional model is used in the implementation of the decomposition algorithms and illustrative test examples are presented. In the 4th chapter the principle of minimum relative entropy (MRE) is introduced, given fully decomposable subset and aggregate constraints, and a detailed analysis is given. In the 5th chapter the MRE principle is used, in conjuction with assymptotic connections to infinite capacity queues, to derive new closed-form approximations for the solution of general central server models. It is shown that the MRE joint state probability, subject to mean queue length (mql) and flow-balance (fb) subset and aggregate constraints, is identical to the exact product-form solution obtained as if the network was separable. In the 6th chapter the MRE solution is extended by using an additional (utilization) constraint in order to approximate the conditional and marginal state probabilities of more general closed QNMs. Algorithms that implement these solutions for certain network configurations are presented. The GE distributional model is used to represent the service and assymptotic flow processes in the network. The algorithms are thoroughly tested and favourable comparisons against exact and other approximate solutions are made. Lastly, suggestions on the way of coupling the units are made, based on extensive experimentation. Chapter 7 applies the concept of subparallelism, introduced by Vantilborgh [VANT 78], in order to propose an extended and universal MRE decomposition algorithm for arbitrary network configurations and rigorous comparisons are carried

out. This algorithm is generalized in the 8th chapter for the case of general queueing networks with multiple servers and several numerical tests are made. Finally in chapter 9 the results of this project are summarized and suggestions for future work are made.

CHAPTER II

DECOMPOSITION SCHEMES FOR QUEUEING NETWORKS

A review on the two major decomposition schemes - the variable aggregation and flow-equivalence - for queueing network models is presented below.

2.1 The variable aggregation decomposition scheme.

Courtois in his monograph [COUR 77], considered stochastic systems of the form:

$$y(t+1) = y(t)Q \qquad (2.1)$$

where y(t) is a row probability vector with elements $y_\ell(t)$ being the unconditional probability of the system being in the state ℓ, (ℓ=1,...,n) out of n possible states, at time t. Q is a stochastic matrix of order n and an element $q_{k\ell}$ of this matrix is the conditional probability that the system is in state ℓ at time t, given that it was in state k at time t-1.

Such a system is called *completely decomposable* when it can be represented by a completely decomposable matrix, i.e. a square matrix such that an identical permutation of rows and columns leaves a set of square matrices on the principal diagonal and zeros everywhere else. *Near-complete decomposability* is defined by replacing the zeros in the above definition by small nonzero numbers.

The *aggregation of variables* is a technique based on the idea that in many complex systems all variables can somehow be clustered into a small number of groups so that: i) interactions among variables of each group can be examined as if interactions between

groups did not exist and ii) interactions among groups can be studied without reference to the interactions within groups. Completely decomposable systems, as defined previously, consist of independent subsystems each of which can be analyzed in isolation without reference to the others. Unfortunately, complex systems rarely belong to this rather trivial category.

Nevertheless Simon and Ando [SIMO 61], investigated circumstances under which variable aggregation still yields satisfactory results when interactions between groups of variables are nonnull but weak compared to interactions within groups. Such systems were qualified as nearly-completely decomposable systems [FISH 62]. This investigation produced two theorems which Courtois applied in systems described by (2.1).

The first theorem asserts that the analysis of a system, under the assumption that it is completely decomposable, will remain approximately valid in all respects in the short run, provided of course that intergroup dependencies are sufficiently weak, as compared to intragroup ones. This merely means that if neglected influences are weak they take a long time to matter much.

The second and most important theorem concerns the long run behaviour of the system and in particular the relative behaviour of the variables within each group. It states that under the near-complete decomposability conditions, when neglected intragroup dependencies have had time to influence the system behaviour, the values of the variables within each group will remain approximately in the same ratio, as if those influences had never existed. So the results obtained in the short run will therefore remain approximately valid in the long run as far as the relative behaviour of the variables of the same group is concerned.

In fact the Simon-Ando theorems are only existence theorems. What

they guarantee is that whatever standard of approximation is required, a nonzero degree of near-complete decomposability always exists which is sufficient to produce results satisfying that standard. Courtois tackled the other end of the problem, i.e. given the characteristics of the system, which standard of approximation can be guaranteed when a multi-level hierarchical aggregation is used. The error analysis that he conducted, revealed that this approximation depends on the degree of coupling and on the indecomposability of these subsystems. Furthermore, he proposed a sufficient condition for near-complete decomposability which serves as a criterion for variable aggregation in the application of this theory in queueing network models.

2.1.1 The exponential network.

The basic model used by Courtois [COUR 77], to demonstrate the application of the ideas described so far in QNMs, has also been studied by Jackson [JACK 63] and Gordon and Newell [GORD 67].

So let's consider a network of L+1 resources $\Sigma_0,\Sigma_1,\ldots,\Sigma_L$, each providing an exponential service with mean service rate μ_ℓ, $\ell=0,\ldots,L$. A customer that completes service at resource Σ_m, applies immediately to resource Σ_ℓ with probability $r_{m\ell}$, $0\leq m,\ell\leq L$. Then for $m=0,\ldots,L$, $\sum_{\ell=0}^{L} r_{m\ell} = 1$.

Let N ($N < +\infty$) be the total number of customers in the network. The state of the system is described by a (L+1)-tuple$(n_0,n_1,\ldots,n_L)$, where n_ℓ is the number of customers present at resource Σ_ℓ. Obviously

$$\sum_{\ell=0}^{L} n_\ell = N.$$

The number of distinguishable states in this system is $\binom{L+N}{N}$. If $p(n_0,n_1,\ldots,n_L,t)$ is the probability that at time t the system is at state $(n_0,n_1,\ldots,n_L)$ then these probabilities satisfy the system of

linear equations:

$$p(n_0,n_1,\ldots,n_L,t+1) = p(n_0,n_1,\ldots,n_L,t)\Big[1-\sum_{\ell=0}^{L} k(n_\ell)\mu_\ell(1-r_{\ell\ell})\Big] + \sum_{\ell=0}^{L}\sum_{\substack{m=0 \\ m\neq\ell}}^{L} k(n_m)p(n_0,\ldots,n_\ell+1,\ldots,n_m-1,\ldots,n_L,t)\mu_\ell r_{\ell m} \qquad (2.2)$$

where

$$k(n_\ell) = \begin{cases} 0 & \text{if } n_\ell=0 \\ 1 & \text{if } n_\ell\neq 0 \end{cases}$$

This system can be written in matrix form as:

$$P(t+1) = P(t)Q \qquad (2.3)$$

This stochastic matrix Q is the matrix of transition probabilities between states which are ordered in lexicographic manner so that the sum $\sum_{\ell=0}^{L} n_\ell N^\ell$ takes increasing values. Let's denote this matrix as:

$$Q = Q(N,L) \qquad (2.4)$$

An example of Q(2,3) is given in figure 2.1., where the term Σ_i in

$n_0n_1n_2n_3$	2000	1100	0200	1010	0110	0020	1001	0101	0011	0002
2000	$1-\sum_1$	μ_0r_{01}	0	μ_0r_{02}	0	0	μ_0r_{03}	0	0	0
1100	μ_1r_{10}	$1-\sum_2$	μ_0r_{01}	μ_1r_{12}	μ_0r_{02}	0	μ_1r_{13}	μ_0r_{03}	0	0
0200	0	μ_1r_{10}	$1-\sum_3$	0	μ_1r_{12}	0	0	μ_1r_{13}	0	0
1010	μ_2r_{20}	μ_2r_{21}	0	$1-\sum_4$	μ_0r_{01}	μ_0r_{02}	μ_2r_{23}	0	μ_0r_{03}	0
0110	0	μ_2r_{20}	μ_2r_{21}	μ_1r_{10}	$1-\sum_5$	μ_1r_{12}	0	μ_2r_{23}	μ_1r_{13}	0
0020	0	0	0	μ_2r_{20}	μ_2r_{21}	$1-\sum_6$	0	0	μ_2r_{23}	0
1001	μ_3r_{30}	μ_3r_{31}	0	μ_3r_{32}	0	0	$1-\sum_7$	μ_0r_{01}	μ_0r_{02}	μ_0r_{03}
0101	0	μ_3r_{30}	μ_3r_{31}	0	μ_3r_{32}	0	μ_1r_{10}	$1-\sum_8$	μ_1r_{12}	μ_1r_{13}
0011	0	0	0	μ_3r_{30}	μ_3r_{31}	μ_3r_{32}	μ_2r_{20}	μ_2r_{21}	$1-\sum_9$	μ_2r_{23}
0002	0	0	0	0	0	0	μ_3r_{30}	μ_3r_{31}	μ_3r_{32}	$1-\sum_{10}$

Figure 2.1 Matrix Q(N,L), N=2, L=3

each diagonal element is equal to the sum of the off-diagonal elements of the corresponding row.

Then the proof of the following lemma, [COUR 77, p. 60], is mainly based on the lexicographic ordering of the states in matrix Q(N,L).

Lemma 2.1. The stochastic matrix Q(N,L), N>0, L>1, may be partitioned among N+1 principal submatrices $Q'(n_L)$, n_L=0,1,...,N. Each submatrix $Q'(n_L)$ is, except for its main diagonal, identical to a matrix $Q(N-n_L,L-1)$. In figure 2.1, solid lines isolate submatrices $Q'(n_3)$, n_3=0,1,2.

Next for every level of aggregation ℓ, ℓ=1,2,...,L-1, an upper bound ω_ℓ, to the maximum degree of coupling ε_ℓ is defined through the following theorem, [COUR 77, p. 61].

Theorem 2.1. If for ℓ=1,2,...,L-1,

$$\omega_\ell = \max_{\substack{n_0,..,n_L \\ \sum n_\ell=N}} \left[\sum_{i=\ell+1}^{L} k(n_i)\mu_i \sum_{m=0}^{\ell} r_{im} + \sum_{i=0}^{\ell} k(n_i)\mu_i \sum_{m=\ell+1}^{L} r_{im} \right] \qquad (2.5)$$

is sufficiently small, then the stochastic matrix Q(N,L), N>0, L>1, defines a (L-1)-level nearly-completely decomposable sustem. Each level of aggregation ℓ consists of $\begin{bmatrix} L-\ell+N \\ L-\ell \end{bmatrix}$ aggregates, which belong to N+1 classes of equivalence $Q(n,\ell)$, n=0,...,N.

The following theorem, [COUR 77, p. 65], is the sufficient condition for (L-1)-level nearly-completely decomposable system.

Theorem 2.2. For the stochastic matrix Q(N,L), N>0, L>1, to be (L-1)-level nearly-completely decomposable, with each level of aggregation ℓ consisting of N+1 distinct equivalence classes of aggregates $Q(n,\ell)$, n=0,...,N, it is sufficient that for ℓ=1,...,L-1

$$\omega_\ell \leq \frac{1}{2}\,(A_\ell+B_\ell) - (A_\ell B_\ell)^{\frac{1}{2}} \cos[\,\pi/(N+1)\,] \qquad (2.6)$$

where ω_ℓ is defined by (2.5) and:

$$A_\ell = \min_{0\leq k\leq \ell-1} (\mu_k r_{k\ell}) \qquad (2.7)$$

$$B_\ell = \mu_\ell \sum_{k=0}^{\ell-1} r_{\ell k} \qquad (2.8)$$

For more details on the above theorems and the concepts involved we refer to Courtois' monograph [COUR 77, p.p. 11-68].

Practically condition (2.6) ensures us that at level ℓ of aggregation the subnetwork that consists of resources $\Sigma_0,\Sigma_1,\ldots,\Sigma_{\ell-1}$ will reach its equilibrium before the system of resources $\Sigma_0,\Sigma_1,\ldots,\Sigma_{\ell-1},\Sigma_\ell$ starts evolving towards its equilibrium. Hence for the analysis of the behaviour of aggregates $Q(n,\ell)$ we can assume that aggregates $Q(m,\ell-1)$, $m=0,\ldots,n$, are in statistical equilibrium or in other words that the distribution of customers among $\Sigma_0,\Sigma_1,\ldots,\Sigma_{\ell-1}$ is approximately stationary. So the behaviour of $Q(n,\ell)$ can be regarded as a convergence toward equilibrium of the customer distribution between resource Σ_ℓ on the one hand and an aggregate resource on the other, which consists of resources $\Sigma_0,\Sigma_1,\ldots,\Sigma_{\ell-1}$.

With each aggregate $Q(n,\ell)$ a queueing system, denoted $M_\ell(N_\ell)$, is associated and is represented as in figure 2.2. In this system N_ℓ customers request alternatively Σ_ℓ and the aggregate resource $\{\Sigma_0,\Sigma_1,\ldots,\Sigma_{\ell-1}\}$. $M_\ell(N_\ell)$ is in state $E_\ell(N_{\ell-1}/N_\ell)$, $N_{\ell-1}=0,1,\ldots,N_\ell$, whenever $N_{\ell-1}$ customers are in the aggregate resource and $n_\ell=N_\ell-N_{\ell-1}$ are present at resource Σ_ℓ.

All interactions between $M_\ell(N_\ell)$ and units $\Sigma_{\ell+1},\ldots,\Sigma_L$ can be disregarded. Hence, $M_\ell(N_\ell)$ can be considered (approximately) as a closed system that no customer can leave or enter. So the population N_ℓ can be regarded as remaining constant. Under the assumption of

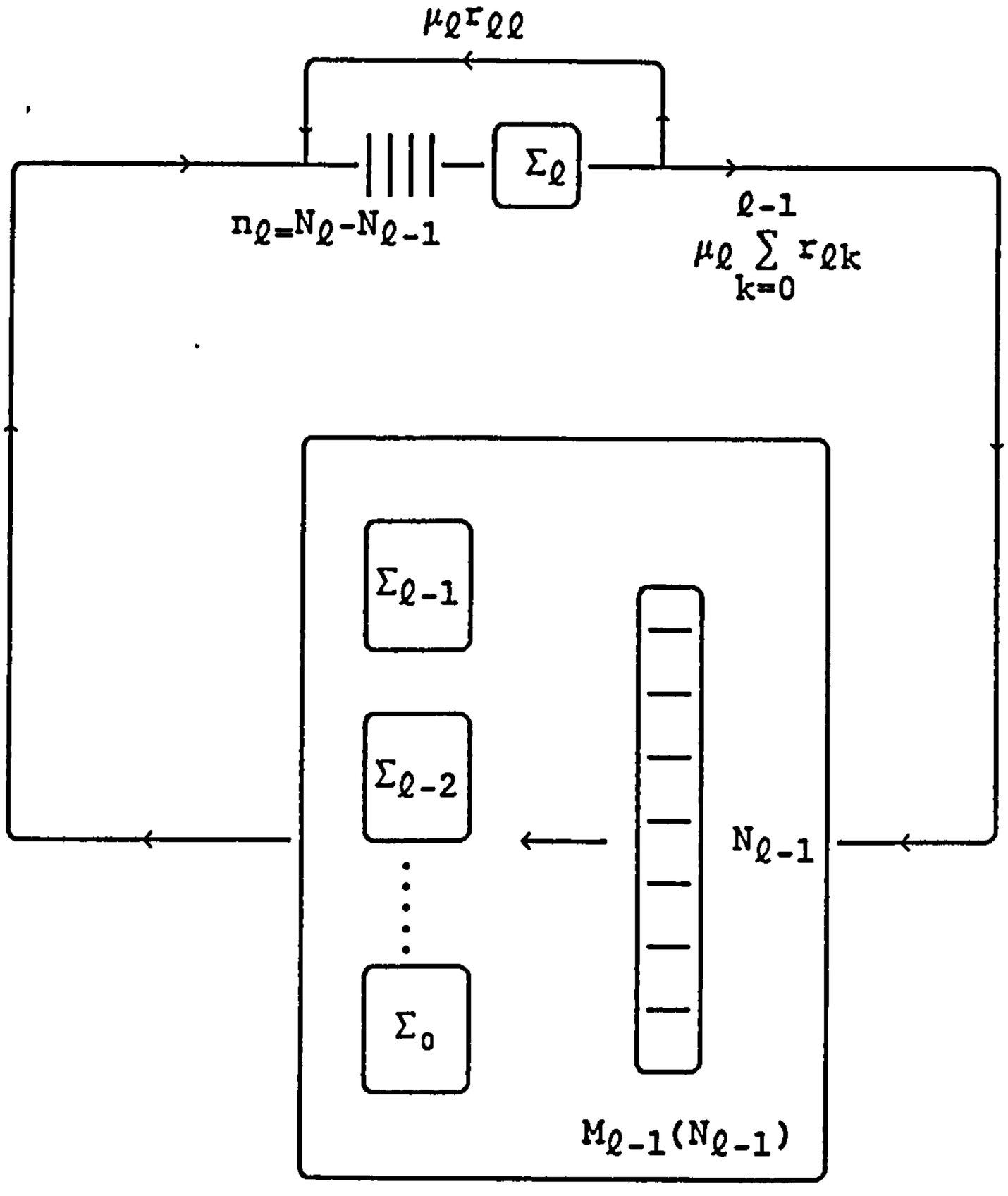

Figure 2.2. Queueing system $M_\ell(N_\ell)$.

irreducibility for $Q(n,\ell)$, there exists a stationary conditional distribution:

$$P_\ell(N_{\ell-1}/N_\ell) = \Pr\{M_\ell(N_\ell) \text{ is in state } E_\ell(N_{\ell-1}/N_\ell)\}$$

$$N_{\ell-1}=0,\ldots,N_\ell \qquad (2.9)$$

which is independent of time and of the initial state of $M_\ell(N_\ell)$. Finally by virtue of this decomposition scheme, the aggregate resource of $M_\ell(N_\ell)$ is nothing but the queueing system $M_{\ell-1}(N_{\ell-1})$ whenever $M_\ell(N_\ell)$ is in state $E_\ell(N_{\ell-1}/N_\ell)$. Thus distribution (2.9) may be evaluated as the solution of an $M/M(n)/1/N_\ell$ two stage cyclic system, where the first M stands for the exponentially distributed service time at resource Σ_ℓ, while M(n) stands for an exponentially distributed aggregate server with load dependent service rates

denoted as $\Psi_{\ell-1,\ell}(N_{\ell-1})$ whenever $N_{\ell-1}$ jobs circulate in the aggregate. It is easy to see that $\Psi_{\ell,k}(N_\ell)$ obeys the recurrence relations:

$$\Psi_{\ell,k}(N_\ell) = [1-P_\ell(N_\ell/N_\ell)]\mu_\ell r_{\ell k} + \sum_{N_{\ell-1}=1}^{N_\ell} P_\ell(N_{\ell-1}/N_\ell)\Psi_{\ell-1,k}(N_{\ell-1})$$

$$k>\ell,\ \ell=1,\ldots,L-1,\ N_\ell=1,\ldots,N \qquad (2.10)$$

and $$\Psi_{0,k}(n_0) = \mu_0 r_{0k} \qquad \forall\ n_0>0 \qquad (2.11)$$

Having derived the conditional distributions $\{ P_\ell(N_{\ell-1}/N_\ell),\ N_{\ell-1}=0,\ldots,N_\ell \}$, for $\ell=1,\ldots,L$, $N_\ell=1,\ldots,N$, let $\alpha_\ell(N_{\ell-1})$, $N_{\ell-1}=0,\ldots,N$, $\ell=1,\ldots,N$, be the unconditional equilibrium probability of $N_{\ell-1}$ jobs being at the aggregate resource $(\Sigma_0,\Sigma_1,\ldots,\Sigma_{\ell-1})$. Then clearly:

$$\alpha_L(N_{L-1}) = P_L(N_{L-1}/N),\ N_{L-1}=0,\ldots,N \qquad (2.12a)$$

and if $P_\ell(n_\ell)$ is the corresponding marginal probability of n_ℓ jobs at resource Σ_ℓ then

$$P_L(n_L) = P_L(N-n_L/N),\ n_L=0,\ldots,N \qquad (2.12b)$$

and for each level $\ell=L-1,\ldots,1$, and for $N_{\ell-1},n_\ell=0,\ldots,N$

$$\alpha_\ell(N_{\ell-1}) = \sum_{N_\ell=N_{\ell-1}}^{N} P_\ell(N_{\ell-1}/N_\ell)\alpha_{\ell+1}(N_\ell) \qquad (2.13a)$$

$$P_\ell(n_\ell) = \sum_{N_\ell=n_\ell}^{N} P_\ell(N_\ell-n_\ell/N_\ell)\alpha_{\ell+1}(N_\ell) \qquad (2.13b)$$

and $$P_0(n_0) = \alpha_1(n_0) \qquad (2.13c)$$

2.1.2 General service time distributions.

Up to now, and through the exponential network, the variable aggregation decomposition framework has been outlined. The most interesting aspect of it is that it can be adjusted and used as an approximation for networks with generally (G) distributed service times.

So let's consider the same network under the assumption that service times are described by an arbitrary (G) distribution. At every level of aggregation ℓ, $\ell=1,\ldots,L$, and for $N_\ell=0,\ldots,N$, the conditional distribution (2.9) can be evaluated using the same queueing system $M_\ell(N_\ell)$. The difference of course is that unit Σ_ℓ has generally distributed service time as well as the units that are represented by the aggregate server (figure 2.2). This imposes the problem of the definition of a distributional form for this aggregate resource. Courtois [COUR 77,p. 81], suggested that this form can be approximated using the exponential model. The justification behind this assumption is based on the fact that it is asymptotically exact. More precisely the output flow from the aggregate resource is nothing else but the superposition of the output processes from the individual resources $\Sigma_0,\Sigma_1,\ldots,\Sigma_{\ell-1}$. Each of these processes is certainly not a Poisson process - in fact it is not even a renewal process [FINC 59]. But if the number of processes is infinite, and under the further assumptions that i) each of them is renewal and ii) they are mutually independent, Cox and Smith [COX 54], have proven that the merged stream forms a Poisson process.

Hence, the conditional distribution (2.9) can be approximated by solving a $G/M(n)/1/N_\ell$ two stage cyclic system, where G stands for the distribution of unit Σ_ℓ and M(n) for the exponentially distributed composite server with load-dependent service rates that are given by (2.10).

An interesting observation is that the fact that all aggregates were analyzed using the same analytical technique is not a constraint imposed by decomposition. So different methods could be applied to analyze each aggregate (heterogenous aggregation [COUR 77, p. 82]). This is most important, for example, at the first level of aggregation (ℓ=1), when the service times involved are generally distributed. There we have to evaluate for N_1=0,...,N distribution $\{P_1(n_0/N_1),\ n_0=0,\ldots,N_1\}$ by isolating units Σ_0 and Σ_1 (figure 2.3). Because none of the units is an aggregate server, any exponential assumption for the distributions involved seems to be a certain source of error. In order to avoid this the conditional distribution has to be evaluated by solving a $G/G/1/N_1$ two stage cyclic system.

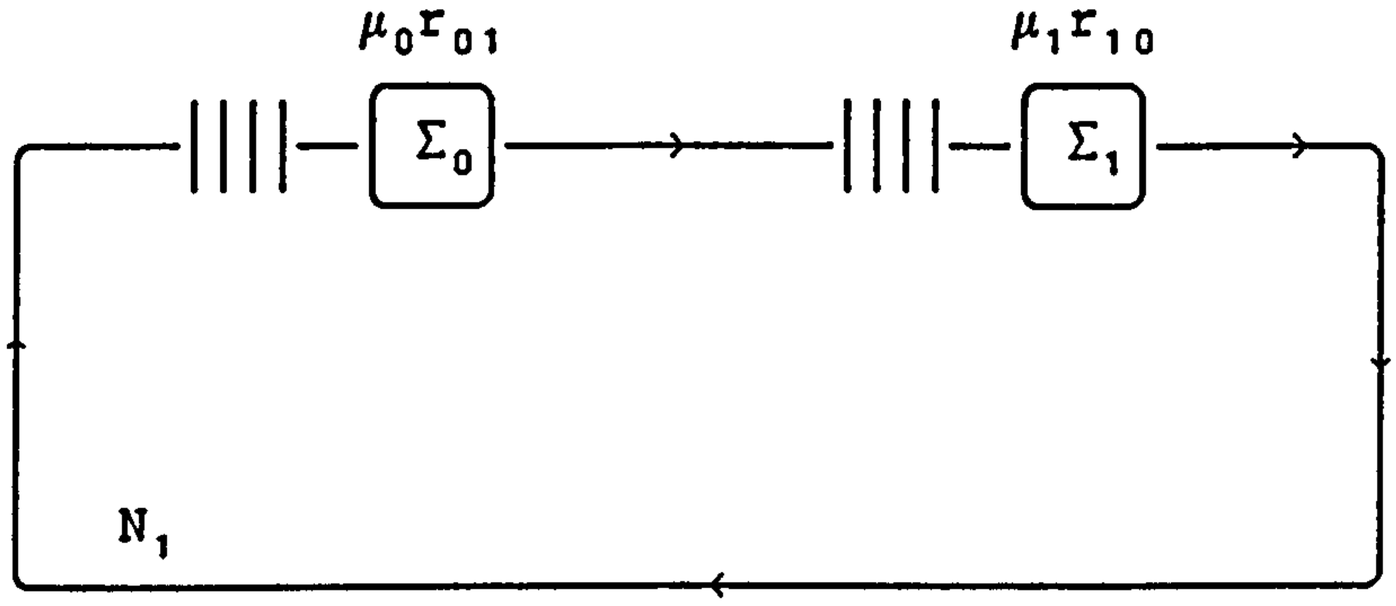

Figure 2.3. Queueing system $M_1(N_1)$.

So the basic steps of a decomposition algorithm for the approximation of the solution of a generally distributed QNM, according to the variable aggregation scheme [COUR 77], are as follows :

Algorithm 2.1.

Step 1. In the first level of aggregation, (ℓ=1), evaluate the conditional distributions $\{P_1(n_0/N_1),\ n_0=0,\ldots,N_1\}$, N_1=1,...,N, ($P_1(0/0)$=1 trivially), using a $G/G/1/N_1$ solution for the two stage cyclic system of figure 2.3 as available. Notice the reduced service rates to be used, i.e. $\mu_0 r_{01}$ for unit Σ_0 and $\mu_1 r_{10}$ for unit Σ_1.

Having evaluated the trivial rates $\Psi_{0,k}(n)=\mu_0 r_{0k}$, $k>0$, $n\neq 0$, calculate the rates $\Psi_{1,,k}(N_1)$, $k>1$, $N_1=0,\ldots,N$ using (2.10).

Step 2. In the ℓth level of aggregation, $\ell=2,\ldots,L$ evaluate the conditional distributions $\{P_\ell(N_{\ell-1}/N_\ell),\ N_{\ell-1}=0,\ldots,N_\ell\}$, $N_\ell=1,\ldots,N$, ($P_\ell(0/0)=1$ trivially), using a $G/M(n)/1/N_\ell$ solution for the system $M_\ell(N_\ell)$ of figure 2.2. The load-dependent rates of the exponential composite server are $\Psi_{\ell-1,\ell}(n)$, $n=1,\ldots,N$, and are known from the previous level, while unit Σ_ℓ has a reduced service rate equal to

$$\mu_\ell \sum_{k=0}^{\ell-1} r_{\ell k}.$$

Lastly, evaluate rates $\Psi_{\ell,k}(N_\ell)$, $k>\ell$, $N_\ell=0,\ldots,N$, using (2.10).

Step 3. Having all the conditional distributions from steps 1 and 2, calculate the marginal distributions $\{P_\ell(n_\ell),\ n_\ell=0,\ldots,N\}$, $\ell=0,\ldots,L$, using relations (2.12)-(2.13).

This algorithm produces the exact solution of a QNM, only when G=M, i.e. service times are exponentially distributed. However, the definition of the service rates involved in every level of aggregation imposes a further restriction on the network configuration that can be tackled using this approach.

2.1.3 Discussion.

This decomposition technique is based on the elaborate and well structured theory of near-complete decomposability. The basic idea of this approach is to create a time-space relation or to partition the state space of a queueing network in groups that reach their equilibrium during a time interval and may be considered at equilibrium from then on.

More precisely, in a multilevel nearly-completely decomposable model, aggregates of successive levels reach their internal equilibrium states at successive time instants T_ℓ, $\ell=1,\ldots,L$. For each such time instant the analysis yields the relative equilibrium

values of the aggregative variables in terms of which the system is described at that level of aggregation.

Courtois work [COUR 77] has produced the sufficient condition for (L-1)-level near-complete decomposability described in theorem 2.2. It concerns the enumeration of the units of the network or in other words the alternative ways of "*coupling*" the units, since generally this method produces different results under different orderings of the network's resources.

This condition would have been very significant if it was also necessary, because as it stands it is of small practical usefulness. More precisely, in QNMs with exponentially distributed service times and in network configurations where algorithm 2.1 produces the exact solution, irrespective of the resources' ordering, conditions 2.6 are rarely satisfied. Nevertheless, this error analysis is of importance since it has been used as a basis to a different approach in the analysis of QNMs, namely bounded aggregation [COUR 84, COUR 86, BALS 87].

There are two conservation laws that any approximate solution of a closed QNM must satisfy. The first concerns the resulting mean queue lengths and statesthat they should add up to N,

$$\sum_{k=0}^{L} \sum_{n=1}^{N} nP_k(n) = N \qquad (2.14)$$

where $P_k(n)$, N, L, are as defined previously. The second law is known as "*flow conservation law*" and concerns the resulting throughputs. It states that for every unit k, k=0,...,L, of the network

$$[1-P_k(0)]\mu_k = \sum_{\ell=0}^{L} [1-P_\ell(0)]\mu_\ell \, r_{\ell k} \qquad (2.15)$$

should be satisfied. When (2.15) holds the network is called "*flow balanced*". This term describes the unconditional requirement that at equilibrium the rate of jobs that depart from each unit must equal

the rate of jobs that arrive at that unit.

It can be easily proved that for the first law (2.14) it is sufficient that the joint probability distribution is well defined and normalized. This is guaranteed in the aggregation-disaggregation scheme that we have described so far. This is not the case though with the second law. The service rates that are introduced by Courtois and used at every level of aggregation do not guarantee that in an arbitrary network configuration, (2.15) is satisfied. Thus, even in an exponential network, algorithm 2.1 does not produce the exact results in all cases.

Another problem that should be addressed at this point concerns the solutions of systems required for the implementation of algorithm 2.1. At the first level of aggregation we need the solution of a $G/G/1/N_1$ system, while at the ℓth level, $\ell=2,\ldots,L$, we need the solution of a $G/M(n)/1/N_\ell$ system. Closed form expressions for such solutions are not available for most general distributions. In certain cases numerical techniques have to be used, which increase the computational cost and decrease the reliability of the algorithm. Thus, even though algorithm 2.1 is described to tackle any general network, is in fact restricted to a small class of distributional forms.

Finally the assumption of an exponentially distributed composite server at every level of aggregation is a potentially significant source of error. However, the choise of a more general distribution to describe these composite servers, simply shifts the problem in the definition of the second and higher moments of this distribution.

2.2 Norton's reduction decomposition scheme.

This second decomposition scheme was introduced by Chandy, Herzog and Woo, [CHAN 75a]. The basic idea was borrowed from the electrical

circuit theory, where it is known as Norton's theorem. The application in the analysis of QNMs was made by analogy.

Given a closed QNM with L+1 resources $\{\Sigma_0,\Sigma_1,\ldots,\Sigma_L\}$ and a fixed number of jobs (level of multiprogramming) N, and assuming that in this particular configuration a subnetwork σ has a common input stream and a common output stream (fig. 2.4), according to this technique, subsystem σ may be replaced by a composite server. This replacement is called aggregation and the subnetwork σ is called aggregate. From the point of view of the complementary network $\sigma^{(c)}$, this composite server, which is a single service center, should behave identically to the aggregate itself. Actually if it is to

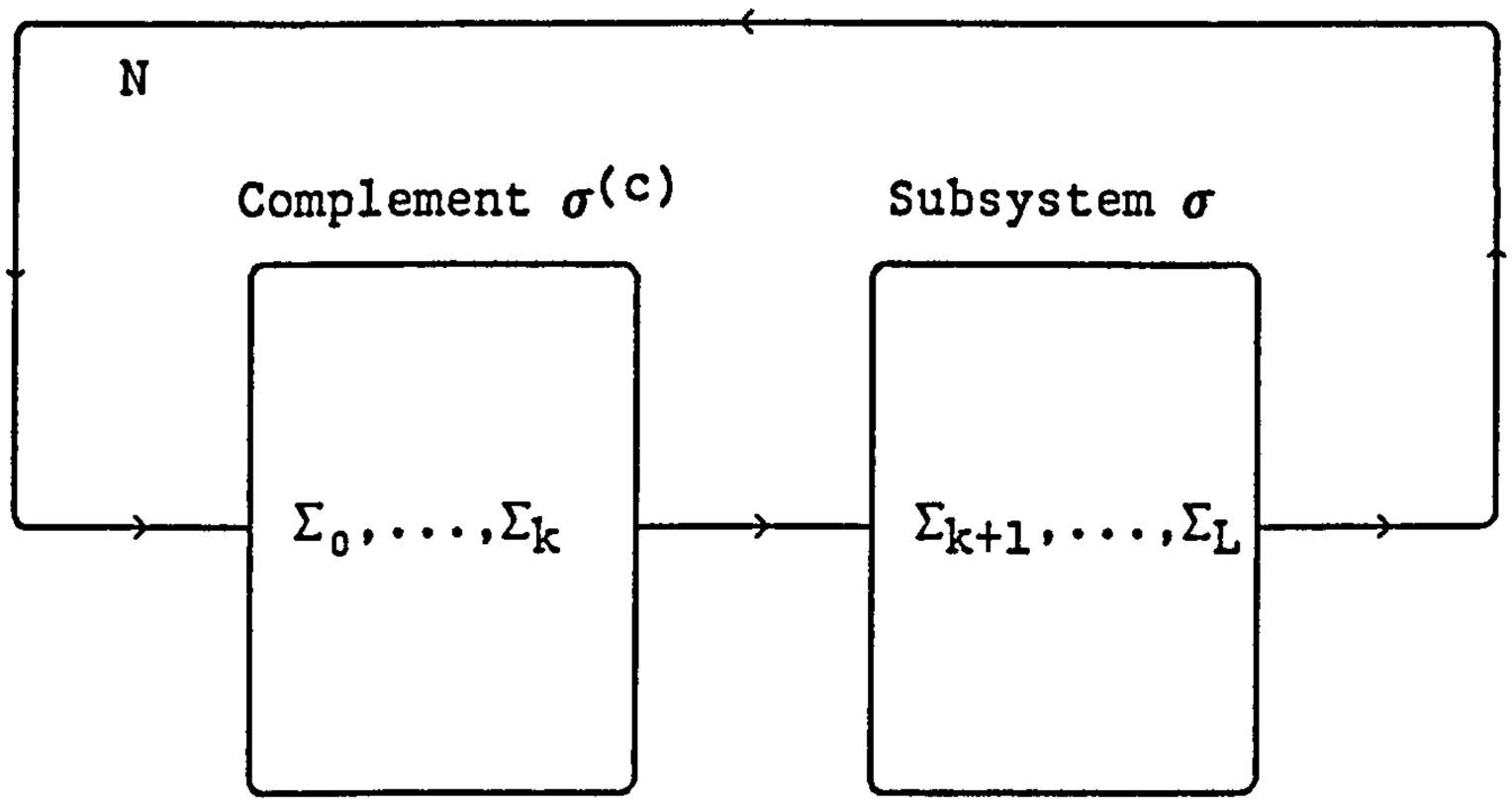

Figure 2.4. A closed QNM.

mimic the exact behaviour of the aggregate, it should have the actual distribution of interdeparture times from the aggregate. It is obvious that such detailed representation of the aggregate is too cumbersome to be of any practical use.

On the other hand this composite server must (minimally) cause the same average delay to jobs passing through it, as those jobs would experience had they actually proceeded through the detailed representation of the aggregate.

So a first approach is to analyze the aggregate (fig. 2.5),

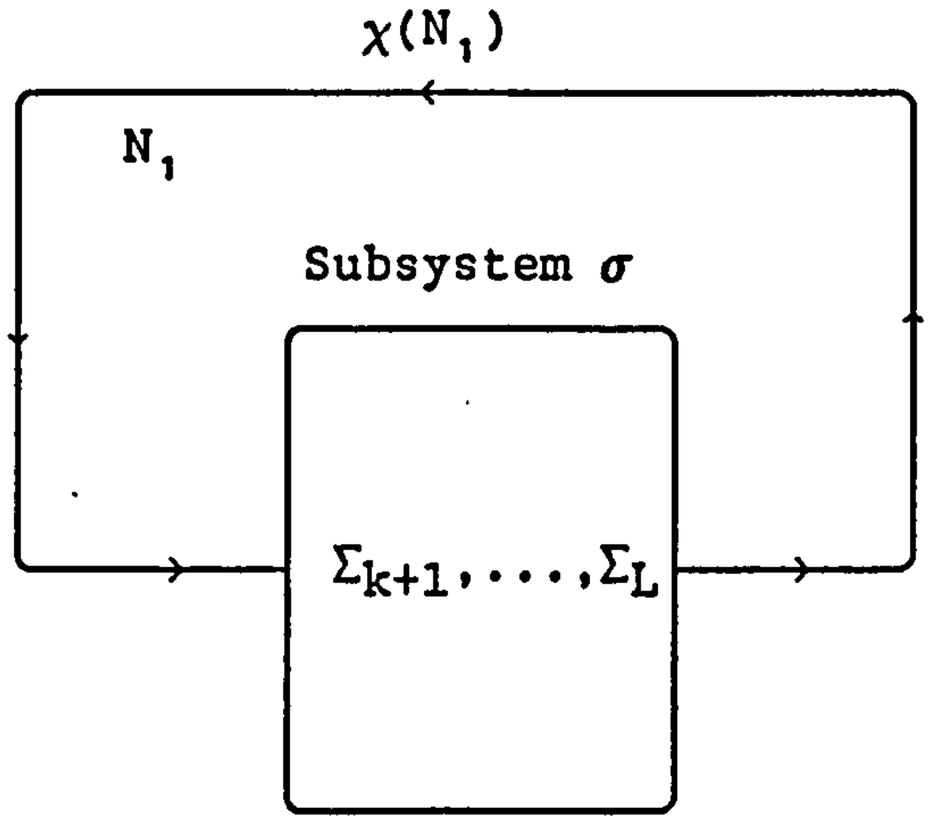

Figure 2.5. Subsystem σ in isolation (first level of aggregation).

in isolation and derive the different throughput rates $\chi(N_1)$, $N_1=1,\ldots,N$, that correspond to different populations N_1 (number of jobs that circulate in the subnetwork).

Then the composite server may be represented as a load-dependent

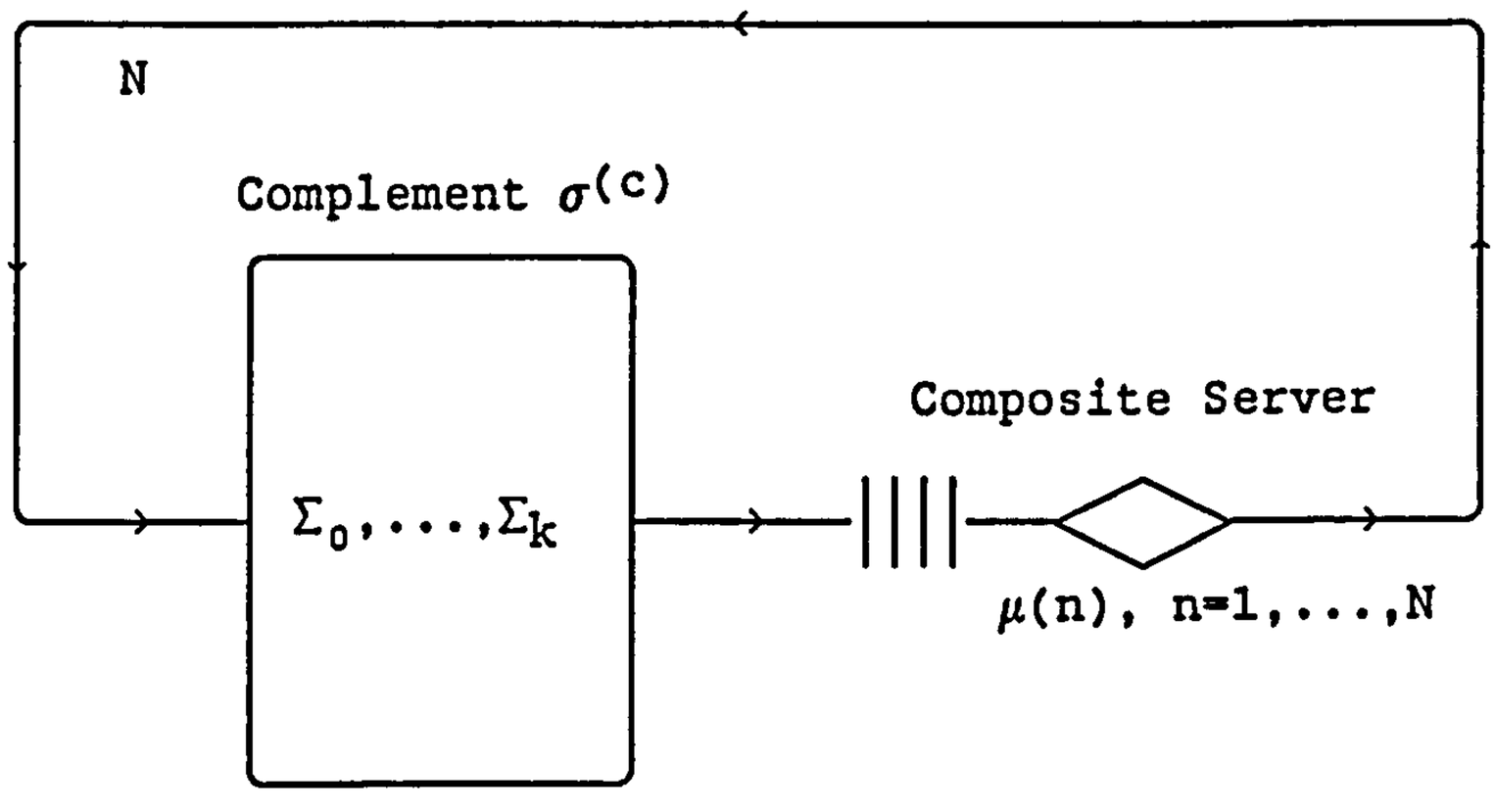

Figure 2.6. Reduced network (2nd level of aggregation).

server (fig. 2.6), with load-dependent service rates $\mu(n)$, $n=1,\ldots,N$, which are exactly the throughputs of the aggregate under the corresponding populations, i.e.

$$\mu(n) = \chi(n) \quad , n=1,\ldots,N \tag{2.16}$$

When a QNM satisfies local balance or is separable the equilibrium state probabilities have the *product form*, as if service

times were exponentially distributed. The conditions of separability can be found in [SAUE 75a]. So in the case of an exponential or separable network the Norton's theorem reduction of a subsystem is exact in that the joint probability distribution of queue lengths at servers not in the subsystem is identical in the original (fig. 2.4) and reduced (fig. 2.6) systems.

In this decomposition scheme the assumption that the output rate of the aggregate depends only on the customers in it, implies the assumption that the aggregate achieves local equilibrium between successive interactions with the complement. In other words the behaviour of the aggregate is independent of its starting condition. This situation occurs if many transitions of jobs between resources of the aggregate occur, before another arrival from the complement takes place. This is most likely to happen when units in the aggregate have service rates that are considerably faster than the service rates in the complement. It is desirable that the aggregate achieves local equilibrium because in that case the average departure rate from the aggregate with a given population in it, will be nearly the equilibrium throughput, regardless of the initial placement of those customers. This is exactly the assumption made in reducing the aggregate to a single service center, whose state is described by the number of customers present.

2.2.1 Generally distributed service times.

In this case the reduced network provides only an approximation to the original network. This is because the step in which the subsystem is studied in isolation, involves the implicit assumption that the input process to the subsystem is identical to its output process (fig. 2.5). This assumption is not valid in networks other than exponential (or more generally separable). Generally, only mean

input and mean output rates are identical. This causes the reduction of Norton's theorem to be inexact.

The first problem here is how to estimate the load-dependent throughputs of the subnetwork. The convinient solution of assuming exponentially distributed service times in the aggregate, as proposed by Chandy et al [CHAN 75b, SAUE 75a], is a major source of error. Another source of error is the definition of the distributional form for the composite server. Similarly to Courtois' proposal, in the variable aggregation scheme, the easy solution is provided once more by the exponential assumption.

However, a step forward towards a more accurate description of the aggregate is the introduction of the squared coefficient of variation, or in other words the involvement of the variability of the aggregate in the approximation [SAUE 75a, SEVC 77]. In this case an estimate for the squared coefficient of variation of the departure process of the aggregate is needed. Several formulas have been introduced as estimates of this coefficient [SAUE 75a, SEVC 77]. For example in the analysis of central server models (fig. 2.7), by Sauer and Chandy [SAUE 75a], where the aggregate is the I/O subsystem (fig. 2.8), the composite server (fig. 2.9), is proposed to have as a squared coefficient of variation C_c^2, the weighted sum of the coefficients of variation of the individual I/O units, with the weights being the I/O transition probabilities.

$$C_c^2 = \sum_{i=1}^{L} r_i C_i^2 \qquad (2.17)$$

where C_i^2, r_i are the squared coefficient of variation and branching or transition probability of unit i respectively, i=0,...,L.

Then according to the result of the above summation a distributional form is chosen for the composite server as :

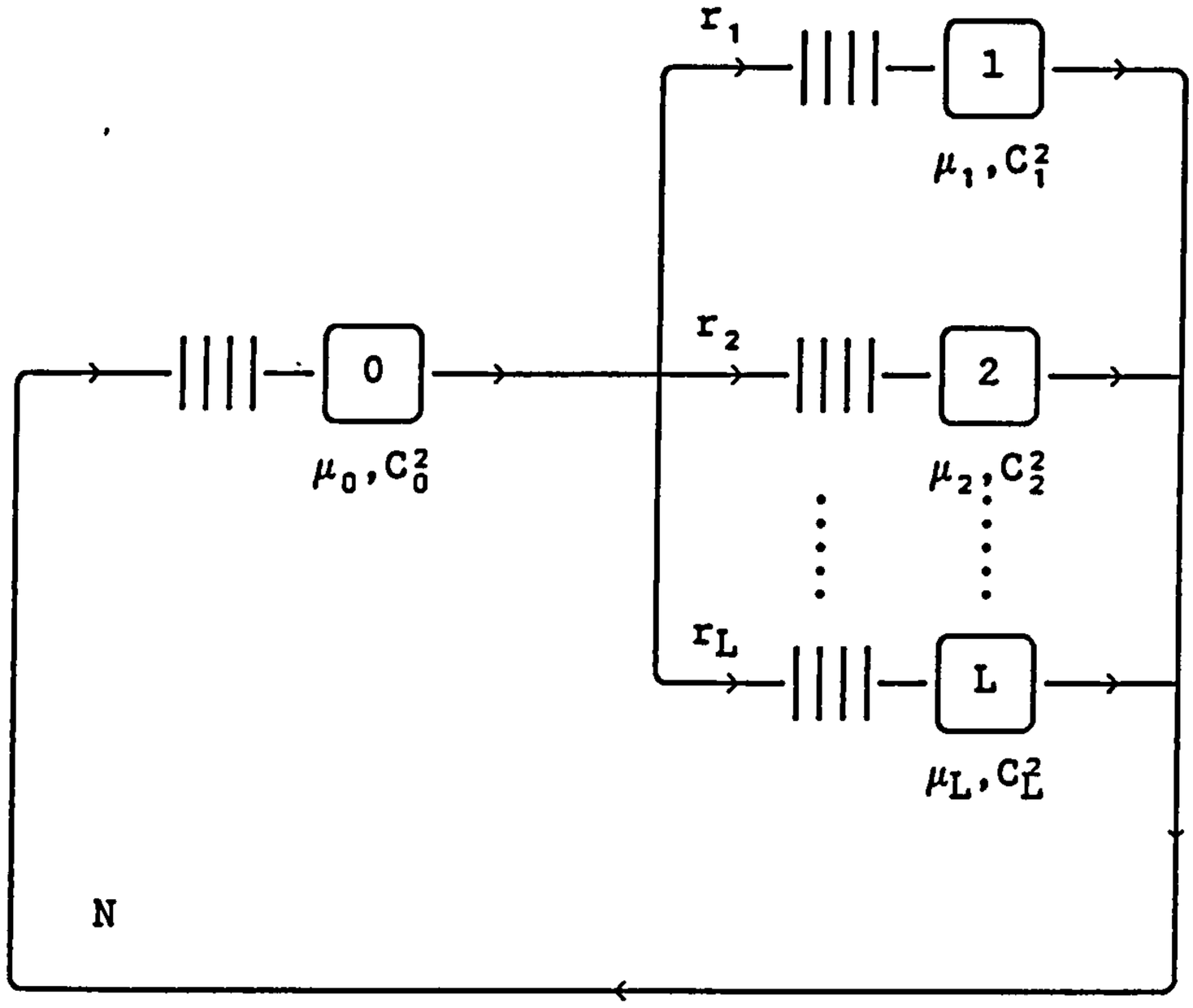

Figure 2.7. Closed central server model.

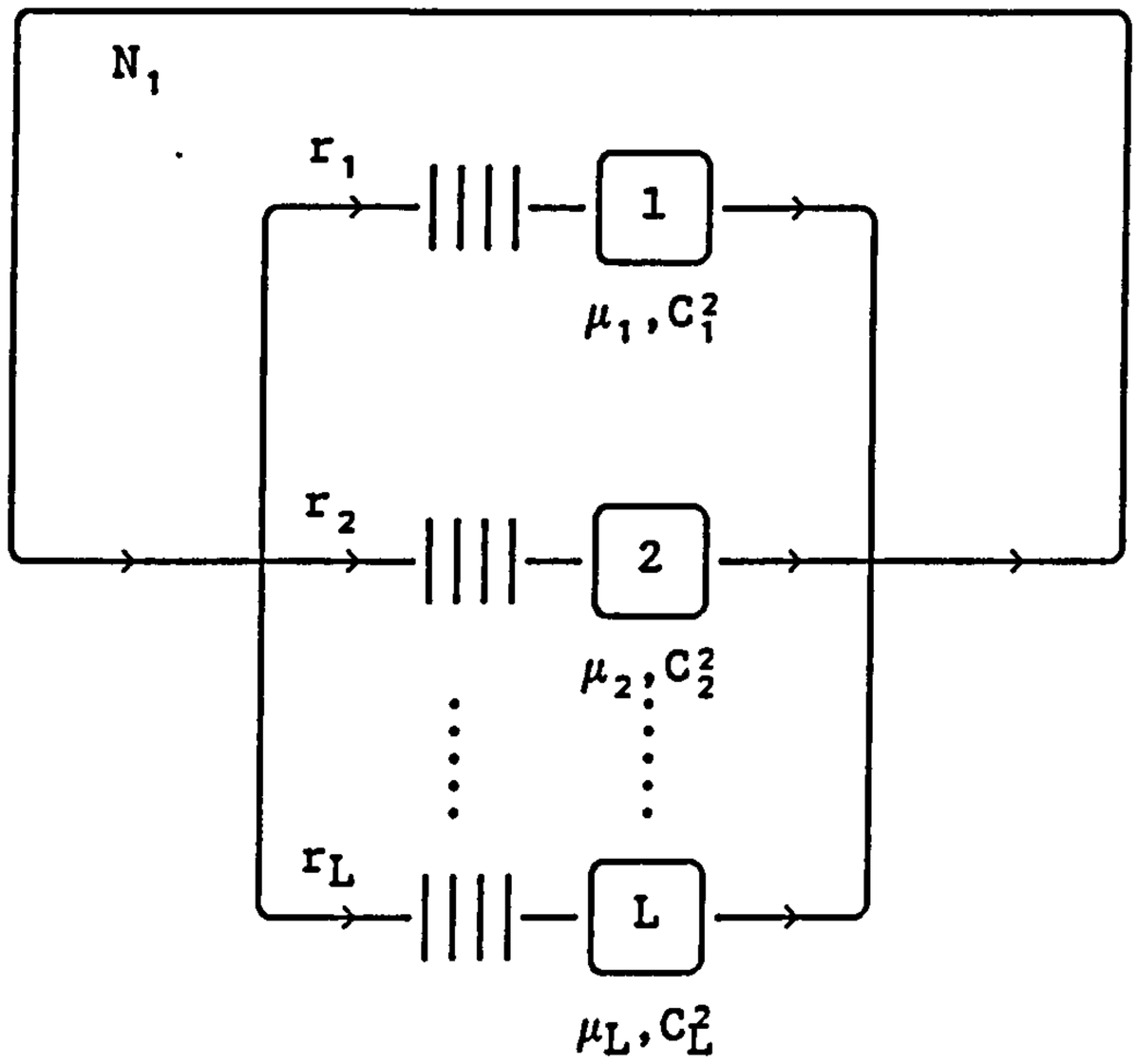

Figure 2.8. I/O susystem in isolation.

1) If $C_c^2 < 1$ a generalized Erlang form with the minimum number of stages necessary to obtain C_c^2 is used.

2) If $C_c^2 > 1$, a standard two stage hyperexponential distribution is used, while

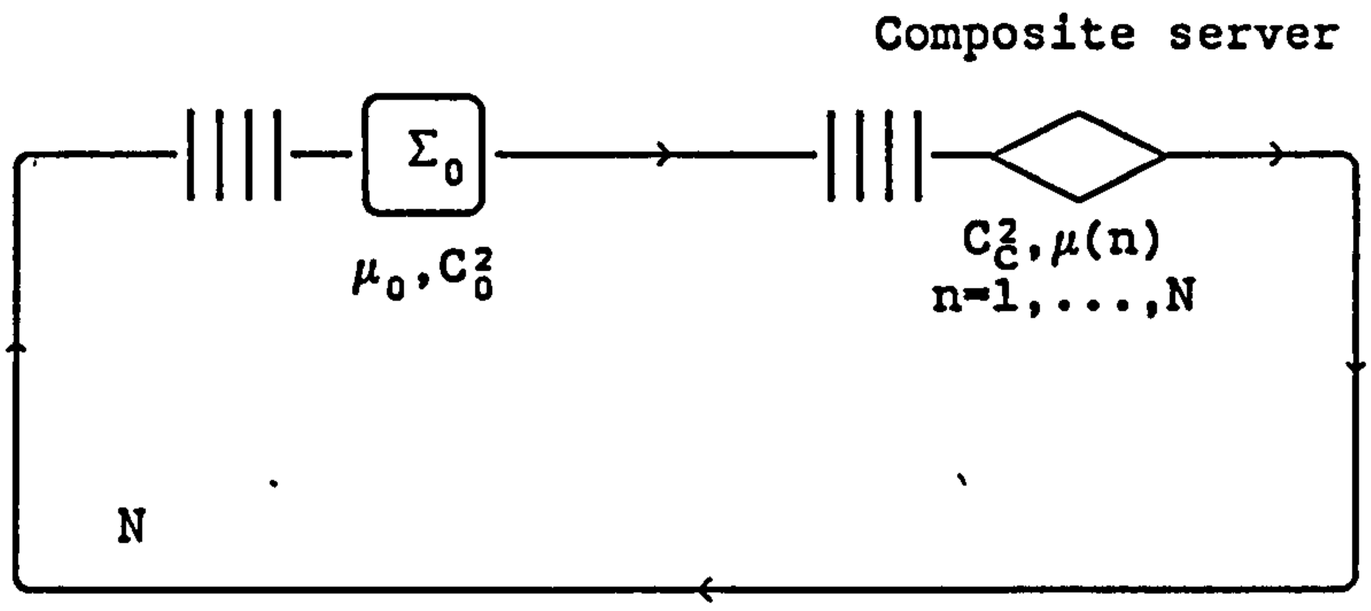

Figure 2.9. Reduced Network (2nd level of aggregation)

3) if C_c^2 is almost equal to 1 an exponential distribution is used.

Then in the second and final level of decomposition, and in case that $C_c^2 \neq 1$, global balance or numerical techniques have to be employed to solve the two stage cyclic network presented in figure 2.9.

The justification behind (2.17) approximate formula is that the weights used (transition probabilities), are directly proportional to the I/O throughputs [SAUE 75a]. It is not difficult, however, to identify reasonable models for which this approach (using (2.17)), does not yield acceptable results [SEVC 77]. Several alternative to (2.17) formulas have been proposed [SEVC 77]. Each of them seems to be successful in a particular class of networks. None of them though seems to be universally acceptable.

2.2.2 Hierarchical application of Norton's reduction scheme.

This technique can be applied hierarchically in a multilevel decomposition of QNMs. Let's consider for example a tandem configuration (fixed routing), with M+1 resources as shown in figure 2.10.

The notation introduced in the variable aggregation scheme may be used here. So in the first level of aggregation the conditional distributions,

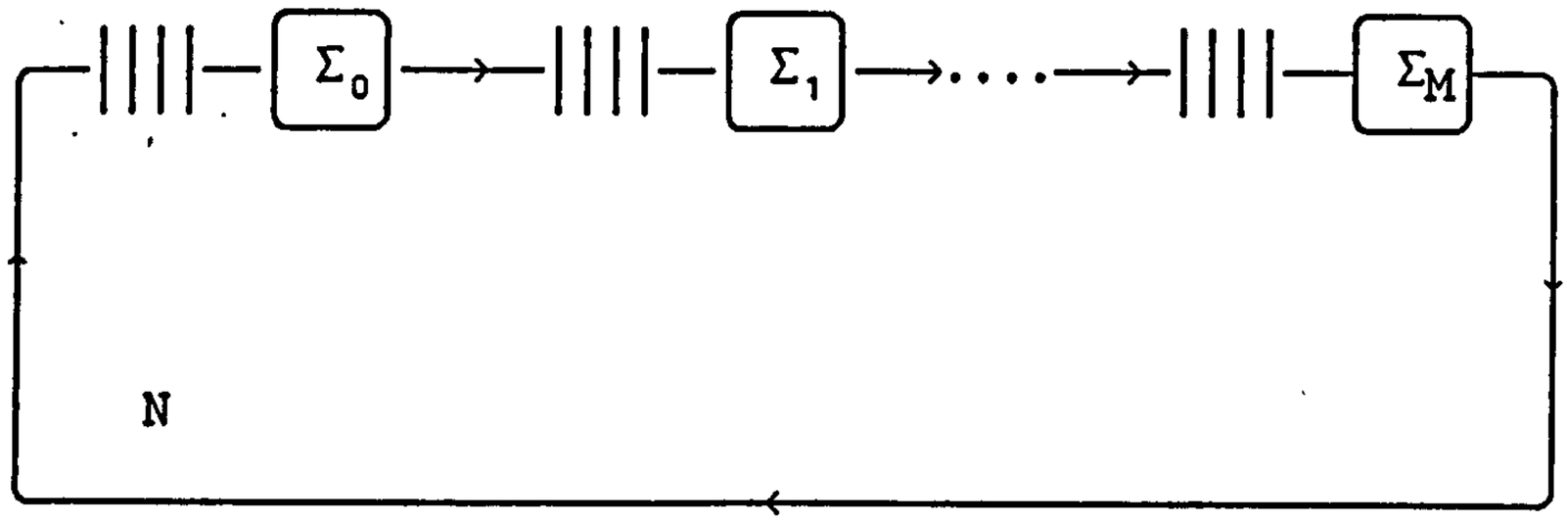

Figure 2.10. Tandem configuration.

$$\{P_1(n_0/N_1),\ n_0=0,\ldots,N_1\},\quad N_1=1,\ldots,N \tag{2.18}$$

may be evaluated, by solving the network of figure 2.11(a). We may recall that $P_1(n_0/N_1)$ denotes the conditional probability of n_0 jobs being present at unit 0, given that N_1 jobs circulate in the first level (fig. 2.11(a))

In the second level of aggregation the distributions

$$\{P_2(N_1/N_2),\ N_1=0,\ldots,N_2\},\ N_2=1,\ldots,N \tag{2.19}$$

may be evaluated by solving the network of figure 2.11(b). We recall that $P_2(N_1/N_2)$ denotes the conditional probability of N_1 jobs being present at the aggregate resource of this level, while N_2 jobs circulate in this 2nd level. The load-dependent rates of the composite server are defined as the load-dependent throughputs of the first level.

Similarly for the ℓth level of aggregation, $\ell=3,\ldots,M-1$. At the Mth level, the conditional distribution

$$\{P_M(N_{M-1}/N),\ N_{M-1}=0,\ldots,N\} \tag{2.20}$$

may be evaluated, using the two stage cyclic network of figure

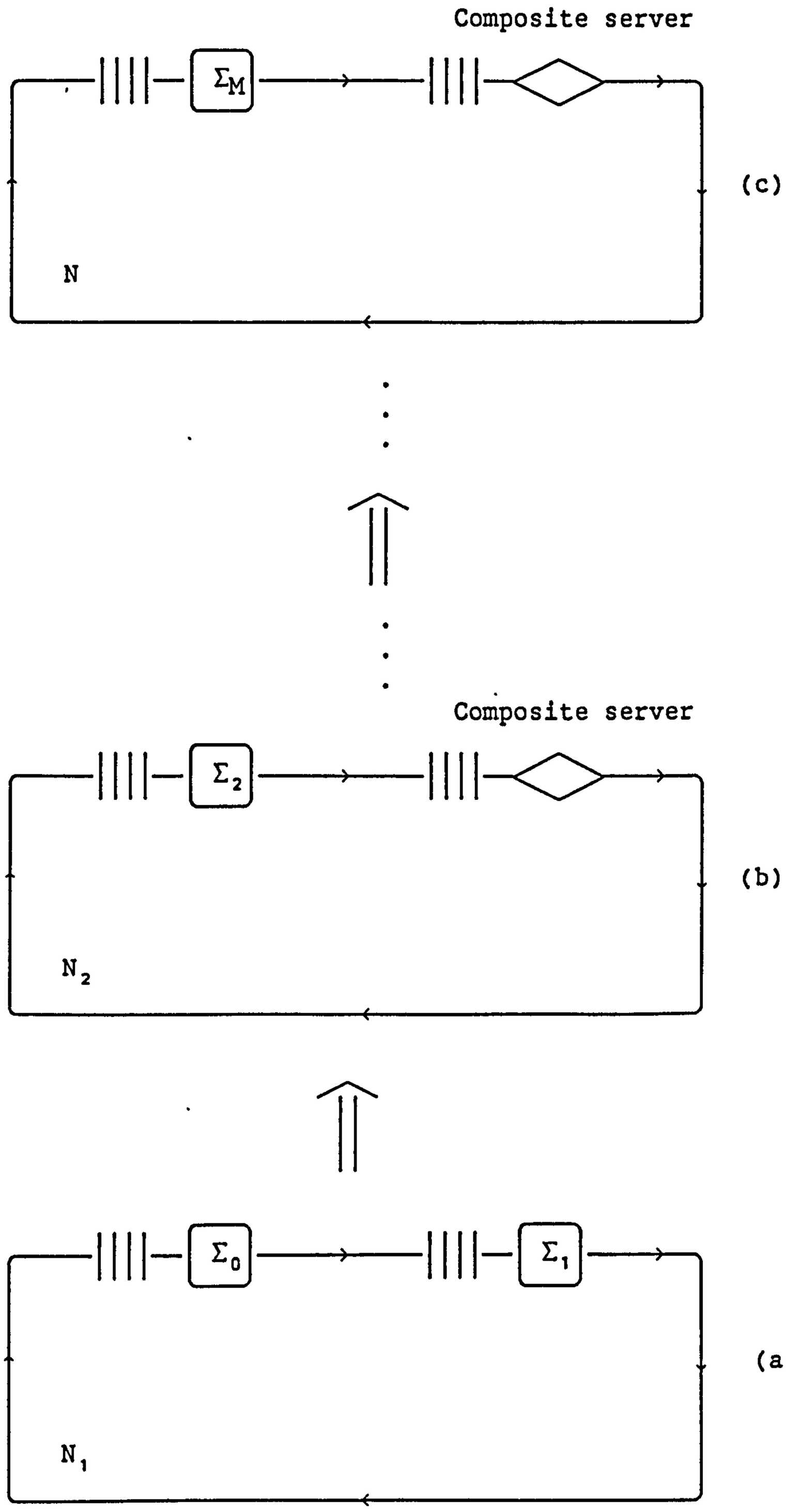

Figure 2.11. Hierarchical decomposition of a tandem configuration

2.11(c). Once more the load-dependent service rates of the composite server are defined as the load-dependent throughputs of the previous level. Lastly, the marginal distributions $\{ P_k(n_k),\ n_k=0,\ldots,N \}$,

k=0,...,M, can be calculated using disaggregating relations (2.12)-(2.13).

2.2.3 Discussion.

Norton's reduction or flow-equivalence decomposition scheme seems to offer a simpler decomposition tool for the analysis of QNMs, in the sense that it is an - by analogy - application of Norton's theorem in queueing network problems and thus easier to comprehend. It is not based on an elaborate theory, as variable aggregation scheme does, yet the concept of local equilibrium is used, for justification purposes.

For general queueing networks, the first problem that this method shares with the variable aggregation approach is the type of solutions required for its implementation. In the demonstration of its hierarchical application, in the previous section (2.2.2), it is clear that G/G/1/n and G/G(n)/1/m or G/M(n)/1/m two stage cyclic types of solutions are to be used. Thus the distributional forms of the service times of the network units, as well as the distributional assumptions for the composite servers, are restricted by the non-existance of efficient solutions for the above systems.

Another problem is the restriction imposed on the network configuration. As it has been mentioned previously, according to Norton's theorem, it is required that the subsystem to be reduced should have a common input stream and a common output stream (fig. 2.4). In fact, this requirement has to do with the flow conservation law mentioned in section 2.1.3. More precisely, if the solution of the subsystem σ in isolation (fig. 2.5), as well as the solution of the reduced system (fig. 2.6), guarantee flow balance, then the resulting solution for the overall system (fig. 2.4) is flow balanced, i.e. satisfies relations (2.15).

2.3 Conclusions.

The two decomposition techniques presented in this chapter, provide an alternative framework for the analysis of QNMs. The variable aggregation scheme advanced by Courtois [COUR 77] introduces a partition of the state space in a multilevel fashion and the aggregative variables (conditional probabilities), defined to describe each level depend on interactions of a single unit and the subsystem of the previous level (fig. 2.2).

In Norton's reduction decomposition scheme a more general decomposition of the state space is introduced, in the sense that it is not necessarily a partition. It has been demonstrated though that this technique may be applied in a multilevel hierarchical fashion, where at each level the interactions between a single unit and the subsystem of the previous level are examined. In such a case a partition of the state space is involved.

Following the decomposition of the state space, states are grouped in what was defined by Courtois as equivalence classes of aggregates. This merely describes the common decomposition assumption that when a subsystem is examined in isolation, the conditional distributions that describe the interactions between units of this subsystem are not dependent on the number of customers present at units that do not belong in this subsystem, but on the number of customers present in the subsystem (i.e. the above distributions are not conditioned on the specific distribution of customers in the complementary network).

Both techniques face the same problem at the level of their implementation, i.e. the availability of efficient solutions for the subsystems to be tackled at every level of aggregation.

Because both techniques decompose the network structurally,

conditions have to be satisfied in order to preserve the flow conservation law. This imposes restrictions in the network configuration, for which such approach is feasible.

The concept of local equilibrium, upon which the justification of these methods is based, is clear in the variable aggregation scheme, where algorithm 2.1 in fact requires that the unit studied in each level communicates directly with the subnetwork and vise versa. It is not clear however, in certain cases like in a tandem configuration, where Norton's reduction scheme is used, how local equilibrium can be achieved, between units that do not interact directly, and still obtain the exact solution in the exponential network. This arises the question of whether the justification of these methods - and especially of the variable aggregation one - restricts their applicability. This point will be addressed and discussed in more detail in the seventh chapter of this thesis.

In the next chapter the maximum entropy (ME) principle will be introduced together with some applications in queueing network analysis. The generalized exponential (GE) distributional model will be also presented, as well as decomposition algorithms for central server models with GE-distributed service times.

CHAPTER III

MAXIMUM ENTROPY FORMALISM AND THE GE DISTRIBUTION

This chapter describes the highlights of maximum entropy formalism and the GE distribution and demonstrates their utility in the implementation of the variable aggregation and Norton's reduction decomposition algorithms as proposed by Courtois [COUR 77] and Chandy et al [CHAN 75a], respectively.

3.1 Maximum Entropy Formalism.

The problem of inferring a probability distribution subject to available information, can be traced to have concerned several leading minds of science and philosophy. According to Jaynes' historical review [JAYN 79], ideas relative to this problem have been found expressed in ancient sources, like Herodotus and Ovennus. Bernoulli examined the assignement of initial numerical values to probability distributions in the absence of any information and stated the basis underlying such initial assignements as an explicit formal principle, named: "*The Principle of Insufficient Reason*". The essence of this principle is that possible events must be considered equally probable, unless we have reason to believe otherwise.

Laplace later produced what is known as "*Bayes Theorem*", which represents the process of "learning by experience", in the sense that a prior probability changes to a posterior one as a result of obtaining new evidence.

Although, in more resent times, with the application of probability analysis in physics, by people like Maxwell, Boltzman and Gibbs, a parallel development to the statistical inference had been taking place, it was not until Shannon [SHAN 48] that the maximum

entropy functional was used as a measure of information and the maximum entropy principle as a method of inference for discrete distributions and a generalization of the principle of Insufficient Reason, [JAYN 79, pp. 37-41].

Further successful applications of the ME principle in statistical mechanics [JAYN 57a, JAYN 57b], motivated further investigation of its properties [KULL 59, JAYN 68, TRIB 69, EMDE 69]. A long standing problem, however, had been the generalization of the principle in continuous distributions. The solution to this problem was given by Shore and Johnson [SHOR 80, JOHN 83], who formally defined the principle of minimum relative (or cross) entropy (MRE). More precisely, they examined the principles of maximum entropy and Kullback's minimum cross-entropy [KULL 59, p. 37], and showed them to be uniquely correct and self-consistent methods of inductive inference. Furthermore, they grouped and proved a number of properties that these methods have [SHOR 81].

3.1.1 The Principle of Maximum Entropy.

Consider a system Q that has a set D of possible discrete states $\{d_0,d_1,d_2,...\}$, which may be finite or countable infinite. Let X be the random variable that describes the state of this system. Suppose that the information about the system consists of constraints on $\{P(d_n),\ d_n \epsilon D\}$ probability distribution, where $P(d_n)$ is the probability that the system is in state d_n, $(P(d_n)=Pr\{X=d_n\})$. It is also assumed that these take the form of mean values of m suitable functions $\{f_1(d_n),\ f_2(d_n),\ ...,\ f_m(d_n)\}$. Since in general the number of possible states is greater than the number of these functions m, there is an infinite set of distributions $\{P(d_n),\ d_n \epsilon D\}$, satisfying these constraints. The problem is which one to choose.

The ME principle states that of all distributions satisfying the

given constraints, the minimally prejudiced one, which should be chosen is the one that maximizes the entropy functional:

$$H(P) = - \sum_{d_n \epsilon D} P(d_n) \log\{P(d_n)\} \qquad (3.1)$$

subject to the constraints,

$$\sum_{d_n \epsilon D} P(d_n) = 1 \qquad (3.2a)$$

$$\sum_{d_n \epsilon D} f_k(d_n) P(d_n) = \langle f_k \rangle \ , \quad k=1,2,\ldots,m \qquad (3.2b)$$

where $\{\langle f_k \rangle, k=1,\ldots,m\}$ are the prescribed mean values defined on the set of functions $\{f_k(d_n), k=1,\ldots,m\}$. This maximization problem is solved using Lagrange's method of undetermined multipliers, leading to the solution:

$$P(d_n) = \frac{1}{Z} \exp\Big\{ - \sum_{k=1}^{m} \lambda_k f_k(d_n) \Big\} \qquad (3.3)$$

where $\{\lambda_k, k=1,\ldots,m\}$ are the Lagrangian multipliers that correspond to the constraints (3.2b) and Z is given by:

$$Z = \exp\{\lambda_0\} = \sum_{d_n \epsilon D} \exp\Big\{ - \sum_{k=1}^{m} \lambda_k f_k(d_n) \Big\} \qquad (3.4)$$

where λ_0 is the Lagrangian multiplier that corresponds to the normalization constraint (3.2a). It can be shown [TRIB 69] that the Lagrangian multipliers $\{\lambda_k, k=1,\ldots,m\}$ satisfy relations:

$$- \frac{\partial \lambda_0}{\partial \lambda_k} = \langle f_k \rangle \ , \quad k=1,\ldots,m \qquad (3.5)$$

Now clearly if (3.4) can be solved analytically, then closed form expressions for multipliers λ_k, k=1,...,m, can be determined, using (3.5). Otherwise, numerical techniques have to be employed in order

to establish values for λ_k, k=1,...,m [JOHN 79].

In an information theoretic context, the ME solution is considered to be the least biased distribution estimate of all solutions that satisfy the given constraints. In other words, since the entropy functional attains its maximum when all states are equally possible (finite state space), the ME solution is the closest to a uniform distribution satisfying the available information.

3.1.2 Queueing applications of ME formalism.

Benes [BENE 65], first proposed use of maximum entropy in statistical mechanical analysis of large scale communication systems. For a telephone system, in which only the expected number of calls in progress is known, Benes derived the ME distribution, which proved to be the equilibrium distribution of an ergodic birth-death Markov process with constant birth and death rates. A direct implication of this is that the ME approximation to the solution of a single open queueing system, subject to the mean number of jobs, is the solution of an M/M/1 system.

Other results followed relating the ME solution - subject to the first moment of the distribution of jobs present in the system (called from now on "*queue length distribution*") - to the exact solution of these systems under the assumption of exponential interarrival and service. Ferdinard [FERD 70], used the principle to derive the equilibrium solution of an M/M/1/N system by analogy with statistical mechanics. Shore [SHOR 78], built an abstract model, from which he determined the ME solution of the M/M/∞ and M/M/∞/N systems.

These results established a new methodology based on information theoretic grounds, rather than the traditional stochastic ones. The numerical value of the first moment of the queue length distribution (qld), which corresponds to the constraint used (mql constraint) in

these first attempts, was not assumed to be a priori known. Yet, the constraint was used to determine the form of the ME solution.

Shore [SHOR 82a], examined the M/G/1 and G/G/1 queues, using as constraints higher moments of the queue length distribution. His approach exploited the fact that moments of the performance distributions are themselves determined by the service and interarrival time moments. Thus, for example in the ME approximation of the M/G/1 system using only the mql constraint, determined by the Pollaczeck-Khinchen formula [KLEI 75, p. 187], he obtained a geometric form of solution for the qld. The problem though that he faced was in determining the ME approximation, using higher moments of the queue length distribution. Because closed form expressions for such solutions are not obtainable, he used an APL function, developed by Johnson [JOHN 79], in order to solve numerically the ME problem. This by itself makes the practicality of his approach questionable.

So, up to this point, even though a connection between the ME approximation and the exponential model has been established, this information theoretic approach had not offered any more than a different interpretation for the robustness of the exponential distribution, [SHOR 82a].

It was then that El-Affendi and Kouvatsos [AFFE 83], examined the M/G/1 queueing system, using as constraints two basic results from classical queueing theory, namely:

$$P_0 = 1-\rho \tag{3.6}$$

$$\langle n \rangle = \sum_{n=1}^{\infty} nP_n = \rho + \frac{\rho^2(1+C_s^2)}{2(1-\rho)} \tag{3.7}$$

where if λ and μ are the mean arrival and service rates respectively, ρ is the utilization of the server, ($\rho=\lambda/\mu$), and C_s^2 is the squared coefficient of variation of the service time. P_n denotes the

equilibrium probability of n customers being present, (queueing or receiving service), at the service facility. Obviously, <n> denotes the first moment of this distribution (mql). Relation (3.6) holds for any G/G/1 queue [KLEI 75, p. 19], while relation (3.7) is the well-known Pollaczek-Khinchin formula for the M/G/1 system, [KLEI 75, p. 187].

The ME solution, subject to the above two constraints and of course the normalization, is of the form:

$$P_n = \begin{cases} 1-\rho & n = 0 \\ (1-\rho)gx^n & n > 0 \end{cases} \qquad (3.8)$$

where g is the coefficient that corresponds to the utilization constraint (3.6), while x is the coefficient that corresdonds to the mql one (3.7).

Using (3.6) and (3.7), El-Affendi and Kouvatsos [AFFE 83], were able to obtain analytic expressions for the above multipliers. The obvious question was, "what kind of approximation does ME solution (3.8) imply ?". In other words how this solution compares with the exact solutions of certain M/G/1-type systems when G is specified, and most important, "is there a distributional form for the service time, for which (3.8) is exact ?". The answer to this last question was, to our view, the most important contribution of this work. ME solution (3.8) is exact when G=GE, i.e. when the service time distribution is assumed to be the generalized exponential one [AFFE 83].

Similar results were established for the G/M/1 [AFFE 83], and G/G/1 [KOUV 83], systems. More precisely, the ME approximate solutions for these systems, subject to the utilization and mql constraints, prove to be exact for G=GE. Thus, an important connection had been established, which can be viewed as an extension of the previous results that concerned the exponential (M)

distributional model, between the ME principle and a more general distribution , which is uniquely specified by its first two moments, and which allows us to involve the coefficient of variation in the analysis of queueing models.

So, the exponential and generalized exponential distributions are related through the entropy formalism, in the sense that open queueing systems of G/G/1 type, which involve these two distributions, i.e. M/M/1, GE/M/1, M/GE/1, GE/GE/1, all have product form solutions, that are also the ME approximations subject to appropriate constraints. This, in an information theoretic context, means that the M/M/1 formulas are the best hypotheses about the G/G/1 systems when the available information is restricted to the first moments of the interarrival and service times, [SHOR 82a], while the GE/GE/1 formulas are the best hypotheses about the G/G/1 systems when information consists of the first and second moments of the interarrival and service times.

Let's now see more closely the definition and some of the properties of the generalized exponential distribution.

3.2 The generalized exponential distribution (GE).

Consider a server, whose service times are independently and identically distributed according to a GE distributional model. Let μ be the mean service rate ($1/\mu$ then is the mean service time) and let C_s^2 be the squared coefficient of variation. The density f(t) and distribution F(t) functions are:

$$f(t) = (1-\tau)u_0(t) + \tau^2\mu e^{-\tau\mu t} \quad , \ t\geq 0 \tag{3.9}$$

$$F(t) = 1 - \tau e^{-\tau\mu t} \quad , \ t\geq 0 \tag{3.10}$$

where

$$\tau = \frac{2}{1+C_s^2} \tag{3.11}$$

and $$u_0(t) = \begin{cases} \infty & t=0 \\ 0 & t\neq 0 \end{cases}$$

$u_0(t)$ is known as the unit impulse function [KLEI 75, p. 342], which describes the fact that the service time random variable (r.v.) S is a mixed one [KLEI 75, p. 373], in the sense that it is a continuous r.v. with discrete, non-zero probability of attaining the value 0, i.e.

$$F(0) = 1-\tau$$

The Laplace-transform $L_S^*(s)$, that characterizes the GE distribution service times is:

$$L_S^*(s) = (1-\tau)+ \tau \frac{\tau\mu}{\tau\mu+s} \qquad (3.12)$$

From the above Laplace-transform (L-T) it can be seen that this server has an exponential branch with mean service rate $\tau\mu$ and a branch without any service delay. Thus, it may be represented as in figure 3.1.

