C O N F O R M A L   S Y M M E T R I E S   I N
S P E C I A L   A N D   G E N E R A L
R E L A T I V I T Y

The derivation and interpretation of conformal symmetries and asymptotic conformal symmetries in Minkowski space-time and in some space-times of general relativity

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PREFACE

The central objective of this work is to present an analysis of the asymptotic conformal Killing vectors in asymptotically-flat space-times of general relativity. This problem has been examined by two different methods; in Chapter 5 the asymptotic expansion technique originated by Newman and Unti [31] leads to a solution for asymptotically-flat space-times which admit an asymptotically shear-free congruence of null geodesics, and in Chapter 6 the conformal rescaling technique of Penrose [54] is used both to support the findings of the previous chapter and to set out a procedure for solution in the general case. It is pointed out that Penrose's conformal technique is preferable to the use of asymptotic expansion methods, since it can be established in a rigorous manner without leading to the possible convergence difficulties associated with asymptotic expansions.

Since the asymptotic conformal symmetry groups of asymptotically flat space-times are generalisations of the conformal group of Minkowski space-time we devote Chapters 3 and 4 to a study of the flat space case so that the results of later chapters may receive an interpretation in terms of familiar concepts. These chapters fulfil a second, equally important, role in establishing local isomorphisms between the Minkowski-space conformal group, SO(2,4) and SU(2,2). The SO(2,4) representation has been used by Kastrup [61] to give a physical interpretation using space-time gauge transformations. This appears as part of the survey of interpretative work in Chapter 7. The SU(2,2) representation of the conformal group has assumed a theoretical prominence in recent years through the work of Penrose [9-11] on twistors. In Chapter 4 we establish contact with twistor ideas by showing that points in Minkowski space-time correspond to certain complex skew-symmetric rank two tensors on the

SU(2,2) carrier space. These objects are, in Penrose's terminology [9], simple skew-symmetric twistors of valence $\begin{bmatrix}2\\0\end{bmatrix}$.

A particularly interesting aspect of conformal objects in space-time is explored in Chapter 8, where we extend the work of Geroch [16] on multipole moments of the Laplace equation in 3-space to the consideration of $\Box\psi = 0$ in Minkowski space-time. This development hinges upon the fact that multipole moment fields are also conformal Killing tensors.

In the final chapter some elementary applications of the results of Chapters 3 and 5 are made to cosmological models which have conformal flatness or asymptotic conformal flatness. In the first class here we have models of the Robertson-Walker type and in the second class we have the asymptotically-Friedmann universes considered by Hawking [73].

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CHAPTER 1

INTRODUCTION TO CONFORMAL SYMMETRIES

1.1 Symmetries in physical theory

The historical development of mathematical physics has shown repeatedly that an appreciation of the space-time symmetry group underlying a theoretical model is of fundamental importance. Firstly, the natural algebraic formalism for the expression and manipulation of theories in space-time will be based on those geometrical objects [1,2]* which have components transforming in a particularly simple way under elements of the symmetry group. We have in mind here the obvious examples provided by 3-vectors in Euclidean 3-space, and Lorentz 4-vectors and tensors in Minkowski space-time, all of which have components that transform according to linear homogeneous transformation laws.

Secondly, the unification of concepts that results from a group-theoretical approach deepens the understanding of the nature of physical laws and provides an important means whereby the transition from classical theories to their quantised counterparts can be made. In this way the passage from classical Hamiltonian dynamics to quantum mechanics, via the correspondence between Poisson brackets and commutators of operators, can be presented in an especially illuminating manner [3]. Even at the purely classical level, the generation of conservation laws from the symmetries of a theory which has field equations derivable from a variational principle [1,4], is seen by many philosophers as a peculiarly profound result.

* Throughout this work, references to sources listed in the bibliography will be given in square parentheses.

Finally, the way in which a given theory is superseded by its "covering theory", (a phrase used by Rohrlich [5]), often reveals an enlargement of the symmetry group, and consequently this device has been used heuristically in the search for new theories. The various classification schemes for fundamental particles [6] show this process in action (although the symmetries considered there are not symmetries of the background space-time), and, to a lesser extent, the hierarchy of Newtonian mechanics, special relativity and general relativity, exhibits the same feature.

In recent years there has been a resurgence of interest in a second mode of generalisation of symmetry groups, the stimulus for which originated in 1910 with some work of Bateman [7] and Cunningham [8] on the Maxwell equations. They showed that these equations are invariant not only under the inhomogeneous Lorentz group (the Poincaré group) but also under the 15-parameter group of conformal transformations of Minkowski space-time. The physical significance of their discovery is still in doubt (in spite of the fact that Chapter 7 of the present work is devoted to this very question), and this accounts for the scant attention that conformal symmetries have received in the intervening years. A short historical survey of the literature is given in the above-mentioned chapter.

It was largely the work of Penrose [9] on the twistor representation of the flat-space conformal group, which appeared in 1967, that revived the interest in conformal invariance, and since that time he and his co-workers have shown that it is not just the symmetries, but also the conformal symmetries that should occupy a central role in our understanding of physics.
In particular, it has been possible to unify the study of zero rest-mass fields of arbitrary spin by recognising that the field equations are conformally-invariant in each case, and therefore lend themselves readily to an analysis in twistor terms [10,11]. Of special interest to relativists is the spin-2 case, which gives the linearised gravitational field. There are difficulties in generalising twistors to arbitrarily curved space-times,

but significant progress has been made in the extension to asymptotically flat space-times [12, 13] , and at present there seems to be some indication that a quantised gravitational theory will emerge.

1.2 Conformal transformations and conformal invariance

The phrase "conformal transformation" has been used by different writers to describe quite dissimilar concepts, and we shall take some care here to define the terms that will be employed in the present work. We distinguish three types of transformation:

(i) the conformal point transformations, or "active" conformal transformations, mapping points x^μ belonging to a region R of space-time to points $\bar{x}^\mu$ belonging to some other region $\bar{R}$, such that

$$\bar{x}^\mu = F^\mu (x^\alpha)$$

is one-to-one and analytic in R, and maps the line element ds at x^μ to

$$d\bar{s} = \sigma(x^\alpha)\, ds$$

at $\bar{x}^\mu$, where $\sigma(x^\alpha)$ is a positive scalar function;*

(ii) the conformal coordinate transformations, or "passive" conformal transformations, which can be generated from conformal point transformations in the manner described below;

* We shall use Greek suffices to range 0,1,2,3, or (where specified) to range 1,2,3,4, in space-time, and use lower-case Roman suffices in any 3-space under consideration. Where other types of suffices (e.g. Roman capitals) are used the convention will be explained on each specific occasion.

(iii) the conformal rescalings, which replace the metric tensor $g_{\mu\nu}$ on a particular manifold by a new metric tensor $\hat{g}_{\mu\nu}$ given by

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} ,$$

where Ω is a positive scalar function, suitably smooth, on the manifold. The metrics $g_{\mu\nu}$, $\hat{g}_{\mu\nu}$ are said to be conformally related.

Except when questions of physical interpretation are investigated (in Chapter 7 of this work), we shall view conformal transformations primarily as point transformations

$$x^{\mu} \rightarrow \bar{x}^{\mu} = F^{\mu}(x^{\alpha}) , \tag{2.1}$$

which are such that

$$d\bar{s}^2 = \sigma^2(x)\, ds^2 , \tag{2.2}$$

or equivalently

$$g_{\alpha\beta}(\bar{x}) \frac{\partial F^{\alpha}}{\partial x^{\mu}} \frac{\partial F^{\beta}}{\partial x^{\nu}} = \sigma^2(x)\, g_{\mu\nu}(x) . \tag{2.3}$$

With this point transformation (2.1) we may associate a coordinate transformation

$$x^{\mu'} = f^{\mu'}(x^{\alpha}) \tag{2.4}$$

according to the following prescription: for all $\underline{x}$ belonging to the domain of (2.1), the image of $\underline{x}$ under the point transformation shall have the same coordinates with respect to the primed frame as $\underline{x}$ has with respect to the unprimed frame. That is,

$$\bar{x}^{\mu'} = x^{\mu}. \tag{2.5}$$

Now

$$\bar{x}^{\mu'} = f^{\mu'}(\bar{x}^{\alpha}) = f^{\mu'}(F^{\alpha}(x^{\beta})) ,$$

so for (2.5) to hold for all $\underline{x}$, it must be the case that f and F are functional inverses of each other. Thus the coordinate transformation that corresponds to the point transformation (2.1) can be written

$$x^{\mu} = F^{\mu}(x^{\alpha'}) \ . \tag{2.6}$$

The effect of (2.6) on the metric tensor can be given by

$$g_{\alpha'\beta'}(\bar{x}') = \frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{\alpha'}} \frac{\partial \bar{x}^{\nu}}{\partial \bar{x}^{\beta'}} g_{\mu\nu}(\bar{x}) = \frac{\partial F^{\mu}(\bar{x}')}{\partial \bar{x}^{\alpha'}} \frac{\partial F^{\nu}(\bar{x}\ \ )}{\partial \bar{x}^{\beta'}} g_{\mu\nu}(\bar{x})$$

$$= \frac{\partial F^{\mu}(x)}{\partial x^{\alpha}} \frac{\partial F^{\nu}(x)}{\partial x^{\beta}} g_{\mu\nu}(\bar{x}) \ , \tag{2.7}$$

where we have used (2.5) . Now, using (2.3), we obtain

$$g_{\alpha'\beta'}(\bar{x}') = \sigma^{-2}(x) g_{\alpha\beta}(x) \ ,$$

and then using (2.5) again gives

$$g_{\alpha'\beta'}(\bar{x}')\, d\bar{x}^{\alpha'} d\bar{x}^{\beta'} = \sigma^{2}(x)\, g_{\alpha\beta}(x) dx^{\alpha} dx^{\beta} \ ,$$

or

$$d\bar{s}'^{2} = \sigma^{2}(x)\, ds^{2} \ .$$

Finally, since $d\bar{s}'^{2} = d\bar{s}^{2}$, we have

$$d\bar{s}^{2} = \sigma^{2}(x)\, ds^{2},$$

which is just the relationship (2.2) between infinitesimal intervals under the point transformation (2.1). Thus, there is no inconsistency in the use of "active" and "passive" viewpoints provided that the above prescription is used to relate transformations in the two schemes.

Turning attention now to the conformal rescalings, given by

$$\hat{g}_{\mu\nu} = \Omega^{2} g_{\mu\nu} \ , \tag{2.8}$$

we can discuss their relationship with the conformal point and conformal coordinate transformations most easily by examining the idea of conformal invariance. Physical theories in a background of flat space-time are usually formulated in such a way that Poincaré invariance is obvious. Now if transformations of the Poincaré group leave the field equations of the theory invariant it can be shown that to prove invariance under the 15-parameter conformal group it is sufficient to show that the theory is invariant under conformal rescalings. In this sense, a theory is said to be conformally invariant if each of the quantities in the theory transforms with a definite conformal weight under conformal rescalings such that the field equations are invariant. (A quantity T is said to have conformal weight n if it transforms according to

$$T \longrightarrow \hat{T} = \Omega^n T$$

under (2.8). The proof of the statement above hinges upon the fact that the generators of symmetry transformations in a space with metric $g_{\mu\nu}$ become, in general, generators of conformal symmetries in the space with a conformally related metric $\hat{g}_{\mu\nu}$. A simple proof of this result appears, in a rather different context, in Chapter 6 of the present work. Thus, in Minkowski space-time the 15 independent generators of conformal symmetries can be produced under conformal rescalings from the 10 independent generators of symmetries.

Of course, invariance of physical theories under conformal rescalings has wider applicability than we have suggested above since the metric tensor $g_{\mu\nu}$ in (2.8) may describe a curved space-time in general relativity. In that case, however, the contact with Poincaré invariance is no longer relevant. Instead one has the result that a theory formulated in a covariant manner will be invariant under any conformal transformations of the form (2.1, 2.2) that the space-time might possess, provided that it is invariant under conformal rescalings. Thus, where conform-invariance is under examination it is sufficient in general relativity to investigate invariance under conformal rescalings. This spirit of approach is obvious, for example, in the work [14] of Fulton, Rohrlich and Witten.

1.3 Motions and conformal motions of space-time

We begin by defining the Lie difference of a geometric object with respect to the transformation

$$x^{\mu} \rightarrow \bar{x}^{\mu} = F^{\mu}(x^{\alpha}) \ . \tag{3.1}$$

Suppose Ψ is a geometric object (with tensor or spinor suffices suppressed), defined throughout space-time, and $\underline{\bar{x}} \in \bar{R}$ is related to $\underline{x} \in R$ according to (3.1). Suppose further that the coordinate systems S and S' are related by

$$x^{\mu} = F^{\mu}(x^{\alpha'}) \ ,$$

where unprimed suffices refer to S, primed suffices refer to S'.

The deformed geometric object $\bar{\Psi}(\underline{x})$, deformed under (3.1), is defined to be that field at $\underline{x} \in R$ whose components with respect to S are equal to the components $\Psi'(\underline{\bar{x}}')$, with respect to S', of the field $\Psi(\underline{\bar{x}})$ at $\underline{\bar{x}} \in \bar{R}$.

The Lie difference of $\Psi(\underline{x})$ with respect to (3.1) is then given by

$$\bar{\Psi}(\underline{x}) - \Psi(\underline{x}) = \Psi'(\underline{\bar{x}}') - \Psi(\underline{x}) \ . \tag{3.2}$$

Of particular interest in our work is the deformed metric tensor $\bar{g}_{\mu\nu}(\underline{x})$, which is given by

$$\bar{g}_{\mu\nu}(\underline{x}) = \frac{\partial F^{\alpha}}{\partial x^{\mu}} \frac{\partial F^{\beta}}{\partial x^{\nu}} \, g_{\alpha\beta}(\underline{\bar{x}}) \ , \tag{3.3}$$

where the analogue of (2.7) has been used. It is convenient now to discuss Killing motions and conformal motions of space-time in terms of the deformed metric tensor. A point transformation (3.1) is, according to the definitions adopted by Yano [2] and elsewhere, a Killing motion or isometry of the space-time if the interval $d\bar{s}$ between two image points $\bar{x}^{\mu}$ and $\bar{x}^{\mu} + d\bar{x}^{\mu}$ is equal to the interval ds between the original points x^{μ} and $x^{\mu} + dx^{\mu}$. As a generalisation of this, a conformal motion is a point transformation of the form (3.1) with the property of mapping ds into a (positive) multiple of itself;

$$d\bar{s}^2 = \sigma^2(x)\; ds^2 \;. \tag{3.4}$$

In the case of a motion, the deformed metric tensor $\bar{g}_{\mu\nu}(x)$ is equal to the original metric tensor $g_{\mu\nu}(x)$, and in the case of a conformal motion we have

$$\bar{g}_{\mu\nu}(x) \;=\; \sigma^2(x)\; g_{\mu\nu}(x) \;. \tag{3.5}$$

It is evident that whereas non-null infinitesimal intervals are not preserved under a conformal motion, ratios of such intervals at a point (and hence angles at a point) are preserved. We note secondly that null intervals map into null intervals under a conformal motion. It is clear also that conformal motions as defined here are identical with the conformal point transformations defined earlier in §1.2 .

1.4 The Killing equations and the conformal Killing equations

Consider the infinitesimal point transformation

$$x^\mu \longrightarrow \bar{x}^\mu = x^\mu + \xi^\mu dt. \tag{4.1}$$

Neglecting terms of order $(dt)^2$, we find that the Lie difference of the metric tensor with respect to (4.1) is given by

$$\begin{aligned}
\bar{g}_{\mu\nu}(x) - g_{\mu\nu}(x) &= \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\frac{\partial \bar{x}^\beta}{\partial x^\nu}\, g_{\alpha\beta}(\bar{x}) \;-\; g_{\mu\nu}(x) \\
&= (\delta^\alpha_\mu + \xi^\alpha_{,\mu}\, dt)\;(\delta^\beta_\nu + \xi^\beta_{,\nu}\; dt)\left[g_{\alpha\beta}(x) + g_{\alpha\beta,\gamma}\,\xi^\gamma\; dt\right] \\
&\qquad - g_{\mu\nu}(x) \\
&= \left[g_{\mu\nu,\gamma}\,\xi^\gamma + g_{\mu\beta}\,\xi^\beta_{,\nu} + g_{\alpha\nu}\,\xi^\alpha_{,\mu}\right] dt \;.^{*}
\end{aligned}$$

* We shall often denote the partial derivative operator $\frac{\partial}{\partial x^\mu}$ by $,\mu$, and reserve ∇_μ or $;\mu$ for covariant derivative with respect to $g_{\mu\nu}$.

Since (4.1) is an infinitesimal transformation, it is usual in this case to call $\bar{g}_{\mu\nu} - g_{\mu\nu}$ the Lie differential of $g_{\mu\nu}$ with respect to (4.1), and to define the Lie derivative, $\mathcal{L}_{\underline{\xi}}\, g_{\mu\nu}$, of $g_{\mu\nu}$ with respect to $\underline{\xi}$ as

$$\mathcal{L}_{\underline{\xi}}\, g_{\mu\nu} = g_{\mu\alpha}\xi^{\alpha}{}_{,\nu} + g_{\alpha\nu}\xi^{\alpha}{}_{,\mu} + g_{\mu\nu,\alpha}\xi^{\alpha} . \qquad (4.2)$$

We note that the Lie differential and the Lie derivative of $g_{\mu\nu}$ are tensors of the same rank as $g_{\mu\nu}$. $\mathcal{L}_{\underline{\xi}}\, g_{\mu\nu}$ can be expressed in terms of covariant derivatives as

$$\mathcal{L}_{\underline{\xi}}\, g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} , \qquad (4.3)$$

which shows explicitly its tensor character.

From the definitions of the previous section it follows that a necessary and sufficient condition for (4.1) to be an infinitesimal conformal motion is

$$\mathcal{L}_{\underline{\xi}}\, g_{\mu\nu} = \tfrac{1}{2}\phi\, g_{\mu\nu} , \qquad (4.4)$$

where the scalar ϕ is given by

$$\phi = \xi^{\mu}{}_{;\mu} . \qquad (4.5)$$

Making use of (4.3), we can write (4.4) in the form

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = \tfrac{1}{2}\phi\, g_{\mu\nu} . \qquad (4.6)$$

The set of equations (4.4) or (4.6) will be known throughout the remainder of this work as the conformal Killing equations. The solutions ξ_{μ} to (4.6) are called conformal Killing vector fields, abbreviated hereafter to "C-K vectors". In the special case $\phi = 0$, (4.6) reduces to the usual form of Killing's equations, the solutions of which are known as Killing vectors.

We shall state, without proof, integrability conditions of (6.4) which will be required in Chapter 2. They are

$$\mathcal{L}_{\underline{\xi}}\, C_{\mu\nu\rho}{}^{\tau} = 0 , \qquad (4.7)$$

and

$$\mathcal{L}_{\underline{\xi}} C_{\mu\nu\rho} + \frac{1}{4} C_{\mu\nu\rho}{}^{\tau} \frac{\partial\phi}{\partial x^{\tau}} = 0 \quad . \tag{4.8}$$

$C_{\mu\nu\rho}{}^{\tau}$ is the Weyl tensor of the space-time and $C_{\mu\nu\rho}$ is defined by

$$C_{\mu\nu\rho} = R_{\rho[\mu;\nu]} - \frac{1}{6} g_{\rho[\mu} R_{;\nu]} \quad , \tag{4.9}$$

where $R_{\mu\nu}$ and R are respectively the Ricci tensor and the Ricci scalar. The derivation of results (4.7) and (4.8) is to be found in the book [2] by Yano.

There is an alternative form of the conformal Killing equations that will be used in the study of asymptotic C-K vectors in asymptotically flat space-times (Chapter 5 of the present work). Define a covariant tensor density $\mathcal{G}_{\mu\nu}$ of weight $-\frac{1}{2}$ by

$$\mathcal{G}_{\mu\nu} = \left[-\det(g_{\alpha\beta})\right]^{-\frac{1}{4}} g_{\mu\nu} \; . \tag{4.10}$$

Then, writing g for $\det(g_{\alpha\beta})$, we have

$$\mathcal{L}_{\underline{\xi}} \mathcal{G}_{\mu\nu} = (-g)^{-\frac{1}{4}} \mathcal{L}_{\underline{\xi}} g_{\mu\nu} - \frac{1}{4} (-g)^{-\frac{5}{4}} g_{\mu\nu} \mathcal{L}_{\underline{\xi}}(-g) \; .$$

But (-g) is a scalar density of weight 2 so

$$\mathcal{L}_{\underline{\xi}} (-g) = \xi^{\alpha}(-g)_{,\alpha} + 2(-g)\,\xi^{\alpha}{}_{,\alpha} = 2(-g)\,\xi^{\alpha}{}_{;\alpha} \, ,$$

and this gives finally

$$(-g)^{\frac{1}{4}} \mathcal{L}_{\underline{\xi}} \mathcal{G}_{\mu\nu} = \mathcal{L}_{\underline{\xi}} g_{\mu\nu} - \frac{1}{2} \xi^{\alpha}{}_{;\alpha}\, g_{\mu\nu} \; . \tag{4.11}$$

The conformal Killing equations (4.4) can now be replaced by

$$\mathcal{L}_{\underline{\xi}} \mathcal{G}_{\mu\nu} = 0 \; , \tag{4.12}$$

with $\mathcal{G}_{\mu\nu}$ defined as in (4.10) .

1.5 Conformal Killing tensors

In this section we establish a link between C-K vectors and first integrals of the null geodesic equations, which generalises in an obvious way to suggest a definition of conformal Killing tensors (C-K tensors). The integral curves of

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0 , \tag{5.1}$$

where s is an affine parameter, are null geodesics if the tangent vector

$$\lambda^\mu = \frac{dx^\mu}{ds} \tag{5.2}$$

satisfies

$$g_{\mu\nu}\lambda^\mu\lambda^\nu = 0 . \tag{5.3}$$

(5.1) can also be written in the form

$$\lambda^\nu\lambda^\mu{}_{;\nu} = 0 . \tag{5.4}$$

The geodesic equations (5.1) or (5.4) are said to admit a first integral of order one if there exists a vector ξ^μ such that

$$\xi_\mu\lambda^\mu = \text{constant}, \tag{5.5}$$

for any λ^μ satisfying (5.3) and (5.4).

It follows from (5.5) that ξ^μ must satisfy

$$\xi_{\mu;\nu}\lambda^\mu\lambda^\nu = 0 ,$$

and this is so if and only if

$$\xi_{(\mu;\nu)} = \phi\, g_{\mu\nu} , \tag{5.6}$$

i.e. if and only if ζ_μ is a C-K vector. *

Generalising to first integrals of higher order we may state that the geodesic equations (5.1) or (5.4) admit a first integral of order n if there exists a tensor $T^{\mu_1\mu_2\cdots\mu_n}$ such that

$$T_{\mu_1\mu_2\cdots\mu_n}\lambda^{\mu_1}\lambda^{\mu_2}\cdots\lambda^{\mu_n} = \text{constant} ,$$

for any λ^μ satisfying (5.3) and (5.4) ; [15,16]. Without loss of generality we take

$$T_{\mu_1\mu_2\cdots\mu_n} = T_{(\mu_1\mu_2\cdots\mu_n)} . \tag{5.7}$$

It follows that $T_{\mu_1\mu_2\cdots\mu_n}$ must satisfy

$$T_{(\mu_1\mu_2\cdots\mu_n;\nu)}\lambda^{\mu_1}\lambda^{\mu_2}\cdots\lambda^{\mu_n}\lambda^\nu = 0 ,$$

and this is so if and only if

$$T_{(\mu_1\mu_2\cdots\mu_n;\nu)} = g_{(\mu_1\mu_2}V_{\mu_3\cdots\mu_n)} \tag{5.8}$$

for some $V_{\mu_3\mu_4\cdots\mu_n}$. We shall call a completely symmetric tensor $T_{\mu_1\mu_2\cdots\mu_n}$ satisfying (5.8) a conformal Killing tensor (C-K tensor) of rank n.

Conformal Killing tensors (in Minkowski space-time) will be examined further in the work on multipole moments that appears in Chapter 8.

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* In (5.6) the brackets around μ,ν indicate symmetrization with respect to those indices, so that we have $\zeta_{(\mu;\nu)} = \frac{1}{2}(\zeta_{\mu;\nu} + \zeta_{\nu;\mu})$.

CHAPTER 2

SOME MATHEMATICAL PRELIMINARIES

2.1 Introduction

The main purpose of this chapter is to present a survey of some of the mathematical techniques that will be used in the chapters that follow. No apology will be made for the juxtaposition of classical topics, such as Lie groups and Spinors, and recently developed techniques like the Newman-Penrose spin coefficients and the $\eth$- operator, since in each case the new ideas have had their birth in the work of the early masters, and it sometimes seems that studies in relativity demand almost as much attention to the early writers as to the modern ones.

However, there has been a rate of progress since 1962 without precedent in the history of the subject, and this was undoubtedly stimulated by the work [17] of Newman and Penrose in which spinor analysis and the calculus of null tetrads in space-time were combined to produce the spin-coefficient formalism. We shall set down some of this material in §7, but with the main objective of applying it to the analysis of the conformal Killing equations. Consequently, the importance of N-P formalism as the starting point for many other avenues of advance will not be spelt out in detail, although most of the basic references will be given.

2.2 Representations of groups

Familiarity with elementary group theory is assumed, but a few of the concepts and definitions that are relevant to representation theory, and are used later in the text, will be summarised here.

If it is possible to establish a homomorphism between a given group G and the group of linear transformations D operating on elements of an n-dimensional vector space V_n (over the complex field in general), we say that the n x n matrices D form a representation (of degree n) of G. The vector space V_n is called the representation space, or carrier space, for this particular representation. Thus, a representation $\mathfrak{D}$ of G is a mapping of elements $g_i \in G$ onto the set $\mathfrak{D} = \{D(i)\}$ of n x n matrices such that if

$$g_i \rightarrow D(i) \quad \text{and } g_j \rightarrow D(j)$$

then

$$g_i g_j \longrightarrow D(i)\, D(j) \ ,$$

where the operation on the left is multiplication in G and the operation on the right is matrix multiplication. We shall require further that the identity element $g_o \in G$ maps onto the n x n identity matrix;

$$g_o \longrightarrow I_n \ ,$$

and then it follows also that

$$g_i^{-1} \rightarrow D^{-1}(i) \ ,$$

and all the matrices of $\mathfrak{D}$ are non-singular. A representation is said to be faithful if the mapping $g_i \rightarrow D(i)$ is an isomorphism.

Two representations $\mathfrak{D} = \{D(i)\}, \mathfrak{F} = \{F(i)\}$ of the same group G are said to be equivalent if there exists a non-singular matrix P such that for all $g_i \in G$,

$$F(i) = P^{-1}\, D(i)\, P \ , \qquad (2.1)$$

where $g_i \rightarrow D(i)$ in the first representation and $g_i \rightarrow F(i)$ in the second representation*. In the representation space V_n, the matrix P in (2.1) defines a change of basis $\underline{\xi} \rightarrow \underline{\eta}$ with

$$\underline{\eta} = P^{-1} \underline{\xi} \ ,$$

* Transformations of the form (2.1) are called similarity transformations.

such that if

$$\xi' = D(i)\,\underline{\xi}$$

in the first representation, then

$$\eta' = F(i)\,\underline{\eta}$$

in the second representation.

We now restrict the discussion to continuous faithful representations of Lie groups. The n x n matrices D(i) appearing in a representation $\mathcal{D}$ have, in general, complex elements, and the dimension of the matrix group $\{D(i)\}$ is the number of independent real parameters required to define a general element of the group as an element of the $2n^2$-dimensional space of all complex n x n matrices. The dimension of the representation $\mathcal{D}$ is evidently the same as the dimension of G.