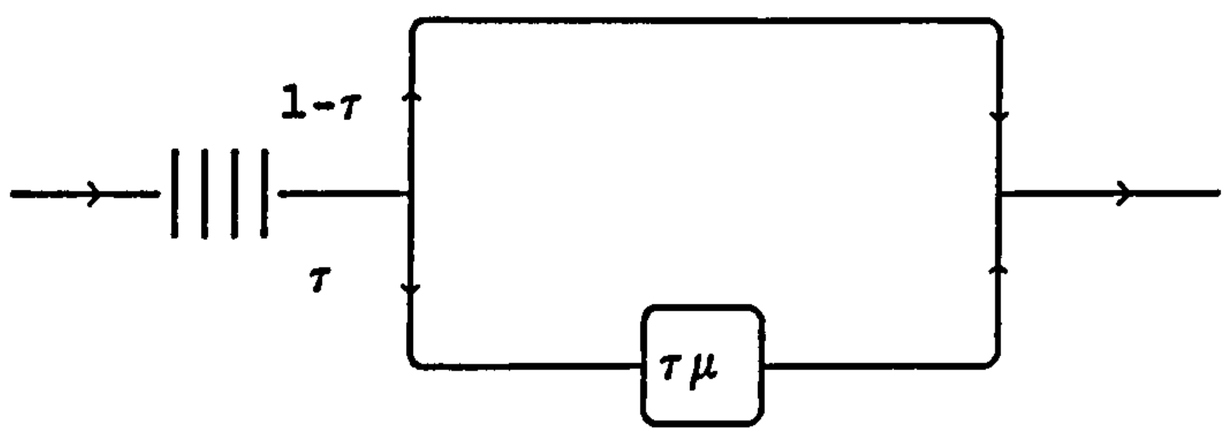

Figure 3.1 GE-distributed server.

Parameter τ is the branch selection probability and from (3.11) it is clear that if $C_S^2 < 1$, then $\tau > 1$ and the distribution becomes improper. Hense, for $C_S^2>1$ this server has two stages. In the first stage the customer selects, according to a Bernoulli trial, the type of service that he will receive. In the second stage, and depending on the selection, he receives, with probability τ, an exponential($\tau\mu$)

service, while, with probability $1-\tau$, he departs without any delay. It is clear that more than one service completions may take place at the same time instant. This classifies the GE distribution as being a bulk type of distribution.

Another observation is that if $C_s^2=1$, then $\tau=1$ and GE reduces to an exponential(μ) distribution. This implies that all exact and approximate solutions, even at the QNM level, should reduce to the known exact solutions of the corresponding exponential systems.

From another point of view GE is an "extremal" member of a family of two-phase models with the same first two moments known [KOUV 88]. For $C_s^2>1$ it is the limiting case of a two-phase hyperexponential (H_2) distribution [KLEI 75, p. 141], (figure 3.2), when the mean service rate of one of the exponential branches tends to infinity ($\mu_1 \longrightarrow \infty$).

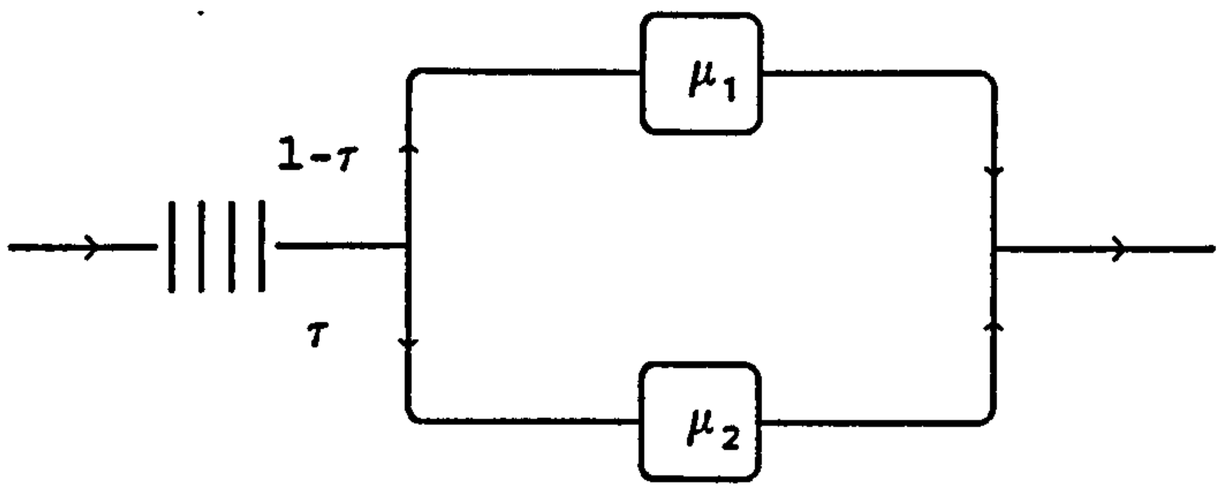

Figure 3.2 Two-phase hyperexponential (H_2) server.

In figure 3.2 μ_1 and μ_2 are the mean service rates of the two exponential branches and they satisfy the relation:

$$\frac{1-\tau}{\mu_1} + \frac{\tau}{\mu_2} = \frac{1}{\mu}$$

where μ and τ are the mean service rate and branch-selection probability. In case that $C_s^2<1$, GE is the limit of another "special" improper hypoexponential (h_2) model. More details on the definitions of these distributions can be found in [KOUV 88].

This viewpoint has been thoroughly exploited in earlier works, by

Kouvatsos on queueing systems that involved the GE distribution, [KOUV 83, KOUV 86a, KOUV 86b]. The exact solution of such systems was derived, using as a starting point the state space and transition rate diagram of the systems in which GE was substituted by a H_2 distribution. This was a convinient way of defining the system of global balance equations to be solved. Then the limits of the parameters of the H_2 distributions were applied in order to transform the H_2 servers to GE ones. In this way several states vanished and both the transition rate diagram and global balance system of equations were simplified. The solution of the system followed. This technique was developed due to the inherent difficulty of determining the transition rates between states of systems that involve bulk-type of distributions.

This "*limiting interpretation*" of GE, which has been justified by the above outlined technique of obtaining the solution of several queueing systems, is to our view a special case of a more general property. The GE distribution may be considered as the limit of any two-phase distribution with an exponential branch and a generally (G) distributed one. The assumption that the mean service rate of this general branch tends to infinity is adequate to produce the GE distribution. For example, this G distribution could be a deterministic one with sufficiently small service time.

3.2.1 The underlying counting process of a GE renewal process.

The above observation is the key to an alternative realization of GE, which proved to be very useful in extending the related to this distribution results. Let's see, in more detail, this alternative approach, by examining the renewal process with GE distributed interevent times. Our interest is to describe the underlying counting process. In fact, since in single queues the arrival process is

assumed to be renewal, let's consider the arrival process with GE interarrival times (figure 3.3), which - for demonstration purposes - is the departing process from a saturated server of GE-type.

Let λ be the mean arrival rate and C_a^2 the squared coefficient of variation of the interarrival times. Then $\sigma = 2/(1+C_a^2)$, similarly to

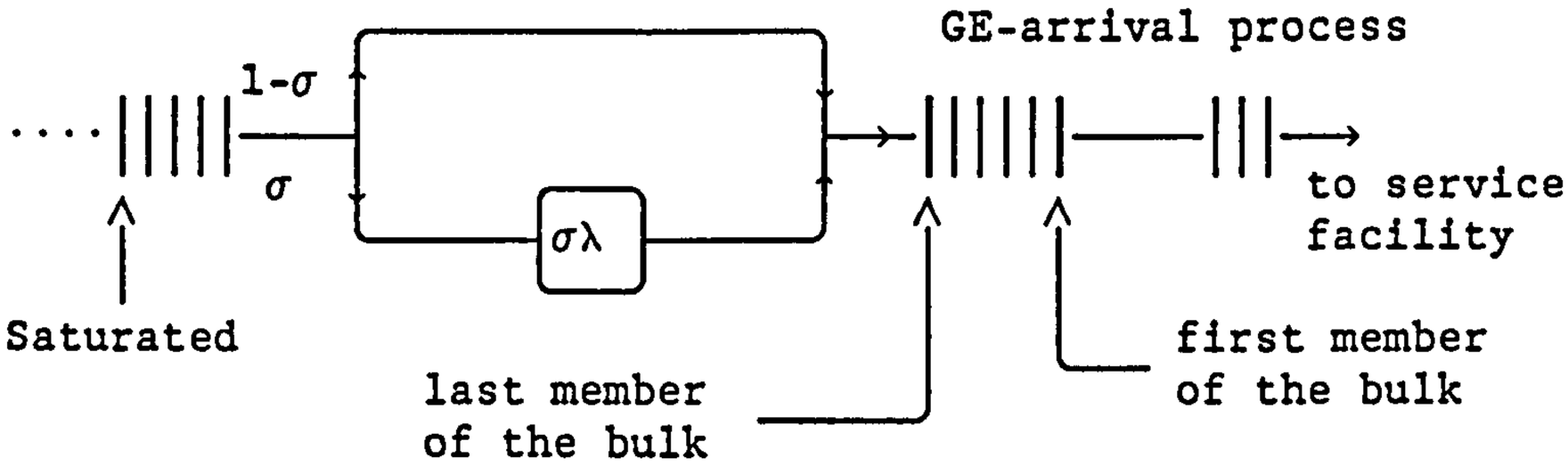

Figure 3.3 GE-arrival (renewal) process.

(3.11). The exponential branch of the server in fig. 3.3 has a mean service rate $\sigma\lambda$.

A first observation is that the departing process from the saturated server consists of bulk departures, and time intervals between bulk occurances are independently and identically distributed according to an exponential($\sigma\lambda$) distribution. This classifies the underlying counting process (arrival process) to be a compound Poisson process, denoted as M^B, where B stands for the distribution of the bulk size.

It is clear, that individual arrivals, who are members of the same bulk, are not separated by time intervals. Still, according to fig. 3.3, and supported by the previous discussion on the limiting interpretation of GE, these bulks may be considered "*ordered*", as if very small time intervals distinguish them physically. It is evident that each bulk has a "*head*" (first member), which has received an exponential($\sigma\lambda$) service, while the rest of the members (if any) are those customers that followed him, up to the point that another

queueing customer chose to receive an exponential($\sigma\lambda$) service.

In this context a number of properties relating to the GE distribution may be proved. Some of them are presented next, and the proofs can be found in Appendix I. So let B be the random variable (r.v.), that describes the number of customers within the same bulk, ($B \geq 1$). Then this r.v. is geometrically distributed, with parameter σ, i.e.,

$$\Pr\{B=k\} = \sigma(1-\sigma)^{k-1} \tag{3.13a}$$

Moreover, consider a tagged customer within a bulk. Let N_f and N_p denote the r.v. that count the number of customers who follow and precede, respectively, the tagged customer within the same bulk. Then random variables N_f and N_p follow a modified geometric distribution with parameter σ, i.e.,

$$\Pr\{N_f=n\} = \Pr\{N_p=n\} = \sigma(1-\sigma)^{n} \tag{3.13b}$$

The result described by (3.13a) may be used to prove that the underlying counting process of a GE renewal process with parameters σ, λ is given by a compound Poisson process as:

$$\Pr\{N(t)=n\} = \begin{cases} \sum_{k=1}^{n} \frac{(\sigma\lambda t)^k}{k!} e^{-\sigma\lambda t} \binom{n-1}{k-1} \sigma^k (1-\sigma)^{n-k} & , n \geq 1 \\ e^{-\sigma\lambda t} & , n=0 \end{cases} \tag{3.14}$$

where N(t) is the r.v. that counts the renewal events (e.g. arrivals) that occur within time t.

The results of this section can be used in a number of queueing problems that involve the GE distribution. For example they may be used in determining the transition rates between states of a system with GE-distributed interarrival and/or service times. Another area of their application is the investigation of the qld distribution of such systems as seen by an arriving or departing customer - let's

from now on call them arriver or departer, respectively.

3.2.2 Arriver's and departer's queue length distributions in queues that involve the GE distribution.

The three, generally different, viewpoints of a simple open queueing system with respect to its queue length distribution are: 1) the observer's with qld $\{P_n,\ n=0,1,2,...\}$, 2) the arriver's with qld $\{P^{(a)}(n),\ n=0,1,2,...\}$ and 3) the departer's with qld $\{P^{(d)}(n),\ n=0,1,2,...\}$. The relation between these three distributions is of particular importance for any queueing system. For example, in order to characterize the interdeparture time intervals and consequently obtain related statistics the departer's qld must be known. Two well-established results describe this relation. The first, [COOP 81, pp. 77-78], states that when jobs arrive in the system in a Poisson fashion (exponentially distributed interarrival times) the arriver's and observer's queue length distributions are identical, i.e.,:

$$\text{Poisson input} \Rightarrow \quad P_n = P^{(a)}(n),\ n=0,1,... \qquad (3.15)$$

The second result, [COOP 81, pp. 185-188], concerns a class of general systems, defined as those where state changes occur one at a time, or in other words, an arrival or departure changes the state of the system by one. In such systems, the arriver's and departer's queue length distributions are identical, i.e.,

$$P^{(a)}(n) = P^{(d)}(n)\ ,\ n=0,1,... \qquad (3.16)$$

Clearly, systems with bulk arrival or/and service processes are excluded from the class of those ones that enjoy this second property. By examining the three definitions closer and by applying the concept of ordered bulks, however, the above result may be extended to include GE distributions.

More precisely, it may be shown, (for a detailed discussion see

Appendix I) that (3.16) is true for any GE/G/c system, i.e. a system with GE-distributed interarrival times, generally distributed service times where G is a non-bulk type of distribution and c (multiple) servers. This result may also be extended to the GE/GE/1 queue. Moreover, for the above two types of queues (GE/G/c and GE/GE/1) it may be proved that the common arriver's and departer's viewpoint is related to the observer's one through the following equation,

$$p^{(d)}(n) = p^{(a)}(n) = \sum_{k=0}^{n} \sigma(1-\sigma)^{n-k} p_k \quad , \; n=0,1,2,... \qquad (3.17)$$

where σ is as defined previously, i.e. the exponential branch selection probability of the GE-distributed arrival process (see fig. 3.3). And if $P_a^*(z)$, $P_d^*(z)$, $P^*(z)$ are the z-transforms of the arriver's, departer's and observer's qlds, respectively, (3.17) implies that these z-transforms satisfy the following relation:

$$P_d^*(z) = P_a^*(z) = \frac{\sigma}{1-(1-\sigma)z} P^*(z) \qquad (3.18)$$

This result can be also proved using level crossing analysis [SHAN 82].

3.2.3 The interdeparture times from a GE/GE/1 queue.

Let T_d be the r.v. that describes the time interval between two consecutive departures from a GE/GE/1 system, and let's denote by $L_d^*(s)$ the Laplace-transform of the interdeparture time distribution. It is not difficult to prove that,

$$L_d^*(s) = [1-P^{(d)}(0)](1-\tau) + [1-P^{(d)}(0)]\,\tau\,\frac{\tau\mu}{\tau\mu+s} +$$

$$+ P^{(d)}(0)\left[\frac{\sigma\lambda}{\sigma\lambda+s}\left[(1-\tau)+\tau\,\frac{\tau\mu}{\tau\mu+s}\right]\right] \qquad (3.19)$$

is true, where σ, λ and τ, μ are the corresponding parameters of the

arrival and service GE processes and are as defined previously. Then using (3.19) and (3.17) it may be shown that the squared coefficient of variation of T_d, C_d^2 ($=Var[T_d]/(E[T_d])^2$) is:

$$C_d^2 = \rho^2 C_s^2 + (1-\rho)C_a^2 + \rho(1-\rho) \qquad (3.20)$$

The above relation will be used frequently and throughout this thesis and can be also found in [KOUV 85].

3.2.4 The splitting and merging of GE processes.

Consider a stream of events, say arrivals, with independently and identically GE-distributed interevent times, that splits at some point into m streams, (fig. 3.4). Let's also assume that the probability of an arrival to select the ith stream is fixed, equal to r_i, i=1,2,...,m.

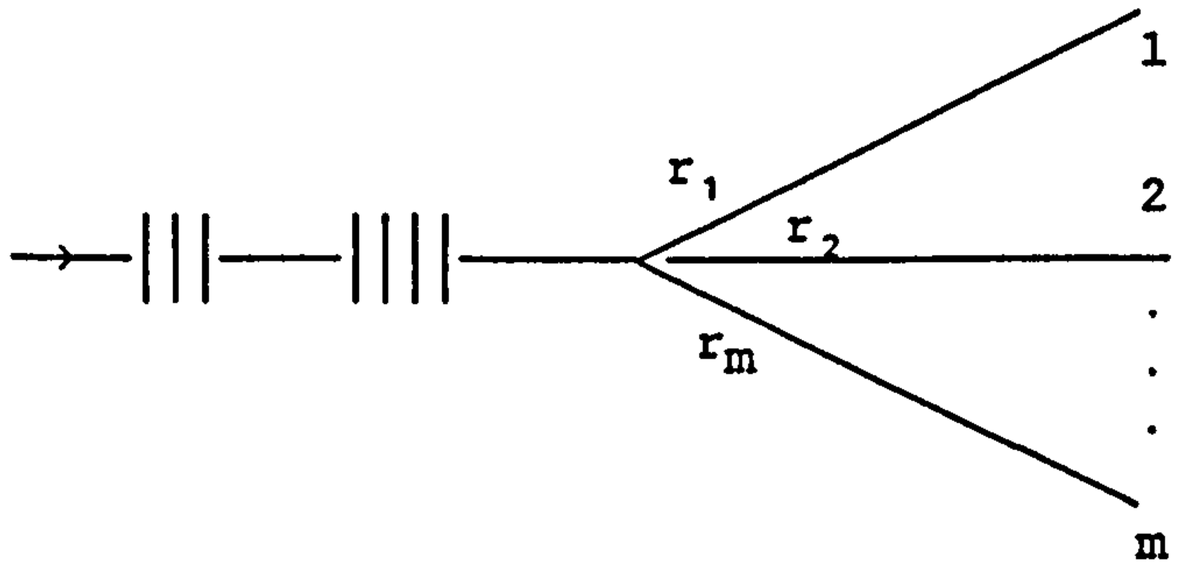

Figure 3.4. Splitting process of a GE-stream.

Let λ, C_a^2 be the mean arrival rate and squared coefficient of variation of the main stream. Then the interevent times of the ith stream, i=1,2,...,m, are independently and identically GE-distributed with parameters λ_i, C_i^2 given by:

$$\lambda_i = \lambda r_i \qquad (3.21a)$$

$$C_i^2 = 1 + r_i(C_a^2-1) \ , \qquad (3.21b)$$

[KOUV 85]. Relation (3.21b) will be refered to as "*the splitting*

formula". Note that in case that $C_a^2=1$ (Poisson stream), (3.23) yields the well-known splitting results of a Poisson stream.

In the merging process of m independent GE-distributed streams, (fig. 3.5), the pooled stream generally is not even a renewal process - except in case that the m streams are identically distributed.

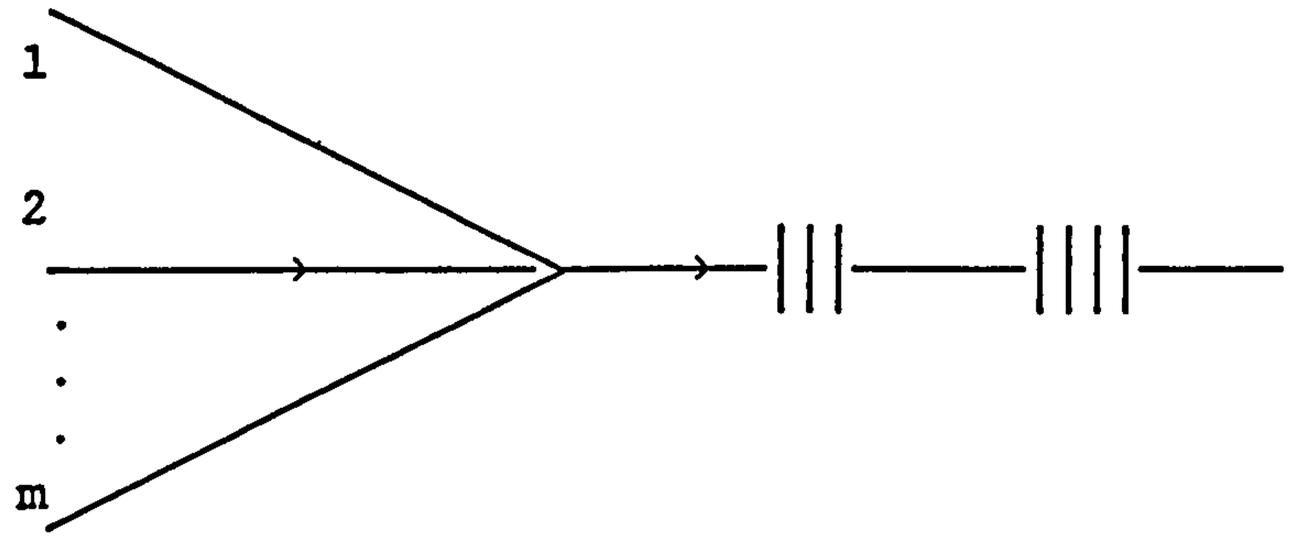

Figure 3.5. Merging of m GE-streams.

Let λ_i, C_i^2 be the parameters of the ith stream, i=1,2,...,m. Then the merged stream is usually approximated by a GE distribution with parameters λ, C_a^2, given by, [KOUV 85, KOUV 86c]:

$$\lambda = \lambda_1+\lambda_2+...+\lambda_m \qquad (3.22a)$$

$$C_a^2 = \frac{\lambda}{\sum_{i=1}^{m} \frac{\lambda_i}{C_i^2+1}} - 1 \qquad (3.22b)$$

Note that (3.22) also yields the exact exponential result in case that $C_i^2=1$, $\forall$ i=1,...,m.

3.2.5 The feedback correction.

In a queueing system with GE-distributed interarrival times, where service time is also $GE(\mu,C_s^2)$-distributed with a feedback probability r, (fig. 3.6), it has been proven by Kouvatsos [KOUV 85], that the server may be equivalently represented by a GE-server without feedback, with parameters μ^*, C_s^{2*}, given by:

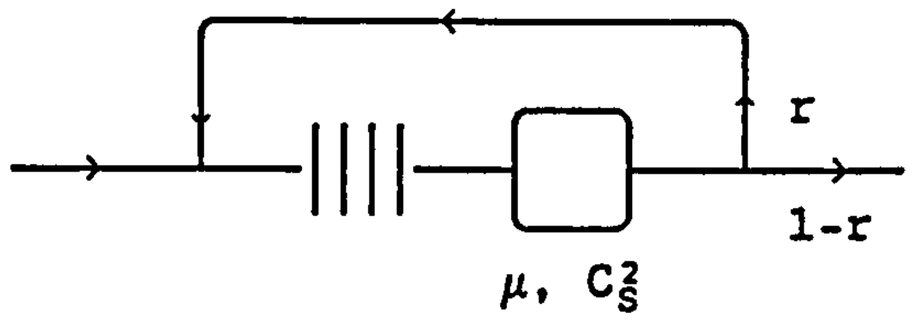

Figure 3.6. GE-server with feedback.

$$\mu^* = \mu(1-r) \tag{3.23a}$$

$$C_S^{2*} = 1 + (1-r)[C_S^2-1] \tag{3.23b}$$

Relations (3.23) are known as the "*feedback correction*". This result is true for GE/GE/1 and GE/GE/1/N queues.

In an open network with GE-servers, formulae (3.20)-(3.23) are used, in order to approximate the first two moments of the network's flow, which is assumed to be GE-distributed. There, even the exact formulae (3.20), (3.21), (3.23), are used in an approximate fashion. Feedback streams, in particular, could be treated as independent streams. This, however, would imply an extra approximation. Thus, in GE-distributed networks, usually the feedback correction precedes any other operation.

At this point it should be mentioned that results presented in this section (and proved in Appendix I using the bulk interpretation of GE) are not new. What may be considered, however, as part contribution of this thesis, is the alternative methodology that was used to prove these results. Furthermore, as mentioned earlier, the bulk interpretation of GE is most useful in deriving the transition rates between states of queueing systems that involve the GE distribution. Using this approach, several new solutions have been established [GEOR 89, TABE 89, XENI 89], that involve load-dependent routing, priority disciplines and blocking.

This new approach will be demonstrated in the following section, where the GE/GE/1/N two-stage cyclic system will be examined in

detail. This system will be tackled, firstly in a conventional, stochastic way and lastly using the ME principle.

3.2.6 The GE/GE/1/N queueing system.

This two stage cyclic system, (fig. 3.7), is of great interest to this thesis for two reasons. Firstly, ideas and properties of its ME solution will be used and extended at the network level, in following chapters, and secondly the first level of any hierarchical decomposition approximate algorithm for general QNMs involves the solution of such a system (see algorithm 2.1 in chapter 2).

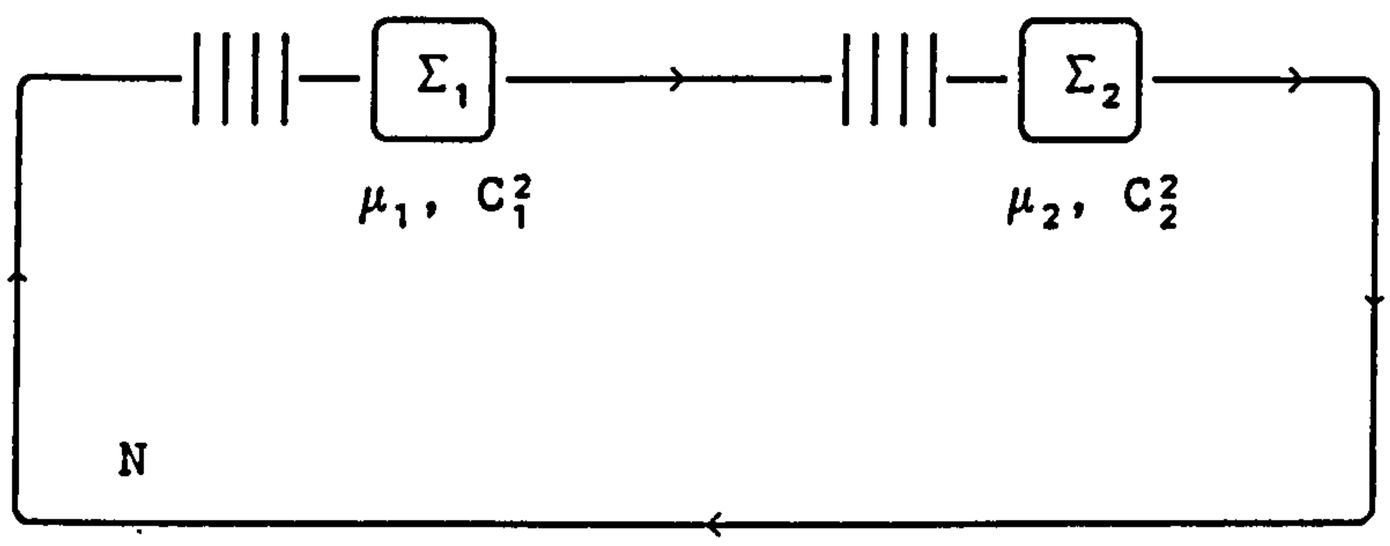

Figure 3.7. The GE/GE/1/N queueing system.

This configuration is the simplest closed queueing network model. Both servers are assumed to be of GE type. Let μ_1, C_1^2 be the mean service rate and square coefficient of variation of server 1 and μ_2, C_2^2 the corresponding parameters of server 2. Let also N be the fixed population that circulates in this system (level of multiprogramming). This system has been solved in the case of generally distributed service times (G/G/1/Nqueue) by Kouvatsos [KOUV 86a] via ME and the solution has been identified as a ME approximation. This solution becomes exact when G=GE.

Let's firstly demonstrate how the global balance system of equations can be defined, using the bulk interpretation of GE. Let $\sigma=2/(1+C_1^2)$ and $\tau=2/(1+C_2^2)$ be the exponential branch selection

probabilities of servers 1 and 2 respectively. The state of the system is n, n=0,1,...,N, whenever n customers are present in the second service facility (server 2) and N-n are in the first (server 1). Note that by the definition of the observer's equilibrium state distribution, the system is in state n, n≥1, whenever the exponential branches of both servers are occupied and n-1 customers are queueing in the second service facility.

Let $P_N(n)$ be the equilibrium probability that the system is at state n, n=0,1,...,N. The, so called, "*dual*" distribution may be defined, which concerns the number of customers present at unit 1. Let's denote this distribution by $\{P_N^*(n),\ n=0,1,...,N\}$. Then clearly:

$$P_N(n) = P_N^*(N-n) \quad , n=0,1,...,N \tag{3.24}$$

Let R_{ij} be the transition rate between states i and j, i,j=0,1,...,N. Then the following lemma determines matrix $R=[R_{ij}]$. Note that R_{ii}, i=0,1,...,N, are not determined, because they are redundant in the system of global balance equations.

Lemma 3.1. The transition rates $\{R_{ij},\ i,j=0,1,...,N,\ i\neq j\}$ between states of a GE/GE/1/N system are:

$$R_{0k} = \sigma\mu_1 \frac{\tau\sigma(1-\sigma)^{k-1}}{1-(1-\sigma)(1-\tau)} \qquad 1\leq k\leq N-1 \tag{3.25a}$$

$$R_{0N} = \sigma\mu_1 \frac{\tau(1-\sigma)^{N-1}}{1-(1-\sigma)(1-\tau)} \tag{3.25b}$$

$$R_{ij} = \sigma^2\mu_1(1-\sigma)^{j-i-1} \qquad \begin{matrix} 1\leq i\leq N-2 \\ i+1\leq j\leq N-1 \end{matrix} \tag{3.25c}$$

$$R_{iN} = \sigma\mu_1(1-\sigma)^{N-i-1} \qquad 1\leq i\leq N-1 \tag{3.25d}$$

$$R_{ij} = \tau^2\mu_2(1-\tau)^{i-j-1} \qquad \begin{matrix} 2\leq i\leq N-1 \\ 1\leq j\leq i-1 \end{matrix} \tag{3.25e}$$

$$R_{i0} = \tau\mu_2(1-\tau)^{i-1} \qquad 1\leq i\leq N-1 \qquad (3.25f)$$

$$R_{N0} = \tau\mu_2 \frac{\sigma(1-\tau)^{N-1}}{1-(1-\sigma)(1-\tau)} \qquad (3.25g)$$

$$R_{Nk} = \tau\mu_2 \frac{\sigma\tau(1-\tau)^{N-k-1}}{1-(1-\sigma)(1-\tau)} \qquad 1\leq k\leq N-1 \qquad (3.25h)$$

Proof. See Appendix I.

Then for k=0,1,...,N, the system of global balance equations is:

$$P_N(k)\sum_{i=0}^{N} R_{ki} = \sum_{i=0}^{N} P_N(i)R_{ik} \qquad (3.26)$$

Since term $P_N(k)R_{kk}$ apears in both sides of the above, it is evident why R_{kk} rate is redundant.

Now this system of equations may be solved, using the method of z-transforms [KLEI 75, pp. 74-75], to obtain, after a considerable amount of non-trivial calculations, the solution of GE/GE/1/N system as:

$$P_N(0) = \frac{1-\rho}{1-\rho^2 x^{N-1}} \qquad (3.27a)$$

$$P_N(k) = P_N(0)\frac{\sigma\tau\rho}{\sigma\rho+\tau(1-\sigma)} x^k \quad , \; k=1,2,...,N-1 \qquad (3.27b)$$

$$P_N(N) = P_N(0)\rho x^{N-1} \qquad (3.27c)$$

where

$$x = \frac{\sigma\rho+\tau(1-\sigma)}{\sigma\rho+\tau(1-\sigma\rho)} = \frac{C_1^2+\rho C_2^2+\rho-1}{C_1^2+\rho C_2^2-\rho+1} \qquad (3.28)$$

and $\rho=\mu_1/\mu_2$.

3.2.7 The ME analysis of the G/G/1/N queueing system.

This section presents a review of the ME analysis of the G/G/1/N queue conducted by Kouvatsos [KOUV 86a].

Constraints assumed here concern the equilibrium queue length distribution, which is denoted as previously $\{P_N(n),\ n=0,1,\ldots,N\}$, and are:

– The normalization.

$$\sum_{k=0}^{N} P_N(k) = 1 \qquad (3.29a)$$

– The mean queue length (mql) constraint.

$$\sum_{k=0}^{N} kP_N(k) = \langle n \rangle_N \qquad (3.29b)$$

– The utilization constraint.

$$\sum_{k=0}^{N} h(k)P_N(k) = U_N \qquad (3.29c)$$

where,

$$h(k) = \begin{cases} 0 & \text{if } k=0 \\ 1 & \text{if } k\neq 0 \end{cases}$$

– The flow-balance (fb) constraint.

$$\sum_{k=0}^{N} f(k)P_N(k) = \Phi_N \qquad (3.29d)$$

where,

$$f(k) = \begin{cases} 0 & \text{if } k\neq N \\ 1 & \text{if } k=N \end{cases}$$

According to (3.3), the ME approximation, subject to the above constraints is:

$$P_N(n) = \frac{1}{Z_N} \exp\{-\lambda_1 n - \lambda_2 h(n) - \lambda_3 f(n)\}$$

where λ_1, λ_2 and λ_3 are the Lagrangian multipliers. Defining $x_2=\exp\{-\lambda_1\}$, $g_2=\exp\{-\lambda_2\}$ and $y_2=\exp\{-\lambda_3\}$, the above becomes:

$$P_N(0) = \frac{1}{Z_N} \tag{3.30a}$$

$$P_N(n) = \frac{1}{Z_N} g_2 x_2^n \quad , n=1,\ldots,N-1 \tag{3.30b}$$

$$P_N(N) = \frac{1}{Z_N} g_2 x_2^N y_2 \tag{3.30c}$$

From now on multipliers x_2, g_2 and y_2 will be refered to as the mql, utilization and fb multipliers respectively. Index 2 indicates that they correspond to the queue length distribution of service station 2 of fig. 3.7. The normalizing constant Z_N is:

$$Z_N = \frac{1-x_2+g_2x_2(1-x_2^{N-1})+(1-x_2)g_2y_2x_2^N}{1-x_2} \tag{3.31}$$

In order to obtain values for the multipliers, an assymptotic connection must be made to an infinite capacity queue. More precisely, every closed network can be viewed as a more general type of network than an open one, in the sense that the information about such a system consists of an additional factor, which is the fixed level of multiprogramming N. Truly, consider a closed network, for which N grows very large. Then the system tends to behave as an open one. The bottleneck device (unit with the highest utilization in a closed network), tends to be saturated and thus becomes an external arrival source (as well as sink) for the rest of the network. Hence, when the exact solution of a closed network is known, the solution of a "*corresponding open network*" is obtainable, by applying th limit ($N \longrightarrow +\infty$) on the former one, (see also [KLEI 75, p. 152]). What is

evident, is that certain parts (or forms) of the closed network's queue length distribution collapse when $N \longrightarrow +\infty$. For example the marginal queue length probabilities of the bottleneck device become zero , for any finite state, as well as any joint probability that includes a finite state of the bottleneck.

In the small network considered here, the condition $x_2<1$ is imposed from (3.31), in order to apply the limit ($N \longrightarrow +\infty$) in (3.30) solution. A further assumption made at this point concerns the invariability of the utilization and mql multipliers g_2, x_2, with respect to N, while it is sufficient to assume that:

$$\lim_{N\to+\infty} y_2 < +\infty$$

so that (3.33) converges as $N \longrightarrow +\infty$. Then:

$$\lim_{N\to+\infty} Z_N = \frac{1-x_2}{1-x_2+g_2x_2}$$

and

$$P_n = \lim_{N\to+\infty} P_N(n) = \frac{1-x_2}{1-x_2+g_2x_2}\, g_2^{h(n)}x_2^n \qquad (3.32)$$

Solution (3.34) may be considered as the ME approximation of a G/G/1 system, which in this case is the corresponding open system of the G/G/1/N queue. Now, constraint (3.29c) maybe written as:

$$1-P_N(0) = U_N$$

and as $N \longrightarrow +\infty$, the above yields:

$$P_0 = 1-\lim_{N\to+\infty} U_N$$

But under the stability conditions assumed previously, this relation is the well-known result [KLEI 75, p. 19], for the utilization of the server in a G/G/1 queue, i.e.:

$$P_0 = 1-\rho_2$$

where $\rho_2=\mu_1/\mu_2$.

Clearly, at this point condition $x_2<1$, assumed initially, so that $\{P_N(n),\ n=0,1,\ldots,N\}$ converges as $N \longrightarrow +\infty$ to a proper distribution, where $0<P_i<1$, $i=0,1,\ldots$, imposes the condition $\rho_2<1$. Thus:

$$x_2<1 \ \Rightarrow \ \rho_2<1$$

Substituting now P_0 from (3.34), equation $P_0=1-\rho_2$ yields:

$$g_2 = \frac{\rho_2(1-x_2)}{x_2(1-\rho_2)} \qquad (3.33)$$

Constraint (3.31b) at the limit ($N \longrightarrow +\infty$) becomes:

$$\sum_{k=0}^{\infty} kP_k = \lim_{N\to+\infty} \langle n\rangle_N = \langle n\rangle$$

Substituting P_k from (3.32) and g_2 from (3.33) the above yields:

$$x_2 = \frac{\langle n\rangle-\rho_2}{\langle n\rangle}$$

In the context of a stable GE/GE/1 queue, the exact mql formula may be derived, by applying spectral methods to solve Lindley's integral equation, and is [KOUV 86a, KOUV 88]:

$$\langle n\rangle = \frac{\rho_2}{2}\left[1 + \frac{C_1^2+\rho_2 C_2^2}{1-\rho_2}\right] \qquad (3.34)$$

Using the above, the expression derived for x_2 yields:

$$x_2 = \frac{C_1^2+\rho_2 C_2^2+\rho_2-1}{C_1^2+\rho_2 C_2^2-\rho_2+1} \qquad (3.35a)$$

then g_2 becomes:

$$g_2 = \frac{2\rho_2}{C_1^2+\rho_2 C_2^2+\rho_2-1} \qquad (3.35b)$$

And then returning to the closed system, the fb equation (3.29d) may

be written as:

$$[1-P_N(N)]\mu_1 = [1-P_N(0)]\mu_2$$

which, using (3.31), (3.33), yields:

$$y_2 = \frac{1-\rho_2}{1-x_2}$$

and substituting x_2 from (3.35a):

$$y_2 = \frac{C_1^2+\rho_2 C_2^2-\rho_2+1}{2} \qquad (3.35c)$$

It is not difficult to prove now that in the case of GE-type servers the solution (3.30) is identical to the exact solution (3.27) of the GE/GE/1/N system.

It is evident from (3.35a) that:

$$\rho_2<1 \Rightarrow x_2<1$$

So, finally

$$x_2<1 \Leftrightarrow \rho_2<1 \qquad (3.36)$$

and thus condition $x_2<1$ is equivalent to the stability condition for this corresponding open system. In case, however, that $x_2>1$, the same constraints may be assumed for the dual distribution $\{P_N^*(n), n=0,1,...,N\}$ defined by (3.24). The form of the solution would be once more:

$$P_N^*(n) = \frac{1}{Z_N^*} g_1^{h(n)} x_1^n y_1^{f(n)} \qquad (3.37)$$

where g_1, x_1, y_1 are the utilization, mql, and fb multipliers, that correspond to the queue length distribution of unit 1, respectively, while Z_N^* is the "*dual normalizing constant*". Using the duality relations (3.24), it is not difficult to prove that:

$$x_1 x_2 = g_1 y_2 = y_1 g_2 = 1 \qquad (3.38)$$

in which case if $x_2>1$, then $x_1<1$. Note also that if $\rho_1=\mu_2/\mu_1$, then the same is true for the relation between ρ_1 and ρ_2, i.e. $\rho_1\rho_2=1$. This precisely describes the fact that the limiting ($N \longrightarrow +\infty$) queue length distribution exists for one of the two queues.

Following the same steps, under the assumption $x_1<1$, it may be easily proven that:

$$x_1 = \frac{C_2^2+\rho_1 C_1^2+\rho_1-1}{C_2^2+\rho_1 C_1^2-\rho_1+1}$$

$$g_1 = \frac{2\rho_1}{C_2^2+\rho_1 C_1^2+\rho_1-1}$$

$$y_1 = \frac{C_2^2+\rho_1 C_1^2-\rho_1+1}{2}$$

It can be verified that the above three relations satisfy (3.38). Thus, solution (3.30) is valid, even when $x_2>1$.

Note that the assumption, on the invariability of multipliers, is exact here. Hence, the solution of this closed system shares the mql and utilization multipliers with its corresponding open system. In fact, even the fb multiplier y_2 (or y_1) is not a function of N. This implies that the GE/GE/1/N queue is governed by its assymptotic behaviour completely.

The invariability assumption is crucial for the ME approximate methods at the network level, [KOUV 86c] for two reasons. Firstly it is the only means of obtaining analytic expressions for the multipliers involved in the ME solution and secondly it allows us to disregard the characteristics of the actual flow in a closed system and use instead the flow of the corresponding open system. This point will be exploited in following chapters.

It is worth noticing that the amount of algebra involved in

obtaining the solution of the GE/GE/1/N system using the ME approach is insignificant compared to the corresponding amount required in the stochastic derivation. So, once more it seems that using this information theoretic method "we get something out of nothing" and we come to recall Jaynes saying, [JAYN 79, p. 31], "...if we can learn how to recognize and remove irrelevant information at the beginning of a problem, we shall be spared having to carry out immense calculations, only to discover at the end that practically everything we calculated was irrelevant to the question we were asking."

A generalization of the solution presented earlier [KOUV 86a] has been derived by Kouvatsos and Almond, [KOUV 86b], and concerns the two stage cyclic configuration with multiple homogeneous or heterogeneous servers, in the sense that servers of the same resource may have different (load-dependent) service rates but the same coefficient of variation, of GE type. This solution is very useful for the conventional implementation of decomposition algorithms described in the previous chapter.

Let's now see several versions of the two major decomposition algorithms as applied to the approximate analysis of queueing networks with GE-distributed service times.

3.3 Approximate decomposition algorithms for central server models with GE-distributed service times.

The central server type of configuration (fig. 2.7), has been chosen, to be used as testing ground for some versions of decomposition algorithms, since this configuration can be tackled using both variable aggregation and Norton's reduction decomposition schemes. The product form solutions of two stage cyclic queues, that involve GE (or exponential) servers with load-dependent service rates [KOUV 86b] provide the building block for the implementation of these

algorithms.

More precisely, five versions of decomposition algorithms will concern us here. Let's firstly outline their implementation.

3.3.1 Approximate decomposition algorithm based on the variable aggregation scheme.

The algorithm chosen to represent the variable aggregation scheme, denoted as COURT, is the one proposed by Courtois for general networks. This algorithm has been presented stepwise in the second chapter (algorithm 2.1, p. 19), for a general (G) type of distribution. Therefore, the only difference here is that G is specified to be the generalized exponential model (G=GE).

The input data, required for this algorithm are presented in fig. 2.7 and they are: the number of units (L+1), the fixed level of multiprogramming (N), for each unit i, i=0,1,...,L, the mean service rate μ_i and squared coefficient of variation C_i^2 of the service time and finally the transition probabilities $\{r_i, i=1,...,L\}$.

In the first level of aggregation (step 1 of algorithm 2.1), the solution (3.30) of a GE/GE/1/N two stage cyclic system is used for the subsystem that consists of the CPU and the first I/O unit. The coefficients of variation used are those of the service times of the two units. Note that the fact that the CPU has a reduced service rate, indicates the option of using a reduced coefficient of variation for this unit, by applying the splitting formula (3.23b) on the coefficient of the service time. Experimentation, however, has not indicated an improvement of the results under this option.

In the ℓth level of aggregation (step 2 of algorithm 2.1), now, $\ell=2,...,L$, the aggregate unit is assumed to be of exponential (M) type, while the ℓth I/O has its given service time characteristics ($GE(\mu_\ell, C_\ell^2)$). The solution used in this level is that of a

GE/M(n)/1/N_Q system, and is a special case of the Kouvatsos-Almond solution, [KOUV 86b].

It must be pointed out that, as it has been mentioned previously, the enumeration of the units plays a significant role in such a hierarchical approach. Different ordering (or way of coupling the units) implies different results. For example, in a central server model with three I/O units, there are six different sets of results. In the tables, in which this algorithm will be tested against the exact solution, only the best approximation is taken under consideration. It is very difficult to provide a criterion on coupling in such approximate decomposition methods based on a mathematical analysis, because the accuracy depends not only on the decomposability of the model, but also on the approximation of the flow. For instance, the assumption of an exponential aggregate server in this version of the algorithm, implies an unpredictable error. Hense, experimentation is the only way of obtaining some indication on which parts of the network should be joined and in which order.

3.3.2 Approximate decomposition algorithms based on Norton's reduction scheme.

The application of this method in a central server model (fig. 2.7), consists of two steps. In the first step the I/O subsystem (fig. 2.8) is solved approximately in isolation for all populations N_1=1,2,...,N. The load-dependent throughputs $\chi(N_1)$ are calculated and used in the second step as the load-dependent service rates of the composite server (fig. 2.9) that represents the subsystem.

As it was mentioned in the second chapter, the accuracy of this method depends mainly on two factors. Firstly on the method used to analyze the I/O subsystem and secondly on the characterization of the distribution (and its parameters) of the composite server. In order

to demonstrate the improvement of the results, subject to the above two factors, four alternative versions of this algorithm are tested, let's denote them as FE_i, i=1,2,3,4. In versions 1 and 2 the subsystem is solved exactly, assuming exponential distribution for all service times. In versions 3 and 4 the subsystem is solved approximately, using the hierarchical decomposition algorithm described in the previous section, i.e. algorithm 2.1 assuming GE-distributed service times. Therefore, an improvement is expected over versions 1 and 2. Another alternative, that would guarantee similar improvement, is to use the universal maximum entropy (UME) algorithm [KOUV 86c], in this first step, as it has been successfully implemented and tested in [OTHM 88].

In versions 1 and 3 now and in the second step, the composite server is assumed to have an axponential distribution, while in versions 2 and 4 a GE distribution is used instead. The coefficient of variation for the composite server, in this later case, is calculated according to formula (2.17), proposed by Sauer and Chandy, [SAUE 75a]. Before summarizing the above into a stepwise description of algorithm FE_4, and for notational convinience, let's reenumerate the units of fig. 2.7 so that the CPU is unit L+1 and the I/Os are units 0 through L.

Algorithm 3.1. (FE_4)

Step 1. Consider the I/O subsystem in isolation (fig. 2.8), i.e. umits 0 up to L.

Step 1.1. Because this subsystem is a fully connected one (every unit communicates with every other unit), the feedback correction is applied to the service characteristics of each server. This correction involves (3.23a) and (3.23b) formulae for the service time parameters, as well as the definition of a new routing matrix with probabilities $\{r_{ij}, i,j=0,...,L\}$ so that for every i=0,...,L:

$$r^*_{ij} = \frac{r_j}{1-r_i} \quad , \; j=0,\ldots,L, \; i\neq j$$

$$r^*_{ii} = 0$$

Step 1.2. To this modified network apply algorithm 2.1 adjusted as in the previous section for GE-distributed service times, in order to evaluate the conditional distributions $\{P_\ell(N_{\ell-1}/N_\ell),\ N_{\ell-1}=0,\ldots,N_\ell\}$, $N_\ell=1,\ldots,N$, $\ell=1,\ldots,L$. Note that in level ℓ, $\ell=2,\ldots,L$, of aggregation the squared coefficient of variation of unit ℓ is reduced, by applying the splitting formula (3.21b) on the coefficient of the service time, with probability $\sum_{k=0}^{\ell-1} r^*_{\ell k}$, in order to approximate the fact that only a proportion of the output flow of unit ℓ is considered at this level, since it is analyzed in connection to the subsystem that consists of units $0,1,\ldots,\ell-1$. Furthermore, for $\ell=1$ (unit 0 versus unit 1) the same modification (3.21b) applies to the coefficients of units 0 and 1 with probabilities r^*_{01} and r^*_{10} respectively, since both service times are assumed GE-distributed.

Step 1.3. Calculate the load-dependent throughputs $\chi(N_L)$, $N_L=1,\ldots,N$, as follows. Let $\chi_\ell(N_L)$, $\ell=0,\ldots,L$ be the throughput of unit ℓ, conditioned on the fact that N_L jobs circulate in the I/O subsystem. Then for $\ell=L$:

$$\chi_L(N_L) = [\,1-P_L(N_L/N_L)\,]\mu_L$$

$$\alpha_L(N_{L-1}) = P_L(N_{L-1}/N_L) \quad , \; N_{L-1}=0,\ldots,N_L$$

and for each level ℓ, $\ell=L-1,\ldots,1$, and for $N_{\ell-1}=0,\ldots,N_L$:

$$\alpha_\ell(N_{\ell-1}) = \sum_{N_\ell=N_{\ell-1}}^{N_L} P_\ell(N_{\ell-1}/N_\ell)\alpha_{\ell+1}(N_\ell)$$

and

$$\chi_\ell(N_L) = \sum_{N_\ell=1}^{N_L} \mu_\ell[\,1-P_\ell(N_\ell/N_\ell)\,]\alpha_{\ell+1}(N_\ell)$$

while

$$\chi_0(N_L) = \mu_0[1-\alpha_1(0)]$$

Then:

$$\chi(N_L) = \sum_{\ell=0}^{L} \chi_\ell(N_L)$$

where the notation used, has been introduced in the 2nd chapter, pp. 16-17.

Step 2. Solve the two stage cyclic system of fig. 2.9, in order to evaluate $\{P_{L+1}(N_L/N),\ N_L=0,\ldots,N\}$ distribution, where $P_{L+1}(N_L/N)$ is the conditional (or marginal, since condition N is redundant) probability that N_L jobs are present in the aggregate server, given that N is the population of jobs. The solution used is the one of a GE/GE(n)/1/N system, [KOUV 86b]. The coefficient of the aggregate server is provided by (2.17), while its load-dependent service rates are exactly the load-dependent throughputs $\chi(N_L)$, $N_L=1,\ldots,N$ of step 1.3.

Step 3. Obtain the marginal distributions, using the same procedure as in step 3 of algorithm 2.1.

Version FE_3 is identical to the above, except that in step 2 the coefficient of the composite server is equal to 1, and the GE/GE(n)/1/N solution reduces to a GE/M(n)/1/N one. Versions FE_1 and FE_2 can be produced from the above algorithm by altering the coefficients of variation of all the I/O units considered in step 2 to be equal to 1. Finally, the difference between FE_1 and FE_2 is the same as the difference between FE_3 and FE_4.

It must be pointed out that in step 1.2, since a hierarchical decomposition algorithm is used to approximate the subsystem, there is a decision to be made on the ordering of the I/Os. These units are ordered with increasing variability, since algorithm 2.1 has been observed to perform better in such a case.

3.3.3 Validation of decomposition algorithms.

The validation of these five approximations is conducted, using an algorithm implemented by Almond [ALMO 88]. This algorithm generates randomly a specified number of networks with GE-distributed service times out of a given population, and solves them exactly, using matrix inversion. Each network is also solved using the above approximate decomposition algorithms as well as the universal maximum entropy (UME) approximate algorithm [KOUV 86c]. Finally, the exact solution of the network, under the assumption of exponential service times, is also tested as an approximation (EXP). These last two algorithms (UME, EXP) have been also implemented by Almond, [ALMO 88]. The approximate results are compared to the exact ones and overall statistics, concerning their accuracy, are computed.

More precisely, the population of networks is specified as follows. M is the number of service centers, N is the fixed level of multiprogramming. For each center i, i=1,...,M, the mean service rate μ_i and squared coefficient of variation C_i^2 are randomly chosen to be one out of four given values, for example $\mu_i \in \{1,3,5,10\}$, $C_i^2 \in \{2,4,10,100\}$. The routing probabilities from each server i to the rest of the network $\{r_{ij}, j=1,...,M\}$ are also randomly defined. An one-to-one correspondence is defined by associating a number to each probability r_{ij}, j=1,...,M. So, if for example $r_{i_1} \leftrightarrow 1$ and $r_{i_2} \leftrightarrow 0.5$, the transition probabilities r_{i_1} and r_{i_2} are selected so that on average r_{i_1} is double that r_{i_2}. Obviously, two centers do not communicate if their corresponding number is 0.

It is evident that the matrix inversion technique, used for the derivation of the exact solution, limits the size of the networks that can be solved. In the experiments carried out, usually $M \in \{3,4,5\}$ and $N \in \{3,4,5\}$.

The error that the approximate solutions imply, is characterized

by computing several appropriate measures. Two of them are chosen to be presented in the tables that follow, namely the utilization error tolerance (UTOL) and the mean queue length error tolerance (NTOL). The first is computed for every network, as the maximum (out of all utilizations) absolute difference between the approximate and exact values. The second is for each network the maximum ratio of the absolute difference over the population N. For each of the above variables, the following statistics are presented:

1) mean value

2) standard deviation from the mean

3) maximum value recorded.

These statistics provide an indication of both the overall accuracy, seen in the mean value and reliability, seen in the standard deviation and the maximum, of the approximation. An error tolerance is considered good if it is less than 0.05 and adequate if it is between 0.05 and 0.1, [CHAN 75b].

3.3.4 Discussion.

The tables 3.1-3.6 presented in Appendix I provide an indication on the performance of the algorithms. Note firstly that versions FE_1 and FE_2 together with the exponential (EXP) algorithm are unacceptable. This is a very good example of the error introduced by the exponential assumption, when analyzing a large part of the network. The only observation is that FE_2, generally, introduces an improvement in the utilizations over FE_1.

Examining now each of the rest of the decomposition algorithms in isolation, the characterization good but unreliable seems to be valid for all three of them (COURT, FE_3, FE_4). This is based on the fact that mean error tolerances are good, but the maximum value sometimes indicate excessive errors, (tables 3.1, 3.4, 3.5, 3.6 for FE_3 and

FE_4, tables 3.2, 3.3, 3.5 for COURT), especially for COURT and FE_3, while FE_4 seems to be more stable in terms of reliability.

However, it should be pointed out that COURT and FE_3 algorithms are related in the sense that FE_3 is an extension of COURT, with respect to the assumptions on the distribution of the composite server (exponential in both), and the method used to analyze the I/O subsystem. It may be noticed that these two algorithms seem to complement one another, i.e. when one fails, the other is quite successful. More specifically, if these two are jointly examined it is noticeable that this approach favours an enumeration of the units, so that higher variability centers are examined in higher levels of aggregation. Hence, the rule "join the high variability unit last" is experimentally implied here. This rule will concern us in later chapters. So, COURT and FE_3 jointly provide a very good approximate algorithm.

Lastly, a few words about the universal maximum entropy (UME) iterative approximate algorithm, which has been developed by Kouvatsos, [KOUV 86c]. Its performance indicates that it is the best of all algorithms tested here, both in accuracy and reliability. It is particularly good in very high and low variability (tables 3.1, 3.5). Its weak point however seems to be a combination of low variability units with a very high variability center (table 3.3). This characterization is in agreement with the one provided by Almond, [ALMO 88], after exhaustive experimentation.

3.4 Review.

In this chapter the ME principle has been presented together with its related generalized exponential distribution. Several properties of this distribution have been outlined and proved in Appendix I using an alternative approach through the concept of ordered bulks.

Some of these properties will be used thoroughly in following chapters. Finally some indicative tests were carried out, concerning the conventional implementation of decomposition algorithms, used as approximations to networks with GE-distributed service times. The results obtained in these tests on the one hand indicate how promising the decomposition approach in the analysis of QNMs is, while on the other hand and together with the very good results obtained by the UME algorithm, provide the motivation for further investigation on decomposition within an information theoretic framework.

In the next chapter the principle of minimum relative entropy (MRE) and its properties will be introduced, followed by a detailed presentation of Shore's work on the principle under the assumption of fully decomposable constraints, [SHOR 82b]. This will provide the information theoretic basis for the application in the analysis of QNMs through decomposition, which is the main contribution of this thesis.

CHAPTER IV

MINIMUM RELATIVE ENTROPY FORMALISM AND THE CONCEPT OF FULLY DECOMPOSABLE CONSTRAINTS

4.1 The principle of Minimum Relative Entropy (MRE).

As mentioned previously (section 3.1) minimum relative (or cross) entropy is a generalization of the maximum entropy principle in continuous distributions. The functional used here is slightly different than the one defined by (3.1). This is due to the fact that adopting (3.1) and just substituting the sum with an integral and the probability distribution with a density function, the resulting inference method does not have the invariance property, according to which the entropy solution should be invariant to a coordinate transformation.

Thus, the modified cross-entropy functional was proposed by Kullback [KULL 59], and examined by Shore and Johnson, [SHOR 80], who axiomatically derived the principle by means of four consistency axioms. Let's firstly define the principle and the notation introduced by Shore and Johnson, [SHOR 80, SHOR 81].

Let X be a continuous random variable with a domain D and a true but unknown density function $f^*(x)$, $x \epsilon D$, which belongs to the class of densities Ω, such that if $h \epsilon \Omega$ then:

$$h(x) \geq 0 \ , \ \forall \ x \ \epsilon \ D$$

and

$$\int_D h(x) \, dx = 1$$

Let $f, g \epsilon \Omega$. The cross-entropy between f(x) and g(x) is defined to be the quantity:

$$H_c(f,g) = \int_D f(x)\log\left\{ \frac{f(x)}{g(x)} \right\} dx \qquad (4.1)$$

Suppose that a prior estimate of $f^*(x)$ is available and let's denote it as g(x). This is usually refered to as "*prior density*", although it is not necessarily an element of Ω, in the sense that it should not strictly satisfy the normalization constraint but a more general one:

$$\int_D g(x)\, dx > 0 \qquad (4.2)$$

Thus, such priors are not necessarily proper densities. In some cases they are regarded as either generalized densities, [KULL 59], or invariant measure functions, [JAYN 68].

Suppose now that new information about the system is available, in the form of linear equality or inequality constraints:

$$\int_D a_k(x) f^*(x)\, dx = \langle a_k \rangle \quad , k=1,\ldots,m \qquad (4.3a)$$

or

$$\int_D b_k(x) f^*(x)\, dx \geq \langle b_k \rangle \quad , k=1,\ldots,n \qquad (4.3b)$$

where $a_k(x)$, k=1,...,m, or $b_k(x)$, k=1,...,n, are suitable functions and $\langle a_k \rangle$, $\langle b_k \rangle$ are given values. Note that constraints (4.3) may equivalently be written with a zero right hand side, using the fact that $f^*(x)$ is normalized. For example, (4.3a) may be written as:

$$\int_D c_k(x) f^*(x)\, dx = 0 \quad , k=1,\ldots,m \qquad (4.3c)$$

where $$c_k(x) = a_k(x) - \langle a_k \rangle$$

The densities that satisfy such constraints always comprise a closed convex subset Υ of Ω in the sense that given any two densities

h_1, $h_2 \epsilon \Upsilon$, then for every real number α such that $0 \leq \alpha \leq 1$, the weighted average $(\alpha h_1(x)+(1-\alpha)h_2(x))$ is also an element of Υ.

The relation $f^* \epsilon \Upsilon$ is denoted by I, and I is refered to as a constraint, while Υ is a constraint set. Evidently constraint I does not fully specify the true density $f^*(x)$. But according to the MRE principle, of all densities $f \epsilon \Upsilon$ the one that minimizes (4.1), subject to constraints (4.3), should be chosen as the best approximation to $f^*(x)$. Once more this MRE solution f(x) is the least biased approximation or the closest, in information theoretic terms, density to g(x), that satisfies the additional information I.

The minimization problem, subject to (4.3a), is solved by Lagrange's method of undetermined multipliers leading to the solution:

$$f(x) = g(x)\exp\left[-\lambda_0 - \sum_{k=1}^{m} \lambda_k a_k(x)\right] \tag{4.4}$$

where λ_k, k=0,...,m, are the Lagrangian multipliers which satisfy:

$$\lambda_0 = \log\left\{ \int_D g(x)\exp\left[-\sum_{k=1}^{m} \lambda_k a_k(x) \right] dx \right\} \tag{4.5a}$$

and

$$-\frac{\partial \lambda_0}{\partial \lambda_k} = \langle a_k \rangle \ , \ k=1,\ldots,m \tag{4.5b}$$

Obviously, if (4.5a) can be solved analytically then closed form expressions for the values of λ_k, k=1,...,m, can be obtained by solving relations (4.5b). Otherwise numerical techniques have to be used for the evaluation of λ_k, k=0,...,m, [JOHN 79].

So given a prior density g(x) and new information I, the procedure whereby a posterior density f(x) is found for which:

$$H_c(f,g) = \min_{f' \epsilon \Upsilon} H_c(f',g)$$

will be expressed by introducing an information operator "⊙" that defines the above as:

$$f = g \odot I \tag{4.6}$$

In some applications, new information about the system is available in the form of subset or decomposed constraints in the following sense. Let the set of all possible states of the system D be partitioned into n disjoint subsets $D_1, D_2, \ldots, D_n$. The probability of being in each subset is expressed as:

$$\int_{D_i} f^*(x)\, dx = a_i \quad , \; i=1,\ldots,n \tag{4.7}$$

where a_i are known values. If M is the set of densities that satisfy (4.7), the notation M will be used for the relation $f^* \epsilon M$, $(M=(f^* \epsilon M))$. Also the conditional density:

$$f^*(x/x \epsilon D_i) = \frac{f^*(x)}{\int_{D_i} f^*(y)\, dy} \tag{4.8}$$

will be denoted as $f^* * D_i$.

If new information is available about $f^* * D_i$, i=1,...,n, which have the form of linear equality or inequality constraints and is denoted by $I_{si} = (f^* * D_i \epsilon \Upsilon_i)$, where Υ_i is a convex subset of Ω_i, the set of all densities defined on D_i, then constraints I_{si} will be called "*subset constraints*". If also the discrete distribution $\{a_i$, i=1,...,n} is not completely specified, but M is defined by a set of constraints I_a, then these constraints will be called "*aggregate constraints*". Finally, information about the system, which consists only of subset and aggregate constraints will be refered to as "*fully decomposable constraints*".

Several properties have been presented and proved by Shore and Johnson in [SHOR 80] and [SHOR 81]. Some of them are presented and

discussed in Appendix II since they provide a better insight of the formalism, while three properties that concern the subset and aggregate constraints, (properties 4, 5 and 6), have been further investigated by Shore, in an attempt to develop an efficient decomposition approach for solving numerically entropy problems [SHOR 82b]. The results of this work are of great importance to this thesis as the potentiality of their application in the analysis of QNMs is evident [SHOR 82b, SHOR 83].

Hence, in the next section the analysis presented by Shore, [SHOR 82b], will be carried out in detail in its original form as well as incorporating a modification in the use of the constraints assumed.

4.2 The MRE principle given fully decomposable subset and aggregate constraints.

Subset Constraints.

Let $D_1, D_2, \ldots, D_n$ be disjoint subsets whose union is D and let f^* and f^**D_j denote the true density on D and true density on the subset D_j respectively. Let $I_{sj}=(f^**D_j \epsilon \Upsilon_j)$ be new information about the conditional density f^**D_j, where $\Upsilon_j \subseteq \Omega_j$ and Ω_j is the set of densities on D_j. In particular suppose that I_{sj} is determined by constraints of the form:

$$I_{sj} : \int_{D_j} \beta_{ij}(x)[f^**D_j](x)\, dx = 0 \quad , i=1,\ldots,K_j \qquad (4.9)$$

Constraints with a non-zero right hand side can always be rewritten in this form.

Now, constraints I_{sj} can also be written as constraints $I^{\dagger}_{sj}$ on the full density f^*, namely $I^{\dagger}_{sj}=(f^* \epsilon \Upsilon^{\dagger}_j)$, where $\Upsilon^{\dagger}_j \subseteq \Omega$ and Ω is the set of densities on D. In particular, constraints (4.9) may be expressed in terms of the full density f^* as:

$$I_{sj}^{\dagger} \; : \; \int_{D} \alpha_{ij}(x) f^{*}(x)\, dx = 0 \quad , \; i=1,\ldots,K_j \tag{4.10}$$

where

$$\alpha_{ij}(x) = \begin{cases} \beta_{ij}(x) & \text{if} \quad x \in D_j \\ 0 & \text{if} \quad x \notin D_j \end{cases} \tag{4.11}$$

Now let's denote:

and $$I_s = I_{s_1} \wedge I_{s_2} \wedge \ldots \wedge I_{sn}$$

$$I_s^{\dagger} = I_{s_1}^{\dagger} \wedge I_{s_2}^{\dagger} \wedge \ldots \wedge I_{sn}^{\dagger}$$

I_s and $I_s^{\dagger}$ will be refered to as subset constraints.

Aggregate Constraints.

Consider the same partition of D and let ψ be a subset aggregation transformation, which for any $f \in \Omega$ produces a discrete distribution $\{[\psi f]_j,\ j=1,\ldots,n\}$ with:

$$[\psi f]_j = \int_{D_j} f(x)\, dx \tag{4.12}$$

Suppose there is new information I_a about the aggregate distribution ψf^{*} of the form:

$$I_a \; : \; \sum_{j=1}^{n} \gamma_{ij} [\psi f^{*}]_j = 0 \quad , \; i=1,\ldots,K_a \tag{4.13}$$

Information I_a can also be expressed as information $I_a^{\dagger}$ about the full density f^{*} as follows:

$$I_a^{\dagger} \; : \; \int_{D} \gamma_i(x) f^{*}(x)\, dx = 0 \quad , \; i=1,\ldots,K_a$$

where

$$\gamma_i(x) = \gamma_{ij} \quad \text{if} \quad x \in D_j \tag{4.15}$$

I_a and $I_a^{\dagger}$ will be refered to as aggregate constraints. When new information I comprises only subset and aggregate constraints, then

$I=I_s \wedge I_a$ or $I^\dagger=I_s^\dagger \wedge I_a^\dagger$ will be refered to as fully decomposable constraints.

Recall now that when new information consists of subset constraints, relative-entropy minimization satisfies the property of weak subset independence (property 5, App. II). So for any prior $g \epsilon \Omega$:

$$(g \odot I_s^\dagger) * D_j = (g * D_j) \odot I_{sj} \tag{4.16}$$

Also when new information consists of aggregate constraints, property 6 (App. II) states that:

$$\psi(g \odot I_a^\dagger) = (\psi g) \odot I_a \tag{4.17}$$

and

$$H_c(g \odot I_a, g) = H_c(\psi(g \odot I_a^\dagger), \psi g) \tag{4.18}$$

Thus, whereas in Appendix II properties 4,5 and 6 concerned cases where information (in the form of constraints) were available about either the conditional densities or the aggregate distribution, here the combination of the two is investigated.

Let's define and compute now the three posterior densities which will concern us from this point on.

1) $g^\dagger = g \odot I_s^\dagger$.

Using (4.4) and (4.10) for all $j=1,\ldots,n$:

$$g^\dagger(x) = g(x)\exp\left[-\lambda - \sum_{j=1}^{n} \sum_{i=1}^{K_j} \lambda_{ij}\alpha_{ij}(x)\right] \tag{4.19}$$

where λ_{ij}, $i=1,\ldots,K_j$, $j=1,\ldots,n$, are the corresponding to $I_s^\dagger$ Lagrangian multipliers, while λ is the normalization multiplier.

2) $h^\dagger = (g \odot I_s^\dagger) \odot I_a^\dagger = g^\dagger \odot I_a^\dagger$.

Once more according to (4.4) and (4.14):

$$h^\dagger(x) = g^\dagger(x)\exp\left[-\theta - \sum_{i=1}^{K_a} \theta_i \gamma_i(x)\right] \tag{4.20a}$$

and using (4.19):

$$h^{\dagger}(x) = g(x)\exp\left[-\lambda - \sum_{j=1}^{n}\sum_{i=1}^{K_j}\lambda_{ij}\alpha_{ij}(x) - \theta - \sum_{i=1}^{K_a}\theta_i\gamma_i(x)\right] \quad (4.20b)$$

where θ, θ_i, $i=1,\ldots,K_a$ are the corresponding Lagrangian multipliers.

3) $f^{\dagger} = g \odot (I_s^{\dagger} \wedge I_a^{\dagger})$.

Clearly, from (4.4) and the definition of $I_s^{\dagger}$ and $I_a^{\dagger}$:

$$f^{\dagger}(x) = g(x)\exp\left[-\varphi - \sum_{j=1}^{n}\sum_{i=1}^{K_j}\eta_{ij}\alpha_{ij}(x) - \sum_{i=1}^{K_a}\xi_i\gamma_i(x)\right] \quad (4.21)$$

where φ, η_{ij}, ξ_i, are the corresponding Lagrangian multipliers.

In this context the following lemma holds, [SHOR 82b].

Lemma 4.1. Relative-Entropy minimization subject to fully decomposable constraints satisfies the following relations:

$$(g \odot (I_s^{\dagger} \wedge I_a^{\dagger})) * D_j = (g * D_j) \odot I_{sj} \quad (4.22)$$

$$\psi(g \odot (I_s^{\dagger} \wedge I_a^{\dagger})) = (\psi(g \odot I_s^{\dagger})) \odot I_a \quad (4.23)$$

$$H_c(f^{\dagger}, g) = H_c(g^{\dagger}, g) + H_c(\psi f^{\dagger}, \psi g^{\dagger}) \quad (4.24)$$

$$H_c(f^{\dagger}, g) = H_c(\psi f^{\dagger}, \psi g) + \sum_{j=1}^{n}[\psi f^{\dagger}]_j H_c(f^{\dagger} * D_j, g * D_j) \quad (4.25)$$

Proof. The proof can be seen in Appendix II.

Consider the case now where instead of (4.9), subset constraints were given by:

$$I_{sj} : \int_{D_j} \beta_{ij}(x)[f^* * D_j](x)\,dx = \langle\beta_{ij}\rangle \quad , i=1,\ldots,K_j \quad (4.26)$$

and using the definition of $f^* * D_j$, (4.26) can be written with respect to the full density $f^*(x)$ as:

$$I_{sj}^{\dagger} : \int_{D} \alpha_{ij}(x) f^*(x)\,dx = \langle\beta_{ij}\rangle[\psi f^*]_j \quad , i=1,\ldots,K_j \quad (4.27)$$

with

$$\alpha_{ij}(x) = \begin{cases} \beta_{ij}(x) & \text{if} \quad x \in D_j \\ 0 & \text{if} \quad x \notin D_j \end{cases} \tag{4.28}$$

Similarly, the aggregate constraints I_a can be of the form:

$$I_a \; : \; \sum_{j=1}^{n} \gamma_{ij}[\psi f^*]_j = \langle \gamma_i \rangle \quad , \; i=1,\ldots,K_a \tag{4.29}$$

which may be written as:

$$I_a^{\dagger} \; : \; \int_D \gamma_i(x) f^*(x) \, dx = \langle \gamma_i \rangle \quad , \; i=1,\ldots,K_a \tag{4.30}$$

with

$$\gamma_i(x) = \gamma_{ij} \quad \text{if} \quad x \in D_j \tag{4.31}$$

It is clear that (4.26) may be written with a zero right hand side. The MRE solution in each subset D_j, $j=1,\ldots,n$ is the same using (4.9) and (4.26) within a multiplicative constant. Since though in the proofs of (4.24) and (4.25) (App. II) the fact that constraints had a zero right hand side was used, it is expected that there is a difference if constraints (4.26)-(4.31) are used. Truly, a modified version of Lemma 4.1 is true if the above constraints are used.

Lemma 4.2. Relative entropy minimization, subject to fully decomposable subset and aggregate constraints defined by (4.26)-(4.31), satisfies the following relations:

$$(g \odot (I_s^{\dagger} \wedge I_a^{\dagger})) * D_j = (g * D_j) \odot I_{sj}$$

$$\psi(g \odot (I_s^{\dagger} \wedge I_a^{\dagger})) = (\psi(g \odot I_s^{\dagger})) \odot I_a$$

$$H_c(f^{\dagger}, g) = H_c(g^{\dagger}, g) + H_c(\psi f^{\dagger}, \psi g^{\dagger}) +$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{K_j} \lambda_{ij} \langle \beta_{ij} \rangle [[\psi g^{\dagger}]_j - [\psi f^{\dagger}]_j] \tag{4.32}$$

$$H_c(f^{\dagger}, g) = H_c(\psi f^{\dagger}, \psi g) + \sum_{j=1}^{n} [\psi f^{\dagger}]_j H_c(f^{\dagger} * D_j, g * D_j)$$

Proof.

So, using non-zero right hand sides in the assumed constraints only affects (4.24). Relation (4.32) is true instead. Relations (4.22), (4.23) and (4.25) are also satisfied here.

In fact in order to prove the first two equations, the same approach, as in Lemma 4.1, should be followed (see App. II). So modifications in the proofs of (4.32) and (4.25) will be pointed out next.

1) Proof of (4.32).

Using (4.19) and (4.21) it follows that:

$$H_c(g^\dagger,g) = \int_D g^\dagger(x)\log\left\{ \frac{g^\dagger(x)}{g(x)} \right\} dx =$$

$$= \int_D g^\dagger(x)\left[-\lambda - \sum_{j=1}^{n}\sum_{i=1}^{K_j}\lambda_{ij}\alpha_{ij}(x)\right] dx \Rightarrow$$

$$H_c(g^\dagger,g) = -\lambda - \sum_{j=1}^{n}\sum_{i=1}^{K_j}\lambda_{ij}<\beta_{ij}>[\psi g^\dagger]_j \qquad (L2.1)$$

Also:
$$H_c(f^\dagger,g) = -\varphi - \sum_{j=1}^{n}\sum_{i=1}^{K_j}\eta_{ij}<\beta_{ij}>[\psi f^\dagger]_j - \sum_{i=1}^{K_a}\xi_i<\gamma_i> \qquad (L2.2)$$

Then from (L1.8) (n.b., (L1.1)-(L1.15) relations can be found in the proof of Lemma 4.1, App. II) and the aggregate constraints (4.29):

$$H_c(\psi f^\dagger,\psi g^\dagger) = \sum_{j=1}^{n} f_j^\dagger \log\left\{ \frac{f_j^\dagger}{g_j^\dagger} \right\} = \sum_{j=1}^{n} f_j^\dagger \left[\lambda-\varphi - \sum_{i=1}^{K_a}\xi_i\gamma_{ij}\right] \Rightarrow$$

$$H_c(\psi f^\dagger,\psi g^\dagger) = \lambda-\varphi - \sum_{i=1}^{K_a}\xi_i<\gamma_i> \qquad (L2.3)$$

Substituting λ from (L2.1) and $-\varphi$ from (L2.2) into (L2.3), and since $\lambda_{ij}=\eta_{ij}$ $\forall$ i,j (see proof of Lemma 4.1):

$$H_c(\psi f^\dagger,\psi g^\dagger) = H_c(f^\dagger,g) - H_c(g^\dagger,g) +$$

$$+\sum_{j=1}^{n}\sum_{i=1}^{K_j}\lambda_{ij}<\beta_{ij}>[[\psi f^\dagger]_j-[\psi g^\dagger]_j]$$

from which (4.32) follows.

2) Proof of (4.25).

Up to (and including) relation (L1.13) the proof is the same as in Lemma 4.1 (App. II). Then using (4.16) and (L1.7):

$$H_c(f^\dagger * D_j, g * D_j) = H_c[[g * D_j]\odot I_{sj}, g * D_j] =$$

$$= \int_{D_j} [[g * D_j]\odot I_{sj}](x)\log\left\{\frac{[[g * D_j]\odot I_{sj}](x)}{[g * D_j](x)}\right\} dx =$$

$$= \int_{D_j} [[g * D_j]\odot I_{sj}](x)\left[-\theta_j - \sum_{i=1}^{K_j}\theta_{ij}\beta_{ij}(x)\right] dx \quad \Rightarrow$$

$$\Rightarrow \qquad H_c(f^\dagger * D_j, g * D_j) = -\theta_j - \sum_{i=1}^{K_j}\theta_{ij}<\beta_{ij}>$$

and since $\eta_{ij}=\lambda_{ij}=\theta_{ij}$ holds, it follows from (L1.8) and (L1.13) that:

$$f_j^\dagger = g_j\exp\left[\theta_j-\varphi - \sum_{i=1}^{K_a}\xi_i\gamma_{ij}\right]$$

and hence that:

$$H_c(\psi f^\dagger,\psi g) = \sum_{j=1}^{n} f_j^\dagger \log\left\{\frac{f_j^\dagger}{g_j}\right\} = \sum_{j=1}^{n} f_j^\dagger \left[\theta_j-\varphi - \sum_{i=1}^{K_a}\xi_i\gamma_{ij}\right] \quad \Rightarrow$$

$$H_c(\psi f^\dagger,\psi g) = \sum_{j=1}^{n} f_j^\dagger\,\theta_j - \sum_{i=1}^{K_a}\xi_i<\gamma_i> - \varphi \qquad \text{(L2.5)}$$

And substituting $-\varphi$ from (L2.2), the above yields:

$$H_c(\psi f^\dagger,\psi g) = \sum_{j=1}^{n} f_j^\dagger\,\theta_j - \sum_{i=1}^{K_a}\xi_i\langle\gamma_i\rangle + H_c(f^\dagger,g) +$$

$$+\sum_{j=1}^{n}\sum_{i=1}^{K_j}\eta_{ij}\langle\beta_{ij}\rangle f_j^\dagger + \sum_{i=1}^{K_a}\xi_i\langle\gamma_i\rangle \Rightarrow$$

$$H_c(f^\dagger,g) = H_c(\psi f^\dagger,\psi g) + \sum_{j=1}^{n} f_j^\dagger\left[-\theta_j - \sum_{i=1}^{K_j}\eta_{ij}\langle\beta_{ij}\rangle\right]$$

and since $\theta_{ij}=\eta_{ij}$ $\forall$ i,j, using (L2.4), (4.25) follows.

Q.E.D.

Now, the following theorem, [SHOR 82b] describes the decomposability of MRE problems when fully decomposable constraints are available.

Theorem 4.1. Relative-Entropy minimization with fully decomposable constraints satisfies the following relation :

$$g\odot(I_s^\dagger\wedge I_a^\dagger) = (g\odot I_s^\dagger)\odot I_a^\dagger \qquad (4.33)$$

Proof. The proof has been given in [SHOR 82b] and for instructive purposes can be seen in Appendix II.

This theorem is true under both types of constraints, i.e. (4.9)-(4.15) or (4.26)-(4.31). As it has been pointed out in App. II (property 8) if I_1, I_2 are two sets of constraints then $g\odot(I_1\wedge I_2)=(g\odot I_1)\odot I_2$ does not hold in general. Thus, the significance of (4.33) is evident and according to this, instead of solving the MRE problem in the full state space of the system, ($g\odot(I_1\wedge I_2)$ option), one may equivalently apply firstly the subset constraints $I_s^\dagger$ and the resulting density may be used as a prior on which to apply the aggregate constraints $I_a^\dagger$. In fact there may be defined a decomposition method, [SHOR 82b], of applying the constraints so that the calculation of the full intermediate density $g^\dagger(x)$ is avoided.

To this end the objective is to define the aggregate distribution

$\{g_j^\dagger,\ j=1,\ldots,n\}$ that results after solving all subset MRE problems. Recall from (L1.2) that:

$$[g^\dagger * D_j](x) = g_j^{\dagger -1} g(x) \exp\left[-\lambda - \sum_{i=1}^{K_j} \lambda_{ij}\beta_{ij}(x)\right] \ ,\ x\epsilon D_j \qquad (4.34)$$

and from (L1.12):

$$[[g*D_j]\odot I_{sj}](x) = g_j^{-1} g(x) \exp\left[-\theta_j - \sum_{i=1}^{K_j} \theta_{ij}\beta_{ij}(x)\right] \ ,\ x\epsilon D_j \qquad (4.35)$$

with $\lambda_{ij}=\theta_{ij}\ \forall\ i,j$, from which (L1.13) followed, i.e.

$$g_j^\dagger = g_j \exp\{-\lambda+\theta_j\} \ \ ,\ j=1,\ldots,n \qquad (4.36)$$

In this last relation g_j are known from the given prior $g(x)$, for each subset D_j, θ_j is the Lagrangian multiplier that corresponds to that subset normalization constraint, and since $g_j^\dagger$, $j=1,\ldots,n$, are normalized, λ is the normalization multiplier. Thus the decomposition method of solving the MRE problem, subject to fully decomposable constraints is as follows:

Step 1. In each subset D_j, $j=1,\ldots,n$, apply the subset constraints I_{sj} and derive $[g*D_j]\odot I_{sj}$ as in (4.35).

Step 2. Using the normalization multipliers θ_j, $j=1,\ldots,n$, construct the prior distribution $\{g_j^\dagger,\ j=1,\ldots,n\}$ from (4.36).

Step 3. Apply the aggregate constraints I_a on the prior of step 2 and derive the final posterior probabilities of occupying each subset D_j, $j=1,\ldots,n$.

The aggregate distribution from step 3 and the conditional densities of step 1 may then be combined, using law of total probability, to obtain the full state space posterior density $g\odot(I_s^\dagger \wedge I_a^\dagger)$.

Remark 4.1. The prior distribution $\{g_j^\dagger,\ j=1,\ldots,n\}$, defined by (4.36) may not be normalized since a further normalization will be

imposed, when the aggregate constraints I_a will be applied.

4.3 Discussion.

In this last section the investigation, carried out by Shore, [SHOR 82b], on the MRE principle given fully decomposable constraints, was extensively presented. An alternative application was also examined, i.e. the case where subset and aggregate constraints have a non-zero right hand side.

As it has been mentioned earlier, the posterior conditional densities, that result after applying the subset information, are the same in both cases within a multiplicative constant. This multiplicative constant, however, which is generally different for each subset, affects the normalization multipliers, used in the derivation of the intermediate prior $\{g_j^\dagger,\ j=1,\ldots,n\}$ (step 2). Consequently, the results are different when applying these two different types of constraints. It has been demonstrated, however, that in both cases the decomposition method of solving the MRE problem is the same.

This further investigation was carried out because in the next chapter and in the application in the analysis of QNMs the second type of constraints will be assumed, i.e. with a non-zero right hand side.

Shore has applied this decomposition technique in developing an alternative one-level "decomposition" algorithm for numerically solving MRE problems. He has successfully demonstrated the computational savings due to this technique, [SHOR 82b], as well as indicated the potential application of this material in QNMs, [SHOR 82b, SHOR 83].

In the following chapter, this approach is applied in a multi-level hierarchical fashion, together with assymptotic

connections to infinite capacity queues, in order to provide a new MRE approximation for the equilibrium queue length distributions of a QNM, based on the decomposition techniques described in the second chapter.

CHAPTER V

A FIRST APPLICATION OF MRE FORMALISM INTO THE HIERARCHICAL DECOMPOSITION OF QNMs

The MRE analysis of the previous chapter (section 4.3) provides a decomposition method of applying fully decomposable constraints corresponding to a partition of the state space rather than the overall state space of a system. Hence, the connection of this approach to the analysis of QNMs through a hierarchical decomposition scheme is clear. Recall that such a decomposition scheme is actually based on a multi-level partition of the state space. In fact, the variable aggregation method can only be applied through such a multi-level partition, while Norton's reduction scheme can be applied based on both single or multi-level partitions (see chapter II, pp. 22-32).

In this first part of the application of MRE formalism in the decomposition of QNMs the study focuses on a specific network configuration, namely the central server model (fig. 5.1), since 1) it is of great interest in the performance analysis and evaluation of computer systems, (e.g. [BUZE 73, SAUE 75a, FERR 78]), and 2) it has an appropriate topology amenable to both variable aggregation and Norton's reduction decomposition schemes. The variable aggregation method is considered first.

5.1 Variable Aggregation decomposition scheme.

For exposition purposes a central server model with two I/O units is examined first (M=2 in fig. 5.1).

5.1.1 A MRE solution for M=2.

<u>The state space.</u>

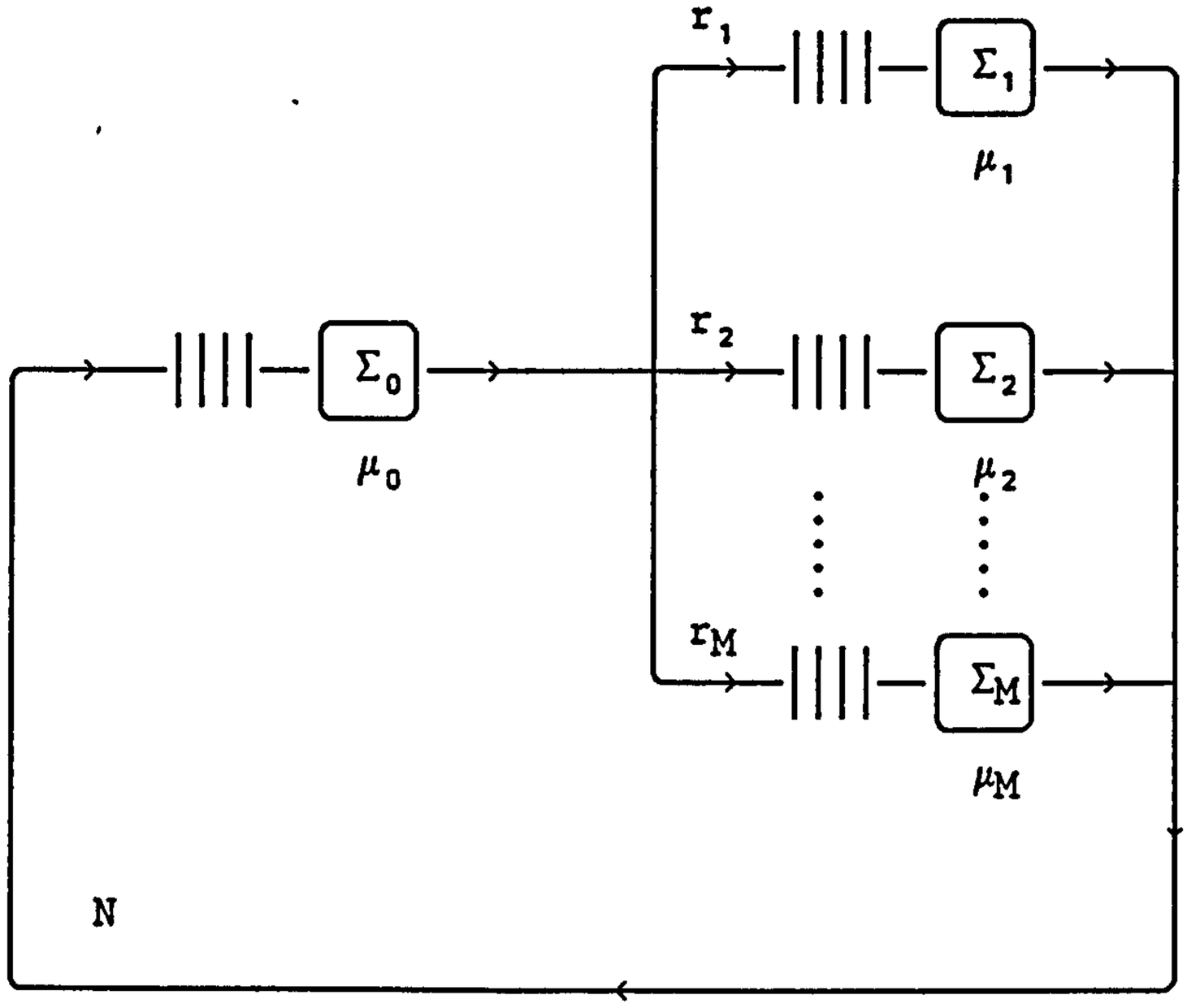

Figure 5.1. Closed central server model.

The state space of this system is $D=\{(n_0,n_1,n_2) \;/\; 0\leq n_0,n_1,n_2\leq N \wedge n_0+n_1+n_2=N\}$, the set of joint queue length states of the three queues involved. The partition of the state space D, also implied by the variable aggregation scheme (see fig. 2.1), is $\{D_0,D_1,\ldots,D_N\}$ with $D_{n_2}=\{(n_0,n_1,n_2) \;/\; 0\leq n_0,n_1\leq N-n_2 \wedge n_0+n_1=N-n_2\}$. This partition is represented in figure 5.2.

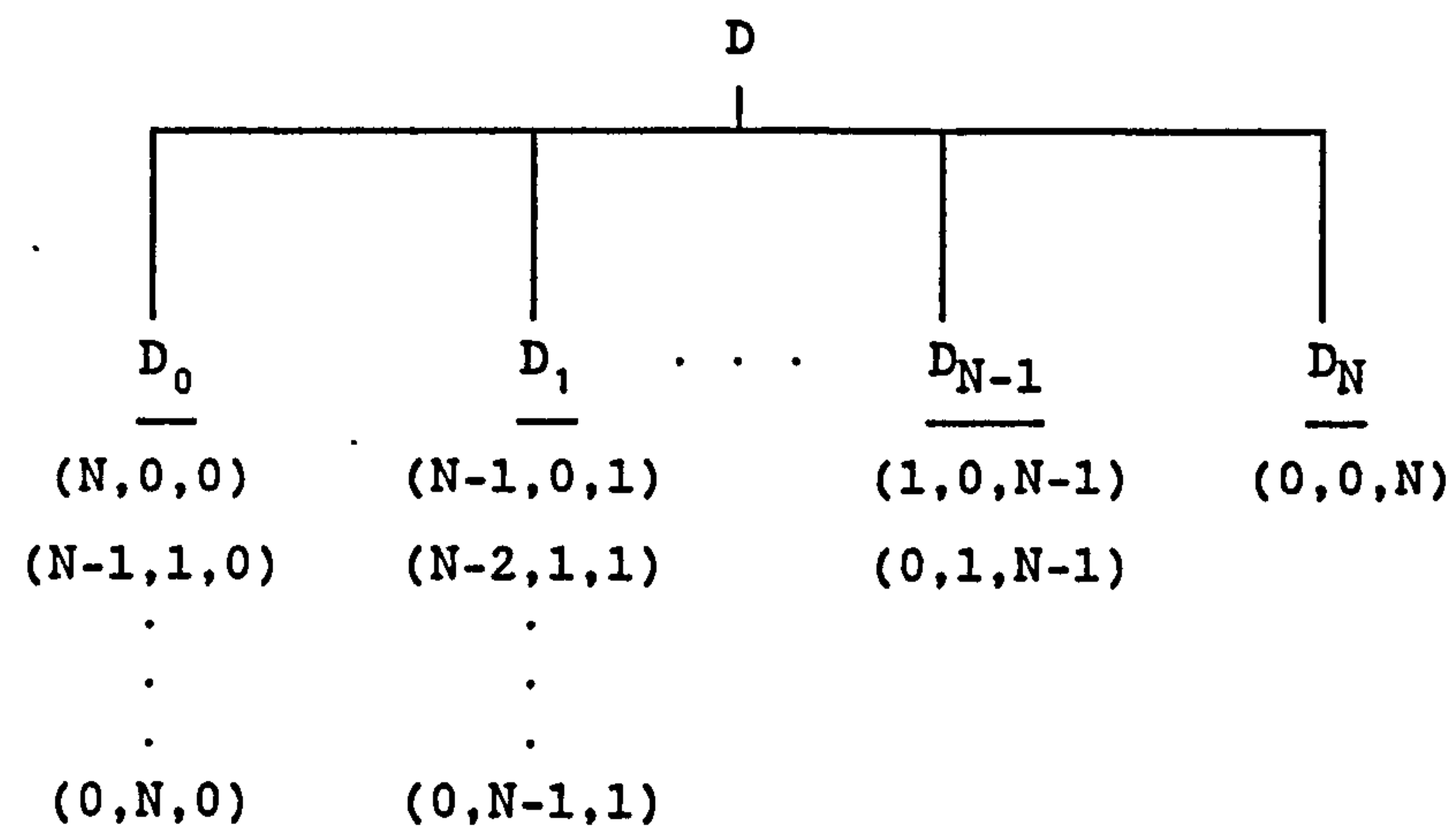

Figure 5.2. Partition of the state space D, for M=2, according to the variable aggregation scheme.

Thus, in this case of M=2 the partition is a single-level one. Each subset D_i contains all possible states of the system under the condition of i customers being present at the second I/O (Σ_2) or N-i customers circulating in the subsystem of the first level, which consists of the CPU and the first I/O, (Σ_0,Σ_1).

Here the joint queue length probability distribution $\{P(n_0,n_1,n_2),\ (n_0,n_1,n_2)\epsilon D\}$ corresponds (notationally) to the overall true or MRE density of the system (f^* or $f^\dagger$) defined in the previous chapter (pages 71 and 77 respectively). To the conditional densities ($f^*{*}D_j$ or $f^\dagger{*}D_j$) the following conditional distributions correspond:

$$\{P_1(n_0/n_2),\ n_0=0,\ldots,N-n_2\}\quad ,\ n_2=0,\ldots,N \tag{5.1}$$

where

$$P_1(n_0/n_2) = \frac{P(n_0,N-n_0-n_2,n_2)}{Pr\{D_{n_2}\}} \tag{5.2}$$

and $Pr\{D_{n_2}\}$ is the probability of occupying the subset D_{n_2}, or in other words the probability that the state of the system is one of the elements of this subset. Note, however, that in this case this aggregate distribution is nothing else but the marginal queue length distribution of the second I/O. Hence:

$$Pr\{D_{n_2}\} = P_2(n_2)\ ,\ n_2=0,1,\ldots,N \tag{5.3}$$

So, having specified distributions (5.1) and (5.3), one may derive the joint probability distribution through relation (5.2). In fact, since the marginal results are more important, one may derive the marginal queue length distributions for queues Σ_0 and Σ_1 using law of total probability, as:

$$P_0(n_0) = \sum_{n_2=0}^{N-n_0} P_1(n_0/n_2)P_2(n_2)\ ,\ n_0=0,\ldots,N \tag{5.4}$$

and

$$P_1(n_1) = \sum_{n_2=0}^{N-n_1} P_1(N-n_2-n_1/n_2)P_2(n_2) \quad , \quad n_1=0,\ldots,N \qquad (5.5)$$

Constraint Information.

Subset Constraints.

In the first level of aggregation and according to the variable aggregation scheme, the conditional distributions (5.1) depend only upon interactions between units Σ_0 and Σ_1, (fig.5.3).

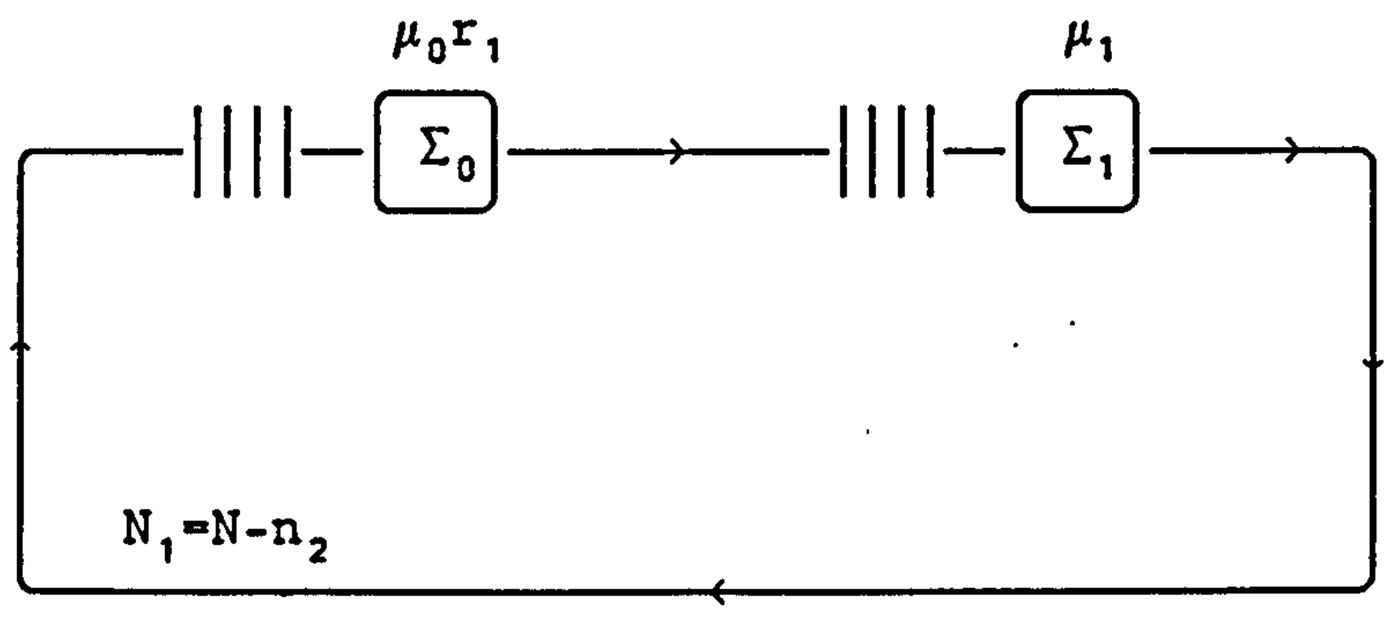

Figure 5.3. Queueing system involved in the 1st level of aggregation.

This subsystem is the central server case of the more general one represented in fig. 2.3. Thus, for each subset D_{n_2}, $n_2=0,\ldots,N-1$, the assumed constraints that concern the conditional distribution (5.1) are:

– The normalization,

$$\sum_{n_0=0}^{N-n_2} P_1(n_0/n_2) = 1 \qquad (5.6a)$$

– The conditional mql, $\langle n_0 \rangle_{N-n_2} (< N-n_2)$

$$\sum_{n_0=0}^{N-n_2} n_0 P_1(n_0/n_2) = \langle n_0 \rangle_{N-n_2} \qquad (5.6b)$$

– The conditional flow-balance (fb) equation,

$$[1-P_1(0/n_2)]\mu_0 r_1 = [1-P_1(N-n_2/n_2)]\mu_1 \qquad (5.6c)$$

expressed by the full buffer conditional state probability:

$$P_1(N-n_2/n_2) = \Phi_{1,N-n_2} \quad , \; 0 < \Phi_{1,N-n_2} < 1 \qquad (5.6d)$$

or

$$\sum_{n_0=0}^{N-n_2} f_1(n_0)P_1(n_0/n_2) = \Phi_{1,N-n_2} \qquad (5.6e)$$

where

$$f_1(n_0) = \begin{cases} 1 & \text{if } n_0=N-n_2 \\ 0 & \text{otherwise} \end{cases} \qquad (5.6f)$$

The case $n_2=N$ (or subset D_N), has not been considered, since trivially $P_1(0/N) = 1$.

Aggregate Constraints.

These constraints concern the marginal distribution of the second level of aggregation $\{P_2(n_2), \; n_2=0,\ldots,N\}$. This distribution, according to the variable aggregation scheme (see fig. 2.2), depends upon interactions between unit Σ_2 and the subsystem of the first level. So the aggregate constraints are:

- The normalization,

$$\sum_{n_2=0}^{N} P_2(n_2) = 1 \qquad (5.7a)$$

- The marginal mql, $\langle n_2 \rangle_N$ $(< N)$

$$\sum_{n_2=0}^{N} n_2 P_2(n_2) = \langle n_2 \rangle_N \qquad (5.7b)$$

- The marginal flow-balance (fb) equation,

$$[1-P_2(0)]\mu_2 = [1-P_0(0)]\mu_0 r_2 \qquad (5.7c)$$

expressed by the full buffer state probability:

$$P_2(N) = \Phi_{2,N},\ 0 < \Phi_{2,N} < 1 \tag{5.7d}$$

or

$$\sum_{n_2=0}^{N} f_2(n_2)P_2(n_2) = \Phi_{2,N} \tag{5.7e}$$

where

$$f_2(n_2) = \begin{cases} 1 & \text{if } n_2=N \\ 0 & \text{otherwise} \end{cases} \tag{5.7f}$$

Note that no assumption was made about the distributional form of the service times of the units involved (simple or composite) and in fact no assumption was made about the type of the composite server in the second level of aggregation. The constraints assumed, however, are similar to the ones used in the ME methods as those have been applied in QNMs analysis. More specifically, aggregate constraints (5.7) are part of the constraints used in the derivation of the GE/GE/1/N solution, in chapter III (section 3.2.7, pp. 53-60), the difference being the utilization constraint used there and the more complex fb constraint used here. Similar comments apply to each set of subset constraints.

In this application no prior knowledge is assumed, in the form of a prior distribution, which is completely equivalent to the assumption of a uniform prior distribution, since the state space is finite. This can be justified by the following Lemma.

Lemma 5.1. Let $D = \{d_1,d_2,\ldots,d_N\}$ be the state space of any system. Let I be new constraint information about the system and assume that the prior distribution, denoted $\{q_1,q_2,\ldots,q_N\}$ is the uniform:

$$q_j = \frac{1}{N}\ ,\ j=1,2,\ldots,N \tag{L1.1}$$

Then minimizing,

$$\sum_{j=1}^{N} p_j \log\left\{ \frac{p_j}{q_j} \right\} \tag{L1.2}$$

functional, subject to I is equivalent to minimizing:

$$\sum_{j=1}^{N} p_j \log\{p_j\} \qquad \text{(L1.3)}$$

Proof.

The functional (L1.2), using (L1.1), may be written as:

$$\sum_{j=1}^{N} p_j \log\{N p_j\} = \log\{N\} + \sum_{j=1}^{N} p_j \log\{p_j\}$$

Now in the above, log{N} is constant, thus minimizing (L1.2), subject to I, is equivalent to minimizing (L1.3).

Q.E.D.

In this small application, however, the uniform prior will be used, in order to further demonstrate its redundancy.

MRE solution.

First level of aggregation.

The set D of possible states here has (N+1)(N+2)/2 elements and each subset D_j, j=0,...,N has (N-j+1) elements. Thus, the prior probability that corresponds to $P_1(n_0/n_2)$ is $1/(N-n_2+1)$, while the one that corresponds to $P_2(n_2)$ is $[2(N-n_2+1)]/[(N+1)(N+2)]$. So, applying the subset constraints (5.6) on the above prior in each subset D_{n_2}, n_2=0,...,N-1, the resulting posterior due to (4.4) is:

$$P_1(n_0/n_2) = \frac{1}{N-n_2+1} \exp\left[-\theta_{0,n_2} - \theta_{1,n_2} n_0 - \theta_{2,n_2} f_1(n_0)\right] \quad , n_0=0,\ldots,N-n_2$$

where θ_{i,n_2}, i=0,1,2, are the Lagrangian multipliers that correspond to constraints (5.6a), (5.6b) and (5.6e) respectively. Substituting $G^*(1,n_2)$ for $\exp[-\theta_{0,n_2}]$, x_0 for $\exp[-\theta_{1,n_2}]$ and y_0 for $\exp[-\theta_{2,n_2}]$ the above becomes:

$$P_1(n_0/n_2) = \frac{1}{G^*(1,N-n_2)} \frac{1}{N-n_2+1} x_0^{n_0} y_0^{f_1(n_0)} \quad , n_0=0,\ldots,N-n_2$$

where $$G^*(1,N-n_2) = \sum_{n_0=0}^{N-n_2} \frac{1}{N-n_2+1} x_0^{n_0} y_0^{f_1(n_0)} \rightarrow$$

$$G^*(1,N-n_2) = \frac{1}{N-n_2+1} \left[\frac{1-x_0^{N-n_2}}{1-x_0} + x_0^{N-n_2} y_0 \right]$$

Thus, it is clear why here the uniform prior is redundant, since it is not a function of n_0. Hence, redifining $G^*(1,N-n_2)$ as:

$$G^*(1,N-n_2) = \frac{1-x_0^{N-n_2}}{1-x_0} + x_0^{N-n_2} y_0 \qquad (5.8)$$

the solution becomes:

$$P_1(n_0/n_2) = \frac{1}{G^*(1,N-n_2)} x_0^{n_0} y_0^{f_1(n_0)}, \quad n_0=0,\ldots,N-n_2 \qquad (5.9)$$

In order to evaluate the multipliers x_0, y_0, an assymptotic connection to an infinite capacity queue has to be established at this point, similarly to the one that took place in the ME analysis of the GE/GE/1/N queue (pp. 55-57). In fact, as it has been implied by the notation, the assumption here is that the mql multiplier x_0 is invariant to N (invariability assumption), while it is sufficient to require that the limit of multiplier y_0, as $N \longrightarrow +\infty$, is not infinity. Let also $x_0 < 1$. Then from (5.8):

$$\lim_{N\to+\infty} G^*(1,N-n_2) = \frac{1}{1-x_0}$$

and

$$\lim_{N\to+\infty} P_1(n_0/n_2) = (1-x_0)x_0^{n_0}, \quad n_0=0,1,\ldots$$

and

$$\lim_{N\to+\infty} P_1(N-n_2/n_2) = 0$$

This last equation expresses the fact that due to the assumption $x_0<1$ the bottleneck device is unit Σ_1 at this level and at the limit ($N\longrightarrow+\infty$) the probability of it being idle is zero. So the fb equation

(5.6c), using the above, becomes:

$$[1-(1-x_0)]\mu_0 r_1 = \mu_1$$

and defining $\rho_0 = \mu_1/\mu_0 r_1$,

$$x_0 = \rho_0 = \frac{\mu_1}{\mu_0 r_1} \qquad (5.10)$$

which is exactly the utilization of server Σ_0 in the corresponding open system, where unit Σ_1 is assumed saturated.

Remark 5.1.

Instead of using the flow information (restricted here to the first moment), to derive the value of x_0, the mql constraint at the limit could be utilized as:

$$\lim_{N\to+\infty} \langle n_0 \rangle_{N-n_2} = \langle n_0 \rangle = \sum_{n_0=1}^{\infty} (1-x_0) n_0 x_0^{n_0} \quad \Rightarrow$$

$$x_0 = \frac{\langle n_0 \rangle}{\langle n_0 \rangle + 1} \qquad (5.11)$$

Then (5.10) follows from (5.11) if $\langle n_0 \rangle$ is substituted by the M/M/1 mql formulae:

$$\langle n_0 \rangle = \frac{\rho_0}{1-\rho_0}$$

So, utilizing the flow information only, in order to derive the multiplier x_0 in the MRE solution at this level, is equivalent to assuming exponentially distributed service and assymptotic interarrival times. However, relation (5.11) is clearly more general than (5.10), in that the MRE solution could be used as an approximation for a more general distribution.

Returning now to the closed system, the fb constraint (5.6c), using (5.9) and definition (5.10), yields:

$$G^*(1,N-n_2) - 1 = \rho_0\left[G^*(1,N-n_2) - x_0^{N-n_2}\, y_0\right]$$

Solving this with respect to y_0 and using (5.8), after some manipulation yields:

$$y_0 = 1$$

Thus, finally,

$$P_1(n_0/n_2) = \frac{1}{G^*(1,N-n_2)}\, x_0^{n_0}\ ,\ n_0=0,\ldots,N-n_2 \qquad (5.12)$$

and

$$G^*(1,N-n_2) = \frac{1-x_0^{N-n_2+1}}{1-x_0} \qquad (5.13)$$

Note that the assumption $x_0<1$, which proved to be equivalent to the stability condition $\rho_0<1$, is responsible for the convergence of $G^*(1,N-n_2)$, given by (5.8), as $N \longrightarrow +\infty$. Recall from (4.36) that the normalizing constants are used to define the prior distribution to be used in the next level of aggregation. Because an assymptotic connection will also take place at that level, the convergence of the prior must be ensured in this level. So if $x_0,\rho_0>1$, one overcomes the problem by assuming the same subset constraints for the "dual" distribution $\{P_1^*(n_1/n_2),\ n_1=0,\ldots,N-n_2\}$, which concerns the number of jobs present at unit Σ_1, (see also GE/GE/1/N analysis, pages 53-60 of 3rd chapter). Following an identical approach it can be shown that:

$$P_1^*(n_1/n_2) = \frac{1}{\tilde{G}^*(1,N-n_2)}\, x_1^{n_1}\ ,\ n_1=0,\ldots,N-n_2$$

and

$$x_0x_1 = \rho_0\rho_1 = 1$$

and

$$\tilde{G}^*(1,N-n_2) = \frac{1-x_1^{N-n_2+1}}{1-x_1}$$

where x_1, ρ_1 and $\tilde{G}^*(1,N-n_2)$ are the corresponding dual mql multiplier

assymptotic utilization and normalizing constant, respectively. Using next the duality relations:

$$P_1(n_0/n_2) = P_1^*(N-n_2-n_0/n_2) \ , \ n_0=0,\ldots,N-n_2$$

it is not difficult to derive $P_1(n_0/n_2)$ using $P_1^*(n_1/n_2)$ and prove that it is once more given by (5.12)-(5.13), which means that solution (5.12) is valid even if $x_0>1$. Furthermore, it is also easy to prove that the dual normalizing constant is related to the original normalizing constant $G^*(1,n)$ as:

$$\tilde{G}^*(1,n) = \frac{1}{x_0^n} G^*(1,n) \quad , \ n=0,1,\ldots$$

So, because one of the two normalizing constants is guaranteed to converge as $N \longrightarrow +\infty$, depending on whether $x_0<1$ or $x_0>1$, the quantity to be used in the definition of the prior distribution is denoted by $G(1,n)$ and is given by:

$$G(1,n) = \min\left[1, \frac{1}{x_0^n} \right] G^*(1,n) \ , \ n=0,1,\ldots \qquad (5.14)$$

Using now the definition (4.36) and taking into account the initial uniform prior, the prior distribution to be used in the next level of aggregation $\{q_2(n_2), \ n_2=0,\ldots,N\}$ is:

$$q_2(n_2) = \frac{1}{Z} \frac{2(N-n_2+1)}{(N+1)(N+2)} \frac{1}{N-n_2+1} G(1,N-n_2) \ , \ n_2=0,\ldots,N$$

where the notational correspondence between the above and definition (4.36) is as follows, Z corresponds to the normalizing factor $\exp\{\lambda\}$, the first ratio corresponds to g_j and the remaining factor corresponds to the normalizing constant $\exp\{\theta_j\}$ of subset D_j. Note that in order to take into account the effect of the uniform initial prior, the relation prior to definition (5.8) was considered as the value of $G^*(1,n)$. The obvious simplification in the above definition

of the prior yields that the only factor that is a function of n_2 is $G(1,N-n_2)$, and since another normalization will take place in the next level of aggregation (see also Remark 4.1), the prior $\{q_2(n_2), n_2=0,...,N\}$ may be redifined as:

$$q_2(n_2) = G(1,N-n_2) \ , \ n_2=0,...,N \tag{5.15}$$

and it will be refered to as "prior", instead of "prior distribution". Note that what has been presented up to now is true for $n_2=0,1,...,N-1$. In the trivial case however that $n_2=N$ (no jobs in the subsystem), no special consideration is given since all formulae reduce to the appropriate results.

Second level of aggregation.

Applying now the aggregate constraints (5.7) on the prior defined by (5.15) the MRE solution is:

$$P_2(n_2) = \frac{1}{G^*(2,N)} G(1,N-n_2)\ x_2^{n_2}\ y_2^{f_2(n_2)}, \ n_2=0,...,N \tag{5.16}$$

where $G^*(2,N)$ is the normalizing constant that corresponds to the normalization constraint, x_2 and y_2 are the mql and fb multipliers that correspond to constraints (5.7b) and (5.7e) respectively and $G(1,N-n_2)$ is the prior defined by (5.15). Applying the normalization constraint to the above yields:

$$G^*(2,N) = \sum_{n_2=0}^{N-1} G(1,N-n_2)x_2^{n_2} + x_2^{N}\ y_2 \tag{5.17}$$

Noting now from (5.14) that:

$$\lim_{N\to+\infty} G(1,N-n_2) = \left[1 - \min\left[\ x_0,\ \tilde{x}_0\ \right]\right]^{-1}$$

where

$$\tilde{x}_0 = x_1 = \frac{1}{x_0}$$

and assuming that $x_2<1$ and that it is invariant to N, while,

$$\lim_{N\to+\infty} y_2 < +\infty$$

and applying the limit ($N \longrightarrow +\infty$),

$$\lim_{N\to+\infty} G^*(2,N) = \frac{1}{1-\min(x_0,\tilde{x_0})} \frac{1}{1-x_2}$$

and

$$\lim_{N\to+\infty} P_2(n_2) = (1-x_2)x_2^{n_2}, \quad n_2=0,1,... \tag{5.18}$$

Note that the convergence of the summation in (5.17) and consequently the convergence of $G^*(2,N)$ to the above described value (as $N \longrightarrow +\infty$) is justified by the Silverman-Toepliz generalization of Cauchy's limit theorem [KNOP 56, pp. 35-36]. Moreover, by the law of total probability (LTP),

$$P_0(0) = \sum_{n_2=0}^{N} P_1(0/n_2)P_2(n_2) \tag{5.19}$$

and therefore,

$$\lim_{N\to+\infty} P_0(0) = 1 - \min(1,x_0) \tag{5.20}$$

Thus, as $N \longrightarrow +\infty$, the fb condition (5.7c) becomes:

$$\min(1,x_0)\mu_0 r_2 = x_2\mu_2$$

which yields:

$$x_2 = \min(1,x_0) \frac{\mu_0 r_2}{\mu_2} \tag{5.21}$$

Let's at this point give some interpretation of what resulted by applying the limit $N \longrightarrow +\infty$. Firstly by assuming $x_2<1$, which is the condition for convergence of (5.17) and consequently the condition of existence of the limit of (5.16) marginal queue length distribution, the chance of unit Σ_2 being the overall bottleneck of the network was eliminated. Next, the geometric form of (5.18) provides a hint of treating assymptotically the queueing system Σ_2 as an M/M/1 one. The input to queue Σ_2 is produced by the CPU (Σ_0) as this is utilized in

the corresponding open system of the previous level, i.e. if Σ_0 was the relative bottleneck of the first level then:

$$\lim_{N\to+\infty} P_0(0) = \lim_{N\to+\infty} P_1(0/n_2) = 0$$

while if unit Σ_1 was the relative bottleneck of the first level then:

$$\lim_{N\to+\infty} P_0(0) = \lim_{N\to+\infty} P_1(0/n_2) = 1 - x_0$$

the combination of the above two relations is described by (5.20). Finally, the resulting value for multiplier x_2, given by (5.21) may be interpreted as the assymptotic utilization of unit Σ_2 in the corresponding open system of this level, i.e. in case that $x_0<1$ (case Σ_1 is the bottleneck of the previous level), (5.21) and (5.10) yield:

$$x_2 = \frac{\mu_1 r_2}{\mu_2 r_1}$$

which is the utilization of Σ_2 in this case's corresponding open system, (fig. 5.4), whereas in case that $x_0>1$, (Σ_0 is the bottleneck

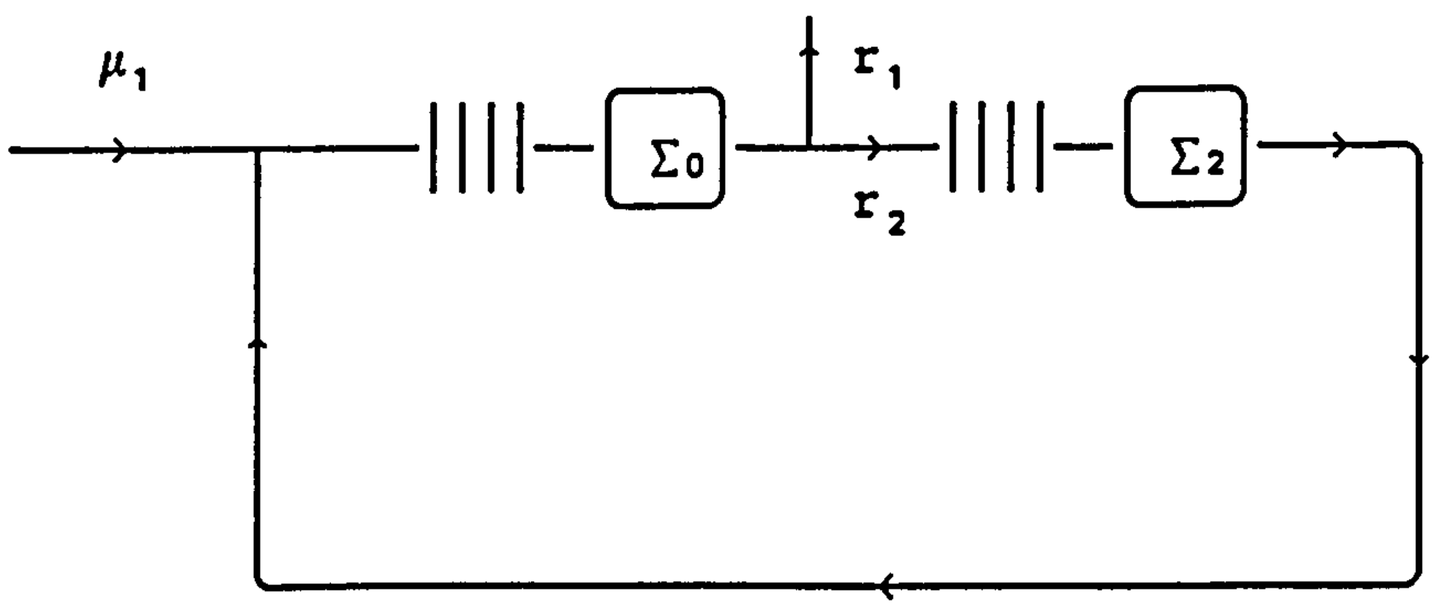

Figure 5.4. Corresponding open system in case Σ_1 is the overall botleneck.

in the previous level), x_2 has the value:

$$x_2 = \frac{\mu_0 r_2}{\mu_2}$$

which is the utilization of Σ_2 in this case's corresponding open system, (fig. 5.5).

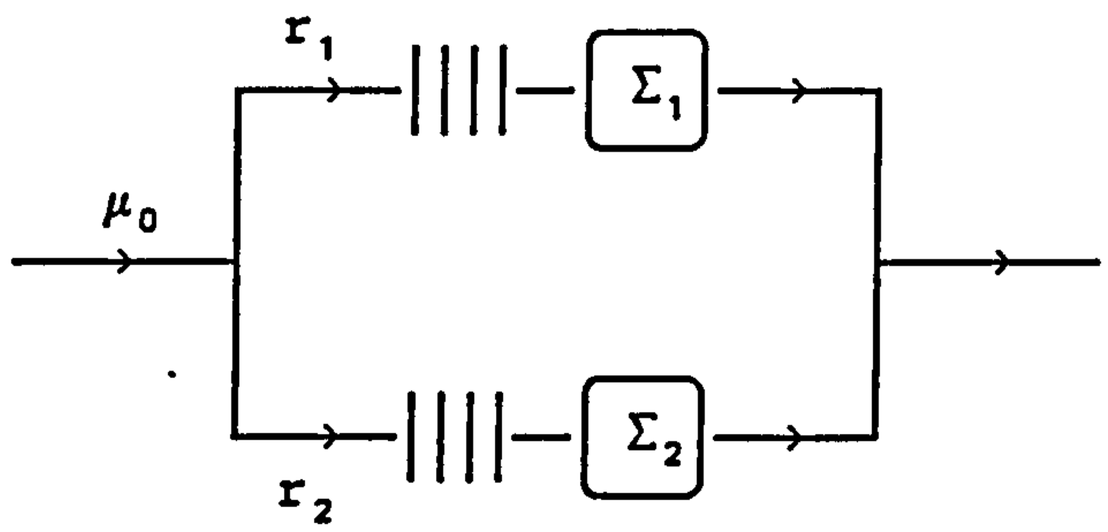

Figure 5.5. Corresponding open system in case Σ_0 is the overall bottleneck.

Returning now to the closed system and using (5.14), (5.16) and (5.19), it can be seen that $P_0(0)$ is given by:

$$P_0(0) = \frac{1}{G^*(2,N)} \sum_{n_2=0}^{N} \min\left[1, \frac{1}{x_0^{N-n_2}} \right] x_2^{n_2} y_2^{f_2(n_2)}$$

and using (5.21), the above and (5.16), the fb constraint (5.7c) may be written as:

$$\left[G^*(2,N) - \sum_{n_2=0}^{N} \min\left[1, \frac{1}{x_0^{N-n_2}} \right] x_2^{n_2} y_2^{f_2(n_2)} \right] x_2 =$$

$$[G^*(2,N) - G(1,N)]\min(1,x_0) \quad \Rightarrow$$

$$x_2\left[\sum_{n_2=0}^{N-1} \sum_{n_0=0}^{N-n_2} x_0^{n_0} \min\left[1, \frac{1}{x_0^{N-n_2}} \right] x_2^{n_2} - \sum_{n_2=0}^{N-1} \min\left[1, \frac{1}{x_0^{N-n_2}} \right] x_2^{n_2} \right] =$$

$$= \left[\sum_{n_2=1}^{N-1} x_2^{n_2} \min\left[1, \frac{1}{x_0^{N-n_2}} \right] \sum_{n_0=0}^{N-n_2} x_0^{n_0} + x_2^{N} y_2 \right] x_0 \min\left[1, \frac{1}{x_0} \right] \Rightarrow$$

$$\sum_{n_2=0}^{N-1} x_2^{n_2+1} \min\left[1, \frac{1}{x_0^{N-n_2}} \right] \sum_{n_0=1}^{N-n_2} x_0^{n_0} = \min(1,x_0) x_2^{N} y_2 +$$

$$+ \sum_{n_2=1}^{N-1} x_2^{n_2} \min\left[1, \frac{1}{x_0^{N-n_2+1}}\right] \sum_{n_0=1}^{N-n_2+1} x_0^{n_0} \quad \Rightarrow$$

$$\sum_{n_2=1}^{N-1} x_2^{n_2} \min\left[1, \frac{1}{x_0^{N-n_2+1}}\right] \sum_{n_0=1}^{N-n_2+1} x_0^{n_0} + x_2^{N} \min\left[1, \frac{1}{x_0}\right] x_0 =$$

$$= \min(1,x_0) x_2^{N} y_2 + \sum_{n_2=1}^{N-1} x_2^{n_2} \min\left[1, \frac{1}{x_0^{N-n_2+1}}\right] \sum_{n_0=1}^{N-n_2+1} x_0^{n_0} \quad \Rightarrow$$

$$y_2 \min(1,x_0) x_2^{N} = x_2^{N} \min(1,x_0) \quad \Rightarrow$$

$$y_2 = 1$$

Thus, once more the flow balance multiplier is one and (5.16), (5.17) become :

$$P_2(n_2) = \frac{1}{G^*(2,N)} G(1,N-n_2)\, x_2^{n_2} \quad , \; n_2=0,\ldots,N \qquad (5.22)$$

$$G^*(2,N) = \sum_{n_2=0}^{N} G(1,N-n_2) x_2^{n_2} \qquad (5.23)$$

When $x_2>1$, constraints (5.7) are assumed for the "dual" distribution $\{P_2^*(N_2),\ N_2=0,\ldots,N\}$, which concerns the number of customers present in the subsystem $(\Sigma_0\Sigma_1)$ rather than unit Σ_2. In fact this type of distribution was used in the second chapter and in the description of the variable aggregation scheme. Obviously,

$$P_2(n_2) = P_2^*(N-n_2) \quad , \; n_2=0,\ldots,N$$

One may proceed in the same fashion to prove the following Lemma.

Lemma 5.2. The MRE solution for the dual distribution $\{P_2^*(N_2),\ N_2=0,\ldots,N\}$ is given by:

$$P_2^*(N_2) = \frac{1}{\tilde{G}^*(2,N)} G(1,N_2) \tilde{x}_2^{N_2} \quad , N_2=0,\ldots,N \tag{5.24}$$

where the normalizing constant obeys:

$$\tilde{G}^*(2,n) = \frac{1}{x_2^n} G^*(2,n) \quad , n=0,1,\ldots \tag{5.25}$$

and the mql multiplier is:

$$\tilde{x}_2 = \frac{1}{x_2} \tag{5.26}$$

where x_2, $G^*(2,n)$ and $G(1,N_2)$ are given by (5.21), (5.23) and (5.14), respectively.

Proof. The proof can be seen in Appendix III.

Using now the duality relations, together with (5.24), in order to derive $\{P_2(n_2), n_2=0,\ldots,N\}$, it can be easily shown that (5.22) is valid even if $x_2>1$. So what has been proven so far can be summarized in the following Lemma.

Lemma 5.3. Consider a central server model with a single CPU (Σ_0) and two I/O units (Σ_1, Σ_2) under a variable aggregation decomposition scheme. Given fully decomposable mql and fb subset and aggregate constraints (5.6), (5.7), respectively and assuming that the corresponding mql multipliers x_0 and x_1 are invariant to the fixed level of multiprogramming N, then:

i. At the first level of aggregation, the conditional MRE solution is:

$$P_1(n_0/n_2) = \frac{1}{G^*(1,N-n_2)} x_0^{n_0} \quad , n_0=0,\ldots,N-n_2 \tag{L3.1}$$

where x_0 and $G^*(1,N-n_2)$ are given by (5.10) and (5.13) respectively. Moreover, the prior $G(1,N-n_2)$ introduced for the next level of decomposition is given by (5.14) and

ii. At the second (and final) level of aggregation, the marginal MRE solution is:

$$P_2(n_2) = \frac{1}{G^*(2,N)} G(1,N-n_2)\, x_2^{n_2} \quad , n_2=0,\ldots,N \qquad (L3.2)$$

where x_2 and $G^*(2,N)$ are given by (5.21) and (5.23) respectively.

Note now that combining (L3.1) and (L3.2) and using (5.14), the joint probability queue length distribution is defined for each possible state $(n_0,N-n_0-n_2,n_2)$ as:

$$P(n_0,N-n_0-n_2,n_2) = \frac{1}{G^*(2,N)} x_0^{n_0} \min\left[1, \frac{1}{x_0^{N-n_2}}\right] x_2^{n_2} \qquad (5.27a)$$

or using Lemma 5.2 and (5.24), it may equivalently be written as:

$$P(n_0,N-n_0-n_2,n_2) = \frac{1}{\tilde{G}^*(2,N)} x_0^{n_0} \min\left[1, \frac{1}{x_0^{N-n_2}}\right] \tilde{x}_2^{n_2} \qquad (5.27b)$$

in case that $x_2>1$. Relations (5.27) represent two alternative ways of expressing the same solution. It is clear in both cases that $G^*(2,N)$ and $\tilde{G}^*(2,N)$ normalizing constants are also the normalizing constants of the MRE solution for the full state space (D) probability distribution.

Let's see more closely now this generalization of the method described so far to an arbitrary size central server model.

5.1.2 A MRE solution for M>2.

Consider the case where there is a third I/O unit in the network. Then whatever took place up to this point can be viewed as conditioned on that $N-n_3$ customers were present in the two I/O subsystem (instead of N), where n_3 is the number of customers present at this third I/O. Naturally, all constraints assumed up to now as well as the notation used have to be conditioned on this additional information about the population. What was up to this point the

second and final level of aggregation now is the intermediate one, while the final level is the one where the third I/O is taken under consideration, and for which level, appropriate constraint information similar to (5.7) have to be assumed. Then the normalizing constants in (5.27a) and (5.27b) define the prior to be used in the third and final level, and since the convergence (as $N \longrightarrow +\infty$) has to be once more ensured, according to (5.25) this prior is:

$$G(2,n) = \min\left[1, \frac{1}{x_2^n} \right] G^*(2,n) \quad , \quad n=0,1,\ldots$$

So, more precisely, the state space D (for M=3) is decomposed into N+1 disjoint subsets D_{n_3}, $n_3=0,1,\ldots,N$, described by $D_{n_3} = \{(n_0,..,n_3)/0 \leq n_i \leq N-n_3,\ i=0,1,2 \wedge \sum_{i=0} n_i = N-n_3\}$. Each subset D_{n_3} may be decomposed into $N-n_3+1$ disjoint subsets D_{n_2,n_3}, $n_3=0,..,N$, $n_2=0,..,N-n_3$, described by $D_{n_2,n_3} = \{(n_0,..,n_3)/0 \leq n_0,n_1 \leq N-n_2-n_3 \wedge n_0+n_1=N-n_2-n_3\}$ (see fig. 5.6).

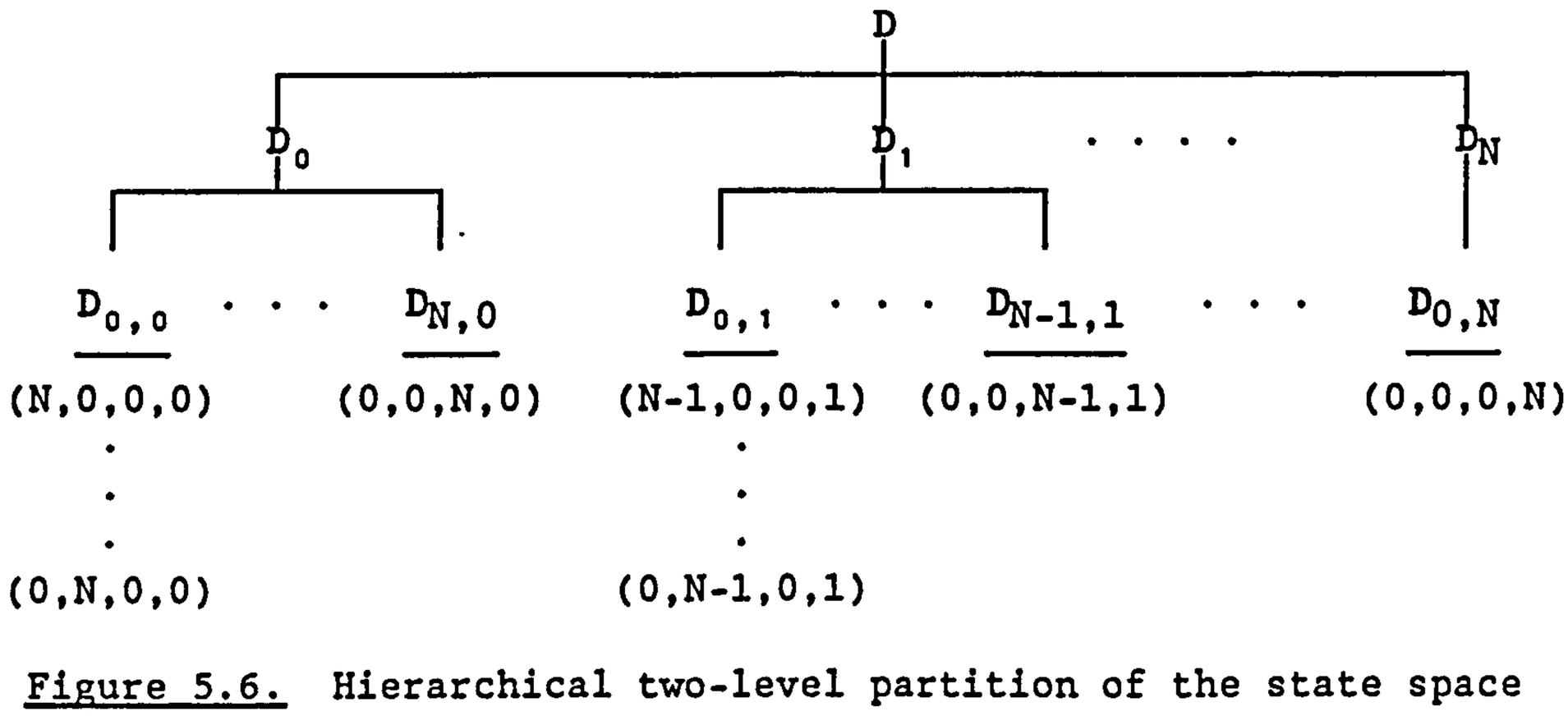

Figure 5.6. Hierarchical two-level partition of the state space D, for M=3.

This partition forms hierarchically three levels of decomposition. The first level comprises from the conditional states in each subset D_{n_2,n_3} with conditional probability $P_1(n_0/n_2,n_3)$ of having n_0 jobs in

the CPU (Σ_0) given that n_2, n_3 jobs are in I/O units Σ_2 and Σ_3, respectively. The second level consists of the conditional states in each subset D_{n_3}, with conditional probability $P_2(n_2/n_3)$ of having n_2 jobs in the I/O unit Σ_2 given that n_3 jobs are in I/O unit Σ_3. Finally the third level refers to the marginal states D_{n_3} with aggregate probability $Pr\{D_{n_3}\}=Pr\{n_3 \text{ jobs in } \Sigma_3\}=P_3(n_3)$. Note that the notation used for the conditional probabilities, whose conditional part describes the precise number of customers present in each unit that do not belong to the subsystem of the corresponding level, merely serves a better representation of the partition involved. Practically however, the conditional part only affects the evaluation of the probabilities on the level of multiprogramming, e.g. distributions $\{P_1(n_0/n_2,n_3\}$, $n_0=0,\ldots,N-n_2-n_3\}$ that have the same sum of conditional parts (i.e. n_2+n_3) all belong to the same class of equivalence and are identical since n_2, n_3 merely define that in the subsystem of the first level circulate $N-n_2-n_3$ customers.

Suppose at each level of decomposition the normalization, mql and fb constraints are known to exist. Note that constraints relating to the conditional probability $P_2(n_2/n_3)$ are clearly considered to be subset constraints with respect to the outer partition $\{D_0,D_1,\ldots,D_N\}$, but they play the role of aggregate constraints at the first decomposition level with respect to the inner partition $\{D_{0,n_3},\ldots,D_{N-n_3,n_3}\}$, $n_3=0,1,\ldots,N$. It is therefore implied that the analytic procedure for a hierarchical decomposition of the state space of a central server model with (M>2) I/O units involves M levels of decomposition and can be described as follows:

Step 1. {1st level of aggregation}

Step 1.1 Apply the subset mql $\langle n_0\rangle_{N-n_2-\ldots-n_M}$ and fb (expressed by $P_1(N-\sum_{k=2}^{M} n_k/n_2,\ldots,n_M)$) constraints to obtain the MRE solution $P_1(n_0/n_2,\ldots,n_M)$, $n_0=0,1,\ldots,N-\sum_{k=2}^{M} n_k$.

Step 1.2 Obtain the prior, $G(1,N-\sum_{k=2}^{M} n_k)$, to be used at the second level of aggregation.

Step 2 For $\ell=2,3,\ldots,M-1$ do {ℓth level of aggregation}

Step 2.1 Apply the subset mql $\langle n_\ell \rangle_{N-n_{\ell+1}-\ldots-n_M}$ and fb (expressed by $P_\ell(N-\sum_{k=\ell+1}^{M} n_k/n_{\ell+1},\ldots,n_M)$) constraints on the prior $G(\ell-1,N-\sum_{k=\ell}^{M} n_k)$ of the previous $(\ell-1)$th level to obtain the MRE solution $P_\ell(n_\ell/n_{\ell+1},\ldots,n_M)$, $n_\ell=0,1,\ldots,N-\sum_{k=\ell+1}^{M} n_k$.

Step 2.2 Obtain the prior, $G(\ell,N-\sum_{k=\ell+1}^{M} n_k)$ to be used at the $(\ell+1)$th level of aggregation.

Step 3. {Mth level of aggregation}

Apply the marginal mql, $\langle n_M \rangle_N$, and fb (expressed by $P_M(N)$) constraints on the prior, $G(M-1,N-n_M)$ of the previous $(M-1)$th level to obtain the marginal MRE solution $P_M(n_M)$, $n_M=0,1,\ldots,N$.

Using this framework the following Theorem is presented.

<u>Theorem 5.1.</u> Consider a general central server model with a single CPU (Σ_0) and M (≥ 2) I/O units ($\Sigma_1,\ldots,\Sigma_M$) under a variable aggregation decomposition scheme. Given fully decomposable mql and fb subset and aggregate constraints (described in steps 1-3) and assuming that $\{x_i,\ i=0,2,\ldots,M\}$ are invariant to the buffer size N, then:

i. At the 1st level of decomposition , the conditional MRE solution is:

$$P_1(n_0/n_2,n_3,\ldots,n_M) = \frac{1}{G^*(1,N-\sum_{k=2}^{M} n_k)}\, x_0^{n_0}, \quad n_0=0,\ldots,N-\sum_{k=2}^{M} n_k \qquad (5.28)$$

where x_0 is given by (5.10) and (for $N_1=N-\sum_{k=2}^{M} n_k$) $G^*(1,N_1)$ is given by

$$G^*(1,N_1) = \sum_{n_0=0}^{N_1} x_0^{n_0} \tag{5.29}$$

Moreover, $G(1,N-\sum_{k=2}^{M} n_k)$ is the prior introduced for the 2nd level of aggregation and is given by (5.14).

ii. At the 2nd level of decomposition, the conditional MRE solution is:

$$P_2(n_2/n_3,\ldots,n_M) = \frac{1}{G^*(2,N-\sum_{k=3}^{M} n_k)}\, G(1,N-\sum_{k=2}^{M} n_k)\, x_0^{n_0} \tag{5.30}$$

$$n_2 = 0,1,\ldots,N-\sum_{k=3}^{M} n_k$$

where x_2 and $G(2,N-\sum_{k=3}^{M} n_k)$ are given by (5.21), (5.23), respectively.

Moreover, $G(2,N-\sum_{k=3}^{M} n_k)$ is the prior introduced for the 3rd level of decomposition and (for $N_2=N-\sum_{k=3}^{M} n_k$) is given by:

$$G(2,N_2) = G^*(2,N_2)\, \min\left[\, 1,\ \frac{1}{x_2^{N_2}} \,\right] \tag{5.31}$$

iii. At the ℓth level of decomposition, $\ell=3,4,\ldots,M-1$, the MRE solution is:

$$P_\ell(n_\ell/n_{\ell+1},\ldots,n_M) = \frac{1}{G^*(\ell,N-\sum_{k=\ell+1}^{M} n_k)}\, G(\ell-1,N-\sum_{k=\ell}^{M} n_k)\, x_\ell^{n_\ell} \tag{5.32}$$

$$n_\ell=0,1,\ldots,N-\sum_{k=\ell+1}^{M} n_k$$

where

$$x_\ell = \min(1,x_{\ell-1})\, \frac{\mu_{\ell-1} r_\ell}{\mu_\ell r_{\ell-1}} \tag{5.33}$$

and for $N_\ell = N - \sum_{k=\ell+1}^{M} n_k$:

$$G^*(\ell,N_\ell) = \sum_{n_\ell=0}^{N_\ell} G(\ell-1,N_\ell-n_\ell)\ x_\ell^{n_\ell} \tag{5.34}$$

Moreover, $\{G(\ell,N-\sum_{k=\ell+1}^{M} n_k),\ n_{\ell+1}=0,1,\ldots,N-\sum_{k=\ell+2}^{M} n_k\}$, is the prior introduced for the $(\ell+1)$th level of aggregation and is given by:

$$G(\ell,N_\ell) = G^*(\ell,N_\ell)\ \min\left[\ 1,\ \frac{1}{x_\ell^{N_\ell}}\ \right] \tag{5.35}$$

iv. At the Mth (and final) level of decomposition, the marginal MRE solution is:

$$P_M(n_M) = \frac{1}{G^*(M,N)}\ G(M-1,N-n_M)\ x_M^{n_M},\quad n_M=0,\ldots,N \tag{5.36}$$

where

$$x_M = \min(1,x_{M-1})\ \frac{\mu_{M-1}r_M}{\mu_M r_{M-1}} \tag{5.37}$$

and

$$G^*(M,N) = \sum_{n_M=0}^{N} G(M-1,N-n_M)\ x_M^{n_M} \tag{5.38}$$

Proof. The proof can be seen in Appendix III.

Moreover, the following corollaries hold.

Corollary 5.1. The MRE joint state probability $P(n_0,n_1,\ldots,n_M)$ of the general central server model of Theorem 5.1 is given by:

$$P(n_0,n_1,\ldots,n_M) = \frac{1}{G(M,N)} \prod_{\substack{i=0\\ i\neq 1}}^{M} U_i \tag{5.39}$$

$$G(M,N) = G^*(M,N) \prod_{\substack{i=0 \\ i\neq 1}}^{M-1} \max(1,x_i^N) \qquad (5.40)$$

$$U_0 = x_0 = \frac{\mu_1}{\mu_0 r_1} \qquad (5.41)$$

and

$$U_i = \frac{\mu_1 r_i}{\mu_i r_1} \quad , \; i=2,3,\ldots,M \qquad (5.42)$$

Proof. The proof can be seen in Appendix III.

Corollary 5.2. The MRE solution of corollary 5.1 has an identical form to the exact product-form solution as if the network was separable.

Proof. The proof can be seen in Appendix III.

5.2 Norton's reduction decomposition scheme.

Consider now the same central server model of figure 5.1 under a Norton's reduction decomposition scheme. The partition of the state space D is applied with respect to the number of customers present at the CPU (Σ_0). Thus, D is decomposed into N+1 disjoint subsets, D_{n_0}, $n_0=0,1,\ldots,N$, described by $D_{n_0}=\{(n_0,n_1,\ldots,n_M) \; / \; 0\leq n_i\leq N-n_0, \; i=1,\ldots,M \; \wedge \sum_{i=1}^{M} n_i=N-n_0\}$, (fig. 5.7). The above partition forms hierarchically two levels of aggregation. The first comprises from the conditional states in each subset D_{n_0}, $n_0=0,1,\ldots,N$, with state probability $P_1(n_1,n_2,\ldots,n_M/n_0)$, of having n_i jobs in I/O unit Σ_i, given that n_0 jobs are at the CPU (Σ_0). The second consists of the marginal states D_{n_0} themselves with probability $Pr\{D_{n_0}\}=Pr\{n_0 \text{ jobs in } \Sigma_0\}=P_0(n_0)$. Note that the same partition was implied in sections 2.2.1 and 3.3.2 where the I/O subsystem was examined in isolation (fig. 2.8) in the first level of aggregation and was substituted by a composite server (fig. 2.9) in the second level.

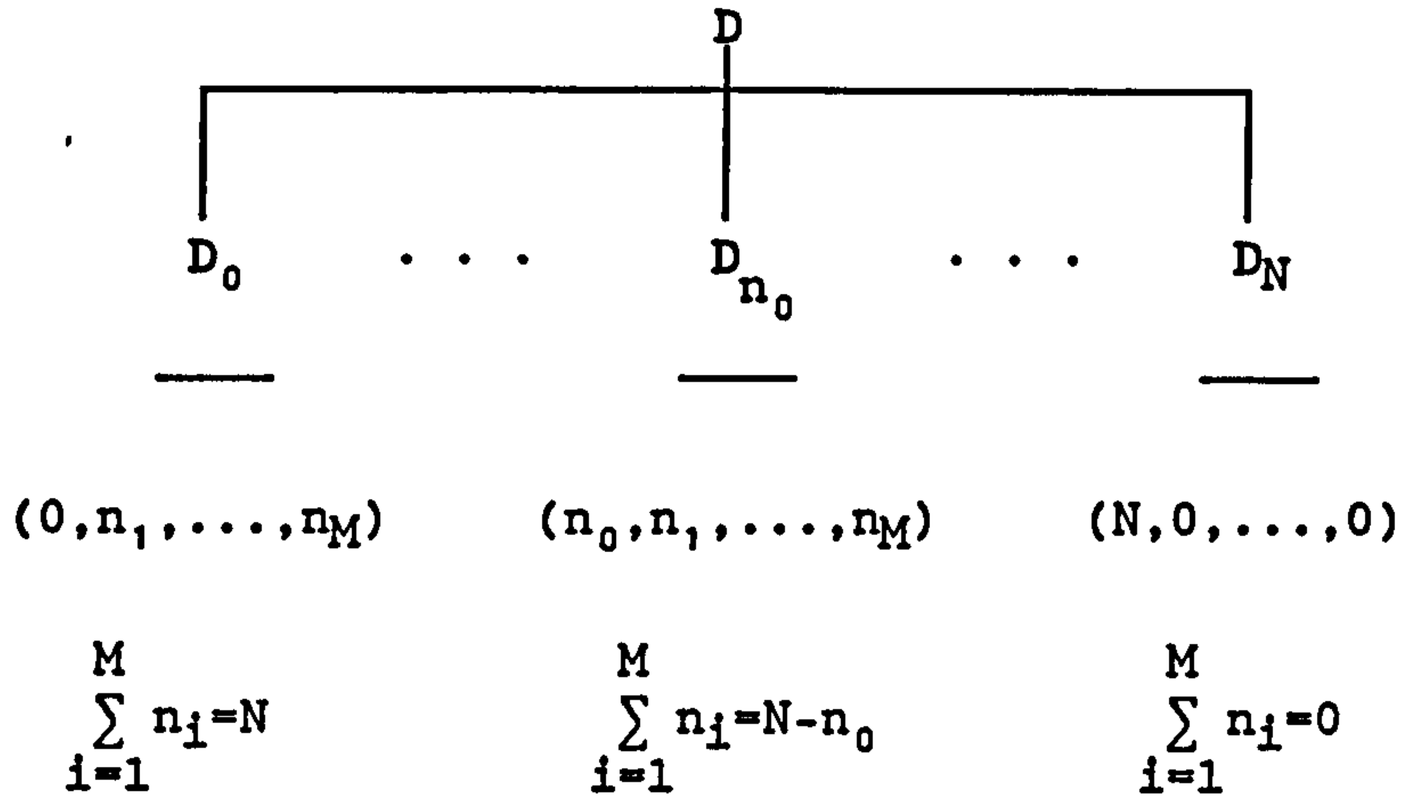

Figure 5.7. Partition of the state space D according to Norton's reduction decomposition scheme.

Constraint Information.

Suppose all that is known about the conditional state probabilities $P_1(n_1,\ldots,n_M/n_0)$ at the first level of aggregation is that they satisfy the following subset constraints, for $n_0=0,1,\ldots,N-1$,

– The normalization.

$$\sum_{(n_0,\ldots,n_M)\epsilon D_{n_0}} P_1(n_1,\ldots,n_M/n_0) = 1 \qquad (5.43a)$$

– The marginal conditional mql $\langle n_i\rangle_{N-n_0}$, $\langle n_i\rangle_{N-n_0}<N-n_0$

$$\sum_{n_i=0}^{N-n_0} n_i P_i(n_i/n_0) = \langle n_i\rangle_{N-n_0} \quad , \; i=1,2,\ldots,M \qquad (5.43b)$$

– The conditional job fb equations, written as:

$$X_i(n_0) = \sum_{j=1}^{M} X_j(n_0) r_i \quad , \; i=1,2,\ldots,M \qquad (5.43c)$$

where X_j is the throughput of I/O unit Σ_j. Constraint (5.43c) can be

expressed via the full buffer state probability, $P_i(N-n_0/n_0)$ as:

$$\sum_{n_i=0}^{N-n_0} f_1(n_i)P_i(n_i/n_0) = \Phi_{i,N-n_0} \tag{5.43d}$$

where

$$f_1(n_i) = \begin{cases} 1 & \text{if } n_i=N-n_0 \\ 0 & \text{otherwise} \end{cases} \tag{5.43e}$$

Moreover, it is assumed that for the marginal state probabilities $\{P_0(n_0),\ n_0=0,1,\ldots,N\}$ at the second level of aggregation, the following aggregate constraints exist:

– The normalization,

$$\sum_{n_0=0}^{N} P_0(n_0) = 1 \tag{5.44a}$$

– The marginal mql, $\langle n_0\rangle_N$ ($< N$)

$$\sum_{n_0=0}^{N} n_0P_0(n_0) = \langle n_0\rangle_N \tag{5.44b}$$

– The marginal flow-balance (fb) equation,

$$[1-P_0(0)]\mu_0 = \sum_{i=1}^{M}[1-P_i(0)]\mu_i \tag{5.44c}$$

expressed by the full buffer state probability:

$$P_0(N) = \Phi_{0,N},\ 0 < \Phi_{0,N} < 1$$

or

$$\sum_{n_0=0}^{N} f_2(n_0)P_0(n_0) = \Phi_{0,N} \tag{5.44d}$$

where

$$f_2(n_0) = \begin{cases} 1 & \text{if } n_0=N \\ 0 & \text{otherwise} \end{cases} \tag{5.44e}$$

The conditional subset $P_1(n_1,\ldots,n_M/n_0)$ and marginal $P_0(n_0)$ probabilities can be analytically determined by making assymptotic connections (as $N \longrightarrow +\infty$) with a related open network and an infinite capacity single queue, respectively.

First level of aggregation.

The conditional ME (or MRE) joint state probability, subject to constraints (5.43) is:

$$P_1(n_1,\ldots,n_M/n_0) = \frac{1}{G^*(1,N-n_0)} \prod_{i=1}^{M} \tilde{x}_i^{n_i} \, y_i^{f_1(n_i)}$$

where $\tilde{x}_i$, y_i, are the mql and fb multipliers respectively, that correspond to unit Σ_i. In order to ensure the convergence of the normalizing constant of the above solution as $N \longrightarrow +\infty$, the above may be written as:

$$P_1(n_2,\ldots,n_M/n_0) = \frac{1}{G(1,N-n_0)} \prod_{i=2}^{M} x_i^{n_i} \, y_i^{f_1(n_i)} \, y_1^{f_1(n_1)} \qquad (5.45)$$

where unit Σ_1 is assumed to be (without loss of generality) the bottleneck of this level. The above transformation is based on the fact that having specified indexes $n_2,\ldots,n_M$, then index n_1 is specified as $n_1=N-n_0-(n_2+\ldots+n_M)$ and thus the number of jobs present at the bottleneck Σ_1 may be excluded from the above conditional state probability, since:

$$P_1(n_1,n_2,\ldots,n_M/n_0) = P_1(n_2,\ldots,n_M/n_0)$$

Thus, the algebraic manipulation that yielded (5.45) guarantees the existance of the limit of (5.45) as $N \longrightarrow +\infty$ or equivalently the convergence of $G(1,N-n_0)$, (see also [KLEI 75, p. 152]).

Then assuming also that multipliers $\{x_i,\ i=2,\ldots,M\}$ are invariant to the population $N-n_0$, an assymptotic connection to an infinite capacity open network consisting of the I/O units $(\Sigma_2,\ldots,\Sigma_M)$, can be

established. The external interarrival-time process of this corresponding open network is generated according to the service time distribution of the bottleneck device Σ_1. The effect of this assymptotic connection may be seen in a different way, as follows. Let $R_s=\{r^*_{ij},\ i,j=1,\ldots,M\}$ be the matrix of routing probabilities of the subsystem. In this particular case (I/O subsystem) $r^*_{ij}=r_j$, $i,j=1,\ldots,M$. Then determining the multipliers x_j, $j=2,\ldots,M$, by applying the fb information at the limit ($N \longrightarrow +\infty$) is completely equivalent to solving for $x_2,\ldots,x_M$ the system,

$$\mathbf{x} = \mathbf{x}R_s \tag{5.46}$$

where vector $\mathbf{x}$ is $\mathbf{x}=(x_1\mu_1, x_2\mu_2, \ldots, x_M\mu_M)$, having specified the value of x_1 as $x_1=1$. Since R is stochastic, system (5.46) has an infinite set of solutions. The solution that corresponds to $x_1=1$ is clearly identical to what is obtained by the above described assymptotic connection. But then returning to the closed system with these values for multipliers x_i, $i=2,\ldots,M$, it is evident that $y_i=1$, $\forall\ i=1,\ldots,M$, because the system is already flow-balanced, since the form of solution (5.45), with x_i, $i=2,\ldots,M$, satisfying (5.46), may be identified as the exact exponential solution of the subsystem, ([KLEI 75, pp. 151-152]).

The prior introduced for the second (and final) level of aggregation is clearly $\{G(1,N-n_0),\ n_0=0,\ldots,N\}$.

Second level of aggregation.

By applying the aggregate constraints (5.44) on the prior introduced in the first level, the MRE solution for the marginal distribution $\{P_0(n_0),\ n_0=0,1,\ldots,N\}$ is given by:

$$P_0(n_0) = \frac{1}{G^*(2,N)}\, G(1,N-n_0)\, x_0^{n_0}\, y_0^{f_2(n_0)},\quad n_0=0,\ldots,N \tag{5.47}$$

where x_0, y_0 are the mql and fb multipliers corresponding to

constraints (5.44b), (5.44d), respectively. From what has been said so far, the first part of the following Theorem has been proven.

Theorem 5.2. Consider a general central server model with a single CPU (Σ_0) and M (≥ 2) I/O units ($\Sigma_1,\ldots,\Sigma_M$) under a Norton's reduction decomposition scheme. Given fully decomposable mql and fb subset and aggregate constraints (5.43) and (5.44) respectively, and assuming that the mql multipliers x_i, i=0,...,N, are invariant to the population N, then:

i. At the first level of decomposition, the MRE (or ME since the uniform prior is ignored) solution, for the conditional queue length joint probability distribution, is given by:

$$P_1(n_1,n_2,\ldots,n_M/n_0) = \frac{1}{G(1,N-n_0)} \prod_{i=2}^{M} x_i^{n_i} \qquad (5.48)$$

where $x_2,\ldots,x_M$ multipliers satisfy the system of equations (5.46) for $x_1=1$, and $G(1,N-n_0)$ is the normalizing constant. The prior introduced for the next level of decomposition is defined by the normalizing constants $\{G(1,N-n_0),\ n_0=0,\ldots,N\}$ since the relative bottleneck at this level is assumed to be unit Σ_1.

ii. At the second level of decomposition, the marginal (or aggregate) MRE solution is:

$$P_0(n_0) = \frac{1}{G^*(2,N)} G(1,N-n_0)\, x_0^{n_0}, \quad n_0=0,\ldots,N \qquad (5.49)$$

where

$$G^*(2,N) = \sum_{n_0=0}^{N} G(1,N-n_0) x_0^{n_0} \qquad (5.50)$$

and

$$x_0 = \frac{1}{\mu_0}\left[\mu_1 + \sum_{i=2}^{M} x_i\mu_i\right] \qquad (5.51)$$

Proof. The proof can be seen in Appendix III.

Moreover, it can be easily shown that the following corollary holds.

Corollary 5.3. The MRE joint state probability $P(n_0,n_1,....,n_M)$ corresponding to the conditional and marginal probabilities of Theorem 5.2 is identical to the exact product-form solution, as if the network was separable.

Proof. The proof can be seen in Appendix III.

5.3 Discussion.

Relative entropy minimization, subject to fully decomposable constraints provides a new universal framework for the implementation of hierarchical decomposition schemes. This technique requires a single (or multi)-level partition of the state space of a QNM and thus particularly favours the state space decomposition implied by the variable aggregation scheme. It has been demonstrated however, that it is applicable through Norton's reduction scheme as well, when this technique involves a partition of the state space. This point is stressed because Norton's reduction method can also be applied via a more general decomposition (not necessarily a partition) of the state space.

MRE, subject to fully decomposable subset and aggregate mql and fb constraints, determines approximately the form of the state probabilities of the QNM. The multipliers involved in this solution were analytically determined via assymptotic connections to infinite capacity systems. This can be viewed as a generalization of a technique used by Kouvatsos [KOUV 86a], which was also reviewed in the analysis of the GE/GE/1/N single queue, in the third chapter. This assymptotic approach, imposed and utilized successively the definition of the prior introduced at every level and used in the following level of decomposition. This way, each mql multiplier was

determined as the multiplier of a solution of a single open queue, whose input was defined by the corresponding open system of the previous level. To this end, the flow balance information was assymptotically used.

5.3.1 The robustnesss of separable queueing networks.

Assymptotic connections is not the only means of determining the multipliers involved. The MRE solution is not necessarily an exact solution of some system and it can be viewed more generally as an approximation. After all no assumption was made about the distributional form of the service times of the network. So Remark 5.1 can be extended to every level of decomposition, and a more general definition can be given to the mql multipliers. A similar attempt was made by Shore [SHOR 82a], in the single M/G/1 queue level. The results obtained from his investigation showed that the approximation of more general queueing systems (like $M/H_2/1$), by using an entropy solution subject to the first moment of the queue length distribution, is not significantly accurate, thus it is not our intention to pursue examining the MRE solution of this chapter as an approximation.

When, however, information about the system is restricted to the first moment of the service times and consequently of the flow in the network, as it has been demonstrated, the MRE solution proves to have an identical form to the solution of the network as if this was separable. Hense, relative entropy minimization, subject to mql and fb fully decomposable constraints, provides an information theoretic justification for the widespread applicability of separable queueing networks.

5.3.2 The equivalency between subset and marginal constraints.

The MRE decomposition implementation can be related to earlier works on ME and general queueing networks. More specifically, the ME joint state probability of a general central server model, subject to the normalization and the marginal mql and fb constraints, has an identical form to the exact product-form solution, as if the network was separable (see [KOUV 85, KOUV 86c]). This is also the case when fully decomposable constraints are used. An explanation of this common attribute can be based on the equivalency between subset and marginal constraints. Let's describe this more precisely through a small example and using the notation of the fourth chapter.

Consider a partition $D_0, D_1, \ldots, D_N$ of the state space D, of an abstract system. Let $\beta(x)$ be a suitable function, well-defined on each subset D_j, j=0,...,N. Suppose the new information about the conditional density function $[f^* * D_j](x)$ consists of the subset constraints I_{sj}, given as:

$$I_{sj} : \int_{D_j} \beta(x)[f^* * D_j](x)\, dx = \langle \beta_j \rangle \quad , \; j=0,\ldots,N \tag{5.52}$$

Constraints I_{sj} can be written in terms of the full density as constraints $I^{\dagger}_{sj}$:

$$I^{\dagger}_{sj} : \int_{D_j} \alpha_j(x) f^*(x)\, dx = \langle \beta_j \rangle f^*_j \quad , \; j=0,\ldots,N \tag{5.53}$$

where

$$f^*_j = \int_{D_j} f^*(x)\, dx \quad , \; j=0,\ldots,N \tag{5.54}$$

and

$$\alpha_j(x) = \begin{cases} \beta(x) & \text{if } x \in D_j \\ 0 & \text{if } x \notin D_j \end{cases} \tag{5.55}$$

If also a prior density g(x) is available, the MRE solution is:

$$f(x) = g(x)\exp\left[-\theta \; - \sum_{j=0}^{N} \lambda_j \alpha_j(x)\right] \;, \; x \epsilon D \qquad (5.56)$$

Let's now assume that for some reason:

$$\lambda_0 = \lambda_1 = \cdots = \lambda_N = \lambda \qquad (5.57)$$

i.e. the Lagrangian multipliers of solution (5.56), that correspond to constraints (5.53), are identical. Then since for every $x \epsilon D$ there exists a unique index $j \epsilon \{0,1,\ldots,N\}$ so that $x \epsilon D_j$, it is implied that:

$$f(x) = g(x)\exp[-\theta \; -\lambda\beta(x)] \;, \; x \epsilon D \qquad (5.58)$$

The MRE solution (5.58) must also satisfy any linear combination of constraints (5.53), namely:

$$\sum_{j=0}^{N} \int_{D} \alpha_j(x) f^*(x)\, dx = \sum_{j=0}^{N} <\beta_j> f_j^* \triangleq <\gamma> \qquad (5.59)$$

Constraint (5.59), using (5.55) can be written as:

$$\int_{D} \beta(x) f^*(x)\, dx = <\gamma> \qquad (5.60)$$

Solving now the MRE problem in the full state space D, subject to constraint (5.60) and of course the normalization, the solution is:

$$f_1(x) = g(x)\exp[-\varphi \; -\xi\beta(x)] \;, \; x \epsilon D$$

The above MRE solution and (5.58) have the same form and satisfy the same constraints, thus they are identical on D, i.e.

$$\lambda = \xi \quad \text{and} \quad \theta = \varphi$$

and $f(x)=f_1(x)$, $\forall \; x \epsilon D$.