The following nomenclature will be adopted to describe the various groups of n x n matrices commonly encountered in representation theory:-

L (n,C): the full linear group, consisting of all n x n matrices with non-zero determinant; dimension $2n^2$;

L (n,R): the real linear group, consisting of all real n x n matrices with non-zero determinant; dimension n^2;

SL (n,C): the unimodular group, consisting of all n x n matrices with determinant equal to unity; dimension $2(n^2-1)$;

SL (n,R): the real unimodular group, consisting of all n x n real matrices with determinant equal to unity; dimension n^2-1;

U (n): the unitary group, consisting of all unitary n x n matrices; dimension n^2;

SU (n): the unimodular unitary group, consisting of all unitary n x n matrices with determinant equal to unity; dimension n^2-1;

O (n): the real orthogonal group, consisting of all real orthogonal n x n matrices; dimension $\frac{1}{2}n(n-1)$;

SO (n): the special orthogonal group (or rotation group), consisting of all real orthogonal n x n matrices with determinant equal to unity; dimension $\frac{1}{2}n(n-1)$.

The definitions of unitarity and orthogonality of matrices are the usual ones:

$U \in U(n)$ satisfies

$$U^{\dagger} U = I_n , \tag{2.1}$$

where I_n is the n x n identity matrix and $\dagger$ denotes hermitian conjugate;

$O \in O(n)$ satisfies

$$\tilde{O} O = I_n , \tag{2.2}$$

where $\tilde{O}$ denotes the transpose of O. In consequence of (2.1), U(n) acting on the space of complex n-dimensional column vectors $\underline{\xi} = \xi^i$, i = 1,2,...n, preserves the hermitian form

$$H = \underline{\xi}^{\dagger} \underline{\xi} = \sum_{i=1}^{n} \bar{\xi}^i \xi^i = \sum_{i=1}^{n} |\xi^i|^2 , \tag{2.3}$$

whilst, in consequence of (2.2), O(n) acting on the space of real n-dimensional column vectors $\underline{\eta} = \eta^i$, i = 1,2,...n, preserves the quadratic form

$$Q = \tilde{\underline{\eta}} \underline{\eta} = \sum_{i=1}^{n} (\eta^i)^2 . \tag{2.4}$$

We shall need to generalise from the positive-definite forms (2.3) and (2.4) to consider indefinite forms which are preserved under certain "pseudo-unitary" and "pseudo-orthogonal" groups. Suppose G is the diagonal n x n matrix with p elements equal to +1 and q(=n-p) elements equal to -1. We shall call

$$H = \underline{\xi}^{\dagger} G \underline{\xi} \tag{2.5}$$

the characteristic hermitian form of U (p,q) and

$$Q = \tilde{\underline{\eta}} G \underline{\eta} \tag{2.6}$$

the characteristic quadratic form of O (p,q). Then matrices $U \in U(p,q)$ must satisfy the pseudo-unitarity condition

$$U^{\dagger} G U = G , \qquad (2.7)$$

and matrices $O \in O(p,q)$ must satisfy the pseudo-orthogonality condition

$$\tilde{O} \; G \; O = G \qquad (2.8)$$

Putting the additional restriction of unimodularity onto U (p,q) and O (p,q) produces the groups SU (p,q) and SO (p,q). Of particular importance in our study of representations of the Minkowski-space conformal group are the pseudo-unitary group SU (2,2) and the pseudo-orthogonal group SO (2,4), but we shall defer the detailed investigation of these groups until Chapter 4. As an immediate application of representation theory we shall look instead at the spinor representation of the homogeneous Lorentz group.

2.3 The spinor representations of the Lorentz group

It is well-known that the full homogeneous Lorentz group decomposes into two parts; the proper Lorentz transformations with determinant equal to +1, and the improper transformations with determinant equal to -1. The proper Lorentz transformations (which include the identity) form a group, which in the terminology of §2.2 is the group SO (1,3). We shall specialise further here to a consideration of the orthochronous proper Lorentz transformations, which have determinant +1 and are future-preserving; (i.e. past and future are not interchanged under transformation). It will be shown that this subgroup of the full Lorentz group is homomorphic with SL (2,C), the group of complex unimodular 2 x 2 matrices.

The construction of this homomorphism has been given by many authors; see for example [18] , and we shall not dwell upon the details here. The Minkowskian coordinates x^{μ} , $\mu = 0,1,2,3,4$, are put into correspondence with 2 x 2 hermitian matrices X by the definition

$$X = x^{\mu} \sigma_{\mu} , \qquad (3.1)$$

where the σ_μ are given by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{3.2}$$

and span the space of 2 x 2 hermitian matrices.

If $A \in SL\ (2,C)$ then

$$X' = A X A^\dagger \tag{3.4}$$

is again hermitian and so may be written in the form

$$X' = x'^\mu \sigma_\mu . \tag{3.5}$$

From the observation that

$$\det X' = \det X \tag{3.6}$$

it follows that

$$(x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 ,$$

so the transformation (3.4) induces a Lorentz transformation on the x^μ. This Lorentz transformation

$$x'^\mu = L^\mu_{\ \nu}\ x^\nu$$

can be computed explicitly from

$$L^\mu_{\ \nu}(A) = \tfrac{1}{2}\ \mathrm{Tr}\ (\sigma_\mu A\, \sigma_\nu\, A^\dagger) , \tag{3.7}$$

where Tr denotes trace, and it can be shown that $L^\mu_{\ \nu}$ is the matrix of an orthochronous Lorentz transformation. It is clear that the homomorphism from SL (2,C) to SO (1,3) is two-to-one (since A and -A both generate the same $L^\mu_{\ \nu}$). We have here a faithful representation of the Lorentz group, but it is two-valued.

Taking the set of matrices $\{A\}$ as our basic representation in SL (2,C) it is straightforward to show that the sets $\{\bar{A}\}$, $\{\tilde{A}^{-1}\}$, $\{A^{\dagger -1}\}$ also form representations of the orthochronous SO (1,3). However, these representations are not all inequivalent since it is easily shown that for all $A \in SL(2,C)$,

$$P^{-1} \tilde{A}^{-1} P = A$$

and

$$P^{-1} A^{\dagger -1} P = \bar{A} ,$$

where

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

On the other hand there is no similarity transformation mapping $\{A\}$ to $\{\bar{A}\}$, so we have here two distinct inequivalent representations. We introduce now two representation spaces of complex 2-vectors which will be called spin spaces. The elements of spin space are called spinors. For the representation $\{A\}$, the spinors

$$\zeta^{A} = \begin{pmatrix} \zeta^{1} \\ \zeta^{2} \end{pmatrix}$$

transform according to

$$\zeta^{A'} = A^{A'}{}_{A} \zeta^{A} , \tag{3.8}$$

and for the representation $\{\bar{A}\}$ the spinors

$$\eta^{\dot{A}} = \begin{pmatrix} \eta^{\dot{1}} \\ \eta^{\dot{2}} \end{pmatrix}$$

transform according to

$$\eta^{\dot{A}'} = \bar{A}^{\dot{A}'}{}_{\dot{A}} \eta^{\dot{A}} . \tag{3.9}$$

The use of undotted and dotted spinor suffices is now the most common notation in the literature of spinors, having appeared originally in many of the early survey articles; see for example [19]. We may set down an invariant scalar product for the space of undotted spinors by noting that the metric spinor ϵ_{AB} must satisfy

$$A^{A'}{}_{C} \, \epsilon_{A'B'} \, A^{B'}{}_{D} = \epsilon_{CD} \tag{3.10}$$

in order to preserve

$$\zeta^C \epsilon_{CD} \xi^D$$

under (3.8). Now any skew-symmetric ϵ_{AB} satisfies (3.10), so without loss of generality we make the choice

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (3.11)$$

for the metric spinor. (3.10) shows that ϵ_{AB} may be thought of as a covariant spinor of rank 2 which maps into itself under spin transformations. A contravariant metric spinor ϵ^{AB} is defined by

$$\epsilon^{AC}\epsilon_{BC} = \delta^A_B , \qquad (3.12)$$

and takes the form

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \qquad (3.13)$$

Raising and lowering of spinor suffices then proceeds according to the rules

$$\zeta^A = \epsilon^{AB}\zeta_B , \qquad (3.14)$$

$$\zeta_A = \zeta^B\epsilon_{BA} , \qquad (3.15)$$

which are consistent with (3.12). The dotted counterparts of (3.11), (3.13) are

$$\epsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \epsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad (3.16)$$

and raising and lowering of dotted suffices follows just the same rules as those for undotted suffices; i.e.

$$\eta^{\dot{A}} = \epsilon^{\dot{A}\dot{B}}\eta_{\dot{B}} , \quad \eta_{\dot{A}} = \eta^{\dot{B}}\epsilon_{\dot{B}\dot{A}} . \qquad (3.17)$$

In the following section we shall examine the link between spinors and tensors, and whilst it may seem at first that we are doing little except to replace each tensor suffix by two spinor suffices it will transpire that the spinor equivalents of some particularly important tensors do in fact take on a simple form, and it is this which makes the spinor formalism an extremely powerful tool.

2.4 Spinor algebra

Although the underlying mathematics of spinor algebra was known to Cartan [20] in 1913, the introduction of spinors as a useful technique in theoretical physics is usually credited to van der Waerden and Infeld [21, 22], who used spinor formalism to study Dirac's equation in flat space-time (1929) and later (1933) developed spinor analysis in a general Riemannian manifold. In more recent years there have appeared two valuable survey articles on spinor theory; the 1953 paper of Bade and Jehle [19] and the excellent account given by Pirani in the Trautmann - Pirani - Bondi book [23]. We shall not attempt here to give a detailed exposition of the algebra of spinors, but will confine ourselves to the most important features of the formalism and the setting down of some results to be used in subsequent chapters.

The spaces of "dotted" and "undotted" 1-spinors were introduced in §2.3, together with the metric 2-spinor which is used for raising and lowering spinor suffices in the ordinary way. In order to relate spinors and tensors we define the mixed quantities $\sigma_\mu^{A\dot{A}}$, $\mu = 0, \ldots, 4$, $A, \dot{A} = 1,2$, which transform as 4-vectors under Lorentz transformations and transform as spinors on the suffices $A, \dot{A}$. The $\sigma_\mu^{A\dot{A}}$ are hermitian;

$$\sigma_\mu^{A\dot{A}} = \bar{\sigma}_\mu{}^{A\dot{A}} \tag{4.1}$$

and are chosen to satisfy

$$\sigma_\mu^{A\dot{A}} \sigma^\mu{}_{B\dot{B}} = \delta^A{}_B \delta^{\dot{A}}{}_{\dot{B}} \quad . \tag{4.2}$$

(4.1) and (4.2) do not define the $\sigma_\mu^{A\dot{A}}$ uniquely, but one permissible set is given by

$$\sigma_0{}^{A\dot{A}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma_1{}^{A\dot{A}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

$$\sigma_2{}^{A\dot{A}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_3{}^{A\dot{A}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}. \tag{4.3}$$

(This is the choice made by Penrose in most of his work on spinors.)
We may add the following relations on the $\sigma_\mu^{A\dot{A}}$, which follow from (4.2);

$$\sigma_\mu^{A\dot{A}} \sigma^\nu_{A\dot{A}} = \delta_\mu^\nu , \qquad (4.4)$$

and

$$\sigma_{\mu}{}^{A}{}_{\dot{A}} \sigma_\nu^{B\dot{A}} + \sigma_\nu{}^{A}{}_{\dot{A}} \sigma_\mu^{B\dot{A}} = g_{\mu\nu} \epsilon^{AB} . \qquad (4.5)$$

This latter equation has been used in some treatments of the theory as the defining equation for the σ_μ^{AA}; (see, for example, [24]).
The spinor equivalent $T^{A\dot{A}\ B\dot{B}\ \cdots}_{C\dot{C}\ \cdots}$ of the tensor $T^{\mu\nu\cdots}_{\rho\cdots}$ is now defined by

$$T^{A\dot{A}\ B\dot{B}\ \cdots}_{C\dot{C}\ \cdots} = \sigma_\mu^{A\dot{A}} \sigma_\nu^{B\dot{B}} \sigma^\rho_{C\dot{C}} \cdots T^{\mu\nu\cdots}_{\rho\cdots} , \qquad (4.6)$$

which inverts to give

$$T^{\mu\nu\cdots}_{\rho\cdots} = \sigma^\mu_{A\dot{A}} \sigma^\nu_{B\dot{B}} \sigma_\rho^{C\dot{C}} \cdots T^{A\dot{A}\ B\dot{B}\ \cdots}_{C\dot{C}\ \cdots} . \qquad (4.7)$$

We shall often write

$$T^{\mu\nu\cdots}_{\rho\cdots} \longleftrightarrow T^{A\dot{A}\ B\dot{B}\ \cdots}_{C\dot{C}}$$

as an abbreviated form of (4.6), (4.7).

It is straightforward to establish that

$$g_{\mu\nu} \leftrightarrow \epsilon_{AB} \epsilon_{\dot{A}\dot{B}} \qquad (4.8)$$

and

$$\delta^\mu_\nu \leftrightarrow \delta_A^{\ B} \delta_{\dot{A}}^{\ \dot{B}} . \qquad (4.9)$$

For a skew-symmetric tensor of rank 2, ($F_{\mu\nu} = -F_{\nu\mu}$), one finds

$$F_{\mu\nu} \leftrightarrow \epsilon_{AB} \bar{\phi}_{\dot{A}\dot{B}} + \phi_{AB} \epsilon_{\dot{A}\dot{B}} , \qquad (4.10)$$

where

$$\phi_{AB} = \tfrac{1}{2} F_{A\dot{A}B}{}^{\dot{A}} \qquad (4.11)$$

is a symmetric 2-spinor.

Finally, for a tensor $R_{\mu\nu\rho\tau}$ with the symmetries of the Riemann tensor, i.e.

$$R_{\mu\nu\rho\tau} = -R_{\nu\mu\rho\tau} = -R_{\mu\nu\tau\rho} = R_{\rho\tau\mu\nu} , \tag{4.12}$$

the spinor equivalent is

$$\begin{aligned} &\Psi_{ABCD}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} + \epsilon_{AB}\Phi_{CD\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} \\ &+\epsilon_{CD}\Phi_{AB\dot{C}\dot{D}}\epsilon_{\dot{A}\dot{B}} + 2\Lambda\left\{\epsilon_{AC}\epsilon_{BD}\epsilon_{\dot{A}\dot{B}}\epsilon_{\dot{C}\dot{D}} + \epsilon_{AB}\epsilon_{CD}\epsilon_{\dot{A}\dot{D}}\epsilon_{\dot{B}\dot{C}}\right\} , \end{aligned} \tag{4.13}$$

where the completely symmetric 4-spinor Ψ_{ABCD} corresponds to the Weyl curvature tensor, $\Phi_{AB\dot{C}\dot{D}}$, satisfying

$$\Phi_{AB\dot{C}\dot{D}} = \Phi_{BA\dot{C}\dot{D}} = \Phi_{AB\dot{D}\dot{C}} = \overline{\Phi}_{\dot{C}\dot{D}AB} ,$$

corresponds to

$$(-\tfrac{1}{2}) \left[R_{\mu\nu} - \tfrac{1}{4} R\, g_{\mu\nu}\right] ,$$

which is essentially the trace-free part of the Ricci tensor, and

$$\Lambda = \frac{1}{24} R.$$

In conclusion, we shall consider the geometrical interpretation of 1-spinors, since this is often a valuable guide to our intuition where applications of spinors to physical theory are concerned. We note first that from (3.14) and (3.15)

$$\xi^A\eta_A = -\xi_A\eta^A ,$$

so that any 1-spinor ξ^A has zero length;

$$\xi^A\xi_A = 0 . \tag{4.14}$$

Then ℓ^μ, defined by

$$\ell^\mu = \sigma^\mu_{A\dot{A}}\lambda^A\mu^{\dot{A}}$$

is a (complex) null vector. If ℓ^μ is to be real it must then have the form

$$\ell^\mu = \pm \sigma^\mu_{A\dot{A}} \lambda^A \bar{\lambda}^{\dot{A}} \quad ; \quad \lambda^A \neq 0 \, , \tag{4.15}$$

where the + sign applies if ℓ^μ is future-pointing and the - sign applies if ℓ^μ is past-pointing. Thus, we may say that, via (4.15), any 1-spinor λ^A defines a real null vector in space-time. This correspondence of 1-spinors with null vectors is not one-to-one since the change of phase

$$\lambda^A \longrightarrow e^{i\alpha} \lambda^A \tag{4.16}$$

leaves (4.15) unchanged. However, λ^A also defines a skew-symmetric tensor $F_{\mu\nu}$ through the correspondence

$$F_{\mu\nu} \longleftrightarrow \epsilon_{AB} \bar{\lambda}_{\dot{A}} \bar{\lambda}_{\dot{B}} + \epsilon_{\dot{A}\dot{B}} \lambda_A \lambda_B \, , \tag{4.17}$$

and if we introduce a second spinor η^A such that $\{\lambda^A, \eta^A\}$ forms a basis for spin space with

$$\lambda_A \eta^A = 1 \, , \tag{4.18}$$

then

$$\epsilon_{AB} = \lambda_A \eta_B - \eta_A \lambda_B \, , \tag{4.19}$$

and $F_{\mu\nu}$, defined by (4.17), can be written

$$F_{\mu\nu} = 2 \, \ell_{[\mu} p_{\nu]} \quad ,$$

where ℓ^μ is given by (4.15) and

$$p_\mu \longleftrightarrow \lambda_A \bar{\eta}_{\dot{A}} + \eta_A \bar{\lambda}_{\dot{A}} \, . \tag{4.20}$$

Under the change of phase (4.16),

$$\eta^A \longrightarrow e^{-i\alpha} \eta^A + k \lambda^A \, , \tag{4.21}$$

where k is some (complex) constant, and then (4.18) and (4.19) are preserved. The effect on p_μ is given by

$$p_\mu \longrightarrow t_\mu + (e^{i\alpha}\bar{k} + e^{-i\alpha}k)\ell_\mu , \qquad (4.22)$$

where

$$t^\mu \longleftrightarrow e^{2i\alpha}\lambda_A \bar{\eta}_{\dot{A}} + e^{-2i\alpha}\bar{\lambda}_{\dot{A}}\eta_A .$$

We find easily that

$$p_\mu \ell^\mu = 0 , \quad p^\mu p_\mu = t^\mu t_\mu = -2 ,$$

and by calculating the scalar product $p^\mu t_\mu$ of the space-like vectors p^μ and t^μ, we find that the angle between p_μ and its image under the phase transformation (4.16) is 2α. These results have led to the conception of a "flag" and "flagpole" structure [25], in which the 1-spinor λ^A defines the null flagpole ℓ^μ according to (4.15), and the flag is the 2-space spanned by ℓ^μ and p^μ, which is also defined by λ^A via (4.17). Then under the phase transformation (4.16) the direction of the flagpole is unchanged but the flag rotates around the flagpole through an angle of 2α.

2.5 Lie groups and Lie algebras

Suppose the elements of a group G depend upon a finite number r of real parameters, so that we write for a general element $g \in G$,

$$g = g(a^1, a^2, \ldots, a^r) ,$$

or, more shortly,

$$g = g(a) = g(a^i), \; i = 1,2,\ldots,r,$$

where the (a^i) are points in some subspace of $\mathbb{R}^r$.
We describe the set of such r-tuples $(a^1, a^2, \ldots, a^r)$ as the group space or parameter space of G. If g(a), g(b) are two elements of G with product given by

$$g(a)\, g(b) = g(c) , \qquad (5.1)$$

then there is a mapping ϕ in the group space such that

$$c = \phi(a,b). \qquad (5.2)$$

The mapping ϕ must possess certain properties in order that the g(a) form a group. Thus, associativity in G becomes

$$\phi(a, \phi(b,c)) = \phi(\phi(a,b), c) . \qquad (5.3)$$

If a_0 corresponds to the identity element of G, then

$$\phi(a, a_0) = \phi(a_0, a) = a , \qquad (5.4)$$

for all points a in the group space. Lastly, since each $g \in G$ possesses an inverse in G, it follows that to each point a in the group space there corresponds a second point $\bar{a}$ with

$$\phi(a, \bar{a}) = \phi(\bar{a}, a) = a_0 . \qquad (5.5)$$

If the mapping ϕ is continuous then G is said to form a topological group. If a topological group is such that any neighbourhood of the group space may be continuously mapped onto a neighbourhood of a Euclidean space we call the group space an r-dimensional manifold. A topological group with a group space that is a manifold is called a Lie group.

The Lie groups considered in this work are of two types. Firstly, we shall often wish to study r-parameter groups of continuous transformations on the coordinates $(x^1, x^2, \ldots x^n)$ in an n-dimensional space. A typical element of such a group may be written as $g(a)$: $x \longrightarrow x'$ with

$$x'^{\mu} = f^{\mu}(x^{\alpha}, a^i) = f^{\mu}(x, a) , \qquad (5.6)$$

where

$$x^{\alpha} = (x^1, x^2, \ldots x^n) ,$$

and

$$a^i = (a^1, a^2, \ldots a^r)$$

are points in coordinate space and group space respectively.

(In this section, Greek suffices will always range 1, 2, ..., n and lower-case Roman will range 1, 2, ..., r. The summation convention will apply as usual.) The functions f^{μ} in (5.6) will be assumed to be analytic in the x^{α} and the a^i. The r-dimensional group space V_r has locally Euclidean properties and constitutes a Riemannian manifold on which a metric interval may be defined. The mapping (5.2) on V_r must possess the properties (5.3,5.4,5.5) in order that the transformations (5.6) form a group. We may remark here that the group of conformal motions in a Riemannian space is a Lie group, a result proved by Kobayashi [26] and mentioned in Yano's book [2].

A second important type of Lie group that will concern us has been mentioned already in §2.2, where we presented the full linear group of transformations in a complex n-dimensional vector space and some of its principal subgroups. In the next part of this section we shall develop the concept of the Lie algebra associated with a given Lie group by reference to Lie groups of this type.

Let us suppose then that the group G consists of linear transformations on a complex n-dimensional vector space. We write g(a) for a general element of G and, without loss of generality, we take

$$a_o = 0$$

as the point in the group space which corresponds to the identity e_G of G. Then

$$g(0) = e_G \; . \qquad (5.7)$$

If a, b are two points in some neighbourhood of the zero in group space we have, to first order in a, b,

$$g(a) = g(0) + \left[\frac{\partial g(a)}{\partial a^i}\right]_{a=0} a^i \; , \qquad (5.8)$$

and

$$g(b) = g(0) + \left[\frac{\partial g(a)}{\partial a^i}\right]_{a=0} b^i \; . \qquad (5.9)$$

Defining

$$X_i = \left[\frac{\partial g(a)}{\partial a^i}\right]_{a=0} \qquad (5.10)$$

as the infinitesimal generators of the group G, we can then write

$$g(a) = e_G + a^i X_i \quad , \qquad (5.11)$$

and

$$g(b) = e_G + b^i X_i \quad . \qquad (5.12)$$

It follows that the product

$$g(a)\, g(b) = g(c)$$

becomes

$$g(c) = e_G + (a^i + b^i) X_i \quad ,$$

(again to first order in the small quantities), and comparing this with

$$g(c) = e_G + c^i X_i$$

gives

$$c^i = a^i + b^i \quad ,$$

so that the mapping ϕ, defined in group space by (5.2) becomes in this case

$$c^i = \phi(a,b) = a^i + b^i + \text{higher order terms.} \qquad (5.13)$$

We show now that the generators X_i , i = 1,2,..., , form a Lie algebra. (For an explicit definition see, for example, [27]). It is evident that the X_i form a vector space $L_r(G)$ of dimension r over the reals, and if we define the commutator of two generators $[X_i, X_j]$ by

$$[X_i, X_j] = X_i X_j - X_j X_i \quad , \qquad (5.14)$$

it remains to show that $[X_i, X_j]$ belongs to $L_r(G)$ and satisfies the Jacobi identity

$$[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0, \qquad (5.15)$$

for all $X_i, X_j, X_k \in L_r(G)$.

From

$$g(c) = g(a)\, g(b) \ ,$$

where $c = \phi(a,b)$ is given by (5.13) it follows that

$$\frac{\partial g(a)}{\partial a^i}\frac{\partial g(b)}{\partial b^j} = \frac{\partial^2 c^k}{\partial a^i \partial b^j}\frac{\partial g(c)}{\partial c^k} + \frac{\partial c^k}{\partial a^i}\frac{\partial c^\ell}{\partial b^j}\frac{\partial^2 g(c)}{\partial c^k \partial c^\ell} \ ,$$

and putting $a = b = 0$ gives

$$X_i\, X_j = \left[\frac{\partial^2 c^k}{\partial a^i \partial b^j}\right]_{\substack{a=0\\b=0}} X_k + \left[\frac{\partial c^k}{\partial a^i}\right]_{a=0}\left[\frac{\partial c^\ell}{\partial b^j}\right]_{b=0}\left[\frac{\partial^2 g(c)}{\partial c^k \partial c^\ell}\right]_{c=0} . \qquad (5.16)$$

Making use of (5.13) to obtain

$$\left[\frac{\partial c^k}{\partial a^i}\right]_{a=0} = \delta^k_i \quad \text{and} \quad \left[\frac{\partial c^\ell}{\partial b^j}\right]_{b=0} = \delta^\ell_j \ ,$$

we rewrite (5.16) as

$$X_i X_j = \left[\frac{\partial^2 c^k}{\partial a^i \partial b^j}\right]_{\substack{a=0\\b=0}} X_k + \left[\frac{\partial^2 g(c)}{\partial c^i \partial c^j}\right]_{c=0} .$$

Then

$$\left[X_i,\, X_j\right] = X_i\, X_j - X_j\, X_i = C^k_{ij}\, X_k \ , \qquad (5.17)$$

where the C^k_{ij} are defined by

$$C^k_{ij} = -C^k_{ji} = \left[\frac{\partial^2 \phi^k(a,b)}{\partial a^i \partial b^j} - \frac{\partial^2 \phi^k(a,b)}{\partial a^j \partial b^i}\right]_{\substack{a=0\\b=0}} . \qquad (5.18)$$

The C^k_{ij} are constants, called the structure constants of the group, and therefore (5.17) shows that $L_r(G)$ is closed under the bracket operation. It is also straightforward to prove that

$$C^h_{ij}\, C^\ell_{hk} + C^h_{jk}\, C^\ell_{hi} + C^h_{ki}\, C^\ell_{hj} = 0 \ , \qquad (5.19)$$

and this is equivalent to the Jacobi identity (5.15).

It is clear from the above construction that to every Lie group there corresponds a Lie algebra. On the other hand, although we know from Lie's original work that if the set of C^k_{ij} are skew-symmetric in i, j and satisfy (5.19) then (5.17) can be solved for the generators X_i, it does not follow that a unique Lie group can be constructed from a given Lie algebra. In fact, only the structure of G in a neighbourhood of its identity is determined by $L_r(G)$, giving rise to what some writers, (e.g. Boerner in [28]), have called a local Lie group. In this sense, two groups with the same Lie algebra are said to be "locally isomorphic".

Construction of the local Lie group from its Lie algebra is carried out using the exponential map from $L_r(G)$ to G defined by

$$g(\alpha) = \exp(\alpha X) = \sum_{N=0}^{\infty} \frac{1}{N!} \alpha^N X^N , \qquad (5.20)$$

where X is an infinitesimal generator, as defined in (5.10), and α is a real parameter. The $g(\alpha)$ given by (5.20) form a 1-parameter subgroup of G. In this way we can generate r 1-parameter subgroups of G by using the r independent infinitesimal generators (5.10). In general it will not be the case that every $g \in G$ is contained in one of these subgroups, but if G is connected (i.e. the group space of G is connected), then certainly g can be written as the product of elements from 1-parameter subgroups.

Because it is often technically simpler to handle Lie algebras rather than Lie groups, they form a useful tool in group-theoretical studies. It is important then to know what inferences one can draw concerning the behaviour of the Lie group just from an investigation of the corresponding Lie algebra. In this connection we quote the following results, which are easily accessible in the literature; see, for example, [29]: -

(i) $[X_i, X_j] = 0$ for all $X_i, X_j \in L_r(G)$

$\Longleftrightarrow$ the connected component of G is Abelian ;

(ii) If $H \subset G$ is an arc-wise connected subgroup of the Lie group G (i.e. every element of H can be joined to e_G by a curve lying wholly in H), then the infinitesimal generators of H form a Lie algebra which is a sub-algebra of $L_r(G)$. Conversely, any sub-algebra of $L_r(G)$ gives rise to an arc-wise connected subgroup of G.

(iii) (the closed subgroup theorem); Let H be a subgroup of G that corresponds to a closed subset of $L_r(G)$, and let H^o be the subset of elements of H that can be joined to the identity by a curve in H. Then H^o is a normal subgroup of H and the factor group H/H^o is discrete. H^o is called the connected component of H.

The brief account of Lie groups and Lie algebras given above contains the basic ideas that will find application in later chapters of this work. We have stopped short of the introduction of the Cartan metric since this will appear in Chapter 8, where its importance in the classification of Lie groups [30] will be emphasised.

2.6 Coordinate systems for flat and asymptotically flat space-times

Some of the subsequent chapters of this work are devoted to the study of conformal Killing vectors in asymptotically flat space-times, and we shall establish here the coordinate system used by Newman and Unti [31] for the description of such space-times, together with the associated null tetrad. It is always possible in the space-times of general relativity to introduce a family of null hypersurfaces

$$u = \text{constant} \,, \tag{6.1}$$

with

$$g^{\mu\nu} \; u_{,\mu} \; u_{,\nu} = 0. \tag{6.2}$$

The tetrad vector ℓ_μ is then defined by

$$\ell_\mu = u_{,\mu} , \qquad (6.3)$$

and u is taken as the first coordinate;

$$x^1 = u .$$

Thevector ℓ^μ is then tangent to a family of null geodesics lying in the u = constant hypersurfaces. An affine parameter r for these geodesics is used as a second coordinate;

$$x^2 = r. \qquad (6.4)$$

Finally, the coordinates x^A, A = 3,4, are used to label the geodesics on each of the u = constant hypersurfaces. With these choices, we have

$$\ell^\mu = (0,1,0,0) \qquad (6.5)$$

and the contravariant metric assumes the form

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & g^{22} & g^{23} & g^{24} \\ 0 & g^{23} & \multicolumn{2}{c}{\multirow{2}{*}{g^{AB}}} \\ 0 & g^{24} & & \end{pmatrix} \qquad (6.6)$$

A second real null vector n^μ, normalised according to

$$\ell_\mu n^\mu = 1 , \qquad (6.7)$$

can be chosen, and two complex null vectors

$$m^\mu = \frac{1}{\sqrt{2}} (a^\mu - i\, b^\mu) \quad ; \quad \bar{m}^\mu = \frac{1}{\sqrt{2}} (a^\mu + i\, b^\mu) \quad , \qquad (6.8)$$

where a^μ, b^μ are real spacelike vectors orthogonal to ℓ^μ and n^μ, and satisfying

$$a_\mu b^\mu = 0 .$$

The tetrad vectors ℓ^{μ}, n^{μ}, m^{μ}, $\bar{m}^{\mu}$ are null and satisfy the following relations;

$$\ell^{\mu} n_{\mu} = - m_{\mu} \bar{m}^{\mu} = 1 ,$$
$$\ell_{\mu} m^{\mu} = \ell_{\mu} \bar{m}^{\mu} = 0 , \quad n_{\mu} m^{\mu} = n_{\mu} \bar{m}^{\mu} = 0 . \tag{6.9}$$

Using an asymptotic expansion technique, Newman and Unti proceed to calculate the leading terms of the $g^{\mu\nu}$ components, expressed as power series in r^{-1}. The reference [31] mentioned above gives full details of this work, which utilises the Newman-Penrose spin coefficient formalism. In particular it is shown that a suitable coordinate transformation on the x^{A} brings the leading terms in g^{AB} to those of a manifestly conformally flat 2-metric.

In Minkowski space-time we shall make use of a number of different coordinate systems, often with the aim of simplifying a particular analytic method. Firstly, one must mention the usual Minkowskian coordinates $x^{\mu} = (t, x, y, z)$, which give the most familiar form of the metric tensor

$$g_{\mu\nu} = \mathrm{diag}\ (1,-1,-1,-1). \tag{6.10}$$

A change to spherical polars (r, θ, ϕ) for the spatial coordinates and

$$u = t - r \tag{6.11}$$

for the first coordinate gives the infinitesimal metric interval in the form

$$ds^2 = du^2 + 2\, du\, dr - r^2 \left[d\theta^2 + \sin^2\theta \ \ d\phi^2 \right] , \tag{6.12}$$

where the range of θ is 0 to π and the range of ϕ is 0 to 2π.

A change of coordinates $(\theta, \phi) \longrightarrow (x^3 , x^4)$ given by

$$\begin{matrix} \cos\theta = -\tanh x^3 \\ \sin\theta = \mathrm{sech}\ x^3 \end{matrix} , \quad \phi = x^4 , \tag{6.13}$$

where x^3 ranges from $-\infty$ to $+\infty$ and x^4 ranges 0 to 2π, brings the metric interval to the form

$$ds^2 = du^2 + 2\,du\,dr - \frac{r^2}{2P^2}\left[(dx^3)^2 + (dx^4)^2\right] , \qquad (6.14)$$

$$\text{where } P = \frac{1}{\sqrt{2}} \cosh x^3 . \qquad (6.15)$$

In this version, the x^3 - x^4 2-space is conformally flat, so (6.14) is the particularisation to Minkowski space-time of the Newman-Unti type of coordinate system described above, The coordinates (u, r, x^3, x^4) relate to (t, x, y, z) according to

$$\left.\begin{aligned} t &= u + r , \\ x &= r \operatorname{sech} x^3 \cos x^4 , \\ y &= r \operatorname{sech} x^3 \sin x^4 , \\ z &= r \tanh x^3 . \end{aligned}\right\} \qquad (6.16)$$

There is a second modification of the angular coordinates (θ, ϕ), that has been particularly important in the development of the $\eth$ operator, to be discussed below in § 2.8 . Under the transformation $(\theta, \phi) \longrightarrow (\zeta, \bar{\zeta})$, where

$$\zeta = e^{i\phi} \cot \tfrac{1}{2}\theta, \quad \bar{\zeta} = e^{-i\phi} \cot \tfrac{1}{2}\theta \qquad (6.17)$$

are the usual stereographic coordinates, the metric takes the form

$$ds^2 = du^2 + 2du\,dr - 4r^2(1 + \zeta\bar{\zeta})^{-2}\, d\zeta\, d\bar{\zeta} . \qquad (6.18)$$

$(\zeta, \bar{\zeta})$ are related to the (x^3, x^4) coordinates of (6.13) by

$$\zeta = e^{-x^3 + ix^4} , \quad \bar{\zeta} = e^{-x^3 - i\,x^4} . \qquad (6.19)$$

2.7 Newman-Penrose spin coefficient formalism

The development of spinor techniques in general relativity, originated by Penrose [24] in 1960, has played a major role in the enormous progress that has been made in the subject during recent years. The use of objects that are tailored to the null geodesic and null-cone structure of space-time has naturally been of especial importance in the study of gravitational radiation, but we should note also that the simplifications achieved in the Einstein field equations by spinor formulations have already led to the discovery of many new solutions. In this field the contributions of Newman and his collaborators have been particularly important, giving us a method of approximate solution based on asymptotic expansion techniques [31], and a number of new exact solutions [32-35]. Although the original 1960 paper of Penrose concentrated upon spinor analysis per se, much of the subsequent work, pioneered by Newman and Penrose [17], combined spinor calculus with the calculus of null tetrads, resulting in the now-familiar presentation of the empty-space Einstein equations as a set of first order equations involving the spin coefficients and the tetrad components of the Weyl tensor. It is from this starting point that most of the advances have been made.

In this essay we shall give a brief introduction to the spin coefficient formalism and subsequently use it in the manner of Collinson and French [36] to set down the conformal Killing equations and their integrability conditions.

Introducing a tetrad of null vectors ℓ_μ, n_μ, m_μ, $\bar{m}_\mu$ with ℓ_μ, n_μ real and m_μ, $\bar{m}_\mu$ complex such that

$$\left.\begin{aligned} &\ell_\mu \ell^\mu = n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = 0 \\ &\ell_\mu n^\mu = - m_\mu \bar{m}^\mu = 1 \\ &\ell_\mu m^\mu = \ell_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0\,, \end{aligned}\right\} \qquad (7.1)$$

one has for the metric tensor

$$g_{\mu\nu} = 2\ell_{(\mu}\, n_{\nu)} - 2m_{(\mu}\, \bar{m}_{\nu)} \,. \tag{7.2}$$

Tetrad notation is introduced by putting

$$Z_{m\mu} = (\ell_\mu,\, n_\mu,\, m_\mu,\, \bar{m}_\mu) \,, \quad m = 1,2,3,4 \,. \tag{7.3}$$

Tetrad indices may be raised or lowered by means of the matrix

$$\eta_{mn} = \eta^{mn} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{7.4}$$

The complex Ricci rotation coefficients $\gamma_m{}^{np}$ are defined by

$$\gamma_m{}^{np} = Z_{m\mu;\nu}\, Z^{n\mu}\, Z^{p\nu} \tag{7.5}$$

and have the symmetry

$$\gamma^{mnp} = -\gamma^{nmp} \,. \tag{7.6}$$

The twelve complex spin coefficients are then defined in terms of the rotation coefficients by

$$\begin{aligned}
\kappa &= \gamma_{131} = \ell_{\mu;\nu}\, m^\mu \ell^\nu \,, \quad \pi = -\gamma_{241} = -n_{\mu;\nu}\bar{m}^\mu \ell^\nu \,, \\
\epsilon &= \tfrac{1}{2}\,(\gamma_{121} - \gamma_{341}) = \tfrac{1}{2}\,(\ell_{\mu;\nu} n^\mu \ell^\nu - m_{\mu;\nu}\bar{m}^\mu \ell^\nu) \,, \\
\rho &= \gamma_{134} = \ell_{\mu;\nu}\, m^\mu \bar{m}^\nu \,, \quad \lambda = -\gamma_{244} = -n_{\mu;\nu}\bar{m}^\mu \bar{m}^\nu \,, \\
\alpha &= \tfrac{1}{2}\,(\gamma_{124} - \gamma_{344}) = \tfrac{1}{2}\,(\ell_{\mu;\nu}\, n^\mu \bar{m}^\nu - m_{\mu;\nu}\, \bar{m}^\mu \bar{m}^\nu) \,, \\
\sigma &= \gamma_{133} = \ell_{\mu;\nu}\, m^\mu m^\nu \,, \quad \mu = -\gamma_{243} = -n_{\mu;\nu}\, \bar{m}^\mu m^\nu \,, \\
\beta &= \tfrac{1}{2}\,(\gamma_{123} - \gamma_{343}) = \tfrac{1}{2}\,(\ell_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu}\, \bar{m}^\mu m^\nu) \,, \\
\nu &= -\gamma_{242} = -n_{\mu;\nu}\, \bar{m}^\mu n^\nu, \quad \tau = \gamma_{132} = \ell_{\mu;\nu} m^\mu n^\nu) \\
\gamma &= \tfrac{1}{2}\,(\gamma_{122} - \gamma_{342}) = \tfrac{1}{2}\,(\ell_{\mu;\nu}\, n^\mu n^\nu - m_{\mu;\nu}\, \bar{m}^\mu n^\nu) \,.
\end{aligned} \tag{7.7}$$

Some of the spin coefficients have straightforward geometrical interpretations, which are often made use of in the construction of specialised coordinate systems. It can be shown that

$$\ell_{\mu;\nu}\ell^{\nu} = (\epsilon+\bar{\epsilon})\ell_{\mu} - \kappa\bar{m}_{\mu} - \bar{\kappa}m_{\mu}, \tag{7.8}$$

so that if

$$\kappa = 0 \tag{7.9}$$

then ℓ_μ is tangent to a congruence of null geodesics, and further, by a change of scale, $\epsilon+\bar{\epsilon}$ can be made zero. Then

$$\ell_{\mu;\nu}\ell^{\nu} = 0 . \tag{7.10}$$

From the two relations

$$\ell^{\mu}{}_{;\mu} = (\epsilon+\bar{\epsilon}) - (\rho+\bar{\rho}) \tag{7.11}$$

and

$$(\text{curl}\,\ell_\mu)^2 \equiv (2\,\ell_{[\mu;\nu]}\,\ell^{\mu;\nu})$$
$$= -(\rho-\bar{\rho})^2 - \kappa[2\,\bar{\tau} - (\alpha+\bar{\beta})] - \bar{\kappa}[2\tau - (\bar{\alpha}+\beta)] \tag{7.12}$$

it follows, on making use of the conditions

$$\kappa = \epsilon+\bar{\epsilon} = 0 ,$$

that

$$\rho = \tfrac{1}{2}\left[-\ell^{\mu}{}_{;\mu} + i\,\text{curl}\,\ell_\mu\right] , \tag{7.13}$$

so that the real part of ρ essentially gives the divergence of the null congruence defined by ℓ_μ. A similar calculation to that used in deriving (7.12) leads to

$$\ell_{(\mu;\nu)}\ell^{\mu;\nu} = 2\sigma\bar{\sigma} - \tfrac{1}{2}\kappa(2\bar{\tau}+\alpha+\bar{\beta}) - \tfrac{1}{2}\bar{\kappa}(2\tau+\bar{\alpha}+\beta) + \tfrac{1}{2}(\rho+\bar{\rho})^2, \tag{7.14}$$

and so, in the case when $\kappa = 0$, we have

$$\sigma\bar{\sigma} = \tfrac{1}{2}\left[\ell_{(\mu;\nu)}\ell^{\mu;\nu} - \tfrac{1}{2}(\ell^{\mu}{}_{;\mu})^2\right] . \tag{7.15}$$

σ is often called the (complex) shear of the ℓ_μ congruence.

Finally, from

$$\ell_{\mu;\nu}\, n^{\nu} = (\gamma+\bar{\gamma})\ell_{\mu} - \bar{\tau}\, m_{\mu} - \tau \bar{m}_{\mu}$$

and the observation that $\gamma + \bar{\gamma}$ can be made zero by a change of scale on ℓ_{μ}, we have

$$\ell_{\mu;\nu} n^{\nu} = - \bar{\tau}\, m_{\mu} - \tau \bar{m}_{\mu},$$

so that τ describes how the direction of ℓ_{μ} changes as we move in the direction of n_{μ}.

The equations above hold unchanged in form if the replacements $\ell_{\mu} \rightarrow n_{\mu}$, $\kappa \rightarrow \nu$, $\rho \rightarrow -\mu$, $\sigma \rightarrow -\lambda$, $\tau \rightarrow \pi$ are made throughout, and then properties of the null congruence defined by n_{μ} may be deduced from the nature of this second set of spin coefficients.

It is usual to make the choice of a tetrad that undergoes parallel propagation along the geodesics of the ℓ_{μ} congruence, and this leads to the conditions

$$\kappa = \epsilon = \pi = 0 . \tag{7.16}$$

If it is further required that ℓ_{μ} be hypersurface orthogonal then we have

$$\rho = \bar{\rho} , \tag{7.17}$$

and finally if ℓ_{μ} is required to be equal to a gradient field then the additional condition is

$$\tau = \bar{\alpha} + \beta . \tag{7.18}$$

For some of the considerations of Chapter 3 it will be important to have the tetrad vectors and spin coefficients for Minkowski space-time with coordinates (u, r, x^3, x^4) as given in (6.13). The tetrad vectors take the form

$$\left.\begin{aligned}
\ell^{\mu} &= (0,1,0,0) , & \ell_{\mu} &= (1,0,0,0) \\
n^{\mu} &= (1, -\tfrac{1}{2},0,0) , & n_{\mu} &= (\tfrac{1}{2},1,0,0) \\
m^{\mu} &= \frac{\cosh x^3}{r\sqrt{2}} (0,0,1,i) , & m_{\mu} &= \frac{-r}{\sqrt{2}\cosh x^3} (0,0,1,i) \\
\bar{m}^{\mu} &= \frac{\cosh x^3}{r\sqrt{2}} (0,0,1,-i) , & \bar{m}_{\mu} &= \frac{-r}{\sqrt{2}\cosh x^3} (0,0,1,-i) ,
\end{aligned}\right\} \qquad (7.19)$$

and the only non-zero spin coefficients are given by

$$\left.\begin{aligned}
\rho &= -\frac{1}{r} , & \mu &= -\frac{1}{2r} \\
\alpha &= \frac{\sinh x^3}{2\sqrt{2}\,r} , & \beta &= -\frac{\sinh x^3}{2\sqrt{2}\,r} .
\end{aligned}\right\} \qquad (7.20)$$

Returning to the rotation coefficients $\gamma_m{}^{np}$ defined in (7.5), we use them to set down the tetrad components R^{mnpq} of the Riemann tensor;

$$\begin{aligned}
R^{mnpq} = \gamma^{mnp;q} - \gamma^{mnq;p} + \gamma_{\ell}{}^{mq}\gamma^{\ell np} - \gamma_{\ell}{}^{mp}\gamma^{\ell nq} \\
+ \gamma^{mn\ell}(\gamma_{\ell}{}^{pq} - \gamma_{\ell}{}^{qp}) .
\end{aligned} \qquad (7.21)$$

Expressing R_{mnpq} in terms of the tetrad components of the Weyl tensor, the Ricci tensor and the scalar curvature results in

$$\begin{aligned}
R_{mnpq} = C_{mnpq} - \tfrac{1}{2}(\eta_{mp} R_{nq} - \eta_{mq} R_{np} + \eta_{nq} R_{mp} - \eta_{np} R_{mq}) \\
- \tfrac{1}{6} R (\eta_{mq}\eta_{np} - \eta_{mp}\eta_{nq})
\end{aligned} \qquad (7.22)$$

In empty space

$$R_{pq} = 0 , \qquad (7.23)$$

so that the Riemann tensor reduces to the Weyl tensor alone. Newman and Penrose define the quantities $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ in terms of tetrad components of the Weyl tensor as follows ;

$$\left.\begin{aligned}
\Psi_0 &= -C_{1313} , \\
\Psi_1 &= -C_{1213} , \\
\Psi_2 &= -\tfrac{1}{2}(C_{1212} - C_{1234}) , \\
\Psi_3 &= C_{1224} , \\
\Psi_4 &= -C_{2424} .
\end{aligned}\right\} \qquad (7.24)$$

Following Collinson and French [36] , we can now set down the conformal Killing equations for empty space-time in Newman-Penrose formalism. From (4.6), (4.7) and (4.8) of Chapter 1, we have, in tetrad notation

$$V_{m;n} + V_{n;m} = \tfrac{1}{2}\phi\,\eta_{mn} + V^{\ell}(\gamma_{m\ell n} + \gamma_{n\ell m}) \qquad (7.25)$$

for the conformal Killing equations, and

$$C_{mnpq;\ell}\,V^{\ell} + C_{\ell npq}\,V^{\ell}{}_{;m} + C_{m\ell pq}\,V^{\ell}{}_{;n} + C_{mn\ell q}\,V^{\ell}{}_{;p} + C_{mnp\ell}\,V^{\ell}{}_{;q}$$

$$= \tfrac{1}{2}C_{mnpq}\,\phi + V^{\ell}\Big\{C_{rnpq}\,({}^{r}_{m\ell} + \gamma^{r}_{\ell m}) + C_{mnpr}\,(\gamma^{r}_{q\ell} + \gamma^{r}_{\ell q}) + C_{mrpq}(\gamma^{r}_{n\ell} + \gamma^{r}_{\ell n})$$

$$+ C_{mnrq}\,(\gamma^{r}_{p\ell} + \gamma^{r}_{\ell p})\Big\} \quad , \qquad (7.26)$$

$$C_{mnp;\ell}\,V^{\ell} + C_{\ell np}\,V^{\ell}{}_{;m} + C_{m\ell p}\,V^{\ell}{}_{;n} + C_{mn\ell}\,V^{\ell}{}_{;p}$$

$$= -C_{mnp}{}^{q}\,\phi_{,q} + V^{\ell}\Big[C_{rnp}\,(\gamma_{m}{}^{r}{}_{\ell} + \gamma^{r}{}_{\ell m}) + C_{mrp}\,(\gamma_{n}{}^{r}{}_{\ell} + \gamma_{\ell}{}^{r}{}_{n})$$

$$+ C_{mnr}\,(\gamma_{p}{}^{r}{}_{\ell} + \gamma^{r}{}_{\ell p})\Big] \qquad (7.27)$$

for the integrability conditions. In our present work we shall make use only of the set of equations (7.25), which become in N-P formalism

$$\begin{aligned}
D V_1 &= (\epsilon+\bar\epsilon)\, V_1 - \bar\kappa V_3 - \kappa V_4 \quad ,\\
\Delta V_2 &= -\,(\gamma+\bar\gamma)\, V_2 - \nu V_3 + \bar\nu\, V_4 \quad ,\\
\delta\, V_3 &= \bar\lambda V_1 - \sigma V_2 - (\bar\alpha-\beta)\, V_3 \quad ,\\
\Delta V_1 + D V_2 - \tfrac{1}{2}\phi &= (\gamma+\bar\gamma)\, V_1 - (\epsilon+\bar\epsilon)\, V_2 + (\pi-\bar\tau)\, V_3 + (\bar\pi-\tau)\, V_4 \, , \qquad (7.28)\\
\delta V_1 + D V_3 &= (\bar\alpha+\beta+\bar\pi)\, V_1 - \kappa V_2 + (\epsilon-\bar\epsilon-\bar\rho)\, V_3 - \sigma V_4 \, ,\\
\delta V_2 + \Delta V_3 &= \bar\nu\, V_1 - (\bar\alpha+\beta+\tau)\, V_2 + (\mu+\gamma-\bar\gamma)\, V_3 + \bar\lambda V_4 \, ,\\
\bar\delta\, V_3 + \delta V_4 + \tfrac{1}{2}\phi &= (\mu+\bar\mu)\, V_1 - (\rho+\bar\rho)\, V_2 + (\alpha-\bar\beta)\, V_3 + (\bar\alpha-\beta)\, \bar V_3 \, ,
\end{aligned}$$

where the operators $D, \Delta, \delta, \bar\delta$ are defined by

$$D \equiv \ell^{\mu}\nabla_{\mu}, \quad \Delta \equiv n^{\mu}\nabla_{\mu}\,, \quad \delta \equiv m^{\mu}\nabla_{\mu}\,, \quad \bar\delta \equiv \bar m^{\mu}\nabla_{\mu}\,.$$

We may note in conclusion that the operators defined above satisfy the following commutator relations when acting on a scalar function ψ;

$$(\Delta D - D\Delta)\psi = [(\gamma + \bar{\gamma}) D + (\epsilon + \bar{\epsilon})\Delta - (\tau + \bar{\pi}) \bar{\delta} - (\bar{\tau} + \pi)\delta] \psi, \qquad (7.29)$$

$$(\delta D - D\delta)\psi = [(\bar{\alpha} + \beta - \bar{\pi}) D + \kappa\Delta - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon}) \delta]\psi, \qquad (7.30)$$

$$(\delta\Delta - \Delta\delta)\psi = [-\bar{\nu} D + (\tau - \bar{\alpha} - \beta)\Delta + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta]\psi, \qquad (7.31)$$

$$(\bar{\delta}\delta - \delta\bar{\delta})\psi = [(\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta]\psi. \qquad (7.32)$$

2.8 The ð operator and the spin -s spherical harmonics

The differential operator ð (usually known as the "thop" operator) appeared first in a paper by Newman and Penrose [37] on the Bondi-Metzner-Sachs group. It was used there in conjunction with the spin-weighted spherical harmonics ${}_sY_{\ell m}$, which for a fixed s form a complete set of orthonormal spin -s functions on the unit sphere. A subsequent survey of the properties of ð and the ${}_sY_{\ell m}$ appeared in the paper [38] by Goldberg et al. We shall rely heavily upon the ð operator formalism in Chapter 6 of this work and the present section gives the pertinent results in preparation for that development.

On the unit sphere in Euclidean 3-space we introduce the angular coordinates θ, ϕ defined in the usual way, so that the position vector of a point on the sphere is given by

$$\underline{r} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta). \qquad (8.1)$$

An orthonormal triad $\{\underline{a}, \underline{b}, \underline{c}\}$ is defined by

$$\begin{aligned} \underline{a} &= (\cos\theta \cos\phi, \cos\theta \sin\phi, -\sin\theta), \\ \underline{b} &= (-\sin\phi, \cos\phi, 0), \\ \underline{c} &= (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta), \end{aligned} \qquad (8.2)$$

so that $\underline{a}$ is tangent to the lines ϕ = constant and $\underline{b}$ is tangent to the lines θ = constant. Construct the complex null vector $\underline{N}$ given by

$$\underline{N} = \frac{1}{\sqrt{2}}(\underline{a} + i\underline{b}), \qquad (8.3)$$

which satisfies

$$\underline{N}.\underline{N} = 0\,, \quad \underline{N}.\bar{\underline{N}} = 1\,.$$

For many purposes the complex stereographic coordinates $(\zeta, \bar{\zeta})$ on the unit sphere make computation more straightforward, and these are defined in terms of (θ, ϕ) by

$$\zeta = e^{i\phi} \cot \tfrac{1}{2}\theta\,, \quad \bar{\zeta} = e^{-i\phi} \cot \tfrac{1}{2}\theta\,. \tag{8.4}$$

If we introduce a complex vector $\underline{M}$ chosen so that Re $\underline{M}$ is tangential to Im ζ = constant, and Im $\underline{M}$ is tangent to Re ζ = constant, it turns out that $\underline{M}$ is related to $\underline{N}$ defined in (8.3) according to

$$\underline{M} = \underline{N}\, e^{-i\phi}\,. \tag{8.5}$$

The metric of the sphere in $(\zeta, \bar{\zeta})$ coordinates is

$$ds^2 = P^{-2}\, d\zeta d\bar{\zeta}\,,$$

where

$$P = \tfrac{1}{2}(1 + \zeta\bar{\zeta})\,, \tag{8.6}$$

and it follows that the complex vector $\underline{M}$ takes the form

$$\underline{M} = -\sqrt{2}\, P \frac{\partial \underline{r}}{\partial \zeta}\,. \tag{8.7}$$

Analytic coordinate transformations of the Möbius type ;

$$\zeta \longrightarrow \zeta'(\zeta) = \frac{a\zeta + b}{c\zeta + d}\,, \quad ad - bc = 1\,,$$

correspond to rotations of the sphere onto itself, and under such a transformation we find

$$P \longrightarrow P' = P \left| \frac{\partial \zeta'}{\partial \zeta} \right| \tag{8.8}$$

and

$$\underline{M} \longrightarrow \underline{M}'\,, \quad \text{where}$$

$$\underline{M} = \frac{\partial \zeta'}{\partial \zeta} \Big/ \left| \frac{\partial \zeta'}{\partial \zeta} \right| \underline{M}'\,. \tag{8.9}$$

From (8.9) it is clear that the coordinate transformation $\zeta \rightarrow \zeta'$ produces a rotation of the $\underline{M}$ vector.

If

$$\underline{M} \longrightarrow \underline{M}' = e^{i\psi}\underline{M} \qquad (8.10)$$

is the result of this coordinate change then we define a quantity η to have spin-weight s if it undergoes the transformation

$$\eta' = e^{is\psi}\eta \quad , \quad s \text{ an integer} , \qquad (8.11)$$

induced by (8.10).

In $(\zeta,\bar{\zeta})$ coordinates we now define the "thop" operator, acting on a quantity η, of spin-weight s, by

$$\eth\eta = 2\,P^{1-s}\frac{\partial}{\partial\bar{\zeta}}(P^{s}\eta) \quad , \qquad (8.12)$$

where η is suitably defined on the unit sphere. The operator $\eth$ is invariant (with spin weight unity) under transformations of the coordinate system that preserve the form (8.6) of the metric. That is,

$$\eth'\eta' \equiv 2\,P'^{\,1-s}\frac{\partial}{\partial\bar{\zeta}'}(P'^{s}\eta') = e^{i(s+1)\psi}\eth\eta , \qquad (8.13)$$

where η is of spin-weight s. Thus $\eth$ has the property of raising the spin-weight by 1.