For a central server model with a single CPU and 2 I/O units the mql constraint for each subset D_j, $j=0,\ldots,N$, implies that $\beta(x)$ is a common function n for all $j=0,\ldots,N$ and by allowing $N \longrightarrow +\infty$, the same

invariant to the buffer size N Lagrangian coefficient, $x_0=\exp\{-\lambda\}$ for all subsets D_j is obtained. Note that the above attribute does not apply to conditional fb constraints (because of the lack of a common function for all D_j, j=0,...,N). But in the case under consideration, the fb constraint proves to be redundant (corresponding fb multiplier equal to 1) and therefore it plays no role in the final solution. Thus, the equivalency between the subset and marginal constraints (5.53) and (5.59), respectively, provides an explanation on why the same product-form approximation for the central server model, is captured by applying either ME formalism directly on the joint queue length distribution, based on marginal mql and fb constraints or MRE formalism based on fully decomposable mql and fb constraints under either variable aggregation or Norton's reduction decomposition schemes.

5.3.3 Extension to more general topologies.

Theorem 5.2 and Corollary 5.3, that concern the application of this new methodology based on Norton's reduction scheme, can be easily extended to cover the case where a single server (like the CPU Σ_0) is examined versus a subsystem with an arbitrary topology, which satisfies the requirement of Norton's theorem, i.e. it has a common input stream and a common output stream. In this way, it can be clearly applied hierarchically in the analysis of cyclic queues in tandem Σ_0, Σ_1, ..., Σ_M, (fig. 5.8), where the subsystems of successive levels are: $\Sigma_0\ \Sigma_1$, $(\Sigma_0\ \Sigma_1)\ \Sigma_2$, ..., $(\Sigma_0\ \Sigma_1 \ldots \Sigma_{M-1})\ \Sigma_M$. The MRE solution subject to fully decomposable mql and fb constraints implies a joint state probability that can be also shown to be identical to the exact solution as if this cyclic system was separable.

Note that for such a type of queueing system, the variable

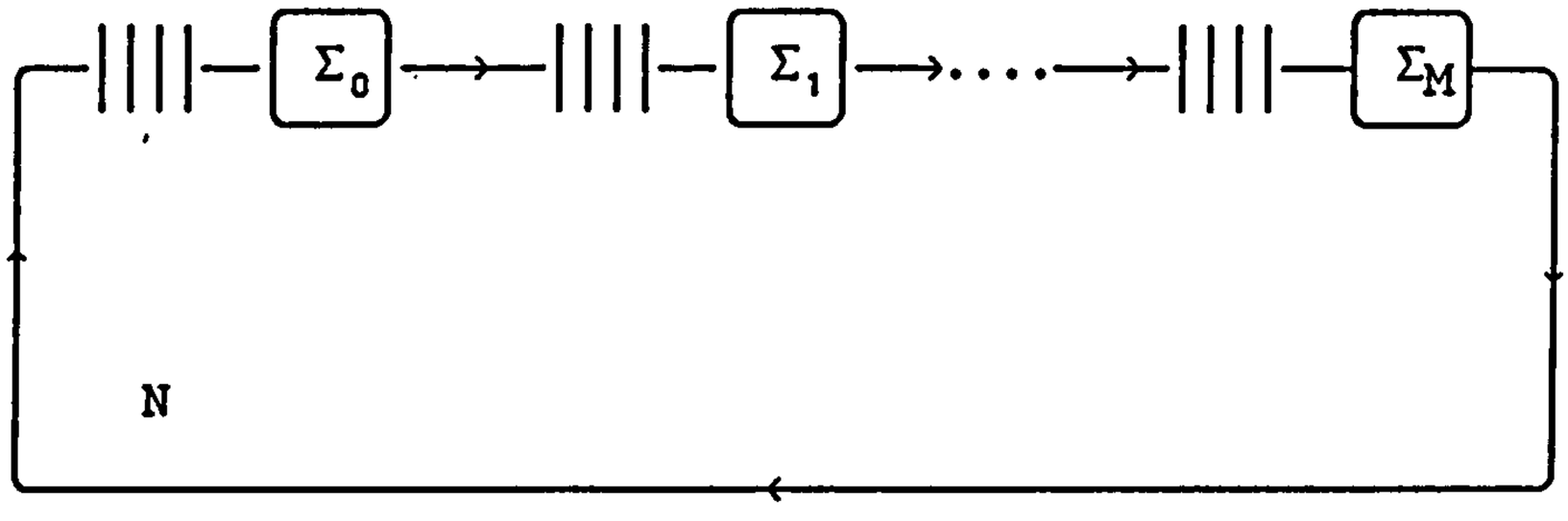

Figure 5.8. Cyclic queues in tandem

aggrregation scheme, as presented by Courtois, [COUR 77], is inaplicable due to the fact that local equilibrium within a subsystem in the context of near complete decomposability is not clearly possible. More precisely Courtois' variable aggregation scheme requires that at every level, the unit that is examined, communicates directly with the subsystem of the previous level. This imposes a severe restriction on the configurations amenable to this scheme. Let's see at this point, more closely and in terms of information available about the system, the cause of these restrictions.

5.3.4 Flow-balance equations under the variable aggregation scheme.

The source of the problems towards this direction can be traced in the decomposition of the flow information. Any approximate solution of a closed network, in order to have any meaningful value, must be flow-balanced. Consider an arbitrary topology network with three units and a routing matrix $R=\{r_{ij},\ i,j=0,1,2\}$. A solution for this network is flow-balanced if it satisfies the following flow equations:

$$\mu_0 U_0 = \sum_{i=0}^{2} U_i \mu_i r_{i0} \qquad (5.61a)$$

$$\mu_1 U_1 = \sum_{i=0}^{2} U_i \mu_i r_{i1} \qquad (5.61b)$$

$$\mu_2 U_2 = \sum_{i=0}^{2} U_i \mu_i r_{i2} \qquad (5.61c)$$

where $$U_i = 1 - P_i(0) \quad , i=0,1,2 \qquad (5.62)$$

are the marginal utilizations of the units of the network and obviously $P_i(0)$ is the marginal probability of unit i being idle. Because of the state space decomposition, the above equations break down to conditional and marginal ones. So in the first level of decomposition (unit Σ_0 versus Σ_1) and according to the variable aggregation scheme (see algorithm 2.1, fig. 2.3), the conditional flow-balance equation is:

$$\mu_0 r_{01} U_0(n_2) = \mu_1 r_{10} U_1(n_2) \qquad (5.63)$$

where $$U_0(n_2) = 1 - P_1(0/n_2) \quad , n_2=0,\ldots,N \qquad (5.64)$$

$$U_1(n_2) = 1 - P_1(N-n_2/n_2) \quad , n_2=0,\ldots,N \qquad (5.65)$$

are the conditional (on the number of jobs present at Σ_2, n_2) utilizations of units Σ_0 and Σ_1. In the second and final level of decomposition the fb equation is:

$$\mu_2 U_2 = \sum_{i=0}^{2} U_i \mu_i r_{i2} \qquad (5.66)$$

where $$U_i = \sum_{n_2=0}^{N} U_i(n_2) P_2(n_2) \quad , i=0,1 \qquad (5.67)$$

and $$U_2 = 1 - P_2(0) \qquad (5.68)$$

are the final marginal utilizations of the units. Relation (5.67) is an application of LTP in order to uncondition the utilizations of the first level. Obviously, (5.66) equation is identical to (5.61c). This is not the case though with the conditional fb equation (5.63). Multiplying both sides of (5.63) with $P_2(n_2)$ and summing over all

n_2=0,...,N, (5.63) yields:

$$\mu_0 r_{01} U_0 = \mu_1 r_{10} U_1 \qquad (5.69)$$

Two significant points arise here. Firstly, that the conditional fb equation (5.63) is also satisfied by the marginal utilizations, irrespective of the values of probabilities $P_2(n_2)$ that were used to uncondition them. Secondly, that equation (5.69) does not in any way guarantee that one of the equations (5.61a) or (5.61b) are satisfied. This guarantee is only true for special cases like the central server model with feedback or fully connected networks with $r_{ij}=r_j$, $\forall$ i, (fig. 5.9). It must be pointed out that this problem also affects the

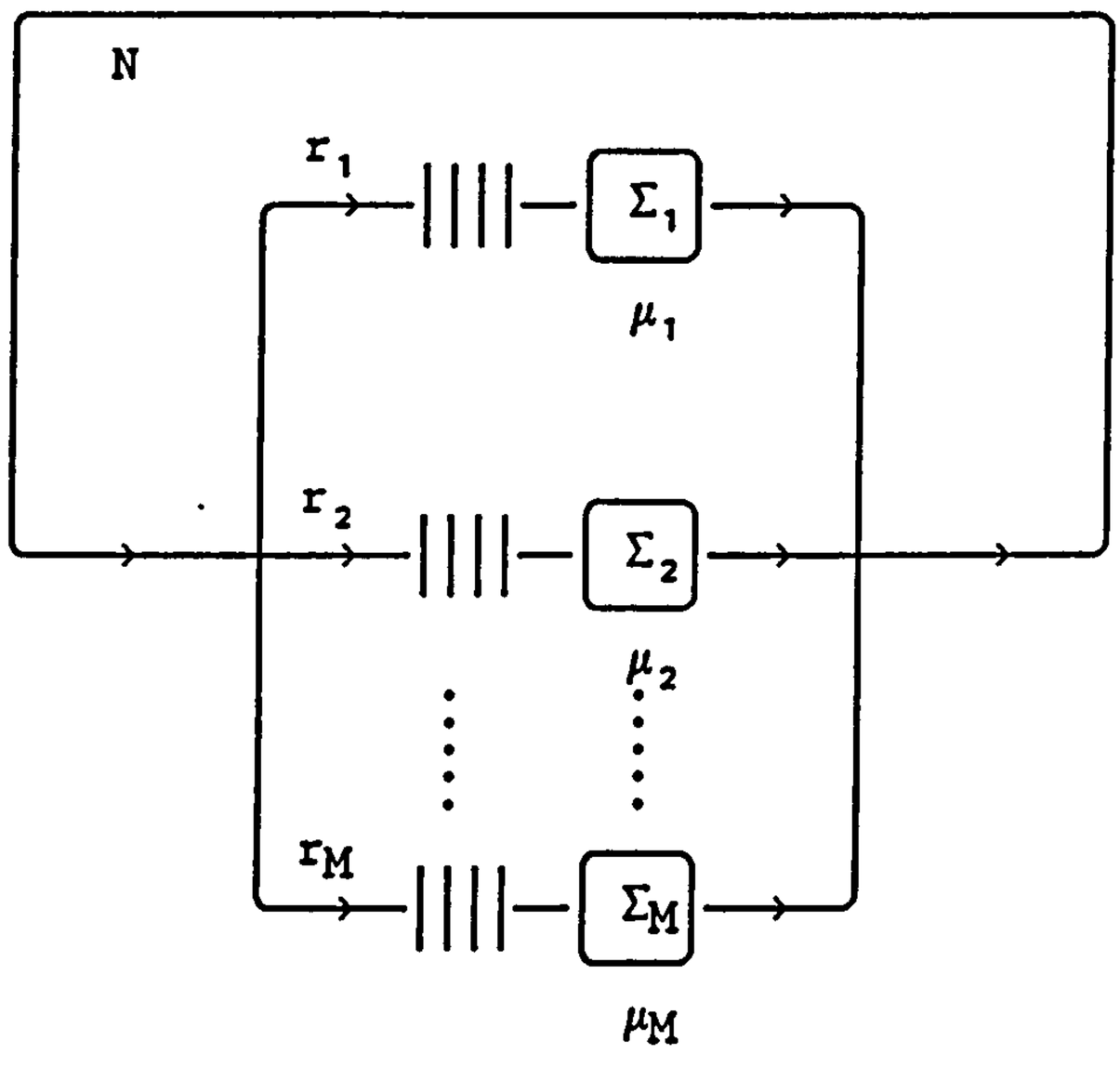

Figure 5.9. Fully connected network with $r_{ij}=r_j$ $\forall$ i=1,...,M.

conventional implementation (algorithm 2.1) of the variable aggregation scheme. It may be easily shown in this small example that the flow information used in our new approach and in algorithm 2.1, result to the same flow-balance equations. Recall from algorithm 2.1 [COUR 77], and under the modified notation used in this chapter, that the load dependent service rates of the subsystem (Σ_0 Σ_1) are defined

as:

$$\Psi_{1,2}(n_2) = [1-P_1(N-n_2/n_2)]\mu_1 r_{12} + [1-P_1(0/n_2)]\mu_0 r_{02}$$

And the fb equation satisfied in the second level is clearly:

$$[1-P_2(0)]\mu_2(1-r_{22}) = \sum_{n_2=0}^{N} \Psi_{1,2}(n_2)P_2(n_2)$$

Using the definition of $\Psi_{1,2}(n_2)$ and identifying the conditional utilizations through (5.64), (5.65) the above becomes:

$$[1-P_2(0)]\mu_2(1-r_{22}) = \mu_1 r_{12} \sum_{n_2=0}^{N} U_1(n_2)P_2(n_2) + \mu_0 r_{02} \sum_{n_2=0}^{N} U_0(n_2)P_2(n_2)$$

and using (5.67) and (5.68) the above yields:

$$U_2\mu_2 = U_0\mu_0 r_{02} + U_1\mu_1 r_{12} + U_2\mu_2 r_{22}$$

which is exactly (5.66) fb equation. So, even though load-dependent composite centers are not explicitely used in our new method, the same fb information is used through the conditional and marginal utilizations and fb equations (see also [COUR 77, pp. 72-73]).

Let's at this point leave this interesting subject, which will be discussed thoroughly in the sixth and seventh chapters, and return to the new MRE solution.

5.3.5 The relative accuracy of the MRE decomposition solution.

As mentioned previously, earlier works on the accuracy of an entropy solution, subject to the mql constraint, as an approximation to more general distributions at the single queue level, [SHOR 82a], do not encourage any further investigation of this form of solution as an approximation. The fact that as it has been presented here the MRE solution is exact for separable networks, means that it is expected to perform well when modelling general distributions, like

H_2, whose parameters classify them to be close to the exponential distribution. However, the relative accuracy of this MRE solution is naturally expected to detiorate as the variability of the service times increases.

In order to further improve the accuracy of the MRE solution, additional subset and aggregate constraints (e.g. utilization constraint) are needed to take into account the service time variability. This extension of the proposed solution is the subject of the following chapter. There, a new and more general form of MRE solution will be introduced, which will serve as an approximation to networks with GE-distributed service times. Several decomposition algorithms will be described and favourable comparisons against the exact solutions and other approximate ones will be made.

CHAPTER VI

A MRE APPROXIMATION INTO THE HIERARCHICAL DECOMPOSITION OF A CLASS OF GENERAL QNMs

In this chapter a generalization of the MRE approximation, described in the previous chapter, will be carried out based on the additional constraint of utilization which will be assumed at every level of decomposition. The ME solution, subject to utilization, mql and fb constraints becomes exact for the GE/GE/1/N queue at the level of the two-stage cyclic queueing network. This solution is a generalization of the ME solution of the previous chapter, which reduces to the exact solution of the M/M/1/N queue. Furthermore, since every level of decomposition is associated to the behaviour of a unit of the network through a set of conditional distributions that describe the interactions of this unit and the subnetwork of the previous level, a parallelism may be drawn between the constraints assumed for the MRE solution of the network in this chapter and the constraints assumed in the ME approximate solution for a general QNM proposed by Kouvatsos, [KOUV 86c]. There the marginal utilization, mql and fb constraints were assumed for each unit, while here the same constraints will concern the conditional distributions at each level of decomposition.

The MRE solution for the GE/GE/1/N system, presented in the third chapter (section 3.2.7, pp. 53-60) will be our starting point and what follows may be viewed as an attempt to use the same approach of evaluating the multipliers of the solution of the simple two stage cyclic system, in evaluating the multipliers of the MRE solution at the network level.

Naturally, it is not expected that the new MRE solution will be

exact for some general distributional form of the service times of the network, however, since it will be built as an extension of the MRE solution of the previous chapter, it is expected to reduce to the exact solution of a separable network under a specific set of parameters, i.e. when information about the flow and the service times of the network is restricted to the first moments.

6.1 A MRE approximate solution of central server models based on the variable aggregation scheme.

Let's initially restrict ourselves in tackling a central server model via the variable aggregation scheme and then generalize the new method to different network topologies via Norton's reduction scheme. For exposition purposes a central server model with 3 I/O units (M=3) will be examined first.

The partition involved here is the same as the one represented in fig. 5.6 and described in pages 103-104 of the previous chapter. The constraints assumed for the conditional and marginal distributions are as follows.

Constraint Information.

Suppose all that is known about each of the conditional distributions $\{P_1(n_0/n_2,n_3),\ n_0=0,\ldots,N-n_2-n_3\}$, $n_2,n_3=0,\ldots,N$ so that $0\leq n_2+n_3<N$, at the first level of decomposition is that the following subset constraints are satisfied:

- The normalization,

$$\sum_{n_0=0}^{N-n_2-n_3} P_1(n_0/n_2,n_3) = 1 \qquad (6.1a)$$

- The conditional mql $\langle n_0\rangle_{N-n_2-n_3}$ $(< N-n_2-n_3)$

$$\sum_{n_0=0}^{N-n_2-n_3} n_0 P_1(n_0/n_2,n_3) = \langle n_0\rangle_{N-n_2-n_3} \qquad (6.1b)$$

- The conditional utilization $\langle U_0 \rangle_{N-n_2-n_3}$, ($< \langle n_0 \rangle_{N-n_2-n_3}$)

$$1 - P_1(0/n_2,n_3) = \langle U_0 \rangle_{N-n_2-n_3}$$

written as:

$$\sum_{n_0=0}^{N-n_2-n_3} f_{1,1}(n_0) P_1(n_0/n_2,n_3) = \langle U_0 \rangle_{N-n_2-n_3} \qquad (6.1c)$$

where $$f_{1,1}(n_0) = \begin{cases} 0 & \text{if } n_0=0 \\ 1 & \text{otherwise} \end{cases}$$

- The conditional fb equation,

$$U_0(1,N-n_2-n_3)\mu_0 r_1 = U_1(1,N-n_2-n_3)\mu_1 \qquad (6.1d)$$

where $$U_0(1,N-n_2-n_3) = 1 - P_1(0/n_2,n_3)$$

and $$U_1(1,N-n_2-n_3) = 1 - P_1(N-n_2-n_3/n_2,n_3)$$

are the conditional utilizations of units Σ_0 and Σ_1 respectively in the first level of decomposition, given that $N-n_2-n_3$ jobs circulate in this level. The fb equation (6.1d) is expressed by the full buffer conditional state probability,

$$P_1(N-n_2-n_3/n_2,n_3) = \langle \Phi_0 \rangle_{N-n_2-n_3}, \quad 0 < \langle \Phi_0 \rangle_{N-n_2-n_3} < 1$$

as:

$$\sum_{n_0=0}^{N-n_2-n_3} f_{1,2}(n_0) P_1(n_0/n_2,n_3) = \langle \Phi_0 \rangle_{N-n_2-n_3} \qquad (6.1e)$$

where $$f_{1,2}(n_0) = \begin{cases} 1 & \text{if } n_0=N-n_2-n_3 \\ 0 & \text{otherwise} \end{cases}$$

Similarly it is assumed that for the conditional state distributions $\{P_2(n_2/n_3),\ n_2=0,\ldots,N-n_3\}$, $n_3=0,1,\ldots,N-1$, at the second level of decomposition, the following subset constraints are satisfied:

- The normalization,

$$\sum_{n_2=0}^{N-n_3} P_2(n_2/n_3) = 1 \qquad (6.2a)$$

– The conditional mql $\langle n_2 \rangle_{N-n_3}$ ($< N-n_3$)

$$\sum_{n_2=0}^{N-n_3} n_2 P_2(n_2/n_3) = \langle n_2 \rangle_{N-n_3} \qquad (6.2b)$$

– The conditional utilization $\langle U_2 \rangle_{N-n_3}$, ($< \langle n_2 \rangle_{N-n_3}$)

$$1 - P_2(0/n_3) = \langle U_2 \rangle_{N-n_3}$$

written as:

$$\sum_{n_2=0}^{N-n_3} f_{2,1}(n_2) P_2(n_2/n_3) = \langle U_2 \rangle_{N-n_3} \qquad (6.2c)$$

where

$$f_{2,1}(n_2) = \begin{cases} 0 & \text{if } n_2=0 \\ 1 & \text{otherwise} \end{cases}$$

– The conditional fb equation,

$$U_0(2,N-n_3)\mu_0 r_2 = U_2(2,N-n_3)\mu_2 \qquad (6.2d)$$

where $U_0(2,N-n_3)$ is the conditional utilization of the CPU (Σ_0) at this second level of aggregation, given that $N-n_3$ jobs circulate in the subsystem of this level, and similarly $U_2(2,N-n_3)$ is the conditional utilization of I/O unit Σ_2 at this level of aggregation under the same population ($N-n_3$). Clearly,

$$U_0(2,N-n_3) = 1 - \sum_{n_2=0}^{N-n_3} P_1(0/n_2,n_3) P_2(n_2/n_3)$$

or

$$U_0(2;N-n_3) = \sum_{n_2=0}^{N-n_3} U_0(1,N-n_2-n_3) P_2(n_2/n_3)$$

and

$$U_2(2,N-n_3) = 1 - P_2(0/n_3)$$

The above fb equation (6.2d) is expressed once more by the full

buffer conditional probability,

$$P_2(N-n_3/n_3) = \langle\Phi_2\rangle_{N-n_3} \quad , \quad 0< \langle\Phi_2\rangle_{N-n_3} <1$$

as:

$$\sum_{n_2=0}^{N-n_3} f_{2,2}(n_2)P_2(n_2/n_3) = \langle\Phi_2\rangle_{N-n_3} \qquad (6.2e)$$

where

$$f_{2,2}(n_2) = \begin{cases} 1 & \text{if } n_2=N-n_3 \\ 0 & \text{otherwise} \end{cases}$$

Finally, it is assumed that for the marginal state distribution $\{P_3(n_3),\ n_3=0,\ldots,N\}$, at the third level of aggregation the following aggregate constraints exist:

- The normalization,

$$\sum_{n_3=0}^{N} P_3(n_3) = 1 \qquad (6.3a)$$

- The marginal mql $\langle n_3\rangle_N$ $(< N)$

$$\sum_{n_3=0}^{N} n_3P_3(n_3) = \langle n_3\rangle_N \qquad (6.3b)$$

- The marginal utilization $\langle U_3\rangle_N$, $(< \langle n_3\rangle_N)$

$$U_3(3,N) = 1 - P_3(0) = \langle U_3\rangle_N$$

written as:

$$\sum_{n_3=0}^{N} f_{3,1}(n_3)P_3(n_3) = \langle U_3\rangle_N \qquad (6.3c)$$

where

$$f_{3,1}(n_3) = \begin{cases} 0 & \text{if } n_3=0 \\ 1 & \text{otherwise} \end{cases}$$

- The marginal fb equation,

$$U_0(3,N)\mu_0 r_3 = U_3(3,N)\mu_3 \qquad (6.3d)$$

where clearly,

$$U_0(3,N) = \sum_{n_3=0}^{N} U_0(2,N-n_3)P_3(n_3) = 1 - P_0(0)$$

is the unconditional (marginal) utilization of the CPU (Σ_0). The fb constraint (6.3d) is expressed by, $P_3(N) = \langle\Phi_3\rangle_N$, $0< \langle\Phi_3\rangle_N <1$, as:

$$\sum_{n_3=0}^{N} f_{3,2}(n_3)P_3(n_3) = \langle\Phi_3\rangle_N \qquad (6.3e)$$

where
$$f_{3,2}(n_3) = \begin{cases} 1 & \text{if } n_3=N \\ 0 & \text{otherwise} \end{cases}$$

The MRE solution, subject to the above constraints, is of the following form.

<u>First level of aggregation.</u>

In the first level of aggregation, the MRE solution, subject to constraints (6.1), is:

$$P_1(n_0/n_2,n_3) = \frac{1}{G^*(1,N-n_2-n_3)}\, x_0^{n_0}\, g_0^{f_{1,1}(n_0)}\, y_0^{f_{1,2}(n_0)} \qquad (6.4)$$

$$n_0=0,1,\ldots,N-n_2-n_3$$

By making an assymptotic connection to the related infinite capacity queue (as $N \longrightarrow +\infty$), one may proceed, as in the third chapter (section 3.2.7, pp. 53-60), to prove that under the invariability assumption about x_0 and g_0, the mql and utilization multipliers respectively, (i.e. assume that these multipliers are not functions of N), the following relations are true:

$$G^*(1,N-n_2-n_3) = 1 + g_0 \sum_{n_0=1}^{N-n_2-n_3-1} x_0^{n_0} + x_0^{N-n_2-n_3} g_0 y_0 \qquad (6.5)$$

$$x_0 = \frac{<n_0>-\rho_0}{<n_0>} \tag{6.6}$$

where $$<n_0> = \lim_{N\to+\infty} <n_0>_{N-n_2-n_3}$$

where ρ_0 is the assymptotic utilization of unit Σ_0 in this first level and is defined as:

$$\rho_0 = \frac{\mu_1}{\mu_0 r_1} \tag{6.7}$$

while $$g_0 = \frac{\rho_0(1-x_0)}{x_0(1-\rho_0)} \tag{6.8}$$

and $$y_0 = \frac{1-\rho_0}{1-x_0} \tag{6.9}$$

The prior to be used at the second level of aggregation is denoted as $\{G(1,N-n_2-n_3),\ n_2=0,\ldots,N-n_3\}$, and given by:

$$G(1,n) = h(1,n)G^*(1,n)\ ,\ n=1,2,\ldots \tag{6.10}$$

where $$h(1,n) = \begin{cases} 1 & \text{if } \rho_0<1 \\ \dfrac{1}{x_0^n g_0 y_0} & \text{if } \rho_0>1 \end{cases}\ ,\ n=1,2,\ldots \tag{6.11}$$

while $G(1,0)=1$ trivially. For more details on the derivation of relations (6.6)-(6.11) we refer to Appendix IV.

Second level of aggregation.

By applying the subset constraints (6.2) on the prior defined in the first level, the MRE solution is:

$$P_2(n_2/n_3) = \frac{G(1,N-n_2-n_3)}{G^*(2,N-n_3)}\, x_2^{n_2}\, g_2^{f_{2,1}(n_2)}\, y_2^{f_{2,2}(n_2)} \tag{6.12}$$

$$n_2=0,1,\ldots,N-n_3$$

and normalizing the above,

$$G^*(2,N-n_3) = G(1,N-n_3) + g_2 \sum_{n_2=1}^{N-n_3-1} G(1,N-n_2-n_3)x_2^{n_2} + x_2^{N-n_3} g_2 y_2 \tag{6.13}$$

where

$$x_2 = \frac{<n_2>-\rho_2}{<n_2>} \tag{6.14}$$

$$<n_2> = \lim_{N\to+\infty} <n_2>_{N-n_3}$$

where ρ_2 is the assymptotic utilization of unit Σ_2 in this second level and is defined as:

$$\rho_2 = \min(1,\rho_0)\ \frac{\mu_0 r_2}{\mu_2} \tag{6.15}$$

while

$$g_2 = \frac{\rho_2(1-x_2)}{x_2(1-\rho_2)} \tag{6.16}$$

$$y_2 = y_2(N_2) = \frac{1}{\mu_2 x_2^{N_2} g_2}\left[\mu_0 r_2\left[U_0(1,N_2)G(1,N_2) + g_2\sum_{n_2=1}^{N_2-1} U_0(1,N_2-n_2)G(1,N_2-n_2)x_2^{n_2}\right] - \mu_2 g_2 \sum_{n_2=1}^{N_2-1} G(1,N_2-n_2)x_2^{n_2}\right] \tag{6.17}$$

The prior to be used at the third level of aggregation is denoted as $\{G(2,N-n_3),\ n_3=0,\ldots,N\}$, and given by:

$$G(2,n) = h(2,n)G^*(2,n)\ ,\ n=1,2,\ldots \tag{6.18}$$

where

$$h(2,n) = \begin{cases} 1 & \text{if } \rho_2<1 \\ \dfrac{1}{x_2^n g_2 y_2} & \text{if } \rho_2>1 \end{cases}\ ,\ n=1,2,\ldots \tag{6.19}$$

and $G(2,0)=1$ trivially.

Third level of aggregation.

In the third and final level of decomposition now, and applying

the marginal constraints (6.3) on the prior defined by (6.18), the MRE solution is:

$$P_3(n_3) = \frac{G(2,N-n_3)}{G^*(3,N)} x_3^{n_3} g_3 f_{3,1}(n_3) y_3 f_{3,2}(n_3), n_3=0,\ldots,N \quad (6.20)$$

and normalizing the above,

$$G^*(3,N) = G(2,N) + g_3 \sum_{n_3=1}^{N-1} G(2,N-n_3)x_3^{n_3} + x_3^N g_3 y_3 \quad (6.21)$$

where

$$x_3 = \frac{\langle n_3\rangle - \rho_3}{\langle n_3\rangle} \quad (6.22)$$

and

$$\langle n_3\rangle = \lim_{N\to+\infty} \langle n_3\rangle_N$$

where ρ_3 is the assymptotic utilization of unit Σ_3 in the final level and is defined as:

$$\rho_3 = \min(1,\rho_2)\frac{\mu_2 r_3}{\mu_3 r_2} \quad (6.23)$$

while

$$g_3 = \frac{\rho_3(1-x_3)}{x_3(1-\rho_3)} \quad (6.24)$$

$$y_3 = y_3(N) = \frac{1}{\mu_3 x_3^N g_3}\left[\mu_0 r_3\left[U_0(2,N)G(2,N) + g_3\sum_{n_3=1}^{N-1} U_0(2,N-n_3)G(2,N-n_3)x_3^{n_3}\right] - \mu_3 g_3 \sum_{n_3=1}^{N-1} G(2,N-n_3)x_3^{n_3}\right] \quad (6.25)$$

Details on the derivation of the mql (x_ℓ), utilization (g_ℓ) and fb (y_ℓ) multipliers as well as details on the definition of the priors $\{G(\ell,n),\ n=0,\ldots,N\}$ for levels $\ell=2,3$ of aggregation can be found in Appendix IV. What is presented in this Appendix may be summarized as

follows. Assuming conditional and marginal mql utilization and fb fully decomposable constraints, MRE principle provides us with a form of solution for the conditional and marginal state probabilities. The mql and utilization multipliers, involved in this solution, are approximated by making assymptotic connections to related infinite capacity systems. This involves the invariability assumption for each pair of mql and utilization multipliers. This assumption could be viewed as considering that at every level these two multipliers are shared by the solution of the closed system and the solution of the corresponding open system. This assumption proved to be exact in the MRE solution of the previous chapter, where only the first moment of the network's flow was used, and is also exact in the two stage cyclic GE/GE/1/N system (see section 3.2.7, pp. 53-60). A somewhat similar assumption has been also made in the implementation of the proposed ME solution for a general network by Kouvatsos [KOUV 86c]. Thus, the fb multiplier is used to "correct" the MRE solution, so that at every level it satisfies the corresponding fb constraint.

Note that the assymptotic utilizations ρ_0, ρ_2 and ρ_3, derived so far, are identical to the mql multipliers x_0, x_2, x_3, respectively, that were established in the MRE solution of Theorem 5.1 in the previous chapter (compare relations (6.7), (6.15) and (6.23) to (5.10), (5.21) and (5.33) respectively). Also note that the same pattern is used in both cases to obtain these values through the use of the fb information assymptotically. This pattern can be described as follows.

At an arbitraty (ℓth) level of aggregation (ℓ=1,2,...,M) the ℓth unit (Σ_ℓ) under consideration, at the limit ($N \longrightarrow +\infty$), receives an input flow, which is generated by the CPU as this is utilized assymptotically in the subsystem of the previous level ($\Sigma_0\Sigma_1...\Sigma_{\ell-1}$). More precisely, if at the (ℓ-1)th level the CPU is the relative

bottleneck (case $\rho_0>1$ and $\rho_k<1$ $\forall$ k=2,...,ℓ-1) the mean arrival rate at unit Σ_ℓ is $\mu_0 r_\ell$, which results to ρ_ℓ being $\rho_\ell=\mu_0 r_\ell/\mu_\ell$. In case, however, that unit Σ_b ($b\neq 0$) is the relative bottleneck, (case b is the largest index for which $\rho_b>1$, i.e. $\rho_{b+1},\ldots,\rho_{\ell-1}<1$), then the assymptotic utilization of the CPU is $\mu_b/\mu_0 r_b$ and the mean arrival rate at unit Σ_ℓ is $(\mu_b/\mu_0 r_b)\mu_0 r_\ell=\mu_b r_\ell/r_b$, which results to ρ_ℓ being $\rho_\ell=\mu_b r_\ell/\mu_\ell r_b$. This pattern is exactly described by definitions (5.33) as well as (6.7), (6.15) and (6.23). The same value for the assymptotic utilization of the CPU could be derived by iteration for the first moment of the flow in the subsystem $(\Sigma_0\Sigma_1\ldots\Sigma_{\ell-1})$, where unit Σ_0 is considered to have a feedback probability equal to the probability of leaving the subsystem $(\Sigma_0\Sigma_1\ldots\Sigma_{\ell-1})$, (i.e. equal to $r_\ell+r_{\ell+1}+\ldots+r_M$), given that the relative bottleneck is known (Σ_b) and for fixed $\rho_b=1$. This last point will be exploited when extending the solution to other network configurations.

At this stage let's describe the proposed approximate MRE solution for a central server model with M I/O units (fig. 5.1) under the variable aggregation scheme. This solution involves M sequential steps, as in Theorem 5.1, and is as follows. Defining:

$$N_\ell = N - \sum_{k=\ell+1}^{M} n_k \quad , \ \ell=1,2,\ldots,M-1$$

Step 1. At the first level of decomposition, the conditional MRE (or ME since the uniform prior is omitted) solution is:

$$P_1(n_0/n_2,\ldots,n_M) = \frac{1}{G^*(1,N_1)}\, x_0^{n_0}\, g_0^{f_{1,1}(n_0)}\, y_0^{f_{1,2}(n_0)} \qquad (6.26)$$

$$n_0=0,1,\ldots,N_1$$

where

$$G^*(1,N_1) = 1 + g_0 \sum_{n_0=1}^{N_1-1} x_0^{n_0} + x_0^{N_1} g_0 y_0 \qquad (6.27)$$

and x_0, g_0, y_0 and ρ_0 are given by (6.6), (6.8), (6.9) and (6.7), respectively. Moreover, the prior $\{G(1,N_2-n_2),\ n_2=0,\ldots,N_2-1\}$ is defined by (6.10)-(6.11), and G(1,0)=1 trivially.

Step 2. At the second level of decomposition, the conditional MRE solution is:

$$P_2(n_2/n_3,\ldots,N_M) = \frac{G(1,N_2-n_2)}{G^*(2,N_2)}\, x_2^{n_2}\, g_2^{f_{2,1}(n_2)}\, y_2^{f_{2,2}(n_2)} \qquad (6.28)$$

$$n_2=0,1,\ldots,N_2$$

where

$$G^*(2,N_2) = G(1,N_2) + g_2 \sum_{n_2=1}^{N_2-1} G(1,N_2-n_2)x_2^{n_2} + x_2^{N_2} g_2 y_2 \qquad (6.29)$$

and x_2, g_2, ρ_2 are given by (6.14), (6.16) and (6.15) respectively, while:

$$y_2 = y_2(N_2) = \frac{1}{\mu_2 x_2^{N_2} g_2}\left[\mu_0 r_2\left[U_0(1,N_2)G(1,N_2) + g_2\sum_{n_2=1}^{N_2-1} U_0(1,N_2-n_2)G(1,N_2-n_2)x_2^{n_2}\right] - \mu_2 g_2 \sum_{n_2=1}^{N_2-1} G(1,N_2-n_2)x_2^{n_2}\right] \qquad (6.30)$$

Moreover, the prior $\{G(2,N_3-n_3),\ n_3=0,\ldots,N_3-1\}$ is defined by (6.18) - (6.19) and G(2,0)=1 trivally.

Step 3 to M-1. At the ℓth level of decomposition, $\ell=3,\ldots,M-1$, the conditional MRE solution is:

$$P_\ell(n_\ell/n_{\ell+1},\ldots,N_M) = \frac{G(\ell-1,N_\ell-n_\ell)}{G^*(\ell,N_\ell)}\, x_\ell^{n_\ell}\, g_\ell^{f_{\ell,1}(n_\ell)}\, y_\ell^{f_{\ell,2}(n_\ell)} \qquad (6.31)$$

$$n_\ell=0,1,\ldots,N_\ell$$

where

$$f_{\ell,1}(n_\ell) = \begin{cases} 0 & \text{if } n_\ell=0 \\ 1 & \text{otherwise} \end{cases}$$

and

$$f_{\ell,2}(n_\ell) = \begin{cases} 1 & \text{if } n_\ell=N_\ell \\ 0 & \text{otherwise} \end{cases}$$

and the normalizing constant is:

$$G^*(\ell,N_\ell) = G(\ell-1,N_\ell) + g_\ell \sum_{n_\ell=1}^{N_\ell-1} G(\ell-1,N_\ell-n_\ell)x_\ell^{n_\ell} + x_\ell^{N_\ell} g_\ell y_\ell \quad (6.32)$$

with

$$x_\ell = \frac{\langle n_\ell\rangle-\rho_\ell}{\langle n_\ell\rangle} \quad (6.33)$$

$$\rho_\ell = \min(1,\rho_{\ell-1}) \frac{\mu_{\ell-1}r_\ell}{\mu_\ell r_{\ell-1}} \quad (6.34)$$

$$g_\ell = \frac{\rho_\ell(1-x_\ell)}{x_\ell(1-\rho_\ell)} \quad (6.35)$$

while

$$y_\ell = y_\ell(N_\ell) = \frac{1}{\mu_\ell x_\ell^{N_\ell} g_\ell} \left[\mu_0 r_\ell \left[U_0(\ell-1,N_\ell)G(\ell-1,N_\ell) + g_\ell \sum_{n_\ell=1}^{N_\ell-1} U_0(\ell-1,N_\ell-n_\ell)G(\ell-1,N_\ell-n_\ell)x_\ell^{n_\ell} \right] - \mu_\ell g_\ell \sum_{n_\ell=1}^{N_\ell-1} G(\ell-1,N_\ell-n_\ell)x_\ell^{n_\ell} \right] \quad (6.36)$$

where $U_0(\ell,N_\ell)$ obeys the recursion:

$$U_0(\ell,N_\ell) = \sum_{n_\ell=0}^{N_\ell} U_0(\ell-1,N_\ell-n_\ell)P_\ell(n_\ell/n_{\ell+1},\ldots,n_M) \quad (6.37)$$

for all $\ell=2,\ldots,M$ and:

$$U_0(1,N_1) = 1 - P_1(0/n_2,\ldots,n_M) \quad (6.38)$$

Clearly, $U_0(M,N)=1-P_0(0)$, with $P_0(0)$ the marginal probability of the CPU being idle. Moreover, the prior $\{G(\ell,N_{\ell+1}-n_{\ell+1}),\ n_{\ell+1}=0,\ldots,N_{\ell+1}\}$

is defined as:

$$G(\ell,N_\ell) = h(\ell,N_\ell)G^*(\ell,N_\ell) \ , \ N_\ell=1,2,... \qquad (6.39)$$

where

$$h(\ell,N_\ell) = \begin{cases} 1 & \text{if } \rho_\ell<1 \\ \dfrac{1}{x_\ell^{N_\ell} g_2 y_2} & \text{if } \rho_\ell>1 \end{cases}, \quad N_\ell=1,2,... \qquad (6.40)$$

Step M. At the Mth level of decomposition, the marginal MRE solution is given by:

$$P_M(N_M) = \frac{G(M-1,N_M-n_M)}{G^*(M,N)} x_M^{n_M} g_M^{f_{M,1}(n_M)} y_M^{f_{M,2}(n_M)} \qquad (6.41)$$

$$n_M=0,1,...,N$$

where x_M, ρ_M, g_M, y_M are given by (6.33)-(6.36) for $\ell=M$.

The above MRE approximate solution for a central server model is a general solution in the sense that no assumption for the distributional form of the service times has been made so far. So any distribution could be used, as long as this distributional commitment allows us to calculate, at every level $\ell=1,2,...,M$, the assymptotic mql $<n_\ell>$ to be used in the evaluation of the multipliers at that level. Thus, this assymptotic mql is in fact the variable via which one can specify the stochastic assumptions about the service and interarrival times of the network. The first moment of the assymptotic flow has been completely specified at every level of decomposition and the values of the assymptotic utilizations $\{\rho_\ell, \ell=0,1,...,M\}$ of the units have been determined. Moreover, the pattern in which these utilizations have been established (see pp. 133-134 and App. IV) gives us a clue on how higher moments of the assymptotic flow can be approximated and used.

In case that the assymptotic mqls are approximated using only the first moments of the interarrival and service times, i.e. $<n_\ell>$, $\ell=0,2,...,M$ are given by:

$$\langle n_\ell \rangle = \frac{\rho_\ell}{1-\rho_\ell} \quad , \forall\ \ell=0,\ldots,M$$

then from (6.50) $x_\ell=\rho_\ell$, while (6.35) and (6.36) yield $g_\ell=y_\ell=1$, and the MRE solution becomes identical to the one described by Theorem 5.1, and which is the exact product form solution as if the network was separable. Moreover, the assymptotic form of the MRE solution at every level when the utilization and fb constraints are not redundant, ($g_\ell, y_\ell \neq 1$), clearly implies a GE form of distribution (see Appendix IV). This is also implied by the constraints assumed and the overall form of the MRE solution. So from now on all service times are assumed to be GE-distributed, while the assymptotic arrival process at every level of decomposition is approximated by a GE renewal process. The second moments of these assymptotic service and interarrival times are approximated as follows.

6.2 The flow approximation.

As mentioned previously, the second moment of the assymptotic flow is calculated following similar steps to the ones used in order to calculate the first moment of this flow. Generally speaking, as mentioned in page 133-134, this calculation involves an iteration at every level of decomposition. So at the ℓth level, given the relative bottleneck of the previous level, this iteration aims in approximating the first and second moments of the flow in the subsystem of the previous level ($\Sigma_0\Sigma_1\ldots\Sigma_{\ell-1}$), given that the bottleneck unit Σ_b, $b\epsilon\{0,1,\ldots,\ell-1\}$, has a constant utilization $\rho_b=1$, and the output flow from this unit is a renewal GE process with squared coefficient of variation (SCV) equal to the SCV of the service time of this unit.

Let's firstly describe the iteration to be used in the central server type of network. Let's name the procedure to be used as

FLOWITERATION1(ℓ), where ℓ stands for the level of decomposition (ℓ=1,2,...,M).

6.2.1 Flow iteration for a central server model under the variable aggregation decomposition scheme.

The notation to be used is as follows.

C_ℓ^2, ℓ=0,1,...,M is the given squared coefficient of variation (SCV) of the service time of unit Σ_ℓ.

$C_{s,\ell}^2$, ℓ=0,1,...,M is the SCV of the service time of unit Σ_ℓ, at the various decomposition levels that this unit is taken under consideration. Note that generally $C_\ell^2 \neq C_{s,\ell}^2$.

$C_{a,\ell}^2$, ℓ=0,1,...,M is the SCV of the assymptotic arrival process at unit Σ_ℓ.

$C_{d,\ell}^2$, ℓ=0,1,...,M is the SCV of the assymptotic departing process from unit Σ_ℓ.

$\lambda_{k,\ell}$, k,ℓ=0,1,...,M is the assymptotic rate of flow that departs unit k and arrives at unit ℓ.

FLOWITERATION1(ℓ) procedure is responsible for calculating the values of ρ_ℓ as well as $C_{a,\ell}^2$ and $C_{s,\ell}^2$, ℓ=0,2,...,M. Given this values the assymptotic mql of unit Σ_ℓ is calculated using the GE/GE/1 mql formulae [KOUV 86a]:

$$<n_\ell> = \frac{\rho_\ell}{2}\left[1 + \frac{C_{a,\ell}^2 + \rho_\ell C_{s,\ell}^2}{1 - \rho_\ell}\right] \qquad (6.42)$$

Substituting (6.42) into (6.33), this later becomes:

$$x_\ell = \frac{C_{a,\ell}^2 + \rho_\ell C_{s,\ell}^2 + \rho_\ell - 1}{C_{a,\ell}^2 + \rho_\ell C_{s,\ell}^2 - \rho_\ell + 1} \qquad (6.43)$$

and similarly (6.35) becomes:

$$g_{\ell} = \frac{2\rho_{\ell}}{C^2_{a,\ell} + \rho_{\ell}C^2_{s,\ell} + \rho_{\ell} - 1} \qquad (6.44)$$

Procedure FLOWITERATION1(ℓ).

First level of aggregation. (ℓ=1).

At this level unit Σ_0 is under consideration, (n.b., symbol "$\leftarrow$" will be used from now on as an assignement operator) as this interacts with I/O unit Σ_1. So firstly from (6.7):

$$\rho_0 \leftarrow \frac{\mu_1}{\mu_0 r_1} \quad , \quad \rho_1 \leftarrow \frac{\mu_0 r_1}{\mu_1} = \frac{1}{\rho_0} \ ,$$

where ρ_1 is the assymptotic utilization of unit Σ_1. Since the service rate of the CPU was taken to be $\mu_0 r_1$, the service time SCV is also modified to be :

$$C^2_{s,0} \leftarrow 1 + r_1[C^2_0 - 1],$$

in order to approximate the fact that only a proportion (r_1) of the CPU's output is taken under consideration here. Recall that the above formula is the splitting (with probability r_1) formula (3.21b). So in fact the CPU is treated as if it had a feedback probability equal to $r_2+r_3+\ldots+r_M = 1-r_1$. In this first level no iteration is needed since without loss of generality unit Σ_1 is assumed to be the bottleneck, and thus:

$$C^2_{a,0} \leftarrow C^2_{s,1} \leftarrow C^2_1$$

The value of ρ_0, however, characterizes the relative bottleneck Σ_b of the first level (this information is useful for the definition of the prior to be used in the next level of aggregation, as well as for the iteration for the assymptotic flow at that level) as:

$$b \leftarrow \begin{cases} 1 & \text{if } \rho_0 \leq 1 \\ 0 & \text{if } \rho_0 > 1 \end{cases}$$

This will be used in the next call of this procedure (for ℓ=2).

<u>ℓth level of aggregation</u> ℓ=2,...,M

Firstly set:

$$\rho_\ell \leftarrow \min(1,\rho_{\ell-1})\frac{\mu_{\ell-1}r_\ell}{\mu_\ell r_{\ell-1}}$$

Here the iteration involves the subsystem $(\Sigma_0\Sigma_1...\Sigma_{\ell-1})$, given that Σ_b, $b\epsilon\{0,1,...,\ell-1\}$, is the relative bottleneck of the previous level. Firstly, in case b=0, no iteration is required, since the CPU (Σ_0) is assymptotically saturated and thus its output is unaffected by the rest of the subsystem $(\Sigma_1...\Sigma_{\ell-1})$. In this case, simply:

$$C^2_{d,0} \leftarrow C^2_{s,0}$$

However, note that since at level ℓ-1 the CPU is examined as it interacts with units $\Sigma_1,...,\Sigma_{\ell-1}$, its service time SCV is given as:

$$C^2_{s,0} = 1 + \sum_{k=1}^{\ell-1} r_k \left[C^2_0 - 1\right]$$

This is justified by the same argument as at the first level of aggregation. Note that according to the above definition of $C^2_{s,0}$, which may be interpreted as a feedback correction, imposed on unit Σ_0, it is reasonable that all transition probabilities r_k, k=1,...,ℓ-1, should be used normalized, i.e. divided by their sum. In case that b=ℓ-1, $(\rho_{\ell-1}>1)$, the utilizations ρ_k, k=0,1,...,ℓ-1, must be readjusted since at the previous level a new bottleneck (unit $\Sigma_{\ell-1}$) was established. In this case and as was described previously (page 134), in this central server network this adjustment involves a new utilization for the CPU (Σ_0) given by:

$$\rho_0 \leftarrow \frac{\mu_b}{\mu_0 r_b}$$

and for each unit Σ_k,

$$\rho_k \leftarrow \frac{\mu_b r_k}{\mu_k r_b} \quad , \; k=1,2,\ldots,\ell-1$$

Note that the above, for $k=\ell-1$, yields $\rho_{\ell-1}=1$. It may be shown that this adjustment is equivalent to dividing all utilizations $\rho_0,\ldots,\rho_{\ell-1}$ by the value of $\rho_{\ell-1}$ that exceeds 1. Finally, in case $b\epsilon\{1,\ldots,\ell-2\}$ the values of the assymptotic utilizations remain unchanged. Having established the appropriate values for ρ_k, $k=0,1,\ldots,\ell-1$, the iteration for the second moment of the flow follows as:

Step 1.

Initialize $C^2_{d,0}$ using (3.20) interdeparture formula as:

$$C^2_{d,0} \leftarrow \rho_0^2 C^2_{s,0} + (1-\rho_0)C_B^2 + \rho_0(1-\rho_0)$$

also, $C^2_{s,b} \leftarrow C^2_{d,b} \leftarrow C_B^2$ and $C^2_{s,k} \leftarrow C_k^2$, $k=1,2,\ldots,\ell-1$

and $$\lambda_{i,0} \leftarrow \rho_i \mu_i \quad , \; i=1,2,\ldots,\ell-1$$

Step 2.

Repeat

For all $k=1,2,\ldots,\ell-1$ do begin

$$C^2_{a,k} \leftarrow 1 + \frac{r_k}{\sum_{i=1}^{\ell-1} r_i}\left[C^2_{d,0} - 1\right]$$

$$C^2_{d,k} \leftarrow \rho_k^2 C^2_{s,k} + (1-\rho_k)C^2_{a,k} + \rho_k(1-\rho_k)$$

end {of loop with index k}

$$C^2_{a,0} \leftarrow \frac{\lambda}{\sum_{i=1}^{\ell-1} \frac{\lambda_{i,0}}{C^2_{d,i}+1}} - 1$$

where

$$\lambda \leftarrow \sum_{i=1}^{\ell-1} \lambda_{i,0}$$

$$C_{d,0}^2 \leftarrow \rho_0^2 C_{s,0}^2 + (1-\rho_0) C_{a,0}^2 + \rho_0(1-\rho_0)$$

Until $C_{d,0}^2$ converges.

$$C_{a,\ell}^2 \leftarrow 1 + \frac{r_\ell}{\sum_{i=1}^{\ell} r_i} [C_{d,0}^2 - 1]$$

Clearly, in step 2 the SCV of the arrival stream at any I/O unit Σ_k, $k=1,2,\ldots,\ell-1$, is approximated by the splitting formula (3.21b), applied on the SCV of the stream that departs the CPU (Σ_0). Then the SCV of the departing stream from the same I/O is approximated using the formula for the interdeparture times from a GE/GE/1 queue (3.20). Note that the bottleneck I/O unit Σ_b is not excluded from this calculation since ρ_b is constant and equal to 1 and thus, the interdeparture formula always yields $C_{d,b}^2 = C_{s,b}^2$, regardless of the SCV of the arriving stream at this unit. Next the SCV of the arriving stream at the CPU (merged stream of all I/O departing streams) is approximated by the merging formula of independent GE streams (3.22b). Finally, the SCV of the departing stream from the CPU is calculated using once more the interdeparture formula (3.20). When the iteration converges, the SCV of the stream that is directed towards unit Σ_ℓ ($C_{a,\ell}^2$) is approximated using the splitting formula (3.21b) on the final value of $C_{d,0}^2$.

The same type of iteration as in step 2 could be used to evaluate the assymptotic utilizations, which instead have been derived analytically. It should be mentioned that the iteration for the second moment of the flow has been also used in the implementation of the ME solution for a general open QNM, [KOUV 85]. The system of equations (with unknowns $C_{d,i}^2$, $i=0,\ldots,\ell-1$) solved in this way is obviously a non-linear one and no guarantee for the convergence of the method is available. However, not a single case of failure has been recorded during years of experimentation involving this type of

iteration.

In this section another, very important, reason for the assymptotic derivation of the two multipliers at every level has been revealed. This way only the assymptotic flow is approximated, which is the flow of an open system. This enables us to use the presented flow formulae for the approximation of the SCVs of the various streams in the network. Such formulae for the approximation of the flow in a closed network are not available and are tediously difficult to derive.

6.3 A MRE decomposition algorithm for central server models based on the variable aggregation scheme.

As mentioned in the previous chapter, (page 104), the notation used here for the conditional probabilities $\{P_\ell(n_\ell/n_{\ell+1},\ldots,n_M)\}$, and which implies a very large state space, is used because it facilitates a better representation of the partition involved. According, however, to the basic decomposition assumption, probability $P_\ell(n_\ell/n_{\ell+1},\ldots,n_M)$ depends only on the sum of its conditional part $(n_{\ell+1}+\ldots+n_M)$, which effectively defines the number of jobs present in the subsystem of the ℓth level. Thus, all distributions:

$$\{P_\ell(n_\ell/n_{\ell+1},\ldots,n_M),\ n_\ell=0,\ldots,N-\sum_{k=\ell+1}^{M} n_k\} \quad ,$$

which have the same sum of conditional parts:

$$N-\sum_{k=\ell+1}^{M} n_k = N_\ell \quad ,$$

are identical, since they belong to the same class of equivalence and may more simply be denoted as:

$$\{P_\ell(n_\ell/N_\ell),\ n_\ell=0,\ldots,N_\ell\},\ N_\ell=0,\ldots,N,\ \ell=1,\ldots,M$$

and $P_\ell(n_\ell/N_\ell)$ is from now on the conditional probability that at level ℓ of aggregation, n_ℓ jobs are present at unit Σ_ℓ, given that N_ℓ jobs circulate in the subsystem of the ℓth level (fig. 2.2). Under this notation, the following algorithm implements the MRE solution presented in this chapter, which is an approximation to the solution of a central server model with GE-distributed service times.

Algorithm 6.1. A MRE decomposition algorithm for central server models with GE-distributed service times.

Input Parameters.

M : number of I/O units $(\Sigma_1,\ldots,\Sigma_M)$.

N : number of jobs.

For unit Σ_i, $i=0,\ldots,M$

μ_i : mean service rate.

C_i^2 : squared coefficient of variation of service time.

For $i=1,\ldots,M$

r_i : transition probability from the CPU (Σ_0) to I/O unit Σ_i.

Step 1. { First level of aggregation }

Step 1.1. { Calculate invariant parameters }

FLOWITERATION1(1)

$$x_0 \leftarrow \frac{C_{a,0}^2 + \rho_0 C_{s,0}^2 + \rho_0 - 1}{C_{a,0}^2 + \rho_0 C_{s,0}^2 - \rho_0 + 1}$$

$$g_0 \leftarrow \frac{2\rho_0}{C_{a,0}^2 + \rho_0 C_{s,0}^2 + \rho_0 - 1}$$

$$y_0 \leftarrow \frac{C_{a,0}^2 + \rho_0 C_{s,0}^2 - \rho_0 + 1}{2}$$

Step 1.2. { Evaluate the conditional distributions }

For N_1=1 to N do begin

Calculate $G^*(1,N_1)$ using (6.27)

$P_1(0/N_1) \leftarrow 1/G^*(1,N_1)$

For n_0=1 , to N_1-1 do

$P_1(n_0/N_1) \leftarrow P_1(0/N_1)g_0x_0^{n_0}$

$P_1(N_1/N_1) \leftarrow P_1(0/N_1)g_0y_0x_0^{N_1}$

end. { of loop with index N_1 }

$P_1(0/0) \leftarrow 1$ and $G^*(1,0) \leftarrow 1$ { trivially }

Step 1.3. { Find the relative bottleneck and evaluate the prior }

Case

$\rho_0 \leq 1$: $b \leftarrow 1$

For N_1=1 to N do

$G(1,N_1) \leftarrow G^*(1,N_1)$

$\rho_0 > 1$: $b \leftarrow 0$

For N_1=1 to N do

$G(1,N_1) \leftarrow G^*(1,N_1)/x_0^{N_1} g_0y_0$

Step 1.4. { Evaluate the conditional utilizations of the CPU }

For N_1=0 to N do

$U_0(1,N_1) \leftarrow 1 - P_1(0/N_1)$

Step 2. { ℓth level of aggregation }

For ℓ=2 to M do begin

Step 2.1. { Evaluate invariant parameters }

FLOWITERATION1(ℓ)

$$x_\ell \leftarrow \frac{C_{a,\ell}^2 + \rho_\ell C_{s,\ell}^2 + \rho_\ell - 1}{C_{a,\ell}^2 + \rho_\ell C_{s;\ell}^2 - \rho_\ell + 1}$$

$$g_\ell \leftarrow \frac{2\rho_\ell}{C_{a,\ell}^2 + \rho_\ell C_{s,\ell}^2 + \rho_\ell - 1}$$

Step 2.2. { Evaluate fb multipliers and conditional probabilities }

For N_ℓ=1 to N do begin

Step 2.2.1. { Apply flow balance correction }

Evaluate $y_\ell = y_\ell(N_\ell)$ using (6.36)

Step 2.2.2. { Evaluate conditional distribution }

Calculate $G^*(\ell,N_\ell)$ using (6.32)

$$P_\ell(0/N_\ell) \leftarrow \frac{G(\ell-1,N_\ell)}{G^*(\ell,N_\ell)}$$

For $n_\ell=1$ to $N_\ell-1$ do

$$P_\ell(n_\ell/N_\ell) \leftarrow \frac{G(\ell-1,N_\ell-n_\ell)}{G^*(\ell,N_\ell)} g_\ell x_\ell^{n_\ell}$$

$$P_\ell(N_\ell/N_\ell) \leftarrow \frac{x_\ell^{N_\ell} g_\ell y_\ell(N_\ell)}{G^*(\ell,N_\ell)}$$

end { of loop with index N_ℓ }

$P_\ell(0/0) \leftarrow 1$ and $G^*(\ell,0) \leftarrow 1$ { trivially }

Step 2.3. { Find the relative bottleneck and evaluate the prior }

Case

$\rho_\ell \leq 1$: For $N_\ell=1$ to N do

$G(\ell,N_\ell) \leftarrow G^*(\ell,N_\ell)$

$\rho_\ell > 1$: $b \leftarrow \ell$

For $N_\ell=1$ to N do

$$G(\ell,N_\ell) \leftarrow G^*(\ell,N_\ell)/x_\ell^{N_\ell} g_\ell y_\ell(N_\ell)$$

Step 2.4. { Evaluate the conditional utilizations of the CPU }

For $N_\ell=0$ to N do

Evaluate $U_0(\ell,N_\ell)$ using (6.37)

end { of loop with index ℓ }

Step 3. { Calculate the marginal distributions }

For $n=0$ to N do begin

$P_M(n) \leftarrow P_M(n/N)$

$P^*_M(n) \leftarrow P_M(N-n/N)$

end

For $\ell=M-1$ downto 2 do begin

For n_ℓ=0 to N do begin

$$P_\ell(n_\ell) \leftarrow \sum_{N_\ell=n_\ell}^{N} P_\ell(n_\ell/N_\ell)P^*_{\ell+1}(N_\ell)$$

$$P^*_\ell(n_\ell) \leftarrow \sum_{N_\ell=n_\ell}^{N} P_\ell(N_\ell-n_\ell/N_\ell)P^*_{\ell+1}(N_\ell)$$

end { of loop with index n_ℓ }

end { of loop with index ℓ }

For n_0=0 to N do

$$P_0(n_0) \leftarrow \sum_{N_1=n_0}^{N} P_1(n_0/N_1)P^*_2(N_1)$$

For n_1=0 to N do

$$P_1(n_1) \leftarrow \sum_{N_1=n_1}^{N} P_1(N_1-n_1/N_1)P^*_2(N_1)$$

In Step 3 of the above, $P_\ell(n_\ell)$ denoted the marginal probability of having n_ℓ jobs at unit Σ_ℓ, while $P^*_\ell(n_\ell)$ was the marginal probability of having n_ℓ jobs present in the subsystem of the ℓth level $(\Sigma_0\Sigma_1\ldots\Sigma_{\ell-1})$.

Having the marginal distributions, all statistics of interest (mqls, utilizations, e.t.c.) may be easily computed. This algorithm may be easily extended to tackle a central server model with feedbacks. It is only required to apply the feedback correction (after the input of the data), to transform (approximately) the network to one without feedbacks. Let's at this point describe the most general case of such a transformation. Consider that the input data for each unit Σ_i of an arbitrary network are, μ^*_i and C^{2*}_i the mean service rate and SCV of the service time. Let also $R^*=\{r^*_{ij}, i,j=0,\ldots,M\}$ be the routing matrix with $r^*_{ii}\neq 0$. Procedure FEEDBACKCORRECTION is as follows:

Procedure FEEDBACKCORRECTION.

For each unit Σ_i, i=0,...,M do begin

$\mu_i \leftarrow (1-r^*_{ii})\mu^*_i$

$C^2_i \leftarrow 1 + (1-r^*_{ii})[C^{2*}_i-1]$

For j=0 to M with j≠i do

$$r_{ij} \leftarrow \frac{r^*_{ij}}{1-r^*_{ii}}$$

$r_{ii} \leftarrow 0$

end

The new parameters $\{\mu_i,\ C^2_i,\ i=0,...,M\}$ and $R=\{r_{ij},\ i,j=0,...,M\}$ describe a network without feedbacks, which is approximately equivalent to the given network. This feedback correction is exact when the network consists of two units (GE/GE/1/N system) and is also exact in case $C^2_i=1$, $\forall$ i=0,...,M, (exponential network) [KOUV 85].

Note now that the form of the MRE solution, described by (6.26), (6.28), (6.31) and (6.41), does not depend in any way on the network configuration. The values of the multipliers involved in this solution, however, are approximated using the conditional and marginal fb constraints. These constraints are clearly responsible for carrying information about the network configuration and in fact in such a way so that a decomposition approach (variable aggregation or Norton's reduction schemes) produces meaningful results, (see discussion in section 5.3.4). Thus, using the same form of MRE solution, let's see now how algorithm 6.1 can be extended to tackle a more general topology of network (amenable to the variable aggregation scheme), which can then be combined with Norton's reduction scheme to produce an alternative way of tackling a central server model, and then also demonstrate how the same form of MRE solution can be used to approximate the solution of a tandem

configuration under a hierarchical application of Norton's reduction method.

6.4 Extension of the MRE solution to more general topologies.

6.4.1 A MRE decomposition algorithm for a type of fully connected networks.

Consider a fully connected network (every unit communicates with every other unit), represented in figure 5.9, with M+1 units enumarated from 0 up to M ($\Sigma_0,\Sigma_1,\ldots,\Sigma_M$). If $R^*=\{r^*_{ij},\ i,j=0,\ldots,M\}$ is the routing matrix of this system, clearly each probability r^*_{ij} does not depend on i, but only on j ($r^*_{ij}=r^*_j$). Consider now the network that results by applying the feedback correction, with routing matrix $R=\{r_{ij},\ i,j=0,\ldots,M\}$. This is a fully connected network without feedbacks, and in fact:

$$r_{ij} = \frac{r^*_j}{1-r^*_i}\ ,\ i,j=0,\ldots,M,\ i\neq j$$

and
$$r_{ii} = 0,\ \forall\ i=0,\ldots,M$$

The conditional fb equations, according to the variable aggregation scheme (fig. 2.3 and fig. 2.2) are as follows. In the first level of decomposition, for $N_1=1,\ldots,N$,

$$\mu_0 r_{01} U_0(1,N_1) = \mu_1 r_{10} U_1(1,N_1) \qquad (6.45)$$

and in the ℓth level of decomposition, $\ell=2,\ldots,M$, for $N_\ell=0,\ldots,N$,

$$\mu_\ell \sum_{k=0}^{\ell-1} r_{\ell k}\ U_\ell(\ell,N_\ell) = \sum_{k=0}^{\ell-1} \mu_k r_{k\ell} U_k(\ell,N_\ell) \qquad (6.46)$$

where $U_k(\ell,N_\ell)$, $k=1,\ldots,M$, $k\leq\ell\leq M$, $N_\ell=1,\ldots,N$, is the conditional utilization for unit Σ_k at the ℓth level of aggregation, given that N_ℓ ($=N-n_{\ell+1}-\ldots-n_M$) jobs circulate at that level, and $U_0(\ell,N_\ell)$ is the

same conditional utilization for unit Σ_0, already defined by (6.37) and (6.38). $U_k(\ell,N_\ell)$, k=2,...,M, obey the recurrence relations:

$$U_k(\ell,N_\ell) = \sum_{n_\ell=0}^{N_\ell} U_k(\ell-1,N_\ell-n_\ell)P_\ell(n_\ell/n_{\ell+1},\ldots,n_M) \ , \ k<\ell \qquad (6.47)$$

and $$U_k(k,N_k) = 1 - P_k(0/n_{k+1},\ldots,n_M) \quad \text{for } k=\ell \qquad (6.48a)$$

while $U_1(\ell,N_\ell)$ obeys (6.47) but not (6.48a). Instead:

$$U_1(1,N_1) = 1 - P_1(N_1/n_2,\ldots,n_M) \qquad (6.48b)$$

with $N_1=N-(n_2+\ldots+n_M)$.

Evidently, the difference with the central server case is that here at the ℓth level of aggregation unit Σ_ℓ communicates with all the units of the subsystem $(\Sigma_0\Sigma_1\ldots\Sigma_{\ell-1})$. So the fb equation (6.46), under the condition that unit Σ_ℓ is not the new relative bottleneck, similarly to the central server case, at the limit ($N \longrightarrow +\infty$) will become:

$$\mu_\ell \sum_{k=0}^{\ell-1} r_{\ell k}\, \rho_\ell = \sum_{k=0}^{\ell-1} \mu_k r_{k\ell} \rho_k \qquad (6.49)$$

with $$\rho_k = \lim_{N\to+\infty} U_k(\ell-1,N_\ell-n_\ell) \ , \ k=0,\ldots,\ell-1$$

So ρ_k, k=0,...,ℓ-1, are the assymptotic utilizations of units $\Sigma_0,\ldots,\Sigma_{\ell-1}$ in case that the relative bottleneck of the previous level (unit Σ_b, $b\epsilon\{0,\ldots,\ell-1\}$) is considered saturated. Applying similar operations to the ones that derived (IV.18), (IV.28), (App. IV) it is revealed that the same pattern is true here. More precisely, in the first level of decomposition, from (6.45) clearly:

$$\rho_0 = \frac{\mu_1 r_{10}}{\mu_0 r_{01}} \ , \quad \rho_1 = \frac{\mu_0 r_{01}}{\mu_1 r_{10}} = \frac{1}{\rho_0}$$

The above value of ρ_0 may be used to evaluate the MRE solution of the

first level (even if $\rho_0>1$). After that, setting:

$$\rho_0 \leftarrow \min(1,\rho_0) \quad \text{and} \quad \rho_1 \leftarrow \min(1,\rho_1)$$

at every level of decomposition $\ell>2$, ρ_ℓ may be computed from (6.49) as:

$$\rho_\ell = \frac{1}{\mu_\ell \sum_{k=0}^{\ell-1} r_{\ell k}} \sum_{k=0}^{\ell-1} \mu_k r_{k\ell} \rho_k$$

as long as ρ_ℓ, $\ell=2,\ldots,\ell-1$, are less or equal to one. If at some level b, however, a utilization ρ_b is greater than one ($\rho_b>1$), then in levels b+1, b+2,... (and as long as $\rho_k \leq 1$, $k=b+1,b+2,\ldots$), utilizations ρ_k, $k \geq b+1$, may be computed by the above, where utilizations $\rho_0,\rho_1,\ldots,\rho_b$ have been modified as:

$$\rho_i \leftarrow \frac{\rho_i}{\rho_b} \quad , \; i=0,\ldots,b$$

At this point the need to extend the notation ρ_k to $\rho_{k,\ell}$ is evident, since at every level of aggregation ℓ all utilizations may be transformed. Let's leave this extension for the next chapter, where the above pattern will be more obviously justifiable. Having defined the assymptotic utilizations an iteration may be applied, similarly to the central server case, in order to approximate the SCV of the assymptotic input flow into unit Σ_ℓ at every level ℓ of decomposition. The notation to be used is the same as in the previous section (page 139), except the introduction of :

$C^2_{i,j}$, $i,j=0,\ldots,M$: the SCV of the stream that departs unit Σ_i and arrives at unit Σ_j.

Let's call the procedure here FLOWITERATION2(ℓ). This can be described as follows.

Procedure FLOWITERATION2(ℓ).

First level of aggregation. (case ℓ=1)

Firstly:

$$\rho_0 \leftarrow \frac{\mu_1 r_{10}}{\mu_0 r_{01}} \quad , \quad \rho_1 \leftarrow \frac{\mu_0 r_{01}}{\mu_1 r_{10}}$$

Similarly to FLOWITERATION1(1), only a proportion of the SCV of the service times of units Σ_0 and Σ_1 is considered here, i.e.:

$$C^2_{s,0} \leftarrow 1 + r_{01}[C_0^2 - 1],$$

and $$C^2_{a,0} \leftarrow C^2_{s,1} \leftarrow 1 + r_{10}[C_1^2 - 1]$$

Note at this point that after using ρ_0 to evaluate the MRE solution of the first level, in case $\rho_0>1$, this utilization has to be set equal to 1 (in the main program), while in case $\rho_0<1$ utilization ρ_1 must be set equal to 1, so that what follows in case ℓ=2 is consistent with case ℓ>2.

ℓth level of aggregation case ℓ=2,3,...,M

Step 1. { Evaluation of ρ_ℓ assuming that variable b has been evaluated in the main program }

Case

b<ℓ-1 : { case $\rho_{\ell-1} \leq 1$, i.e. unit $\Sigma_{\ell-1}$ is not the relative bottleneck of the previous level }

$$\rho_\ell \leftarrow \frac{1}{\mu_\ell \sum_{k=0}^{\ell-1} r_{\ell k}} \sum_{k=0}^{\ell-1} \mu_k r_{k\ell} \rho_k$$

b=ℓ-1 : { case $\rho_{\ell-1}>1$ }

begin

For i=0 to ℓ-1 do

$$\rho_i \leftarrow \frac{\rho_i}{\rho_b}$$

$$\rho_\ell \leftarrow \frac{1}{\mu_\ell \sum_{k=0}^{\ell-1} r_{\ell k}} \sum_{k=0}^{\ell-1} \mu_k r_{k\ell} \rho_k$$

end

Step 2. { Evaluation of $C^2_{a,\ell}$ }

Step 2.1. { Define the SCVs of the service times to be used in the iteration }

For i=0 to ℓ-1 do begin

$$s_i \leftarrow \sum_{k=0}^{\ell-1} r_{ik}$$

$$C^2_{s,i} \leftarrow 1 + s_i \left[C^2_i - 1\right]$$

end

Step 2.2. { Define the rate of flow λ_{ij} between unit Σ_i and Σ_j }

For i=0 to ℓ-1 do

For j=0 to ℓ do

$$\lambda_{ij} \leftarrow \mu_i \rho_i r_{ij}$$

Step 2.3. { Initialize }

For i=0 to ℓ-1 do begin

$$C^2_{a,i} \leftarrow 1 + \frac{r_{bi}}{s_b}\left[C^2_{s,b} - 1\right]$$

$$C^2_{d,i} \leftarrow \rho_i^2 C^2_{s,i} + (1-\rho_i) C^2_{a,i} + \rho_i(1-\rho_i)$$

end

Step 2.4. { Iterate }

Repeat

For i=0 to ℓ-1 do

For j=0 to ℓ-1 do

$$C^2_{i,j} \leftarrow 1 + \frac{r_{ij}}{s_i}\left[C^2_{d,i} - 1\right]$$

For i=0 to ℓ-1 do begin

$$C_{a,i}^2 \leftarrow \frac{\lambda_i}{\sum_{j=0}^{\ell-1} \frac{\lambda_{j,i}}{C_{j,i}^2 + 1}} - 1$$

where

$$\lambda_i \leftarrow \sum_{j=0}^{\ell-1} \lambda_{j,i}$$

$$C_{d,i}^2 \leftarrow \rho_i^2 C_{s,i}^2 + (1-\rho_i) C_{a,i}^2 + \rho_i(1-\rho_i)$$

end

Until $C_{d,i}^2$, $i=0,\ldots,\ell-1$ converge

Step 2.5. { Evaluate $C_{a,\ell}^2$ and $C_{s,\ell}^2$ }

For $i=0$ to $\ell-1$ do

$$C_{i,\ell}^2 \leftarrow 1 + \frac{r_{i\ell}}{s_i+r_{i\ell}} [C_{d,i}^2 - 1]$$

$$C_{a,\ell}^2 \leftarrow \frac{\lambda_\ell}{\sum_{i=0}^{\ell-1} \frac{\lambda_{i,\ell}}{C_{i,\ell}^2 + 1}} - 1$$

where

$$\lambda_\ell \leftarrow \sum_{i=0}^{\ell-1} \lambda_{i,\ell}$$

$$C_{s,\ell}^2 \leftarrow 1 + \sum_{k=0}^{\ell-1} r_{\ell k} [C_k^2 - 1]$$

Having approximated the assymptotic flow, it remains to solve for the fb multiplier $y_\ell = y_\ell(N_\ell)$, $\ell=2,\ldots,M$. This can be done by solving (6.46) with respect to y_ℓ, using (6.47), (6.31) and (6.32). So after some manipulation:

$$y_\ell(N_\ell) = \frac{1}{x_\ell^{N_\ell} g_\ell \mu_\ell \sum_{k=0}^{\ell-1} r_{\ell k}} \left[\sum_{k=0}^{\ell-1} \mu_k r_{k\ell} \left[U_k(\ell-1,N_\ell) G(\ell-1,N_\ell) + \right. \right.$$

$$\left. \left. + g_\ell \sum_{n_\ell=1}^{N_\ell-1} U_k(\ell-1,N_\ell-n_\ell) G(\ell-1,N_\ell-n_\ell) x_\ell^{n_\ell} \right] \right] -$$

$$- \frac{1}{x_\ell^{N_\ell}} \sum_{n_\ell=1}^{N_\ell-1} G(\ell-1, N_\ell-n_\ell) x_\ell^{n_\ell} \qquad (6.50)$$

Now the algorithm that implements the MRE solution for this type of network (fig. 5.9) is as follows.

Algorithm 6.2. A MRE decomposition algorithm for a fully connected type of network configuration.

Input Parameters.

M+1 : number of units $(\Sigma_0, \Sigma_1, \ldots, \Sigma_M)$

N : number of jobs.

For unit Σ_i, i=0,...,M

μ_i^* : mean service rate.

C_i^{2*} : squared coefficient of variation of service time.

For i,j=0,...,M

r_{ij}^* : transition probability from unit Σ_i to unit Σ_j.

Step 1. { feedback correction transforming μ_i^*, C_i^{2*}, r_{ij}^* to μ_i, C_i^2 and r_{ij} }

FEEDBACKCORRECTION

From this point on, the implementation is identical to algorithm 6.1 (steps 1 through 3), except in the following points:

1) Procedure FLOWITERATION2(ℓ) is called in the place of procedure FLOWITERATION1(ℓ).

2) In step 2.2.1 of algorithm 6.1, the multiplier $y_\ell = y_\ell(N_\ell)$ is calculated using (6.50) instead of (6.36).

3) In step 1.4, the additional conditional utilization $U_1(1,N_1)$ of unit Σ_1 is evaluated using (6.48b), while at every level ℓ (ℓ>1) and in step 2.4, additionally to $U_0(\ell,N_\ell)$, all conditional utilizations $U_k(\ell,N_\ell)$, k=1,2,...,ℓ-1, are calculated using (6.47) and also $U_\ell(\ell,N_\ell)$ using (6.48a).

6.4.2 An alternative decomposition approach for central server models.

An alternative algorithm may be proposed now, for the central server model. In fact, it is just needed to extend algorithm 6.2 by a single level, in which the CPU is joined to the I/O subsystem. So consider once more a central server model with a single CPU and M I/O units. Let's change the enumeration of the units here so that the I/O units are units Σ_0, Σ_1,...., Σ_{M-1}, and the CPU is unit Σ_M. Isolating the subsystem $(\Sigma_0\Sigma_1...\Sigma_{M-1})$ by partitioning the state space with respect to the queue length state of the CPU (unit Σ_M), algorithm 6.2 can tackle hierarchically this I/O subsystem (levels 1,2,...,M-1 of aggregation) and at the Mth level the CPU is joined, assuming the same type of constraints, which yield the same form of MRE solution, the only difference from the previous levels being that the fb constraint in this last level is of the form:

$$\mu_M^* U_M(M,N) = \sum_{k=0}^{M-1} U_k(M,N)\mu_k^* \qquad (6.51)$$

where μ_k^*, k=0,1,...,M, are the given service rates of the units. This different fb equation imposes a modification on the procedure FLOWITERATION2(ℓ). More precisely, in order to calculate the first and second moment of the assymptotic flow generated by the I/O subsystem and directed to the CPU, steps 1 and 2.5 of procedure FLOWITERATION2(ℓ) have to be slightly modified. Let's call this new procedure FLOWITERATION3(ℓ). Then this may be described as follows:

Procedure FLOWITERATION3(ℓ).

This procedure is identical to FLOWITERATION2(ℓ) everywhere except steps 1 and 2.5. Firstly, in step 1, the formula for the assymptotic utilization ρ_ℓ is not valid in case ℓ=M. So in this case formula,

$$\rho_M \leftarrow \frac{1}{\mu_M^*} \sum_{k=0}^{M-1} \rho_k \mu_k^*$$

is used, which results by using (6.51) assymptotically. Similarly, in step 2.5 the operations differ when ℓ=M. Thus, step 2.5 may be described as follows:

Step 2.5.

Case

$\ell < M$: Proceed as in step 2.5 of FLOWITERATION2(ℓ).

ℓ=M : begin { Calculate $C_{a,M}^2$ }

For i=0 to M-1 do begin

$\lambda_{i,M} \leftarrow \rho_i \mu_i$

$C_{i,M}^2 \leftarrow C_{d,i}^2$

end

$$C_{a,M}^2 \leftarrow \frac{\lambda_M}{\sum_{i=0}^{M-1} \frac{\lambda_{i,M}}{C_{i,M}^2 + 1}} - 1$$

where

$$\lambda_M \leftarrow \sum_{i=0}^{M-1} \lambda_{i,M}$$

$$C_{s,M}^2 \leftarrow C_M^{2*}$$

Moreover, using (6.51), (6.41), (6.32) and (6.47), the fb multiplier y_M=y_M(N) is given as:

$$y_M = \frac{1}{\mu_M^* x_M^N g_M} \left[\sum_{k=0}^{M-1} \mu_k^* \sum_{n_M=0}^{N-1} U_k(M-1,N-n_M) G(M-1,N-n_M) x_M^{n_M} g_M^{f_{M,1}(n_M)} \right] -$$

$$- \frac{1}{x_M^N} \sum_{n_M=1}^{N-1} G(M-1,N-n_M) x_M^{n_M} \qquad (6.52)$$

Let's now describe this alternative algorithm, using algorithm 6.2 as a procedure with parameters M (number of units) and N

(population), i.e. as it has been described previously it should be denoted as Algorithm 6.2(M,N). Let's also assume that in this central server model the CPU (Σ_M) may have a feedback, which is very common in such a type of configuration.

Algorithm 6.3. Alternative MRE algorithm for central server models.

Input Parameters.

M : number of I/O units, (total number of M+1 units).

N : number of jobs.

For unit Σ_i, i=0,...,M

μ_i^* : mean service rate.

C_i^{2*} : squared coefficient of variation of service time.