It is convenient to define also a second operator $\bar{\eth}$ by

$$\bar{\eth}\eta = 2\,P^{1+s}\frac{\partial}{\partial\zeta}(P^{-s}\eta) \quad , \qquad (8.14)$$

acting on a spin-weight s quantity, and it is seen that $\bar{\eth}$ has the property of lowering the spin-weight by 1.

Transforming back to spherical polars (θ,ϕ) we find that (8.12) and (8.14) then take the forms

$$\eth\eta = -(\sin\theta)^{s}\left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right]\left\{(\sin\theta)^{-s}\eta\right\} \qquad (8.15)$$

and

$$\bar{\eth}\eta = -(\sin\theta)^{-s}\left[\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right]\left\{(\sin\theta)^{s}\eta\right\} \quad , \qquad (8.16)$$

where once again η is a quantity having spin-weight s.

Finally, we wish to express the $\eth$ operator formalism in the (x^3, x^4) coordinates introduced in § 2.5, since it is this coordinate system that will be employed in Chapter 6. Operating on the quantity η of spin weight s we have now

$$\eth\eta = -\cosh x^3\left(\frac{\partial\eta}{\partial x^3} + i\frac{\partial\eta}{\partial x^4}\right) - s\eta\sinh x^3 \qquad (8.17)$$

and

$$\bar{\eth}\eta = -\cosh x^3\left(\frac{\partial\eta}{\partial x^3} - i\frac{\partial\eta}{\partial x^4}\right) + s\eta\sinh x^3 \, . \qquad (8.18)$$

The operators $\eth$ and $\bar{\eth}$ do not, in general, commute, and we can show that

$$(\bar{\eth}\eth - \eth\bar{\eth})\eta = 2s\eta \, , \qquad (8.19)$$

where η is of spin weight s.

It is natural to use the property of the $\eth$ operator mentioned at (8.13) to generate, from the usual spherical harmonics $Y_{\ell m}(\theta,\phi)$, new sets of spin-weighted functions, to be denoted by ${}_sY_{\ell m}(\theta,\phi)$, and defined according to

$${}_sY_{\ell m}(\theta,\phi) = \begin{cases} \left[\dfrac{(\ell-s)!}{(\ell+s)!}\right]^{\frac{1}{2}} \eth^{s}\, Y_{\ell m}(\theta,\phi) & , 0 \leq s \leq \ell \, , \\ \left[\dfrac{(\ell+s)!}{(\ell-s)!}\right]^{\frac{1}{2}} (-1)^{s}\, \bar{\eth}^{-s}\, Y_{\ell m}(\theta,\phi) & , -\ell \leq s \leq 0 . \end{cases} \qquad (8.20)$$

For $|s| > \ell$, ${}_sY_{\ell m}$ is not defined. For a fixed value of s the ${}_sY_{\ell m}$ form a complete orthonormal set for the expansion of continuous, spin-weight s functions defined on the unit sphere.

Other properties of the spin-s spherical harmonics are:

$$\text{(i)} \quad {}_s\bar{Y}_{\ell m} = (-1)^{m+s} \, {}_{-s}Y_{\ell\, -m} \quad ; \tag{8.21}$$

(ii) For all s, with $|s| \leq \ell$,

$$\eth \, {}_sY_{\ell m} = \left[(\ell-s)(\ell+s+1)\right]^{\frac{1}{2}} \, {}_{s+1}Y_{\ell m} \quad , \tag{8.22}$$

$$\text{(iii)} \quad \bar{\eth} \, {}_sY_{\ell m} = -\left[(\ell+s)(\ell-s+1)\right]^{\frac{1}{2}} \, {}_{s-1}Y_{\ell m} \quad , \tag{8.23}$$

$$\text{(iv)} \quad \bar{\eth}\eth \, {}_sY_{\ell m} = -(\ell-s)(\ell+s+1) \, {}_sY_{\ell m} \quad , \tag{8.24}$$

$$\text{(v)} \quad \eth\bar{\eth} \, {}_sY_{\ell m} = -(\ell+s)(\ell-s+1) \, {}_sY_{\ell m} \quad . \tag{8.25}$$

More general versions of (ii) - (v) can be proved, giving

$$\text{(vi)} \quad \eth^p \, {}_sY_{\ell m} = \left[(\ell-s)(\ell-s-1)\ldots(\ell-s-p+1).(\ell+s+1)(\ell+s+2)\ldots(\ell+s+p)\right]^{\frac{1}{2}} \times {}_{s+p}Y_{\ell m} \, , \tag{8.26}$$

$$\text{(vii)} \quad \bar{\eth}^q \, {}_sY_{\ell m} = (-1)^q \left[(\ell+s)(\ell+s-1)\ldots(\ell+s-q+1).(\ell-s+1)(\ell-s+2)\ldots(\ell-s+q)\right]^{\frac{1}{2}} \times {}_{s-q}Y_{\ell m} \, , \tag{8.27}$$

$$\text{(viii)} \quad \bar{\eth}^q\eth^p \, {}_sY_{\ell m} = (-1)^q \frac{(\ell+s+p)!}{(\ell-s-p)!} \left[\frac{(\ell-s)!\,(\ell-s-p+q)!}{(\ell+s)!\,(\ell+s+p-q)!}\right]^{\frac{1}{2}} \times {}_{s+p-q}Y_{\ell m} \, , \tag{8.28}$$

$$\text{(ix)} \quad \eth^p\bar{\eth}^q \, {}_sY_{\ell m} = (-1)^q \frac{(\ell-s+q)!}{(\ell+s-q)!} \left[\frac{(\ell+s)!\,(\ell+s-q+p)!}{(\ell-s)!\,(\ell-s+q-p)!}\right]^{\frac{1}{2}} \times {}_{s-q+p}Y_{\ell m} \, . \tag{8.29}$$

For fixed s, the ${}_sY_{\ell m}$ are orthogonal with respect to ℓ and m as the following orthogonality relation reveals;

$$\int {}_sY_{\ell m}(\theta,\phi)\, {}_s\bar{Y}_{\ell' m'}\, dS = \delta_{\ell\ell'}\,\delta_{mm'} , \qquad (8.30)$$

where $dS = \sin\theta\, d\theta\, d\phi$, and integration is over the unit sphere.

The completeness relation of the ${}_sY_{\ell m}$ for a fixed integer s is given by

$$\sum_{\ell}\ \sum_{m=-\ell}^{+\ell} {}_s\bar{Y}_{\ell m}(\theta,\phi)\ {}_sY_{\ell m}(\theta',\phi') = \delta(\cos\theta - \cos\theta')\,\delta(\phi-\phi'). \qquad (8.31)$$

In conclusion, we write out some of the spin-s spherical harmonics in the (x^3, x^4) coordinates defined in §2.6 above, since these appear in later chapters of this work.

s = 0

$${}_0Y_{00} = \frac{1}{\sqrt{4\pi}} ,$$

$${}_0Y_{11} = -\sqrt{\frac{3}{8\pi}}\ \text{sech}\ x^3\ e^{ix^4} ,$$

$${}_0Y_{10} = \sqrt{\frac{3}{4\pi}}\ \tanh x^3 ,$$

$${}_0Y_{1-1} = \sqrt{\frac{3}{8\pi}}\ \text{sech}\ x^3\ e^{-ix^4} ,$$

s = 1, ℓ = 1

$${}_1Y_{11} = -\frac{1}{4}\sqrt{\frac{3}{\pi}}\ e^{ix^4}(\tanh x^3 + 1) ,$$

$${}_1Y_{10} = -\sqrt{\frac{3}{8\pi}}\ \text{sech}\ x^3 ,$$

$${}_1Y_{1-1} = \frac{1}{4}\sqrt{\frac{3}{\pi}}\ e^{-ix^4}(\tanh x^3 - 1) ,$$

s = -1, ℓ = 1

$${}_{-1}Y_{11} = \frac{1}{4}\sqrt{\frac{3}{\pi}}\ e^{ix^4}(\tanh x^3 - 1) ,$$

$${}_{-1}Y_{10} = \sqrt{\frac{3}{8\pi}}\ \text{sech}\ x^3 ,$$

$${}_{-1}Y_{1-1} = -\frac{1}{4}\sqrt{\frac{3}{\pi}}\ e^{-ix^4}(\tanh x^3 + 1) .$$

The following combinations of harmonics are also important:-

$$\frac{i}{2}\left[{}_0Y_{11} + {}_0Y_{1\,-1}\right] = \sqrt{\frac{3}{8\pi}}\ \mathrm{sech}\ x^3 \sin x^4 \quad ,$$

$$-\frac{1}{2}\left[{}_0Y_{11} - {}_0Y_{1-1}\right] = \sqrt{\frac{3}{8\pi}}\ \mathrm{sech}\ x^3 \cos x^4 \quad ,$$

$$i\left[{}_1Y_{11} + {}_1Y_{1-1}\right] = \sqrt{\frac{3}{4\pi}}\ (\tanh x^3 \sin x^4 - i \cos x^4) \ ,$$

$$-\left[{}_1Y_{11} - {}_1Y_{1-1}\right] = \sqrt{\frac{3}{4\pi}}\left(\tanh x^3 \cos x^4 + i \sin x^4\right),$$

$$-i\left[{}_{-1}Y_{11} + {}_{-1}Y_{1-1}\right] = \sqrt{\frac{3}{4\pi}}(\tanh x^3 \sin x^4 + i \cos x^4) \quad ,$$

$$\left[{}_{-1}Y_{11} - {}_{-1}Y_{1-1}\right] = \sqrt{\frac{3}{4\pi}}\ (\tanh x^3 \cos x^4 - i \sin x^4)\ .$$

From (8.24) we see that the ${}_sY_{\ell m}$ are eigenfunctions of the operator $\bar{\eth}\eth$. Taking the particular case s = 0, and transforming to (x^3, x^4) coordinates, the eigenvalue equation becomes

$$\cosh^2 x^3 \left[\frac{\partial^2}{\partial x^{3\,2}} + \frac{\partial^2}{\partial x^{4\,2}}\right] {}_0Y_{\ell m} \quad = \quad -\ell(\ell+1)\ {}_0Y_{\ell m} \quad ,$$

in which form it appears in Chapter 5 of the present work.

2.9 C-K vectors in Euclidean 3-space

As a simple example of the use of the $\eth$-operator formalism we shall here solve the conformal Killing equations in Euclidean 3-space. The spin -s spherical harmonics ${}_sY_{\ell m}$ will be used in the analysis and to facilitate this we choose spherical polar coordinates $(x^1, x^2, x^3) = (r,\theta,\phi)$, so that the 3-space metric g_{ij} takes the form

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} . \qquad (9.1)$$

The only non-vanishing Christoffel symbols are

$$\Gamma^1_{22} = -r \,, \qquad \Gamma^1_{33} = -r \sin^2\theta \,,$$
$$\Gamma^2_{12} = \frac{1}{r} = \Gamma^3_{13} \,, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta \,, \tag{9.2}$$
$$\Gamma^3_{23} = \cot\theta$$

Selecting a triad of null vectors n^i, m^i, $\bar{m}^i$ according to

$$n^i = \delta^i_1 \,,$$
$$m^i = \frac{1}{r\sqrt{2}} \left(\delta^i_2 + \frac{i}{\sin\theta} \delta^i_3 \right), \tag{9.3}$$
$$\bar{m}^i = \frac{1}{r\sqrt{2}} \left(\delta^i_2 - \frac{i}{\sin\theta} \delta^i_3 \right) ,$$

enables us to express any vector $\mathfrak{J}_i$ in the form

$$\mathfrak{J}_i = \mathfrak{J}_+ \bar{m}_i + \mathfrak{J}_- m_i + \mathfrak{J}_0 n_i \,, \tag{9.4}$$

where

$\mathfrak{J}_+ = \mathfrak{J}_i m^i$ is of spin weight + 1,

$\mathfrak{J}_0 = \mathfrak{J}_i n^i$ is of spin weight 0,

and $\mathfrak{J}_- = \mathfrak{J}_i \bar{m}^i$ is of spin weight -1.

The $\eth$-operator is connected with covariant differentiation with respect to g_{ij} and the following results express this relationship in an explicit manner:

$$\mathfrak{J}_{i;j} n^i m^j = -\frac{1}{r\sqrt{2}} \eth \mathfrak{J}_0 - \frac{1}{r} \mathfrak{J}_+ \,,$$
$$\mathfrak{J}_{i;j} n^i \bar{m}^j = -\frac{1}{r\sqrt{2}} \bar{\eth} \mathfrak{J}_0 - \frac{1}{r} \mathfrak{J}_- \,,$$
$$\mathfrak{J}_{i;j} m^i m^j = -\frac{1}{r\sqrt{2}} \eth \mathfrak{J}_+ \,, \qquad \mathfrak{J}_{i;j} \bar{m}^i \bar{m}^j = -\frac{1}{r\sqrt{2}} \bar{\eth} \mathfrak{J}_- \,,$$
$$\mathfrak{J}_{i;j} m^i \bar{m}^j = -\frac{1}{r\sqrt{2}} \bar{\eth} \mathfrak{J}_+ + \frac{1}{r} \mathfrak{J}_0 \,,$$

$$\mathfrak{J}_{i;j}\,\bar{m}^i m^j = -\frac{1}{r\sqrt{2}}\,\eth\mathfrak{J}_- + \frac{1}{r}\mathfrak{J}_0 \quad ,$$

$$\mathfrak{J}_{i;j}\, n^i n^j = \frac{\partial}{\partial r}\mathfrak{J}_0, \quad \mathfrak{J}_{i;j}\, m^i n^j = \frac{\partial}{\partial r}\mathfrak{J}_+ , \quad \mathfrak{J}_{i;j}\,\bar{m}^i n^j = \frac{\partial}{\partial r}\mathfrak{J}_- \;.$$

The conformal Killing equations,

$$\mathfrak{J}_{\mu;\nu} + \mathfrak{J}_{\nu;\mu} = \tfrac{1}{2}\lambda\, g_{\mu\nu} \;,$$

where $\lambda = \lambda\,(r,\theta,\phi)$ is the usual conformal factor, can now be re-written using the above relationships, to obtain

$$\frac{\partial}{\partial r}\mathfrak{J}_0 = \frac{1}{4}\lambda\,(r,\theta,\phi) \quad , \tag{9.5}$$

$$\eth\mathfrak{J}_0 = \sqrt{2}\,(r\frac{\partial}{\partial r}\mathfrak{J}_+ - \mathfrak{J}_+) \quad , \tag{9.6}$$

$$\bar{\eth}\mathfrak{J}_0 = \sqrt{2}\,(r\frac{\partial}{\partial r}\mathfrak{J}_- - \mathfrak{J}_-) \quad , \tag{9.7}$$

$$\eth\mathfrak{J}_+ = 0 \quad , \tag{9.8}$$

$$\bar{\eth}\mathfrak{J}_- = 0 \quad , \tag{9.9}$$

$$\bar{\eth}\mathfrak{J}_+ + \eth\mathfrak{J}_- = 2\sqrt{2}\,\mathfrak{J}_0 - \frac{r}{\sqrt{2}}\lambda(r,\theta,\phi)\;. \tag{9.10}$$

It is a straightforward matter to integrate these equations. Frequent use is made of the properties of $\eth$, especially those given in (8.22) and (8.23).

Using (9.8) and the fact that $\mathfrak{J}_+$ is of spin weight 1 gives

$$\mathfrak{J}_+ = a(r)\;{}_1Y_{11} + b(r)\,{}_1Y_{10} + c(r)\;{}_1Y_{1-1}\;, \tag{9.11}$$

where a,b,c are arbitrary complex functions of r. $\mathfrak{J}_i$ is a real vector, so

$$\bar{\mathfrak{J}}_+ = \mathfrak{J}_-$$

leads to

$$\zeta_- = \bar{c}(r)\ {}_{-1}Y_{11} - \bar{b}(r)\ {}_{-1}Y_{10} + \bar{a}(r)\ {}_{-1}Y_{1-1} \tag{9.12}$$

which satisfies (9.9) identically.

From (9.6) and (9.8) we have

$$\bar{\eth}\,\eth\,\zeta_0 = 0.$$

Noting that ζ_0 is real and of spin weight zero then gives

$$\zeta_0 = d_{00}(r)\ {}_0Y_{00} + d_{11}(r)\ {}_0Y_{11} + d_{10}(r)\ {}_0Y_{10} + d_{1-1}(r)\ {}_0Y_{1-1}\ , \tag{9.13}$$

where d_{00}, d_{10} are real and the complex functions d_{11} , d_{1-1} satisfy

$$\bar{d}_{11} = -d_{1-1}\,. \tag{9.14}$$

Substitution of (9.11) and (9.13) into (9.6), followed by comparing coefficients in the ${}_sY_{\ell m}$ gives

$$\begin{aligned} d_{11}(r) &= r\,\dot{a}(r) - a(r) \quad , \\ d_{10}(r) &= r\,\dot{b}(r) - b(r) \quad , \\ d_{1-1}(r) &= r\,\dot{c}(r) - c(r) \quad , \end{aligned}$$

where $\cdot$ denotes $\frac{d}{dr}$. Using (9.14) leads to the restriction

$$c(r) = -\,\bar{a}(r) + Fr\ , \tag{9.15}$$

where $a(r)$ is a complex function of r and F is a complex constant. Similarly, using the fact that d_{10} is real leads to

$$b(r) = u(r) + i\,Er\ , \tag{9.16}$$

where $u(r)$ is a real function of r and E is a real constant.

Substitution of (9.5) into (9.10) gives

$$\bar{\eth}\zeta_+ + \eth\zeta_- = 2\sqrt{2}\,\zeta_0 - 2\sqrt{2}\,r\frac{\partial\zeta_0}{\partial r}\quad .$$

Using (9.11 - .13), (9.15) and (9.16) leads to the following equations for $a(r)$, $u(r)$ and $d_{oo}(r)$;

$$r\ddot{a} - \dot{a} + \tfrac{1}{2}\bar{F} = 0 \quad ,$$
$$r\ddot{u} - \dot{u} = 0 \quad ,$$
$$d_{oo} - r\dot{d}_{oo} = 0 \quad ,$$

from which

$$a(r) = \tfrac{1}{2}Ar^2 + \tfrac{1}{2}\bar{F}r + C \, ,$$
$$u(r) = \tfrac{1}{2}Br^2 + D,$$
$$d_{oo}(r) = Gr \, ,$$

where F, A, C are complex constants and B, D, G are real constants.

The final solutions take the form

$$\mathfrak{J}_+ = (\tfrac{1}{2}Ar^2 + \tfrac{1}{2}\bar{F}r + C)\;{}_1Y_{11} + (\tfrac{1}{2}Br^2 - iEr + D)\;{}_1Y_{10} + (-\tfrac{1}{2}\bar{A}r^2 + \tfrac{1}{2}Fr - \bar{C})\;{}_1Y_{1-1} ,$$

$$\mathfrak{J}_- = (-\tfrac{1}{2}Ar^2 + \tfrac{1}{2}\bar{F}r - C)\;{}_{-1}Y_{11} + (-\tfrac{1}{2}Br^2 + iEr - D)\;{}_{-1}Y_{10} + (\tfrac{1}{2}\bar{A}r^2 + \tfrac{1}{2}Fr + \bar{C})\;{}_{-1}Y_{1-1}$$

$$\mathfrak{J}_o = Gr\;{}_0Y_{00} + (\tfrac{1}{2}Ar^2 - C)\;{}_0Y_{11} + (\tfrac{1}{2}Br^2 - D)\;{}_0Y_{10} + (-\tfrac{1}{2}\bar{A}r^2 + \bar{C})\;{}_0Y_{1-1} \, .$$

There are three complex parameters A, C, F, and four real parameters B, D, E, G in this solution, giving, as we should expect, a 10-parameter family of C-K vectors.

CHAPTER 3

MINKOWSKI SPACE-TIME - THE INFINITESIMAL CONFORMAL MOTIONS

3.1 Introduction

This chapter lays the foundation for subsequent development by solving the conformal Killing equations in Minkowski space-time with N-U type coordinates. A basis for the space of conformal Killing vectors is used to construct the operators in the Lie algebra of the conformal group and the homomorphisms of this Lie algebra with the algebras of SO(2,4) and SU(2,2) are exhibited. The consequent local isomorphisms between the Minkowski space conformal group, SO(2,4) and SU(2,2), are of importance in the physical interpretation of the conformal group and in the construction of twistors, each of which forms the subject of a later chapter in this work.

3.2 The conformal Killing equations and their solution

In Chapter 2 we presented the conformal Killing equations in N-P formalism and also gave the tetrad vectors and spin coefficients for Minkowski space with N-U type coordinates. This material is now used to find the general conformal Killing vector ξ^{μ} of Minkowski space. The tetrad components of ξ^{μ} are written in the form

$$V_{\underline{1}} = \xi_{\mu} l^{\mu} = \xi^{\mu} l_{\mu} = \xi^{1} , \quad (2.1a)$$

$$V_{\underline{2}} = \xi_{\mu} n^{\mu} = \xi^{\mu} n_{\mu} = \tfrac{1}{2}\xi^{1} + \xi^{2}, \quad (2.1b)$$

$$V_{\underline{3}} = \xi_{\mu} m^{\mu} = \xi^{\mu} m_{\mu} = \frac{-r}{\sqrt{2}\cosh x^{3}} (\xi^{3} + i\xi^{4}), \quad (2.1c)$$

$$V_{\underline{4}} = \xi_{\mu} \bar{m}^{\mu} = \xi^{\mu} \bar{m}_{\mu} = \frac{-r}{\sqrt{2}\cosh x^{3}} (\xi^{3} - i\xi^{4}). \quad (2.1d)$$

The appropriate conformal Killing equations are

$$D\,V_{1} = 0 \tag{2.2a}$$

$$D\,V_{2} + \Delta V_{1} = \tfrac{1}{2}\,\emptyset \tag{2.2b}$$

$$D\,V_{3} + \delta V_{1} = \frac{1}{r} V_{3} \tag{2.2c}$$

$$\Delta V_{2} = 0 \tag{2.2d}$$

$$\delta V_{3} = -\frac{\sinh x^3}{r\sqrt{2}} V_{3} \tag{2.2e}$$

$$\delta V_{2} + \Delta V_{3} = -\frac{1}{2r} V_{3} \tag{2.2f}$$

$$\bar{\delta} V_{3} + \delta V_{4} = \frac{2}{r} V_{2} - \tfrac{1}{2}\,\emptyset - \frac{1}{r} V_{1} + \frac{\sinh x^3}{r\sqrt{2}} V_{3} + \frac{\sinh x^3}{r\sqrt{2}} \bar{V}_{3}, \quad (2.2g)$$

(2.2g)

where the notation of Chapter 2 for differential operators has been used.

We have also

$$\emptyset = \xi^{\mu}_{;\mu} = \xi^{\mu}_{,\mu} + \frac{\xi^{\mu}(-g)_{,\mu}}{2(-g)},$$

and putting $-g = r^4 \operatorname{sech}^4 x^3$ we find

$$\emptyset = \xi^{\mu}_{,\mu} + \frac{2}{r}\xi^2 - 2\,\xi^3 \tanh x^3 . \tag{2.3}$$

From (2.2a) we have immediately

$$\xi^1 = \overset{o}{\xi}{}^1(u, x^A) , \tag{2.4}$$

where we use ^{o} to indicate functions independent of r.

It follows then from (2.2c) that

$$\xi^3 = \overset{o}{\xi}{}^3 - r^{-1} \cosh^2 x^3\, \overset{o}{\xi}{}^1_{,3} \tag{2.5}$$

$$\text{and } \xi^4 = \overset{o}{\xi}{}^4 - r^{-1} \cosh^2 x^3\, \overset{o}{\xi}{}^1_{,4}. \tag{2.6}$$

Using (2.4, 5, 6) in (2.3) gives

$$\emptyset = \frac{\partial}{\partial r}\xi^2 + \frac{2}{r}\xi^2 + \overset{o}{\xi}{}^1_{,1} + \overset{o}{\xi}{}^3_{,3} + \overset{o}{\xi}{}^4_{,4} - 2\,\overset{o}{\xi}{}^3 \tanh x^3 - r^{-1}\cosh^2 x^3\left(\overset{o}{\xi}{}^1_{,33} + \overset{o}{\xi}{}^1_{,44}\right). \tag{2.6a}$$

(2.2b) can now be written as

$$\frac{\partial}{\partial r}\xi^2 = \tfrac{1}{2}\,\emptyset - \overset{o}{\xi}{}^1_{,1} ,$$

and substitution of the above expression for $\emptyset$ leads to

$$\frac{\partial}{\partial r}\xi^2 - \frac{2}{r}\xi^2 = A + r^{-1}B, \tag{2.7}$$

where $$A = \overset{\circ}{\xi}{}^3_{,3} + \overset{\circ}{\xi}{}^4_{,4} - \overset{\circ}{\xi}{}^1_{,1} - 2\,\overset{\circ}{\xi}{}^3 \tanh x^3 \tag{2.8}$$

and $$B = -\cosh^2 x^3 (\overset{\circ}{\xi}{}^1_{,33} + \overset{\circ}{\xi}{}^1_{,44}) . \tag{2.9}$$

(2.7) integrates to give

$$\xi^2 = \overset{\circ}{\xi}{}^2 r^2 - Ar - \tfrac{1}{2}B . \tag{2.10}$$

This completes the integration of the "radial equations" (2.2 a,b,c). The remaining equations of the set (2.2) give conditions on the r-independent functions appearing in (2.4, 5, 6, 10).

(2.2d) gives

$$\frac{\partial \xi^2}{\partial r} = \overset{\circ}{\xi}{}^1_{,1} + 2\,\xi^2_{,1} \quad ,$$

and substitution of (2.10), followed by equating coefficients in powers of r, leads to three conditions;

$$\overset{\circ}{\xi}{}^2_{,1} = 0 \; , \tag{2.11}$$

$$\overset{\circ}{\xi}{}^2 + A_{,1} = 0, \tag{2.12}$$

$$A + \overset{\circ}{\xi}{}^1_{,1} - B_{,1} = 0. \tag{2.13}$$

In a similar way, (2.2e) leads to

$$\overset{\circ}{\xi}{}^3_{,3} - \overset{\circ}{\xi}{}^4_{,4} = 0, \tag{2.14}$$

$$\overset{\circ}{\xi}{}^3_{,4} + \overset{\circ}{\xi}{}^4_{,3} = 0, \tag{2.15}$$

$$\cosh x^3 \, (\overset{\circ}{\xi}{}^1_{,33} - \overset{\circ}{\xi}{}^1_{,44}) + 2 \sinh x^3 \; \overset{\circ}{\xi}{}^1_{,3} = 0, \tag{2.16}$$

and $$\cosh x^3 \; \overset{\circ}{\xi}{}^1_{,34} + \sinh x^3 \; \overset{\circ}{\xi}{}^1_{,4} = 0. \tag{2.17}$$

(2.2f) gives

$$2\,\overset{\circ}{\xi}{}^1_{,3} - B_{,3} = 0, \tag{2.18}$$

$$2\,\overset{\circ}{\xi}{}^1_{,4} - B_{,4} = 0, \tag{2.19}$$

$$\overset{\circ}{\xi}{}^1_{,31} - A_{,3} = 0, \tag{2.20}$$

$$\overset{\circ}{\xi}{}^1_{,41} - A_{,4} = 0, \tag{2.21}$$

$$\cosh^2 x^3 \; \overset{\circ}{\xi}{}^2_{,3} - \overset{\circ}{\xi}{}^3_{,1} = 0, \tag{2.22}$$

and $$\cosh^2 x^3 \; \overset{\circ}{\xi}{}^2_{,4} - \overset{\circ}{\xi}{}^4_{,1} = 0. \tag{2.23}$$

Finally, (2.2g) yields the condition

$$-\xi^3_{,3} - \xi^4_{,4} + 2\,\xi^3 \tanh x^3 = \frac{2}{r}\,\xi^2 - \tfrac{1}{2}\,\emptyset . \tag{2.24}$$

From (2.10) and (2.6a),

$$\emptyset = 4r\,\overset{\circ}{\xi}{}^2 - 2A + 2\,\overset{\circ}{\xi}{}^1_{,1} .$$

Then the right-hand side of (2.24) becomes

$$\frac{2}{r}\xi^2 - \tfrac{1}{2}\phi = -A - r^{-1}B - \overset{\circ}{\xi}{}^1{}_{,1}$$
$$= -\overset{\circ}{\xi}{}^3{}_{,3} - \overset{\circ}{\xi}{}^4{}_{,4} + 2\overset{\circ}{\xi}{}^3 \tanh x^3 + r^{-1}\cosh^2 x^3 (\overset{\circ}{\xi}{}^1{}_{,33} + \overset{\circ}{\xi}{}^1{}_{,44}),$$

where (2.8), (2.9) have been employed. We can easily show that the left-hand side of (2.24) is equal to the same expression, and hence that (2.24) is identically satisfied.

In summary then, there are thirteen equations (2.11) - (2.23) giving restrictions on the form of $\overset{\circ}{\xi}{}^\mu$, A, B that appear in (2.4, 5, 6, 10). The details of the integration of these equations are relegated to Appendix 1. The final result, depending on 15 real parameters, gives the general conformal Killing vector ξ^μ as

$$\xi^1 = \operatorname{sech} x^3 \sin x^4 [au^2 + bu + c] + \operatorname{sech} x^3 \cos x^4 [du^2 + eu + f]$$
$$+ \tanh x^3 [gu^2 + hu + j] + [ku^2 + \ell u + m] \quad ,$$
$$\xi^2 = -\operatorname{sech} x^3 \sin x^4 [a(u^2 + 2ur + 2r^2) + b(u + r) + c] - \operatorname{sech} x^3 \cos x^4 \times$$
$$[d(u^2 + 2ur + 2r^2) + e(u + r) + f] - \tanh x^3 [g(u^2 + 2ur + 2r^2)$$
$$+ h(u + r) + j] + 2kr(u + r) + \ell r,$$
$$\xi^3 = \sinh x^3 \sin x^4 [a(2u + r^{-1}u^2) + b(1 + r^{-1}u) + cr^{-1}]$$
$$+ \sinh x^3 \cos x^4 [d(2u + r^{-1}u^2) + e(1 + r^{-1}u) + fr^{-1}] - g(2u + r^{-1}u^2) - h(1 + r^{-1}u)$$
$$- jr^{-1} + n\cosh x^3 \cos x^4 - p\cosh x^3 \sin x^4,$$
$$\xi^4 = \cosh x^3 \cos x^4 [-a(2u + r^{-1}u^2) - b(1 + r^{-1}u) - cr^{-1}]$$
$$+ \cosh x^3 \sin x^4 [d(2u + r^{-1}u^2) + e(1 + r^{-1}u) + fr^{-1}] + n\sinh x^3 \sin x^4$$
$$+ p\sinh x^3 \cos x^4 + q .$$

It is convenient for subsequent manipulation and physical interpretation to have the conformal Killing vectors expressed in terms of Minkowskian coordinates (t, x, y, z). The transformation from N-U type coordinates has been given in Chapter 2 and leads to the following components for the general conformal Killing vector;

$$\xi^1 = -2aty - by - 2dtx - ex + 2gtz - hz + k(t^2 + x^2 + y^2 + z^2) + \ell t + m, \quad (2.25a)$$

$$\xi^2 = -2axy - d(t^2 + x^2 - y^2 - z^2) - et - f + 2gxz + 2kxt + \ell x + nz - qy, \quad (2.25b)$$

$$\xi^3 = -a(t^2 + y^2 - x^2 - z^2) - bt - c - 2dxy + 2gyz + 2kty + \ell y - pz + qx, \quad (2.25c)$$

$$\xi^4 = -2ayz - 2dxz + g(t^2 + z^2 - x^2 - y^2) + ht + j + 2ktz + \ell z - nx + py. \quad (2.25d)$$

We select the following 15 vectors as a basis for the space of conformal Killing vectors; (the table below gives contravariant components)

$\underset{\sim}{\xi}_0 = (1,0,0,0)$,

$\underset{\sim}{\xi}_1 = (0,1,0,0)$,

$\underset{\sim}{\xi}_2 = (0,0,1,0)$,

$\underset{\sim}{\xi}_3 = (0,0,0,1)$,

$\underset{\sim}{\xi}_4 = (x,t,0,0)$,

$\underset{\sim}{\xi}_5 = (y,0,t,0)$,

$\underset{\sim}{\xi}_6 = (z,0,0,t)$,

$\underset{\sim}{\xi}_7 = (0,y,-x,0)$,

$\underset{\sim}{\xi}_8 = (0,z,0,-x)$,

$\underset{\sim}{\xi}_9 = (0,0,z,-y)$,

$\underset{\sim}{\xi}_{10} = (t,x,y,z)$,

$\underset{\sim}{\xi}_{11} = (\frac{1}{2}[t^2 + x^2 + y^2 + z^2], tx, ty, tz)$,

$\underset{\sim}{\xi}_{12} = (xt, \frac{1}{2}[x^2 + t^2 - y^2 - z^2], xy, xz)$,

$\underset{\sim}{\xi}_{13} = (yt, yx, \frac{1}{2}[y^2 + t^2 - x^2 - z^2], yz)$,

$\underset{\sim}{\xi}_{14} = (zt, zx, zy, \frac{1}{2}[z^2 + t^2 - x^2 - y^2])$.

3.3 The Lie algebra of the flat-space conformal group.

From the basis vectors given above we construct the following 15 differential operators

$$X_A = \xi_A^{\mu} \frac{\partial}{\partial x^{\mu}} \quad , A = 0,1,\ldots,14. \qquad (3.1)$$

Each operator generates a finite conformal transformation given by

$$\bar{x}^{\mu} = \exp(\epsilon_A X_A) x^{\mu}, \qquad (3.2)$$

where $\exp(\epsilon X) = I + \epsilon X + \frac{\epsilon^2}{2!} X^2 + \ldots$

and ϵ_A is an infinitesimal parameter. These finite transformations will be written down later and their physical significance will be discussed. For the present we are concerned only with the X_A operators. It may be demonstrated that these form a Lie algebra.

The commutator of two such operators is defined in the usual way:-

$$[X_A, X_B] = X_A X_B - X_B X_A \qquad (3.3)$$

The commutation relations have been calculated and are presented in the table on page 58. It is straightforward to verify that the vector space of the X_A gives a Lie algebra under the bracket operation defined in (3.3) (The axioms for a Lie algebra appeared in Chapter 2.) Apart from changes of nomenclature the commutator table given here is identical with the table of Hill [39].

The structure of the above Lie algebra is neatly expressed in tensor formalism if we introduce the following elements as a basis [40] :-

$$L_{\mu\nu} = 2i\, x_{[\mu}\, \partial_{\nu]} \;, \tag{3.4a}$$

$$P_\mu = i\, \partial_\mu \;, \tag{3.4b}$$

$$D = i\, x^\mu \partial_\mu \;, \tag{3.4c}$$

$$K_\mu = i\left[2\, x_\mu\, x^\alpha \partial_\alpha - (x^\alpha\, x_\alpha)\, \partial_\mu\right] . \tag{3.4d}$$

The operators X_A of (3.1) are related to those of (3.4) according to

$$L_{01} = iX_7 \;, \qquad L_{02} = iX_8 \;, \qquad L_{03} = iX_9 \;,$$

$$L_{12} = iX_4 \;, \qquad L_{13} = iX_5 \;, \qquad L_{23} = iX_6 \;,$$

$$P_\mu = iX_\mu \;, \; \mu = 0,1,2,3, \qquad D = -iX_{10} \;,$$

$$K_0 = iX_{11} \;, \; K_1 = -iX_{12} \;, \; K_2 = -iX_{13} \;, \; K = -iX_{14} \;.$$

Commutation relations for the operators $X_A = \xi^\mu_A \partial_\mu$.

$[X_0, X_1] = 0$						
$[X_0, X_2] = 0$	$[X_1, X_2] = 0$					
$[X_0, X_3] = 0$	$[X_1, X_3] = 0$	$[X_2, X_3] = 0$				
$[X_0, X_4] = 0$	$[X_1, X_4] = -X_2$	$[X_2, X_4] = X_1$	$[X_3, X_4] = 0$			
$[X_0, X_5] = 0$	$[X_1, X_5] = -X_3$	$[X_2, X_5] = 0$	$[X_3, X_5] = X_1$	$[X_4, X_5] = X_6$		
$[X_0, X_6] = 0$	$[X_1, X_6] = 0$	$[X_2, X_6] = -X_3$	$[X_3, X_6] = X_2$	$[X_4, X_6] = -X_5$	$[X_5, X_6] = X_4$	
$[X_0, X_7] = X_1$	$[X_1, X_7] = X_0$	$[X_2, X_7] = 0$	$[X_3, X_7] = 0$	$[X_4, X_7] = X_8$	$[X_5, X_7] = X_9$	$[X_6, X_7] = 0$
$[X_0, X_8] = X_2$	$[X_1, X_8] = 0$	$[X_2, X_8] = X_0$	$[X_3, X_8] = 0$	$[X_4, X_8] = -X_7$	$[X_5, X_8] = 0$	$[X_6, X_8] = X_9$
$[X_0, X_9] = X_3$	$[X_1, X_9] = 0$	$[X_2, X_9] = 0$	$[X_3, X_9] = X_0$	$[X_4, X_9] = 0$	$[X_5, X_9] = -X_7$	$[X_6, X_9] = -X_8$
$[X_0, X_{10}] = X_0$	$[X_1, X_{10}] = X_1$	$[X_2, X_{10}] = X_2$	$[X_3, X_{10}] = X_3$	$[X_4, X_{10}] = 0$	$[X_5, X_{10}] = 0$	$[X_6, X_{10}] = 0$
$[X_0, X_{11}] = 2X_{10}$	$[X_1, X_{11}] = 2X_7$	$[X_2, X_{11}] = 2X_8$	$[X_3, X_{11}] = 2X_9$	$[X_4, X_{11}] = 0$	$[X_5, X_{11}] = 0$	$[X_6, X_{11}] = 0$
$[X_0, X_{12}] = 2X_7$	$[X_1, X_{12}] = 2X_{10}$	$[X_2, X_{12}] = -2X_4$	$[X_3, X_{12}] = -2X_5$	$[X_4, X_{12}] = X_{13}$	$[X_5, X_{12}] = X_{14}$	$[X_6, X_{12}] = 0$
$[X_0, X_{13}] = 2X_8$	$[X_1, X_{13}] = 2X_4$	$[X_2, X_{13}] = 2X_{10}$	$[X_3, X_{13}] = -2X_6$	$[X_4, X_{13}] = -X_{12}$	$[X_5, X_{13}] = 0$	$[X_6, X_{13}] = X_{14}$
$[X_0, X_{14}] = 2X_9$	$[X_1, X_{14}] = 2X_5$	$[X_2, X_{14}] = 2X_6$	$[X_3, X_{14}] = 2X_{10}$	$[X_4, X_{14}] = 0$	$[X_5, X_{14}] = -X_{12}$	$[X_6, X_{14}] = -X_{13}$

$[X_7, X_8] = -X_4$						
$[X_7, X_9] = -X_5$	$[X_8, X_9] = -X_6$					
$[X_7, X_{10}] = 0$	$[X_8, X_{10}] = 0$	$[X_9, X_{10}] = 0$				
$[X_7, X_{11}] = X_{12}$	$[X_8, X_{11}] = X_{13}$	$[X_9, X_{11}] = X_{14}$	$[X_{10}, X_{11}] = X_{11}$			
$[X_7, X_{12}] = X_{11}$	$[X_8, X_{12}] = 0$	$[X_9, X_{12}] = 0$	$[X_{10}, X_{12}] = X_{12}$	$[X_{11}, X_{12}] = 0$		
$[X_7, X_{13}] = 0$	$[X_8, X_{13}] = X_{11}$	$[X_9, X_{13}] = 0$	$[X_{10}, X_{13}] = X_{13}$	$[X_{11}, X_{13}] = 0$	$[X_{12}, X_{13}] = 0$	
$[X_7, X_{14}] = 0$	$[X_8, X_{14}] = 0$	$[X_9, X_{14}] = X_{11}$	$[X_{10}, X_{14}] = X_{14}$	$[X_{11}, X_{14}] = 0$	$[X_{12}, X_{14}] = 0$	$[X_{13}, X_{14}] = 0$

In terms of the basis (3.4) the commutation relations become

$$[L_{\kappa\lambda}, L_{\mu\nu}] = i\,(g_{\lambda\mu} L_{\kappa\nu} - g_{\kappa\mu} L_{\lambda\nu} + g_{\kappa\nu} L_{\lambda\mu} - g_{\lambda\nu} L_{\kappa\mu}) \,, \tag{3.5a}$$

$$[P_\lambda, L_{\mu\nu}] = i\,(g_{\lambda\mu} P_\nu - g_{\lambda\nu} P_\mu) \,, \tag{3.5b}$$

$$[K_\lambda, L_{\mu\nu}] = i\,(g_{\lambda\mu} K_\nu - g_{\lambda\nu} K_\mu) \,, \tag{3.5c}$$

$$[K_\mu, P_\nu] = 2i\,(g_{\mu\nu} D - L_{\mu\nu}) \,, \tag{3.5d}$$

$$[P_\mu, P_\nu] = 0 \,, \tag{3.5e}$$

$$[K_\mu, K_\nu] = 0 \,, \tag{3.5f}$$

$$[D, P_\mu] = i\,P_\mu \,, \tag{3.5g}$$

$$[D, K_\mu] = -i\,K_\mu \,, \tag{3.5h}$$

$$[D, L_{\mu\nu}] = 0. \tag{3.5i}$$

The finite transformations generated by the X_A have been given by many writers e.g. [41,42] . We shall take them in the form

$$\bar{x}^\mu = L^\mu{}_\nu\, x^\nu, \text{ where } L^\mu{}_\nu \; L_{\mu\rho} = g_{\nu\rho} \,, \tag{3.6}$$

$$\bar{x}^\mu = x^\mu + a^\mu \,, \tag{3.7}$$

$$\bar{x}^\mu = \rho\, x^\mu, \; \rho \text{ constant } (> 0) \,, \tag{3.8}$$

$$\bar{x}^\mu = \frac{x^\mu - (x.x)\, c^\mu}{1 - 2\,\underline{c}.\underline{x} + (\underline{c}.\underline{c})\,(\underline{x}.\underline{x})} \,, \tag{3.9}$$

where we use $\underline{u}.\underline{v} = u^\alpha v_\alpha$ for the Minkowski scalar product. In subsequent work we shall use C to denote this group of transformations.

The homogeneous Lorentz group (3.6) is generated by X_4, X_5, X_6 (Lorentz rotations) and X_7, X_8, X_9 (spatial rotations). The translations (3.7) are generated by X_0, X_1, X_2, X_3. The dilatations (3.8) are generated by X_{10} and finally the "special conformal transformations" (3.9) are generated by X_{11}, X_{12}, X_{13}, X_{14}.

(3.6) and (3.7) together give the inhomogeneous Lorentz group (also known as the Poincaré group), and this group, augmented by the dilatations (3.8), gives the Weyl group.

Whereas the physical significance of the Poincaré group is well understood, the interpretation of the dilatations and (most especially) the conformal transformations (3.9), has been debated often in the literature, e.g. [41,43]. We shall discuss some of the suggestions in Chapter 7 of the present work.

3.4 The homomorphism between SO(2,4) and the flat-space conformal group.

In this section we shall demonstrate that there exists a homomorphism between the Lie algebras of SO(2,4) and the Minkowski space conformal group C. Let y^A, A = 0,...,5 be the coordinates of a 6-dimensional space with "metric tensor" G_{AB} given by

$$G_{AB} = \text{diag}\ (1,\ 1,\ -1,\ -1,\ -1,\ -1). \qquad (4.1)$$

The group of transformations on the y^A that preserves the quadratic form $G_{AB}\ y^A\ y^B$ is the psudo-orthogonal group SO(2,4). We use G_{AB} to lower suffices in the usual way, and then we may write

$$M_{AB} = 2i\ \ y_{[A}\partial_{B]} \qquad (4.2)$$

$$A,B=0,\ldots,5$$

for the generators of infinitesimal transformations of SO(2,4). The commutation relations between the M_{AB} may be expressed in the form

$$[M_{AB},\ M_{CD}] = i(G_{BC}\ M_{AD} - G_{AC}\ M_{BD} + G_{AD}\ M_{BC} - G_{BD}\ M_{AC}) \qquad (4.3)$$

We demonstrate that a homomorphism exists between the Lie algebras of C and SO(2,4) by exhibiting the correspondence between the operators (3.4) and (4.2) in an explicit manner.

$$P_0 \leftrightarrow M_{10} - M_{20} \qquad K_0 \leftrightarrow M_{10} + M_{20}$$

$$P_1 \leftrightarrow -M_{13} + M_{23} \qquad K_1 \leftrightarrow -M_{13} - M_{23}$$

$$P_2 \leftrightarrow M_{15} - M_{25} \qquad K_2 \leftrightarrow M_{15} + M_{25}$$

$$P_3 \leftrightarrow M_{14} - M_{24} \qquad K_3 \leftrightarrow M_{14} + M_{24}$$

$$L_{01} \leftrightarrow -M_{03} \qquad L_{12} \leftrightarrow -M_{35}$$

$$L_{02} \leftrightarrow M_{05} \qquad L_{13} \leftrightarrow -M_{34} \qquad D \leftrightarrow M_{12}$$

$$L_{03} \leftrightarrow M_{04} \qquad L_{14} \leftrightarrow -M_{45}$$

It is straightforward then to verify that the commutation relations (3.5) become identical with (4.3).

From the homomorphism of the Lie algebras of C and SO(2,4) it follows that there exists a local isomorphism between the corresponding Lie groups. This will be of importance in Chapter 7 when the question of physical interpretation is discussed.

3.5 The homomorphism between SU(2,2) and SO(2,4)

In this section we shall consider SU(2,2) as the group of 4 x 4 complex matrices U satisfying

$$U^{\dagger} G U = G \quad , \tag{5.1}$$

and

$$\det U = 1 \quad , \tag{5.2}$$

where $\dagger$ denotes the hermitian conjugate and

$$G = \text{diag}\ (1, -1, -1, 1) \ . \tag{5.3}$$

Matrices of SU(2,2), acting on the vector space of complex 4-vectors $\underline{\xi}$, preserve the hermitian form

$$\underline{\xi}'^{\dagger} G \underline{\xi}' = \underline{\xi}^{\dagger} G \underline{\xi} \quad , \tag{5.4}$$

where

$$\underline{\xi}' = U \underline{\xi}$$

Let

$$U = I + \epsilon V \tag{5.5}$$

represent (to first order in the infinitesimal parameter ϵ), an infinitesimal operator of SU(2,2). Then the "pseudo-unitary" condition (5.1) requires

$$(I + \epsilon V)^{\dagger} G (I + \epsilon V) = G,$$

from which follows the condition

$$\epsilon\ (GV + V^{\dagger} G) + O(\epsilon^2) = 0 \ .$$

We demand then that the generators V of the infinitesimal transformations satisfy

$$V^{\dagger} = -GVG^{-1} \quad . \tag{5.6}$$

From the condition of unimodularity (5.2) we find, on using the well-known identity [44] ,

$$\det\,(I + \epsilon V) = 1 + \epsilon\, \mathrm{Tr}\, V + O(\epsilon^2),$$

that the matrices V must be trace-free;

$$\mathrm{Tr}\, V = 0 \quad . \tag{5.7}$$

The set of 4 x 4 complex matrices satisfying (5.6) and (5.7) forms a 15-dimensional vector space over the reals. We now write down a basis $\{V_a\}$, a = 1,2,...,15 for this space and note that with a bracket operation defined by

$$[V_a, V_b] = V_a V_b - V_b V_a$$

the space forms a Lie algebra. Then we show that this algebra is homomorphic with the Lie algebra of SO(2,4) by giving explicitly the correspondence between the basis elements in the two algebras.

As a guide in selecting a basis for the Lie algebra of SU(2,2) it was helpful to first write down the sixteen 4 x 4 matrices that span the complete matrix ring M_4. This was done simply from an examination of the Clifford algebra C_4, since it is known [28] that the matrix representation of C_4 is isomorphic with the full matrix ring M_4.

We utilise the Pauli matrices and the 2 x 2 identity matrix, labelled as follows,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.8}$$

in order to condense the expression of 4 x 4 matrices as Kronecker products.

A basis $\{V_a\}$ for the Lie algebra of SU(2,2) is

$$\begin{array}{lll} V_1 = \frac{1}{2}\, \sigma_1 \times \sigma_0 & V_6 = \frac{i}{2}\, \sigma_2 \times \sigma_1 & V_{11} = \frac{1}{2}\, \sigma_0 \times \sigma_1 \\ V_2 = \frac{1}{2}\, \sigma_2 \times \sigma_0 & V_7 = \frac{i}{2}\, \sigma_2 \times \sigma_2 & V_{12} = \frac{1}{2}\, \sigma_0 \times \sigma_2 \\ V_3 = \frac{1}{2}\, \sigma_3 \times \sigma_1 & V_8 = \frac{-i}{2}\, \sigma_1 \times \sigma_1 & V_{13} = \frac{1}{2}\, \sigma_1 \times \sigma_3 \\ V_4 \;\; \frac{1}{2}\, \sigma_3 \times \sigma_2 & V_9 = \frac{-i}{2}\, \sigma_1 \times \sigma_2 & V_{14} = \frac{1}{2}\, \sigma_2 \times \sigma_3 \\ V_5 = \frac{-i}{2}\, \sigma_3 \times \sigma_0 & V_{10} = \frac{-i}{2}\, \sigma_0 \times \sigma_3 & V_{15} = \frac{-i}{2}\, \sigma_3 \times \sigma_3 \end{array} \tag{5.9}$$

where x denotes Kronecker product of matrices.

In calculating commutators of the V_a it is useful to have the following relations available:-

$$\begin{aligned}
[\sigma_j \times \sigma_k, \ \sigma_\ell \times \sigma_m] &= -2i \ (\delta_{km}\epsilon_{j\ell n}\sigma_n \times \sigma_0 + \delta_{j\ell}\epsilon_{kmn}\sigma_0 \times \sigma_n) \ , \\
[\sigma_j \times \sigma_0, \ \sigma_\ell \times \sigma_\nu] &= -2i\,\epsilon_{j\ell n}\,\sigma_n \times \sigma_\nu \ , \\
[\sigma_0 \times \sigma_k, \ \sigma_\nu \times \sigma_m] &= -2i\,\epsilon_{kmn}\ \sigma_\nu \times \sigma_n \ , \\
[\sigma_0 \times \sigma_k, \ \sigma_\ell \times \sigma_0] &= 0 \ ,
\end{aligned} \qquad (5.10)$$

where $\nu = 0,\ldots,3$ and i, j etc. = 1,2,3.

In deriving (5.10) one makes use of the following properties of the Pauli matrices:-

$$[\sigma_j, \sigma_k] = -2i\,\epsilon_{jk\ell}\,\sigma_\ell \ ,$$

and

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\,\delta_{jk}\sigma_0 \ .$$

The relations (5.10) make it clear that the V_a form a Lie algebra under the bracket operation

$$[V_a, V_b] = V_a V_b - V_b V_a .$$

The correspondence of the V_a with the operators M_{AB} of SO(2,4) is given by

$$\begin{array}{lll}
V_1 \leftrightarrow -iM_{12} & V_6 \leftrightarrow -M_{15} & V_{11} \leftrightarrow -iM_{04} \\
V_2 \leftrightarrow -M_{23} & V_7 \leftrightarrow -M_{14} & V_{12} \leftrightarrow iM_{05} \\
V_3 \leftrightarrow M_{25} & V_8 \leftrightarrow -iM_{35} & V_{13} \leftrightarrow iM_{03} \\
V_4 \leftrightarrow M_{24} & V_9 \leftrightarrow -iM_{34} & V_{14} \leftrightarrow -M_{01} \\
V_5 \leftrightarrow M_{13} & V_{10} \leftrightarrow -iM_{45} & V_{15} \leftrightarrow M_{02}
\end{array} \qquad (5.11)$$

Using (5.10) it is straightforward to verify that the commutation relations of the V_a are precisely those of the Lie algebra of SO(2,4) given in (4.3). We have shown that the Lie algebra of SU(2,2) is homomorphic with the Lie algebra of SO(2,4). This indicates that there exists local isomorphism between the groups SU(2,2) and SO(2,4), a fact which has been fruitfully exploited by Penrose in his development of twistor algebra [9 - 12] .

CHAPTER 4

MINKOWSKI SPACE-TIME - THE FINITE CONFORMAL TRANSFORMATIONS

4.1 Introduction

This chapter looks in detail at the finite transformations of the Minkowski-space conformal group. The work of section 4.2 had its origin in some remarks of Gürsey [45], but the present treatment gives a mathematically deeper approach and some detailed forms of the composition properties that are believed not to have appeared previously in the literature. The remaining sections are devoted to two important representations of the Minkowski-space conformal group, arising from the homomorphisms that exist between C, SO(2,4) and SU(2,2). Once again, the treatment here is more comprehensive than in previous accounts, especially in the work on the SU(2,2) representation.

4.2 Composition properties of the conformal transformations

The infinitesimal operators of the Lie algebra considered in Chapter 3 generate a sub-group of the full conformal group of Minkowski space; viz that part of the full group connected with the identity. It is this restricted conformal group that we study here. The elements of this transformation group are given in the form

$$\mathcal{L}(L) : \quad x'^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}, \text{ where } L^{\mu}{}_{\nu}\, L_{\mu\rho} = g_{\nu\rho}\,, \tag{2.1}$$

$$\mathcal{J}(\underline{a}) : \quad x'^{\mu} = x^{\mu} + a^{\mu}\,, \tag{2.2}$$

$$\mathcal{D}(\rho) : \quad x'^{\mu} = \rho\, x^{\mu}, \quad \rho \text{ constant}, \tag{2.3}$$

$$\mathcal{S}(\underline{c}) : \quad x'^{\mu} = \frac{x^{\mu} - c^{\mu}(\underline{x}.\underline{x})}{\sigma(x)}\,, \text{ where } \sigma(x) = 1 - 2\,\underline{c}.\underline{x} + c^2 x^2. \tag{2.4}$$

The notation $\underline{u}.\underline{v} = u^{\alpha} v_{\alpha}$ for Minkowski scalar product has been used, and u^2 denotes $\underline{u}.\underline{u}$.

(2.1) are the transformations of the homogeneous Lorentz group. (2.1) together with the translations (2.2) give the inhomogeneous Lorentz group (often called the Poincaré group). Including the dilatations (2.3) with the previous transformations gives the group generally referred to as the Weyl group. Finally, including the special conformal transformations (2.4) with the transformations of the Weyl group gives the conformal group (restricted in the sense explained above).

As examples of transformations of the full conformal group which are excluded from the restricted group we give the reflections and the "inversion" transformation

$$\mathcal{I} : \quad x'^0 = \frac{x^0}{x^\alpha x_\alpha} \quad , \qquad x'^i = \frac{-x^i}{x^\alpha x_\alpha} \quad . \tag{2.5}$$

ansformation is not continuous with the identity and consequently has no infinitesimal generator in the Lie algebra. However, the $\mathcal{I}$ transformation is important in providing a link between the special conformal transformations and the translations, since we can show that

$$\mathcal{S}(c) = \mathcal{I}\, \mathcal{T}(-c^0, c^i)\, \mathcal{I} \tag{2.6}$$

(a proof is given in Appendix 2).

It may be verified that the operations $\mathcal{D}, \mathcal{T}, \mathcal{L}$ commute (with, however, different values of the parameters).