For i=0,...,M-1

r_i^* : For i=0,...,M-1 it is the transition probability from the CPU (Σ_M) to I/O unit Σ_i, while for i=M it is the feedback probability of the CPU.

Step 1. { Feedback correction for the CPU }

$\mu_M^* \leftarrow \mu_M^*(1-r_M^*)$

$C_M^{2*} \leftarrow 1 + (1-r_M^*)[C_M^{2*} - 1]$

For i=0 to M-1 do

$r_i^* \leftarrow r_i^*/(1-r_M^*)$

Step 2. { Prepare the routing matrix for the I/O subsystem in isolation and then apply the feedback correction }

For i=0 to M-1 do

For j=0 to M-1 do

$r_{ij}^* \leftarrow r_j^*$

FEEDBACKCORRECTION { for units $\Sigma_0,...,\Sigma_{M-1}$ in isolation }

Step 3. { Solve the I/O subsystem }

Algorithm 6.2(M-1,N)

Step 4. { Mth level of aggregation where the CPU is joined to the I/O subsystem }

Step 4.1. { Evaluate invariant parameters }

FLOWITERATION3(M)

$$x_M \leftarrow \frac{C_{a,M}^2 + \rho_M C_{s,M}^2 + \rho_M - 1}{C_{a,M}^2 + \rho_M C_{s,M}^2 - \rho_M + 1}$$

$$g_M \leftarrow \frac{2\rho_M}{C_{a,M}^2 + \rho_M C_{s,M}^2 + \rho_M - 1}$$

Step 4.2. { Evaluate fb multiplier and marginal probabilities }

Step 4.2.1. { Apply fb correction }

Evaluate $y_M = y_M(N)$ using (6.52)

Step 4.2.2. { Evaluate marginal distribution }

Calculate $G^*(M,N)$ using (6.32)

$$P_M(0) \leftarrow \frac{G(M-1,N)}{G^*(M,N)}.$$

For n_M=1 to N-1 do

$$P_M(n_M) \leftarrow \frac{G(M-1,N-n_M)}{G^*(M,N)} g_M^{n_M} x_M$$

$$P_M(N) \leftarrow \frac{x_M^N \; g_M y_M(N)}{G^*(M,N)}$$

Step 5. { Calculate the marginal distributions }

For n=0 to N do

$P_M^*(n) \leftarrow P_M(N-n)$

For ℓ=M-1 downto 2 do begin

For n_ℓ=0 to N do begin

$$P_\ell(n_\ell) \leftarrow \sum_{N_\ell=n_\ell}^{N} P_\ell(n_\ell/N_\ell) P_{\ell+1}^*(N_\ell)$$

$$P_\ell^*(n_\ell) \leftarrow \sum_{N_\ell=n_\ell}^{N} P_\ell(N_\ell - n_\ell/N_\ell) P_{\ell+1}^*(N_\ell)$$

end { of loop with index n_ℓ }

end { of loop with index ℓ }

For n_0=0 to N do

$$P_0(n_0) \leftarrow \sum_{N_1=n_0}^{N} P_1(n_0/N_1)P_2^*(N_1)$$

For n_1=0 to N do

$$P_1(n_1) \leftarrow \sum_{N_1=n_1}^{N} P_1(N_1-n_1/N_1)P_2^*(N_1)$$

So algorithm 6.3 is a natural extension of algorithm 6.2 and in fact it is nothing else but the implementation of the same form of MRE solution as in algorithm 6.1. In both cases (algorithm 6.1 and 6.3) the same form of information is assumed for all units of the central server model, the difference being on the partition of the state space which results by the different enumeration of the units. All algorithms presented so far, provide the exact solution (regardless of the enumeration of the units) when the network has exponential service times, i.e. in case all SCVs are equal to 1, while for a non-exponential network, the enumeration of the units plays a key role in the results, since a different partition of the state space implies a different approximate solution.

For instance, algorithm 6.1 is based on a hierarchical partition which always involves the CPU at the first level of aggregation and thus, for a central server model with two I/O units (M=2), there are two alternative ways of coupling the network resources, namely $((\Sigma_0\Sigma_1)\Sigma_2)$ or $((\Sigma_0\Sigma_2)\Sigma_1)$, while in case M=3 there are six such different ways. This type of partition was proposed by Courtois [COUR 77], in his variable aggregation scheme. Algorithm 6.3 is also based on a hierarchical partition of the state space, where the CPU is joined always at the last level of aggregation. This also involves different ways of coupling the I/O units, which for a general network yield different results. This type of partition is justified by

Norton's reduction scheme.

. . . So these two decomposition methods merely provide the form of conditional and marginal flow-balance information to be used in our new method so that the resulting MRE solution is marginally flow-balanced, (see also section 5.3.4). Their difference however, from our point of view, lies only in the different physical interpretation and justification of the alternative partitions that they propose. Hence, they seem to be unified under the more general concept of a decomposition scheme based on a hierarchical partition of the state space, which yields the same form of MRE solution for both methods.

Remark 6.1.

Note at this point that the way that the second moment of the assymptotic flow was calculated at every level of decomposition in these three algorithms is not unique. This calculation provides only an approximation for this second moment, hence many alternatives may be proposed and in fact have been tested. Let's for example firstly examine procedure FLOWITERATION3(ℓ), which was used in algorithm 6.3. Note that at the last level of aggregation the related open network of the I/O subsystem (fig. 5.9) is transformed to a network without feedbacks before the iteration is applied. This transformation is exact with respect to the first moment of the flow, while it is an approximate one with respect to the second moment, at the level of a network with GE-distributed service times and with more than two units. According to this feedback transformation a unit with service parameters μ_k^*, C_k^{2*} and feedback probability r_{kk} is substituted by a unit with service parameters μ_k, $C_{s,k}^2$ (as in fig. 6.1) and without a feedback stream.

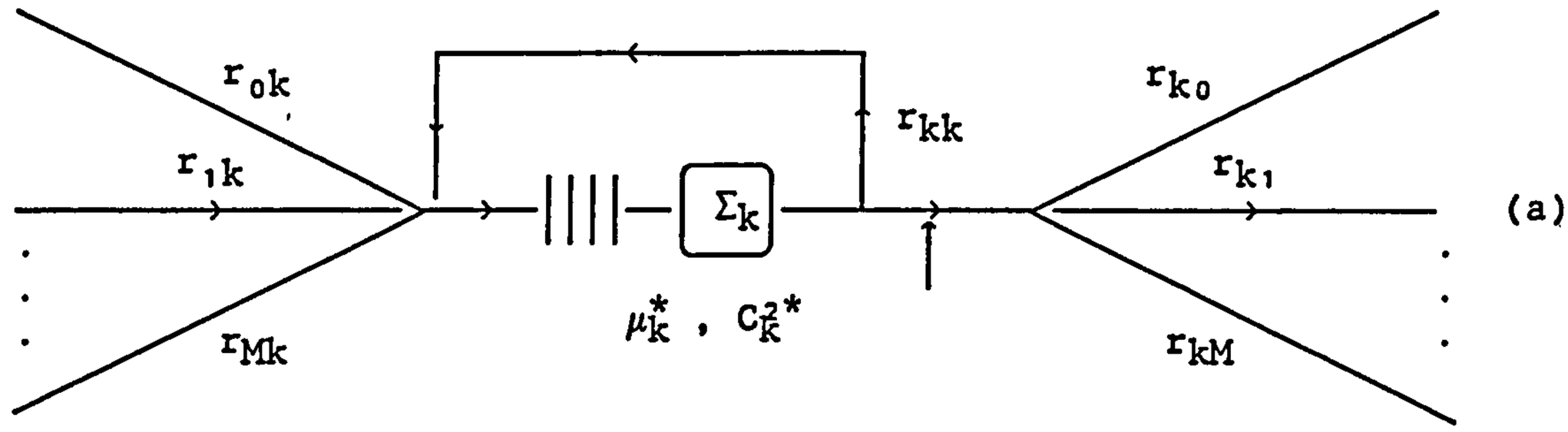

Figure 6.1. Feedback transformation.

In such iterative methods, if one is interested to approximate the second moment of the flow that departs unit Σ_k and is directed towards the rest of the network, (pointed by an arrow in fig. 6.1(a)), it has been experimentally proven that the feedback correction must be firstly applied. An explanation for this is that as a consequence of the correction the feedback stream is eliminated and thus an extra assumption of renewality as well as an extra application of the approximate merging formula (3.22b) is avoided. Then the above second moment is approximated by the SCV of the flow that departs unit Σ_k of fig 6.1b. This way however, one does not obtain any information about the feedback flow, which may be needed in certain cases, like the one described above, where the CPU intercepts all links of the I/O subsystem, even the feedback streams. As FLOWITERATION3(ℓ) was presented, these feedback streams are not

taken under consideration at all and the second moment of the assymptotic flow into the CPU is calculated by the merging of the remaining streams. Hence, in this particular case an improvement of the results should be expected when the iteration in the I/O subsystem is applied without the feedback corrections and in fact tests support this speculation. However, this improvement is not very significant and this is most probably due to the extra assumptions made about the feedback streams.

A similar point may be made for procedure FLOWITERATION1(ℓ) used in algorithm 6.1. Note that there, the service time SCV of the CPU at the ℓth level of aggregation is modified by the splitting formula and this transformation may be seen as a feedback correction with feedback probability the probability of leaving the subsystem ($\Sigma_0\Sigma_1...\Sigma_\ell$) of that level. Since at the next level of aggregation the characteristics of the assymptotic flow that departs unit Σ_0 and is directed towards unit $\Sigma_{\ell+1}$ will be required, one may also consider the feedback stream of the CPU as an individual stream. Note however, that only a proportion ($r_{\ell+1}$) of this feedback stream is directed towards unit $\Sigma_{\ell+1}$. This could be the reason why, as tests showed, FLOWITERATION1(ℓ), as it was presented in this thesis, should be used. More generally, thorough testing of the algorithms presented so far indicates that the accuracy of the MRE solution is not significantly sensitive to such alterations on the calculation of the assymptotic flow parameters.

Let's see next, how the same MRE solution can be used to tackle a tandem configuration, where this assymptotic flow is much more easier to calculate and issues as the above do not raise.

6.4.3 A MRE decomposition algorithm for a tandem type of QNM.

Consider a network with fixed routing (tandem configuration),

with M+1 units ($\Sigma_0,\Sigma_1,\ldots,\Sigma_M$) represented in figure 2.10, and with a fixed level of multiprogramming N. A hierarchical multilevel partition of the state space may decompose the analysis of this network into M levels of aggregation, as shown in figure 2.11, and where at every level the composite server is nothing else but the subsystem of the previous level. The form of the MRE solution of this chapter may be used here as well, the only difference once more being the fb information. From fig. 2.11 clearly the fb constraint of the first level is,

$$\mu_0 U_0(1,N_1) = \mu_1 U_1(1,N_1) \tag{6.53a}$$

while at the ℓth level of decomposition, $\ell=2,3,\ldots,M$,

$$\mu_\ell U_\ell(\ell,N_\ell) = \mu_{\ell-1} U_{\ell-1}(\ell,N_\ell) \tag{6.53b}$$

where $U_k(\ell,N_\ell)$ are the conditional utilizations, defined already by (6.47)-(6.48) and μ_k, $k=0,\ldots,M$ are the given mean service rates. As usual these fb constraints do not affect the form of the MRE solution but the values of the multipliers involved. Note now that because unit ℓ communicates with the subsystem $(\Sigma_0\Sigma_1\ldots\Sigma_{\ell-1})$ only through unit $\Sigma_{\ell-1}$, the approximation of the assymptotic flow parameters and the fb multipliers $y_\ell(N_\ell)$, $\ell=2,\ldots,M$, $N_\ell=0,\ldots,N$, is simplified significantly here. Firstly, (6.53b) may be solved with respect to $y_\ell(N_\ell)$, using (6.47), (6.48a), (6.31) and (6.32) and thus:

$$y_\ell(N_\ell) = \frac{1}{\substack{N_\ell \\ x_\ell \quad g_\ell\mu_\ell}} \left[\mu_{\ell-1} \left[U_{\ell-1}(\ell-1,N_\ell)G(\ell-1,N_\ell) + \right.\right.$$

$$\left.\left. + \; g_\ell \sum_{n_\ell=1}^{N_\ell-1} U_{\ell-1}(\ell-1,N_\ell-n_\ell)G(\ell-1,N_\ell-n_\ell)x_\ell^{n_\ell} \right]\right] -$$

$$- \frac{1}{x_\ell^{N_\ell}} \sum_{n_\ell=1}^{N_\ell-1} G(\ell-1, N_\ell-n_\ell) x_\ell^{n_\ell} \qquad (6.54)$$

Note that in the above only the conditional utilizations of the unit joined at the previous level are needed and so recursion (6.47) may be avoided. Now with respect to the assymptotic parameters, from (6.53a), clearly:

$$\rho_0 = \frac{\mu_1}{\mu_0} \quad , \quad \rho_1 = \frac{\mu_0}{\mu_1} = \frac{1}{\rho_0}$$

and obviously, since the mean service rates of units Σ_0, Σ_1 have not been modified, the SCVs of the service times should also remain as given. Then at the second level of aggregation and since unit Σ_2 communicates only with unit Σ_1, the assymptotic mean arrival rate at the former is the output rate of the later as this is utilized assymptotically at the previous level, i.e. in subsystem $(\Sigma_0\Sigma_1)$. Thus, if the relative bottleneck at the previous level is unit Σ_1 (b=1) the arrival rate at unit Σ_2 is μ_1, while if b=0 then this rate is $\rho_1\mu_1$. Similarly, at the ℓth level of aggregation:

$$\rho_\ell = \frac{\mu_{\ell-1}\min(1,\rho_{\ell-1})}{\mu_\ell} \quad , \ \ell=2,\ldots,M$$

The SCV of the assymptotic arrival stream at unit Σ_ℓ may be approximated using the same thought, i.e. if b=ℓ-1 then $C^2_{\bar{a},\ell} = C^2_{\ell-1}$, while if b$\neq\ell$-1 (b$\epsilon\{0,1,\ldots,\ell-2\}$) then simply,

$$C^2_{\bar{a},b+1} = C^2_{\bar{d},b} = C^2_b$$

and for k=b+2,....,ℓ-1

$$C^2_{\bar{a},k+1} = C^2_{\bar{d},k} = \rho_k^2 C_k^2 + (1-\rho_k) C^2_{\bar{a},k} + \rho_k(1-\rho_k)$$

So no iteration is required in this case. Then procedure FLOWITERATION4(ℓ) may be described as follows.

Procedure FLOWITERATION4(ℓ).

In case ℓ=1 begin

$$\rho_0 \leftarrow \frac{\mu_1}{\mu_0} \quad , \quad \rho_1 \leftarrow \frac{\mu_0}{\mu_1}$$

$$C^2_{a,0} \leftarrow C^2_{s,1} \leftarrow C^2_1$$

$$C^2_{s,0} \leftarrow C^2_0$$

end

In case ℓ>1 begin

$$\rho_\ell \leftarrow \frac{\mu_{\ell-1}\min(1,\rho_{\ell-1})}{\mu_\ell}$$

$$C^2_{a,b+1} \leftarrow C^2_{d,b} \leftarrow C^2_b$$

For k=b+1 to 1-1 do

$$C^2_{a,k+1} \leftarrow C^2_{d,k} \leftarrow \rho_k^2 C^2_k + (1-\rho_k)C^2_{a,k} + \rho_k(1-\rho_k)$$

$$C^2_{s,\ell} \leftarrow C^2_\ell$$

end

Let's call the algorithm that implements the MRE solution for this type of network algorithm 6.4. Then this is the same as algorithm 6.1 except in the following points:

Algorithm 6.4. { As algorithm 6.1 except: }

1) In the input parameters section, no transition probabilities are required.

2) In place of FLOWITERATION1(ℓ), procedure FLOWITERATION4(ℓ) should be called.

3) Step 1.4 should be as follows:

For N_1=0 to N do

$U_1(1,N_1) \leftarrow 1 - P_1(N_1/N_1)$

4) Step 2.2.1 should be as follows:

Evaluate $y_\ell = y_\ell(N_\ell)$ using (6.54)

5) Step 2.4 should be as follows:

For N_ℓ=0 to N do

$$U_\ell(\ell,N_\ell) \leftarrow 1 - P_\ell(0/N_\ell)$$

6.5 Validation of MRE decomposition algorithms.

The validation of these algorithms was conducted using once more Almond's, [ALMO 88], testing algorithm, which has been described in section 3.3.3. The error tolerances characterize the accuracy and reliability of the MRE decomposition algorithms at the level of a network with GE-distributed service times. The results from two alternative approximations are included in the tables presented in this thesis. The first approximation is the solution of the network as if this was separable (denoted as EXP), which may be obtained by any of the entropy algorithms when all SCVs are considered equal to 1. The results from this algorithm merely indicate the large differences occuring when variability is present but ignored. The second algorithm is the very successful universal maximum entropy algorithm, [KOUV 86c], (denoted as UME), whose validation, [ALMO 88], has shown that it is the best approximate algorithm, for a network with GE-distributed service times, out of a number of available approximations, [CHAN 75b, SEVC 77, SHUM 77, COUR 77, TRIP 79, MARI 79, ZAHO 83, AGRA 84, WHIT 84].

At this point another issue should be briefly addressed, and that concerns the types of QNMs that can be modelled using a GE form of distribution, with respect to the values of the service times SCVs involved. Evidently, the GE distribution becomes improper when its SCV is less than 1 (see also page 42 of the third chapter). This means that the distribution does not represent a proper service facility in such a case. The use, however, of improper types of distributions to model service times with SCVs less than one is not new. Sauer [SAUE 75b] and Kouvatsos, [KOUV 86c] have used such types

of distributions (improper H_2 and GE respectively), as heuristic approximations, successfully in the past. In the case of GE, the exact solution of a queueing system (even network), which involves service times SCVs less than one, is the solution of the same form of system of global balance equations used for a proper network. What of course this solution lacks is a phisical representation of the improper servers. Note also that for such improper coefficients restrictions should be imposed because, as it can be seen in formulae (6.43) and (6.44), the denominators of the invariant multipliers are not guaranteed to be non-zero, in which case the algorithms that implement the MRE solution may crash. The same failure may occur in the derivation of the exact solution. Because the likelihood of this failure increases as the coefficients decrease, [ALMO 88], in the test that follow the values for the SCVs of the service times will be greater or equal to 0.5. Otherwise, these improper cases will not receive any special attention, since, as the results show, there is not any particular difference between such proper and improper networks as far as the behaviour of the algorithms is concerned. More details on this issue can be found in [KOUV 86a, KOUV 86c]. Tests of the MRE decomposition algorithms showed them to be very powerful means of approximating network statistics of interest like mean queue lengths, utilizations, response times, e.t.c.. The first problem that these algorithms face is that they do not provide always acceptable marginal queue length distributions. This is due to the fact that in some cases the fb correction attempted through the fb multipliers and after the derivation of the invariant multipliers, at every level of decomposition, fails in the sense that these fb multipliers, as it can be seen from relations (6.36), (6.50), (6.52) and (6.54), are not guaranteed to be positive. This problem is not associated to the existence of a proper MRE solution under the constraints assumed,

since this existence is guaranteed when information is not contradictory. Note however, that the evaluation of the multipliers involved in this solution is done in a clearly approximate fashion and this, to our view, is the source of the problem. This failure in flow-balancing the system results in negative values for certain conditional and marginal probabilities, i.e. at level $\ell \geq 2$ and with population N_ℓ the probability $P_\ell(N_\ell/N_\ell)$ may be affected. In most of the cases, however, the absolute values of these negative probabilities are very close to zero and thus they do not affect severely the calculation of the marginal statistics. The frequency of occurance of this failure varies with respect to the network configuration and most importantly with respect to the way of coupling the units of the network. In fact it has been recorded that in tandem configurations (algorithm 6.4) this failure phenomenon is very rare, which indicates that the invariability assumption concerning the mql and utilization multipliers is not in general the source of error. This frequency increases in central server type of networks. The difference in the way of tackling these two types of networks is that in the former case no iteration is involved in the approximation of the assymptotic flow and only the formula for the SCV of the interdeparture times from a GE/GE/1 queue is used, while in the later case two more approximate formulae are used. Recall that the interdeparture and splitting formulae are exact and wherever they are used the only assumption made is that the streams involved are renewal GE-distributed ones, while the merging formula is only an approximation even under the assumption of independent renewal GE-distributed merging streams.

Another important point, that has been recorded, is that regardless of the network configuration, there is always a way of coupling the units of the network so that this failure can be

avoided. This claim is based on extensive experimentation and it is difficult to prove, because of the complexity of the solution and the expression for the fb multipliers involved. Strangely enough, the above behaviour cannot be used in determining which is the best way of coupling the units of the network, since in some cases the results obtained from an analysis, which yields a negative probability (of small magnitude), are better than the ones obtained by a different enumeration of the units, which avoids the negative fb multiplier.

Let's put aside this issue at this point (the behaviour of the algorithms with respect to this problem will be reported in every individual test) and let's concern ourselves with the most important problem that approximate decomposition methods face, which is the different results obtained under different orderings of the network units.

6.5.1 The problem of ordering the units when an approximate hierarchical decomposition method is used.

As mentioned previously (page 161), when a decomposition method, based on a hierarchical multilevel partition of the state space, is used to analyze approximately a QNM, the resulting solution is not unique, but depends on the order that the units of the network are joined together and thus effectively on the enumeration of the units. The question to be answered is whether there are some criteria to help us decide on which is the best enumeration out of all possible ones. A proper investigation, based on an error analysis, faces the large complexity of these models at the level of the matrix of transition rates between states of the system, which is quite difficult even to describe analytically, when a bulk distribution like GE is used. Efforts have been made to determine heuristic rules that would point out the best results in each case, but this has been

proven to be difficult. Experience gained from exhaustive experimentation allows us, however, to use the MRE decomposition algorithms in a way that guarantees always good and reliable results. To this end, the algorithm developed by Almond, [ALMO 88], proved to be a very helpful tool, since the experimentation was not restricted to a small number of networks, instead valid conclusions may be drawn for each set of network parameters because of the large sample of networks (usually 200) solved for each network population. In this chapter only two types of networks will be considered, i.e. the tandem and central server configurations. Let's proceed now with the validation of algorithm 6.4.

6.5.2 QNMs with tandem type of configuration.

For such networks algorithm 6.4 is used. Note that for a network with four units (M=3) there are twelve different ways of coupling them (let's call them sequences 1 to 12 from now on), namely, $\{(((\Sigma_0\Sigma_1)\Sigma_2)\Sigma_3)$, $(((\Sigma_0\Sigma_1)\Sigma_3)\Sigma_2)$, $(((\Sigma_0\Sigma_2)\Sigma_1)\Sigma_3)$, $(((\Sigma_0\Sigma_2)\Sigma_3)\Sigma_1)$, $(((\Sigma_0\Sigma_3)\Sigma_1)\Sigma_2)$, $(((\Sigma_0\Sigma_3)\Sigma_2)\Sigma_1)$, $(((\Sigma_1\Sigma_2)\Sigma_0)\Sigma_3)$, $(((\Sigma_1\Sigma_2)\Sigma_3)\Sigma_0)$, $(((\Sigma_1\Sigma_3)\Sigma_0)\Sigma_2)$, $(((\Sigma_1\Sigma_3)\Sigma_2)\Sigma_0)$, $(((\Sigma_2\Sigma_3)\Sigma_0)\Sigma_1)$, $(((\Sigma_2\Sigma_3)\Sigma_1)\Sigma_0)\}$, where for example $(((\Sigma_0\Sigma_1)\Sigma_2)\Sigma_3)$ implies that at the first level of aggregation, units Σ_0 and Σ_1 are taken under consideration, while at the second and third levels units Σ_2 and Σ_3 are joined, respectively. Let's from now on denote this sequence as SEQ(0,1,2,3). Algorithm 6.4 as described here can tackle only four sequences, namely SEQ(0,1,2,3), SEQ(1,2,3,0), SEQ(2,3,0,1) and SEQ(3,0,1,2). It is not difficult, however, to extend it so that it can tackle the rest of them.

Comparisons of the MRE solution against the exact one showed that this approximation is very powerful and favours service times with high variability. It was evident, from the initial stages of this

investigation, that when units of the network have different SCVs then the rule "join the unit with the high variability last", (already mentioned in section 3.3.4), should be followed, i.e. the units should be ordered with increasing variability. In fact the larger the difference between the SCVs of the network, the stronger the rule becomes. In case that some (or all) units have the same SCVs, in this type of tandem configuration, then the rule "join the faster unit first" should be followed. These two rules will from now on determine the proposed order of coupling, whose results appear first in the associated tables 6.1-6.10 (Appendix IV). Additionally to this proposed sequence, the results obtained out of ordering the units with decreasing mean service rates are presented and denoted as SEQ(msr). Note that the proposed sequence is not claimed to be the best out of all possible, but it may be characterized as a good and reliable approximation. It should be mentioned also that results obtained out of all different sequences do not differ widely. The best sequence in every test, provides very good results, which in many cases indicate outstanding accuracy (less than 1% absolute relative error in utilizations and mean queue lengths). Usually, some of the sequences are quite close to the best one, providing good and reliable approximations, while some of them may be characterized as good (less than 0.05 average error tolerances) but unreliable (maximum error tolerances, usually in the interval [0.05, 0.25]). The proposed according to the above rules sequence proves to be consistently the best or close to the best one.

In tables 6.1-6.3 (Appendix IV), the proposed sequence is SEQ(msr). It can be seen that the results vary from being very good (table 6.1) at low variability networks to outstanding (table 6.3) when variability is large. This improvement, as variability increases, can also be seen in tables 6.4-6.10. Very good results are

also obtained when a unit with high variability is joined to units with low variability (table 6.10). In fact this type of combination has been proven to be the weak point of the universal maximum entropy algorithm (UME), which, as it can be seen in table 6.10, appears to be unreliable. In general the MRE decomposition algorithm 6.4 is comparable to the UME one and thus the best available approximation for tandem configurations of networks with GE-distributed service times. Note that not a single case of flow-balance failure has been recorded in the proposed sequences presented in these tables (a total of 2000 networks) as well as in all tests carried out (over 15000 networks) for such type of networks. Finally, it should be mentioned that in case units of the network have feedback streams, then the ordering of the units should be made posterior to the feedback transformation of the network.

6.5.3 QNMs with central server type of configuration.

For this type of networks, either algorithm 6.1 or algorithm 6.3 is used, once more depending on the order of coupling. Note firstly that here, due to the fact that the two algorithms are based on the two existing decomposition schemes, not all possible sequences produce meaningful results. Thus, the only sequences to be examined are those where the CPU (unit Σ_0) is joined at the first or at the last level of decomposition. This restricts our attention, for example in a network with 3 I/O units, to the following nine (instead of twelve) sequences {SEQ(0,1,2,3), SEQ(0,1,3,2), SEQ(0,2,1,3), SEQ(0,2,3,1), SEQ(0,3,1,2), SEQ(0,3,2,1), SEQ(1,2,3,0), SEQ(1,3,2,0), SEQ(2,3,1,0)}. This implies that it is not always possible to order the units with increasing variability, which is once more a valid rule aiming at a good and reliable approximation when SCVs differ.

The routing of such a system is not fixed but random, thus a

different criterion has been tested successfully in case that all SCVs are equal. This criterion is in fact the minimization of the upper bound ω_ℓ to the maximum degree of coupling at level ℓ of aggregation, (ℓ=1,2,...,M-1), that appears on the left hand side of (2.6) and is defined by (2.5). Quantity ω_ℓ may be interpreted as the maximum rate at which the subsystem of the ℓth level of aggregation, (units $\Sigma_0,\Sigma_1,\ldots,\Sigma_\ell$), interacts with the rest of the system, (units $\Sigma_{\ell+1},\Sigma_{\ell+2},\ldots,\Sigma_M$). For a central server model with N≥M+1, quantity ω_ℓ is simply given as:

$$\omega_\ell = \mu_0 \sum_{m=\ell+1}^{M} r_m + \sum_{k=\ell+1}^{M} \mu_k \quad , \ \ell=1,\ldots,M-1$$

It is not difficult to see that minimizing the above, at every level of decomposition, is equivalent to ordering the I/O units with decreasing values for quantities $\{\mu_0 r_k + \mu_k,\ k=1,2,\ldots,M\}$. Thus, SEQ(msr), in the tables associated to the central server type of configuration, denotes the sequence which is defined by assigning to each I/O unit k, k=1,2,...,M, the quantity $\mu_0 r_k+\mu_k$ and then reenumerating them so that these values are in decreasing order. In tables 6.11-6.20 (Appendix IV), only the proposed sequence appears and the results characterize the MRE algorithms for this type of networks as good and reliable approximations. Note that in all tests the CPU was considered to have a feedback probability, denoted as r_0, and the results are once more comparable to the ones obtained by the UME algorithm, except in certain cases where low variability units are combined with high variability ones (tables 6.15, 6.19), where the MRE decomposition solution seems to perform noticeably better.

The frequency at which the flow-balance failures appear is 17.8% in the proposed sequences and out of 2000 networks used in tables 6.11-6.20. This significant increase over the tandem configuration,

however, does not seem to affect the resulting statistics significantly. It is also worth reporting that with respect to this frequency there has been observed a large difference between algorithms 6.1 and 6.3. It seems that when the CPU is joined last, (algorithm 6.3), very few negative fb multipliers occur, (in these tests, 10.1% in general, 2.3% when this sequence is proposed), as opposed to the case where the CPU is always present at the first level of decomposition.

6.6 Discussion.

In this chapter the form of the MRE solution, subject to fully decomposable mean queue length, utilization and flow-balance constraints, was adopted and used, in conjuction with assymptotic connections to related infinite capacity systems, to approximate the equilibrium marginal queue length distributions of a queueing network, which then can be used to obtain statistics of interest.

The algorithms implemented for tandem and central server types of network configurations prove to be very good and reliable tools at the level of a network with GE-distributed service times, while they provide the exact product form solution when the network is separable. The reliability of these algorithms very much depends, however, on the order of coupling the units of the network. Extensive experimentation has shown us the way to use these algorithms in a very good, if not the best, and reliable fashion. To this end it has been observed that variability plays the first role, irrespective of the network configuration. The rule "join the unit with the high variability last" is in intuitive agreement with the principles of near-complete decomposability, since servers with high variability are rather unreliable and their interaction rates with the rest of the network are certainly negatively affected by this. Recall that

near-complete decomposability suggests that units, which interact more strongly, should be joined at lower levels of decomposition. Near-complete decomposability, however, cannot explain why similar behaviour is observed in tandem configurations, where units proposed to be joined at the first level of decomposition for example, may not communicate directly in the actual network and thus local equilibrium cannot be clearly justified.

This new method is based on a multilevel partition of the network's state space and because of that its applicability at this point is restricted to all network configurations that are amenable to the variable aggregation scheme and to some configurations amenable to a hierarchical application of Norton's reduction scheme. Unfortunately, the variable aggregation scheme, as was presented by Courtois, [COUR 77], can tackle a very small variety of network topologies, the most general being, to our knowledge, the particular type of fully connected network, where units are connected in parallel, and which is represented in figure 5.9. This is due to the fact that in order to apply the ideas of near-complete decomposability and local equilibrium, Courtois required that a unit is joined at a level of decomposition ℓ should communicate directly with the subsystem of that level (units $\Sigma_0,\Sigma_1,\ldots,\Sigma_{\ell-1}$). To this end, the routing probabilities are unaffected, and at every level the service time characteristics of the units are altered. These alterations effectively aim to produce a marginally flow-balanced system by satisfying at every level appropriate conditional flow balance equations. The most general form of such equations presented so far is given by relations (6.45) and (6.46). Multiplying both sides of (6.46) by $P^*_\ell(N_\ell)$, which denotes the unconditional probability of having N_ℓ jobs present at the subsystem $(\Sigma_0\Sigma_1\ldots\Sigma_{\ell-1})$ of the ℓth level, and summing over all $N_\ell=0,\ldots,N$, it follows that:

$$\mu_{\ell} \sum_{k=0}^{\ell-1} r_{\ell k} U_{\ell}(M,N) = \sum_{k=0}^{\ell-1} \mu_k r_{k\ell} U_k(M,N) \qquad (6.55)$$

for $\ell=2,3,\ldots,M$, and performing a similar operation, (6.45) yields:

$$\mu_0 r_{01} U_0(M,N) = \mu_1 r_{10} U_1(M,N) \qquad (6.56)$$

So the conditional fb equations are also satisfied by the marginal resulting utilizations, regardless of the actual values of $P^*_{\ell}(N_{\ell})$, $N_{\ell}=0,\ldots,N$. However, in order the system to be marginally flow-balanced these marginal utilizations should satisfy, for every $\ell=0,1,\ldots,M$, equation,

$$\mu_{\ell} U_{\ell}(M,N) = \sum_{\substack{k=0 \\ k\neq\ell}}^{M} \mu_k r_{k\ell} U_k(M,N) \qquad (6.57)$$

So the problem is clear. Simply, the conditional fb equations do not generally guarantee an overall flow-balanced system. In fact, only the last unit Σ_M is always flow-balanced.

In the following chapter, the generalization of the MRE decomposition approach to an arbitrary network configuration will be described, using the concept of subparallelism. This concept was defined by Vantilborgh, [VANT 78], in his effort to introduce a different definition of every level of aggregation, aiming at extending the exactness of the decomposition solution, described by Courtois [COUR 77], at the level of the exponential network, to an arbitrary network configuration.

CHAPTER VII

MRE HIERARCHICAL DECOMPOSITION OF GENERAL QNMs WITH ARBITRARY CONFIGURATION

7.1 The concept of subparallelism.

Vantilborgh [VANT 78], examined the conditions under which aggregation yields exact results. To this end his work was naturally restricted to exponential networks. His contribution, however, in the extention of the applicability of the variable aggregation scheme to arbitrary network configurations was, to our view, far more important. His starting point was the material presented by Courtois [COUR 77, chapters IV and V] on the analysis of exponential networks by aggregation. Part of this material is presented in the second chapter of this thesis, and the algorithm proposed by Courtois is algorithm 2.1 under the assumption of exponentially distributed service times (i.e. for G=M).

As it has been pointed out so far, at the ℓth level of aggregation, $\ell=1,2,\ldots,M$, the definitions of the rates at which unit Σ_ℓ interacts with the subsystem of that level ($\Sigma_0\Sigma_1\ldots\Sigma_{\ell-1}$) (see fig. 2.2 and relations (2.10)-(2.11)) are not generally adequate to provide the exact results in the case of an exponential network. It has been demonstrated, from the viewpoint of our new method and the flow-balance constraints used which are the same flow-balance constraints satisfied in the conventional implementation of the variable aggregation scheme, that this fb information does not guarantee a marginally flow-balanced system. In order to overcome this problem, Vantilborgh had to redefine the queueing system of each level of decomposition as presented in figure 7.1 and denote it as $M_\ell(N_\ell,R(\ell))$. The difference between this system and the one proposed

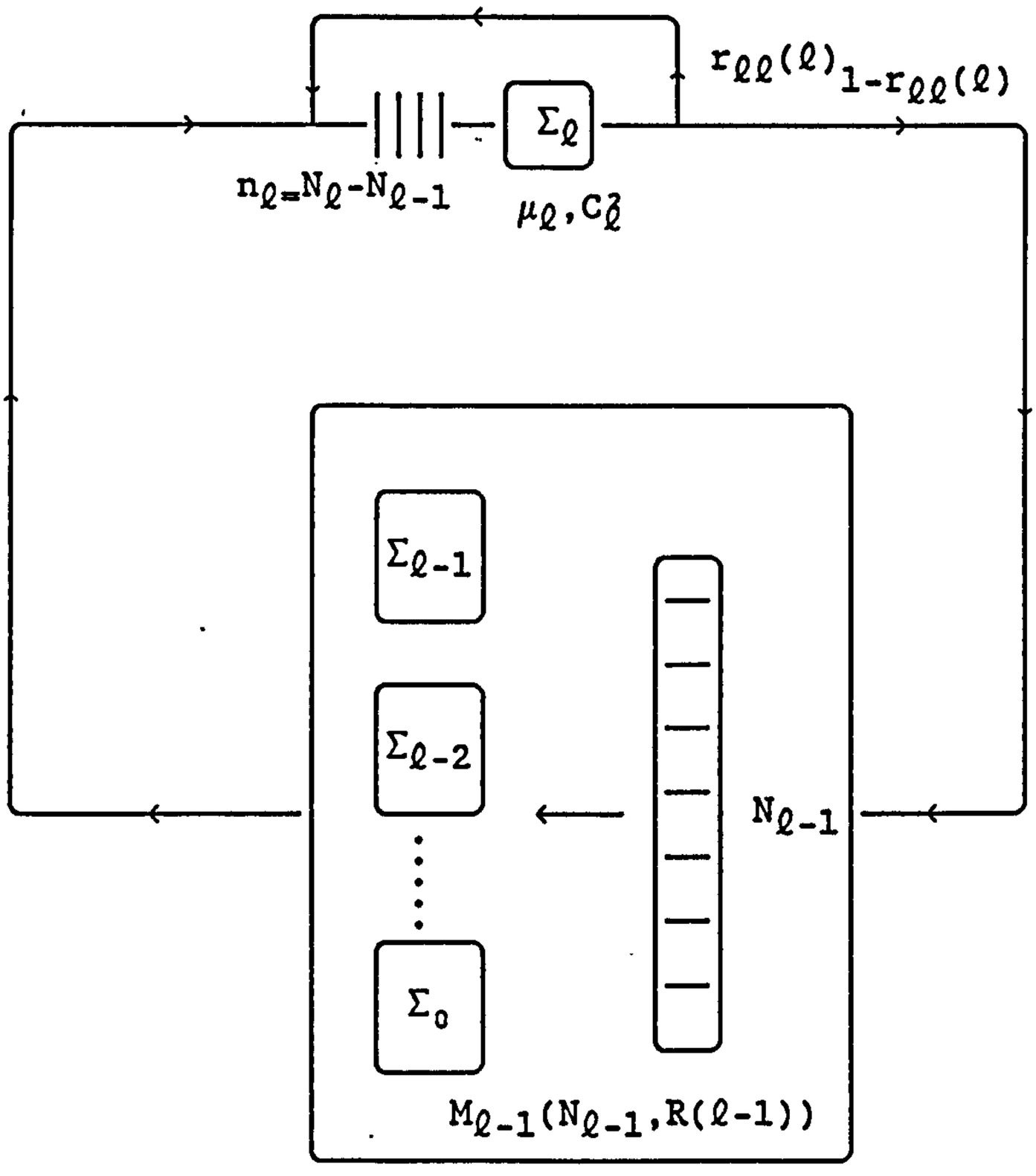

Figure 7.1. Queueing system $M_\ell(N_\ell,R(\ell))$.

by Courtois (fig. 2.2) is that in this case all units $\Sigma_0,\Sigma_1,\ldots,\Sigma_\ell$ have their original service times characteristics. What uniquely identifies this ℓth level is a corresponding routing matrix $R(\ell)$. This matrix (dependent on the level ℓ of aggregation) describes the configuration of the subsystem of figure 7.1, or in other words how units $\Sigma_0,\Sigma_1,\ldots,\Sigma_\ell$ are connected in this level. So the stochastic square matrix $R(\ell)$ is of order $(\ell+1)$ and its entry $r_{ij}(\ell)$ is the probability that a customer, upon service completion at Σ_i, applies immediately to Σ_j, $i,j=0,1,\ldots,\ell$. Clearly, $R(M)\triangleq R$ is the original routing matrix of the network. The load-dependent rates of the composite server in figure 7.1 have been redefined by Vantilborgh [VANT 78], so that according to our notation, at every level of aggregation the conditional flow-balance equation to be satisfied is given as:

$$U_\ell(\ell,N_\ell)\mu_\ell[1-r_{\ell\ell}(\ell)] = \sum_{k=0}^{\ell-1} \mu_k r_{k\ell}(\ell) U_k(\ell,N_\ell) \quad , \qquad (7.1)$$

$\ell=1,\ldots,M$, $N_\ell=0,\ldots,N$, where the conditional utilizations $U_k(\ell,N_\ell)$, $k=0,1,\ldots,\ell$, are defined by (6.37), (6.38), (6.47) and (6.48).

The concept of subparallelism has been introduced by Vantilborgh, [VANT 78], and may be described as follows: Consider a sequence of matrices $\{R(1), R(2),\ldots, R(M)\}$, where $R(\ell)$, $\ell=1,\ldots,M$, is defined as above and their corresponding left Perron-Frobenius (P-F) eigenvectors $\{t(1), t(2),\ldots, t(M)\}$. Each of these eigenvectors is the solution of the matrix equation:

$$t(\ell) = t(\ell)R(\ell) \quad , \ \ell=1,\ldots,M \qquad (7.2)$$

Consider also a set of functions $\{\varphi_{km},\ k=2,..,M,\ m<k\}$ so that:

$$\varphi_{km} : R^{k+1} \longrightarrow R^{m+1} : \varphi_{km}(t(k)) \triangleq \varphi_{km}((t_0(k),t_1(k),\ldots,t_k(k))) = (t_0(k),t_1(k),\ldots,t_m(k)) \qquad (7.3)$$

where $t(k) = (t_0(k),t_1(k),\ldots,t_k(k))$, $k=1,2,\ldots,M$

<u>Definition 7.1.</u> Two vectors $t(k)$, $t(m)$, $m<k$, are called subparallel when:

$$\varphi_{km}(t(k)) = \nu t(m) \qquad (7.4)$$

for some constant $\nu \epsilon R$, or in other words, when the first $m+1$ coordinates of $t(k)$ are proportional to the coordinates of $t(m)$.

<u>Definition 7.2.</u> The sequence of matrices $\{R(1), R(2),\ldots, R(M)\}$ is said to satisfy the subparallelism condition if every two P-F eigenvectors $t(k)$, $t(m)$, $k=2,\ldots,M$, $m<k$, are subparallel.

According to Vantilborgh, the necessary and sufficient condition for the variable aggregation scheme to yield exact results for an exponential network, is that the routing matrices $\{R(1), R(2),\ldots, R(M)\}$ that correspond to the M levels of aggregation should satisfy

the subparallelism condition [VANT 78].

The problem that the above condition overcomes, is actually the problem of marginally flow balancing the network, which was discussed in sections 5.3.4. and 6.6. and which has more general implications than merely the exactness of the variable aggregation scheme in exponential networks. This condition seems to clear the way for methods that are based on a hierarchical partition of the network's state space, so that they can tackle arbitrary network configurations. Let's see this more closely. Our claim is that if the conditional fb equations, to be used at every level of decomposition in order to define the values of the fb multipliers involved in the MRE solution, are given by (7.1) and the sequence of the matrices $\{R(1),R(2),...,R(M)\}$ satisfies the subparallelism condition, then the resulting marginal distributions imply a marginally flow balanced system. Let's define at this point as $t(\ell,N_\ell) = (t_0(\ell,N_\ell),t_1(\ell,N_\ell),...,t_\ell(\ell,N_\ell))$ the vector of conditional throughputs at level ℓ of aggregation given that N_ℓ jobs circulate at that level. Clearly,

$$t_k(\ell,N_\ell) = \mu_k U_k(\ell,N_\ell) \ , \qquad (7.5)$$

$k=0,1,...,\ell$, $\ell=1,2,...,M$, $N_\ell=0,1,...,N$. Then evidently $t_k(M,N)$, $k=0,1,...,M$, are the marginal throughputs and the system of equations (6.57) may be written in matrix form as:

$$t(M,N) = t(M,N)R(M) \qquad (7.6)$$

and the vector $t(M,N)$ may be specified by the above equation within a multiplicative constant. In the first level of aggregation now (case $\ell=1$) equation (7.1) may be written as:

$$U_0(1,N_1)\mu_0 = \mu_0 r_{00}(1)U_0(1,N_1) + \mu_1 r_{10}(1)U_1(1,N_1)$$

and is satisfied for all $N_1=0,1,...,N$. Thus multiplying both sides of

the above by $P_1^*(N_1)$, the unconditional probability of having N_1 jobs present at the subsystem $(\Sigma_0\Sigma_1)$, and summing over all $N_1=0,1,\ldots,N$, it follows that:

$$\mu_0 U_0(M,N) = \mu_0 U_0(M,N) r_{00}(1) + \mu_1 U_1(M,N) r_{10}(1)$$

and denoting as $t(1)$ $(=(t_0(M,N),t_1(M,N)))$ the vector of the marginal throughputs for units Σ_0 and Σ_1 it follows that the matrix equation

$$t(1) = t(1)R(1)$$

is satisfied, irrespective of the values of $P_1^*(N_1)$, $N_1=0,\ldots,N$ probabilities. At the ℓth level of aggregation now, $\ell=2,\ldots,M-1$, performing a similar operation on equation (7.1), it follows that :

$$\mu_\ell U_\ell(M,N) = \sum_{k=0}^{\ell} \mu_k U_k(M,N) r_{k\ell}(\ell)$$

$$t_\ell(M,N) = \sum_{k=0}^{\ell} t_k(M,N) r_{k\ell}(\ell)$$

So defining as $t(\ell)=(t_0(M,N),t_1(M,N),\ldots,t_\ell(M,N))$ the vector of the marginal throughputs for units $\Sigma_0,\ldots,\Sigma_\ell$, the above equation and the fact that vector $t(\ell-1)=(t_0(M,N),t_1(M,N),\ldots,t_{\ell-1}(M,N))$ is the P–F eigenvector of $R(\ell-1)$, i.e.

$$t(\ell\text{-}1) = t(\ell\text{-}1)R(\ell\text{-}1)$$

in conjuction with the hypothesis that matrices $R(1),\ldots,R(M)$ satisfy the subparallelism condition, imply that:

$$t(\ell) = t(\ell)R(\ell)$$

Similarly for $\ell=M$ equation (7.1) yields

$$t(M) = t(M)R(M)$$

where $t(M)$ is the same vector as $t(M,N)$ involved in (7.6). Thus, the

system is marginally flow-balanced.

So defining this sequence of matrices {R(1),...,R(M)} that satisfy the subparallelism condition and using (7.1) equation to flow-balance the MRE solution at every level of aggregation extends our new method to an arbitary network configuration. It remains to define the way of establishing these subparallel matrices, and furthermore to establish some queueing interpretation to this algebraic manipulation. This problem was formulated by Vantilborgh, [VANT 78], as follows: Given a stochastic matrix R(ℓ) of order (ℓ+1) with unknown P-F eigenvector, define a stochastic matrix R(ℓ-1) of order ℓ, whose P-F eigenvector is subparallel to the one that corresponds to R(ℓ), without calculating any of them, of course. This problem is related to the Gaussian elimination method of solving linear equations and, according to Vantilborgh, a way of solving it is to partition R(ℓ) as follows:

$$R(\ell) = \begin{bmatrix} S(\ell) & C(\ell) \\ D(\ell) & r_{\ell\ell}(\ell) \end{bmatrix}$$

where $r_{\ell\ell}(\ell)$ is a simple entry of R(ℓ), while C(ℓ)is the column of elements above $r_{\ell\ell}(\ell)$, D(ℓ) consists of the elements of the last row of R(ℓ) up to and not including $r_{\ell\ell}(\ell)$ and S(ℓ) is the matrix of the remaining entries of R(ℓ), which is of order ℓ. Then the required matrix is simply given as:

$$R(\ell-1) = S(\ell) + C(\ell)D(\ell)/(1-r_{\ell\ell}(\ell)) \qquad (7.7)$$

and R(ℓ-1) is also stochastic. So given an original routing matrix R≙R(M), one may apply (7.7) recursively and define the required sequence {R(1),R(2),...,R(M)}.

However, this algebraic manipulation gives us no clue of the resulting subsystem configuration at every level of aggregation, and

furthermore the way of defining such matrices is not unique, [VANT 78]. It is reasonable to decide on choosing (or rejecting) (7.7), by requiring that it yields the familiar structures that have already being used in the previous chapter and in specific network configurations. So let's see through some examples the effect of (7.7) in central server and tandem types of configurations.

Example 7.1. Let's consider a central server model with 3 I/O units (M=3) and without feedback streams. The routing matrix for this system is :

$$R \triangleq R(3) = \begin{bmatrix} 0 & r_1 & r_2 & r_3 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Applying (7.7) recursively, it follows that:

$$R(2) = \begin{bmatrix} r_3 & r_1 & r_2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad R(1) = \begin{bmatrix} r_2+r_3 & r_1 \\ 1 & 0 \end{bmatrix}$$

Clearly, these matrices exactly imply the subsystems already used in our analysis. In the first level of aggregation (units Σ_0 and Σ_1), the feedback probability r_2+r_3 of the CPU is equal to the probability that the job upon service completion will depart the subsystem of this level. The same interpretation may be given to the feedback probability r_3 of the CPU at the second level of aggregation. Recall that algorithm 6.1 treats this model in exactly the same fashion, i.e. at every level, the CPU has modified service time parameters according to the feedback correction with the above feedback probability. In case that the CPU had an original feedback probability r_0, the only difference in R(2) and R(1) would have been that this probability would have been added to the above feedback

probabilities, i.e. r_3 in R(2) would have been r_0+r_3 and r_2+r_3 would have been $r_0+r_2+r_3$.

Note now that by reenumerating the units so that the CPU is unit Σ_3 instead of Σ_0, the routing matrix R(3) becomes:

$$R \triangleq R(3) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ r_1 & r_2 & r_3 & 0 \end{bmatrix}$$

Applying (7.7) on the above, it follows that:

$$R(2) = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_1 & r_2 & r_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

which implies that at the second level of aggregation the I/O subsystem (the CPU is shorted) is considered in isolation. So by altering the enumeration of the units it is possible to short any unit and thus the nine different sequences, considered in the previous chapter for central server models may be produced. Additionally to those sequences, another three may be produced here, namely SEQ(1,2,0,3), SEQ(1,3,0,2) and SEQ(2,3,0,1), which were not amenable to algorithms 6.1 and 6.3. Generally, for a network with (M+1) units $(\Sigma_0,\Sigma_1,\ldots,\Sigma_M)$ all the different (M+1)!/2 sequences may be produced.

Example 7.2. Let's consider now a tandem configuration with four units (M=3) and routing matrix,

$$R \triangleq R(3) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which corresponds to a tandem configuration. Applying (7.7) recursively on R(3), it follows that:

$$R(2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad R(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence, it is also clear that the subsystems implied by the above sequence of matrices are the ones used in algorithm 6.4.

So from these two examples it follows that this method of obtaining the matrices that satisfy the subparallelism condition has been used implicitly in the previous chapter, in order to implement the form of the MRE decomposition solution, which, by the way, is unaffected by what has been discussed so far in this chapter. The multipliers involved in this solution have to be evaluated using this new and most general methodology.

7.2 A MRE decomposition algorithm for an arbitrary network configuration.

As it was implied above, the form of the MRE solution to be used here is the same as the one implemented in the previous chapter. The steps to evaluate the invariant mql and utilization multipliers as well as the load-dependent fb ones are similar to those followed in the case of algorithms 6.1-6.4. Firstly, the approximation of the assymptotic flow is considered.

7.2.1 The flow approximation.

In this case, at every level of aggregation ℓ, ℓ=1,2,...,M, the subsystem $(\Sigma_0\Sigma_1...\Sigma_{\ell-1})$ is a perfectly described network, and this is most convenient when it comes to justifying the steps followed to approximate the assymptotic flow that departs the subsystem and is directed towards unit Σ_ℓ.

The first concern is to calculate the assymptotic utilization of unit Σ_ℓ, denoted here as $\rho_\ell(\ell)$. Under the assumption that $\rho_\ell(\ell)<1$, equation (7.1) yields at the limit ($N \longrightarrow +\infty$),

$$\mu_\ell[1-r_{\ell\ell}(\ell)]\rho_\ell(\ell) = \sum_{k=0}^{\ell-1} \mu_k\rho_k(\ell-1)r_{k\ell}(\ell) \qquad (7.8)$$

where $\rho_k(\ell-1)$, $k=0,1,\ldots,\ell-1$, are the assymptotic utilizations of the previous level $(\ell-1)$, i.e. they are a solution of the system:

$$t(\ell-1) = t(\ell-1)R(\ell-1)$$

where $t_k(\ell-1)=\rho_k(\ell-1)\mu_k$, $k=0,\ldots,\ell-1$, (note that since μ_k,$k=0,\ldots,M$, are known the actual unknowns in the above system are the variables $\rho_k(\ell-1)$, $k=0,\ldots,\ell-1$). The space of vectors that satisfy the above is of dimension 1 (vectors are parallel) and according to our assymptotic approach the solution of interest is the one where $\rho_b(\ell-1)$ is fixed equal to 1, where index $b\epsilon\{0,1,\ldots,\ell-1\}$ identifies the relative bottleneck of that level. Now, because of the subparallelism condition, vector $(\rho_0(\ell-1),\rho_1(\ell-1),\ldots,\rho_{\ell-1}(\ell-1),\rho_\ell(\ell))$, where $\rho_\ell(\ell)$ is defined to satisfy (7.8), is, within a multiplicative constant, the left P-F eigenvector of $R(\ell)$. So, in the next $((\ell+1)$th) level, when the vector $(\rho_0(\ell),\rho_1(\ell),\ldots,\rho_\ell(\ell))$ will be required so that no utilization exceeds 1, if $\rho_\ell(\ell)\not\leq 1$, (i.e. the relative bottleneck at levels $(\ell-1)$ and ℓ is the same), $(\rho_0(\ell),\rho_1(\ell),\ldots,\rho_\ell(\ell))$ is the same as $(\rho_0(\ell-1),\rho_1(\ell-1),\ldots,\rho_{\ell-1}(\ell-1),\rho_\ell(\ell))$, while if $\rho_\ell(\ell)>1$ and since the above two vectors are subparallel, the vector of utilizations to be used in this $(\ell+1)$th level is $(\rho_0(\ell-1)/\rho_\ell(\ell),\rho_1(\ell-1)/\rho_\ell(\ell),\ldots,\rho_{\ell-1}(\ell-1)/\rho_\ell(\ell),1)$. Thus, formula (7.8) will be used to determine $\rho_\ell(\ell)$ at every level ℓ, with conjuction to the following recursive formula for the assymptotic utilizations of the previous level:

$$(\rho_0(\ell),\rho_1(\ell),\ldots,\rho_\ell(\ell)) = \begin{cases} (\rho_0(\ell-1),\rho_1(\ell-1),\ldots,\rho_\ell(\ell)) & \text{if } \rho_\ell(\ell)\leq 1 \\ (\frac{\rho_0(\ell)}{\rho_\ell(\ell)},\frac{\rho_1(\ell)}{\rho_\ell(\ell)},\ldots,\frac{\rho_{\ell-1}(\ell-1)}{\rho_\ell(\ell)},1) & \text{if } \rho_\ell(\ell)>1 \end{cases} \quad (7.9)$$

for $\ell=2,3,\ldots,M-1$. The above formula has been also used in algorithms 6.1-6.4, only sometimes implicitly since the specification of the network's configuration, allowed us to express the above pattern in a different, more easily justifiable but equivalent manner.

Note also that at every level of aggregation, the vector of utilizations (assymptotic or not) that corresponds to the left P-F eigenvector of the routing matrix $R(\ell)$, is invariant to the feedback transformation. This invariance, however, is restricted to the first moment of the flow, and does not hold for the second moment. Due to the definition of the sequence of matrices $\{R(1),R(2),\ldots,R(M)\}$, it is very likely that a unit without a feedback probability at some level ℓ may have a feedback probability at level $\ell-1$. So the instance at which the feedback transformation is applied is important. For example, in case that the original matrix $R(M)$ has feedback probabilities, different results are obtained if a feedback transformation is applied before (7.7) is used, compared to the results obtained if (7.7) is applied prior to a feedback transformation. The decision on which route to follow in the algorithm that is to be presented in this chapter is based on the reasonable requirement that it should provide at least similar (if not identical) results, to the ones obtained by algorithms 6.1-6.4. To this end, the first step is to define a "level-dependent" feedback transformation on the original service time characteristics, which in this algorithm remained unchanged, and which for unit Σ_i, $i=0,\ldots,M$ are denoted as μ_i^* and C_i^2. This transformation is responsible for defining μ_i and $C_{s,i}^2$ the mean and SCV of the service times at unit

Σ_i, to be used at the corresponding level, as well as the modified routing matrix $R(\ell)$.

Procedure FEEDBACKCORRECTION(ℓ).

```
For  i=0  to  ℓ  do  begin
  For  j=0  to  ℓ  do
    if  i≠j  then   r_ij(ℓ) ← r_ij(ℓ)/(1-r_ii(ℓ))
    μ_i ← μ*_i(1-r_ii(ℓ))
    C²_s,i ← 1 + [1-r_ii(ℓ)][C²_i - 1]
    r_ii(ℓ) ← 0
  end
```

So, for example, in the first level of aggregation, the above procedure is called, to define the service parameters for units Σ_0 and Σ_1 and thus create the two stage cyclic system without feedback streams, that was used so far, and which is represented in fig. 2.3.

Now, procedure FLOWITERATION(ℓ) is to be called at level (ℓ+1), ℓ=1,2,...,M-1, and is responsible for iterating for the assymptotic flow parameters of the subsystem ($\Sigma_0\Sigma_1...\Sigma_\ell$), in the usual fashion, in order to produce the SCV of the assymptotic arrival stream at unit $\Sigma_{\ell+1}$, as well as the assymptotic utilization of this unit.

Procedure FLOWITERATION(ℓ).

Step 1. { Prepare the assymptotic utilizations to be used }

```
Case
ℓ = 1 : { Case iteration concerns units Σ_0 and Σ_1 }
        ρ_b ← 1
ℓ > 1 : { Case iteration concerns units Σ_0,Σ_1,...,Σ_ℓ }
        if  b = ℓ  then
          For  i=0  to  ℓ  do
            ρ_i ← ρ_i/ρ_b
```

Step 2. { Initialize λ_{ij}, $C^2_{a,i}$, $C^2_{d,i}$ }

For i=0 to ℓ do begin

For j=0 to ℓ do

$\lambda_{ij} \leftarrow \rho_i \mu_i r_{ij}(\ell)$

$C^2_{a,i} \leftarrow 1 + r_{bi}(\ell)[C^2_{s,b} - 1]$

$C^2_{d,i} \leftarrow \rho_i^2 C^2_{s,i} + (1-\rho_i) C^2_{a,i} + \rho_i(1-\rho_i)$

end

Step 3. { Iterate }

Repeat

For i=0 to ℓ do

For j=0 to ℓ do

$C^2_{i,j} \leftarrow 1 + r_{ij}(\ell)[C^2_{d,i} - 1]$

For i=0 to ℓ do begin

$$C^2_{a,i} \leftarrow \frac{\lambda_i}{\sum_{j=0}^{\ell} \frac{\lambda_{j,i}}{C^2_{j,i} + 1}} - 1$$

where

$$\lambda_i \leftarrow \sum_{j=0}^{\ell} \lambda_{j,i}$$

$C^2_{d,i} \leftarrow \rho_i^2 C^2_{s,i} + (1-\rho_i) C^2_{a,i} + \rho_i(1-\rho_i)$

end

Until $C^2_{d,i}$, i=0,...,ℓ converge

Step 4. { Prepare network parameters, according to feedback transformation on R(ℓ+1) }

FEEDBACKCORRECTION(ℓ+1)

Step 5. { Evaluate assymptotic utilization of unit $\Sigma_{\ell+1}$ }

$$\rho_{\ell+1} \leftarrow \frac{1}{\mu_{\ell+1}} \sum_{i=0}^{\ell} \rho_i \mu_i r_{i,\ell+1}(\ell+1)$$

Step 6. { Calculate $C^2_{a,\ell+1}$ }

For i=0 to ℓ do begin

$\lambda_{i,\ell+1} \leftarrow \rho_i \mu_i r_{i,\ell+1}(\ell+1)$

$$C^2_{i,\ell+1} \leftarrow 1 + r_{i,\ell+1}(\ell+1)[C^2_{d,i} - 1]$$

end

$$C^2_{a,\ell+1} \leftarrow \frac{\lambda_{\ell+1}}{\sum_{j=0}^{\ell} \frac{\lambda_{j,\ell+1}}{C^2_{j,\ell+1} + 1}} - 1$$

where

$$\lambda_{\ell+1} \leftarrow \sum_{j=0}^{\ell} \lambda_{j,\ell+1}$$

Note that in step 5 of the above procedure, formula (7.8) was used, since at that stage $\mu_{\ell+1}$ has the value $\mu^*_{\ell+1}(1-r_{\ell+1,\ell+1}(\ell+1))$, because procedure FEEDBACKCORRECTION(ℓ+1) has just been called. Note also that the routing matrix R(ℓ) was used in the iteration, while the links between the subsystem ($\Sigma_0\Sigma_1...\Sigma_\ell$) and unit $\Sigma_{\ell+1}$ were specified by the last column of matrix R(ℓ+1) as it should be done, according to the assymptotic form (7.8) of the fb equation (7.1).

Solving next (7.71) with respect to $y_\ell = y_\ell(N_\ell)$, in the usual fashion, using (6.47), (6.31) and (6.32), after some manipulation, it follows that:

$$y_\ell(N_\ell) = \frac{1}{x_\ell^{N_\ell} g_\ell \mu_\ell} \left[\sum_{k=0}^{\ell-1} \mu_k r_{k\ell}(\ell) \left[U_k(\ell-1,N_\ell)G(\ell-1,N_\ell) + g_\ell \sum_{n_\ell=1}^{N_\ell-1} U_k(\ell-1,N_\ell-n_\ell)G(\ell-1,N_\ell-n_\ell)x_\ell^{n_\ell} \right] \right] -$$

$$- \frac{1}{x_\ell^{N_\ell}} \sum_{n_\ell=1}^{N_\ell-1} G(\ell-1,N_\ell-n_\ell)x_\ell^{n_\ell} \qquad (7.10)$$

Note also that in the above, the feedback transformation on R(ℓ) is assumed to have taken place before the calculation of $y_\ell(N_\ell)$. Let's proceed now with the description of the proposed algorithm.

7.2.2 The decomposition algorithm.

Algorithm 7.1. A MRE decomposition algorithm for an arbitrary network configuration.

Input Parameters.

M+1 : number of units, $(\Sigma_0,\Sigma_1,\ldots,\Sigma_M)$.

N : number of jobs.

For unit Σ_i, i=0,...,M

μ_i^* : mean service rate.

C_i^2 : squared coefficient of variation of service time.

For i=0,...,M-1

$r_{ij}(M)$: Original (corresponding to routing matrix R(M)) transition probability from unit Σ_i to unit Σ_j.

Step 1. { Create sequence of routing matrices R(1), R(2),..., R(M) using (7.7) }

For ℓ=M-1 downto 1 do

For i=0 to ℓ do

For j=0 to ℓ do

$$r_{ij}(\ell) \leftarrow r_{i,\ell+1}(\ell+1)r_{\ell+1,j}(\ell+1)/(1-r_{\ell+1,\ell+1}(\ell+1))$$

From this point on the algorithm is identical to steps 1-3 of algorithm 6.1 except the following differences:

1) In step 1.1 of algorithm 6.1, instead of calling procedure FLOWITERATION1(1), the following three commands should be executed,

FEEDBACKCORRECTION(1)

$\rho_0 \leftarrow \mu_1/\mu_0$

$\rho_1 \leftarrow \mu_0/\mu_1$

2) In step 1.4, the additional conditional utilization $U_1(1,N_1)$ of unit Σ_1 is evaluated using (6.48b), while at every level ℓ (ℓ>1) and in step 2.4, additionally to $U_0(\ell,N_\ell)$, all conditional utilizations $U_k(\ell,N_\ell)$, k=1,2,...,ℓ-1, are calculated using (6.47) and also $U_\ell(\ell,N_\ell)$ using (6.48a).

3) In step 2.1 the call of FLOWITERATION1(ℓ) is replaced by FLOWITERATION(ℓ-1).

4) In step 2.2.1 the fb multiplier y_ℓ=$y_\ell(N_\ell)$ is calculated using (7.10).

Reviewing the basic steps of the above algorithm, it can be seen that the operations for deriving the routing matrices, which define the various subsystems, are carried out prior to any feedback transformation on the original (or in fact any intermediate) matrix. Having established the sequence {R(1),R(2),...,R(M)}, starting with R(1) at the first level of aggregation and moving upwards, at level ℓ, the matrix R(ℓ-1) and the feedback transformation applied on it define the service characteristics of units Σ_0, Σ_1,...,$\Sigma_{\ell-1}$ to be used in the iteration that approximates $C^2_{d,i}$, i=0,1,...,ℓ-1. From that point on matrix R(ℓ), transformed by procedure FEEDBACKCORRECTION and its related service times characteristics for units Σ_0, Σ_1,...,Σ_ℓ, are used. The effect of this is that algorithm 7.1 gives exactly the same results as algorithms 6.1 and 6.4 for central server and tandem network configurations, while it provides similar (but not identical) results to algorithm 6.4 for central server models, due to a difference associated to that in algorithm 6.2 an initial feedback transformation is applied on the I/O subsystem. Naturally, this algorithm provides the exact product form solution when the network has exponentially distributed service times, regardless of the enumeration of the network's units, while it can tackle any type of network configuration and under any different ordering of the units.

7.2.3 Validation of the MRE decomposition algorithm.

The accuracy of algorithm 7.1 has been already demonstrated in tables 6.1-6.20, and in central server and tandem types of networks.

It remains to investigate its performance in the most general case of fully connected networks, i.e. networks where every unit may communicate with every other unit.

Once more, variability dominates the selection of the proposed enumeration of the units. Thus, this ordering is done with respect to increasing "actual" variability, where this term means that it is not the original service time SCVs that are taken under consideration but the SCVs that result after the feedback transformation. This has to be clear, since in such networks it is very common to have non-zero feedback probabilities.

In case that all actual SCVs are identical, the decision is once more based on maximizing the interaction rates at every level of aggregation. More precisely, in the first level of aggregation, to every different pair of units (Σ_i,Σ_j), $i,j\in\{0,1,\ldots,M\}$ an average interaction rate is associated and given as:

$$\mu_i^* r_{ij}(M) + \mu_j^* r_{ji}(M)$$

Then the pair of units to be considered at the first level is the one with maximum interaction rate. At level ℓ, $\ell>1$, of decomposition now, and given that the subsystem $(\Sigma_0,\ldots,\Sigma_{\ell-1})$ has been defined, to each of the units Σ_j, $j=\ell,\ldots,M$, an average interaction rate (with the subsystem) is associated and given as:

$$\mu_j^* \sum_{i=0}^{\ell-1} r_{ij}(M) + \sum_{i=0}^{\ell-1} \mu_i^* r_{ij}(M)$$

Then the unit to be joined to the subsystem $(\Sigma_0\ldots\Sigma_{\ell-1})$ is the one with maximum such rate. In tables 7.1-7.8 the results according to the proposed sequence are presented. If the selection of this sequence was made with respect to variability, the sequence is denoted as SEQ(SCV), while in the opposite case it is denoted as SEQ(msr). Tables 7.1-7.10 (App. V) are indicative of the performance

of algorithm 7.1 in networks with low (tables 7.1, 7.3, 7.8), high (tables 7.2, 7.5, 7.6) and mixed (tables 7.4, 7.7, 7.9, 7.10) variability service times. The results are generally very good, showing that the MRE is a good and reliable approximation. As compared to the results obtained by the UME algorithm (see App. V), it is evident that every one of these two algorithms performs better in certain types of networks, but the general impression is that they are comparable. More precisely, UME algorithm seems to be slightly superior in low variability service times (tables 7.1, 7.3), while the MRE algorithm is better when variability increases (tables 7.4, 7.5, 7.7). In tables 7.9 and 7.10 the configuration used is that of a central server model. The results produced are identical to the ones obtained by sequences SEQ(1,2,0,4) and SEQ(1,3,0,4,2) respectively and they are the best, compared to all possible sequences. It is interesting to note that in both cases these optimum sequences cannot be produced neither by algorithm 6.1 nor by algorithm 6.3 of the previous chapter, since the CPU (unit Σ_0) is not joined at the first or at the last level of aggregation. This demonstrates the clear advantage of algorithm 7.1 over these algorithms. Finally, in the proposed sequences of tables 7.1-7.10, the frequency of occurance of flow-balance failures is close to 16%.

Because the new MRE algorithm can be applied to any type of network configuration, it may be compared to a number of existing algorithms that have been validated in a comparative study conducted by Almond, [ALMO 88], as mentioned in the previous chapter (page 168). In particular it would be interesting to examine its performance as this is compared to the one of the universal maximum entropy (UME), [KOUV 86c] (see also App. V), which proved to be the best out of all major approximation methods, at the level of a network with GE-distributed service times [ALMO 88]. To this end, the second part

of this validation may be described as follows. The effects of three metaproperties on the approximation are to be examined, namely, the effects of "load balance", "nearness to product form" and "connectivity". Firstly, a base population of networks is defined, for which M=2, N=3, $\mu_i \epsilon \{2,4,6,8\}$, $C_i^2 \epsilon \{0.5,1,10,100\}$, i=0,1,2, each unit communicates with every other unit except itself (fuly connected networks without feedback streams) and the average values of the routing probabilities are identical. To examine the effect of load balance, samples were taken from the following populations,

Load balance

1) as base population but $\mu_0 \epsilon \{8,16,24,32\}$

2) base population

3) as base population but $\mu_0 \epsilon \{0.5,1,1.5,2\}$

The effect of the other two metaproperties considered is investigated using the following populations,

Nearness to product form

1) base population

2) as base population but $C_0^2=1$

3) as base population but $C_0^2=C_1^2=1$

Connectivity

1) Simply connected (tandem)

2) base population

3) fully connected with feedbacks.

The performance of the algorithm is characterized by the percentage error in mean system response time (SRT), denoted as "absDiffSRT%". Quantity SRT is defined as:

$$SRT = \frac{N}{\mu_0 U_0}$$

where U_0 is the marginal utilization of unit Σ_0. Note that the above definition has the meaning of the real average response time only for

certain network configurations. Nevertheless, it may be considered as a performance measure for an arbitrary configuration as well, since it is used for comparison of an approximation versus the exact result. In figure 7.2 the mean and standard deviation of the variable absDiffSRTZ are displayed in stacked bars. The height of the shaded portion of the bar gives the mean value of the variable. The standard deviation from the mean is indicated by the additional height of the non-shaded section of the bar. The three network populations used to test the load balance, have been defined with respect to the load of unit Σ_0 and the three bars correspond to a "light" or "balanced" or "heavy" load of this unit. Similarly, the three bars that appear in the test of the "Nearness to product form" metaproperty, correspond to the number of units fixed as exponential (none or one or two units), while in the tests of "Connectivity", the three bars correspond to the units' connectivity and simple stands for tandem, fully stands for the base population, while fully+FB stands for fully connected networks without feedback streams. For each metaproperty, two graphs are presented and describe the performance of algorithm 7.1 (denoted as MRE) and the performance of the universal maximum entropy algorithm (denoted as UME). Also figure 7.3 presents exactly the same tests as figure 7.2 the only difference being that N=7 for all network populations used.

The MRE algorithm 7.1 is once more represented by a proposed in each case sequence. The data used for these tests imply three different types of networks with respect to the values of the SCVs, namely, 1) networks where all three units have different SCVs, 2) networks where two units have the same SCV and 3) networks where all three units have the same SCV. In cases 1 and 3, the sequence that was used was defined by the rules already presented in this section. In the second case, if this common SCV for the two units was smaller

than the one of the third unit then the units with the smaller SCV were joined in the first level of decomposition. In the opposite case, the decision on which of the two units with the same SCV will join the third unit in the first level of aggregation was based on the maximum interaction rate.

It is evident, from figures 7.2, 7.3, that algorithm 7.1, in conjunction with the above described selection of the enumeration of the units, performs much better than the UME algorithm in these tests. It should be pointed out, however, that the type of networks used are, to our knowledge, the ones where UME algorithm appears to be unreliable (i.e. there is a combination of units with very low and very high service time variability). Thus, there is an unreliable performance by this algorithm recorded in the connectivity test of figure 7.2 as well as in the load balance and connectivity tests of figure 7.3. The performance of algorithm 7.1 reaches outstanding levels in all tests and this classifies the new algorithm as one of the best available approximations for this type of networks and at the level of the GE distribution.

7.2.4 Computational cost.

The number of operations - that is multiplications and divisions - performed by algorithm 7.1 has been estimated to be approximately,

$$M^3 + N^2 + N^2M + N(N+1)\left[2(M-2) + \frac{M(M+1)}{2}\right] + \sum_{\ell=3}^{M} [\text{itsno}(\ell-1)[\ell(2\ell+1)+7]]$$

where itsno(ℓ) is the number of iterations performed by procedure FLOWITERATION(ℓ) at the ℓth level of aggregation, ℓ=2,3,...,M. Generally, algorithms that are based in such hierarchical decomposition schemes are computationally more expensive than

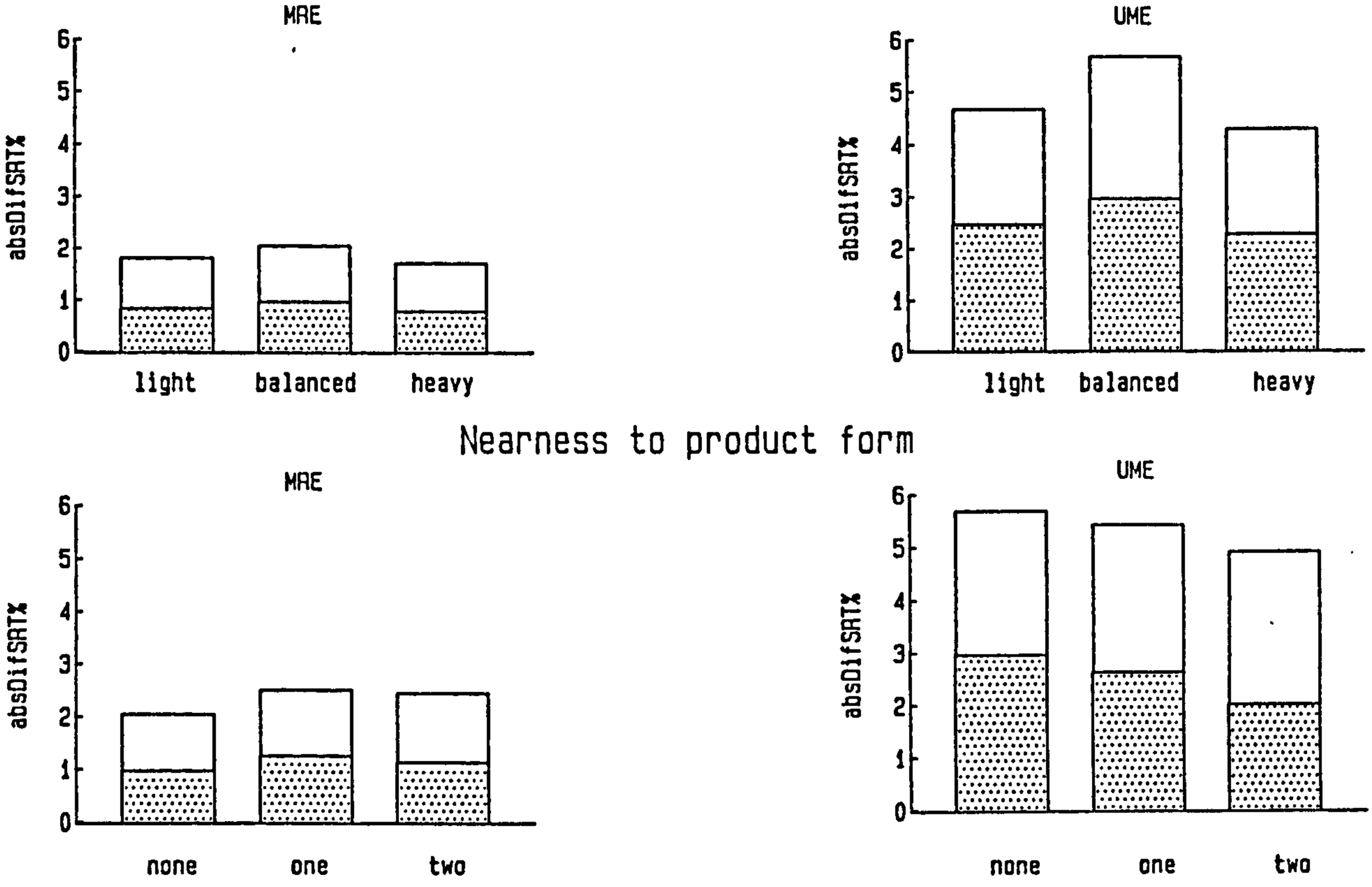

Figure 7.2. Parametric performance of the MRE and UME approximations.

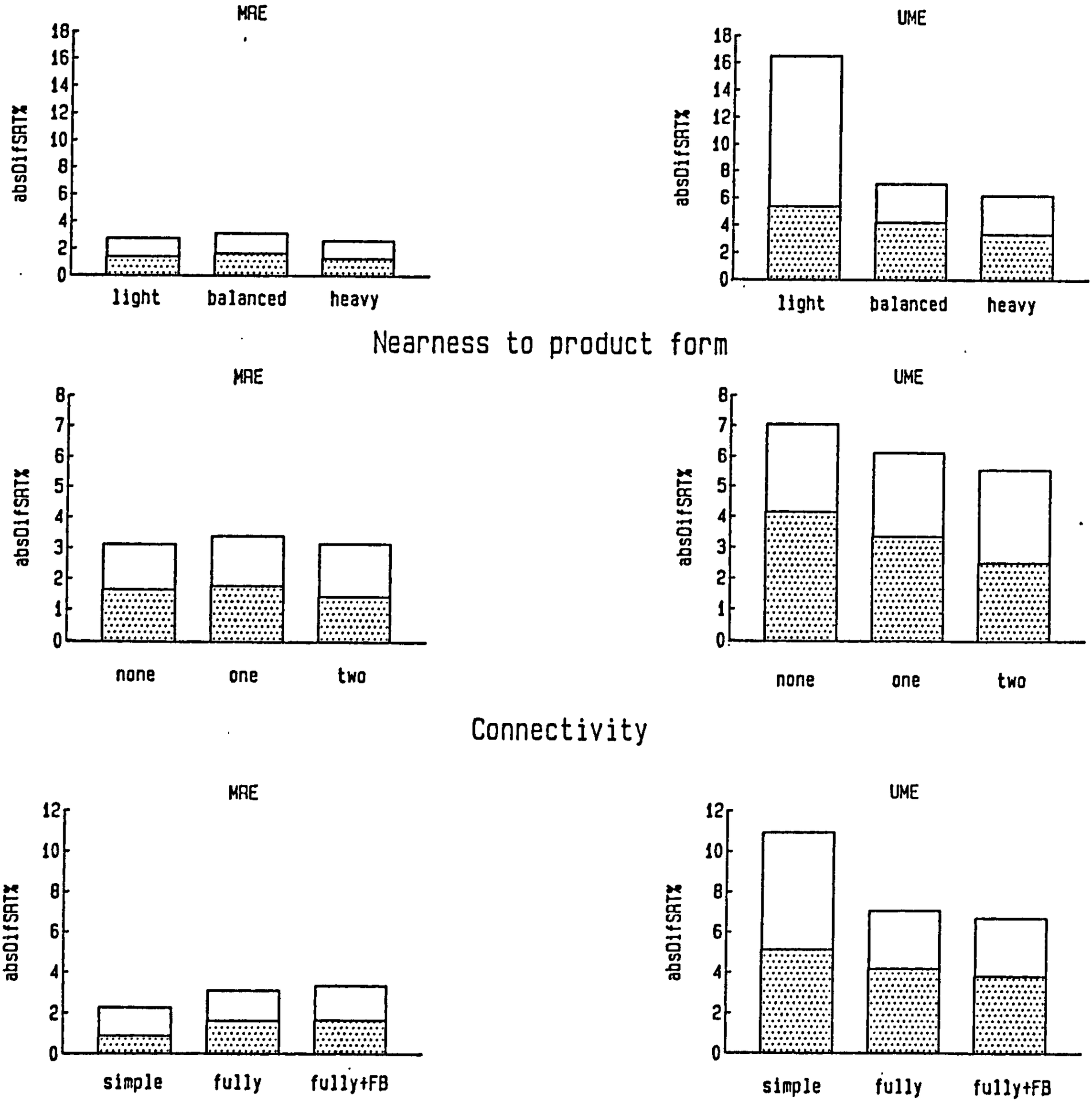

Figure 7.3. Parametric performance of the MRE and UME approximations.

algorithms based on convolution techniques, but overall they are comparable [COUR 77, p. 87]. For algorithm 7.1 some execution times that have been recorded for varying network sizes (with respect to M and N), are presented in figure 7.4. The system used was a Sun 4/280 using Sun OS 4.0 software. These execution times are justifiable by the above estimate since two terms in it are of order M^3 and one of order $M^2 \times N^2$ (note that the summation only involves M and is of order M^3 - at least - if the itsno(ℓ) parameter is not considered). Hence this decomposition algorithm favours networks with small number of units M and large customer population N.

M \ N	10	30	50
10	1.583	1.983	2.616
15	7.166	7.816	9.233
20	12.783	13.966	16.283

Figure 7.4. Table of execution times in CPU seconds.

The problem that seems to be of more importance in such decomposition algorithms is their comparatively large memory requirements. Note that in order to obtain the marginal queue length distributions, an $M \times N^2$ size array has to be used for the conditional distributions, as well as an $M^2 \times N$ size one for the conditional utilizations (or load-dependent rates in the conventional implementation). In the case of networks with small number of units (M) but large population of jobs (N), some space may be saved at the cost of determining only the marginal utilizations and mean queue lengths, without calculating the marginal distributions. As it has been mentioned, after the last level of aggregation, the values of $U_k(M,N)$, $k=0,1,\ldots,M$, are exactly the marginal utilizations. In a similar fashion the conditional mql of unit Σ_k, $\langle n_k \rangle(\ell, N_\ell)$ may be

defined, conditioned on the level ℓ of decomposition and the population N_ℓ that circulates at that level. So for the first level,

$$<n_0>(1,N_1) = \sum_{n_0=1}^{N_1} n_0 P_1(n_0/N_1) \qquad (7.11a)$$

and

$$<n_1>(1,N_1) = \sum_{n_1=1}^{N_1} n_1 P_1(N_1-n_1/N_1) \qquad (7.11b)$$

and at the ℓth level of decomposition, $\ell>1$,

$$<n_\ell>(\ell,N_\ell) = \sum_{n_\ell=1}^{N_1} n_\ell P_\ell(n_\ell/N_\ell) \qquad (7.12)$$

and

$$<n_k>(\ell,N_\ell) = \sum_{N_{\ell-1}=1}^{N_\ell} <n_k>(\ell-1,N_{\ell-1}) P_\ell(N_\ell-N_{\ell-1}/N_\ell), \qquad (7.13)$$

$k=0,1,\ldots,\ell-1$. Definitions (7.11)-(7.12) and recursion (7.13), provide as final values of $<n_k>(M,N)$, $k=0,1,\ldots,M$, the actual marginal mean queue lengths. This way the conditional mqls occupy an array of size $M^2 \times N$. Moreover, comparing algorithm 7.1 to the algorithms of the previous chapter, note that the former requires an extra M^3 memory spaces for storage of the M routing matrices $\{R(1),R(2),\ldots,R(M)\}$, in the present implementation. However, since matrix $R(\ell)$, $\ell=1,2,\ldots,M$, is of use only to the ℓth level of decomposition, the algorithm can be easily modified to use the same array for $R(\ell)$ and $U_k(\ell,N_\ell)$ by defining a common matrix of size $M^2 \times \max(M,N)$. This involves using procedure FLOWITERATION(ℓ) one level of aggregation earlier than it is done in the presented version of the algorithm, i.e. at level ℓ call FLOWITERATION(ℓ) instead of FLOWITERATION(ℓ-1).

Exact solutions for networks of similar size to those used in figure 7.4 are not numerically obtainable. Simulation has to be used,

which, under the departmental restrictions on the CPU time, proved to be inadequate to provide satisfactory equilibrium convergence for large networks, using the simulation package QNAP-2, [VERA 84]. So this validation study is restricted to what has been presented so far for relatively small networks, based on the argument that extrapolating trends seen in small networks may indicate the expected performance in larger networks, [SHUM 76, TRIP 79].

7.3 Discussion.

In this chapter, the concept of subparallelism, [VANT 78], has been used to extend the applicability of the MRE decomposition solution to arbitrary network configurations. This concept unifies the different hierarchical decomposition schemes, used in the previous chapter, under the umbrella of a more general one, i.e. a decomposition method that is based on a hierarchical multilevel partition of the network's state space, and which is essential for the application of MRE with fully decomposable constraints. The need for this unification was clear in the previous chapter and the fact that different sequences amenable to either the variable aggregation technique or the Norton's reduction one, appeared to complement each other in terms of accuracy of the approximation, clearly pointed this way.

The resulting algorithm proves to be a very useful tool when service times are modelled according to the GE distribution and thus contributes in avoiding exponential assumptions and allows us to examine the effect of variability. Roughly, its overall credibility is comparable to that of UME algorithm. To this end, the MRE algorithm via the application of the GE distribution is expected to define (as in the case of UME [WALS 84, KOUV 86a, KOUV 86c]) important pessimistic (optimistic) performance bounds on throughputs

over corresponding global balance solutions based on two-phase distributional models satisfying the same first two moments for the service times with SCVs greater than 1 (less than 1). Furthermore, compared to the UME algorithm, algorithm 7.1 seems to offer an alternative, which proves to be very useful for types of networks where the former is unreliable, like tandem configurations with a combination of low and high variability service times. In addition, this decomposition algorithm avoids complicated iterations, used in the UME algorithm to calculate the assymptotic flow (subject to the fixed population mean technique, see App. V and [KOUV 86c]) and to flow balance the solution, which are not guaranteed to converge. In algorithm 7.1 the only iteration involved is the one that approximates the assymptotic flow, using a related open network at every level. Flow-balancing the closed network via the fb multipliers, even though complicated is analytic. The prize to be paid for these relatively simple and efficient techniques (compared to the ones used in the UME algorithm) is that sometimes a failure in flow balance occurs, which mainly affects the marginal queue length probabilities rather than the network statistics, due to the fact that in most of the cases these resulting negative probabilities are of very small absolute value. Consequently, in networks with other than tandem configuration, utilizations and average statistics should be mostly trusted. Our view is that the source of this problem can be traced in the assymptotic flow used in conjunction with the approximate formulae involved in its calculation (see also discussion on an alternative version of the UME algorithm in App. V).

Up to this point, the same form of MRE solution was used, which is associated with single server networks, in the sense that the same form of constraints were assumed for all units of the network and these constraints have been also used in the past in single server

systems like the G/G/1 and G/G/1/N queues,[KOUV 86a]. The result of this can be viewed at every level of aggregation, where the assymptotic form of the related conditional distributions, which describe the interactions between unit Σ_ℓ and the subsystem $(\Sigma_0\Sigma_1...\Sigma_{\ell-1})$, proves to be identical to the one of a GE/GE/1 queue. So each unit of the network is treated almost independently in its corresponding level of aggregation. This implies that by changing the form of constraints assumed in every level, to the form of constraints used for the maximum entropy derivation of the $GE_1/1/GE_2/c/N$ system (this notation implies a two stage cyclic system with a single and a multiple (c) server) it may be feasible to extend algorithm 7.1 to tackle networks with multi-server queues. This extension is the subject of the following chapter.

CHAPTER VIII

AN EXTENTION TO GENERAL QNMs WITH MULTIPLE-SERVERS

Observing the MRE decomposition solution (see chapter VI, pp. 134–137) at the ℓth level of aggregation, it may be seen that the information assumed concerns the ℓth unit of the network Σ_ℓ and the form of the solution is identical to that of a simple GE/GE/1/N system, apart from the prior used and the fb multiplier which is responsible for adjusting this solution so that it satisfies a relatively more complex fb constraint. The limit (as $N \longrightarrow +\infty$) of this solution suggests that assymptotically, unit Σ_ℓ should be treated as a GE/GE/1 queue where the arrival process is generated by the subsystem $(\Sigma_0 \ldots \Sigma_{\ell-1})$ as this is treated also assymptotically. In this sense the GE/GE/1 solution is used as a building block and the generalization of the MRE approximation to the multi-server case seems to be conceptually simple. It certainly involves altering this building block to the multi-server one i.e. GE/GE/c_ℓ, where c_ℓ is the number of servers of unit Σ_ℓ, ℓ=0,1,...,M.

8.1 The GE/GE/c and GE/1/N||GE/c/N systems

The problems that this generalization faces, all arise from the fact that this more complex building block, has not been solved analytically in the maximum entropy fashion that the GE/GE/1 system has. Assuming the "existence" of appropriate mean value constraints, maximum entropy formalism has been used by Kouvatsos [KOUV 86b] to "guess" the form of the solution, while the actual values of the multipliers have been derived by solving the system of global balance equations. Let's see now the constrains assumed and the solution of the GE/GE/c and GE/1/N||GE/c/N systems, as these were presented by

Kouvatsos, [KOUV 86b].

For the GE/GE/c system, shown in figure 8.1, let P_n, n=0,1,... be the queue length distribution of the GE multiple (c) server. Then the

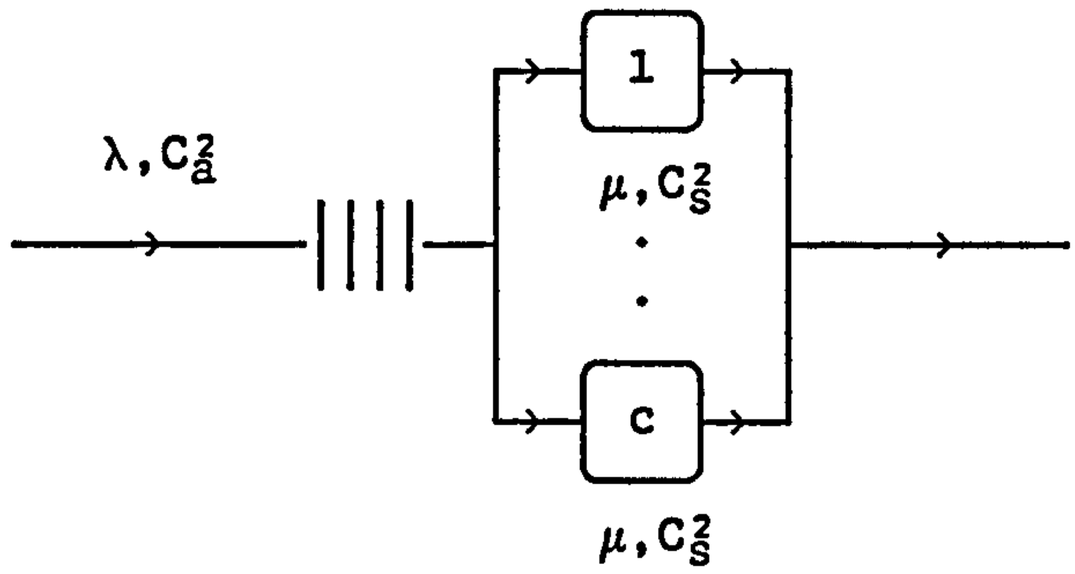

Figure 8.1. The GE/GE/c queue.

following constrains, concerning this distribution, are assumed to be known:

- The normalization,

$$\sum_{n=0}^{\infty} P_n = 1 \qquad (8.1a)$$

- The probabilities $P(\tilde{n}\geq j)=U(j)$, $U(j)\epsilon(0,1)$, $j=1,2,\ldots,c$, written as:

$$\sum_{n=0}^{N} h_j(n)P_n = U(j) \qquad (8.1b)$$

where

$$h_j(n) = \begin{cases} 1 & \text{if } n\geq j \\ 0 & \text{if } n<j \end{cases} \qquad (8.1c)$$

where $\tilde{n}$ is the random variable that describes the queue length.

- The mean waiting length, $\langle n_q \rangle$,

$$\sum_{n=0}^{N} n_q(n)P_n = \langle n_q \rangle \quad , \qquad (8.1d)$$

where
$$n_q(n) = \begin{cases} n-c & \text{if } n \geq c \\ 0 & \text{if } n<c \end{cases} \tag{8.1e}$$

The ME solution for P_n, n=0,1,..., is of the form:

$$P_n = g_n P_{n-1} \ , \ n=1,\ldots,c \tag{8.2a}$$

and
$$P_n = x P_{n-1} \ , \ n>c \tag{8.2b}$$

with
$$P_0 = \frac{1}{Z} = \left[1 + \sum_{n=1}^{c-1} \prod_{j=1}^{n} g_j + \frac{\prod_{j=1}^{c} g_j}{1-x} \right]^{-1} \tag{8.2c}$$

where g_j, j=1,2,...,c, are the multipliers that correspond to constraints (8.1b), x is the multiplier that corresponds to the mean waiting length constraint (8.1d), and Z is the normalizing constant. The values of the multipliers that make this ME solution exact for the GE/GE/c system are as follows, [KOUV 86b],

$$g_j = \frac{(\lambda\beta_2 + (j-1)\mu\alpha_2\beta_1)\alpha_2}{j\mu\alpha_2(1-\alpha_1\beta_1)} \ , \ j=1,\ldots,c-1 \tag{8.3}$$

$$g_c = \frac{(\lambda\beta_2 + (c-1)\mu\alpha_2\beta_1)\alpha_2}{\lambda\beta_2\alpha_1 + c\mu\alpha_2} \tag{8.4}$$

$$x = \frac{\lambda\beta_2 + c\mu a_2\beta_1}{\lambda\beta_2\alpha_1 + c\mu\alpha_2} \tag{8.5}$$

$$\alpha_1 = \frac{C_s^2 - 1}{C_s^2 + 1} \ , \ \alpha_2 = 1 - \alpha_1 \tag{8.6}$$

$$\beta_1 = \frac{C_a^2 - 1}{C_a^2 + 1} \ , \ \beta_2 = 1 - \beta_1 \tag{8.7}$$

Note that in order to be closer to our familiar GE/GE/1 system, nothing prevents us from assuming a mean queue length constraint instead of the mean waiting length one, in which case P_n, n=1,...,c, given by (8.2a) is additionally multiplied by x, while g_j, j=1,...,c, given by (8.3)-(8.4), is devided by x and becomes:

$$g_j = \frac{(\lambda\beta_2 + (j-1)\mu\alpha_2\beta_1)\alpha_2}{j\mu\alpha_2(1-\alpha_1\beta_1)} \frac{1}{x} \quad , \quad j=1,\ldots,c-1 \tag{8.8}$$

and

$$g_c = \frac{(\lambda\beta_2 + (c-1)\mu\alpha_2\beta_1)\alpha_2}{\lambda\beta_2\alpha_1 + c\mu\alpha_2} \frac{1}{x} = \frac{(\lambda\beta_2 + (c-1)\mu\alpha_2\beta_1)\alpha_2}{\lambda\beta_2 + c\mu\alpha_2\beta_1} \tag{8.9}$$

Similar constraints are assumed for the queue length distribution of the GE/1/N||GE/c/N system, (represented in figure 8.2), denoted as $P_N(n)$, $n=0,1,\ldots,N$, where in addition a fb constraint is assumed as:

$$\sum_{n=0}^{N} f(n)P_N(n) = \Phi_N \ , \tag{8.10}$$

where

$$f(n) = \begin{cases} 0 & \text{if } n<N \\ 1 & \text{if } n=N \end{cases} \tag{8.11}$$

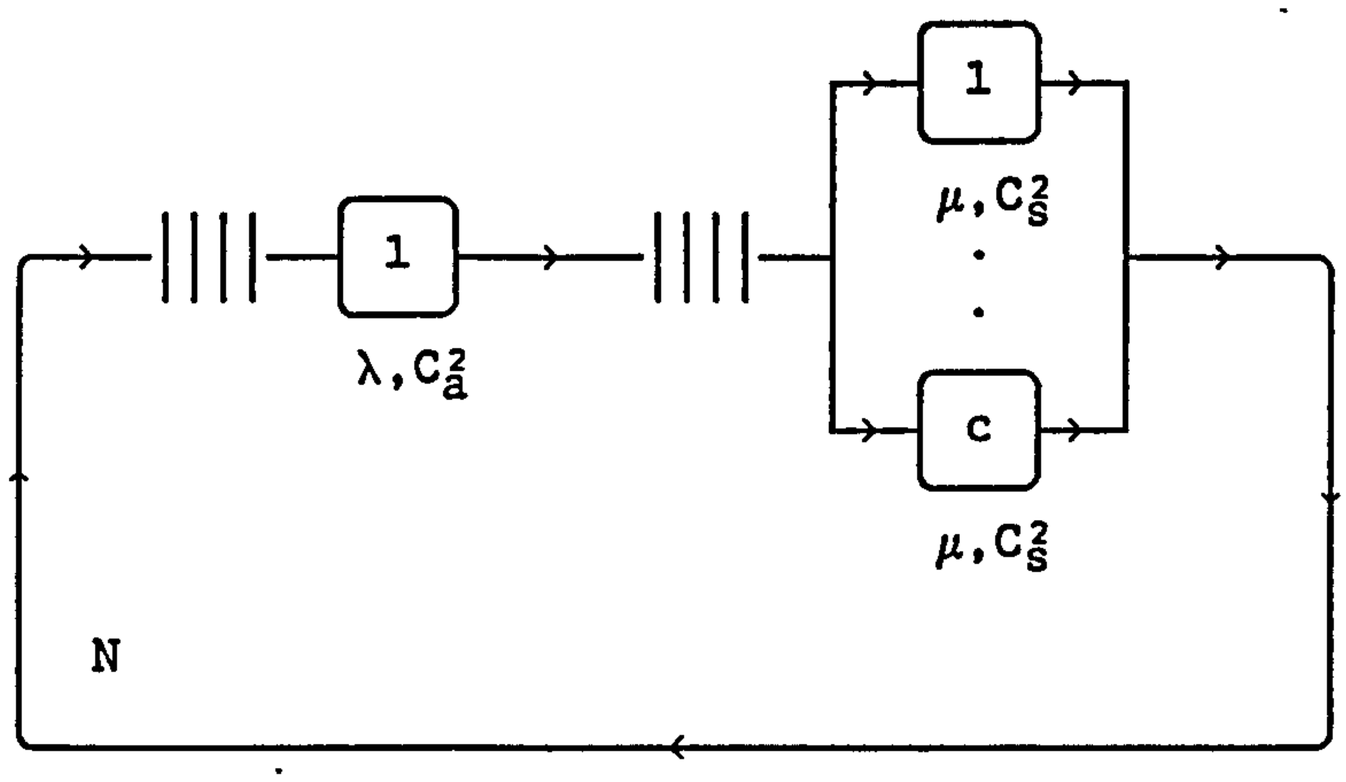

Figure 8.2. The GE/1/N||GE/c/N queue.

The ME solution is of the form:

$$P_N(0) = \frac{1}{Z_N}$$

$$P_N(n) = \frac{1}{Z_N} \prod_{j=1}^{n} g_j \, x^n \quad , \quad n=1,2,\ldots,c$$

$$P_N(n) = \frac{1}{Z_N} \prod_{j=1}^{c} g_j \, x^n \quad , \quad n=c+1,\ldots,N-1$$

$$P_N(N) = \frac{1}{Z_N} \prod_{j=1}^{c} g_j \, x^N \, y \quad , \tag{8.12}$$

where y is the extra fb multiplier. Then g_j, j=1,...,c, and x are given by (8.8), (8.9) and (8.5) respectively, while Z_N may be derived by applying the normalization constraint on distribution (8.12). The fb multiplier proves to be given by:

$$y = \frac{\lambda\beta_2\alpha_1 + c\mu\alpha_2}{c\mu\alpha_2\beta_2} \tag{8.13}$$

So the solution of this two stage cyclic system shares the same invariant (to the fixed level of multiprogramming N) multipliers g_j, j=1,...,c, and x with the solution of the related infinite capacity GE/GE/c system. Multiplier y may be derived by solving the fb equation:

$$\lambda[1-P_N(N)] = \sum_{k=1}^{c-1} k\mu P_N(k) + c\mu \sum_{k=c}^{N} P_N(k) \tag{8.14}$$

or alternatively, it may be derived as the invariant utilization multiplier of the dual distribution $P_N^*(n)$, n=0,...,N, which concerns the queue length of the single server facility. The form of the dual ME solution, subject to appropriate constraints, as described by Kouvatsos, [KOUV 86b], is:

$$P_N^*(0) = \frac{1}{Z_N^*}$$

$$P_N^*(n) = \frac{1}{Z_N^*} g' x'^{n} \quad , \; n=1,2,...,N-c$$

$$P_N^*(n) = \frac{1}{Z_N^*} g' \, x'^{n} \prod_{j=1}^{n+c-N} y'_j \quad , \; n=N-c+1,2,...,N \tag{8.15}$$

where g', x' and y'_j , j=1,...,c, are corresponding Lagrangian coefficients. The above solution at the limit $N \longrightarrow +\infty$ and under the

condition of stability $x'<1$ (or $c\mu<\lambda$), which implies that the bottleneck unit is the multiple server one, becomes:

$$P^*(n) = \frac{1}{Z^*}\, g'^{h(n)}\, x'^{n} \quad , n=0,1,\ldots$$

where $$Z^* = \lim_{N\to+\infty} Z_N^*$$

and $$h(n) = \begin{cases} 0 & \text{if } n=0 \\ 1 & \text{if } n>0 \end{cases}$$

Note that the above solution is of the GE/GE/1 form, and naturally so, since assymptotically the multiple server is considered saturated and thus produces a renewal GE-distributed arrival stream with parameters $(c\mu, C_s^2)$. So applying (8.14) at the limit, it easily follows that:

$$g' = \frac{\rho'(1-x')}{x'(1-\rho')}$$

where $$\rho' = \frac{c\mu}{\lambda}$$

and using the mql constraint assymptotically,

$$x' = \frac{<n>-\rho'}{<n>}$$

where $$<n> = \frac{\rho'}{2}\left[\, 1 + \frac{C_s^2 + \rho' C_a^2}{1-\rho'} \,\right]$$

is the mean queue length of the GE/GE/1 system. Substituting x' into g' it follows that:

$$g' = \frac{2\rho'}{C_s^2 + \rho' C_a^2 + \rho' - 1}$$

Using also the duality relations $P_N(n)=P_N^*(N-n)$, $n=0,\ldots,N$, it may be seen that (8.15) implies (8.12) and also that:

$$g'y = xx' = g_j y'_{c+1-j} = 1 \quad , \; j=1,2,\ldots,c$$

It may be verified that the above expression for g' implies the same value for y as relation (8.13) does. Note at this point that the definition of the constraints and subsequently the form (8.15) for the dual distribution $P^*_N(n)$, n=0,...,N, was arbitrary. This definition, however, is of particular importance to our decomposition method, because it specifies the relation between the "dual" normalizing constants, which are used to define the prior at every level of aggregation. Using the fact that $P^*_N(0)=P_N(N)$, (8.12) and (8.15) yield:

$$\frac{1}{Z^*_N} = \frac{1}{Z_N} \prod_{j=1}^{c} g_j \; x^N \; y \quad \Rightarrow$$

$$Z^*_N = \frac{Z_N}{\prod_{j=1}^{c} g_j \; x^N \; y} = \frac{1}{P_N(N)} \tag{8.16}$$

Using this as a definition for the "dual" prior in the MRE solution, (in the first level of aggregation, or even in a higher one, since the presence of a prior in the form of the solution, does not affect the manipulations presented so far), proved not to be sufficient in order to obtain the exact exponential solution of the network. So relation (8.16) and subsequently (8.15) form of "dual" distribution is not the proper one - in a decomposition sense - even though it appears to be the natural extension of the single server case. It has been found instead that the following definition of the dual solution should be used:

$$P^*_N(0) = \frac{1}{Z^*_N} \prod_{j=1}^{c} y'_j$$

$$P^*_N(n) = \frac{1}{Z^*_N} \prod_{j=1}^{c} y'_j \; g' x'^{n} \quad , \; n=1,\ldots,N-c$$

$$P_N^*(n) = \frac{1}{Z_N^*} \prod_{j=1}^{N-n} y_j' \; g'x'^{n} \quad , \; n=N-c+1,\ldots,N-1$$

$$P_N^*(N) = \frac{1}{Z_N^*} \; g'x'^{N} \tag{8.17}$$

which implies that:

$$xx' = g'y = 1 \tag{8.18a}$$

and

$$y_j' = g_j \quad , \; j=1,\ldots,c \tag{8.18b}$$

$$Z_N^* = \frac{Z(N)}{x^N y} = \frac{\prod_{j=1}^{c} y_j'}{P_N(N)} \tag{8.19}$$

This effectively implies that in the single server case and in the first level of aggregation what was defined to be the flow balance constraint, should have in fact been the utilization constraint of the dual distribution, i.e. instead of assuming that the "last" probability $P_N(N)$ is known, it could have been assumed that the sum of all probabilities $P_N(0)+P_N(1)+\ldots+P_N(N-1)$ is given, which is in fact the same information, presented in a different manner. That is why the MRE solution presented in the sixth chapter would not be affected by this alteration. Even though this observation has no practical consequence, it provides a different interpretation to the results. More precisely, recall that in the single server case, in the first level of aggregation, the fb multiplier proved to be invariant to the population N_1 of that level, while in all higher levels it was a load-dependent one. It can be seen now that this was due to the fact that the GE/GE/1/N system does not in fact need a fb multiplier to adjust it so that it satisfies the fb equation. It is sufficient to assume the mean queue length constraint and similar utilization constraints for both servers. Then by applying the limit to both dual queue length distributions these utilizations may be determined assymptotically. The resulting solution (astonishingly !)

satisfies the fb condition of the closed two stage cyclic system regardless of the actual population of jobs that circulate in it. Then of course, the fb multipliers $y_\ell(N_\ell)$, $\ell=2,\ldots,M$, are responsible for adjusting the MRE solution so that it expresses a marginally flow balanced network. These multipliers, however, do not play the same role as the first level ones, since in all levels, $\ell=2,\ldots,M$, unit Σ_ℓ is examined as it interacts with a subsystem and not with a simple unit.

Note that according to form (8.17), even solution (8.12) should be written as:

$$P_N(0) = \frac{1}{Z_N}\, g'$$

$$P_N(n) = \frac{1}{Z_N}\, g' \prod_{j=1}^{n} g_j\, x^n \quad , \quad n=1,2,\ldots,c$$

$$P_N(n) = \frac{1}{Z_N}\, g' \prod_{j=1}^{c} g_j\, x^n \quad , \quad n=c+1,\ldots,N-1$$

$$P_N(N) = \frac{1}{Z_N} \prod_{j=1}^{c} g_j\, x^N \quad , \tag{8.20}$$

As in the single server case though, the normalizing constants defined by (8.12) and (8.20) differ in a multiplicative constant and in case that these are used to define a prior at some level of decomposition they make no difference since the corresponding fb multipliers of the following level also differ in this same multiplicative constant, which consequently appears as a factor of the normalizing constant of that level and thus is cancelled. This is only due to the fact that the above multiplier g' is the utilization multiplier that corresponds to a single server.

So in level ℓ of decomposition, $\ell=2,3,\ldots,M$, the constraints assumed to be known for distribution $\{P_\ell(n_\ell/N_\ell),\ n_\ell=0,\ldots,N_\ell\}$, $N_\ell=1,\ldots,N$, and which are associated with unit Σ_ℓ (multiple (c_ℓ)

server), are, 1) the probabilities $P_\ell(n_\ell \geq n/N_\ell)$, $n=1,2,\ldots,\min(N_\ell,c_\ell)$, 2) the mean queue length constraint and 3) the fb constraint, expressed with respect to probability $P_\ell(N_\ell/N_\ell)$ as in the single server case. Then the form of the MRE solution is:

$$P_\ell(n_\ell/N_\ell) = \frac{G(\ell-1,N_\ell-n_\ell)}{G^*(\ell,N_\ell)} x_\ell^{n_\ell} \prod_{j=1}^{\min(c_\ell,n_\ell,N_\ell)} g_{\ell,j}\, y_\ell^{f_\ell(n_\ell)} \qquad (8.21)$$

$$n_\ell=0,1,\ldots,N_\ell$$

where
$$f_\ell(n_\ell) = \begin{cases} 0 & \text{if } n_\ell<N_\ell \\ 1 & \text{if } n_\ell=N_\ell \end{cases}$$

Clearly, $G(\ell-1,n_\ell)$, $n_\ell=0,\ldots,N_\ell$, is the prior defined in the previous level of aggregation, while $g_{\ell,j}$, $j=1,\ldots,\min(c_\ell,N_\ell)$, x_ℓ and $y_\ell \equiv y_\ell(N_\ell)$ are the multipliers corresponding to the constraints assumed. Also $G^*(\ell,N_\ell)$ is the normalizing constant. The flow balance equation to be used in the ℓth level, $\ell=2,\ldots,M$, for population N_ℓ is,

$$\sum_{k=1}^{\min(N_\ell,c_\ell)-1} k\mu_\ell P_\ell(k/N_\ell) + \min(N_\ell,c_\ell)\mu_\ell \sum_{k=\min(N_\ell,c_\ell)}^{N_\ell} P_\ell(k/N_\ell) =$$

$$= \sum_{i=0}^{\ell-1} \left[\sum_{k=1}^{\min(N_\ell,c_i)-1} k\mu_i \sum_{j=k}^{N_\ell} U_{i,\ell-1}(k,j) P_\ell(N_\ell-j/N_\ell) + \right.$$

$$\left. + \mu_i \min(N_\ell,c_i) \sum_{j=\min(N_\ell,c_i)}^{N_\ell} U_{i,\ell-1}(\min(N_\ell,c_i),N_\ell) P_\ell(N_\ell-j/N_\ell) \right] r_{i\ell}(\ell) \qquad (8.22)$$

where $r_{i\ell}(\ell)$ is, as denoted in the previous chapter, an element of the ℓth level routing matrix $R(\ell)$, and $U_{i,\ell}(k,N_\ell)$ is the probability that at level ℓ of aggregation, unit Σ_i, $i=0,\ldots,\ell-1$, has k jobs present, given that N_ℓ jobs circulate at that level, $k=1,2,\ldots,c_\ell$,

j=k,k+1,...,N. These probabilities satisfy the following relations:

$$U_{0,1}(n,N_1) = P_1(n/N_1) \ , \ n=1,\ldots,\min(N_1,c_0-1) \qquad (8.23a)$$

$$U_{0,1}(c_0,N_1) = \sum_{n=c_0}^{N_1} P_1(n/N_1) \ , \ \text{for } N_1>c_0 \qquad (8.23b)$$

$$U_{1,1}(n,N_1) = P_1(N_1-n/N_1) \ , \ n=1,\ldots,\min(N_1,c_1-1) \qquad (8.23c)$$

$$U_{1,1}(c_1,N_1) = \sum_{n=c_1}^{N_1} P_1(N_1-n/N_1) \ , \ \text{for } N_1>c_1 \qquad (8.23d)$$

$$U_{\ell,\ell}(n,N_\ell) = P_\ell(n/N_\ell) \ , \ n=1,\ldots,\min(N_\ell,c_\ell-1) \qquad (8.23e)$$

$$U_{\ell,\ell}(c_\ell,N_\ell) = \sum_{n=c_\ell}^{N_\ell} P_\ell(n/N_\ell) \ , \ \text{for } N_\ell>c_\ell \qquad (8.23f)$$

$$U_{i,\ell}(n,N_\ell) = \sum_{j=n}^{N_\ell} U_{i,\ell-1}(n,j)P_\ell(N_\ell-j/N_\ell) \ , \qquad (8.23g)$$

$$n=1,\ldots,\min(c_\ell,N_\ell), \ N_\ell=1,\ldots,N$$

Let's consider now the first level of aggregation, which is the most complicated one in terms of the form of the MRE solution, because both servers there may be multiple.

8.2 The $GE_1/c_1/N||GE_0/c_0/N$ system.

The exact solution of this system, which is represented in figure 8.3, is as follows, [KOUV 86b]. Let $P_N(n)$, n=0,...,N, be the queue length distribution of the second unit (GE_2). Then,

$$P_N(n) = \frac{u(n-1)}{d(\ n\ )} P_N(n-1) \ , \ n=1,\ldots,N \qquad (8.24a)$$

where

$$u(n) = \begin{cases} c_1\mu_1\beta_2 + c_0\mu_0\alpha_2\beta_1 & \text{if } c_0 \leq n \leq N-c_1 \\ (N-n)\mu_1\beta_2(1-\alpha_1\beta_1) & \text{if } n>\max(c_0-1,N-c_1) \\ [\,c_1\mu_1\beta_2+n\mu_0\alpha_2\beta_1\,]\alpha_2 & \text{if } n<\min(c_0,N-c_1+1) \\ (N-n)\mu_1\beta_2\alpha_2 & \text{if } N-c_1<n<c_0 \end{cases}$$

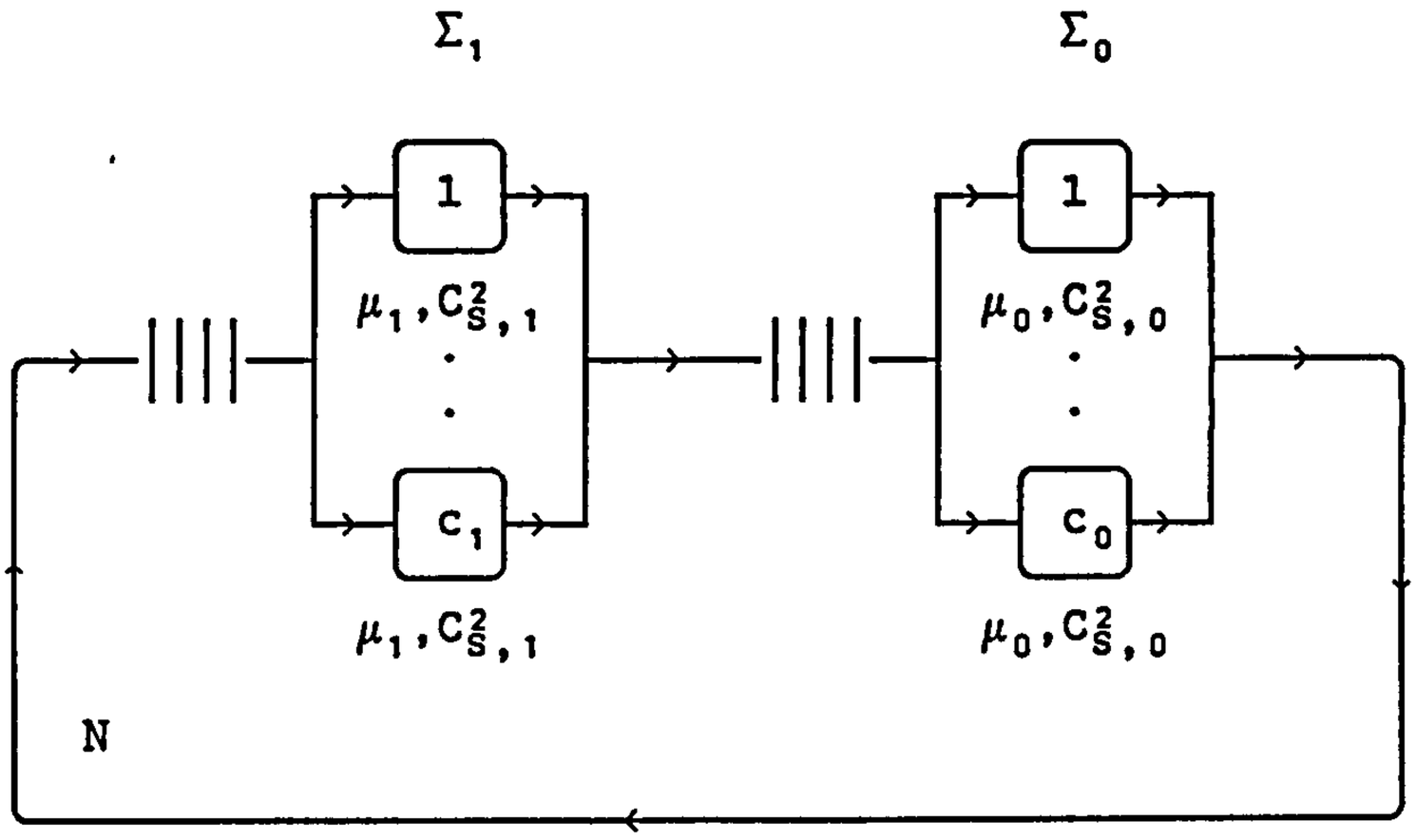

Figure 8.3. The two station $GE_1/c_1/N||GE_0/c_0/N$ cyclic queue.

$$d(n) = \begin{cases} c_1\mu_1\beta_2\alpha_1 + c_0\mu_0\alpha_2 & \text{if } c_0 \leq n \leq N-c_1 \\ [(N-n)\mu_1\beta_2\alpha_1 + c_0\mu_0\alpha_2]\beta_2 & \text{if } n > \max(c_0-1, N-c_1) \\ n\mu_0\alpha_2(1-\alpha_1\beta_1) & \text{if } n < \min(c_0, N-c_1+1) \\ n\mu_0\alpha_2\beta_2 & \text{if } N-c_1 < n < c_0 \end{cases}$$

with

$$\alpha_1 = \frac{C_{s,0}^2 - 1}{C_{s,0}^2 + 1} \quad , \quad \alpha_2 = 1 - \alpha_1$$

$$\beta_1 = \frac{C_{s,1}^2 - 1}{C_{s,1}^2 + 1} \quad , \quad \beta_2 = 1 - \beta_1$$

and

$$P_N(0) = \left[1 + \sum_{n=1}^{N} \prod_{j=1}^{n} \frac{u(j-1)}{d(j)} \right]^{-1} \qquad (8.24b)$$

The above solution, even though it is the exact one for the two stage cyclic system under consideration, does not reveal much about the constraints that would interpret it as a ME solution. This information is of course essential to our decomposition approach, since it is the only way of defining properly the normalizing constant to be used in the definition of the prior. The straight forward extension of the constraints assumed in the case of the GE/1/N||GE/c/N system, which lead to (8.20) form of solution (or the form (8.17) for the dual distribution) implies that in case $c_0+c_1<N$ the ME solution for the system of figure 8.3 is:

$$P_N(0) = \frac{1}{Z_N} \prod_{j=1}^{c_1} y_j$$

$$P_N(n) = \frac{1}{Z_N} \prod_{j=1}^{c_1} y_j \prod_{j=1}^{n} g_j \, x_0^n \quad , \; n=1,\ldots,c_0$$

$$P_N(n) = \frac{1}{Z_N} \prod_{j=1}^{c_1} y_j \prod_{j=1}^{c_0} g_j \, x_0^n \quad , \; n=c_0+1,\ldots,N-c_1$$

$$P_N(n) = \frac{1}{Z_N} \prod_{j=1}^{N-n} y_j \prod_{j=1}^{c_0} g_j \, x_0^n \quad , \; n=N-c_1+1,\ldots,N-1$$

$$P_N(N) = \frac{1}{Z_N} \prod_{j=1}^{c_0} g_j \, x_0^N \qquad (8.25)$$

By analogy, the dual distribution $P_N^*(n)$, $n=0,\ldots,N$, is of the form:

$$P_N^*(0) = \frac{1}{Z_N^*} \prod_{j=1}^{c_0} g'_j$$

$$P_N^*(n) = \frac{1}{Z_N^*} \prod_{j=1}^{c_0} g'_j \prod_{j=1}^{n} y'_j \, x_1^n \quad , \; n=1,\ldots,c_1$$

$$P_N^*(n) = \frac{1}{Z_N^*} \prod_{j=1}^{c_0} g'_j \prod_{j=1}^{c_1} y'_j \, x_1^n \quad , \; n=c_1+1,\ldots,N-c_0$$

$$P_N^*(n) = \frac{1}{Z_N^*} \prod_{j=1}^{N-n} g'_j \prod_{j=1}^{c_1} y'_j \, x_1^n \quad , \; n=N-c_0+1,\ldots,N-1$$

$$P_N^*(N) = \frac{1}{Z_N^*} \prod_{j=1}^{c_1} y'_j \, x_1^N \qquad (8.26)$$

and using the duality relations $P_N(n)=P_N^*(N-n)$, $n=0,\ldots,N$, it may be easily shown that:

$$x_0 x_1 = 1$$

and

$$g_j = g'_j \quad , \; j=1,\ldots,c_0$$

$$y_j = y'_j \quad , \; j=1,\ldots,c_1$$

Multipliers g_j, y_j and x_0 can be easily derived assymptotically, as

the multipliers of the related infinite capacity queues as follows. Under the assumptions of invariance for g_j, $j=1,\ldots,c_0$, y_j, $j=1,\ldots,c_1$, and x_0, and of the stability condition $x_0<1$, the limit of (8.25) (as $N \longrightarrow +\infty$) is:

$$P(0) = \frac{1}{Z}$$

$$P(n) = \frac{1}{Z} \prod_{j=1}^{n} g_j \, x_0^n \quad , \quad n=1,\ldots,c_0$$

$$P(n) = \frac{1}{Z} \prod_{j=1}^{c_0} g_j \, x_0^n \quad , \quad n=c_0+1,\ldots$$

where $$P(n) = \lim_{N\to+\infty} P_N(n) \quad , \quad n=0,1,\ldots$$

and $$Z = \left[\prod_{j=1}^{c_1} y_j \right]^{-1} \lim_{N\to+\infty} Z_N$$

The above solution may be identified as the solution of a GE/GE/c_0 queue, where the input process is generated by the saturated multiple server Σ_1, whose output is a renewal GE process with parameters $(c_1\mu_1, C^2_{s,1})$. Then multipliers g_j, $j=1,\ldots,c_0$, and x_0 are given by similar relations to (8.8) and (8.5), respectively, i.e.,

$$g_j = \frac{(c_1\mu_1\beta_2 + (j-1)\mu_0\alpha_2\beta_1)\alpha_2}{j\mu_0\alpha_2(1-\alpha_1\beta_1)} \frac{1}{x_0} \quad , \quad j=1,\ldots,c_0-1 \qquad (8.27)$$

and

$$g_{c_0} = \frac{(c_1\mu_1\beta_2 + (c_0-1)\mu_0\alpha_2\beta_1)\alpha_2}{c_1\mu_1\beta_2\alpha_1 + c_0\mu_0\alpha_2} \frac{1}{x_0} \qquad (8.28)$$

$$x_0 = \frac{c_1\mu_1\beta_2 + c_0\mu_0\alpha_2\beta_1}{c_1\mu_1\beta_2\alpha_1 + c_0\mu_0\alpha_2} \qquad (8.29)$$

$$\alpha_1 = \frac{C^2_{s,0} - 1}{C^2_{s,0} + 1} \quad , \quad \alpha_2 = 1 - \alpha_1$$

$$\beta_1 = \frac{C^2_{s,1} - 1}{C^2_{s,1} + 1} \quad , \quad \beta_2 = 1 - \beta_1$$

Making analogous assumptions about the dual solution (8.26) next,

multipliers y_j, $j=1,\ldots,c_1$, and x_1 may be identified as the multipliers of a GE/GE/c_1 queue, where the arrival GE renewal process has parameters $(c_0\mu_0, C_0^2)$. So,

$$y_j = \frac{(c_0\mu_0\alpha_2 + (j-1)\mu_1\beta_2\alpha_1)\beta_2}{j\mu_1\beta_2(1-\alpha_1\beta_1)} \frac{1}{x_1}, \quad j=1,\ldots,c_1-1 \tag{8.30}$$

and

$$y_{c_1} = \frac{(c_0\mu_0\alpha_2 + (c_1-1)\mu_1\beta_2\alpha_1)\beta_2}{c_0\mu_0\alpha_2\beta_1 + c_1\mu_1\beta_2} \frac{1}{x_1} \tag{8.31}$$

and

$$x_1 = \frac{1}{x_0} = \frac{c_0\mu_0\alpha_2 + c_1\mu_1\beta_2\alpha_1}{c_0\mu_0\alpha_2\beta_1 + c_1\mu_1\beta_2} \tag{8.32}$$

Let's demonstrate now that the form of solution just described is identical to the exact one, given by (8.24), in case that $N>c_0+c_1$, with a small example.

Example 8.1. The $GE_1/2/5||GE_2/2/5$ solution.

According to (8.25) and (8.27)-(8.32), the following relations are true:

$$\frac{P_5(1)}{P_5(0)} = g_1x_0 = \frac{2\mu_1\beta_2\alpha_2}{\mu_0\alpha_2(1-\alpha_1\beta_1)}$$

$$\frac{P_5(2)}{P_5(1)} = g_2x_0 = \frac{(2\mu_1\beta_2 + \mu_0\alpha_2\beta_1)\alpha_2}{2\mu_1\beta_2\alpha_1 + 2\mu_0\alpha_2}$$

$$\frac{P_5(3)}{P_5(2)} = x_0 = \frac{2\mu_1\beta_2 + 2\mu_0\alpha_2\beta_1}{2\mu_1\beta_2\alpha_1 + 2\mu_0\alpha_2}$$

$$\frac{P_5(4)}{P_5(3)} = \frac{x_0}{y_2} = \frac{2\mu_0\alpha_2\beta_1 + 2\mu_1\beta_2}{(2\mu_0\alpha_2 + \mu_1\beta_2\alpha_1)\beta_2}$$

$$\frac{P_5(5)}{P_5(4)} = \frac{x_0}{y_1} = \frac{\mu_1\beta_2(1-\alpha_1\beta_1)}{2\mu_0\alpha_2\beta_2}$$

Now, according to (8.24) the exact solution satisfies the following relations:

$$\frac{P_5(1)}{P_5(0)} = \frac{u(0)}{d(1)} = \frac{2\mu_1\beta_2\alpha_2}{\mu_0\alpha_2(1-\alpha_1\beta_1)}$$

$$\frac{P_5(2)}{P_5(1)} = \frac{u(1)}{d(2)} = \frac{(2\mu_1\beta_2 + \mu_0\alpha_2\beta_1)\alpha_2}{2\mu_1\beta_2\alpha_1 + 2\mu_0\alpha_2}$$

$$\frac{P_5(3)}{P_5(2)} = \frac{u(2)}{d(3)} = \frac{2\mu_1\beta_2 + 2\mu_0\alpha_2\beta_1}{2\mu_1\beta_2\alpha_1 + 2\mu_0\alpha_2}$$

$$\frac{P_5(4)}{P_5(3)} = \frac{u(3)}{d(4)} = \frac{2\mu_0\alpha_2\beta_1 + 2\mu_1\beta_2}{(2\mu_0\alpha_2 + \mu_1\beta_2\alpha_1)\beta_2}$$

$$\frac{P_5(5)}{P_5(4)} = \frac{u(4)}{d(5)} = \frac{\mu_1\beta_2(1-\alpha_1\beta_1)}{2\mu_0\alpha_2\beta_2}$$

So the two solutions are identical. Then the form (8.25) implies that:

$$Z_N = \frac{\prod_{j=1}^{c_1} y_j}{P_N(0)} \qquad (8.33)$$

while the "dual" normalizing constant, according to (8.26) is:

$$Z_N^* = \frac{\prod_{j=1}^{c_0} g_j}{P_N(N)} \qquad (8.34)$$

where y_j, $j=1,\ldots,c_1$, are given by (8.30)-(8.31) and g_j, $j=1,\ldots,c_0$, are given by (8.27)-(8.28).

Furthermore it has been found that the solution described by (8.25), (8.27)-(8.32) is the exact in the following cases:

1) Both servers are exponentially distributed,

2) both servers are GE-distributed and one of them is a single server,

3) both servers are GE-distributed and $c_0+c_1 \leq N+1$, $c_0, c_1 \geq 2$.

This solution is not exact when both servers are GE-distributed and $c_0+c_1>N+1$. In the algorithm that will be described in this chapter, the normalizing constants of the first level of aggregation will be defined according to (8.33)-(8.34), despite this problem. The good results obtained by this algorithm indicate that it may be the case that (8.33)-(8.34) are correct and the problem occurs in the form of

other probabilities then $P_N(0)$ and $P_N(N)$ in case that $c_0+c_1>N+1$.

Let's now proceed to describe the approximation of the flow to be used in the algorithm.

8.3 The flow approximation.

In levels $\ell=2,\ldots,M$, of aggregation, as in the single server case (algorithm 7.1), the assymptotic flow arriving at unit Σ_ℓ, is calculated by iteration in the subsystem $(\Sigma_0\Sigma_1\ldots\Sigma_{\ell-1})$ using once more a saturated bottleneck unit. This subsystem then is an open network and the first moment of the flow in such a network with multiple servers, (each of them having c_i homogeneous servers with mean service rate μ_i and SCV $C^2_{s,i}$, $i=0,1,\ldots,\ell-1$), is exactly the same as the first moment of the flow in a network with the same routing matrix $R(\ell)$ where each multiple server $\Sigma_i(c_i,\mu_i)$ is substituted by a single server with mean service rate $c_i\mu_i$. This is not the case, however, with the second moment of this assymptotic flow. In the absence of an appropriate formula for the SCV of the interdeparture times from a multi-server unit of GE-type, it has to be assumed that each multiple server behaves as a single one with mean service rate $c_i\mu_i$ and SCV equal to the SCV $C^2_{s,i}$ of each of the c_i servers. This type of approximation, which is used in the UME algorithm as well, will be refered to as the "heavy traffic approximation", since the multiple server is considered either empty or with all its servers busy. This effectively means that the procedure which iterates for the first two moments of the assymptotic flow is identical to the one used in algorithm 7.1 (procedure FLOWITERATION(ℓ)) the only differences being that 1) each centre Σ_i, $i=0,\ldots,\ell$, involved has service rate equal to $c_i\mu_i$, and 2) it is not only the utilization ρ_ℓ that is of interest to us here but the assymptotic input rate λ_ℓ at unit Σ_ℓ, which is to be used in the

definition of the multipliers, and which is given as:

$$\lambda_{\ell} = \sum_{k=0}^{\ell-1} \rho_k c_k \mu_k r_{k\ell}(\ell) \qquad (8.35)$$

λ_{ℓ} also defines the assymptotic utilization ρ_{ℓ} of unit Σ_{ℓ} as:

$$\rho_{\ell} = \frac{\lambda_{\ell}}{c_{\ell}\mu_{\ell}}$$

Then, similarly to the single server case, λ_{ℓ} and $C^2_{a,\ell}$ the assymptotic mean arrival rate and SCV are used to evaluate the invariant multipliers $g_{\ell,j}$, j-1,...,c_{ℓ}, and x_{ℓ} involved in (8.21).

Note at this point that in the ℓth level of decomposition the subsystem $(\Sigma_0\Sigma_1...\Sigma_{\ell})$ is solved for some populations N_{ℓ} less than c_{ℓ} and thus there is an option of evaluating the multipliers within the loop with index N_{ℓ} instead of considering them to be completely invariant. In other words, there is an option of assuming that unit Σ_{ℓ} has N_{ℓ} servers instead of c_{ℓ} when $N_{\ell}<c_{\ell}$. The difference of course is that $g_{\ell,j}$, j=1,...,min(N_{ℓ},c_{ℓ})-1 are calculated using (8.8) which is slightly different than (8.9) used to calculate $g_{\ell,j}$ for j=min(N_{ℓ},c_{ℓ}). Multiplier x_{ℓ} may also be dependent on N_{ℓ}, in the above sense. Despite the fact that it seems intuitively sound to consider that centre Σ_{ℓ} has min(N_{ℓ},c_{ℓ}) servers, the results do not reduce to the exact exponential ones under this option. It appears that server Σ_{ℓ} must be always treated as a c_{ℓ}-multiple server, in order to preserve the exactness at the level of the exponential network. This is probably due to the fact that by altering the structure of the server, its assymptotic behaviour is altered as well, i.e. there is not a unique related open network at every level of decomposition. Note also that in the exponential case only multiplier x_{ℓ} is responsible for this difference in the results, since in this case (8.8) for j=c and (8.9) reduce to identical values. On the other hand

it has been observed that the results detiorate if the utilization multipliers are evaluated independent of N_ℓ. Thus, in the algorithm that follows, the mql multiplier is considered to be completely invariant and is given as:

$$x_\ell = \frac{\lambda_\ell \beta_2 + c_\ell \mu_\ell a_2 \beta_1}{\lambda_\ell \beta_2 \alpha_1 + c_\ell \mu_\ell \alpha_2}$$

and the utilization multipliers are given as:

$$g_{\ell,j} = \frac{(\lambda_\ell \beta_2 + (j-1)\mu_\ell \alpha_2 \beta_1)\alpha_2}{j\mu_\ell \alpha_2 (1-\alpha_1 \beta_1)} \frac{1}{x_\ell} \quad , \quad j=1,\ldots,\min(c_\ell, N_\ell)-1$$

$$g_{\ell,\min(N_\ell,c_\ell)} = \frac{(\lambda_\ell \beta_2 + (\min(N_\ell,c_\ell)-1)\mu \alpha_2 \beta_1)\alpha_2}{\lambda_\ell \beta_2 \alpha_1 + \min(N_\ell,c_\ell)\mu_\ell \alpha_2} \frac{1}{x_\ell}$$

...

where

$$\alpha_1 = \frac{C_{s,\ell}^2 - 1}{C_{s,\ell}^2 + 1} \quad , \quad \alpha_2 = 1 - \alpha_1$$

$$\beta_1 = \frac{C_{a,\ell}^2 - 1}{C_{a,\ell}^2 + 1} \quad , \quad \beta_2 = 1 - \beta_1$$

and λ_ℓ is given by (8.35).

8.4 Extension of the MRE decomposition algorithm to multi-server networks.

The algorithm that is the extension of algorithm 7.1 to multi-server queueing networks may be described as follows.

Algorithm 8.1.

Input Parameters.

M+1 : number of units $(\Sigma_0, \Sigma_1, \ldots, \Sigma_M)$

N : number of jobs.

For unit Σ_i, i=0,...,M

μ_i^* : mean service rate.

c_i : number of servers.

C_i^2 : squared coefficient of variation of the service time of each of

the c_i servers.

For i,j=0,...,M

$r_{ij}(M)$: transition probability from unit Σ_i to unit Σ_j.

Procedure CREATEPARTITION

```
 begin
  For  ℓ=M-1  downto  1  do
    For  i=0  to  ℓ  do
      For  j=0  to  ℓ  do
        r_ij(ℓ) ← r_i;ℓ+1(ℓ+1) r_ℓ+1,j(ℓ+1) / (1-r_ℓ+1,ℓ+1(ℓ+1))
 end.
```

Step 1. { First level of aggregation }

CREATEPARTITION

Step 1.1. { Calculate invariant parameters }

FEEDBACKCORRECTION(1)

$$\rho_0 \leftarrow \frac{c_1\mu_1}{c_0\mu_0} \quad , \quad \rho_1 \leftarrow 1/\rho_0$$

$$\alpha_1 \leftarrow \frac{C_{s,0}^2-1}{C_{s,0}^2+1} \quad , \quad \alpha_2 \leftarrow 1-\alpha_1$$

$$\beta_1 \leftarrow \frac{C_{s,1}^2-1}{C_{s,1}^2+1} \quad , \quad \beta_2 \leftarrow 1-\beta_1$$

$$x_0 \leftarrow \frac{c_1\mu_1\beta_2 + c_0\mu_0\alpha_2\beta_1}{c_1\mu_1\beta_2\alpha_1 + c_0\mu_0\alpha_2}$$

For j=1 to c_0-1 do

$$g_{0,j} \leftarrow \frac{(c_1\mu_1\beta_2 + (j-1)\mu_0\alpha_2\beta_1)\alpha_2}{j\mu_0\alpha_2(1-\alpha_1\beta_1)} \frac{1}{x_0}$$

$$g_{0,c_0} \leftarrow \frac{(c_1\mu_1\beta_2 + (c_0-1)\mu_0\alpha_2\beta_1)\alpha_2}{c_1\mu_1\beta_2\alpha_1 + c_0\mu_0\alpha_2} \frac{1}{x_0}$$

$$x_1 \leftarrow 1/x_0$$

For j=1 to c_1-1 do

$$g_{1,j} \leftarrow \frac{(c_0\mu_0\alpha_2 + (j-1)\mu_1\beta_2\alpha_1)\beta_2}{j\mu_1\beta_2(1-\alpha_1\beta_1)} \frac{1}{x_1}$$

$$g_{1,c_1} \leftarrow \frac{(c_0\mu_0\alpha_2 + (c_1-1)\mu_1\beta_2\alpha_1)\beta_2}{c_0\mu_0\alpha_2\beta_1 + c_1\mu_1\beta_2} \frac{1}{x_1}$$

Step 1.2. { Evaluate the conditional distributions }

For N_1=1 to N do

For n_0=0 to N_1 do

Evaluate $P_1(n_0/N_1)$ using the exact solution (8.24)

$P_1(0/0) \leftarrow 1$ { trivially }

Step 1.3. { Find the bottleneck and evaluate the prior }

Case

$\rho_0 \leq 1$: begin

$b \leftarrow 1$

For N_1=1 to N do

$$G(1,N_1) \leftarrow \frac{\prod_{j=1}^{\min(N_1,c_1)} g_{1,j}}{P_1(0/N_1)}$$

end

$\rho_0 > 1$: begin

$b \leftarrow 0$

For N_1=1 to N do

$$G(1,N_1) \leftarrow \frac{\prod_{j=1}^{\min(N_1,c_0)} g_{0,j}}{P_1(N_1/N_1)}$$

end

$G(1,0) \leftarrow 1$ { trivially }

Step 1.4. { Evaluate the conditional utilization probabilities for units $\dot{\Sigma}_0$, Σ_1 }

For N_1=1 to N do begin

For k=1 to $\min(N_1,c_0)$ do

$$U_{0,1}(k,N_1) \leftarrow \begin{cases} P_1(k/N_1) & \text{if } k<c_0 \\ \sum_{j=c_0}^{N_1} P_1(j/N_1) & \text{if } k=c_0 \end{cases}$$

For k=1 to $\min(N_1,c_1)$ do

$$U_{1,,1}(k,N_1) \leftarrow \begin{cases} P_1(N_1-k/N_1) & \text{if } k<c_1 \\ \sum_{j=c_0}^{N_1} P_1(N_1-j/N_1) & \text{if } k=c_1 \end{cases}$$

Step 2. { ℓth level of aggregation }

For ℓ=2 to M do begin

Step 2.1. { Evaluate invariant parameters }

FLOWITERATION(ℓ-1) { Returns the values of λ_ℓ and $C^2_{a,\ell}$ }

$$\alpha_1 \leftarrow \frac{C^2_{s,\ell} - 1}{C^2_{s,\ell} + 1} \quad , \quad \alpha_2 \leftarrow 1-\alpha_1$$

$$\beta_1 \leftarrow \frac{C^2_{a,\ell} - 1}{C^2_{a,\ell} + 1} \quad , \quad \beta_2 \leftarrow 1-\beta_1$$

$$x_\ell \leftarrow \frac{\lambda_\ell\beta_2 + c_\ell\mu_\ell a_2\beta_1}{\lambda_\ell\beta_2\alpha_1 + c_\ell\mu_\ell\alpha_2}$$

Step 2.2. { Evaluate utilization, fb multipliers and conditional distributions }

For N_ℓ=1 to N do begin

Step 2.2.1. { Evaluate utilization multipliers }

For j=1 to $\min(N_\ell,c_\ell)-1$ do

$$g_{\ell,j} \leftarrow \frac{(\lambda_\ell\beta_2 + (j-1)\mu_\ell\alpha_2\beta_1)\alpha_2}{j\mu_\ell\alpha_2(1-\alpha_1\beta_1)} \frac{1}{x_\ell}$$

$$g_{\ell,\min(N_\ell,c_\ell)} \leftarrow \frac{(\lambda_\ell\beta_2 + (\min(N_\ell,c_\ell)-1)\mu\alpha_2\beta_1)\alpha_2}{\lambda_\ell\beta_2\alpha_1 + \min(N_\ell,c_\ell)\mu_\ell\alpha_2} \frac{1}{x_\ell}$$

Step 2.2.2. { Apply fb correction }

{ Evaluate $P_\ell(n_\ell/N_\ell)$, n_ℓ=0,...,N_ℓ, without normalization and fb }

$P_\ell(0/N_\ell) \leftarrow G(\ell-1,N_\ell)$

For n_ℓ=1 to N_ℓ do

$$P_\ell(n_\ell/N_\ell) \leftarrow G(\ell-1,N_\ell-n_\ell) \prod_{j=1}^{\min(n_\ell,c_\ell)} g_{\ell,j}\, x_\ell^{n_\ell}$$

{ Calculate fb multiplier $y_\ell(N_\ell)$ }

$$y_{\ell}(N_{\ell}) \leftarrow \frac{S_1 - S_2}{S_3}$$

where S_1 is the right-hand side of (8.22) fb equation and S_2, S_3 are calculated as follows,

Case

$c_{\ell}<N_{\ell}$: begin

$$S_2 \leftarrow \sum_{k=1}^{c_{\ell}-1} k\mu_{\ell}P_{\ell}(k/N_{\ell}) + c_{\ell}\mu_{\ell} \sum_{k=c_{\ell}}^{N_{\ell}-1} P_{\ell}(k/N_{\ell})$$

$$S_3 \leftarrow c_{\ell}\mu_{\ell}P_{\ell}(N_{\ell}/N_{\ell})$$

end

$c_{\ell}>N_{\ell}$: begin

$$S_2 \leftarrow \sum_{k=1}^{N_{\ell}-1} k\mu_{\ell}P_{\ell}(k/N_{\ell})$$

$$S_3 \leftarrow N_{\ell}\mu_{\ell}P_{\ell}(N_{\ell}/N_{\ell})$$

end

Step 2.2.3. { Incorporate fb correction and normalize }

$$P_{\ell}(N_{\ell}/N_{\ell}) \leftarrow P_{\ell}(N_{\ell}/N_{\ell})y_{\ell}(N_{\ell})$$

For $n_{\ell}=0$ to N_{ℓ} do

$$P_{\ell}(n_{\ell}/N_{\ell}) \leftarrow \frac{P_{\ell}(n_{\ell}/N_{\ell})}{\sum_{k=0}^{N_{\ell}} P_{\ell}(n_{\ell}/N_{\ell})}$$

Step 2.2.4. { Evaluate the prior }

Case

$$\rho_{\ell}\leq 1 : G(\ell,N_{\ell}) \leftarrow \frac{G(\ell-1,N_{\ell})}{P_{\ell}(0/N_{\ell})}$$

$$\rho_{\ell}>1 : G(\ell,N_{\ell}) \leftarrow \frac{\prod_{j=1}^{\min(N_{\ell},c_{\ell})} g_{\ell,j}}{P_{\ell}(N_{\ell}/N_{\ell})}$$

end { of loop with index N_{ℓ} }

$P_{\ell}(0/0) \leftarrow 1$, $G(\ell,0) \leftarrow 1$ { trivially }

Step 2.3. { Evaluate conditional utilization probabilities }

For $i=0$ to $\ell-1$ do

For $N_\ell=1$ to N do

For $k=1$ to $\min(N_\ell, c_i)$ do

$$U_{i,\ell}(k,N_\ell) \leftarrow \sum_{j=k}^{N_\ell} U_{i,\ell-1}(k,j) P_\ell(N_\ell - j/N_\ell)$$

For $N_\ell=1$ to N do

For $k=1$ to $\min(N_\ell, c_\ell)$ do

$$U_{\ell,\ell}(k,N_\ell) \leftarrow \begin{cases} P_\ell(k/N_\ell) & \text{if } k<c_\ell \\ \sum_{j=c_\ell}^{N_\ell} P_\ell(j/N_\ell) & \text{if } k=c_\ell \end{cases}$$

end { of loop with index ℓ }

Step 3. { Evaluate marginal probabilities }

This step is identical to step 3 of algorithm 6.1.

8.5 Validation of the MRE decomposition algorithm for multi-server networks.

Algorithm 8.1 provides the exact solution when the network has exponentially distributed service times, while it is the extension of algorithm 7.1 presented in the previous chapter for single server networks. The experimentally established rules for determining the enumeration of the units of the network, defined in the previous chapter, seem to be valid for this algorithm as well. So once more it is the variability that determines the way of coupling when the SCVs of the network's centres differ. The rule "join the unit with high variability last" is used for such networks and in the tests that follow the corresponding sequence is denoted as SEQ(SCV). In case that the decision cannot be based on this rule (case all SCVs are identical and units do not have feedback streams), the enumeration is

made as described in the two previous chapters according to the highest interaction rate, which here is defined using the full capacity mean service rate ($c_i\mu_i$) for centre Σ_i with c_i servers, i=0,...,M. The corresponding sequence is denoted as SEQ(msr).

In tables 8.1-8.10 (App. VI), algorithm 8.1 is tested versus the exact solution and the UME approximate algorithm - as this later has been extended by Kouvatsos and Almond to multi-server networks - in three types of network configurations, i.e. tandem (tables 8.1-8.2), central server (tables 8.3-8.4) and fully connected (tables 8.5-8.10) ones, and for various combinations of the networks' parameters. Its performance is good although not as good if judged by the standards set by algorithm 7.1. This was expected since the "heavy traffic approximation" imposes extra assumptions. The effect of this approximation is also noticeable in the performance of the UME algorithm. So both algorithms may be characterized as good and reliable approximations but not as reliable as their corresponding single- server versions, since sometimes the maximum error tolerances exceed the bounds of "good performance". It is also worth noticing the improved performance of the exponential solution as compared to the results in the single-server networks, especially when the centre with high variability is a multiple server one (see tables 8.3, 8.5) indicating possibly that this centre tends to behave as an exponential one. An explanation for this may be that the output from a multi-server unit is closer to a pooled stream, which in turn is closer to an exponential flow process regardless of the distribution of the merging streams [COX 54]. When the high variability unit is a single server one, however, (see tables 8.2, 8.4, 8.10), its performance detiorates rapidly to the poor standards demonstrated in table 6.1-6.20 and 7.1-7.10.

Returning to algorithm 8.1, the fb failures recorded in these

tests were 23% and the patterns of occurance are similar to the ones observed in the single server case, i.e. failures are rare (0% in these tests) in the case of tandem configuration, while their frequency increases in central server and fully connected networks. This increase in the above frequency may be associated with the poorer approximation of the assymptotic flow as well as the assumptions made about the form of the MRE solution in order to define the prior in the first level.

8.6 Discussion.

In this chapter an extension of the MRE decomposition approximate algorithm to cover the multiple server operation in networks of queues has been presented. It is interesting to observe that the structure of the adopted decomposition scheme allows us to examine one unit at every level of aggregation and thus it is possible to incorporate constraints that correspond to certain type of ME solution that is appropriate for a certain type of server. In the single server case for instance at every level of aggregation the constraints were close in form to the ones used in the GE/GE/1, GE/GE/1/N systems, while in the multiple server case these forms of constraints were substituted by the ones used in the $GE_1/GE_2/c_2$, $GE_1/1/N||/GE_2/c_2/N$ queues. The difficulty, however, here is that the form of solution adopted in the first level of decomposition, and which is essential for the proper definition of the prior, is not exact for all combinations of the parameters c_0, c_1, N_1. So an improvement is expected if some other, more successful form is used. A certain improvement is also expected if the "heavy traffic approximation" is substituted by a proper iteration for multi-server open networks. This of course requires the exact formula for the SCV of the interdeparture times from a GE/GE/c system. These problems are

open to suggestions and future work.

CHAPTER IX

CONCLUSIONS AND FUTURE WORK

9.1 Thesis summary.

In this thesis we have proposed a new, information theoretic approach for the implementation of decomposition schemes for the exact and approximate analysis of closed queueing networks with a single class of customers, FCFS scheduling and single or multiple servers, using for the first time the concept of Minimum Relative Entropy with fully decomposable constrains initially proposed by Shore [SHOR 82b].

In the second chapter, the two major decomposition techniques have been presented as applied in a hierarchical, multi-level fashion for the analysis of queueing network models. The restrictions imposed on the network configuration by these methods have been pointed out.

In the third chapter, the ME principle was introduced together with the generalized expontential distribution (GE). Given the first two moments of the interarrival and service times, the GE distribution appears to be the best hypothesis made - in an information theoretic sense - at the level of some simple queueing systems. This may be an explanation for the success of the ME solution adopted by the UME approximate algorithm at the network level. The high standards of accuracy set by this algorithm provided the motivation as well as the target in seeking for an alternative approach via decomposition. The GE/GE/1/N queue was presented in detail since it was the building block to this investigation in more than one sense. Firstly, the solution of this system had to be used in the first level of aggregation, and secondly (and most importantly), the assumption of invariance for the mql and

utilization coefficients and their analytic derivation via an assymptotic connection to an infinite capacity queue were the steps that were adopted and generalized at each level of aggregation in order to approximate the associated mql and utilization multipliers of that level.

In the fourth chapter Shore's [SHOR 82b] method for solving the MRE optimization problem was introduced, subject to fully decomposable constraints corresponding to a single level partition of the state space of the system. This kind of analysis was used as an information theoretic building block in later chapters for the multi-level hierarchical decomposition of general QNMs.

In the fifth chapter a set of fully decomposable subset and aggregate mean queue length and flow-balance constraints were assumed in the context of a multi-level partition implied by the variable aggregation decomposition scheme. By aplying an assymptotic approach in order to calculate analytically the mql multiplier at every level of decomposition, it is shown that the MRE decomposition approximate solution is identical to the exact solution of the network as if this was separable. There only the first moment of the flow was used through the conditional and marginal flow-balance equations, which carry the same flow information as the equations satisfied by the variable aggregation scheme [COUR 77], under the definition of the service rates of the queues (composite or not), at every level of aggregation, proposed by Courtois. In fact because these definitions were adopted (but expressed in a different form using the level as well as load-dependent utilizations of the CPU), the demonstration of the exactness of the MRE soluton for separable networks was restricted to the central server network configuration. The information theoretic approach gave us a first alternative and very revealing view of the problem faced by the variable aggregation

method, as described by Courtois [COUR 77], (section 5.3.4.). More general network configurations could be tackled using Norton's reduction decomposition scheme, yet restrictions were imposed by the fact that the MRE method could be applied only via a partition of the network's state space and not a general decomosition amenable to Norton's technique.

In the sixth chapter a generalization of the MRE form of solution was presented by adding the utilization constraint at every level of decomposition. This brought the form of MRE solution to be similar - in an information theoretic sense - to the one assumed by Kouvatsos [KOUV 86a, KOUV 86b] for queues with generally distributed service times. The decomposition algorithms that implemented this solution proved to be as successful in terms of accuracy as the UME algorithm, for queueing networks with GE-distributed service times. Still, however, none of them was applicable to QNMs with arbitrary configuration. At this stage the flow conservation problem inherited in Courtois methodology, and which was extended in applications of Norton's reduction scheme, was fully exposed (section 6.6), and its interpretation via an information theoretic standpoint was truly unique.

Equally unique was the view of the solution given to this problem by Vantilborgh [VANT 78]. The concept of subparallelism introduced by him was used in the seventh chapter to preserve flow conservation and produce a universal decomposition algorithm for arbitrary network configurations, based on a hierarchical partition of the network's state space.

Finally, in the eighth chapter, and for completeness purposes, a generalization of this algorithm was proposed in order to tackle networks with multiple servers of GE-type.

9.2 Discussion and future work.

The MRE approximate solutions proposed in this thesis, could be considered more generally as an approximation for networks with arbitrarily (G) distributed service times, as indeed is the case with the ME approximate solution used in the UME algorithm. Alternative flow and assymptotic mql formulae can be used as appropriate. The validation of algorithms at the level of different than GE distributions could be a subject of future work.

The tools developed by this investigation in a way impose (and thus restrict) the selection of a general form of distribution. For the GE-type of distribution, however, the tests showed that these tools are very good and reliable. The GE by itself is a most appropriate distribution for modelling flow and service times of bulky and bursty nature and which is the case in certain types of queueing networks, e.g. computer communication networks. It remains an open problem whether distributions defined by higher moments can be identified as entropy distributions, in which case this decomposition approach can be easily generalized (given of course that appropriate flow formulae are available).

The success of the MRE decomposition algorithm should not be attributed only to its successful product form, but to the estimation of the assymtotic flow as well. Note that this assymptotic flow was not used to approximate the actual flow of the closed network, due to the fact that we've assumed that the mql and utilization multipliers at all levels are common in the solution of the closed system and the solution of its related open one (as $N \longrightarrow +\infty$). Thus in the future this way of defining the mql and utilization multipliers could be used in the UME entropy as well and in place of the fixed-population-mean technique (FPM) [KOUV 86c]. This modification could provide an alternative version of this algorithm, certainly

computationally more efficient and possibly as accurate as the original UME approximation (for more details on this see Appendix V). A joint attempt with my colleague John Almond to develop such a new version produced promising results. Further work is needed however.

This attempt revealed another interesting point to our investigation. When the FPM technique was substituted by the assymptotic one proposed here and used in the decomposition algorithms, failures in flow balancing the network were recorded without influencing the average statistics significantly, as has been the case with our algorithms. The practical difference between the FPM technique and the assymptotic one is that the flow in the "pseudo open" network [KOUV 86c] is more moderate than the flow of the related open network (the bottleneck in the former case is never saturated). Hence, an interesting subject for future investigation could be the effect that the FPM technique has on these decomposition algorithms and in particular with respect to the fb failures.

Further work is also needed towards the derivation of the exact formula for the SCV of the interdeparture times from a GE/GE/c queue. This formula could significantly improve the performance of algorithm 8.1 for multi-server networks with GE-distributed service times. To this end, the methodology introduced in chapter 3 and in the corresponding Appendix I, for the derivation of the qld as seen by a departing job, may be found useful.

APPENDIX I

ANALYTIC RESULTS (CHAPTER III)

The bulk interpretation of the GE distribution is used to provide proofs for the analytic results presented in chapter 3.

Corollary I.1 The bulk size B is geometrically distributed, with parameter σ, i.e.,

$$\Pr\{B=k\} = \sigma(1-\sigma)^{k-1} \qquad (I.1)$$

Proof.

Consider the head of the bulk, that at a time instant completes his exponential($\sigma\lambda$) service. The size of this bulk is one (B=1), if the next customer in the queue chooses the exponential branch with probability σ. Thus, $\Pr\{B=1\}=\sigma$. Then, the bulk size is k>1, if k-1 customers follow the head of the bulk, through the null branch, i.e. without service, while the kth customer in the queue selects the exponential branch. Hence,

$$\Pr\{B=k\} = \sigma(1-\sigma)^{k-1} \quad , \; k=1,2,...$$

Q.E.D.

Clearly, r.v. B is defined by a sequence of independent Bernoulli trials, since service times are independently distributed, by assumption.

Consider next a tagged customer within a bulk. Let N_f and N_p denote the r.v. that count the number of customers who follow and precede, respectively, the tagged customer within the same bulk.

Corollary I.2. Random variables N_f and N_p follow a modified geometric distribution with parameter σ, i.e.,

$$\Pr\{N_f=n\} = \Pr\{N_p=n\} = \sigma(1-\sigma)^n$$

Proof.

Both results follow from corollary I.1 and the memoryless property of the geometric distribution, [FELL 68, p. 328].

Q.E.D.

Let N(t) be the r.v. that counts the arrivals that occur within time t.

Theorem I.1. The underlying counting process of a GE renewal process (arrival process) with parameters σ,λ is given by a compound Poisson process as:

$$\Pr\{N(t)=n\} = \begin{cases} \sum_{k=1}^{n} \frac{(\sigma\lambda t)^k}{k!} e^{-\sigma\lambda t} \binom{n-1}{k-1} \sigma^k (1-\sigma)^{n-k} & , n\geq 1 \\ e^{-\sigma\lambda t} & , n=0 \end{cases} \qquad (I.2)$$

Proof..

The probability of not having any arrivals within time t is equal to the probability of not having any bulk occurances during time t. Since time intervals between bulk occurances are exponentially($\sigma\lambda$) distributed, it follows that:

$$\Pr\{N(t)=0\} = e^{-\sigma\lambda t} \qquad (T1.1)$$

Let's denote by K(t) the number of bulk occurances during time t. Then K(t) follows a Poisson distribution and:

$$\Pr\{K(t)=n\} = \frac{(\sigma\lambda t)^k}{k!} e^{-\sigma\lambda t} \qquad (T1.2)$$

If $\Pr\{N(t)=n/K(t)=k\}$, $k\leq n$, is the conditional probability that n arrivals will occur given that k bulk arrivals occur during time t, then clearly for $n\geq 1$:

$$\Pr\{N(t)=n\} = \sum_{k=1}^{n} \Pr\{N(t)=n/K(t)=k\}\Pr\{K(t)=k\} \qquad (T1.3)$$

Given now that K(t)=k, N(t) is the sum of k random variables, each of them following a geometric(σ) distribution. Thus, Pr{N(t)=n/K(t)=k} is described by a negative binomial distribution, [FELL 68, p. 165]:

$$\Pr\{N(t)=n/K(t)=k\} = \binom{n-1}{k-1} \sigma^k (1-\sigma)^{n-k} \qquad \text{(T1.4)}$$

Substituting (T1.2) and (T1.4) into (T1.3) the result follows.

Q.E.D.

An alternative proof of Theorem I.1, following a more classical approach, can be found in [XENI 89].

Let's now examine, how this concept of "ordered" bulks can be used in order to relate the "*outside observer's*" equilibrium queue length distribution of a queueing system, to the "*arriver's*" and "*departer's*" ones.

There are three, generally different, viewpoints of a queueing system, in the sense that there are three different definitions of the equilibrium queue length distribution. These are:

1) The observer's point of view, which corresponds to a queue length probability distribution $\{P_n,\ n=0,1,2,...\}$, with P_n defined as the proportion of time that the system spends in state n (n is the number of jobs present in the system), in the long run of course.

2) The arriver's point of view, with probability distribution $\{P^{(a)}(n),\ n=0,1,2,...\}$, where $P^{(a)}(n)$ is the proportion of arriving jobs that find the system at state n.

3) The departer's point of view, with probability distribution $\{P^{(d)}(n),\ n=0,1,2,...\}$, where $P^{(d)}(n)$ is the proportion of departing jobs that leave the system at state n.

There are two well-established results concerning the relation between these three distributions. The first, [COOP 81, pp. 77-78], states that, when jobs arrive in the system in a Poisson fashion (exponentially distributed interarrival times), the arriver's and

observer's queue length distributions are identical, thus:

$$\text{Poisson input} \Rightarrow P_n = P^{(a)}(n),\ n=0,1,... \qquad (I.3)$$

The second result, [COOP 81, pp. 185-188], concerns a class of general systems, defined as those where state changes occur one at a time, or in other words, an arrival or departure changes the state of the system by one. In such systems, the arriver's and departer's queue length distributions are identical, i.e.,

$$p^{(a)}(n) = p^{(d)}(n)\ ,\ n=0,1,... \qquad (I.4)$$

It is evident that the only requirement for (I.4) to be true, is that service and arrival processes are not bulk processes. So, systems which involve the GE distribution do not have this property. Let's see how, by examining the three definitions closer and by applying the concept of ordered bulks, the above result may be extended to include GE distributions.

The M/GE/1 queue.