Explicitly, we have,

$$\mathcal{L}(L)\, \mathcal{T}(\underline{a}) = \mathcal{T}(L^\mu{}_\nu a^\nu)\, \mathcal{L}(L) \tag{2.7}$$

(or equivalently, $\mathcal{T}(\underline{b})\, \mathcal{L}(L) = \mathcal{L}(L)\, \mathcal{T}(b^\mu L_\mu{}^\nu)$),

$$\mathcal{L}(L)\, \mathcal{D}(\rho) = \mathcal{D}(\rho)\, \mathcal{L}(L) \ , \tag{2.8}$$

$$\mathcal{D}(\rho)\, \mathcal{T}(\underline{a}) = \mathcal{T}(\rho \underline{a})\, \mathcal{D}(\rho). \tag{2.9}$$

The inversion $\mathcal{I}$ commutes with $\mathcal{D}$ and $\mathcal{L}$ but not with $\mathcal{T}$.

We have

$$\mathcal{I}\, \mathcal{D}(\rho) = \mathcal{D}(\tfrac{1}{\rho})\, \mathcal{I} \quad , \tag{2.10}$$

and $$\mathcal{L}(L^\mu{}_\nu)\, \mathcal{I} = \mathcal{I}\, \mathcal{L}(K^\mu{}_\nu) \quad , \tag{2.11}$$

where $$K^\mu{}_\nu = \begin{pmatrix} L^0{}_0 & -L^0{}_j \\ -L^i{}_0 & L^i{}_j \end{pmatrix} .$$

It follows that the special conformal transformations $\mathcal{S}$ commute with $\mathcal{D}$ and $\mathcal{L}$ but not with $\mathcal{J}$.

We note also the following results on products of transformations;

$$\mathcal{J}(\underline{a}_1)\,\mathcal{J}(\underline{a}_2) = \mathcal{J}(\underline{a}_1 + \underline{a}_2) \quad , \tag{2.12}$$

$$\mathcal{D}(\rho_1)\,\mathcal{D}(\rho_2) = \mathcal{D}(\rho_1\rho_2) \quad , \tag{2.13}$$

$$\mathcal{L}(L)\,\mathcal{L}(M) = \mathcal{L}(L^{\mu}{}_{\alpha}M^{\alpha}{}_{\nu}) \ , \tag{2.14}$$

$$\mathcal{I}\,\mathcal{I} = I, \text{ the identity transformation,}$$

$$\text{and } \mathcal{S}(\underline{a})\,\mathcal{S}(\underline{b}) = \mathcal{S}(\underline{a} + \underline{b}) \ . \tag{2.15}$$

Although $\mathcal{I}$ does not commute with $\mathcal{J}$, it is possible to express a transformation $\mathcal{I}\mathcal{J}\mathcal{I}$ in the form $\mathcal{L}\mathcal{J}\mathcal{D}\mathcal{I}\mathcal{J}$ in which $\mathcal{I}$ appears once only. The parameters of operations in this identity are obtained after a tedious calculation (given in Appendix 2);

$$\mathcal{I}\,\mathcal{J}(a^{\mu})\,\mathcal{I} = \mathcal{L}(L^{\mu}{}_{\nu})\,\mathcal{J}\left(\frac{a^{\mu}}{a^{\alpha}a_{\alpha}}\right)\mathcal{D}\left(\frac{-1}{a^{\alpha}a_{\alpha}}\right)\mathcal{I}\,\mathcal{J}\left(\frac{a^{0}}{a^{\alpha}a_{\alpha}}, \frac{-a^{i}}{a^{\alpha}a_{\alpha}}\right), \tag{2.16}$$

$$\text{where} \quad L^{\mu}{}_{\nu} = \begin{pmatrix} \dfrac{2\,a^{0}\,a_{0}}{a^{\alpha}a_{\alpha}} - 1 & \dfrac{2\,a^{0}\,a_{j}}{a^{\alpha}a_{\alpha}} \\ -\dfrac{2\,a^{i}\,a_{0}}{a^{\alpha}a_{\alpha}} & \delta^{i}{}_{j} - \dfrac{2\,a^{i}\,a_{j}}{a^{\alpha}a_{\alpha}} \end{pmatrix} . \tag{2.17}$$

$L^{\mu}{}_{\nu}$ can be shown to be the matrix of a Lorentz transformation.

The most general conformal transformation of our restricted conformal group may be written in the form

$$\mathcal{C} = \mathcal{D}\mathcal{L}\mathcal{J}\mathcal{I}\mathcal{J} , \tag{2.18}$$

since it is clear that $\mathcal{C}$ takes the form $\mathcal{D}\mathcal{L}\mathcal{K}$, where $\mathcal{K}$ has one of the following decompositions; $(\mathcal{J}\mathcal{S})^{n}$, $\mathcal{S}(\mathcal{J}\mathcal{S})^{n}$, $(\mathcal{J}\mathcal{S})^{n}\mathcal{J}$, $\mathcal{S}(\mathcal{J}\mathcal{S})^{n}\mathcal{J}$, and it is straightforward to show that each of these expressions for $\mathcal{K}$ reduces to the form $\mathcal{L}\mathcal{D}\mathcal{J}\mathcal{I}\mathcal{J}$. With $\mathcal{C}$ in the form (2.18) the explicit dependence of the general conformal transformation on 15 parameters is easily displayed:-

$\mathcal{C} = \mathcal{D}(\rho)\,\mathcal{L}(L^{\mu}{}_{\nu})\,\mathcal{J}(b^{\alpha})\,\mathcal{I}\,\mathcal{J}(a^{\beta})$ maps $x^{\mu} \to x'^{\mu}$ according to

$$x'^{0} = \frac{\rho}{x^{2} + 2\underline{a}.\underline{x} + a^{2}}\Big\{L^{0}{}_{0}\big[x^{0} + a^{0} + b^{0}(x^{2} + 2\underline{a}.\underline{x} + a^{2})\big] + L^{0}{}_{i}\big[-x^{i} - a^{i} + b^{i}(x^{2} + 2\underline{a}.\underline{x} + a^{2})\big]\Big\} \quad ,$$

$$x'^{i} = \frac{\rho}{x^2 + 2\underline{a}.\underline{x} + a^2} \Big\{ L^i{}_o\left[x^o + a^o + b^o(x^2 + 2\underline{a}.\underline{x} + a^2)\right]$$

$$+ L^i{}_j\left[-x^j - a^j + b^j\ (x^2 + 2\underline{a}.\underline{x} + a^2)\right]\Big\} \quad . \qquad (2.19)$$

The law of composition for two-$\mathcal{C}$ transformations

$$\mathcal{C}_1 = \mathcal{D}(\rho_1)\mathcal{L}(\underset{(1)}{L}{}^\mu{}_\nu)\mathcal{T}(\underset{(1)}{b}{}^\mu)\,\mathcal{J}\,\mathcal{T}(\underset{(1)}{a}{}^\mu)$$

and $\mathcal{C}_2 = \mathcal{D}(\rho_2)\mathcal{L}(\underset{(2)}{L}{}^\mu{}_\nu)\mathcal{T}(\underset{(2)}{b}{}^\mu)\,\mathcal{J}\,\mathcal{T}(\underset{(2)}{a}{}^\mu)$,

is given by

$$\mathcal{C}_2\mathcal{C}_1 = \mathcal{D}\left(\frac{-\rho_2}{\rho_1\, m^\alpha m_\alpha}\right)\mathcal{L}(\underset{(2)}{L}{}^\mu{}_\alpha\, K^\alpha{}_\beta\ \ M^\beta{}_\nu)\,\mathcal{T}(-m^\mu - \rho_1\,(m^\sigma m_\sigma)\,\underset{(2)}{b}{}^\alpha K_\alpha{}^\beta\, M_\beta{}^\mu) \times$$

$$\mathcal{J}\ \mathcal{T}\left(\underset{(1)}{a}{}^o + \frac{m^o}{m^\alpha m_\alpha}\ ,\ \underset{(1)}{a}{}^i - \frac{m^i}{m^\alpha m_\alpha}\right),$$

where $m^\alpha = \dfrac{\underset{(2)}{a}{}^\beta\,\underset{(1)}{L}{}_\beta{}^\alpha}{\rho_1} + \underset{(1)}{b}{}^\alpha$, $\quad K^\mu{}_\nu = \begin{pmatrix} \underset{(1)}{L}{}^o{}_o & -\underset{(1)}{L}{}^o{}_i \\ -\underset{(1)}{L}{}^i{}_o & \underset{(1)}{L}{}^i{}_j \end{pmatrix}$, (2.20)

and $\quad M^\mu{}_\nu = \begin{pmatrix} \dfrac{2m^om_o}{m^\alpha m_\alpha} - 1 & \dfrac{2m^om_j}{m^\alpha m_\alpha} \\ -\dfrac{2m^im_o}{m^\alpha m_\alpha} & \delta^i{}_j - \dfrac{2m^im_j}{m^\alpha m_\alpha} \end{pmatrix}$.

The "physical" sub-groups of the conformal group do not emerge from (2.18) in a straightforward way. We find that the $\mathcal{L}$, $\mathcal{T}$, and $\mathcal{D}$ transformations appear as certain limiting cases;

$$\mathcal{L}(L^\mu{}_\nu) = \lim_{\underline{m}\to 0}\left[\mathcal{D}\left(\frac{-1}{m^\alpha m_\alpha}\right)\mathcal{L}(L^\mu{}_\alpha M^\alpha{}_\nu)\,\mathcal{T}(-m^\alpha)\,\mathcal{J}\,\mathcal{T}\left(\frac{m^o}{m^\alpha m_\alpha}\ ,\ -\frac{m^i}{m^\alpha m_\alpha}\right)\right] \ ,$$

$$\mathcal{T}(a^\mu) = \lim_{\underline{m}\to 0}\left[\mathcal{D}\left(-\frac{1}{m^\alpha m_\alpha}\right)\mathcal{L}(M^\mu{}_\nu)\,\mathcal{T}(-m^\alpha)\,\mathcal{J}\,\mathcal{T}\left(a^o + \frac{m^o}{m^\alpha m_\alpha}\ ,\ a^i - \frac{m^i}{m^\alpha m_\alpha}\right)\right] \ ,$$

$$\mathcal{D}(\rho) = \lim_{\underline{m}\to 0}\left[\mathcal{D}\left(\frac{-\rho}{m^\alpha m_\alpha}\right)\mathcal{L}(M^\mu{}_\nu)\,\mathcal{T}(-m^\alpha)\,\mathcal{J}\,\mathcal{T}\left(\frac{m^o}{m^\alpha m_\alpha}\ ,\ \frac{-m^i}{m^\alpha m_\alpha}\right)\right] \ , \qquad (2.21)$$

where $M^\mu{}_\nu$ and m^μ have been given in (2.20).

Finally, the special conformal transformations $\mathcal{S}$ arise as a particular case of (2.18) in the form

$$\mathcal{S}(-m^o, m^i) = \mathcal{D}\left(\frac{-1}{m^\alpha m_\alpha}\right)\mathcal{L}(M^\mu{}_\nu)\,\mathcal{T}(-m^\alpha)\,\mathcal{J}\,\mathcal{T}\left(\frac{m^o}{m^\alpha m_\alpha}\ ,\ \frac{-m^i}{m^\alpha m_\alpha}\right) \ . \qquad (2.22)$$

The above results show that if we wish to retain contact with the physical significance of the parameters in a conformal transformation, it is convenient to write

$$\mathcal{C} = \mathcal{D}\left(\frac{-\rho}{m^\alpha m_\alpha}\right)\mathcal{L}\,(L^\mu{}_\alpha M^\alpha{}_\nu)\,\mathcal{J}(-m^\alpha)\,\mathcal{S}\,\mathcal{J}\left(\frac{m^0}{m^\alpha m_\alpha}, -\frac{m^i}{m^\alpha m_\alpha}\right) \qquad (2.23)$$

for the general form of the transformation.

4.3 A representation of the flat-space conformal group in a projective space of five dimensions.

We set up a mapping from Minkowski space $\mathcal{M}$ onto a certain hyperquadric Q in the five-dimensional projective space P_5. The usual Minkowskian coordinates x^μ, $\mu = 0 \ldots,3$ are adopted in $\mathcal{M}$ and we choose homogeneous coordinates (ξ^μ, ξ^4, ξ^5), $\mu = 0,\ldots,3$ in P_5. Consider the mapping $x^\mu \rightarrow (\xi^\mu, \xi^4, \xi^5)$ given by

$$\frac{\xi^\mu}{\xi^5} = x^\mu \quad , \qquad \frac{\xi^4}{\xi^5} = x^\mu x_\mu \quad , \qquad (3.1)$$

where the scalar product uses the Minkowski space metric $g_{\mu\nu}$ = diag (1,-1,-1,-1) Evidently the hyperquadric $Q \subset P_5$ is given by

$$\xi^\mu \xi_\mu - \xi^4 \xi^5 = 0. \qquad (3.2)$$

We consider now the image in Q of the hypersphere

$$(x^\mu - a^\mu) \quad (x_\mu - a_\mu) \quad = k \qquad (3.3)$$

in $\mathcal{M}$. Under (3.1) we find that (3.3) becomes

$$\alpha_\mu \xi^\mu - \tfrac{1}{2}\,(\xi^4 \alpha^5 + \xi^5 \alpha^4) \quad + \tfrac{1}{2} k\, \alpha^5 \xi^5 = 0 \;, \qquad (3.4)$$

where $\underline{\alpha}$ is the image in Q of the point a^μ in $\mathcal{M}$. Thus, the image in P_5 of the hypersphere (3.3) is the intersection of Q with the hyperplane (3.4). The null-cones of $\mathcal{M}$ are given by (3.3) with k = 0, so the image in P_5 of the null-cone with vertex a^μ in $\mathcal{M}$ is given by the intersection of Q with the hyperplane

$$\alpha_\mu \xi^\mu - \tfrac{1}{2}\,(\xi^4 \alpha^5 + \xi^5 \alpha^4) = 0 \quad , \qquad (3.5)$$

where $\underline{\alpha}$ is the image in Q of a^μ in $\mathcal{M}$. We note that (3.5) is just the hyperplane tangent to Q at the point $(\alpha^\mu, \alpha^4, \alpha^5)$.

The following simple change of coordinates in P_5;

$$\xi^0 = \eta^1,\ \xi^1 = \eta^2,\ \xi^2 = \eta^3,\ \xi^3 = \eta^4,\ \xi^4 = \eta^5 + \eta^0,\ \xi^5 = \eta^5 - \eta^0, \tag{3.6}$$

brings (3.2) to the form

$$(\eta^0)^2 + (\eta^1)^2 - (\eta^2)^2 - (\eta^3)^2 - (\eta^4)^2 - (\eta^5)^2 = 0 \quad , \tag{3.7}$$

which shows the importance of the group O(2,4) in the algebraic structure we are here investigating. The homomorphism between the conformal group of Minkowski space and the group SO(2,4) has been demonstrated in Chapter 3. We shall exploit this fact in the following paragraphs.

The conformal transformations in $\mathcal{M}$ arise in P_5 as the linear homogeneous transformations on the η^A, $A = 0,1,\ldots,5$, which leave (3.7) invariant. These transformations take the form

$$\eta^{A'} = \Gamma^{A'}{}_A\, \eta^A \quad ; \quad A,A' = 0,1,\ldots,5, \tag{3.8}$$

with

$$\Gamma^{A'}{}_A\; G_{A'B'}\; \Gamma^{B'}{}_B = \varkappa\, G_{AB}\ , \tag{3.9}$$

where $\varkappa$ is a real constant and

$$G_{AB} = G_{A'B'} = \mathrm{diag}\ (1,1,-1,-1,-1,-1). \tag{3.10}$$

We follow a method similar to that of Dirac [46], by identifying a particular point of Q with the point at infinity in $\mathcal{M}$. Our choice is the point (1,0,0,0,0,1), where the coordinates in P_5 are now the η^A defined in (3.6). It will be shown that the transformations in P_5 that leave this point invariant correspond to the $\mathcal{L}$, $\mathcal{J}$ and $\mathcal{D}$ transformations in $\mathcal{M}$, whilst transformations in P_5 that do not possess this property correspond to $\mathcal{J}$ and $\mathcal{S}$ transformations in $\mathcal{M}$.

The point (1,0,0,0,0,1) of Q is invariant under a transformation of the form (3.8) if and only if the polar hyperplane of this point with respect to Q is invariant. This hyperplane is given by

$$\eta^5 - \eta^0 = 0. \tag{3.11}$$

We shall find the most general transformation (3.8) which preserves (3.7) and (3.11). Suffices A, B etc. range over 0,1,...,5 ; suffices a,b etc. range over 1,...,4 and we shall use the Minkowskian metric g_{ab} = diag (1,-1,-1,-1) to lower small Roman suffices in the usual way. Firstly, we set down the most general transformation that preserves the expressions $G_{AB}\eta^A\eta^B$ and $\eta^5-\eta^0$. It has the form

$$\begin{aligned} \eta^{a'} &= \Lambda^{a'}{}_a\eta^a + (\eta^5-\eta^0)\,\delta^{a'}{}_a\tau^a , \\ \eta^{5'}-\eta^{0'} &= \eta^5-\eta^0 , \\ \eta^{5'}+\eta^{0'} &= (\eta^5+\eta^0) + 2\Lambda_{a'b}\,\delta^{a'}{}_a\tau^a\eta^b + (\eta^5-\eta^0)\,\tau^a\tau_a , \end{aligned} \tag{3.12}$$

where

$$\Lambda^{a'}{}_a\, g_{a'b'}\Lambda^{b'}{}_b = g_{ab} . \tag{3.13}$$

The general transformation that preserves the expression $G_{AB}\eta^A\eta^B$ and the equation (3.11) is given by the product of transformation (3.12) with a transformation of the form

$$\begin{aligned} \eta^{a'} &= \delta^{a'}{}_a\eta^a , \\ \eta^{5'}-\eta^{0'} &= \lambda(\eta^5-\eta^0) , \\ \eta^{5'}+\eta^{0'} &= \lambda^{-1}(\eta^5+\eta^0) , \end{aligned} \tag{3.14}$$

where λ is a real-valued parameter.

The most general transformation that preserves the equations (3.7) and (3.11) is given by the product of transformations (3.12) and (3.14) with a transformation of the form

$$\eta^{A'} = \mu\,\delta^{A'}{}_A\eta^A , \tag{3.15}$$

where μ is a real constant.

In order to examine the transformations induced in $\mathcal{M}$ by the transformations (3.12, 14, 15) in Q we re-introduce the Minkowskian coordinates x^μ given by

$$x^0 = \frac{\eta^1}{\eta^5-\eta^0} , \quad x^1 = \frac{\eta^2}{\eta^5-\eta^0} , \quad x^2 = \frac{\eta^3}{\eta^5-\eta^0} , \quad x^3 = \frac{\eta^0}{\eta^5-\eta^0} . \quad (3.16$$

It is straightforward then to show that (3.12) induces an inhomogeneous Lorentz transformation in $\mathcal{R}$ and that (3.14) induces a dilatation in $\mathcal{M}$. (3.15) merely expresses the fact that homogeneous coordinates in P_5 are defined only up to a factor; it induces no transformation of the Minkowski coordinates. Thus we see that the transformations (3.8) which leave the point (1,0,0,0,0,1) invariant induce the Weyl group in Minkowski space $\mathcal{M}$. Of the transformations which do not leave the point (1,0,0,0,0,1) invariant we say little here, except to remark that the "inversion" transformation $\mathcal{J}$ is induced by the transformation

$$\begin{aligned} \eta^{1'} = -\eta^1, \quad \eta^{2'} = \eta^2, \quad \eta^{3'} = \eta^3, \quad \eta^{4'} = \eta^4, \\ \eta^{5'} - \eta^{0'} = -(\eta^5 + \eta^0), \\ \eta^{5'} + \eta^{0'} = -(\eta^5 - \eta^0), \end{aligned} \tag{3.17}$$

in Q, and that the transformation in Q which induces a special conformal transformation $\mathcal{S}$ in $\mathcal{M}$ may be written down by making use of (3.12), (3.17) and (2.6).

Since the point (1,0,0,0,0,1) of Q is identified with the point at infinity in Minkowski space, we can identify the intersection of Q and the hyperplane $\eta^5 - \eta^0 = 0$ (or $\xi^5 = 0$) as the null-cone of the point at infinity. In this sense we have "compactified" the Minkowski space to produce the manifold described by Penrose [47] and used by him in his development of the algebra of twistors.

4.4 A representation of the flat-space conformal group in a space of complex skew-symmetric tensors.

Consider the action of the elements of the matrix group SU(2,2) on the space Z of complex 4-vectors. Let $z^\mu, \mu = 0,\ldots,3$ be the components of $\underline{z} \in Z$. Under a transformation $U^{\mu'}_{\ \mu} \in SU(2,2)$, the z^μ transform according to

$$z^{\mu'} = U^{\mu'}_{\ \mu} z^\mu, \tag{4.1}$$

where

$$\bar{U}^{\mu'}_{\ \mu} G_{\mu'\nu'} U^{\nu'}_{\ \nu} = G_{\mu\nu}, \tag{4.2}$$

$$\det U^{\mu'}_{\ \mu} = 1, \tag{4.3}$$

and

$$G_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,1). \tag{4.4}$$

The invariant hermitian form of SU(2,2) is

$$\bar{z}^{\mu}_{(1)} G_{\mu\nu} z^{\nu}_{(2)} = \bar{z}^{0}_{(1)} z^{0}_{(2)} - \bar{z}^{1}_{(1)} z^{1}_{(2)} - \bar{z}^{2}_{(1)} z^{2}_{(2)} + \bar{z}^{3}_{(1)} z^{3}_{(2)}, \; z_{(1)}, z_{(2)} \in Z.$$

From Z we generate the space $\widehat{Z^{(2)}}$ of skew-symmetric rank 2 tensors in the usual way [48] . The components with respect to a suitable basis, of a typical element $q \in \widehat{Z^{(2)}}$ may be written

$$q^{\mu\nu} = z^{\mu}_{(1)} z^{\nu}_{(2)} - z^{\nu}_{(1)} z^{\mu}_{(2)} , \qquad z_{(1)}, z_{(2)} \in Z. \tag{4.5}$$

The transformation (4.1) in Z induces the transformation $q \rightarrow q'$ in $\widehat{Z^{(2)}}$ with

$$q^{\mu'\nu'} = U^{\mu'}_{\mu} \; U^{\nu'}_{\nu} \; q^{\mu\nu}. \tag{4.6}$$

The product transformation W U of U: $\underline{z} \rightarrow \underline{z}' = U\underline{z}$ and W: $\underline{z} \rightarrow \underline{z}' = W\underline{z}$ in Z induces in $\widehat{Z^{(2)}}$ the transformation

$$q^{\mu'\nu'} = (WU)^{\mu'}_{\mu} \; (WU)^{\nu'}_{\nu} \; q^{\mu\nu}. \tag{4.7}$$

There is a certain invariant of $\widehat{Z^{(2)}}$ preserved under the map induced in $\widehat{Z^{(2)}}$ by any unimodular map of Z into itself. We consider the well-known identity

$$\epsilon_{\mu'\nu'\rho'\sigma'} U^{\mu'}_{\mu} \; U^{\nu'}_{\nu} \; U^{\rho'}_{\rho} \; U^{\sigma'}_{\sigma} = \epsilon_{\mu\nu\rho\sigma} \det (U^{\alpha}_{\beta}), \tag{4.8}$$

given, for example, in [49] . Multiplying both sides of (4.8) by $z^{\mu}_{(1)} z^{\nu}_{(2)} z^{\rho}_{(1)} z^{\sigma}_{(2)}$, where $z_{(1)}, z_{(2)}$ are arbitrary vectors of Z, gives

$$\epsilon_{\mu'\nu'\rho'\sigma'} \; z^{\mu'}_{(1)} \; z^{\nu'}_{(2)} \; z^{\rho'}_{(1)} \; z^{\sigma'}_{(2)} = \det(U^{\alpha}_{\beta}) \, \epsilon_{\mu\nu\rho\sigma} \, z^{\mu}_{(1)} z^{\nu}_{(2)} z^{\rho}_{(1)} z^{\sigma}_{(2)}, \tag{4.9}$$

where use has been made of (4.1).

$$\begin{aligned} \text{Now } \epsilon_{\mu\nu\rho\sigma} \, z^{\mu}_{(1)} z^{\nu}_{(2)} z^{\rho}_{(1)} z^{\sigma}_{(2)} &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \, z^{\rho}_{(1)} z^{\sigma}_{(2)} (z^{\mu}_{(1)} z^{\nu}_{(2)} - z^{\nu}_{(1)} z^{\mu}_{(2)}) \\ &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \, z^{\rho}_{(1)} z^{\sigma}_{(2)} q^{\mu\nu} \\ &= \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \, q^{\mu\nu} q^{\rho\sigma} \\ &= 2 \; (q^{01} \, q^{23} + q^{02} \, q^{31} + q^{03} \, q^{12}) , \end{aligned}$$

so we see from (4.8) that when $U^{\mu'}_{\mu}$ is unimodular, the expression

$$Q(q) \equiv - (q^{01} \, q^{23} + q^{02} \, q^{31} + q^{03} \, q^{12}) \tag{4.10}$$

is invariant under the transformation induced in $\widehat{Z^{(2)}}$ by $U^{\mu'}_{\mu}$ in Z.

The representation space that we introduce now will be shown to carry a representation of SO(2,4), and so, because of the homomorphism between this group and the conformal group C, it also carries a representation of C. Define the sub-space $\Lambda \subset \widehat{Z^{(2)}}$ to consist of those elements $q \in \widehat{Z^{(2)}}$ with components satisfying

$$q^{23} = \overline{q^{01}} , \quad q^{13} = - \overline{q^{02}} , \quad q^{12} = -\overline{q^{03}} . \qquad (4.11)$$

We show that Λ is an invariant sub-space under maps induced in $\hat{Z}^{(2)}$ by operations of SU(2,2) in Z.

The general matrix element of SU(2,2) may be decomposed as the product of six "elementary" matrices, to be defined below, and a certain diagonal matrix. By "elementary" matrix we mean one that describes a transformation confined to a particular plane of the space Z. For example, U_{12} denotes the operator of rotation in the $z^1 - z^2$ plane. Recalling the way in which the invariant hermitian form of SU(2,2) has been defined, we see that the elementary operations of the group are rotations U_{03}, U_{12} and pseudo-rotations T_{01}, T_{02}, T_{13}, T_{23}, the terminology here being that which is normally employed in descriptions of the Lorentz group operators (See for example [51])

Typical elementary matrices of the two types are

$$U_{12}(j,\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & j & -ke^{-i\phi} & 0 \\ 0 & ke^{i\phi} & j & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_{01}(c,\psi) = \begin{pmatrix} c & se^{-i\psi} & 0 & 0 \\ se^{i\psi} & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,$$

where $j^2 + k^2 = 1$ and $c^2 - s^2 = 1$.

Explicitly, the general element U of SU(2,2) may be written

$$U = D(\delta_1, \delta_2, \delta_3)\, T_{23}(c_4, \psi_4)\, U_{12}(j_2, \phi_2)\, T_{13}(c_3, \psi_3)\, T_{01}(c_2, \psi_2)\, T_{02}(c_1, \psi_1)\, U_{03}(j_1, \phi_1) , \qquad (4.12)$$

where $D(\delta_1, \delta_2, \delta_3) = \mathrm{diag}\,(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}, e^{-i(\delta_1 + \delta_2 + \delta_3)})$. (4.13)

A generalisation of results of Murnaghan [50] for unitary groups to the case of the pseudo-unitary SU(2,2) has been invoked here. One may check by direct calculation that each elementary matrix in the decomposition (4.12) induces in $\hat{Z}^{(2)}$ a transformation that leaves the sub-space Λ invariant. It follows then from the way in which a product of maps in Z induces a composite map in $\hat{Z}^{(2)}$ (equation 4.7) that the general matrix U of SU(2,2) induces a map of $\hat{Z}^{(2)}$ which leaves the sub-space Λ invariant.

For $q \in \Lambda$ the quadratic form $Q(q)$ takes the form

$$Q(q) = -|q^{01}|^2 - |q^{02}|^2 + |q^{03}|^2. \qquad (4.14)$$

Introducing real variables η^A, $A = 0,1,\ldots,5$ by

$$q^{01} = \eta^2 + i\eta^3, \quad q^{02} = \eta^4 + i\eta^5, \quad q^{03} = \eta^0 + i\eta^1, \qquad (4.15)$$

puts $Q(\eta)$ in the form

$$Q(\eta) = G_{AB}\,\eta^A\,\eta^B, \qquad A,B, = 0,1,\ldots,5, \qquad (4.16)$$

with $$G_{AB} = \mathrm{diag}\,(1,1,-1,-1,-1,-1). \qquad (4.17)$$

We have shown that any transformation of SU(2,2) on Z induces on $\hat{Z}^{(2)}$ a transformation that leaves invariant the sub-space Λ, and maps elements q of Λ in such a way that the real variables η^A defined in (4.15) transform under O(2,4). In the following section we shall display the matrices of SU(2,2) that correspond to conformal transformations of particular physical importance.

4.5 Finite transformations of SU(2,2)

We now label infinitesimal operators in SU(2,2) by the X_A, $A = 0,1,\ldots,14$, used in the work on the conformal group C in Chapter 3. This should not lead to confusion at this stage, and has the advantage that the physical interpretation (which is easily made for the operators of C) is carried over into SU(2,2). Combining results from Sections 3.3, 3.4 and 3.5 of Chapter 3, we write

$$\left.\begin{aligned} X_0 &= -\tfrac{1}{2}\,(\sigma_3 + i\sigma_2) \times \sigma_3 \\ X_1 &= \tfrac{1}{2}\,(\sigma_3 + i\sigma_2) \times \sigma_0 \\ X_2 &= -\tfrac{1}{2}\,(\sigma_2 - i\sigma_3) \times \sigma_1 \\ X_3 &= -\tfrac{1}{2}\,(\sigma_2 - i\sigma_3) \times \sigma_2 \end{aligned}\right\} (5.1) \qquad \left.\begin{aligned} X_4 &= \tfrac{i}{2}\,\sigma_1 \times \sigma_1 \\ X_5 &= \tfrac{i}{2}\,\sigma_1 \times \sigma_2 \\ X_6 &= \tfrac{i}{2}\,\sigma_0 \times \sigma_3 \end{aligned}\right\} (5.2),$$

$$\left.\begin{aligned} X_7 &= \tfrac{1}{2}\,\sigma_1 \times \sigma_3 \\ X_8 &= -\tfrac{1}{2}\,\sigma_0 \times \sigma_2 \\ X_9 &= \tfrac{1}{2}\,\sigma_0 \times \sigma_1 \end{aligned}\right\} \quad (5.3), \qquad X_{10} = -\tfrac{1}{2}\,\sigma_1 \times \sigma_0 \quad (5.4),$$

$$\left.\begin{aligned} X_{11} &= \tfrac{1}{2}\,(\sigma_3 - i\,\sigma_2) \times \sigma_3 \\ X_{12} &= -\tfrac{1}{2}\,(\sigma_3 - i\,\sigma_2) \times \sigma_0 \\ X_{13} &= \tfrac{1}{2}\,(\sigma_2 + i\,\sigma_3) \times \sigma_1 \\ X_{14} &= \tfrac{1}{2}\,(\sigma_2 + i\,\sigma_3) \times \sigma_2 \end{aligned}\right\} \quad (5.5)$$

,

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

σ_0 is the 2x2 identity matrix and x denotes Kronecker product. We recall that (5.1) generate translations, (5.2) and (5.3) generate spatial rotations and Lorentz rotations respectively, (5.4) generates dilatations, and (5.5) generate the special conformal transformations.

It is straightforward to show that

$$X_0^2 = X_1^2 = X_2^2 = X_3^2 = X_{11}^2 = X_{12}^2 = X_{13}^2 = X_{14}^2 = 0, \qquad (5.6)$$

and $$X_4^2 = X_5^2 = X_6^2 = -X_7^2 = -X_8^2 = -X_9^2 = -X_{10}^2 = -\tfrac{1}{4} I_4,$$

where I_4 is the 4x4 identity matrix. These properties of the infinitesimal operators of SU(2,2) make it a simple matter to calculate the operators of finite transformations. We have already (in §4 of this chapter) written down the elementary operators of SU(2,2); the present approach however, yields directly the operators of SU(2,2) corresponding to the physical aspects of the conformal transformations. Without giving the complete list, we cite the following typical cases:-

(i) the translation operators are given by

$$Y_0 = \exp(2\epsilon X_0) = \begin{pmatrix} 1-\epsilon & 0 & \epsilon & 0 \\ 0 & 1+\epsilon & 0 & -\epsilon \\ -\epsilon & 0 & 1+\epsilon & 0 \\ 0 & \epsilon & 0 & 1-\epsilon \end{pmatrix} ; \; Y_1 = \exp(2\epsilon X_1) = \begin{pmatrix} 1+\epsilon & 0 & -\epsilon & 0 \\ 0 & 1+\epsilon & 0 & -\epsilon \\ \epsilon & 0 & 1-\epsilon & 0 \\ 0 & \epsilon & 0 & 1-\epsilon \end{pmatrix}$$

$$Y_2 = \exp(2\epsilon X_2) = \begin{pmatrix} 1 & i\epsilon & 0 & -i\epsilon \\ i\epsilon & 1 & -i\epsilon & 0 \\ 0 & i\epsilon & 1 & -i\epsilon \\ i\epsilon & 0 & -i\epsilon & 1 \end{pmatrix} ; \; Y_3 = \exp(2\epsilon X_3) = \begin{pmatrix} 1 & -\epsilon & 0 & \epsilon \\ \epsilon & 1 & -\epsilon & 0 \\ 0 & -\epsilon & 1 & \epsilon \\ \epsilon & 0 & -\epsilon & 1 \end{pmatrix} .$$

(ii) a typical spatial rotation operator is given by

$$Y_4 = \exp(2\epsilon X_4) = \begin{pmatrix} j & 0 & 0 & ik \\ 0 & j & ik & 0 \\ 0 & ik & j & 0 \\ ik & 0 & 0 & j \end{pmatrix} , \quad \text{where } \begin{matrix} j = \cos\epsilon \\ k = \sin\epsilon \end{matrix} ;$$

(iii) a typical Lorentz rotation is given by

$$Y_7 = \exp(2\epsilon X_7) = \begin{pmatrix} c & 0 & s & 0 \\ 0 & c & 0 & -s \\ s & 0 & c & 0 \\ 0 & -s & 0 & c \end{pmatrix} , \quad \text{where } \begin{matrix} c = \cosh\epsilon \\ s = \sinh\epsilon \end{matrix} ;$$

(iv) the dilatation operator is

$$Y_{10} = \exp(2\epsilon X_{10}) = \begin{pmatrix} c & 0 & -s & 0 \\ 0 & c & 0 & -s \\ -s & 0 & c & 0 \\ 0 & -s & 0 & c \end{pmatrix} , \quad \text{where } \begin{matrix} c = \cosh\epsilon \\ s = \sinh\epsilon \end{matrix} .$$

(v) To exhibit the operators of the special conformal transformations we make use of the relations

$$X_{11} = -X_0^{\dagger}, \; X_{12} = -X_1^{\dagger}, \; X_{13} = -X_2^{\dagger}, \; X_{14} = -X_3^{\dagger},$$

which are easily derived from (5.1). It follows that

$$Y_{11} = (Y_0{}^{\dagger})^{-1}, \; Y_{12} = (Y_1{}^{\dagger})^{-1}, \; Y_{13} = (Y_2{}^{\dagger})^{-1}, \; Y_{14} = (Y_3{}^{\dagger})^{-1},$$

and Y_0, Y_1, Y_2, Y_3 have been given above. The calculations make use of the matrix identities $(A \times B)^{\dagger} = A^{\dagger} \times B^{\dagger}$, $\exp(A^{\dagger}) = (\exp A)^{\dagger}$ and

$$\exp(-A) = (\exp A)^{-1}.$$

It is instructive here to establish contact with the formalism of Penrose in his work on twistor algebra [9]. We have taken

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

as the matrix of the invariant hermitian form of SU(2,2), whereas Penrose makes the choice

$$G_P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The unitary matrix

$$P = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad (5.7)$$

serves to transform G into G_p according to

$$G_p = P^{\dagger} G P.$$

This is equivalent to the change of basis

$$Z \longrightarrow P^{\dagger} Z$$

in the representation space. The matrix operators Y_A that we have introduced above assume the forms given by Penrose if we transform according to

$$Y_A \longrightarrow P^{\dagger} Y_A P.$$

CHAPTER 5

ASYMPTOTICALLY FLAT SPACE-TIMES - THE KILLING VECTORS AND CONFORMAL KILLING VECTORS

5.1 Introduction.

The work of this chapter uses the asymptotic expansion technique first developed by Newman and Unti [31] for the study of asymptotically flat space-times in general relativity. An important aspect of their work concerned the asymptotic Killing symmetries possessed by metrics of this type, where it was found that the inhomogeneous Lorentz group did not emerge as the asymptotic symmetry group. Instead, a larger symmetry group, consisting of the inhomogeneous Lorentz group augmented by the addition of the infinite set of so-called "super-translations", was discovered. This group has become known as the B-M-S (Bondi-Metzner-Sachs) group or G.B.M. (generalised Bondi-Metzner) group.

The present work extends the previous considerations to an analysis of asymptotic conformal Killing symmetries. The appropriate equations are set down and although the most general case is not solved here (see, however, Chapter 6 for a fuller discussion), we obtain an interesting result (believed to be new) for metrics which are asymptotically shear-free.

5.2 Asymptotically flat space-times; the tensor density $\mathfrak{g}_{\mu\nu}$

The components of the contravariant metric tensor have been given in [31]. Our calculation requires the covariant components which are given below:-

$$g_{11} = -a_0 - a_1 r^{-1} + 0(r^{-2}) , \qquad g_{12} = 1 ,$$

$$g_{1A} = \frac{b_2{}^A}{2P^2} + \frac{r^{-1}}{4P^4}(2P^2 b_3{}^A + b_2{}^B d_3{}^{AB}) + 0(r^{-2}) ,$$

$$g_{22} = 0, \quad g_{2A} = 0 .$$

(A,B = 3,4 and summation over repeated suffices is to be understood)

$$g_{33} = \frac{-r^2}{2P^2} - \frac{r}{4P^4} d_3^{33} - \frac{1}{8P^6}\left\{2P^2 d_4^{33} - \det(d_3^{AB})\right\}$$
$$- \frac{r^{-1}}{16P^8}\left\{4P^4 d_5^{33} + 4P^2 d_3^{33} d_4^{33} - d_3^{33} \det(d_3^{AB})\right\} + O(r^{-2}) ,$$

$$g_{34} = \frac{-r}{4P^4} d_3^{34} + O(1) ,$$

$$g_{44} = \frac{-r^2}{2P^2} - \frac{r}{4P^4} d_3^{44} - \frac{1}{8P^6}\left\{2P^2 d_4^{44} - \det(d_3^{AB})\right\}$$
$$- \frac{r^{-1}}{16P^8}\left\{4P^4 d_5^{44} + 4P^2 d_3^{44} d_4^{44} - d_3^{44} \det(d_3^{AB})\right\} + O(r^{-2}) .$$

It follows that

$$[-\det(g_{\mu\nu})]^{-\frac{1}{4}} = \sqrt{2}\, Pr^{-1} - \frac{\sqrt{2}}{16P^3} r^{-3}\left\{2P^2(d_4^{33} + d_4^{44}) - \det(d_3^{AB})\right\}$$
$$- \frac{\sqrt{2}}{16P^3} r^{-4}\left\{2P^2(d_5^{33} + d_5^{44}) - (d_3^{33} d_4^{44} + d_4^{33} d_3^{44})\right\} + O(r^{-5}).$$

In the above work we have adopted the notation used by Newman and Unti:-

$$a_0 = -2P^2\left[\frac{\partial^2}{\partial x^{3\,2}} + \frac{\partial^2}{\partial x^{4\,2}}\right] \log P , \qquad a_1 = -2 \,\mathrm{Re}\, \Psi_2^{\circ} ;$$

$$b_2^{\,3} = -\mathrm{Re}\left\{2P^4 \nabla(\bar{\sigma}^{\circ}/P^2)\right\} , \qquad b_2^{\,4} = \mathrm{Im}\left\{2P^4 \nabla(\bar{\sigma}^{\circ}/P^2)\right\} ,$$

$$b_3^{\,3} = \mathrm{Re}\left\{4P\left[\tfrac{1}{3}\Psi_1^{\circ} + P^3 \sigma^{\circ} \nabla(\bar{\sigma}^{\circ}/P^2)\right]\right\} , \qquad b_3^{\,4} = \mathrm{Im}\left\{4P\left[\tfrac{1}{3}\Psi_1^{\circ} + P^3\sigma^{\circ}\nabla(\bar{\sigma}^{\circ}/P^2)\right]\right\} ,$$

$$d_3^{33} = -d_3^{44} = 2P^2(\sigma^{\circ} + \bar{\sigma}^{\circ}) , \qquad d_3^{34} = -2iP^2(\sigma^{\circ} - \bar{\sigma}^{\circ}) ,$$
$$d_4^{33} = d_4^{44} = -6P^2\sigma^{\circ}\bar{\sigma}^{\circ} , \qquad d_4^{34} = 0.$$

The coefficients d_5^{AB} were not calculated in [31], but they appear in a recent paper by Unti [52], where a computer program is used to extend the original Newman and Unti calculations to higher order terms.

The components of the covariant tensor density

$$\mathfrak{g}_{\mu\nu} \equiv [-\det(g_{\alpha\beta})]^{-\frac{1}{4}} g_{\mu\nu}$$

are found to be:-

$$\mathfrak{g}_{11} = -a_0 P\sqrt{2}\, r^{-1} - a_1 P\sqrt{2}\, r^{-2} + O(r^{-3}) , \qquad \mathfrak{g}_{12} = \sqrt{2}\, Pr^{-1} + O(r^{-3}) ,$$

$$\mathfrak{g}_{1A} = \frac{b_2^{\,A}}{\sqrt{2}P} r^{-1} + \frac{r^{-2}}{2\sqrt{2}P^2}(2P^2 b_3^{\,A} + b_2^{\,B} d_3^{AB}) + O(r^{-3}) ,$$

$$\mathfrak{g}_{22} = \mathfrak{g}_{2A} = 0 ,$$

$$g_{33} = \frac{-r}{\sqrt{2}P} - \frac{d_3^{33}}{2\sqrt{2}P^3} + \frac{\sqrt{2}}{32P^5} V_1^{33} r^{-1} + \frac{\sqrt{2}}{64P^7} V_2^{33} r^{-2} + O(r^{-3}) \quad ,$$

$$g_{34} = -\frac{d_3^{34}}{2\sqrt{2}P^3} + O(r^{-2}) \quad ,$$

$$g_{44} = \frac{-r}{\sqrt{2}P} - \frac{d_3^{44}}{2\sqrt{2}P^3} + \frac{\sqrt{2}}{32P^5} V_1^{44} r^{-1} + \frac{\sqrt{2}}{64P^7} V_2^{44} r^{-2} + O(r^{-3}) \quad ,$$

where $V_1^{AA} = 3\det(d_3^{CD}) - 4P^2 d_4^{AA}$

and $V_2^{AA} = 3d_3^{AA}\det(d_3^{CD}) - 12P^2 d_3^{AA} d_4^{AA} + 4P^4(d_5^{33} + d_5^{44}) - 16P^4 d_5^{AA}$

(no summation over A).

5.3 The equations for asymptotic C-K vectors

The Lie derivative of $g_{\mu\nu}$ with respect to ξ^μ is

$$\mathcal{L}_\xi g_{\mu\nu} \equiv g_{\mu\nu,\alpha}\xi^\alpha + g_{\alpha\nu}\xi^\alpha_{,\mu} + g_{\mu\alpha}\xi^\alpha_{,\nu} - \tfrac{1}{2} g_{\mu\nu}\xi^\alpha_{,\alpha} \tag{3.1}$$

and the conformal Killing equations are ([2], and Chapter 2)

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \ . \tag{3.2}$$

Reference to the form of the $g_{\mu\nu}$ components suggests the following set of equations for the asymptotic C-K vectors in asymptotically flat space:-

$$\mathcal{L}_\xi g_{11} = O(r^{-1}) \ , \qquad \mathcal{L}_\xi g_{1A} = O(r^{-1}) \ , \tag{3.3}$$

$$\mathcal{L}_\xi g_{12} = O(r^{-1}) \ , \quad \mathcal{L}_\xi g_{22} = 0 \ , \quad \mathcal{L}_\xi g_{2A} = 0 \ , \tag{3.4}$$

$$\mathcal{L}_\xi g_{34} = O(1) \ , \tag{3.5}$$

together with equations derived by comparing coefficients in

$$\left.\begin{aligned} \mathcal{L}_\xi g_{33} &= Qr + R + O(r^{-1}) \ , \\ \text{and } \mathcal{L}_\xi g_{44} &= Qr - R + O(r^{-1}) \ . \end{aligned}\right\} \tag{3.6}$$

We can in fact impose a stronger condition on $\mathcal{L}_\xi g_{12}$ and in what follows we replace the above equation by

$$\mathcal{L}_\xi g_{12} = O(r^{-2}) \ . \tag{3.4'}$$

Our method generalises that of Newman and Unti [31], in which they consider the conditions necessary to preserve the asymptotic form of the metric components in asymptotically flat spaces.

First, we can solve the "radial" equations (3.4), (3.4').

$\mathcal{L}_{\xi} g_{22} = 0$ leads immediately to

$$\xi^1 = \overset{\circ}{\xi}{}^1(u,x^A) \quad , \tag{3.7}$$

and the two equations $\mathcal{L}_{\xi} g_{2A} = 0$ give, after a little manipulation,

$$\xi^3 = \overset{\circ}{\xi}{}^3 - 2P^2 \overset{\circ}{\xi}{}^1_{,3}\, r^{-1} - \tfrac{1}{2} r^{-2}\left[d_3^{44} \overset{\circ}{\xi}{}^1_{,3} - d_3^{34} \overset{\circ}{\xi}{}^1_{,4}\right] + O(r^{-3}) \tag{3.8}$$

$$\xi^4 = \overset{\circ}{\xi}{}^4 - 2P^2 \overset{\circ}{\xi}{}^1_{,4}\, r^{-1} - \tfrac{1}{2} r^{-2}\left[d_3^{33} \overset{\circ}{\xi}{}^1_{,4} - d_3^{34} \overset{\circ}{\xi}{}^1_{,3}\right] + O(r^{-3}) . \tag{3.9}$$

Here and in what follows, the $^{\circ}$ indicates a function independent of r (it is omitted on P, d^{AB} etc. which are obviously independent of r).

Assuming that ξ^2 admits of expansion in positive and negative powers of r and then using $\mathcal{L}_{\xi} g_{12} = O(r^{-2})$ leads to a solution for ξ^2 of the form

$$\xi^2 = \overset{\circ}{\xi}{}^2 r^2 - Ar - \tfrac{1}{2}B + Cr^{-1} + O(r^{-2}) \tag{3.10}$$

where A satisfies

$$A = -\overset{\circ}{\xi}{}^1_{,1} + \overset{\circ}{\xi}{}^3_{,3} + \overset{\circ}{\xi}{}^4_{,4} - \frac{2}{P} P_{,3} \overset{\circ}{\xi}{}^3 - \frac{2}{P} P_{,4} \overset{\circ}{\xi}{}^4 . \tag{3.11}$$

The coefficients $\overset{\circ}{\xi}{}^2$, A, B are of course not the same functions as in flat space; we use the same notation here as in Chapter 3 so that direct comparison is possible.

The remaining Lie derivative equations put restrictions on the coefficients in the expansions (3.7) - (3.10). We use the equations (3.3) first.

$\mathcal{L}_{\xi} g_{11} = O(r^{-1})$ leads to two conditions

$$\overset{\circ}{\xi}{}^2_{,1} = 0 \quad , \tag{3.12}$$

$$\overset{\circ}{\xi}{}^2 = \frac{1}{a_o} A_{,1} \quad . \tag{3.13}$$

Then $\mathcal{L}_{\xi} g_{13} = O(r^{-1})$ gives

$$2P^2 \overset{\circ}{\xi}{}^2_{,3} - \overset{\circ}{\xi}{}^3_{,1} = 0 \quad , \tag{3.14}$$

$$\overset{\circ}{\xi}{}^1_{,3,1} - A_{,3} - \frac{b_2^3}{P^2} \overset{\circ}{\xi}{}^2 - \frac{1}{4P^4}\left[d_3^{33} \overset{\circ}{\xi}{}^3_{,1} + d_3^{34} \overset{\circ}{\xi}{}^4_{,1}\right] = 0, \tag{3.15}$$

and in a similar way $\mathcal{L}_{\xi} g_{14} = O(r^{-1})$ gives

$$2P^2 \overset{\circ}{\xi}{}^2_{,4} - \overset{\circ}{\xi}{}^4_{,1} = 0 \quad , \tag{3.16}$$

$$\overset{\circ}{\xi}{}^1_{,4,1} - A_{,4} - \frac{b_2^4 \overset{\circ}{\xi}{}^2}{P^2} - \frac{1}{4P^4}\left[d_3^{34} \overset{\circ}{\xi}{}^3_{,1} + d_3^{44} \overset{\circ}{\xi}{}^4_{,1}\right] = 0 . \tag{3.17}$$

(3.5) gives the condition

$$\overset{\circ}{\xi}{}^3_{,4} + \overset{\circ}{\xi}{}^4_{,3} = \frac{d_3^{34}}{2P^2} \overset{\circ}{\xi}{}^2 . \tag{3.18}$$

Finally, we use the two equations of (3.6). The calculations are somewhat tedious but comparing coefficients of the various powers of r in $\mathcal{L}_{\underline{\xi}} G_{33}$ and $\mathcal{L}_{\underline{\xi}} G_{44}$ leads to

$$\overset{\circ}{\xi}{}^{3}{}_{,3} - \overset{\circ}{\xi}{}^{4}{}_{,4} = \frac{d_3{}^{33}}{2P^2} \overset{\circ}{\xi}{}^{2} \quad , \tag{3.19}$$

$$\frac{B}{\sqrt{2}P} - \frac{\sqrt{2}}{8P^5} V_1{}^{33} \overset{\circ}{\xi}{}^{2} + \sqrt{2}\, P\left[\overset{\circ}{\xi}{}^{1}{}_{,3,3} + \overset{\circ}{\xi}{}^{1}{}_{,4,4}\right] - \frac{d_3{}^{33}}{\sqrt{2}P^3} (\overset{\circ}{\xi}{}^{3}{}_{,3} - \overset{\circ}{\xi}{}^{4}{}_{,4})$$
$$- \frac{d_3{}^{34}}{\sqrt{2}P^3} (\overset{\circ}{\xi}{}^{4}{}_{,3} + \overset{\circ}{\xi}{}^{3}{}_{,4}) = 0 \, . \tag{3.20}$$

(3.19) comes from comparing coefficients of r; (3.20) comes from the coefficients of r^0. (3.20) can be written in a more convenient form by substitution from (3.18), (3.19).

Then we have

$$\frac{B}{\sqrt{2}P} + \sqrt{2}P(\overset{\circ}{\xi}{}^{1}{}_{,3,3} + \overset{\circ}{\xi}{}^{1}{}_{,4,4}) - \frac{\overset{\circ}{\xi}{}^{2}}{4\sqrt{2}P^5}\left[\det(d_3{}^{AB}) - 4P^2 d_4{}^{33}\right] = 0 \, . \tag{3.21}$$

5.4 Deductions from the equations

The work of Newman and Unti on asymptotic Killing symmetries has shown that the asymptotic symmetry group in asymptotically flat space-times is a "kinematic" structure, in the sense that it is independent of the nature of the gravitational field persisting in the space. However, this observation does not hold with respect to the asymptotic conformal symmetries. We see from the equations of §3 that the gravitational field influences the conformal symmetries via the asymptotic shear σ^0 which appears in $d_3{}^{AB}$ and $d_4{}^{AB}$. It is for this reason that we restrict our attention in this chapter to the case $\sigma^0 = 0$, for which the conformal symmetry group reassumes its "kinematic" status. There is one general result which may be stated here, since it follows in a straightforward way from the above equations; a necessary condition for the existence of asymptotic conformal symmetries is that

$$\frac{\partial^2}{\partial u^2}\sigma^0 = 0 \, .$$

5.5 A particular case; the asymptotic Killing vectors

The equations for C.K. vectors

$$\mathcal{L}_{\underline{\xi}} g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \xi^{\alpha}_{;\alpha} = 0 \tag{5.1}$$

reduce to Killing's equations when the additional restriction

$$\xi^{\alpha}_{;\alpha} = 0 \tag{5.2}$$

is imposed. We shall show that the inclusion of this restriction in our analysis leads to the results on asymptotic motions given in [31] .

We have shown previously that

$$\mathcal{L}_{\underline{\xi}} g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \xi^{\alpha}_{;\alpha} = (-g)^{\frac{1}{4}} \mathcal{L}_{\underline{\xi}} \mathcal{G}_{\mu\nu} \tag{5.3}$$

where $\mathcal{G}_{\mu\nu} = (-g)^{-\frac{1}{4}} g_{\mu\nu}$

and $g = \det (g_{\mu\nu})$.

When $\xi^{\alpha}_{;\alpha} = 0$ (5.3) becomes

$$\mathcal{L}_{\underline{\xi}} g_{\mu\nu} = (-g)^{\frac{1}{4}} \mathcal{L}_{\underline{\xi}} \mathcal{G}_{\mu\nu} \tag{5.4}$$

Suppose for the moment that $\mathcal{L}_{\underline{\xi}} g_{\alpha\beta} \neq 0$, where we refer now to the particular (α,β) component of the tensor equation.

Then if the component $g_{\alpha\beta}$ satisfies

$$g_{\alpha\beta} = 0(r^{n})$$

we have $\mathcal{G}_{\alpha\beta} = 0(r^{n-1})$, using $(-g)^{-\frac{1}{4}} = 0(r^{-1})$.

The relevant Lie derivative equation in our analysis would be

$$\mathcal{L}_{\underline{\xi}} \mathcal{G}_{\alpha\beta} = 0 \;(r^{n-1})$$

from which (5.4) gives

$$\mathcal{L}_{\underline{\xi}} g_{\alpha\beta} = 0(r^{n}) \quad . \tag{5.5}$$

(5.5) is just the corresponding Newman-Unti equation.

On the other hand, if the (α,β) component of the tensor equation is $\mathcal{L}_{\underline{\xi}} g_{\alpha\beta} = 0$ it is clear (again from (5.4)) that we must take $\mathcal{L}_{\underline{\xi}} \mathcal{G}_{\alpha\beta} = 0$ in order to extract the Newman-Unti result.