In this system, since arrivals form a Poisson stream, (I.3) is true. From the above discussion though, it is not clear whether (I.4) holds. It is evident however, that (I.4) is true for any $M/H_2/1$ system, even for a system where μ_1 (see fig. 3.2), grows very large, in which case the H_2 server tends to behave as a GE one. Departures form groups (bulks), with minute time intervals between members of the same group. What, however, makes (I.4) true is that a member of a bulk, which let not be the last, considers,at the time instant of its departure, the next member of the same bulk as being still in the system. So, actual time between departures is not essential, it is merely used in the hypothesis of (I.4), in order to separate individual departures and thus define, as just stated, the point of view of a departer. It does not matter, whatsoever, that time

intervals between individual departures may tend to be zero. Thus, (I.4) result may be applied, even when the H_2 is substituted with a GE, under the concept of ordered bulks, in the sense that a multiple (bulk) departure at a time instant is considered ordered and thus analyzed to the individual departures that form it. This way the important point, that an individual departer considers members of his bulk as being still in the system, at the instant of his departure, is preserved.

The question that arises here is, "since result (I.4) can so easily be extended to bulk systems, what is the interpretation of the restriction imposed by the hypothesis?". The answer is that there is another viewpoint for a departer, that may be defined in such a system. More precisely, it concerns the view of the system by the bulk. Let's denote by $P^{(bd)}(n)$ the probability that a departing bulk leaves n customers in the system. This view is being shared by all members of the same bulk and clearly $P^{(bd)}(n) \neq P^{(d)}(n)$. It is exactly this view that the restriction intends to exclude from satisfying (I.4). In fact, the following corollary relates distribution $\{P^{(bd)}(n),\ n=0,1,...\}$ to the observer's one.

Corollary I.3 In a M/GE/1 queue, distribution $\{P^{(bd)}(n),\ n=0,1,...\}$ is related to the observer's one $\{P_n,\ n=0,1,...\}$ through the following relations:

$$P^{(bd)}(0) = 1 - [\tau(1-\rho)+\rho](1-P_0)$$

$$P^{(bd)}(n) = [\tau(1-\rho)+\rho]P_n \quad , \ n \geq 1$$

where τ and μ are the parameters of the GE server ($\tau = 2/(1+C_s^2)$) ρ is the utilization of the server, i.e., $\rho = \lambda/\mu$, where λ is the parameter of the exponentially distributed interarrival times.

Proof.

Consider the "head" (first member) of a departing bulk. Let N_d be

the number of customers that he leaves in the system under his point of view, i.e., $P^{(d)}(n)=Pr\{N_d=n\}$, while N_{bd} is the number of customers that the bulk, in which he belongs, leaves in the system, i.e., $P^{(bd)}(n)=Pr\{N_{bd}=n\}$. Then by law of total probability, for $n\geq 1$:

$$P^{(bd)}(n) = \sum_{k=n}^{\infty} Pr\{N_{bd}=n/N_d=k\}P^{(d)}(k) \qquad (C3.1)$$

Now since:

$$P^{(d)}(k) = P^{(a)}(k) = P_k \quad , k=0,1,2,...$$

(C3.1) may be expressed with respect to the observer's distribution, which in a M/GE/1 system is, [AFFE 83]:

$$P_k = (1-\rho)\tau\left[\frac{\rho}{\tau(1-\rho)+\rho}\right]^k \quad , k\geq 1 \qquad (C3.2)$$

It is also clear, from the definition of N_{bd} and N_d that the event $\{N_{bd}=n/N_d=k\}$ is equivalent to that k-n customers follow the head of the bulk. And since this last number is decided by a sequence of independent Bernoulli trials:

$$Pr\{N_{bd}=n/N_d=k\} = \tau(1-\tau)^{k-n} \qquad (C3.3)$$

Substituting (C3.2), (C3.3) into (C3.1):

$$P^{(bd)}(n) = \sum_{k=n}^{\infty} \tau(1-\tau)^{k-n}(1-\rho)\tau\left[\frac{\rho}{\tau(1-\rho)+\rho}\right]^k \quad , k\geq 1$$

which after some manipulation yields:

$$P^{(bd)}(n) = [\tau(1-\rho)+\rho](1-\rho)\tau\left[\frac{\rho}{\tau(1-\rho)+\rho}\right]^n$$

and using (C3.2):

$$P^{(bd)}(n) = [\tau(1-\rho)+\rho]P_n \quad , n\geq 1 \qquad (C3.4)$$

The value of $P^{(bd)}(0)$ follows, using (C3.4) and the fact that

$\{P^{(bd)}(n),\ n=0,1,...\}$ is normalized.

Q.E.D.

In order to obtain the interdeparture time distribution, however, $\{P^{(d)}(n),\ n=0,1,2,...\}$ distribution should be used or equivalently the observer's one $\{P_n,\ n=0,1,2,...\}$.

The GE/G/c system.

Let's examine now, another class of systems, where the arrival process is GE-distributed, and there is an arbitrary c ($c \geq 1$), number of servers with an arbitrary but continuous (non-bulk) service time distribution G. Here, in order to obtain the interdeparture distribution and the corresponding statistics, the departer's queue length distribution should be used. Due to the assumption of a continuous G distribution, there is a unique definition for the departer's viewpoint, since there are not bulk departures. The problem here has been transfered to the arriver's distribution. Similarly to the previous section, there are two points of view for the arriver, under the concept of ordered bulks. The first and most useful is related to the limiting interpretation of GE. Under this view, the arriver considers members of his own bulk that precede him (if any) as being already in the system at the instant of his arrival. The corresponding distribution is denoted as $\{P^{(a)}(n),\ n=0,1,...\}$. So the individual arriver considers as state n, the number of customers that were already queueing just before the bulk arrival, plus these members of his own bulk that move ahead of him, as if minute time intervals separated them. This point of view is identical to the departer's one, i.e.,

$$P^{(a)}(n) = P^{(d)}(n) \quad , n=0,1,2,... \qquad (I.5)$$

as a consequence of the limiting interpretation and result (I.4). Hence, the problem here is to relate this arriver's distribution to

the observer's one $\{P_n, n=0,1,...\}$.

The second point of view for the arriver, which is being also shared by all members of the same bulk, is the number of customers that were present in the system just before the bulk arrival instant. Let's denote the associated distribution as $\{P^{(ba)}(n), n=0,1,...\}$. Since bulks arrive in a poisson fashion, result (I.3) may be applied and thus this second viewpoint is identical to the observer's one, i.e.,

$$P^{(ba)}(n) = P_n \quad , n=0,1,2,... \tag{I.6}$$

Corollary I.4. In a GE/G/c system, where G is a non-bulk distribution, thearriver's and departer's queue length distributions are related to the observer's ones by:

$$P^{(d)}(n) = P^{(a)}(n) = \sum_{k=0}^{n} \sigma(1-\sigma)^{n-k}P_k \quad , n=0,1,2,... \tag{I.7}$$

and if $P_a^*(z)$, $P_d^*(z)$ and $P^*(z)$ are the z-transforms that correspond to these distributions:

$$P_d^*(z) = P_a^*(z) = \frac{\sigma}{1-(1-\sigma)z} P^*(z) \tag{I.8}$$

Proof.

The first part of the equations (I.7), (I.8) is exactly (I.5). It remains to express $P^{(a)}(n)$ with respect to $P^{(ba)}(n)$. Let N_a be the r.v. that corresponds to $\{P^{(a)}(n), n=0,1,...\}$ distribution, i.e., $P^{(a)}(n)=Pr\{N_a=n\}$. Let also N_{ba} be the r.v. that corresponds to $\{P^{(ba)}(n), n=0,1,...\}$, i.e., $P^{(ba)}(n)=Pr\{N_{ba}=n\}$. Then:

$$P^{(a)}(n) = \sum_{k=0}^{n} Pr\{N_a=n/N_{ba}=k\}P^{(ba)}(k) \tag{C4.1}$$

But the conditional probability $Pr\{N_a=n/N_{ba}=k\}$, $k \leq n$, is exactly the

probability that n-k customers precede this individual arriver in the same bulk, and from corollary I.2:

$$Pr\{N_a=n/N_{ba}=k\} = \sigma(1-\sigma)^{n-k} \ , \ k\leq n, \ n=0,1,... \qquad (C4.2)$$

Substituting (C4.2) into (C4.1) and using (I.6), (I.7) follows.

Then, using the definition of the z-transform:

$$P_a^*(z) = \sum_{k=0}^{\infty} z^k p^{(a)}(k)$$

and substituting $P^{(a)}(k)$ by the summation in (I.7), (I.8) follows after some manipulation.

Q.E.D.

The GE/GE/1 system.

This system is the most complicated one, since all points of view defined so far are needed here. So, the notation, introduced in the previous two sections, is used to express the two different viewpoints of the arriver and the two of the departer.

Here, in order to derive the interdeparture time distribution, the only case that a departer leaves an empty system must be the case where this departer has spent some time in the system and at the point of his departure there is no other customer queueing behind him. This is important, because in the situation where a bulk arrives to find the system empty, several customers may depart without delay, before someone chooses the exponential branch and forces those who follow him to queue. Consider one of those departers (that depart at the same time instant of their arrival). Such a departer must consider the following members of his own bulk as already being in the system at the instant of his departure. So, an extra assumption must be made here, that all members of a bulk are considered to have arrived, in the case of an empty system, before the first customer departs. This, under the limiting interpretation, would happen if the

minute time intervals between members of the same arriving bulk were assumed much smaller than the corresponding ones of a departing bulk. Thus, under this assumption, the distribution to be used in the derivation of the interdeparture time distribution, is $\{P^{(d)}(n), n=0,1,...\}$, which is identical to $\{P^{(a)}(n), n=0,1,...\}$ as defined previously, i.e.:

$$p^{(a)}(n) = p^{(d)}(n) \quad , n=0,1,2,... \qquad (I.9)$$

And since the above assumption implies a situation similar to the GE/G/c system of the previous section, the arriver's point of view, involved in (I.9), is related to the observer's one by (I.7) and (I.8). Thus, corollary I.4 holds for the GE/GE/1 system. This result can be also proved using level crossing analysis [SHAN 82].

Corollary I.5. Let T_d be the r.v. that describes the time between two consecutive departures from a GE/GE/1 system. Let C_d^2 be the coefficient of variation of this interdeparture time ($C_d^2=Var[T_d]/(E[T_d])^2$. Then:

$$C_d^2 = \rho^2 C_s^2 + (1-\rho)C_a^2 + \rho(1-\rho) \qquad (I.10)$$

where if λ, C_a^2 and μ, C_s^2 are the parameters of the GE-distributed interarrival and service times respectively, ρ is the server's utilization, $\rho=\lambda/\mu$.

Proof.

Let $L_d^*(s)$, $s\geq 0$, be the Laplace-transform of the interdeparture time distribution. Consider a departure at some time instant. In case that the departer leaves an empty system, with probability $P^{(d)}(0)$, the time up to the next departure is the sum of the remaining interarrival time and a GE-distributed service time. The remaining interarrival time is clearly exponentially distributed, with parameter $\sigma\lambda$. This property is usually refered to as the

"*pseudo-memoryless*" property of GE [KOUV 88].

Now, in case that the departer leaves a non-empty system, with probability $1-P^{(d)}(0)$, the time up to the next departure is a GE-distributed service time. Thus, using total Laplace-transform:

$$L_d^*(s) = P^{(d)}(0)\left[\frac{\sigma\lambda}{\sigma\lambda+s}\left[(1-\tau)+\tau\frac{\tau\mu}{\tau\mu+s}\right]\right] + \left[1-P^{(d)}(0)\right]\left[(1-\tau)+\tau\frac{\tau\mu}{\tau\mu+s}\right] \quad (C5.1)$$

where $\tau=2/(1+C_s^2)$ and $\sigma=2/(1+C_a^2)$. And isolating the only term that is constant, (C5.1) may be written as:

$$L_d^*(s) = \left[1-P^{(d)}(0)\right](1-\tau) + \left[1-P^{(d)}(0)\right]\tau\,\frac{\tau\mu}{\tau\mu+s} +$$

$$+ P^{(d)}(0)\left[\frac{\sigma\lambda}{\sigma\lambda+s}\left[(1-\tau)+\tau\frac{\tau\mu}{\tau\mu+s}\right]\right] \quad (C5.2)$$

where term $\left[1-P^{(d)}(0)\right](1-\tau)$ is clearly the probability of having a zero interdeparture time. From (C5.1) it can be seen that:

$$E[T_d] = P^{(d)}(0)\left[\frac{1}{\sigma\lambda}+\frac{1}{\mu}\right] + \left[1-P^{(d)}(0)\right]\frac{1}{\mu} \quad (C5.3)$$

while $P^{(d)}(0)$ is given by (I.19) as $P^{(d)}(0)=\sigma P_0$, where P_0 is the observer's probability of an idle system and is equal to $1-\rho$, [KOUV 88]. Thus:

$$P^{(d)}(0) = \sigma P_0 = \sigma(1-\rho) \quad (C5.4)$$

Then using (C5.4), (C5.3) becomes:

$$E[T_d] = \frac{1}{\mu} + \frac{1-\rho}{\lambda} = \frac{1}{\mu} + \frac{\mu-\lambda}{\lambda\mu} = \frac{1}{\lambda} \quad (C5.5)$$

as expected, under the assumed conditions of stability. Now, differentiating (C5.2) twice and for s=0, after some manipulation:

$$E[T_d^2] = \frac{d^2L_d^*(s)}{ds^2}\Big|_{s=0} = \frac{2}{\tau\mu^2} + \frac{2(1-\rho)}{\sigma\lambda^2} + \frac{2(1-\rho)}{\lambda\mu} \quad (C5.6)$$

Then substituting (C5.5) and (C5.6) into the definition:

$$C_d^2 = \frac{E[T_d^2] - (E[T_d])^2}{(E[T_d])^2}$$

after some manipulation:

$$C_d^2 = \rho^2 C_s^2 + (1-\rho)C_a^2 + \rho(1-\rho)$$

and the proof is complete.

Q.E.D.

This result can be also found in [KOUV 85].

Proof of Lemma 3.1 of chapter 3.

Let's firstly derive R_{0k}, for k=1,2,...,N-1. So the system initially is considered to be in state 0, hense all jobs are present at unit 1, where the exponential($\sigma\mu_1$) branch is occupied. The state will change when there will be a bulk departure from unit 1. This will occur with rate $\sigma\mu_1$, thus R_{0k} is of the form:

$$R_{0k} = \sigma\mu_1 q$$

where q here is the probability that this bulk departure will eventually alter the state of unit 2 to k. Note that at time t, and depending on the size of the arriving bulk, many events may take place instantenously before the exponential branches of both servers are once more occupied and the system settles in a new state.

The size of the bulk that departs unit 1 and arrives at unit 2 is of size n, n=1,2,...,N, with probability $\sigma(1-\sigma)^{n-1}$. In the case of R_{0k}, clearly, this bulk size must be at least k, $n \geq k$. Let's firstly examine the case that $k \leq n \leq N-1$. Under this hypothesis N-n jobs remain

at unit 1 and its exponential branch is occupied. Then the probability that k of those n jobs will remain in unit 2 is equal to the probability that the first n-k will depart unit 2, through the null branch and the (n-k+1)th job will select the exponential($\tau\mu_2$) service. This will happen with probability $\tau(1-\tau)^{n-k}$. Thus, the first term of q is:

$$\sum_{n=k}^{N-1} \sigma(1-\sigma)^{n-1}\tau(1-\tau)^{n-k}$$

Now in the case that n=N, i.e. the bulk consists of all jobs in the system, there are several options to be examined. Let's firstly assume that ℓ, $1 \leq \ell \leq k$, of those customers remain at unit 2, after the first passage of the bulk through this unit. This occurs with probability $\tau(1-\tau)^{N-\ell}$. The remaining N-ℓ jobs will return to unit 1, which is empty, and it must be required that k-ℓ of those depart once more without service. This probability is $\sigma(1-\sigma)^{k-\ell}$.

In case now that ℓ=0, i.e. the whole bulk departs unit 2 without delay, (probability $(1-\tau)^N$), and returns to the empty unit 1, it must be required that the head of the bulk departs unit 1 without delay, (probability $(1-\sigma)$), so that the situation is identical to the initial one, i.e. the probability that from this point the system will settle at state k is exactly q. Thus:

$$q = \sum_{n=k}^{N-1} \sigma(1-\sigma)^{n-1}\tau(1-\tau)^{n-k} +$$

$$+ (1-\sigma)^{N-1}\left[\sum_{\ell=1}^{k} \tau(1-\tau)^{N-\ell}\sigma(1-\sigma)^{k-\ell} + (1-\tau)^N(1-\sigma)q\right]$$

from which:

$$q = \frac{1}{1-(1-\tau)^N(1-\sigma)^N}\left[\sum_{n=k}^{N-1}\sigma\tau(1-\sigma)^{n-1}(1-\tau)^{n-k} + \right.$$

$$\left. + (1-\sigma)^{N-1}\sum_{\ell=1}^{k}\sigma\tau(1-\tau)^{N-\ell}(1-\sigma)^{k-\ell}\right]$$

which after some manipulation yields:

$$q = \frac{\tau\sigma(1-\sigma)^{k-1}}{1-(1-\sigma)(1-\tau)}$$

Thus,
$$R_{0k} = \sigma\mu_1\frac{\tau\sigma(1-\sigma)^{k-1}}{1-(1-\sigma)(1-\tau)}$$

Similarly, in order to derive R_{0N}, it is clear that the bulk size must be n=N, (prbability $(1-\sigma)^{N-1}$). Then if ℓ, $1\leq\ell\leq N$ of those jobs remain at unit 2 after the first passage of the bulk through this unit (probability $\tau(1-\tau)^{N-\ell}$), it must be required that the remaining $N-\ell$ jobs, that arrive at the empty unit 1, should depart without delay (probability $(1-\sigma)^{N-\ell}$). While in case $\ell=0$, i.e. the whole bulk departs unit 2 (probability $(1-\tau)^N$), it must be required that the head of the bulk selects the null branch (probability $(1-\sigma)$), so that the probability that from this point the system will settle at state N is q. Thus, for R_{0N}, probability q is:

$$q = (1-\sigma)^{N-1}\left[\sum_{\ell=1}^{N}\tau(1-\tau)^{N-\ell}(1-\sigma)^{N-\ell} + (1-\tau)^N(1-\sigma)q\right]$$

from which:

$$q = \frac{\tau(1-\sigma)^{N-1}}{1-(1-\sigma)^N(1-\tau)^N}\sum_{\ell=1}^{N}(1-\tau)^{N-\ell}(1-\sigma)^{N-\ell}$$

and after some manipulation:

$$q = \frac{\tau(1-\sigma)^{N-1}}{1-(1-\sigma)(1-\tau)}$$

and thus:

$$R_{0N} = \sigma\mu_1 \frac{\tau(1-\sigma)^{N-1}}{1-(1-\sigma)(1-\tau)}$$

Now, for $1 \leq i \leq N-2$ and $i+1 \leq j \leq N-1$ the derivation of R_{ij} is simpler. Clearly once more:

$$R_{ij} = \sigma\mu_1 q$$

where probability q here is defined accordingly. In this case the bulk that departs unit 1 and arrives at unit 2 can only have size j-i. Thus:

$$q = \sigma(1-\sigma)^{j-i-1}$$

and

$$R_{ij} = \sigma^2\mu_1(1-\sigma)^{j-i-1}$$

while for R_{iN}, $1 \leq i \leq N-1$, the bulk must have size N-i, hence:

$$q = (1-\sigma)^{N-i-1}$$

and

$$R_{iN} = \sigma\mu_1(1-\sigma)^{N-i-1}$$

Now consider the same problem for the dual distribution $\{P_N^*(n), n=0,1,\ldots,N\}$, where the unknown rates are denoted as R_{ij}^*. The same arguments would be used and the same results (3.25a)-(3.25d) would be derived in a "dual" fashion, i.e. the form of these relations would be the same, only μ_2 and τ would interchange roles with μ_1 and σ respectively. Thus:

$$R_{0k}^* = \tau\mu_2 \frac{\sigma\tau(1-\tau)^{k-1}}{1-(1-\tau)(1-\sigma)} \qquad 1 \leq k \leq N-1$$

$$R_{0N}^* = \tau\mu_2 \frac{\sigma(1-\tau)^{N-1}}{1-(1-\tau)(1-\sigma)}$$

$$R_{ij}^* = \tau^2\mu_2(1-\tau)^{j-i-1}$$

$$R^{*}_{iN} = \tau\mu_2(1-\tau)^{N-i-1}$$

and of course:

$$R_{ij} = R^{*}_{N-i,N-j} \quad , \; i,j=0,1,\ldots,N$$

Using the above relations, formulae (3.25e)-(3.25h) follow.

Q.E.D.

TABLES 3.1-3.6 (CHAPTER III)

Table 3.1: Central Server Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	1.6, 1, 2, 5	10, 12, 11, 13	—
1	1.2, 3.3, 2.2, 5.2	20, 15, 25, 30	0.526
2	1, 3.7, 2.1, 5.6	35, 40, 50, 45	0.263
3	0.6, 1.7, 3.1, 6.6	55, 60, 65, 70	0.211

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
COURT	UTOL	0.0042	0.0037	0.0196
	NTOL	0.0077	0.0042	0.0224
FE_1	UTOL	0.1116	0.0634	0.3204
	NTOL	0.0696	0.0275	0.1422
FE_2	UTOL	0.0527	0.0426	0.2179
	NTOL	0.0697	0.0325	0.1486
FE_3	UTOL	0.0375	0.0259	0.1089
	NTOL	0.0152	0.0119	0.0495
FE_4	UTOL	0.0255	0.0109	0.0580
	NTOL	0.0109	0.0054	0.0286
UME	UTOL	0.0074	0.0049	0.0246
	NTOL	0.0042	0.0025	0.0122
EXP	UTOL	0.2711	0.0706	0.3812
	NTOL	0.1014	0.0369	0.1534

Table 3.2: Central Server Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	1.6, 2.2, 2, 4.8	70, 80, 90, 100	—
1	1.2, 3.3, 2.5, 6.2	60, 65, 55, 50	0.556
2	1, 2.7, 2, 4.6	45, 40, 30, 35	0.278
3	0.6, 1.7, 2.6, 6.6	25, 20, 15, 10	0.167

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
COURT	UTOL	0.0170	0.0145	0.0708
	NTOL	0.0176	0.0077	0.0456
FE_1	UTOL	0.0751	0.0339	0.1999
	NTOL	0.0763	0.0273	0.1308
FE_2	UTOL	0.0683	0.0319	0.1808
	NTOL	0.0769	0.0277	0.1307
FE_3	UTOL	0.0065	0.0054	0.0271
	NTOL	0.0094	0.0066	0.0332
FE_4	UTOL	0.0078	0.0050	0.0291
	NTOL	0.0106	0.0070	0.0376
UME	UTOL	0.0086	0.0064	0.0305
	NTOL	0.0077	0.0041	0.0213
EXP	UTOL	0.3230	0.0408	0.3814
	NTOL	0.1014	0.4250	0.1600

Table 3.3: Central Server Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	1.2, 2.4, 2, 4.8	25, 35, 50, 80	—
1	1, 1.3, 2.5, 5.2	8, 8, 8, 8	0.526
2	1, 0.7, 0.5, 1.6	7, 6, 5, 5	0.316
3	0.3, 0.7, 2.6, 1.6	4, 1, 3, 2	0.158

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
COURT	UTOL	0.0394	0.0257	0.1237
	NTOL	0.0299	0.0118	0.0691
FE_1	UTOL	0.0410	0.0244	0.1172
	NTOL	0.0386	0.0129	0.0661
FE_2	UTOL	0.0347	0.0214	0.1052
	NTOL	0.0406	0.0134	0.0695
FE_3	UTOL	0.0094	0.0084	0.0447
	NTOL	0.0146	0.0079	0.0444
FE_4	UTOL	0.0068	0.0057	0.0336
	NTOL	0.0171	0.0085	0.0448
UME	UTOL	0.0188	0.0112	0.0529
	NTOL	0.0219	0.0121	0.0618
EXP	UTOL	0.2150	0.0307	0.2684
	NTOL	0.0486	0.0219	0.0946

Table 3.4: Central Server Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	1.2, 2.4, 2, 4.8	4, 3, 2, 1	—
1	1, 1.3, 1.5, 4.2	5, 6, 6.5, 7	0.526
2	1, 0.7, 0.5, 1.6	8, 9, 9, 10	0.263
3	0.3, 0.7, 2.6, 1.6	15, 19, 25, 40	0.211

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
COURT	UTOL	0.0041	0.0038	0.0227
	NTOL	0.0053	0.0033	0.0196
FE_1	UTOL	0.1600	0.0475	0.2575
	NTOL	0.0443	0.0188	0.0859
FE_2	UTOL	0.0520	0.0395	0.1635
	NTOL	0.0651	0.0194	0.1081
FE_3	UTOL	0.0686	0.0284	0.1666
	NTOL	0.0315	0.0170	0.0714
FE_4	UTOL	0.0287	0.0127	0.0658
	NTOL	0.0206	0.0078	0.0406
UME	UTOL	0.0143	0.0092	0.0472
	NTOL	0.0093	0.0041	0.0197
EXP	UTOL	0.1973	0.0312	0.2701
	NTOL	0.0445	0.0209	0.0963

Table 3.5: Central Server Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	1.2, 2.4, 2.9, 4.8	8, 7, 6, 5	—
1	1, 1.3, 1.5, 4.2	1, 1.5, 1.8, 2	0.5
2	1, 0.7, 0.5, 1.6	3.2, 3.6, 4, 4.5	0.25
3	0.3, 0.7, 2.6, 1.6	2.2, 2.5, 2.7, 3	0.25

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
COURT	UTOL	0.0132	0.0125	0.0800
	NTOL	0.0218	0.0097	0.0484
FE_1	UTOL	0.0332	0.0241	0.1074
	NTOL	0.0213	0.0082	0.0427
FE_2	UTOL	0.0181	0.0155	0.0752
	NTOL	0.0272	0.0095	0.0471
FE_3	UTOL	0.0116	0.0102	0.0593
	NTOL	0.0077	0.0043	0.0241
FE_4	UTOL	0.0152	0.0077	0.0387
	NTOL	0.0123	0.0048	0.0260
UME	UTOL	0.0062	0.0046	0.0272
	NTOL	0.0098	0.0064	0.0233
EXP	UTOL	0.1201	0.0246	0.1666
	NTOL	0.0295	0.0127	0.0592

Table 3.6: Central Server Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	1.2, 2.4, 0.9, 4.8	4, 3, 2, 5	—
1	1, 1.3, 1.5, 4.2	6, 8.5, 9.8, 11	0.556
2	1, 0.7, 0.5, 1.8	13.2, 12.6, 14, 15.5	0.278
3	0.3, 0.7, 2.6, 1.6	25.2, 20.5, 17.7, 30	0.167

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
COURT	UTOL	0.0035	0.0031	0.0206
	NTOL	0.0054	0.0032	0.0153
FE_1	UTOL	0.1524	0.0525	0.2539
	NTOL	0.0483	0.0184	0.0893
FE_2	UTOL	0.0554	0.0420	0.1643
	NTOL	0.0618	0.0212	0.1075
FE_3	UTOL	0.0665	0.0268	0.1482
	NTOL	0.0269	0.0151	0.0598
FE_4	UTOL	0.0249	0.0120	0.0550
	NTOL	0.0157	0.0066	0.0322
UME	UTOL	0.0120	0.0071	0.0324
	NTOL	0.0070	0.0033	0.0155
EXP	UTOL	0.2116	0.0366	0.2701
	NTOL	0.0521	0.0213	0.0973

APPENDIX II (CHAPTER IV)

Properties of the MRE principle.

The following properties have been presented and proved by Shore and Johnson in [SHOR 80] and [SHOR 81].

Property 1 : Uniqueness.

The posterior $f=g\odot I$ is unique for any prior $g \in \Omega$ and new information $I=(f^* \in \Upsilon)$, where $\Upsilon \subseteq \Omega$.

Proof. (see [SHOR 80]).

According to this property, if the same problem is solved twice in exactly the same way, the same answer is expected to result both times, or in other words the relative entropy minimization yields a unique solution.

Property 2. : Invariance.

Let Γ be a coordinate transformation from $x \in D$ to $y \in D'$ with:

$$(\Gamma f)(y) = J^{-1} f(x)$$

where J is the Jacobian $J=\partial(y)/\partial(x)$. Let $\Gamma\Omega$ be the set of densities Γf corresponding to densities $f \in \Omega$. Let $(\Gamma\Upsilon) \subseteq (\Gamma\Omega)$ correspond to $\Upsilon \subseteq \Omega$. Then for any prior $g \in \Omega$ and new information $I=(f^* \in \Upsilon)$:

$$(\Gamma g)\odot(\Gamma I) = \Gamma(g\odot I) \qquad \text{(P2.1)}$$

and

$$H_c[\Gamma(g\odot I), \Gamma g] = H_c[g\odot I, g] \qquad \text{(P2.2)}$$

Proof. (see [SHOR 80])

Equation (P2.1) states that the same answer is obtained when one solves the inference problem in two different coordinate systems, in that the posteriors in the two systems are related by the coordinate transformation. Moreover, the cross-entropy, which represents the "information distance" between the posteriors and the priors, has the same value in both coordinate systems.

Property 3. : System Independence.

Consider two systems with sets D_1 and D_2 of states and probability densities $f_1^* \epsilon \Omega^{(1)}$ and $f_2^* \epsilon \Omega^{(2)}$. Let $g_1 \epsilon \Omega^{(1)}$ and $g_2 \epsilon \Omega^{(2)}$ be prior densities for the two systems and suppose there is new information $I_1=(f_1^* \epsilon \Upsilon^{(1)})$ and $I_2=(f_2^* \epsilon \Upsilon^{(2)})$ available about the two systems, where $\Upsilon^{(1)} \subseteq \Omega^{(1)}$ and $\Upsilon^{(2)} \subseteq \Omega^{(2)}$. Then:

$$(g_1 g_2) \odot (I_1 \wedge I_2) = (g_1 \odot I_1)(g_2 \odot I_2) \qquad \text{(P3.1)}$$

$$H_c[f_1(x_1)f_2(x_2), g_1(x_1)g_2(x_2)] = H_c(f_1(x_1), g_1(x_1)) + H_c(f_2(x_2), g_2(x_2)) \qquad \text{(P3.2)}$$

Proof. (see [SHOR 80]).

This property states that it does not matter whether one accounts for independent information about two systems separately or together in terms of a joint density. Whether the two systems are in fact independent is irrelevant, the property applies as long as there are independent priors and independent information.

An application of this property can be seen in the analysis of open queueing networks, [KOUV 85], where the assumed new information concerns each queueing system of the network separately and where no prior information is considered available. Then because of this property, the resulting product form solution implies no correlation between queues of the network. This implication is exact in the case of exponentially distributed service times, it is only an approximation, however, in the case of a general network. This property can be easily generalized to n systems.

Property 4. : Subset Independence.

Let $D_1, D_2, \ldots, D_n$ be disjoint sets whose union is D and let $g \epsilon \Omega$ be any known prior. For each set D_i, let $I_i=(f^* * D_i \epsilon \Upsilon_i)$ be new information about the conditional density $f^* * D_i$, where $\Upsilon_i \subseteq \Omega_i$ and Ω_i is the set of densities on D_i, $i=1,\ldots,n$. Let $M=(f^* \epsilon M)$ be new

information giving the probability of being in each of the n subsets, where M is the set of densities that satisfy:

$$\int_{D_i} f^*(x)\, dx = m_i \quad , \; i=1,\ldots,n$$

for each subset D_i, where m_i, $i=1,\ldots,n$, are the given values. Then if $I=I_1 \wedge I_2 \wedge \ldots \wedge I_n$ the following relations hold:

$$[\, g \odot (I \wedge M)\,] * D_i = (g * D_i) \odot I_i \qquad \text{(P4.1)}$$

$$H_c[\, g \odot (I \wedge M), g\,] = \sum_{i=1}^{n} m_i H_c(f_i, g_i) + \sum_{i=1}^{n} m_i \log\left\{ \frac{m_i}{d_i} \right\} \qquad \text{(P4.2)}$$

where $g_i = g * D_i$, $f_i = f * D_i$, $f = g \odot (I \wedge M)$ and,

$$d_i = \int_{D_i} g(x)\, dx \quad , \; i=1,\ldots,n$$

In fact under this notation (P4.1) can be written as:

$$f_i = g_i \odot I_i$$

Proof. (see [SHOR 80]).

This property states that whether one treats an independent subset of system states in terms of a separate conditional density or in terms of the full system density one ends up with the same result for the MRE posterior conditional density for x, given $x \in D_i$, $i=1,\ldots,n$.

This property concerns cases in which the set of states decomposes into disjoint subsets D_i and new information I_i is obtained about the conditional densities $f^* * D_i$ in each subset. One way of accounting for this information is to obtain a conditional posterior $f_i = (g * D_i) \odot I_i$ from each conditional prior $g * D_i$ directly. The alternative is to obtain a posterior $f = g \odot I$ for the entire system, where $I = I_1 \wedge I_2 \wedge \ldots \wedge I_n$. The two results should be related by $f * D_i = f_i$.

Moreover, suppose that the probability of being in each of the n subsets m_i is also known, i=1,...,n. Clearly, taking this information (M) into account should not affect the conditional posterior densities that result from taking I into account.

These four properties, presented up to now, are in fact the four consistency axioms, based on which Shore and Johnson proved the following theorem.

Theorem II.1. Let a functional $H(f,g)$ satisfy properties 1-4. Then H is equivalent to the cross-entropy functional H_c, defined by (4.1).

This effectively implies that given a prior density and new constraint information about a system there is a unique posterior distribution that can be chosen by a procedure that satisfies the four axioms. This unique posterior is the solution of the minimum relative entropy problem. Some further properties of the MRE principle are:

Property 5. : Weak subset independence.

Let $D_1, D_2, \ldots, D_n$ be a partition of D. Let new information I comprise information about the conditional densities $f^* * D_i$. Thus:

$$I = \bigwedge_{i=1}^{n} I_i \quad \text{and} \quad I_i = (f^* * D_i \epsilon \Upsilon_i)$$

where $\Upsilon_i \subseteq \Omega_i$ and Ω_i is the set of densities on D_i. Let:

$$d_i = \int_{D_i} g(x)\, dx \quad , \; i=1,\ldots,n$$

denote the prior probability of being in each subset D_i. Then the following relations hold:

$$(g \odot I) * D_i = (g * D_i) \odot I_i \qquad \text{(P5.1)}$$

$$H_c(g\odot I,g) = \sum_{i=1}^{n} r_i H_c(f_i,g_i) + \sum_{i=1}^{n} r_i \log\left\{ \frac{r_i}{d_i} \right\} \qquad \text{(P5.2)}$$

where $g_i=g*D_i$, $f_i=f*D_i$, $f=g\odot I$ and r_i are the posterior probabilities of being in each subset:

$$r_i = \int_{D_i} f(x)\, dx \quad , \; i=1,\ldots,n$$

Proof. (see [SHOR 81]).

This property refers to the same case as property 4. The only difference is that there is not any assumption made about the probabilities of being in each subset D_i. So r_i, $i=1,\ldots,n$, is considered to be the posterior probabilities derived from $f=g\odot I$, i.e. after deriving f, using only I.

Property 6. : Subset Aggregation.

Let $D_1,D_2,\ldots,D_n$ be disjoint subsets whose union is D. Let ψ be a transformation which aggregates the states in each subset D_i such that, for any $f \in \Omega$, $f'=\psi f$ is a discrete distribution with:

$$f'(x_i) = \psi f(x_i) = \int_{D_i} f(x)\, dx$$

where x_i is a discrete state corresponding to $x \in D_i$. Suppose new information $I'=((\psi f^*)\epsilon\Upsilon')$ is obtained about the aggregate distribution ψf^*, where $\Upsilon'$is the convex constraint set of discrete distributions. Then for any prior $g \in \Omega$, the relations:

$$g*D_i = (g\odot I)*D_i \qquad \text{(P6.1)}$$

$$(\psi g)\odot I' = \psi(g\odot I) \qquad \text{(P6.2)}$$

$$H_c[\psi(g\odot I),\psi g] = H_c(g\odot I,g) \qquad \text{(P6.3)}$$

all hold, where $I=\psi^{-1}I'$ is the information I' expressed in terms of f^* instead of in terms of ψf^*. That is $I=(f^*\epsilon(\psi^{-1}\Upsilon'))$, where $\psi^{-1}\Upsilon' \subseteq \Omega$

is the set of densities f such that $\psi f \in \Upsilon'$.

Proof. (see [SHOR 81]).

Relation (P6.1) states that in the absense of any information about the conditional densities on the subsets D_i, i=1,...,n, the prior and posterior conditional densities are the same. Thus, information about the aggregate distribution $\{\psi f^*(x_i), i=1,...,n\}$ only affects the discrete probabilities of occupying each subset D_i. In fact this new information I′ may be incorporated either using it at that discrete level or by expressing it with respect to the full density $f^*(x)$. The result, according to (P6.2) is the same.

Property 7. : Idempotence.

For any prior $g \in \Omega$, and new information $I=(f^* \epsilon \Upsilon)$, $\Upsilon \subseteq \Omega$, the following relation holds:

$$(g \odot I) \odot I = g \odot I \qquad \text{(P7.1)}$$

Proof. (see [SHOR 81]).

Its interpretation is obvious. Taking the same information into account twice has, as required, the same effect as taking it into account once.

Property 8. : Null effect of redundant information.

Let constraints I_1 and I_2 be given as, $I_1=(f^* \epsilon \Upsilon_1)$ and $I_2=(f^* \epsilon \Upsilon_2)$ for constraint sets $\Upsilon_1, \Upsilon_2 \subseteq \Omega$. If $(g \odot I_1) \epsilon \Upsilon_2$ then:

$$g \odot I_1 = (g \odot I_1) \odot (I_1 \wedge I_2) = (g \odot I_1) \odot I_2 = g \odot (I_1 \wedge I_2) \qquad \text{(P8.1)}$$

Proof. (see [SHOR 81]).

When entropy is applied for infering a probability distribution, redundant information do not impose any additional consideration since they drop out of the calculation automatically, according to this property. It is worthy pointing out that the last part of the above equation $(g \odot (I_1 \wedge I_2) = (g \odot I_1) \odot I_2)$ is not generally true for any I_1, I_2. In this case, however, where I_2 is redundant, this equality

holds.

Property 9.': Triangle inequality.

For any h ϵ Υ and g ϵ Ω:

$$H_c(h,g) \geq H_c(h,f) + H_c(f,g) \qquad \text{(P9.1)}$$

where f=g⊙I, I=($f^*\epsilon\Upsilon$). When I is determined by a finite set of equality constraints only, equality holds in (P9.1).

Proof. (see [SHOR 81]).

Property 10. : "Relative-Entropy closeness" inequality.

$$H_c(f^*,g\odot I) \leq H_c(f^*,g) \qquad \text{(P10.1)}$$

holds, with equality if and only if g⊙I=g, i.e. I is redundant information.

Proof. (see [SHOR 81]).

This property states that the posterior density f=g⊙I is always closer to the true density f^*, in the relative-entropy sense, than is the prior density g(x).

The following two properties only apply in the presence of equality constraints.

Property 11. : Intermediate posteriors I

Let information about a system consist of $I_1=(f^*\epsilon\Upsilon_1)$ and $I_2=(f^*\epsilon\Upsilon_2)$, where $\Upsilon_1,\Upsilon_2\subseteq\Omega$ are constraint sets with a non-empty intersection ($\Upsilon_1\cap\Upsilon_2\neq\emptyset$). Suppose that I_1 is determined by a set of equality constraints only. Then the following relations hold:

$$(g\odot I_1)\odot(I_1\wedge I_2) = g\odot(I_1\wedge I_2) \qquad \text{(P11.1)}$$

$$H_c(f,g) = H_c(f,f_1) + H_c(f_1,g) \qquad \text{(P11.2)}$$

where $f=g\odot(I_1\wedge I_2)$ and $f_1=g\odot I_1$.

Proof. (see [SHOR 81]).

This property states that intermediate posteriors can be used as priors in computing final posteriors, without affecting the result of

the analysis.

Property 12. : Intermediate posteriors II

Suppose there are two underlying probability densities $f_1^*(x)$ and $f_2^*(x)$. Let I_1 and I_2, respectively, denote the sets of equality constraints:

$$\int_D a_i(x) f_1^*(x)\, dx = \langle a_i^{(1)} \rangle \ , \ i=1,\ldots,m$$

$$\int_D a_i(x) f_2^*(x)\, dx = \langle a_i^{(2)} \rangle \ , \ i=1,\ldots,s$$

where $s \geq m$. Then:

$$(g \odot I_1) \odot I_2 = g \odot I_2 \qquad \text{(P12.1)}$$

holds. Moreover, if $\lambda_k^{(1)}$, $\lambda_k^{(1,2)}$ and $\lambda_k^{(2)}$ are the Lagrangian multipliers associated with $f_1 = g \odot I_1$, $f_{1,2} = f_1 \odot I_2$ and $f_2 = g \odot I_2$ respectively, then:

$$\lambda_k^{(2)} = \lambda_k^{(1)} + \lambda_k^{(1,2)} \ , \ k=1,\ldots,m \qquad \text{(P12.2)}$$

$$\lambda_k^{(2)} = \lambda_k^{(1,2)} \ , \ k=m+1,\ldots,s \qquad \text{(P12.3)}$$

and

$$H_c(f_2,g) = H_c(f_2,f_1) + H_c(f_1,g) + \sum_{r=1}^{m} \lambda_r^{(1)} (a_r^{(1)} - a_r^{(2)}) \qquad \text{(P12.4)}$$

also holds.

Proof. (see [SHOR 81]).

Property 12 applies to situations in which $f_1^*(x)$ or estimates of it are considered to be good estimates of $f_2^*(x)$. If I_2 is determined in part by expectations of the same functions as I_1, but with different expected values, then the results of taking I_1 into account are completely wiped out by subsequently taking I_2 into account. Generalization to n different densities $f_i^*(x)$ is straight-forward. Note that if $s \geq m$ and $a_r^{(1)} = a_r^{(2)}$ for $r=1,\ldots,m$, this property reduces

to property 11.

Some more properties of the MRE principle can be found in [SHOR 81]. Note that most of the properties are also satisfied by the ME principle, which is the discrete case of MRE principle in the absense of any prior distribution. A small exception that affects the axiomatic derivation of the ME principle can be found in [JOHN 83]. It should be also mentioned here that in case the prior distribution is the uniform (finite state space), then it is possible to completely ignore this prior information, the results of the ME maximization are completely unaffected by this.

Proof of Lemma 4.1 (chapter 4)

1) Proof of (4.22).

Let:

$$f_j^\dagger = [\psi f^\dagger]_j = \int_{D_j} f^\dagger(x)\,dx \quad , \; j=1,\ldots,n$$

then equations (4.21), (4.11) and (4.15) yield:

$$[f^\dagger * D_j](x) = f_j^{\dagger -1} g(x) \exp\left[-\varphi - \sum_{i=1}^{K_j} \eta_{ij}\beta_{ij}(x) - \sum_{i=1}^{K_a} \xi_i \gamma_{ij}\right] \quad , \; x \epsilon D_j \qquad (L1.1)$$

Similarly if:

$$g_j^\dagger = [\psi g^\dagger]_j = \int_{D_j} g^\dagger(x)\,dx \quad , \; j=1,\ldots,n$$

then equations (4.19) and (4.11) yield:

$$[g^\dagger * D_j](x) = g_j^{\dagger -1} g(x) \exp\left[-\lambda - \sum_{i=1}^{K_j} \lambda_{ij}\beta_{ij}(x)\right] \quad , \; x \epsilon D_j \qquad (L1.2)$$

Denoting now, for every j=1,...,n:

$$F_j^{\dagger} = f_j^{\dagger -1}\exp\left[-\varphi - \sum_{i=1}^{K_a}\xi_i\gamma_{ij}\right] \qquad \text{(L1.3)}$$

and substituting (L1.3) into (L1.1):

$$[f^{\dagger} * D_j](x) = F_j^{\dagger} g(x)\exp\left[-\sum_{i=1}^{K_j}\eta_{ij}\beta_{ij}(x)\right] \quad , \; x\epsilon D_j \qquad \text{(L1.4)}$$

In the same way, for :

$$G_j^{\dagger} = g_j^{\dagger -1}\exp(-\lambda) \qquad \text{(L1.5)}$$

(L1.5) into (L1.2) yields:

$$[g^{\dagger} * D_j](x) = G_j^{\dagger} g(x)\exp\left[-\sum_{i=1}^{K_j}\lambda_{ij}\beta_{ij}(x)\right] \quad , \; x\epsilon D_j \qquad \text{(L1.6)}$$

Since both (L1.4) and (L1.6) have the same form, satisfy the same subset constraints (4.10) and integrate to unity on D_j, it follows that they are equal everywhere on D_j. Thus:

or
$$f^{\dagger} * D_j = g^{\dagger} * D_j \quad , \; j=1,\ldots,n$$

$$(g \odot (I_s^{\dagger} \wedge I_a^{\dagger})) * D_j = (g \odot I_s^{\dagger}) * D_j \quad , \; j=1,\ldots,n \qquad \text{(L1.7)}$$

holds as well as $\eta_{ij}=\lambda_{ij}$ and $F_j^{\dagger}=G_j^{\dagger}$. Using (L1.7) and (4.16), (4.22) follows.

Note that in the proof of (4.22) the fact that constraints (4.9) and (4.13) all have a zero right hand side was never used.

2) <u>Proof of (4.23).</u>

Since $F_j^{\dagger}=G_j^{\dagger}$, definitions (L1.3) and (L1.5) yield:

$$f_j^{\dagger -1}\exp\left[-\varphi - \sum_{i=1}^{K_a}\xi_i\gamma_{ij}\right] = g_j^{\dagger -1}\exp(-\lambda) \quad \Rightarrow$$

$$\Rightarrow \qquad f_j^\dagger = g_j^\dagger \exp\left[\lambda-\varphi - \sum_{i=1}^{K_a} \xi_i \gamma_{ij}\right] \tag{L1.8}$$

Now, solving the discrete relative-entropy problem for $(\psi g^\dagger)\odot I_a$, from (4.4), the solution is:

$$[(\psi g^\dagger)\odot I_a]_j = g_j^\dagger \exp\left[-\nu - \sum_{i=1}^{K_a} \epsilon_i \gamma_{ij}\right] \tag{L1.9}$$

where ν and ϵ_i are the Lagrangian multipliers corresponding to a normalization constraint and to the aggregate constraints (4.13) respectively. From (L1.8), $f_j^\dagger$ also satisfies these constraints and since (L1.8) and (L1.9) have the same form it follows that they are equal. That is:

or
$$\psi f^\dagger = (\psi g^\dagger)\odot I_a$$

$$\psi(g\odot(I_s^\dagger \wedge I_a^\dagger)) = (\psi(g\odot I_s^\dagger))\odot I_a$$

which proves (4.23). Note that this proof also followed irrespective of the fact that constraints (4.9) and (4.13) all have zero right hand side.

3) <u>Proof of (4.24).</u>

Recall from (4.19) and (4.21) that:

$$g^\dagger(x) = g(x)\exp\left[-\lambda - \sum_{j=1}^{n} \sum_{i=1}^{K_j} \lambda_{ij}\alpha_{ij}(x)\right]$$

$$f^\dagger(x) = g(x)\exp\left[-\varphi - \sum_{j=1}^{n} \sum_{i=1}^{K_j} \eta_{ij}\alpha_{ij}(x) - \sum_{i=1}^{K_a} \xi_i \gamma_i(x)\right]$$

Using (4.19) and since $g^\dagger(x)$ satisfies the subset constraints:

$$H_c(g^\dagger, g) = \int_D g^\dagger(x)\log\left\{\frac{g^\dagger(x)}{g(x)}\right\} dx =$$

$$= \int_D g^\dagger(x)\left[-\lambda - \sum_{j=1}^{n}\sum_{i=1}^{K_j}\lambda_{ij}\alpha_{ij}(x)\right] dx = -\lambda\int_D g^\dagger(x)\,dx = -\lambda$$

where in the above equation the fact that constraints (4.10) have a zero right hand side was used. So:

$$H_c(g^\dagger,g) = -\lambda \qquad \text{(L1.10)}$$

Similarly, using (4.21) and the subset and aggregate constraints:

$$H_c(f^\dagger,g) = \int_D f^\dagger(x)\log\left\{\frac{f^\dagger(x)}{g(x)}\right\} dx =$$

$$= \int f^\dagger(x)\left[-\varphi - \sum_{j=1}^{n}\sum_{i=1}^{K_j}\eta_{ij}\alpha_{ij}(x) - \sum_{i=1}^{K_a}\xi_i\gamma_i(x)\right] dx \Rightarrow$$

$$\Rightarrow \quad H_c(f^\dagger,g) = -\varphi \qquad \text{(L1.11)}$$

Also from equation (L1.8) and the aggregate constraints (4.14):

$$H_c(\psi f^\dagger,\psi g^\dagger) = \sum_{j=1}^{n} f_j^\dagger \log\left\{\frac{f_j^\dagger}{g_j^\dagger}\right\} = \sum_{j=1}^{n} f_j^\dagger\left[\lambda-\varphi - \sum_{i=1}^{K_a}\xi_i\gamma_{ij}\right] \Rightarrow$$

$$\Rightarrow \quad H_c(\psi f^\dagger,\psi g^\dagger) = \lambda-\varphi$$

Combining the above three results it follows that:

$$H_c(f^\dagger,g) = H_c(g^\dagger,g) + H_c(\psi f^\dagger,\psi g^\dagger)$$

which is (4.24).

3) <u>Proof of (4.25).</u>

By definition:

$$[g*D_j](x) = g_j^{-1}g(x)$$

Minimizing $H_c[[g*D_j]\odot I_{sj},[g*D_j]]$, subject to the normalization and subset constraints, yields:

$$[[g*D_j]\odot I_{sj}](x) = g_j^{-1}g(x)\exp\left[-\theta_j - \sum_{i=1}^{K_j}\theta_{ij}\beta_{ij}(x)\right] \ , \ x\epsilon D_j \qquad (L1.12)$$

Due to (4.16), it follows from (L1.2), (L1.12) that $\lambda_{ij}=\theta_{ij}$ and:

$$g_j^{\dagger} = g_j\exp\{-\lambda+\theta_j\} \qquad (L1.13)$$

provided that the constraint functions $\beta_{ij}(x)$ are linearly independent. Now using (4.16) and (L1.7), it follows that:

$$H_c(f^{\dagger}*D_j, g*D_j) = H_c[[g*D_j]\odot I_{sj}, g*D_j] =$$

$$= -\int_{D_j} [[g*D_j]\odot I_{sj}](x)\log\left\{\frac{[[g*D_j]\odot I_{sj}](x)}{[g*D_j](x)}\right\}dx =$$

$$= -\int_{D_j} [[g*D_j]\odot I_{sj}](x)\left[-\theta_j - \sum_{i=1}^{K_j}\theta_{ij}\beta_{ij}(x)\right]dx \quad \Rightarrow$$

$$\Rightarrow \qquad H_c(f^{\dagger}*D_j, g*D_j) = -\theta_j \qquad (L1.14)$$

and since $\eta_{ij}=\lambda_{ij}=\theta_{ij}$ holds, it follows from (L1.8) and (L1.13) that:

$$f_j^{\dagger} = g_j\exp\left[\theta_j-\varphi - \sum_{i=1}^{K_a}\xi_i\gamma_{ij}\right]$$

and hence that:

$$H_c(\psi f^{\dagger}, \psi g) = \sum_{j=1}^{n} f_j^{\dagger}\log\left\{\frac{f_j^{\dagger}}{g_j}\right\} = \sum_{j=1}^{n} f_j^{\dagger}\left[\theta_j-\varphi - \sum_{i=1}^{K_a}\xi_i\gamma_{ij}\right] \ \Rightarrow$$

$$H_c(\psi f^{\dagger}, \psi g) = \sum_{j=1}^{n} f_j^{\dagger}\,\theta_j - \varphi \qquad (L1.15)$$

Equations (L1.14), (L1.15) and (L1.11) produce (4.25). At this point the proof is complete.

Q.E.D.

Proof of Theorem 4.1 (chapter 4).

Using the subset aggregation property (4.17), it is clear that:

$$[\psi(g\odot I_s^\dagger)]\odot I_a = \psi((g\odot I_s^\dagger)\odot I_a^\dagger)$$

This and relation (4.23) yield:

$$\psi(g\odot(I_s^\dagger\wedge I_a^\dagger)) = \psi((g\odot I_s^\dagger)\odot I_a^\dagger) \qquad (T1.1)$$

In order to prove (4.33), it remains to prove that:

$$[g\odot(I_s^\dagger\wedge I_a^\dagger)]*D_j = [(g\odot I_s^\dagger)\odot I_a^\dagger]*D_j \qquad (T1.2)$$

for every j=1,2,...,n. Then (4.33) follows from (T1.1) and (T1.2).

Using (4.11) and (4.15), relation (4.20b) yields:

$$[h^\dagger*D_j](x) = h_j^{\dagger-1}g(x)\exp\left[-\lambda-\theta-\sum_{j=1}^{n}\sum_{i=1}^{K_j}\lambda_{ij}\beta_{ij}(x)-\sum_{i=1}^{K_a}\theta_i\gamma_{ij}\right]$$

where

$$h_j^\dagger = [\psi h^\dagger]_j = \int_{D_j} h^\dagger(x)\,dx$$

So if:

$$H_j^\dagger = h_j^{\dagger-1}\exp\left[-\lambda-\theta-\sum_{i=1}^{K_a}\theta_i\gamma_{ij}\right]$$

the above may be written as:

$$[h^\dagger*D_j](x) = H_j^\dagger g(x)\exp\left[-\sum_{i=1}^{K_j}\lambda_{ij}\beta_{ij}(x)\right] \quad , \; x\epsilon D_j \qquad (T1.3)$$

Equations (L1.6) and (T1.3) differ only in the leading factor $G_j^\dagger$ or $H_j^\dagger$. Since they both integrate to unity, it follows that $G_j^\dagger=H_j^\dagger$ and:

$$g^\dagger*D_j = h^\dagger*D_j$$

or

$$(g\odot I_s^\dagger)*D_j = [(g\odot I_s^\dagger)\odot I_a^\dagger]*D_j$$

Equation (T1.2) now follows from the above and (L1.7).

Q.E.D.

APPENDIX III (CHAPTER V)

ANALYTIC RESULTS

Proof of Lemma 5.2.

In this case the MRE solution of this level is of the form,

$$P_2^*(N_2) = \frac{1}{\tilde{G}^*(2,N)} G(1,N_2) \tilde{x}_2^{N_2} \tilde{y}_2^{f_2(N_2)}, \quad N_2=0,\ldots,N \qquad (L2.1)$$

where normalization yields:

$$\tilde{G}^*(2,N) = \sum_{N_2=0}^{N} G(1,N_2) \tilde{x}_2^{N_2} \tilde{y}_2^{f_2(N_2)} \Rightarrow$$

$$\tilde{G}^*(2,N) = \sum_{N_2=0}^{N} \tilde{x}_2^{N_2} \tilde{y}_2^{f_2(N_2)} \min\left[1, \frac{1}{x_0^{N_2}}\right] \sum_{n_0=0}^{N_2} x_0^{n_0} \qquad (L2.2)$$

Assuming that $\tilde{x}_2<1$ and it is invariant to N and that,

$$\lim_{N\to+\infty} \tilde{y}_2 < +\infty$$

the probability of the CPU being idle in this level is:

$$P_0(0) = \sum_{N_2=0}^{N} P_1(0/N-N_2) P_2^*(N_2) =$$

$$= \sum_{N_2=0}^{N} \frac{1}{G^*(1,N_2)} \frac{1}{\tilde{G}^*(2,N)} G^*(1,N_2) \min\left[1, \frac{1}{x_0^{N_2}}\right] \tilde{x}_2^{N_2} \tilde{y}_2^{f_2(N_2)} \Rightarrow$$

$$P_0(0) = \frac{1}{\tilde{G}^*(2,N)} \sum_{N_2=0}^{N} \tilde{x}_2^{N_2} \tilde{y}_2^{f_2(N_2)} \min\left[1, \frac{1}{x_0^{N_2}}\right] \qquad (L2.3)$$

Thus,

$$\tilde{G}^*(2,N)[1-P_0(0)] = \sum_{N_2=0}^{N} \tilde{x}_2^{N_2} \tilde{y}_2^{f_2(N_2)} \min\left[1, \frac{1}{x_0^{N_2}}\right] \sum_{n_0=0}^{N_2} x_0^{n_0} -$$

$$- \sum_{N_2=0}^{N} \tilde{x}_2^{N_2} \, y_2^{f_2(N_2)} \min\left[1, \frac{1}{x_0^{N_2}} \right] \quad \Rightarrow$$

$$\tilde{G}^*(2,N)[1-P_0(0)] = \sum_{N_2=1}^{N} \tilde{x}_2^{N_2} \, y_2^{f_2(N_2)} \min\left[1, \frac{1}{x_0^{N_2}} \right] \sum_{n_0=1}^{N_2} x_0^{n_0} \qquad \text{(L2.4)}$$

$$\lim_{N\to+\infty} \tilde{G}^*(2,N)[1-P_0(0)] = \sum_{N_2=1}^{\infty} \tilde{x}_2^{N_2} \min\left[1, \frac{1}{x_0^{N_2}} \right] \sum_{n_0=1}^{N_2} x_0^{n_0} =$$

$$\tilde{x}_2 x_0 \min\left[1, \frac{1}{x_0} \right] \sum_{N_2=1}^{\infty} \tilde{x}_2^{N_2-1} \min\left[1, \frac{1}{x_0^{N_2-1}} \right] \sum_{n_0=0}^{N_2} x_0^{n_0} =$$

$$\tilde{x}_2 \min(1,x_0) \sum_{N_2=0}^{\infty} \tilde{x}_2^{N_2} \min\left[1, \frac{1}{x_0^{N_2}} \right] \sum_{n_0=0}^{N_2} x_0^{n_0}$$

and using (L2.2) at the limit ($N \longrightarrow +\infty$) it follows that:

$$\lim_{N\to+\infty} \tilde{G}^*(2,N)[1-P_0(0)] = \tilde{x}_2 \min(1,x_0) \lim_{N\to+\infty} \tilde{G}^*(2,N)$$

from which, since by assumption

$$\lim_{N\to+\infty} \tilde{G}^*(2,N) < +\infty$$

it follows that:

$$\lim_{N\to+\infty} [1-P_0(0)] = \tilde{x}_2 \min(1,x_0) \qquad \text{(L2.5)}$$

The fb constraint (5.7c) expressed with respect $P_2^*(N_2)$ is:

$$[1-P_0(0)]\mu_0 r_2 = [1-P_2^*(N)]\mu_2 \qquad \text{(L2.6)}$$

and noticing that:

$$\lim_{N\to+\infty} P_2^*(N) = \lim_{N\to+\infty} P_2(0) = 0$$

applying the limit $N \longrightarrow +\infty$ on (L2.6) and using (L2.5), yields:

$$\tilde{x}_2 = \frac{1}{\min(1,x_0)} \frac{\mu_2}{\mu_0 r_2} = \frac{1}{x_2} \qquad \text{(L2.7)}$$

thus, (5.26) is proven.

Note at this point that when seeking for the limit in (L2.5) it is essential to use this dual distribution since a problem of convergence occurs if $\{P_2(n_2),\ n_2=0,\ldots,N\}$ is used under the assumption $x_2>1$. It is also worth noticing that using (L2.7) into (L2.5) yields:

$$\lim_{N\to+\infty} [1-P_0(0)] = \frac{\mu_2}{\mu_0 r_2}$$

which is exactly the resulting utilization of the CPU in the corresponding open system of this level, under the present assumption that unit Σ_2 is the bottleneck of this level, (fig. III.1).

Returning now to the closed system and from (L2.1):

$$P_2^*(N) = \frac{1}{\tilde{G}^*(2,N)} G(1,N)\ \tilde{x}_2^N\ \tilde{y}_2$$

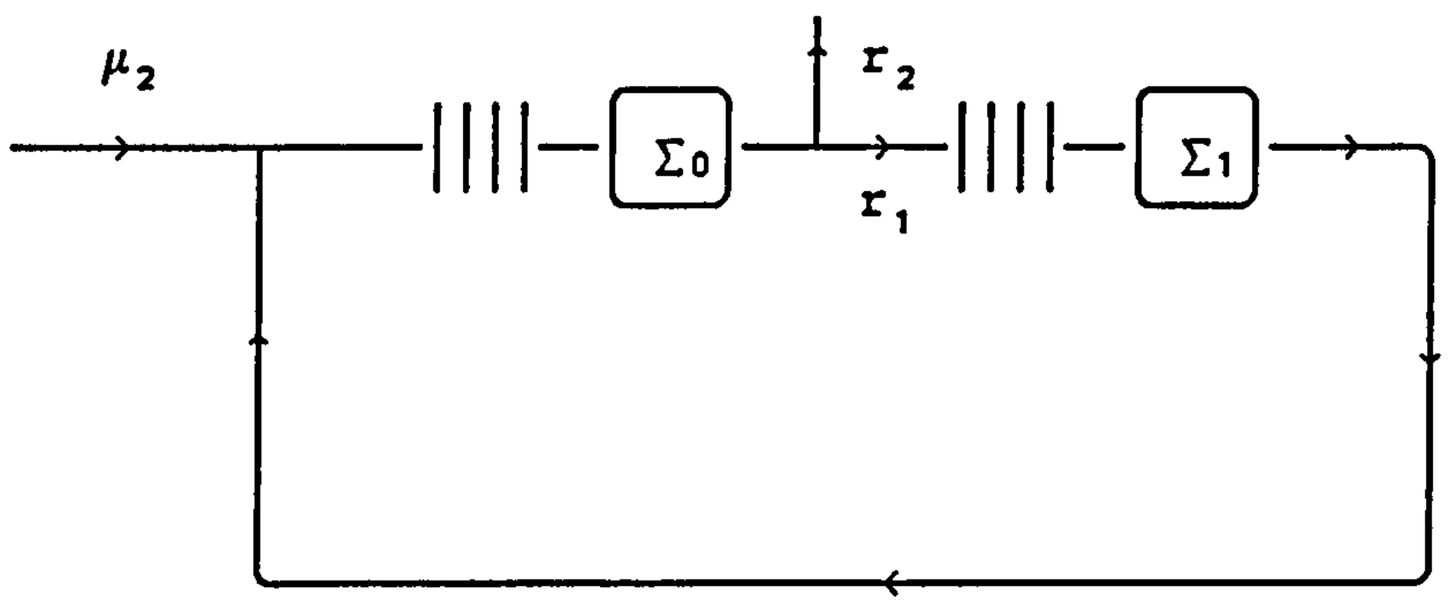

Figure III.1. Corresponding open system in case Σ_2 is the overall botleneck.

and using this and (L2.2),

$$\tilde{G}^*(2,N)[1-P_2(N)] = \sum_{N_2=0}^{N-1} \tilde{x}_2^{N_2} \min\left[1, \frac{1}{x_0^{N_2}}\right] \sum_{n_0=0}^{N_2} x_0^{n_0} \qquad \text{(L2.8)}$$

Then the fb constraint (L2.6), using (L2.4) and (L2.8) becomes:

$$\mu_0 r_2 \sum_{N_2=1}^{N} \tilde{x}_2^{N_2} \tilde{y}_2^{f_2(N_2)} \min\left[1, \frac{1}{x_0^{N_2}}\right] \sum_{n_0=1}^{N_2} x_0^{n_0} =$$

$$\mu_2 \sum_{N_2=0}^{N-1} \tilde{x}_2^{N_2} \min\left[1, \frac{1}{x_0^{N_2}}\right] \sum_{n_0=0}^{N_2} x_0^{n_0} \quad \Rightarrow$$

$$\mu_0 r_2 \min(1,x_0) \sum_{N_2=1}^{N} \tilde{x}_2^{N_2} \tilde{y}_2^{f_2(N_2)} \min\left[1, \frac{1}{x_0^{N_2-1}}\right] \sum_{n_0=0}^{N_2-1} x_0^{n_0} =$$

$$\mu_2 \sum_{N_2=0}^{N-1} \tilde{x}_2^{N_2} \min\left[1, \frac{1}{x_0^{N_2}}\right] \sum_{n_0=0}^{N_2} x_0^{n_0} \qquad \text{(L2.7)} \quad \Rightarrow$$

$$\tilde{x}_2^{N} \tilde{y}_2 \min\left[1, \frac{1}{x_0^{N-1}}\right] \sum_{n_0=0}^{N-1} x_0^{n_0} +$$

$$\sum_{N_2=1}^{N-1} \tilde{x}_2^{N_2} \min\left[1, \frac{1}{x_0^{N_2-1}}\right] \sum_{n_0=0}^{N_2-1} x_0^{n_0} =$$

$$\sum_{N_2=1}^{N} \tilde{x}_2^{N_2} \min\left[1, \frac{1}{x_0^{N_2-1}}\right] \sum_{n_0=0}^{N_2-1} x_0^{n_0}$$

and noting that the right hand side of the above equation may be written as:

$$\tilde{x}_2^{N} \min\left[1, \frac{1}{x_0^{N-1}}\right] \sum_{n_0=0}^{N-1} x_0^{n_0} + \sum_{N_2=1}^{N-1} \tilde{x}_2^{N_2} \min\left[1, \frac{1}{x_0^{N_2-1}}\right] \sum_{n_0=0}^{N_2-1} x_0^{n_0}$$

it follows that:

$$\tilde{y}_2 = 1 \qquad \text{(L2.9)}$$

Because of (L2.9), relations (5.24) and (5.25) easily follow.

Q.E.D.

Proof of Theorem 5.1.

Expressions involved in parts i. and ii. follow directly by performing similar operations to those carried out for Lemma 5.3. Parts iii. and iv. can be proven by induction. Assuming that relations (5.32)-(5.35) hold up to the ℓth level of decomposition, it will be shown that these relations are also valid at the $(\ell+1)$th level. To this end, at the $(\ell+1)$th level, the MRE solution, subject to the prior introduced at the ℓth level and the subset mql and fb constraints, is clearly given by:

$$P_{\ell+1}(n_{\ell+1}/n_{\ell+2},\ldots,n_M) = \frac{G(\ell,N-\sum_{k=\ell+1}^{M} n_k)}{G^*(\ell+1,N-\sum_{k=\ell+2}^{M} n_k)} x_{\ell+1}^{n_{\ell+1}} y_{\ell+1}^{f_{\ell+1}(n_{\ell+1})} \qquad \text{(T1.1)}$$

$$n_{\ell+1} = 0,\ldots,N-\sum_{k=\ell+2}^{M} n_k$$

where

$$f_{\ell+1}(n_{\ell+1}) = \begin{cases} 1 & \text{if } n_{\ell+1}=N-\sum_{k=\ell+2}^{M} n_k \\ 0 & \text{otherwise} \end{cases}$$

Let's at this point, in order to simplify the notation, define:

$$N_\ell = N-\sum_{k=\ell+1}^{M} n_k \quad , \ \ell=1,\ldots,M-1 \qquad \text{(T1.2)}$$

So obviously $N \longrightarrow +\infty$ implies $N_\ell \longrightarrow +\infty$. Normalizing (T1.1) now yields:

$$G^*(\ell+1,N_{\ell+1}) = \sum_{n_{\ell+1}=0}^{N_{\ell+1}-1} G(\ell,N_{\ell+1}-n_{\ell+1})\, x_{\ell+1}^{n_{\ell+1}} + x_{\ell+1}^{N_{\ell+1}} y_{\ell+1}$$

Using (5.35), (5.34) recursively, as well as (T1.2), the above may be

written as:

$$G^*(\ell+1,N_{\ell+1}) = \sum_{n_{\ell+1}=0}^{N_{\ell+1}-1} \min\left[1, \frac{1}{x_\ell^{N_\ell}} \right] x_{\ell+1}^{n_{\ell+1}} \times$$

$$\times \sum_{n_\ell=0}^{N_\ell} \min\left[1, \frac{1}{x_{\ell-1}^{N_{\ell-1}}} \right] x_\ell^{n_\ell} \times \cdots \times \sum_{n_2=0}^{N_2} \min\left[1, \frac{1}{x_0^{N_1}} \right] x_2^{n_2} \times$$

$$\times \sum_{n_0=0}^{N_1} x_0^{n_0} + x_{\ell+1}^{N_{\ell+1}} y_{\ell+1} \qquad \text{(T1.3)}$$

At this point let's derive the limit,

$$\lim_{N\to+\infty} \left[\min\left[1, \frac{1}{x_0^{N_1}} \right] \sum_{n_0=0}^{N_1} x_0^{n_0} \right] = A$$

Considering the two possible cases ($x_0<1$ and $x_0>1$) and defining $\tilde{x}_0 = 1/x_0$, the above limit is:

$$A = \frac{1}{1-\min(x_0,\tilde{x}_0)}$$

Similarly it may be proven that:

$$\lim_{N\to+\infty} \left[\min\left[1, \frac{1}{x_m^{N_m}} \right] \sum_{n_m=0}^{N_m} x_m^{n_m} \right] = \frac{1}{1-\min(x_m,\tilde{x}_m)} \quad , \; m=2,3,\ldots,\ell$$

where $\tilde{x}_m=1/x_m$. Thus, if it is firstly assumed that $x_{\ell+1}<1$ and is invariant of N, as well as:

$$\lim_{N\to+\infty} y_{\ell+1} < +\infty$$

(T1.3) yields:

$$\lim_{N\to+\infty} G^*(\ell+1,N_{\ell+1}) = \frac{1}{1-x_{\ell+1}} \left[(1-\min(x_0,\tilde{x}_0)) \prod_{k=2}^{\ell} (1-\min(x_k,\tilde{x}_k)) \right] \qquad \text{(T1.4)}$$

Note that at every level of aggregation the convergence of the limit of $G^*(\ell,N_\ell)$, $\ell=2,\ldots,M$, is justified by the generalization of Cauchy's limit theorem [KNOP 56, pp. 35-36]. Expanding now $G(\ell,N_\ell)$ similarly to (T1.3) and applying the same limit ($N \longrightarrow +\infty$), it yields:

$$\lim_{N\to+\infty} G(\ell,N_\ell) = (1-\min(x_0,\tilde{x}_0)) \prod_{k=2}^{\ell} (1-\min(x_k,\tilde{x}_k)) \qquad \text{(T1.5)}$$

and noting from (T1.2) that $N_\ell=N_{\ell+1}-n_{\ell+1}$, it follows from (T1.1), (T1.4), (T1.5) that:

$$\lim_{N\to+\infty} P_{\ell+1}(n_{\ell+1}/n_{\ell+2},\ldots,n_M) = (1-x_{\ell+1})x_{\ell+1}^{n_{\ell+1}}, \quad n_{\ell+1}=0,1,\ldots \qquad \text{(T1.6)}$$

At this point let's find the probability of the CPU being idle at this level of aggregation, denoted as $\Pr\{\text{CPU idle}/n_{\ell+2},\ldots,n_M\}$. Using LTP:

$$\Pr\{\text{CPU idle}/n_{\ell+2},\ldots,n_M\} = \sum_{n_{\ell+1}=0}^{N_{\ell+1}} \Pr\{\text{CPU idle}/n_{\ell+1},\ldots,n_M\} \times$$

$$\times P_{\ell+1}(n_{\ell+1}/n_{\ell+2},\ldots,n_M)$$

Let $$q = \lim_{N\to+\infty} \Pr\{\text{CPU idle}/n_{\ell+1},\ldots,n_M\}$$

Probability q was used at the previous level of aggregation in the limit of the fb constraint at that level as:

$$(1-q)\mu_0 r_\ell = \mu_\ell \min(1,x_\ell)$$

which yields:

$$q = 1 - \frac{\mu_\ell}{\mu_0 r_\ell} \min(1,x_\ell) \qquad \text{(T1.7)}$$

and since from (T1.6):

$$\lim_{N\to+\infty} \Pr\{\text{CPU idle}/n_{\ell+2},\ldots,n_M\} = \sum_{n_{\ell+1}=0}^{\infty} q(1-x_{\ell+1})x_{\ell+1}^{n_{\ell+1}} = q \qquad \text{(T1.8)}$$

the fb constraint of this (ℓ+1)th level at the limit ($N \longrightarrow +\infty$) can be written as:

$$(1-q)\mu_0 r_{\ell+1} = x_{\ell+1}\mu_{\ell+1} \quad \Rightarrow$$

$$x_{\ell+1} = \frac{\mu_\ell r_{\ell+1}}{\mu_{\ell+1} r_\ell} \min(1, x_\ell)$$

which is exactly (5.33).

Then returning to the closed system, the value of the fb multiplier $y_{\ell+1}$ may be determined using the fb equation,

$$[1-\Pr\{\text{CPU idle}/n_{\ell+2},\ldots,n_M\}]\mu_0 r_{\ell+1} = [1-P_{\ell+1}(0/n_{\ell+2},\ldots,n_M)]\mu_{\ell+1}$$

(T1.9)

Note that $\Pr\{\text{CPU idle}/n_{\ell+2},\ldots,n_M\}$, using LTP, is given by:

$$\Pr\{\text{CPU idle}/n_{\ell+2},\ldots,n_M\} = \sum_{n_{\ell+1}=0}^{N_{\ell+1}} \sum_{n_\ell=0}^{N_\ell} \cdots \sum_{n_2=0}^{N_2} P_1(0/n_2,\ldots,n_M) \times$$

$$\times P_2(n_2/n_3,\ldots,n_M) \times \cdots \times P_\ell(n_\ell/n_{\ell+1},\ldots,n_M) P_{\ell+1}(n_{\ell+1}/n_{\ell+2},\ldots,n_M)$$

Using (T1.1), (5.32), (5.34) and (5.35) and after some manipulation, the above may be written as:

$$\Pr\{\text{CPU idle}/n_{\ell+2},\ldots,n_M\} = \frac{1}{G^*(\ell+1,N_{\ell+1})} \left[\sum_{n_{\ell+1}=0}^{N_{\ell+1}-1} \min\left[1, \frac{1}{x_\ell^{N_\ell}}\right] x_{\ell+1}^{n_{\ell+1}} \times \right.$$

$$\times \sum_{n_\ell=0}^{N_\ell} \min\left[1, \frac{1}{x_{\ell-1}^{N_{\ell-1}}}\right] x_\ell^{n_\ell} \times \cdots \times \sum_{n_2=0}^{N_2} \min\left[1, \frac{1}{x_0^{N_1}}\right] x_2^{n_2} +$$

$$\left. + x_{\ell+1}^{N_{\ell+1}} y_{\ell+1} \right] \qquad \text{(T1.10)}$$

Relation (5.33) yields:

$$x_{\ell+1} = \frac{\mu_\ell r_{\ell+1}}{\mu_{\ell+1} r_\ell} \min(1,x_\ell) \Rightarrow x_{\ell+1} = \min(1,x_\ell) \frac{\mu_\ell}{\mu_0 r_\ell} \frac{\mu_0 r_{\ell+1}}{\mu_{\ell+1}} \Rightarrow$$

$$\frac{\mu_0 r_{\ell+1}}{\mu_{\ell+1}} = \frac{x_{\ell+1}}{\min(1,x_\ell)} \frac{\mu_0 r_\ell}{\mu_\ell} \qquad \text{(T1.11)}$$

Note that (T1.11) is recursive and holds for all indexes 3,4,...,ℓ,ℓ+1. From (5.21) also,

$$\frac{\mu_0 r_2}{\mu_2} = \frac{x_2}{\min(1,x_0)} \qquad \text{(T1.12)}$$

Thus, using (T1.11) and (T1.12) it follows that:

$$\frac{\mu_0 r_{\ell+1}}{\mu_{\ell+1}} = \frac{x_{\ell+1} x_\ell \cdot \cdot \cdot x_3 x_2}{\min(1,x_\ell)\min(1,x_{\ell-1}) \cdots \min(1,x_2)\min(1,x_0)} \Rightarrow$$

$$\frac{\mu_0 r_{\ell+1}}{\mu_{\ell+1}} = \frac{x_{\ell+1}}{\min(1,x_0)} \prod_{k=2}^{\ell} \frac{x_k}{\min(1,x_k)}$$

and using the fact that $\max(1,x)=x/\min(1,x)$, $\forall\ x\epsilon R$,

$$\frac{\mu_0 r_{\ell+1}}{\mu_{\ell+1}} = x_{\ell+1} \frac{\max(1,x_0)}{x_0} \prod_{k=2}^{\ell} \max(1,x_k)$$

Then using the above, the fb constraint (T1.9) may be written as:

$$[1 - \Pr\{\text{CPU idle}/n_{\ell+2},\ldots,n_M\}]\, x_{\ell+1} =$$

$$= [1 - P_{\ell+1}(0/n_{\ell+2},\ldots,n_M)] \frac{x_0}{\max(1,x_0) \prod_{k=2}^{\ell} \max(1,x_k)}$$

and using the fact that $\min(1,1/x)=1/\max(1,x)$, $\forall\ x\epsilon R-\{0\}$, the above is written as:

$$[1 - \Pr\{\text{CPU idle}/n_{\ell+2},\ldots,n_M\}]\, x_{\ell+1} =$$

$$= [1 - P_{\ell+1}(0/n_{\ell+2},\ldots,n_M)]x_0 \min\left[1, \frac{1}{x_0}\right] \prod_{k=2}^{\ell} \min\left[1, \frac{1}{x_k}\right] \quad \text{(T1.13)}$$

Then using (T1.10) and (T1.3), the above may be solved with respect to $y_{\ell+1}$, which after some manipulation proves to be:

$$y_{\ell+1} = 1 \quad \text{(T1.14)}$$

This implies that both (5.32) and (5.34) are true under the hypothesis $x_{\ell+1}<1$. In case, however, that $x_{\ell+1}>1$, the dual distribution $P^*_{\ell+1}(N_a/n_{\ell+2},\ldots,n_M)$, $N_a=0,1,\ldots,N_{\ell+1}$ may be used, which concerns the number of customers N_a that are present in the subsystem (or composite server) of level $\ell+1$ and which satisfy the duality relations:

$$P^*_{\ell+1}(N_a/n_{\ell+2},..,n_M) = P_{\ell+1}(N_{\ell+1}-N_a/n_{\ell+2},..,n_M) \ , \ N_a=0,\ldots,N_{\ell+1} \quad \text{(T1.15)}$$

where $N_{\ell+1}$ is given by (T1.2). Applying the same form of subset constraints, the MRE solution for this dual distribution will be:

$$P^*_{\ell+1}(N_a/n_{\ell+2},\ldots,n_M) = \frac{G(\ell,N_a)}{\tilde{G}^*(\ell+1,N_{\ell+1})} \tilde{x}_{\ell+1}^{N_a} \tilde{y}_{\ell+1}^{f_{\ell+1}(N_a)} \quad \text{(T1.16)}$$

$$N_a=0,1,\ldots,N_{\ell+1}$$

One can proceed in a similar fashion as previously and verify (as in Lemma 5.2) that:

$$\tilde{x}_{\ell+1} = \frac{1}{x_{\ell+1}} \quad , \quad \tilde{y}_{\ell+1} = 1 \quad \text{(T1.17)}$$

Then this solution, through relations (T1.15), completely specifies distribution $\{P_{\ell+1}(n_{\ell+1}/n_{\ell+2},\ldots,n_M), n_{\ell+1}=0,1,\ldots,N_{\ell+1}\}$, and in fact proves that (5.32)-(5.34) are also valid in case that $x_{\ell+1}>1$. Moreover, it is easy to see that the dual normalizing constant satisfies:

$$\tilde{G}^*(\ell+1,N_{\ell+1}) = \frac{1}{x_{\ell+1}^{N_{\ell+1}}} G^*(\ell+1,N_{\ell+1}) \qquad \text{(T1.18)}$$

And since the convergence of the prior introduced here, must be guaranteed, definition (5.35) follows. This completes the proof of the Theorem.

Q.E.D.

Proof of Corollary 5.1.

By applying the general multiplicative rule on the conditional and marginal probabilities of Theorem 5.1, the joint state probability $P(n_0,n_1,\ldots,n_M)$ can be expressed as:

$$P(n_0,n_1,..,n_M) = P_1(n_0/n_2,..,n_M)P_2(n_2/n_3,..,n_M)\cdots P_{M-1}(n_{M-1}/n_M)P_M(n_M)$$

and using (5.32), (5.35), the above may be written as:

$$P(n_0,n_1,\ldots,n_M) = \frac{1}{G^*(M,N)} x_0^{n_0} \min\left[1, \frac{1}{x_0^{N_1}}\right] x_2^{n_2} \min\left[1, \frac{1}{x_2^{N_2}}\right] \times$$

$$\times \cdots \times x_{M-1}^{n_{M-1}} \min\left[1, \frac{1}{x_{M-1}^{N_{M-1}}}\right] x_M^{n_M} \qquad \text{(C1.1)}$$

where

$$N_\ell = N - \sum_{k=\ell+1}^{M} n_k \qquad \text{(C1.2)}$$

Using the fact that $[\min(1,x)]^k=\min(1,x^k)$, (C1.1) may be written as:

$$P(n_0,n_1,\ldots,n_M) = \frac{1}{G^*(M,N)} x_0^{n_0}\left[\min\left[1,\frac{1}{x_0}\right]\right]^{N_1} x_2^{n_2}\left[\min\left[1,\frac{1}{x_2}\right]\right]^{N_2} \times$$

$$\times \cdots \times x_{M-1}^{n_{M-1}}\left[\min\left[1, \frac{1}{x_{M-1}}\right]\right]^{N_{M-1}} x_M^{n_M}$$

and since $\min(1,1/x)=1/\max(1,x)$,

$$P(n_0,n_1,\ldots,n_M) = \frac{1}{G^*(M,N)} x_0^{n_0} [\max(1,x_0)]^{-N_1} x_2^{n_2} [\max(1,x_2)]^{-N_2} \times$$

$$\times \cdots \times x_{M-1}^{n_{M-1}} [\max(1,x_{M-1})]^{-N_{M-1}} x_M^{n_M}$$

Then using (C1.2) and defining G(M,N) by (5.40), the above becomes:

$$P(n_0,n_1,\ldots,n_M) = \frac{1}{G(M,N)} U_0^{n_0} U_2^{n_2} \cdots U_{M-1}^{n_{M-1}} U_M^{n_M} \qquad \text{(C1.3)}$$

with

$$U_k = x_k \prod_{\substack{i=0 \\ i\neq 1}}^{k-1} \max(1,x_i) \quad , \; k=2,\ldots,M \qquad \text{(C1.4)}$$

and

$$U_0 = x_0 \qquad \text{(C1.5)}$$

Using now (5.33) and (C1.4), for k=2,...,M it follows that:

$$U_k = \prod_{\substack{i=0 \\ i\neq 1}}^{k-2} \max(1,x_i) \max(1,x_{k-1}) \min(1,x_{k-1}) \frac{\mu_{k-1}r_k}{\mu_k r_{k-1}} =$$

$$\prod_{\substack{i=0 \\ i\neq 1}}^{k-2} \max(1,x_i) \, x_{k-1} \frac{\mu_{k-1}r_k}{\mu_k r_{k-1}} =$$

$$\prod_{\substack{i=0 \\ i\neq 1}}^{k-3} \max(1,x_i) \max(1,x_{k-2}) \min(1,x_{k-2}) \frac{\mu_{k-2}r_{k-1}}{\mu_{k-1}r_{k-2}} \frac{\mu_{k-1}r_k}{\mu_k r_{k-1}} =$$

$$\prod_{\substack{i=0 \\ i\neq 1}}^{k-3} \max(1,x_i) \, x_{k-2} \frac{\mu_{k-2}r_k}{\mu_k r_{k-2}} = \ldots$$

$$\max(1,x_0)x_2 \frac{\mu_2 r_k}{\mu_k r_2} = x_0 \frac{\mu_0 r_2}{\mu_2} \frac{\mu_2 r_k}{\mu_k r_2} = \frac{\mu_1}{\mu_0 r_1} \frac{\mu_0 r_k}{\mu_k} \Rightarrow$$

$$U_k = \frac{\mu_1 r_k}{r_1 \mu_k} \quad , \; k=2,3,\ldots,M$$

Q.E.D.

Proof of Corollary 5.2.

When the network is separable the exact solution [KLEI 75, p. 152] is given by:

$$P(n_0,n_1,\ldots,n_M) = \frac{1}{Z(M,N)} \prod_{i=0}^{M} V_i^{n_i} \qquad (C2.1)$$

where Z(M,N) is the normalizing constant and V_i, i=0,1,...,M are coefficients that satisfy the fb equations (for a central server model):

$$\mu_i V_i = \mu_0 V_0 r_i \ , \ i=1,2,\ldots,M \qquad (C2.2)$$

Solution (C2.1) may be written as:

$$P(n_0,n_1,\ldots,n_M) = \frac{V_1^N}{Z(M,N)} \prod_{\substack{i=0 \\ i\neq 1}}^{M} \tilde{V}_i^{n_i} \qquad (C2.3)$$

where

$$\tilde{V}_i = \frac{V_i}{V_1} \ , \ i=0,2,\ldots,M \qquad (C2.4)$$

Substituting V_i from (C2.2) into (C2.4) it follows that $\tilde{V}_0$ and $\tilde{V}_i$, i=2,...,M, are given by (5.41) and (5.42) respectively. Thus, (5.39) is identical to (C2.3).

Q.E.D.

Proof of Theorem 5.2.

Part i. has been already proven by what has been described prior to this Theorem. In part ii. the MRE solution is given by (5.47), and applying the normalization constraint,

$$G^*(2,N) = \sum_{n_0=0}^{N} G(1,N-n_0) x_0^{n_0} y_0^{f_2(n_0)} \qquad (T2.1)$$

Note now that:

$$\lim_{N\to+\infty} G(1,N-n_0) = \left[\prod_{k=2}^{M} (1-x_k) \right]^{-1} \qquad \text{(T2.2)}$$

since from what has been said, x_k, k=2,...,M are nothing else but the utilizations of the corresponding open system of the first level (see also [KLEI 75, p. 152]). Then using (T2.2), and assuming that $x_0<1$,

$$\lim_{N\to+\infty} G^*(2,N) = \left[(1-x_0) \prod_{k=2}^{M} (1-x_k) \right]^{-1} \qquad \text{(T2.3)}$$

and

$$\lim_{N\to+\infty} P_0(n_0) = (1-x_0)x_0^{n_0} \quad , \; n_0=0,1,\ldots \qquad \text{(T2.4)}$$

Now if $P_k(n_k)$ and $P_k(n_k/n_0)$ are the marginal queue length probabilities of unit k, k=1,...,M, unconditional and conditional respectively, the fb constraint (5.44c) may be written as:

$$[1-P_0(n_0)]\mu_0 = \sum_{k=1}^{M} \left[1- \sum_{n_0=0}^{N} P_k(0/n_0)P_0(n_0)\right]\mu_k \qquad \text{(T2.5)}$$

and noticing that:

$$\lim_{N\to+\infty} P_1(0/n_0) = 0$$

and

$$\lim_{N\to+\infty} P_k(0/n_0) = 1-x_k \quad , \; k=2,\ldots,M$$

the limit of (T2.5) yields:

$$x_0\mu_0 = \mu_1 + \sum_{k=2}^{M} \left[1-(1-x_k) \sum_{n_0=0}^{\infty} (1-x_0)x_0^{n_0} \right]\mu_k \quad \Rightarrow$$

$$x_0 = \frac{1}{\mu_0} \left[\mu_1 + \sum_{k=2}^{M} x_k\mu_k \right]$$

which is (5.51).

Returning now to the closed system, $P_k(0)$ is given by:

$$P_k(0) = \sum_{n_0=0}^{N} P_k(0/n_0)P_0(n_0) \quad \Rightarrow$$

$$P_k(0) = \frac{1}{G^*(2,N)} \left[\sum_{n_0=0}^{N-1} x_0^{n_0} G(1,N-n_0)P_k(0/n_0) + x_0^{N} y_0 \right] \qquad \text{(T2.6)}$$

Then (5.47), (T2.1) (T2.6) and the fb equation (5.44c), after some manipulation yield:

$$\mu_0 x_0^{N} y_0 + \mu_0 \sum_{n_0=1}^{N-1} G(1,N-n_0)x_0^{n_0} = \sum_{n_0=0}^{N-1} x_0^{n_0} G(1,N-n_0) \sum_{k=1}^{M} [1-P_k(0/n_0)]\mu_k \qquad \text{(T2.7)}$$

it is not difficult to prove now that for every k=2,...,M, the following relation is true, (see also [FERR 78, p. 206]):

$$1-P_k(0/n_0) = x_k[1-P_1(0/n_0)] \quad , k=2,...,M \qquad \text{(T2.8)}$$

Incorporating (T2.8) into (T2.7) and identifying x_0 by (5.51), it follows that:

$$x_0^{N} y_0 = x_0 \sum_{n_0=0}^{N-1} x_0^{n_0} G(1,N-n_0)[1-P_1(0/n_0)] - \sum_{n_0=1}^{N-1} G(1,N-n_0)x_0^{n_0} \qquad \text{(T2.9)}$$

Denoting as:

$$N_i = N - n_0 - \sum_{k=i+1}^{M} n_k \quad , i=2,...,M \qquad \text{(T2.10)}$$

it may be proved that:

$$1-P_1(0/n_0) = \frac{1}{G(1,N-n_0)} \sum_{n_M=0}^{N_M} x_M^{n_M} \sum_{n_{M-1}=0}^{N_{M-1}} x_{M-1}^{n_{M-1}} \cdots \sum_{n_2=0}^{N_2} x_2^{n_2} \qquad \text{(T2.11)}$$

Then substituting $[1-P_1(0/n_0)]G(1,N-n_0)$, after some manipulation it follows that:

$$y_0 = 1$$

This proves (5.49) and (5.50).

In case that $x_0>1$, the same aggregate constraints may be assumed for the dual distribution of $\{P_0(n_0),\ n_0=0,\ldots,N\}$, i.e. the marginal distribution of the number of jobs present in the I/O subsystem $(\Sigma_1,\ldots,\Sigma_M)$ rather than unit Σ_0. Following this route, it can be shown that (5.49) is valid even if $x_0>1$.

Q.E.D.

Proof of Corollary 5.3.

Combining (5.48) and (5.49), the joint state probability is clearly:

$$P(n_0,n_1,\ldots,n_M) = \frac{1}{G^*(2,N)} \prod_{k=0}^{M} x_k^{n_k} \qquad \text{(C3.1)}$$

where x_0 is given by (5.51), while x_i, $i=2,\ldots,M$ and $x_1=1$ satisfy (5.46). So the MRE solution of Theorem 5.2 implies a form of solution for the joint state probability, which is identical to the solution of the network as if this was separable. Let R be the routing matrix of the original network. Then:

$$R = \begin{bmatrix} 0 & r_1 & r_2 & \cdots & r_M \\ 1 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \text{(C3.2)}$$

while matrix R_s as defined previously is:

$$R_s = \begin{bmatrix} r_1 & r_2 & r_3 & \cdots & r_M \\ r_1 & r_2 & r_3 & \cdots & r_M \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ r_1 & r_2 & r_3 & \cdots & r_M \end{bmatrix} \qquad \text{(C3.3)}$$

It is required to be shown that vector $(x_0\mu_0,\mu_1,x_2\mu_2,\ldots,x_M\mu_M)$ is a solution of the system:

$$y = yR \tag{C3.4}$$

where $y=(y_0,y_1,\ldots,y_M)$, given that vector $(\mu_1,x_2\mu_2,\ldots,x_M\mu_M)$ is a solution of the system:

$$\tilde{y}_s = \tilde{y}_s R_s \tag{C3.5}$$

with $\tilde{y}_s=(\tilde{y}_1,\ldots,\tilde{y}_M)$. More precisely, it must be shown that the following equations are satisfied,

$$x_0\mu_0 = \mu_1 + x_2\mu_2 + \cdots + x_M\mu_M \tag{C3.6a}$$

$$\mu_1 = x_0\mu_0 r_1 \tag{C3.6b}$$

and

$$x_k\mu_k = x_0\mu_0 r_k \quad , \; k=2,\ldots,M \tag{C3.6c}$$

Equation (C3.6a) obviously is satisfied because of (5.51). Equation (C3.6b), using (5.51) may be written as:

$$\mu_1 = [\mu_1 + x_2\mu_2 + \cdots + x_M\mu_M]r_1$$

But this is satisfied, since it is the first equation of the system:

$$(\mu_1,x_2\mu_2,\ldots,x_M\mu_M) = (\mu_1,x_2\mu_2,\ldots,x_M\mu_M)R_s$$

Similarly, equations (C3.6c) are satisfied.

Q.E.D.

APPENDIX IV (CHAPTER VI)

Analytic derivation of the MRE solution subject to constraints (6.1)-(6.3).

First level of aggregation.

In the first level of aggregation, the MRE solution, subject to constraints (6.1), is:

$$P_1(n_0/n_2,n_3) = \frac{1}{G^*(1,N-n_2-n_3)}\, x_0^{n_0}\, g_0^{f_{1,1}(n_0)}\, y_0^{f_{1,2}(n_0)} \qquad \text{(IV.1)}$$

$$n_0=0,1,\ldots,N-n_2-n_3$$

By making an assymptotic connection to the related infinite capacity queue (as $N \longrightarrow +\infty$), one may proceed, as in the third chapter (section 3.2.7, pp. 53-60), to prove that under the invariability assumption about the mql and utilization multipliers (i.e. assume that these multipliers are not functions of N), as well as under the stability condition ($x_0<1$) for the corresponding open queue, the following relations are true:

$$G^*(1,N-n_2-n_3) = 1 + g_0 \sum_{n_0=1}^{N-n_2-n_3-1} x_0^{n_0} + x_0^{N-n_2-n_3} g_0 y_0 \qquad \text{(IV.2)}$$

$$x_0 = \frac{<n_0>-\rho_0}{<n_0>} \qquad \text{(IV.3)}$$

where $$<n_0> = \lim_{N\to+\infty} <n_0>_{N-n_2-n_3}$$

and

$$\rho_0 = \frac{\mu_1}{\mu_0 r_1} \qquad \text{(IV.4)}$$

$$g_0 = \frac{\rho_0(1-x_0)}{x_0(1-\rho_0)} \qquad \text{(IV.5)}$$

$$y_0 = \frac{1-\rho_0}{1-x_0} \qquad \text{(IV.6)}$$

In case that $x_0>1$, one may assume that the same subset constraints (6.1) concern the dual distribution $\{P_1^*(n_1/n_2,n_3),\ n_1=0,\ldots,N-n_2-n_3\}$, which in this first level describes the number of jobs present at unit Σ_1 rather than Σ_0. Obviously,

$$P_1(n_0/n_2,n_3) = P_1^*(N-n_0-n_2-n_3/n_2,n_3)\ ,n_0=0,\ldots,N-n_2-n_3 \qquad (IV.7)$$

The MRE solution for this dual distribution would then be:

$$P_1^*(n_1/n_2,n_3) = \frac{1}{\tilde{G}^*(1,N-n_2-n_3)}\ \tilde{x}_0^{\,n_1}\ \tilde{g}_0^{\,f_{1,1}(n_1)}\ \tilde{y}_0^{\,f_{1,2}(n_1)} \qquad (IV.8)$$

$$n_1=0,\ldots,N-n_2-n_3$$

And following a similar approach, it is easy to prove (see also section 3.2.7, pp. 58-59) that:

$$\tilde{x}_0\tilde{x}_0 = g_0\tilde{y}_0 = y_0\tilde{g}_0 = 1 \qquad (IV.9)$$

where x_0, g_0, y_0, are given by (IV.3), (IV.5) and (IV.6), respectively. Then using the duality relations (IV.7), it is easy to see that solution (IV.1)-(IV.6) is valid even if $x_0>1$. Now, the prior defined in this level by the normalizing constants, must converge as $N \longrightarrow +\infty$, because another assymptotic connection will take place in the next level of aggregation. Using (IV.7) for $n_0=0$ it follows that:

$$P_1(0/n_2,n_3) = P_1^*(N-n_2-n_3/n_2,n_3) \Rightarrow$$

$$\frac{1}{G^*(1,N-n_2-n_3)} = \frac{1}{\tilde{G}^*(1,N-n_2-n_3)}\ \tilde{x}_0^{\,N-n_2-n_3}\ \tilde{g}_0\tilde{y}_0 \Rightarrow$$

$$\tilde{G}^*(1,N-n_2-n_3) = G^*(1,N-n_2-n_3)\ \frac{1}{x_0^{\,N-n_2-n_3}\ g_0y_0} \qquad (IV.10)$$

Thus, the prior, denoted as $\{G(1,N-n_2-n_3),\ n_2=0,\ldots,N-n_3\}$, and given by:

$$G(1,n) = \begin{cases} G^*(1,n) & \text{if } \rho_0<1 \\ \tilde{} & \\ \tilde{G}^*(1,n) & \text{if } \rho_0>1 \end{cases} , \quad n=1,2,... \qquad \text{(IV.11)}$$

may be written as:

$$G(1,n) = h(1,n)G^*(1,n) , \quad n=1,2,... \qquad \text{(IV.12)}$$

where

$$h(1,n) = \begin{cases} 1 & \text{if } \rho_0<1 \\ & \\ \dfrac{1}{n} & \text{if } \rho_0>1 \end{cases} , \quad n=1,2,...$$

$$x_0\ g_0 y_0$$

Note that in the above relations the case n=0 is excluded. In fact, this case, where no jobs are present in the subsystem of the first level ($n_2+n_3=N$), is also excluded in the assumed subset constraints. This is because trivially $P_1(0/n_2,N-n_2)=1$ and,

$$G^*(1,0) = \tilde{G}^*(1,0) = 1$$

while the formulae that describe the solution of this level do not reduce to the above expected values. So the value of the prior for n=0 is G(1,0)=1, and in fact using the same argument $G(\ell,0)=1\ \forall\ \ell$ = level of aggregation. Note also that all multipliers involved in solution (IV.1) proved to be invariant to N. This will not be the case, however, with the fb multipliers of higher levels of aggregation.

Second level of aggregation.

By applying the subset constraints (6.2) on the prior defined in the first level, the MRE solution is:

$$P_2(n_2/n_3) = \frac{G(1,N-n_2-n_3)}{G^*(2,N-n_3)}\ x_2^{n_2}\ g_2^{f_{2,1}(n_2)}\ y_2^{f_{2,2}(n_2)} \qquad \text{(IV.13)}$$

$$n_2=0,1,\ldots,N-n_3$$

and normalizing the above,

$$G^*(2,N-n_3) = G(1,N-n_3) + g_2 \sum_{n_2=1}^{N-n_3-1} G(1,N-n_2-n_3)x_2^{n_2} + x_2^{N-n_3} g_2 y_2 \qquad \text{(IV.14)}$$

Assuming now that x_2, g_2 are invariant to N, also that:

$$\lim_{N\to+\infty} y_2 < +\infty$$

and that $x_2<1$, which is the condition of convergence of (IV.14) and thus the condition of existance of the limit (as $N \longrightarrow +\infty$) of distribution (IV.13), and applying the limit on (IV.14) and (IV.13), clearly,

$$\lim_{N\to+\infty} G^*(2,N-n_3) = \lim_{N\to+\infty} G(1,N) \frac{1-x_2+g_2x_2}{1-x_2} \qquad \text{(IV.15)}$$

and

$$\lim_{N\to+\infty} P_2(n_2/n_3) = \frac{1-x_2}{1-x_2+g_2x_2} x_2^{n_2} g_2^{f_{2,1}(n_2)}, \quad n_2=0,1,\ldots \qquad \text{(IV.16)}$$

Note that the convergence of the summation in (IV.14) is once more ensured by the generalization of Cauchy's limit theorem, (see [KNOP 56, pp. 35-36]). The above has the form of a GE/GE/1 solution. Recalling now from the previous level of aggregation that:

$$\lim_{N\to+\infty} G^*(1,N) = \frac{1-x_0+g_0x_0}{1-x_0}$$

and substituting g_0 from (IV.5), the above yields:

$$\lim_{N\to+\infty} G^*(1,N) = \frac{1}{1-\rho_0}$$

and since similarly from the dual solution of the first level follows that:

$$\lim_{N\to+\infty} \tilde{G}^*(1,N) = \frac{1}{1-\frac{1}{\rho_0}}$$

it is evident from (IV.11) that:

$$\lim_{N\to+\infty} G(1,N) = \left[1-\min\left[\rho_0, \frac{1}{\rho_0} \right] \right]^{-1} \qquad \text{(IV.17)}$$

Let's now derive quantity $1-U_0(2,N-n_3)$, which is exactly the probability of the CPU (Σ_0) being idle, given that n_3 jobs are present at I/O unit Σ_3.

$$1-U_0(2,N-n_3) = \Pr\{\text{Cpu idle}/n_3\} = \sum_{n_2=0}^{N-n_3} P_1(0/n_2,n_3)P_2(n_2/n_3)$$

Using (IV.1), (IV.13) and (IV.12), the above becomes:

$$1-U_0(2,N-n_3) = \frac{1}{G^*(2,N-n_3)} \sum_{n_2=0}^{N-n_3} h(1,N-n_2-n_3)x_2^{n_2} g_2^{f_{2,1}(n_2)} y_2^{f_{2,2}(n_2)}$$

Since from (IV.12),

$$\lim_{N\to+\infty} h(1,N) = \begin{cases} 1 & \text{if } \rho_0<1 \\ 0 & \text{if } \rho_0>1 \end{cases}$$

(IV.17) and (IV.15) yield:

$$\lim_{N\to+\infty} U_0(2,N-n_3) = \min(1,\rho_0) \qquad \text{(IV.18)}$$

The above relation has a clear interpretation. Under the assumption that unit Σ_2 is not the relative bottleneck of this level, then the assymptotic utilization of the CPU is one if this was the relative bottleneck of the previous level, while it is ρ_0 if the relative bottleneck of the previous level was I/O unit Σ_1. Then the fb equation (6.2d), at the limit ($N \longrightarrow +\infty$) becomes:

$$\min(1,\rho_0)\mu_0 r_2 = \left[1- \frac{1-x_2}{1-x_2+g_2x_2} \right]\mu_2$$

and defining,

$$\rho_2 = \min(1,\rho_0)\,\frac{\mu_0 r_2}{\mu_2} \qquad \text{(IV.19)}$$

the above flow equation may be solved with respect to g_2, which proves to be:

$$g_2 = \frac{\rho_2(1-x_2)}{x_2(1-\rho_2)} \qquad \text{(IV.20)}$$

Applying next the mql constraint (6.2b) at the limit ($N \longrightarrow +\infty$), using (IV.16) and (IV.20) it follows that:

$$x_2 = \frac{<n_2>-\rho_2}{<n_2>} \qquad \text{(IV.21)}$$

where $$<n_2> = \lim_{N\to+\infty} <n_2>_{N-n_3}$$

Note that substituting g_2 from (IV.20) into the assymptotic normalizing constant of (IV.16), this becomes:

$$\lim_{N\to+\infty} P_2(n_2/n_3) = (1-\rho_2)x_2^{n_2}\, g_2^{f_{2,1}(n_2)}, \quad n_2=0,1,\ldots$$

Examining definition (IV.19) more carefully, the quantity $\min(1,\rho_0)\mu_0$ may be identified as the assymptotic throughput of the CPU as this is utilized assymptotically in the subsystem of the previous level ($\Sigma_0\Sigma_1$). Note also that ρ_0, ρ_2 appeared in the previous chapter as well, (there they were denoted as x_0, x_2), in relations (5.10) and (5.21), where only the first moment of the flow was taken under consideration.

Returning now to the closed system, the fb equation (6.2d) may be used to derive the value of y_2 as follows:

$$U_0(2,N-n_3)\mu_0 r_2 = U_2(2,N-n_3)\mu_2 \quad \Rightarrow$$

$$\mu_0 r_2 \sum_{n_2=0}^{N-n_3} U_0(1,N-n_2-n_3)P_2(n_2/n_3) = \mu_2[1-P_2(0/n_3)]$$

Noting now that for $n_2=N-n_3$, $U_0(1,0)=1-P_1(0/N-n_3,n_3)=0$, the left hand side of the above is known since it does not involve the unknown multiplier y_2 at all and since $U_0(1,N-n_2-n_3)$ are known from the previous level of aggregation. Substituting $P_2(n_2/n_3)$, $n_2=0,\ldots,N-n_3-1$, from (IV.13) the above becomes:

$$\mu_0 r_2\left[U_0(1,N-n_3)G(1,N-n_3) + g_2 \sum_{n_2=1}^{N-n_3-1} U_0(1,N-n_2-n_3)G(1,N-n_2-n_3)x_2^{n_2} \right] =$$

$$\mu_2[G^*(2,N-n_3) - G(1,N-n_3)]$$

Substituting $G^*(2,N-n_3)$ from (IV.14), the right hand side of the above becomes:

$$\mu_2\left[g_2 \sum_{n_2=1}^{N-n_3-1} G(1,N-n_2-n_3)x_2^{n_2} + x_2^{N-n_3} g_2 y_2 \right]$$

Thus, the fb equation finally yields:

$$\mu_2 x_2^{N-n_3} g_2 y_2 =$$

$$\mu_0 r_2\left[U_0(1,N-n_3)G(1,N-n_3) + g_2 \sum_{n_2=1}^{N-n_3-1} U_0(1,N-n_2-n_3)G(1,N-n_2-n_3)x_2^{n_2} \right] -$$

$$- \mu_2 g_2 \sum_{n_2=1}^{N-n_3-1} G(1,N-n_2-n_3)x_2^{n_2} \qquad \text{(IV.22)}$$

Equation (IV.22) provides the value of y_2. So, $y_2=y_2(N-n_3)$ proved to be a function of $N-n_3$, in fact all fb multipliers, except the one in the first level of aggregation, in an arbitrary network are load-dependent. Notation $y_2(N-n_3)$, however, will be used only when there is a chance of confusion. Normally, y_2 will be used, (and generally y_ℓ, $\ell=2,3,\ldots$).

In case now that $x_2>1$, which implies that unit Σ_2 is the relative

bottleneck ($\rho_2>1$) at this level, let's assume, as in the first level of aggregation, that constraints (6.2) concern the dual distribution $\{P_2^*(N_a/n_3),\ N_a=0,\ldots,N-n_3\}$, which describes the number of jobs present in the subsystem $(\Sigma_0\Sigma_1)$ rather than unit Σ_2. Clearly the duality relations:

$$P_2(n_2/n_3) = P_2^*(N-n_2-n_3/n_3)\ ,\ n_2=0,\ldots,N-n_3 \qquad \text{(IV.23)}$$

are true. The MRE solution for this distribution is:

$$P_2^*(N_a/n_3) = \frac{G(1,N_a)}{\tilde{G}^*(2,N-n_3)}\ \tilde{x}_2^{N_a}\ \tilde{g}_2^{f_{2,1}(N_a)}\ \tilde{y}_2^{f_{2,2}(N_a)} \qquad \text{(IV.24)}$$

$$N_a=0,1,\ldots,N-n_3$$

Using relations (IV.23), it is easy to prove that:

$$\tilde{x}_2 x_2 = \tilde{g}_2 y_2 = \tilde{y}_2 g_2 = 1 \qquad \text{(IV.25)}$$

Relation (IV.25) cannot be derived in the usual fashion, i.e. with an assymptotic connection to an infinite capacity system, as was the case in the first level of aggregation and in every level of the MRE solution of the previous chapter. This difficulty occurs because multiplier $\tilde{g}_2$, according to (IV.25) is expected to depend on $N-n_3$ and thus an invariability assumption for this multiplier is not valid in this case. Hence, at this point an assumption must be made. It is assumed that solution (IV.13) and the values of the multipliers given by (IV.20), (IV.21) and (IV.22) are valid even when $x_2>1$. Note that this assumption does not concern the form of the MRE solution but merely the values of the multipliers involved. Furthermore, it proves to be correct in the GE/GE/1/N system and in the MRE solution of the previous chapter.

What interests us at this point is how the above assumption affects the assymptotic utilization of the CPU at this level. This is because the definition of the prior up to now was such that at the

next level of decomposition the invariant multipliers could be evaluated assymptotically. So in case $x_2,\rho_2>1$ the normalizing constant in (IV.24), which defines the prior, and the assumption of that form of solution are expected to alter the assymptotic utilization of the CPU, from $\min(1,\rho_0)$ that was in the first level, to $\mu_2/\mu_0 r_2$ as was also the case in the previous chapter (see App. III, pp. A-38, A-39). Let's see whether this is true at this point.

So, let $x_2,\rho_2>1$, in which case $\tilde{x}_2<1$, then from (IV.22) y_2 is:

$$y_2 = \frac{\rho_2}{\min(1,\rho_0)} \left[\frac{U_0(1,N-n_3)G(1,N-n_3)}{x_2^{N-n_3} g_2} + \right.$$

$$\left. + \sum_{n_2=1}^{N-n_3-1} U_0(1,N-n_2-n_3)G(1,N-n_2-n_3)\tilde{x}_2^{N-n_2-n_3} \right] -$$

$$- \sum_{n_2=1}^{N-n_3-1} G(1,N-n_2-n_3)\tilde{x}_2^{n_2}$$

and thus,

$$\lim_{N\to+\infty} y_2(N-n_3) = \frac{\rho_2}{\min(1,\rho_0)} \sum_{N_a=1}^{\infty} U_0(1,N_a)G(1,N_a)x_2^{N_a} - \sum_{N_a=1}^{\infty} G(1,N_a)x_2^{N_a} \qquad \text{(IV.26)}$$

The normalizing constant in (IV.24), using (IV.25) at the limit is:

$$\lim_{N\to+\infty} \tilde{G}^*(2,N-n_3) = 1 + \lim_{N\to+\infty} \frac{1}{y_2(N-n_3)} \sum_{N_a=1}^{N-n_3} G(1,N_a)\tilde{x}_2^{N_a} \qquad \text{(IV.27)}$$

The conditional utilization of the CPU in this second level is:

$$U_0(2,N-n_3) = \sum_{n_2=0}^{N-n_3} U_0(1,N-n_2-n_3)P_2^*(N-n_2-n_3/n_3) =$$

$$= \frac{1}{\tilde{G}^*(2,N-n_3)} \left[U_0(1,N-n_3)G(1,N-n_3)\tilde{g}_2\tilde{x}_2^{N-n_3}\tilde{y}_2 + \right.$$

$$\left. + \tilde{g}_2 \sum_{N_a=1}^{N-n_3-1} U_0(1,N_a)G(1,N_a)\tilde{x}_2^{N_a} \right]$$

And applying the limit on the above, using (IV.27),

$$\lim_{N\to+\infty} U_0(2,N-n_3) = \frac{\sum_{N_a=1}^{\infty} U_0(1,N_a)G(1,N_a)\tilde{x}_2^{N_a}}{\lim_{N\to+\infty} y_2(N-n_3) + \sum_{N_a=1}^{\infty} G(1,N_a)\tilde{x}_2^{N_a}}$$

and using (IV.26), it follows that:

$$\lim_{N\to+\infty} U_0(2,N-n_3) = \frac{\min(1,\rho_0)}{\rho_2} = \frac{\mu_2}{\mu_0 r_2} \qquad \text{(IV.28)}$$

which is clearly the assymptotic utilization of the CPU in the subsystem $(\Sigma_0\Sigma_1\Sigma_2)$ under the assumption that unit Σ_2 is the relative bottleneck in this level, (see fig. III.1, App. III). Thus, once more assuming that the prior is defined by:

$$G(2,n) = \begin{cases} G^*(2,n) & \text{if } \rho_2<1 \\ \tilde{G}^*(2,n) & \text{if } \rho_2>1 \end{cases} \quad , \; n=1,2,\ldots$$

will allow us to derive the mql and utilization multipliers in the following level of aggregation assymptotically. Using (IV.23) and (IV.25) it is not difficult to see that:

$$\tilde{G}^*(2,n) = \frac{1}{x_2^n g_2 y_2} G^*(2,n) \quad , \; n=1,2,\ldots$$

and thus

$$G(2,n) = h(2,n)G^*(2,n) \quad , \; n=1,2,\ldots \qquad \text{(IV.29)}$$

where

$$h(2,n) = \begin{cases} 1 & \text{if } \rho_2<1 \\ \dfrac{1}{x_2^{n} g_2 y_2} & \text{if } \rho_2>1 \end{cases}, \quad n=1,2,\ldots \qquad \text{(IV.30)}$$

In the third level of decomposition now, and applying the marginal constraints (6.3) on the prior defined by (IV.29), the MRE solution is:

$$P_3(n_3) = \frac{G(2,N-n_3)}{G^*(3,N)} x_3^{n_3} g_3^{f_{3,1}(n_3)} y_3^{f_{3,2}(n_3)}, n_3=0,\ldots,N \qquad \text{(IV.31)}$$

and normalizing the above,

$$G^*(3,N) = G(2,N) + g_3 \sum_{n_3=1}^{N-1} G(2,N-n_3) x_3^{n_3} + x_3^{N} g_3 y_3 \qquad \text{(IV.32)}$$

And applying the limit on the above two, under the usual assumptions of invariability about x_3, g_3 and that:

$$\lim_{N\to+\infty} y_3 < +\infty$$

as well as the assymptotic stability condition $x_3<1$, it follows that:

$$\lim_{N\to+\infty} P_3(n_3) = \frac{1-x_3}{1-x_3+g_3x_3} x_3^{n_3} g_3^{f_{3,1}(n_3)}, \quad n_3=0,1,\ldots \qquad \text{(IV.33)}$$

Under the assumption that unit Σ_3 is not the new bottleneck of the system, the marginal utilization of the CPU (Σ_0),

$$U_0(3,N) = \sum_{n_3=0}^{N} U_0(2,N-n_3) P_3(n_3)$$

is assymptotically ($N \longrightarrow +\infty$) independent of unit Σ_3 and combining (IV.18) and (IV.28), it is given as:

$$\lim_{N\to+\infty} U_0(3,N) = \lim_{N\to+\infty} U_0(2,N-n_3) = \min(1,\rho_2) \frac{\mu_2}{\mu_0 r_2} \qquad \text{(IV.34)}$$

Then the fb condition (6.3d) at the limit ($N \longrightarrow +\infty$) yields:

$$\min(1,\rho_2)\ \frac{\mu_2}{\mu_0 r_2}\ \mu_0 r_3 = \left[\ 1- \frac{1-x_3}{1-x_3+g_3x_3}\ \right]\mu_3$$

and defining:

$$\rho_3 = \min(1,\rho_2)\ \frac{\mu_2 r_3}{\mu_3 r_2} \qquad \text{(IV.35)}$$

the above flow equation yields once more:

$$g_3 = \frac{\rho_3(1-x_3)}{x_3(1-\rho_3)} \qquad \text{(IV.36)}$$

and applying the mql constraint (6.3b) at the limit ($N \longrightarrow +\infty$), it may be easily shown that:

$$x_3 = \frac{<n_3>-\rho_3}{<n_3>} \qquad \text{(IV.37)}$$

where

$$<n_3> = \lim_{N\to+\infty} <n_3>_N$$

Returning now to the closed system, the fb equation (6.3d) may be written as:

$$\mu_0 r_3 \sum_{n_3=0}^{N} U_0(2,N-n_3)P_3(n_3) = \mu_3[1-P_3(0)]$$

Noting that $U_0(2,0)=0$, the left hand side of the above becomes:

$$\mu_0 r_3\ \frac{1}{G^*(3,N)}\ \left[\ U_0(2,N)G(2,N) + g_3 \sum_{n_3=1}^{N-1} U_0(2,N-n_3)G(2,N-n_3)x_3^{\,n_3}\ \right]$$

where everything is known, except $G^*(3,N)$. Using (IV.31) and (IV.32), the right hand side of the above fb equation becomes:

$$\mu_3\ \frac{1}{G^*(3,N)}\ \left[\ g_3 \sum_{n_3=1}^{N-1} G(2,N-n_3)x_3^{\,n_3} + x_3^{\,N}\ g_3y_3\ \right]$$

Substituting into the fb equation, it follows that:

$$\mu_3 x_3^N g_3 y_3 = \mu_0 r_3 \left[U_0(2,N)G(2,N) + g_3 \sum_{n_3=1}^{N-1} U_0(2,N-n_3)G(2,N-n_3)x_3^{n_3} \right] - \\ - \mu_3 g_3 \sum_{n_3=1}^{N-1} G(2,N-n_3)x_3^{n_3} \qquad \text{(IV.38)}$$

from which the value of the fb multiplier $y_3=y_3(N)$ follows. As in the previous level of decomposition, the above described solution is assumed to be valid even when $x_3>1$.

TABLES 6.1-6.20 (CHAPTER VI)

Table 6.1 : Tandem Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

UNIT i	mean service rate	sq. coef. of variation
0	1, 2, 3, 4	2, 2, 2, 2
1	1, 2, 3, 4	2, 2, 2, 2
2	1, 2, 3, 4	2, 2, 2, 2
3	1, 2, 3, 4	2, 2, 2, 2

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(msr)	UTOL	0.0052	0.0039	0.0153
	NTOL	0.0037	0.0014	0.0067
UME	UTOL	0.0011	0.0006	0.0038
	NTOL	0.0042	0.0020	0.0084
EXP	UTOL	0.0960	0.0077	0.1086
	NTOL	0.0248	0.0159	0.0467

Table 6.2 : Tandem Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation
0	1, 2, 3, 4	15, 15, 15, 15
1	1, 2, 3, 4	15, 15, 15, 15
2	1, 2, 3, 4	15, 15, 15, 15
3	1, 2, 3, 4	15, 15, 15, 15

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(msr)	UTOL	0.0017	0.0014	0.0047
	NTOL	0.0019	0.0011	0.0039
UME	UTOL	0.0034	0.0028	0.0117
	NTOL	0.0050	0.0027	0.0119
EXP	UTOL	0.2748	0.0291	0.3075
	NTOL	0.0513	0.0325	0.1092

Table 6.3 : Tandem Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation
0	1, 2, 3, 4	50, 50, 50, 50
1	1, 2, 3, 4	50, 50, 50, 50
2	1, 2, 3, 4	50, 50, 50, 50
3	1, 2, 3, 4	50, 50, 50, 50

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(msr)	UTOL	0.0006	0.0005	0.0018
	NTOL	0.0007	0.0005	0.0017
UME	UTOL	0.0016	0.0012	0.0054
	NTOL	0.0021	0.0011	0.0052
EXP	UTOL	0.3144	0.0345	0.3544
	NTOL	0.0615	0.0380	0.1299

Table 6.4 : Tandem Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

UNIT i	mean service rate	sq. coef. of variation
0	1, 2, 3, 4	0.5, 0.6, 0.7, 0.9
1	1, 2, 3, 4	1, 1.3, 1.5, 1.8
2	1, 2, 3, 4	1.9, 2.2, 2.4, 2.5
3	1, 2, 3, 4	2.7, 2.9, 3, 3.3

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3)	UTOL	0.0189	0.0118	0.0533
	NTOL	0.0122	0.0050	0.0213
SEQ(msr)	UTOL	0.0230	0.0179	0.0917
	NTOL	0.0175	0.0108	0.0743
UME	UTOL	0.0039	0.0042	0.0215
	NTOL	0.0151	0.0073	0.0400
EXP	UTOL	0.0593	0.0323	0.1248
	NTOL	0.0239	0.0077	0.0392

Table 6.5 : Tandem Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation
0	5, 6, 7, 8	0.5, 0.6, 0.7, 0.9
1	1, 2, 3, 4	1, 1.3, 1.5, 1.8
2	1, 2, 3, 4	1.9, 2.2, 2.4, 2.5
3	8, 9, 10, 11	2.7, 2.9, 3, 3.3

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3)	UTOL	0.0094	0.0072	0.0315
	NTOL	0.0098	0.0044	0.0175
SEQ(msr)	UTOL	0.0114	0.0116	0.0533
	NTOL	0.0136	0.0074	0.0436
UME	UTOL	0.0034	0.0036	0.0133
	NTOL	0.0102	0.0062	0.0236
EXP	UTOL	0.0470	0.0273	0.0892
	NTOL	0.0199	0.0087	0.0345

Table 6.6 : Tandem Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation
0	1, 2, 3, 4	1, 2, 3, 4
1	10, 11, 13, 15	10, 15, 18, 19
2	1, 2, 3, 4	20, 25, 28, 29
3	16, 17, 18, 19	30, 35, 36, 38

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3)	UTOL	0.0073	0.0026	0.0121
	NTOL	0.0067	0.0027	0.0123
SEQ(msr)	UTOL	0.0077	0.0078	0.0288
	NTOL	0.0069	0.0061	0.0248
UME	UTOL	0.0118	0.0082	0.0288
	NTOL	0.0163	0.0059	0.0328
EXP	UTOL	0.2371	0.0306	0.2908
	NTOL	0.0654	0.0280	0.1001

Table 6.7 : Tandem Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation
0	1, 2, 3, 4	10, 12, 13, 14
1	1, 2, 3, 4	20, 25, 28, 29
2	1, 2, 3, 4	30, 35, 38, 39
3	1, 2, 3, 4	50, 55, 56, 58

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3)	UTOL	0.0027	0.0013	0.0061
	NTOL	0.0023	0.0015	0.0069
SEQ(msr)	UTOL	0.0021	0.0020	0.0104
	NTOL	0.0023	0.0029	0.0071
UME	UTOL	0.0035	0.0032	0.0153
	NTOL	0.0023	0.0011	0.0162
EXP	UTOL	0.3012	0.0332	0.3450
	NTOL	0.0592	0.0353	0.1250

Table 6.8 : Tandem Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation
0	1, 2, 3, 4	10, 12, 13, 14
1	9, 10, 1, 5	20, 25, 28, 29
2	7, 9, 30, 4	30, 35, 38, 39
3	1, 2, 3, 4	50, 55, 56, 58

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3)	UTOL	0.0013	0.0016	0.0078
	NTOL	0.0016	0.0018	0.0089
SEQ(msr)	UTOL	0.0037	0.0038	0.0127
	NTOL	0.0024	0.0018	0.0085
UME	UTOL	0.0052	0.0036	0.0149
	NTOL	0.0070	0.0034	0.0151
EXP	UTOL	0.2924	0.0266	0.3393
	NTOL	0.0720	0.0368	0.1263

Table 6.9 : Tandem Configuration.

TEST DATA

M = 4, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation
0	1, 2, 3, 4	5, 6, 7, 9
1	10, 12, 13, 14	10, 15, 16, 17
2	10, 12, 13, 14	20, 22, 23, 25
3	10, 12, 13, 14	30, 35, 38, 39
4	1, 2, 3, 4	110, 110, 120, 100

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3,4)	UTOL	0.0004	0.0003	0.0016
	NTOL	0.0015	0.0008	0.0040
SEQ(msr)	UTOL	0.0126	0.0112	0.0360
	NTOL	0.0072	0.0052	0.0209
UME	UTOL	0.0200	0.0075	0.0340
	NTOL	0.0218	0.0077	0.0354
EXP	UTOL	0.2902	0.0256	0.3332
	NTOL	0.0778	0.0330	0.1241

Table 6.10 : Tandem Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

UNIT i	mean service rate	sq. coef. of variation
0	1, 2, 3, 4	1, 1.5, 2, 2.5
1	1, 2, 3, 4	3.5, 3.8, 4, 4.5
2	1, 2, 3, 4	5, 5, 5.5, 6
3	1, 2, 3, 4	65, 70, 90, 100

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3)	UTOL	0.0075	0.0038	0.0208
	NTOL	0.0049	0.0026	0.0155
SEQ(msr)	UTOL	0.0344	0.0343	0.1474
	NTOL	0.0305	0.0203	0.0787
UME	UTOL	0.0476	0.0206	0.0964
	NTOL	0.0509	0.0220	0.1026
EXP	UTOL	0.2718	0.0377	0.3416
	NTOL	0.0840	0.0291	0.1362

Table 6.11 : Central Server Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	4, 5, 6, 7	2, 2, 2, 2	0.4
1	1, 2, 3, 4	2, 2, 2, 2	0.2
2	1, 2, 3, 4	2, 2, 2, 2	0.2
3	1, 2, 3, 4	2, 2, 2, 2	0.2

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(msr)	UTOL	0.0069	0.0048	0.0187
	NTOL	0.0061	0.0029	0.0142
UME	UTOL	0.0036	0.0021	0.0101
	NTOL	0.0046	0.0017	0.0079
EXP	UTOL	0.0601	0.0099	0.0761
	NTOL	0.0117	0.0066	0.0284

Table 6.12 : Central Server Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	7, 8, 9, 10	15, 15, 15, 15	0.385
1	1, 2, 3, 4	15, 15, 15, 15	0.231
2	1, 2, 3, 4	15, 15, 15, 15	0.231
3	1, 2, 3, 4	15, 15, 15, 15	0.154

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(msr)	UTOL	0.0096	0.0066	0.0278
	NTOL	0.0096	0.0036	0.0215
UME	UTOL	0.0064	0.0054	0.0257
	NTOL	0.0075	0.0020	0.0126
EXP	UTOL	0.2677	0.0377	0.3154
	NTOL	0.0709	0.0343	0.1275

Table 6.13 : Central Server Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	7, 8, 9, 10	60, 60, 60, 60	0.4
1	1, 2, 3, 4	60, 60, 60, 60	0.24
2	1, 2, 3, 4	60, 60, 60, 60	0.16
3	1, 2, 3, 4	60, 60, 60, 60	0.2

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(msr)	UTOL	0.0038	0.0025	0.0116
	NTOL	0.0043	0.0017	0.0083
UME	UTOL	0.0032	0.0023	0.0136
	NTOL	0.0035	0.0011	0.0067
EXP	UTOL	0.3460	0.0400	0.4052
	NTOL	0.0865	0.0456	0.1672

Table 6.14 : Central Server Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	4, 8, 12, 17	1, 2, 3, 3.5	0.455
1	1, 2, 3, 4	4, 4.5, 5, 5.5	0.227
2	0.5, 1, 1.5, 2.5	6, 6.5, 6.8, 7	0.227
3	1, 2, 3, 4	7.2, 7.8, 8, 8.5	0.091

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3)	UTOL	0.0112	0.0094	0.0484
	NTOL	0.0107	0.0061	0.0327
UME	UTOL	0.0175	0.0111	0.0494
	NTOL	0.0132	0.0055	0.0244
EXP	UTOL	0.1471	0.0454	0.2150
	NTOL	0.0546	0.0231	0.0872

Table 6.15 : Central Server Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	4, 5, 6, 8	10, 40, 70, 100	0.185
1	1, 2, 3, 4	0.5, 0.5, 0.6, 0.6	0.37
2	0.5, 1, 1.5, 2	0.7, 0.7, 0.8, 0.8	0.222
3	0.5, 1, 1.5, 2	0.85, 0.85, 0.9, 0.9	0.222

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(1,2,3,0)	UTOL	0.0283	0.0131	0.0639
	NTOL	0.0123	0.0081	0.0514
UME	UTOL	0.0328	0.0225	0.0845
	NTOL	0.0418	0.0293	0.1099
EXP	UTOL	0.1365	0.0693	0.2724
	NTOL	0.0903	0.0337	0.1449

Table 6.16 : Central Server Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	7, 8, 9, 10	5, 6, 8, 10	0.37
1	0.5, 1, 1.5, 2	15, 18, 19, 20	0.296
2	0.5, 1, 1.5, 2	25, 26, 28, 30	0.185
3	0.5, 1, 1.5, 2	34, 36, 37, 38	0.148

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3)	UTOL	0.0058	0.0046	0.0262
	NTOL	0.0048	0.0023	0.0135
UME	UTOL	0.0078	0.0047	0.0344
	NTOL	0.0080	0.0027	0.0141
EXP	UTOL	0.2583	0.0435	0.3078
	NTOL	0.0649	0.0284	0.1110

Table 6.17 : Central Server Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	7, 8, 10, 14	10, 14, 18, 20	0.357
1	1, 2, 3, 4	40, 45, 50, 55	0.25
2	0.5, 1, 1.5, 2	60, 65, 70, 75	0.214
3	1, 1.5, 2, 2.5	80, 85, 90, 100	0.179

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3)	UTOL	0.0027	0.0022	0.0120
	NTOL	0.0029	0.0016	0.0076
UME	UTOL	0.0062	0.0036	0.0310
	NTOL	0.0046	0.0014	0.0080
EXP	UTOL	0.2948	0.0411	0.3474
	NTOL	0.0772	0.0372	0.1283

Table 6.18 : Central Server Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	7, 8, 12, 13	80, 90, 95, 100	0.526
1	1, 1.5, 2, 2.5	1, 2, 3, 4	0.211
2	1, 2, 3, 4	5, 6, 6.5, 7	0.158
3	0.5, 1, 1.5, 2	7.5, 8, 9, 9.5	0.105

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(1,2,3,0)	UTOL	0.0123	0.0063	0.0324
	NTOL	0.0090	0.0039	0.0226
UME	UTOL	0.0131	0.0078	0.0349
	NTOL	0.0182	0.0097	0.0445
EXP	UTOL	0.2036	0.0277	0.2618
	NTOL	0.0548	0.0197	0.0932

Table 6.19 : Central Server Configuration.

TEST DATA

M = 4, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	10, 12, 15, 20	10, 20, 90, 100	0.385
1	0.5, 1, 2, 3	0.5, 0.5, 0.6, 0.6	0.231
2	1, 2, 2.5, 3.5	0.65, 0.65, 0.7, 0.7	0.154
3	1, 1.5, 2.5, 3	0.75, 0.75, 0.8, 0.8	0.154
4	0.5, 1, 1.5, 2	0.85, 0.9, 0.9, 0.95	0.077

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(1,2,3,0)	UTOL	0.0240	0.0092	0.0450
	NTOL	0.0114	0.0067	0.0306
UME	UTOL	0.0191	0.0176	0.0629
	NTOL	0.0290	0.0220	0.0867
EXP	UTOL	0.0615	0.0425	0.1732
	NTOL	0.0633	0.0316	0.1241

Table 6.20 : Central Server Configuration.

TEST DATA

M = 4, N = 3, Test No = 200

UNIT i	mean service rate	sq. coef. of variation	average r_i
0	8, 12, 15, 25	1, 2, 2.5, 3	0.417
1	0.5, 2, 4, 4.5	3.5, 4, 5, 6	0.25
2	0.5, 1, 1.5, 2	7, 8, 9, 10	0.167
3	0.5, 1, 2, 2.5	11, 12, 13, 14	0.125
4	0.5, 0.8, 1, 1.5	30, 50, 70, 100	0.042

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(0,1,2,3)	UTOL	0.0123	0.0077	0.0483
	NTOL	0.0074	0.0034	0.0230
UME	UTOL	0.0168	0.0087	0.0395
	NTOL	0.0112	0.0041	0.0232
EXP	UTOL	0.2028	0.0411	0.2803
	NTOL	0.0522	0.0199	0.1030

APPENDIX V (CHAPTER VII)

Notes on the Universal Maximum Entropy algorithm.

The universal maximum entropy algorithm (UME), [KOUV 86c], provides an approximation for general closed queueing networks with single servers and multiple job classes. This algorithm has been extended to tackle multi-server networks of queues. The performance of UME algorithm in single class, single (or multiple) server networks is of particular interest to this thesis because it has been found to be the best approximate method for networks with GE-distributed service times, [ALMO 88] and because it shares several ideas with our new MRE decomposition algorithm. So the basic steps of this algorithm will be briefly presented so that similarities and differences to the MRE decomposition algorithm (algorithm 7.1 of 7th chapter) become clear. Moreover, using ideas born and developed in our decomposition approach, a new, modified version of the UME algorithm will be proposed, which is computationally more efficient whilst it provides comparable accuracy to the original version of this algorithm.

UME algorithm belongs to a category of approximations which assume a product form of solution for the equilibrium joint queue length distribution and then use generalized convolution techniques in order to implement an efficient way of obtaining the marginal queue length distributions. This class of approximations uses a completely different basis than the one used in decomposition techniques whose starting point is the decomposition of the network's state space into conditional and marginal distributions, implying a structural decomposition of the network as oposed to "convolution" type of methods which always consider the network as a whole. The difference can be viewed if the Jacksonian product form solution for

exponential networks [JACK 63] - as this has been implemented by Buzen [BUZE ,73], [BRUE 80] - is compared to the solution proposed by Courtois [COUR 77] for the same type of networks. However, it is interesting not only to observe the differences but the similarities as well. The solution proposed by Courtois - as it has been extended by Vantilborgh [VANT 78] - provides an alternative exact method for an arbitrary network configuration of exponential type. So two different in principle methods converge to identical results. A parallelism may be drawn between the above differences/similarities and the corresponding ones between the UME and the MRE decomposition algorithms for general QNMs. Of course the results obtained from these later two algorithms are not identical, because the second moment approximation of the flow is involved and thus they are classified as approximations. Still their performances are not far apart. Let's see now the UME algorithm more closer.

The mean value type of constraints assumed, concern the marginal equilibrium queue length distributions of the network's queues $\{P_k(n),\ n=0,\ldots,N\}$, $k=1,\ldots,M$ - except for the normalization constraint which applies to the joint probability distribution of the system - and for each centre k, $k=0,\ldots,M$, they are:

- The normalization

$$\sum_{\bar{n}\epsilon S(M,N)} P(\bar{n}) = 1$$

where $\bar{n}=(n_0,n_1,\ldots,n_M)$ is the vector that describes the joint queue length distribution and S(M,N) is the state space consisting of all possible such vectors, i.e.

$$S(M,N) = \{(n_0,n_1,\ldots,n_M)/0\leq n_i\leq N,\ i=0,\ldots,M,\ \wedge \sum_{i=0}^{M} n_i=N\}$$

- The utilization constraint

$$\sum_{n=0}^{N} h(n)P_k(n) = U_k$$

where

$$h(n) = \begin{cases} 0 & \text{if } n=0 \\ 1 & \text{if } n>0 \end{cases}$$

- The mean queue length constraint

$$\sum_{n=0}^{N} nP_k(n) = \langle n_k \rangle$$

Thus, the constraints (except the normalization) consist of independent information about each queue and according to property 3 (system independence) of Appendix II (also in [SHOR 81]) the resulting ME solution is of product form and given as:

$$P(\bar{n}) = \frac{1}{Z(M,N)} \prod_{i=0}^{M} g_i^{h(n_i)} x_i^{n_i} \qquad (V.1)$$

where g_i, x_i, are multipliers that correspond to the utilization and mql constraints assumed for unit i, i=0,...,M, respectively, and Z(M,N) is the normalizing constant. So the ME formalism, and according to the property of system independence, proposes that each queue of the network should be treated independently with respect to the utilization and mql constraints, which provide the familiar form of the GE/GE/1 solution for each centre. It is only the normalization constraint that carries information about the correlation between the queues.

UME algorithm adopts the concept of the "pseudo open " network in order to evaluate what was considered to be, throughout this thesis, the assymptotic flow. This related network - as the above descriptive term reveals - is something between an open and a closed network. It

has the same transition probabilities, service rates and coefficients of variation as the original closed network. Its state space, however, is that of an open network (infinite capacity queueing centres) satisfying the additional constraint,

$$\sum_{i=0}^{M} <n_i> = N$$

The above constraint with respect to the first moment of the flow for example provides the missing equation to the matrix form of equation:

$$t = tR$$

where $t=(t_0,t_1,\ldots,t_M)$ is the vector of throughputs and R is the routing matrix of the network. In the case of decomposition, instead of the above constraint, it was assumed that $t_b=\mu_b$ where b was the bottleneck of the system. Hence, the iteration for the two moments of the flow in the "pseudo open" network is identical to the iteration performed in the Mth level of aggregation of algorithm 7.1, except of course the above difference.

This method is called the "Fixed-Population-Mean" (FPM) technique and was introduced by Whitt [WHIT 84]. The flow approximated by this method is not exactly assymptotic since the resulting moments depend on N via the above FPM constraint. Practically this means that this flow is more moderate than the one considered in the corresponding open network, since in the "pseudo open" network the bottleneck is not saturated. In the case of a network with GE-distributed service times the resulting from this iteration first and second moments are used to evaluate multipliers g_i, x_i, i=0,...,M, through the usual formulae also applied in decomposition.

Having calculated the mean queue length and utilization multipliers, the solution does not satisfy the flow balance equations of the closed network. So after the first part of the UME algorithm a

second part follows, where the solution is forced to satisfy these fb equations by evaluating via an iterative technique appropriate fb multipliers y_i, i=0,...,M. These multipliers can be viewed as corresponding to appropriate fb constraints, in which case the ME form of solution (V.1) may be rewritten as:

$$P(\bar{n}) = \frac{1}{Z(M,N)} \prod_{i=0}^{M} g_i^{h(n_i)} x_i^{n_i} y_i^{h(n_i)} \qquad (V.2)$$

In fact there are two versions for this second step of the algorithm. Alternatively to (V.2) where multiplier y_i appears in the same probabilities as the utilization multiplier g_i does, the exponent of y_i, i=0,...,M, in (V.2) may be defined do be funtion $f(n_i)$ where:

$$f(n_i) = \begin{cases} 1 & \text{if } n_i=N \\ 0 & \text{if } n_i \neq N \end{cases}$$

Under the above alternative approach the procedure of determining these fb multipliers is non-iterative and thus computationally more efficient [ALMO 88]. For more details on this part of the UME algorithm we refer to [KOUV 86c].

<u>An alternative version of the UME approximate algorithm.</u>

Let's return to the first part of the algorithm which is of interest to us and where an alteration may be proposed aiming to avoid the FPM technique which involves a computationally expensive Newton-Raphson iteration. Let's start by assuming the same form (V.1) of ME solution for the joint queue length distribution of the network. Without loss of generality let's also assume that the bottleneck unit of the network is unit Σ_0. Then naturally the limit of all joint probabilities $P(n_0,n_1,....,n_M)$ with $n_0<+\infty$ as $N \longrightarrow +\infty$ is zero, hence in order to make the assymptotic connection, the state space must be transformed as well as the form (V.1) of solution, so

that the limit as $N \longrightarrow +\infty$ exists. This transformation, involves redefining the joint queue length state probability as $P^*(n_1,n_2,\ldots,n_M)$ by considering that unit Σ_0 has $N-n_0-\ldots-n_M$ jobs present, i.e.,

$$P^*(n_1,n_2,\ldots,n_M) = P(N-\sum_{k=1}^{M} n_k,n_1,\ldots,n_M),\ 0\leq n_k\leq N,\ k=1,\ldots,M,$$
$$\text{and}\quad 0\leq \sum_{k=1}^{M} n_k \leq N$$

Then rewritting the ME solution (V.1) for this equivalent form of distribution and for $N_0=N-n_1-\ldots-n_M$,

$$P^*(n_1,\ldots,n_M) = \frac{1}{Z(M,N)}\, g_0^{h(N_0)}\, x_0^{N_0} \prod_{i=1}^{M} g_i^{h(n_i)}\, x_i^{n_i} \Rightarrow$$

$$P^*(n_1,\ldots,n_M) = \frac{1}{Z^*(M,N)}\, g_0^{h(N_0)} \prod_{i=1}^{M} g_i^{h(n_i)} \left[\frac{x_i}{x_0}\right]^{n_i}$$

or

$$P^*(n_1,\ldots,n_M) = \frac{1}{Z^*(M,N)}\, g_0^{h(N_0)} \prod_{i=1}^{M} g_i^{h(n_i)}\, x_i'^{n_i}$$

where

$$Z^*(M,N) = \frac{Z(M,N)}{x_0^N} \quad \text{and} \quad x_i' = \frac{x_i}{x_0}\ ,\ i=1,\ldots,M$$

Assuming that g_i, x_i, $i=0,\ldots,M$, are completely invariant to N, and as $N \longrightarrow +\infty$, the above solution tends to the solution of the corresponding open system where the external arrival source, as well as the sink of the network is the saturated bottleneck unit Σ_0. The form of this limiting distribution may be identified as the ME solution for the corresponding open system and according to it each queue may be treated approximately as a GE/GE/1 queue and the parameters of the assymptotic flow can be approximated with a similar iteration to the one used in the last level of algorithm 7.1. The first two moments of this assymptotic flow may then be used in the

usual fashion to specify the values of multipliers g_i, x_i', i=1,...,M. A problem arises with the definition of the utilization multiplier g_0 that corresponds to the bottleneck unit Σ_0. In order to preserve the exactness of this version of the UME algorithm in the case of the smallest closed network, i.e. the two stage cyclic GE/GE/1/N queue, the appropriate assymptotic input flow to unit Σ_0 seems to be the flow that is calculated in the subsystem ($\Sigma_1 \ldots \Sigma_M$) (complementary to Σ_0 subnetwork), which is defined by shorting unit Σ_0 in a decomposition fashion. More precisely, formulae (7.7) may be applied on the routing matrix R(M+1) of the system and define the appropriate routing matrix R(M) for the subsystem ($\Sigma_1 \Sigma_2 \ldots \Sigma_M$). Then the iteration used in algorithm 7.1 may be used on this subsystem, to approximate the first two moments of the streams that depart units Σ_1, Σ_2,..., Σ_M (using a saturated relative bottleneck unit Σ_b, $b \epsilon \{1,2,\ldots,M\}$. Finally, the original routing matrix R(M+1) should be used to evaluate the first and second moments of the flow that arrives at unit Σ_0.

Having evaluated multipliers g_0, g_i, x_i, i=1,...,M, solution (V.2) may be flow balanced by any of the two methods used in the original version of UME (iterative or non-iterative) and mentioned previously. This alternative version is exact in the case of the exponential network as well as at the level of the GE/GE/1/N two stage cyclic queue, while the original UME algorithm provides only an approximation for the GE/GE/1/N system due to the FPM technique. Moreover, it is computationally more efficient and simpler to implement. An experimental version of this algorithm has been programmed by my colleague J. Almond and the results were comparable to the ones obtained by the original UME algorithm.

The only problem recorded was that failures in flow balancing the network in the second part of the algorithm occured in the form of

negative fb multipliers (when using the non-iterative fb method) producing negative probabilities of small magnitude, which did not affect the average statistics significantly. Recall that this same problem occured in the MRE decomposition algorithms. This strengthens the viewpoint that these fb failures are due to the assumption of invariance for the utilization and mql multipliers and the excessive assymptotic flow used in their evaluation.

TABLES 7.1-7.10 (CHAPTER VII)

Table 7.1 : Fully Connected Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

i	μ_i	C_i^2	r_{i0}	r_{i1}	r_{i2}	r_{i3}
						average
0	1, 2, 3, 5	2, 2, 2, 2	0	0.3	0.5	0.2
1	1, 2, 3, 5	2, 2, 2, 2	0.5	0	0.4	0.1
2	1, 2, 3, 5	2, 2, 2, 2	0	0.6	0	0.4
3	1, 2, 3, 5	2, 2, 2, 2	0.1	0.2	0.7	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(msr)	UTOL	0.0099	0.0065	0.0321
	NTOL	0.0063	0.0029	0.0224
UME	UTOL	0.0074	0.0038	0.0159
	NTOL	0.0030	0.0014	0.0080
EXP	UTOL	0.0674	0.0136	0.0876
	NTOL	0.0215	0.0101	0.0419

Table 7.2 : Fully Connected Configuration.

TEST DATA

M = 4, N = 3, Test No = 200

i	μ_i	C_i^2	average r_{i_0}	r_{i_1}	r_{i_2}	r_{i_3}	r_{i_4}
0	1, 4, 5, 8	40, 40, 40, 40	0	0.6	0.2	0	0.2
1	2, 3, 4, 7	40, 40, 40, 40	0.3	0	0.3	0.1	0.3
2	1, 3, 6, 8	40, 40, 40, 40	0.5	0.2	0	0	0.3
3	2, 4, 6, 7	40, 40, 40, 40	0	0	0.8	0	0.2
4	3, 5, 6, 7	40, 40, 40, 40	0.2	0	0.1	0.7	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(msr)	UTOL	0.0056	0.0055	0.0282
	NTOL	0.0051	0.0020	0.0115
UME	UTOL	0.0064	0.0039	0.0284
	NTOL	0.0028	0.0014	0.0084
EXP	UTOL	0.2868	0.0330	0.3376
	NTOL	0.0694	0.0385	0.1248

Table 7.3 : Fully Connected Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

i	μ_i	C_i^2	r_{i0}	r_{i1}	r_{i2}	r_{i3}
						average
0	1, 2, 3, 4	0.5, 0.7, 1, 1.3	0	0	0.6	0.4
1	1, 2, 3, 4	1.5, 1.8, 2, 2.2	0.5	0.2	0.3	0
2	1, 2, 3, 4	2.5, 2.8, 3, 3.3	0	0.6	0	0.4
3	1, 2, 3, 4	3.5, 3.8, 4, 4.5	0.1	0.2	0.4	0.3

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0122	0.0107	0.0550
	NTOL	0.0124	0.0072	0.0340
UME	UTOL	0.0100	0.0068	0.0387
	NTOL	0.0046	0.0026	0.0164
EXP	UTOL	0.0782	0.0297	0.1385
	NTOL	0.0268	0.0110	0.0523

Table 7.4 : Fully Connected Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

i	μ_i	C_i^2	r_{i0}	r_{i1}	r_{i2}	r_{i3}
						average
0	1, 3, 5, 8	3, 4, 5, 6	0	0.2	0.7	0.1
1	1, 3, 5, 8	10, 12, 15, 18	0.2	0	0.5	0.3
2	1, 3, 5, 8	22, 25, 27, 30	0.8	0	0	0.2
3	1, 3, 5, 8	35, 45, 50, 55	0.1	0.7	0.2	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0081	0.0055	0.0285
	NTOL	0.0064	0.0036	0.0242
UME	UTOL	0.0149	0.0093	0.0429
	NTOL	0.0103	0.0052	0.0255
EXP	UTOL	0.2677	0.0673	0.3639
	NTOL	0.0879	0.0385	0.1484

Table 7.5 : Fully Connected Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

i	μ_i	C_i^2	r_{i0}	r_{i1}	r_{i2}	r_{i3}
						average
0	1, 3, 5, 7	10, 12, 15, 17	0.1	0	0	0.9
1	1, 3, 5, 7	20, 23, 26, 29	0.5	0.2	0.3	0
2	1, 3, 5, 7	33, 35, 37, 40	0.4	0	0.3	0.3
3	1, 3, 5, 7	45, 50, 60, 70	0.2	0.4	0	0.4

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0044	0.0044	0.0239
	NTOL	0.0041	0.0022	0.0115
UME	UTOL	0.0074	0.0051	0.0285
	NTOL	0.0053	0.0030	0.0201
EXP	UTOL	0.2523	0.0611	0.3314
	NTOL	0.0696	0.0300	0.1200

Table 7.6 : Fully Connected Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

i	μ_i	C_i^2	r_{i0}	r_{i1}	r_{i2}	r_{i3}
						average
0	1, 2, 3, 4	20, 25, 30, 35	0.2	0	0.6	0.2
1	1, 2, 3, 4	40, 45, 50, 55	0.4	0	0.4	0.2
2	1, 2, 3, 4	65, 70, 80, 90	0.1	0.4	0.2	0.3
3	1, 2, 3, 4	100,110,120,130	0	0.7	0.3	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0022	0.0019	0.0125
	NTOL	0.0036	0.0016	0.0092
UME	UTOL	0.0063	0.0026	0.0130
	NTOL	0.0024	0.0011	0.0059
EXP	UTOL	0.3091	0.0252	0.3519
	NTOL	0.0709	0.0342	0.1304

Table 7.7 : Fully Connected Configuration.

TEST DATA

M = 4, N = 3, Test No = 200

i	μ_i	C_i^2	r_{i0}	r_{i1}	r_{i2}	average r_{i3}	r_{i4}
0	2, 4, 6, 8	1, 1.5, 2, 2.5	0.1	0.2	0	0	0.7
1	2, 4, 6, 8	4, 5, 6, 7	0.2	0.1	0.1	0.1	0.5
2	2, 4, 6, 8	9, 9.5, 10, 11	0.4	0.2	0.3	0.1	0
3	2, 4, 6, 8	12, 13, 14, 15	0	0.2	0.4	0.2	0.2
4	2, 4, 6, 8	90, 100, 110, 120	0.5	0.3	0	0	0.2

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0086	0.0067	0.0365
	NTOL	0.0062	0.0038	0.0209
UME	UTOL	0.0143	0.0085	0.0384
	NTOL	0.0299	0.0106	0.0576
EXP	UTOL	0.2237	0.0313	0.2743
	NTOL	0.0501	0.0256	0.0939

Table 7.8 : Fully Connected Configuration.

TEST DATA

M = 3, N = 4, Test No = 200

i	μ_i	C_i^2	average r_{i0}	r_{i1}	r_{i2}	r_{i3}
0	2, 3, 8, 10	0.5,0.6,0.7,0.8	0	0.3	0.2	0.5
1	1, 2, 7, 11	1.5, 2, 2.5, 3	0.4	0	0.2	0.4
2	3, 5, 6, 15	3.5, 4, 4.5, 5	0.5	0.3	0.1	0.1
3	2, 4, 8, 9	5.5, 6, 7, 7.5	0.2	0.2	0.6	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0178	0.0127	0.0609
	NTOL	0.0121	0.0067	0.0441
UME	UTOL	0.0207	0.0134	0.0599
	NTOL	0.0092	0.0052	0.0274
EXP	UTOL	0.1105	0.0465	0.2044
	NTOL	0.0440	0.0154	0.0766

Table 7.9 : Central Server Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

i	μ_i	C_i^2	r_{i0}	r_{i1}	r_{i2}	r_{i3}
						average
0	4, 7, 8, 10	15, 20, 25, 30	0.3	0.3	0.2	0.2
1	1, 2, 3, 4	1, 1.5, 2, 2.5	1	0	0	0
2	1, 2, 3, 4	3, 4, 4.5, 5	1	0	0	0
3	1, 2, 3, 4	40, 70, 75, 90	1	0	0	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0135	0.0068	0.0409
	NTOL	0.0062	0.0035	0.0212
UME	UTOL	0.0079	0.0054	0.0348
	NTOL	0.0145	0.0067	0.0381
EXP	UTOL	0.2144	0.0420	0.2908
	NTOL	0.0544	0.0203	0.1015

Table 7.10 : Central Server Configuration.

TEST DATA

M = 4, N = 3, Test No = 200

i	μ_i	C_i^2	r_{i0}	r_{i1}	r_{i2}	r_{i3}	r_{i4}
							average
0	5, 6, 9, 10	5, 5.5, 7, 9	0.2	0.3	0.2	0.2	0.1
1	1, 2, 3, 4	0.5, 1, 1.5, 1.7	1	0	0	0	0
2	1, 2, 3, 4	35, 40, 42, 48	1	0	0	0	0
3	1, 2, 3, 4	2, 2.5, 3, 4	1	0	0	0	0
4	1, 2, 3, 4	12, 15, 18, 20	1	0	0	0	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0183	0.0077	0.0361
	NTOL	0.0101	0.0050	0.0319
UME	UTOL	0.0095	0.0070	0.0396
	NTOL	0.0163	0.0079	0.0347
EXP	UTOL	0.1739	0.0422	0.2648
	NTOL	0.0508	0.0177	0.0898

APPENDIX VI (CHAPTER VIII)

TABLES 8.1-8.10

Table 8.1 : Tandem Configuration.

TEST DATA

M = 2, N = 5, Test No = 200

i	$c_i\mu_i$	C_i^2	c_i	average r_{i_0}	r_{i_1}	r_{i_2}
0	1, 3, 5, 7	3, 3, 3, 3	2	0	1	0
1	2, 4, 6, 6.5	3, 3, 3, 3	2	0	0	1
2	0.8,2.5,3.4,4.5	3, 3, 3, 3	1	1	0	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(msr)	UTOL	0.0097	0.0058	0.0222
	NTOL	0.0063	0.0024	0.0122
UME	UTOL	0.0115	0.0053	0.0205
	NTOL	0.0067	0.0041	0.0162
EXP	UTOL	0.0949	0.0321	0.1240
	NTOL	0.0273	0.0159	0.0594

Table 8.2 : Tandem Configuration.

TEST DATA

M = 2, N = 3, Test No = 200

i	$c_i\mu_i$	C_i^2	c_i	average r_{i0}	r_{i1}	r_{i2}
0	2, 4, 6, 8	1, 3, 5, 6	3	0	1	0
1	2, 4, 6.5, 7	5, 6, 7, 8	3	0	0	1
2	2.5,4.5,6.5,8.5	30, 40, 60, 100	1	1	0	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0115	0.0040	0.0269
	NTOL	0.0165	0.0052	0.0353
UME	UTOL	0.0070	0.0037	0.0171
	NTOL	0.0104	0.0061	0.0242
EXP	UTOL	0.0653	0.0355	0.1742
	NTOL	0.0888	0.0318	0.1460

Table 8.3 : Central Server Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

i	$c_i\mu_i$	C_i^2	c_i	r_{i0}	r_{i1}	r_{i2}	r_{i3}
							average
0	8, 10, 15, 20	30, 40, 50, 100	3	0.5	0.3	0.1	0.1
1	1, 2, 3, 4	0.5,0.6,0.7,0.8	1	1	0	0	0
2	1, 2, 3, 4	1, 1.2, 1.5, 2	1	1	0	0	0
3	0.5, 1, 2, 3	2.2, 2.5, 3, 4	1	1	0	0	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0041	0.0039	0.0204
	NTOL	0.0056	0.0055	0.0292
UME	UTOL	0.0256	0.0176	0.0791
	NTOL	0.0186	0.0138	0.0586
EXP	UTOL	0.0165	0.0149	0.0610
	NTOL	0.0110	0.0082	0.0337

Table 8.4 : Central Server Configuration.

TEST DATA

M = 3, N = 4, Test No = 120

i	$c_i\mu_i$	C_i^2	c_i	r_{i0}	r_{i1}	r_{i2}	r_{i3}
							average
0	4, 6, 8, 12	1, 2, 3, 4	4	0	0.6	0.3	0.1
1	1, 2, 3, 4	6, 7, 8, 9	1	1	0	0	0
2	1, 2, 3, 4	10, 11, 12, 14	1	1	0	0	0
3	0.5, 1, 2, 3	17, 20, 22, 23	1	1	0	0	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0142	0.0108	0.0448
	NTOL	0.0142	0.0101	0.0526
UME	UTOL	0.0077	0.0061	0.0299
	NTOL	0.0106	0.0071	0.0344
EXP	UTOL	0.1389	0.0443	0.2402
	NTOL	0.0719	0.0289	0.1293

Table 8.5 : Fully Connected Configuration.

TEST DATA

M = 2, N = 3, Test No = 200

i	$c_i\mu_i$	C_i^2	c_i	average r_{i0}	r_{i1}	r_{i2}
0	1, 2, 3, 4	2, 4, 6, 8	2	0.1	0.4	0.5
1	1, 2, 3, 4	10, 14, 15, 16	2	0.6	0	0.4
2	1, 2, 3, 4	30, 50, 80, 90	2	0.2	0.7	0.1

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0096	0.0088	0.0533
	NTOL	0.0056	0.0043	0.0288
UME	UTOL	0.0109	0.0075	0.0357
	NTOL	0.0109	0.0046	0.0213
EXP	UTOL	0.0436	0.0131	0.0778
	NTOL	0.0151	0.0068	0.0334

Table 8.6 : Fully Connected Configuration.

TEST DATA

M = 2, N = 5, Test No = 100

i	$c_i\mu_i$	C_i^2	c_i	average r_{i0}	r_{i1}	r_{i2}
0	1, 2, 3, 4	5, 6, 7, 8	4	0	0.3	0.7
1	1, 2, 3, 4	1, 2, 3, 4	1	0.5	0.1	0.4
2	1, 2, 3, 4	0.5,0.6,0.7,0.9	1	0.3	0.4	0.3

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0050	0.0051	0.0218
	NTOL	0.0049	0.0044	0.0206
UME	UTOL	0.0163	0.0146	0.0597
	NTOL	0.0145	0.0138	0.0587
EXP	UTOL	0.0336	0.0255	0.1015
	NTOL	0.0164	0.0121	0.0461

Table 8.7 : Tandem Configuration.

TEST DATA

M = 2, N = 4, Test No = 200

i	$c_i\mu_i$	C_i^2	c_i	average r_{i_0}	r_{i_1}	r_{i_2}
0	1, 2, 3, 4	2, 2, 2, 2	3	0.2	0.2	0.6
1	6, 7, 8, 9	3, 3, 3, 3	1	0.5	0	0.5
2	5, 6, 7, 8	4, 4, 4, 4	1	0.7	0.2	0.1

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0064	0.0059	0.0319
	NTOL	0.0052	0.0053	0.0282
UME	UTOL	0.0040	0.0026	0.0152
	NTOL	0.0028	0.0015	0.0077
EXP	UTOL	0.0437	0.0215	0.0931
	NTOL	0.0278	0.0130	0.0541

Table 8.8 : Fully Connected Configuration.

TEST DATA

M = 3, N = 4, Test No = 100

i	$c_i\mu_i$	C_i^2	c_i	average r_{i0}	r_{i1}	r_{i2}	r_{i3}
0	1, 2, 3, 4	0.5,0.8,1,1.5	1	0	0.4	0.4	0.2
1	1, 2, 3, 4	2, 3, 4, 5	1	0.7	0	0.1	0.2
2	1, 2, 3, 4	10, 11, 12, 15	1	0.4	0.3	0	0.3
3	1, 2, 3, 4	50, 60, 70, 80	4	0.2	0.4	0.4	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0192	0.0137	0.0527
	NTOL	0.0139	0.0118	0.0477
UME	UTOL	0.0171	0.0128	0.0495
	NTOL	0.0164	0.0102	0.0495
EXP	UTOL	0.0947	0.0468	0.2013
	NTOL	0.0456	0.0143	0.0809

Table 8.9 : Fully Connected Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

i	$c_i\mu_i$	C_i^2	c_i	r_{i0}	r_{i1}	r_{i2}	r_{i3}
							average
0	4, 6, 8, 10	10, 15, 18, 20	1	0	0.1	0.2	0.7
1	4, 6, 8, 10	25, 30, 40, 45	1	0.5	0	0.5	0
2	4, 6, 8, 10	50, 55, 60, 65	1	0.3	0.4	0	0.3
3	4, 6, 8, 10	7, 8, 9, 10	3	0.2	0.3	0.3	0.2

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0146	0.0082	0.0362
	NTOL	0.0247	0.0074	0.0480
UME	UTOL	0.0071	0.0037	0.0157
	NTOL	0.0154	0.0066	0.0343
EXP	UTOL	0.1508	0.0397	0.2795
	NTOL	0.1043	0.0179	0.1350

Table 8.10 : Fully Connected Configuration.

TEST DATA

M = 3, N = 3, Test No = 200

i	$c_i\mu_i$	C_i^2	c_i	average r_{i0}	r_{i1}	r_{i2}	r_{i3}
0	2, 4, 6, 8,	5, 5.5, 6, 6.5	3	0	0.4	0.2	0.4
1	2, 4, 6, 8	10, 12, 13, 14	1	0	0	0.7	0.3
2	2, 4, 6, 8	0.6,0.8,1,1.5	1	0.8	0.2	0	0
3	2, 4, 6, 8	30, 40, 50, 70	1	0.1	0.4	0.5	0

RESULTS

Approximation	Variable	Mean	Stand. dev.	Maximum
SEQ(SCV)	UTOL	0.0172	0.0145	0.0428
	NTOL	0.0259	0.0241	0.0587
UME	UTOL	0.0160	0.0093	0.0484
	NTOL	0.0186	0.0104	0.0541
EXP	UTOL	0.1238	0.0492	0.2543
	NTOL	0.0695	0.0201	0.1155

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