In our case then, all the equations of § 5.3 may be adopted except that

$$\mathcal{L}_{\underline{\xi}} \mathcal{G}_{12} = 0 \tag{5.6}$$

must replace $\mathcal{L}_{\underline{\xi}} \mathcal{G}_{12} = 0(r^{-2})$. The effect of this is to leave (3.10) and (3.11) unaltered and to give the extra condition

$$B + 2P^{2}(\overset{0}{\xi}{}^{1}_{,3,3} + \overset{0}{\xi}{}^{1}_{,4,4}) = 0 \; . \tag{5.7}$$

Using (5.2), which sets the conformal factor equal to zero, gives rise to three more conditions. We calculate $\xi^{\alpha}{}_{;\alpha}$ from

$$\xi^{\alpha}{}_{;\alpha} = \xi^{\alpha}{}_{,\alpha} + \frac{1}{2}(-g)^{-1}\xi^{\alpha}\frac{\partial}{\partial x^{\alpha}}(-g)$$

and find that

$$\xi^{\alpha}{}_{;\alpha} = 4r\overset{\circ}{\xi}{}^{2} + \left\{\overset{\circ}{\xi}{}^{1}{}_{,1} + \overset{\circ}{\xi}{}^{3}{}_{,3} + \overset{\circ}{\xi}{}^{4}{}_{,4} - 3A - \frac{2}{P}\left[\overset{\circ}{\xi}{}^{3}P_{,3} + \overset{\circ}{\xi}{}^{4}P_{,4}\right]\right\}$$
$$+ r^{-1}\left\{-B + \frac{\overset{\circ}{\xi}{}^{2}}{4P^{4}}\left[\det(d_{3}{}^{AB}) - 4P^{2}d_{4}{}^{33}\right] - 2P^{2}\left[\overset{\circ}{\xi}{}^{1}{}_{,3,3} + \overset{\circ}{\xi}{}^{1}{}_{,4,4}\right]\right\} + O(r^{-2}).$$

Then (5.2) yields

$$\overset{\circ}{\xi}{}^{2} = 0 \quad , \tag{5.8}$$

$$\overset{\circ}{\xi}{}^{1}{}_{,1} + \overset{\circ}{\xi}{}^{3}{}_{,3} + \overset{\circ}{\xi}{}^{4}{}_{,4} - 3A - \frac{2}{P}\left[\overset{\circ}{\xi}{}^{3}P_{,3} + \overset{\circ}{\xi}{}^{4}P_{,4}\right] = 0 \quad , \tag{5.9}$$

and $$\frac{B}{\sqrt{2}P} + \sqrt{2}\,P\left[\overset{\circ}{\xi}{}^{1}{}_{,3,3} + \overset{\circ}{\xi}{}^{1}{}_{,4,4}\right] - \frac{\overset{\circ}{\xi}{}^{2}}{4\sqrt{2}P^{5}}\left[\det(d_{3}{}^{AB}) - 4P^{2}d_{4}{}^{33}\right] = 0\,. \tag{5.10}$$

This last equation is not new; it is just (3.21), obtained previously from consideration of $\mathcal{L}_{\xi}g_{33}$ and $\mathcal{L}_{\xi}g_{44}$.

We can now exhibit the Newman-Unti equations as a particular case in our analysis. Using (5.8) in (3.13) - (3.20) gives

$$A_{,1} = 0 \quad , \tag{5.11}$$

$$\overset{\circ}{\xi}{}^{3}{}_{,1} = \overset{\circ}{\xi}{}^{4}{}_{,1} = 0 \quad , \tag{5.12}$$

$$\overset{\circ}{\xi}{}^{1}{}_{,3,1} - A_{,3} = 0 \quad , \tag{5.13}$$

$$\overset{\circ}{\xi}{}^{1}{}_{,4,1} - A_{,4} = 0 \quad , \tag{5.14}$$

$$\overset{\circ}{\xi}{}^{3}{}_{,3} = \overset{\circ}{\xi}{}^{4}{}_{,4} \quad , \tag{5.15}$$

$$\overset{\circ}{\xi}{}^{3}{}_{,4} = -\overset{\circ}{\xi}{}^{4}{}_{,3} \quad , \qquad (5.15)$$

$$B = -2P^{2}\left[\overset{\circ}{\xi}{}^{1}{}_{,3,3} + \overset{\circ}{\xi}{}^{1}{}_{,4,4}\right] \,. \tag{5.17}$$

(3.11) and (5.9) give

$$\overset{\circ}{\xi}{}^{1}{}_{,1} = A \quad , \tag{5.18}$$

and consequently (5.13), (5.14) are identically satisfied. We are led to asymptotic Killing vectors of the Newman-Unti form:-

$$\xi^{1} = \overset{\circ}{\xi}{}^{1}(u,x^{A}) \tag{5.19}$$

$$\xi^{2} = -\overset{\circ}{\xi}{}^{1}{}_{,1}\, r + P^{2}\left[\overset{\circ}{\xi}{}^{1}{}_{,3,3} + \overset{\circ}{\xi}{}^{1}{}_{,4,4}\right] + O(r^{-1}) \tag{5.20}$$

$$\xi^{3} = \overset{\circ}{\xi}{}^{3}(x^{A}) - 2r^{-1}P^{2}\overset{\circ}{\xi}{}^{1}{}_{,3} + O(r^{-2}) \tag{5.21}$$

$$\xi^{4} = \overset{\circ}{\xi}{}^{4}(x^{A}) - 2r^{-1}P^{2}\overset{\circ}{\xi}{}^{1}{}_{,4} + O(r^{-2}) \quad , \tag{5.22}$$

where

$$\overset{\circ}{\xi}{}^{3}{}_{,3} = \overset{\circ}{\xi}{}^{4}{}_{,4} \tag{5.23}$$

$$\overset{\circ}{\xi}{}^{3}{}_{,4} = -\overset{\circ}{\xi}{}^{4}{}_{,3} \tag{5.24}$$

$$\overset{\circ}{\xi}{}^{1}{}_{,1,1} = 0 \quad . \tag{5.25}$$

(5.23), (5.24) reflect the conformal flatness of the leading terms in g_{AB} and (5.25) prevents the reappearance of an $O(r)$ term in the transformed g (Recall that the present treatment, unlike that of Newman and Unti, exploits coordinate freedom to make $g^{22} = O(1)$ before the calculation of the symmetries).

5.6 Asymptotically shear free metrics - their asymptotic Killing vectors

In this and the following section we consider metrics for which $\sigma^0 = 0$; i.e. the asymptotic shear vanishes. We make the specific choice $P = \frac{1}{\sqrt{2}} \cosh x^3$, which corresponds to choosing coordinates (x^3, x^4) so that the metric g_{AB} is asymptotically the metric of a 2-sphere. This facilitates comparison with the work of Chapter 3. One of our principal aims is to see what symmetries in the asymptotically-flat spaces are inherited from Minkowski space.

We exhibit the Killing vectors first since a knowledge of these will assist in the interpretation of the conformal Killing vectors derived in 5.7.

The equations (5.19) - (5.25) become

$$\xi^1 = \overset{\circ}{\xi}{}^{1}(u,x^A) \tag{6.1}$$

$$\xi^2 = -\overset{\circ}{\xi}{}^{1}{}_{,1}\, r + \frac{1}{2}\cosh^2 x^3 \left[\overset{\circ}{\xi}{}^{1}{}_{,3,3} + \overset{\circ}{\xi}{}^{1}{}_{,4,4} \right] + O(r^{-1}) \tag{6.2}$$

$$\xi^3 = \overset{\circ}{\xi}{}^{3}(x^A) - r^{-1}\cosh^2 x^3\, \overset{\circ}{\xi}{}^{1}{}_{,3} + O(r^{-2}) \tag{6.3}$$

$$\xi^4 = \overset{\circ}{\xi}{}^{4}(x^A) - r^{-1}\cosh^2 x^3\, \overset{\circ}{\xi}{}^{1}{}_{,4} + O(r^{-2}) \quad , \tag{6.4}$$

where

$$\overset{\circ}{\xi}{}^{3}{}_{,3} = \overset{\circ}{\xi}{}^{4}{}_{,4} \tag{6.5}$$

$$\overset{\circ}{\xi}{}^{3}{}_{,4} = -\overset{\circ}{\xi}{}^{4}{}_{,3} \tag{6.6}$$

$$\overset{\circ}{\xi}{}^{1}{}_{,1,1} = 0 \ . \tag{6.7}$$

Further, when $\sigma^0 = 0$ we have $g^{22} = -1 + O(r^{-1})$, so the $O(1)$ term of $\mathcal{L}_\xi\, g^{22}$ must vanish. This leads to

$$-B_{,1} + 2\overset{\circ}{\xi}{}^{1}{}_{,1} = 0 \tag{6.8}$$

where, as usual, $B = -\cosh^2 x^3 \left[\overset{\circ}{\xi}{}^{1}{}_{,3,3} + \overset{\circ}{\xi}{}^{1}{}_{,4,4} \right]$.

Substitution of (5.18) into (5.9) gives

$$\overset{\circ}{\xi}{}^{3}{}_{,3} + \overset{\circ}{\xi}{}^{4}{}_{,4} - 2\,\overset{\circ}{\xi}{}^{1}{}_{,1} - 2\,\overset{\circ}{\xi}{}^{3}\tanh x^3 = 0 . \qquad (6.9)$$

To solve the above equations we put

$$\overset{\circ}{\xi}{}^{1} = u\,\alpha(x^A) + K(x^A) \quad , \qquad (6.10)$$

so that (6.7) is satisfied and

$$\overset{\circ}{\xi}{}^{1}{}_{,1} = \alpha(x^A) \ .$$

(6.8) becomes an equation for $\alpha(x^A)$ of the form

$$\cosh^2 x^3 \left[\alpha_{,3,3} + \alpha_{,4,4}\right] = -2\alpha \quad , \qquad (6.11)$$

which has solution

$$\alpha(x^A) = \sum_m \alpha_m \ {}_0Y_{1m}(x^A) \quad ; \quad \alpha_m \ \text{constants} .$$

(see Chapter 2 for details of the properties of the ${}_sY_{\ell m}$)

We put

$$\alpha(x^A) = b\sin x^4 \operatorname{sech} x^3 + e\cos x^4 \operatorname{sech} x^3 + h\tanh x^3 \ , \qquad (6.12)$$

where the naming of coefficients here will make the connection with the work of Chapter 3 obvious.

Furthermore, we can expand $K(x^A)$ in the form

$$K(x^A) = m + c\sin x^4 \operatorname{sech} x^3 + f\cos x^4 \operatorname{sech} x^3 + j\tanh x^3 + \tau(x^A)$$

where $\quad \tau(x^A) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{+\ell} \tau_{\ell m} Y_{\ell m}(x^A)$

and $\tau_{\ell m}$ are constants.

We see then that ξ^1 and ξ^2 may be written as

$$\xi^1 = \hat{\xi}^1 + \tau(x^A)$$

$$\xi^2 = \hat{\xi}^2 + \tfrac{1}{2}\cosh^2 x^3 \left(\tau_{,3,3} + \tau_{,4,4}\right) + O(r^{-1}) \ ,$$

where $\hat{\xi}^1$, $\hat{\xi}^2$ are components of the Minkowski space Killing vectors given in Chapter 3.

Using (6.5) and (6.9) leads to the following equation for $\overset{\circ}{\xi}{}^{3}$:

$$\overset{\circ}{\xi}{}^{3}{}_{,3} - \overset{\circ}{\xi}{}^{3}\tanh x^3 = \overset{\circ}{\xi}{}^{1}{}_{,1} \quad ,$$

which has solution

$$\overset{\circ}{\xi}{}^{3} = b\sin x^4 \sinh x^3 + e\cos x^4 \sinh x^3 - h + \lambda(x^4)\cosh x^3 \quad ,$$

where $\lambda(x^4)$ is a function of x^4 only.

Using (6.5) and (6.6) then gives

$$\overset{\circ}{\xi}{}^3 = (b \sin x^4 + e \cos x^4) \sinh x^3 + (n \cos x^4 - p \sin x^4) \cosh x^3 - h\,, \qquad (6.13)$$

$$\overset{\circ}{\xi}{}^4 = (-b \cos x^4 + e \sin x^4) \cosh x^3 + (n \sin x^4 + p \cos x^4) \sinh x^3 + q. \qquad (6.14)$$

It follows that our complete solution is

$$\xi^1 = \hat{\xi}^1 + \tau(x^A) \quad , \qquad (6.15)$$

$$\xi^2 = \hat{\xi}^2 + \frac{1}{2} \cosh^2 x^3 \; (\tau_{,3,3} + \tau_{,4,4}) + O(r^{-1}), \qquad (6.16)$$

$$\xi^3 = \hat{\xi}^3 - r^{-1} \cosh^2 x^3 \; \tau_{,3} + O(r^{-2}), \qquad (6.17)$$

$$\xi^4 = \hat{\xi}^4 - r^{-1} \cosh^2 x^3 \; \tau_{,4} + O(r^{-2}), \qquad (6.18)$$

where $\hat{\xi}^\mu$ denotes the components of the Minkowski space Killing vectors (as given in Chapter 3), and

$$\tau(x^A) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{+\ell} \tau_{\ell m\; 0}Y_{\ell m} (x^A) \quad ; \; \tau_{\ell m} \text{ constants}. \qquad (6.19)$$

The infinite group generated by the Killing vectors (6-15) - (6.18) consists of the inhomogeneous Lorentz transformations, augmented by the "super-translations" [37] which are generated by the spherical harmonics in $\tau(x^A)$. In the literature the asymptotic symmetry group of asymptotically flat space-time is generally referred to as the B.M.S. (Bondi-Metzner-Sachs) group. We have specialised our analysis to the case of metrics with $\sigma_0 = 0$ so that the results will be of direct use in the work of the next section.

5.7 Asymptotically shear-free metrics - their asymptotic C-K vectors

It was noted in § 5.4 that the asymptotic conformal symmetries do not emerge as a kinematic structure in general asymptotically-flat space-times. Instead, the gravitational field influences the C-K vectors in a profound way via the asymptotic shear σ_0. In this section we consider the case where σ_0 vanishes; (i.e. we are considering metrics which are asymptotically shear-free). It is found that the asymptotic C-K vectors are those of the B.M.S ; group, together with the generators of the dilatation and special conformal transformations inherited from Minkowski space. There are no "super-conformal" transformations present; that is, there is nothing that generalises the special conformal transformations in the way that the super-translations generalise the translations.

With the coordinate system of 5.6 the relevant equations from 5.3 become

$$\overset{o}{\xi}{}^{2}_{,1} = 0 \qquad (7.1a)$$

$$\overset{o}{\xi}{}^{2} = -A_{,1} \qquad (7.1b)$$

$$\cosh^2 x^3 \; \overset{o}{\xi}{}^{2}_{,3} - \overset{o}{\xi}{}^{3}_{,1} = 0 \qquad (7.1c)$$

$$\cosh^2 x^3 \; \overset{o}{\xi}{}^{2}_{,4} - \overset{o}{\xi}{}^{4}_{,1} = 0 \qquad (7.1d)$$

$$\overset{o}{\xi}{}^{3}_{,3} - \overset{o}{\xi}{}^{4}_{,4} = 0 \qquad (7.1e)$$

$$\overset{o}{\xi}{}^{3}_{,4} + \overset{o}{\xi}{}^{4}_{,3} = 0 \qquad (7.1f)$$

$$\overset{o}{\xi}{}^{1}_{,1,3} - A_{,3} = 0 \qquad (7.1g)$$

$$\overset{o}{\xi}{}^{1}_{,1,4} - A_{,4} = 0 \quad , \qquad (7.1h)$$

together with

$$\xi^1 = \overset{o}{\xi}{}^{1}(u,x^A) \qquad (7.2a)$$

$$\xi^2 = \overset{o}{\xi}{}^{2} r^2 - Ar - \tfrac{1}{2}B + O(r^{-1}) \qquad (7.2b)$$

$$\xi^3 = \overset{o}{\xi}{}^{3} - \cosh^2 x^3 \; \overset{o}{\xi}{}^{1}_{,3} \; r^{-1} + O(r^{-2}) \qquad (7.2c)$$

$$\xi^4 = \overset{o}{\xi}{}^{4} - \cosh^2 x^3 \; \overset{o}{\xi}{}^{1}_{,4} \; r^{-1} + O(r^{-2}) \quad , \qquad (7.2d)$$

where $$A = \overset{o}{\xi}{}^{3}_{,3} + \overset{o}{\xi}{}^{4}_{,4} - \overset{o}{\xi}{}^{1}_{,1} - 2\,\overset{o}{\xi}{}^{3} \tanh x^3 \qquad (7.3)$$

and $$B = -\cosh^2 x^3 \left[\overset{o}{\xi}{}^{1}_{,3,3} + \overset{o}{\xi}{}^{1}_{,4,4} \right] . \qquad (7.4)$$

In the particular case under consideration here it is seen that since $a_0 = -1$ the r^{-1} coefficients in $\mathcal{G}_{11}$ and $\mathcal{G}_{12}$ become identical. Furthermore, we have ensured by our condition (7.3) on A that the $O(r^{-1})$ term of $\mathcal{L}_{\xi}\mathcal{G}_{12}$ vanishes. We must therefore impose a further condition to make the $O(r^{-1})$ term in $\mathcal{L}_{\xi}\mathcal{G}_{11}$ vanish.

This condition takes the form

$$A - B_{,1} + \overset{o}{\xi}{}^{1}_{,1} = 0 \quad , \qquad (7.5)$$

where (7.3) has been used to simplify the algebra.

We solve the equations (7.1a - h) to find the four functions $\overset{o}{\xi}{}^{\mu}$.

From (7.1a,b) it follows that

$$A = \alpha(x^3,x^4) + u\,\beta(x^3,x^4) \qquad (7.6)$$

and $$\overset{o}{\xi}{}^{2} = -\beta(x^3,x^4) \quad , \qquad (7.7)$$

where α, β are arbitrary functions of x^3,x^4.

Using (7.1 g,h) gives

$$\overset{\circ}{\xi}{}^1 = u\alpha(x^3,x^4) + \tfrac{1}{2}u^2\beta(x^3,x^4) + \gamma(u) + \chi(x^3,x^4) \quad , \tag{7.8}$$

where $\gamma(u)$, $\chi(x^3,x^4)$ are arbitrary functions of their arguments.

From (7.1 c,d) it follows that

$$\overset{\circ}{\xi}{}^3 = -u\cosh^2 x^3\,\beta_{,3} + f(x^3,x^4) \tag{7.9}$$

$$\overset{\circ}{\xi}{}^4 = -u\cosh^2 x^3\,\beta_{,4} + g(x^3,x^4) \quad , \tag{7.10}$$

and then (7.1 e,f) lead to conditions on β, f and g in the form

$$\cosh x^3\,(\beta_{,3,3} - \beta_{,4,4}) + 2\sinh x^3\beta_{,3} = 0 \tag{7.11}$$

$$\frac{\partial}{\partial x^3}\left[\cosh x^3\beta_{,4}\right] = 0 \tag{7.12}$$

$$f_{,3} - g_{,4} = 0 \tag{7.13}$$

$$f_{,4} + g_{,3} = 0 \quad . \tag{7.14}$$

(7.12) gives

$$\beta(x^3,x^4) = \lambda(x^4)\,\mathrm{sech}\,x^3 + \mu(x^3) \tag{7.15}$$

where λ,μ are arbitrary functions, and substitution of this into (7.11) gives

$$\lambda_{,4,4} + \lambda = C_0$$

$$\mu_{,3,3}\cosh x^3 + 2\mu_{,3}\sinh x^3 = C_0 \quad ,$$

where C_0 is a constant. We find the solutions for λ,μ immediately:-

$$\lambda = C_1\cos x^4 + C_2\sin x^4 + C_0$$

$$\mu = -C_0\,\mathrm{sech}\,x^3 + C_3\tanh x^3 + C_4 \quad ,$$

where C_0, C_1, C_2, C_3, C_4 are arbitrary constants.

(7.15) gives

$$\beta(x^3,x^4) = (C_1\cos x^4 + C_2\sin x^4)\,\mathrm{sech}\,x^3 + C_3\tanh x^3 + C_4 \,, \tag{7.16}$$

and hence, from (7.9), (7.10), we find

$$\overset{\circ}{\xi}{}^3 = u\left[\sinh x^3\,(C_1\cos x^4 + C_2\sin x^4) - C_3\right] + f(x^3,x^4) \tag{7.17}$$

$$\overset{\circ}{\xi}{}^4 = u\cosh x^3\,(C_1\sin x^4 - C_2\cos x^4) + g(x^3,x^4) \quad , \tag{7.18}$$

where f and g satisfy the conditions

$$f_{,3} = g_{,4} \quad ; \quad f_{,4} = -g_{,3} \quad . \tag{7.19}$$

It now follows from (7.7), (7.8) and (7.16) that

$$\overset{\circ}{\xi}{}^1 = \tfrac{1}{2}u^2\left[(C_1\cos x^4 + C_2\sin x^4)\,\mathrm{sech}\,x^3 + C_3\tanh x^3 + C_4\right] + u\alpha(x^A) + \gamma(u) + \chi(x^A) \tag{7.20}$$

$$\overset{\circ}{\xi}{}^2 = -(C_1 \cos x^4 + C_2 \sin x^4) \operatorname{sech} x^3 - C_3 \tanh x^3 - C_4 \quad . \tag{7.21}$$

Straightforward calculation gives

$$A\ (u,x^A) = u\left[(C_1 \cos x^4 + C_2 \sin x^4) \operatorname{sech} x^3 + C_3 \tanh x^3 + C_4\right] + \alpha(x^A) \tag{7.22}$$

and

$$B = u^2\left[(C_1 \cos x^4 + C_2 \sin x^4) \operatorname{sech} x^3 + C_3 \tanh x^3\right] - \cosh^2 x^3\left[u\ (\alpha_{,3,3} + \alpha_{,4,4}) + (K_{,3,3} + K_{,4,4})\right]. \tag{7.23}$$

The remaining equations (7.3) and (7.5) give

$$\gamma_{,1} = -2\, C_4\, u + 2\ell \tag{7.24}$$

$$f_{,3} - f(x^A) \tanh x^3 = \alpha(x^A) + \ell \tag{7.25}$$

$$\cosh^2 x^3\left[\alpha_{,3,3} + \alpha_{,4,4}\right] = -\, 2\,\alpha(x^A) - 2\ell \quad , \tag{7.26}$$

where ℓ is a constant. Using the $\eth$ operator formalism shows that (7.26) may be written as

$\bar{\eth}\eth\alpha = -2\alpha - 2\ell$, which has the solution

$\alpha(x^A) = \sum_m \alpha_{m_0} Y_{1m} - \ell$, where the

α_m are constants. We shall take $\alpha(x^A)$ in the form

$$\alpha\ (x^A) = (b \sin x^4 + e \cos x^4) \operatorname{sech} x^3 + h \tanh x^3 - \ell\ . \tag{7.27}$$

Solving (7.25) now gives

$$f\ (x^A) = b \sinh x^3 \sin x^4 + e \sinh x^3 \cos x^4 - h + w(x^4) \cosh x^3 ,$$

where w is a function of x^4 only. Using (7.19) leads to

$$f(x^A) = b \sinh x^3 \sin x^4 + e \sinh x^3 \cos x^4 + n \cosh x^3 \cos x^4 - p \cosh x^3 \sin x^4 - h \quad , \tag{7.28}$$

$$g(x^A) = -b \cosh x^3 \cos x^4 + e \cosh x^3 \sin x^4 + n \sinh x^3 \sin x^4 + p \sinh x^3 \cos x^4 + q \quad , \tag{7.29}$$

where n, p, q are arbitrary constants.

Finally, (7.24) gives

$$\gamma\,(u) = -\, C_4\, u^2 + 2\ell u + m \quad , \tag{7.30}$$

where m is a further arbitrary constant.

We now write down the ξ^μ components by using (7.2a-d)

$$\xi^1 = \tfrac{1}{2}u^2\left[(C_1 \cos x^4 + C_2 \sin x^4) \operatorname{sech} x^3 + C_3 \tanh x^3 - C_4\right] + u\left[(b \sin x^4 + e \cos x^4) \operatorname{sech} x^3 + h \tanh x^3 + \ell\right] + m + K(x^A) \quad ,$$

$$\xi^2 = -\tfrac{1}{2} C_2 \operatorname{sech} x^3 \sin x^4 (u^2 + 2ur + 2r^2) - \tfrac{1}{2} C_1 \operatorname{sech} x^3 \cos x^4 (u^2 + 2ur + 2r^2)$$
$$-\tfrac{1}{2} C_3 \tanh x^3 (u^2 + 2ur + 2r^2) - C_4 r(u + r)$$
$$- (u + r) \left[(b \sin x^4 + e \cos x^4) \operatorname{sech} x^3 + h \tanh x^3\right] + r\ell$$
$$+ \frac{1}{2} \cosh^2 x^3 (K_{,3,3} + K_{,4,4}) + O(r^{-1}) ,$$
$$\xi^3 = \tfrac{1}{2} C_2 \sinh x^3 \sin x^4 (2u + r^{-1} u^2) + \tfrac{1}{2} C_1 \sinh x^3 \cos x^4 (2u + r^{-1} u^2)$$
$$- \tfrac{1}{2} C_3 (2u + r^{-1} u^2)$$
$$+ (1 + r^{-1} u) \sinh x^3 (b \sin x^4 + e \cos x^4) - h (1 + r^{-1} u)$$
$$+ n \cosh x^3 \cos x^4 - p \cosh x^3 \sin x^4 - r^{-1} \cosh^2 x^3 K_{,3} + O(r^{-2}) ,$$
$$\xi^4 = -\tfrac{1}{2} C_2 \cosh x^3 \cos x^4 (2u + r^{-1} u^2) + \tfrac{1}{2} C_1 \cosh x^3 \sin x^4 (2u + r^{-1} u^2)$$
$$- (1 + r^{-1} u) \cosh x^3 (b \cos x^4 - e \sin x^4) + n \sinh x^3 \sin x^4$$
$$+ p \sinh x^3 \cos x^4 + q$$
$$- r^{-1} \cosh^2 x^3 K_{,4} + O(r^{-2}) .$$

To examine the relation of the conformal symmetry group in this case to the corresponding group in Minkowski space we make the following relabelling of parameters; $C_2 \to 2a$, $C_1 \to 2d$, $C_3 \to 2g$, $C_4 \to -2k$. Further, we expand the arbitrary function $K(x^A)$ in spin-zero spherical harmonics as follows

$$K(x^A) = c \operatorname{sech} x^3 \sin x^4 + f \operatorname{sech} x^3 \cos x^4 + j \tanh x^3 + \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \tau_{\ell m\, 0}Y_{\ell m}(x^A),$$

where $\tau_{\ell m}$ are constants and ${}_0Y_{\ell m}(x^A)$ are the usual spherical harmonics. Without loss of generality we have set the coefficient of ${}_0Y_{00}$ equal to zero; the constant term in (7.30) absorbs any additive constant in $K(x^A)$.

With these replacements made we find

$$\xi^1 = \hat{\xi}^1 + \tau(x^A) , \qquad (7.31a)$$
$$\xi^2 = \hat{\xi}^2 + \tfrac{1}{2} \cosh^2 x^3 [\tau_{,3,3} + \tau_{,4,4}] + O(r^{-1}) , \qquad (7.31b)$$
$$\xi^3 = \hat{\xi}^3 - r^{-1} \cosh^2 x^3 \tau_{,3} + O(r^{-2}) , \qquad (7.31c)$$
$$\xi^4 = \hat{\xi}^4 - r^{-1} \cosh^2 x^3 \tau_{,4} + O(r^{-2}) , \qquad (7.31d)$$

where $\hat{\xi}^\mu$ are the Minkowski space C.K vectors already given in Chapter 3, and

$$\tau(x^A) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \tau_{\ell m\, 0}Y_{\ell m}(x^A) , \qquad (7.32)$$

where the $\tau_{\ell m}$ are constants.

It is clear from the results of 5.6 that the spherical harmonics ($\ell \geq 2$) that appear in the expansion (7.32) of $\tau(x^A)$ are merely generators of super-translations already present in the B.M.S. group.

The "true conformal" part of the symmetry group contains nothing that is not inherited directly from Minkowski space; there are no "super-conformal" transformations.

CHAPTER 6

PENROSE'S CONFORMAL TECHNIQUE - ITS APPLICATION TO ASYMPTOTIC CONFORMAL KILLING VECTORS

6.1 Introduction

The earliest work on asymptotic properties in curved space-times made use of asymptotic expansion techniques to achieve its results. The contributions of Newman, Penrose, Unti and others in this area of development have already been reviewed in earlier chapters. Objections to this method of approach were partly resolved by some work of Dixon [53] which showed that consideration of higher order terms in the N-U expansions added no further restrictions to the solutions, but the fundamental criticisms concerning convergence of the expansions and topological questions were not met. The conformal technique of Penrose [54] avoids these objections by introducing a second space-time $\mathcal{R}$, conformal to the physical space-time $\tilde{\mathcal{R}}$, in which a hypersurface of points at infinity can be treated as readily as any finite hypersurface. Since zero rest-mass field equations are conform-invariant the asymptotic properties of gravitational fields in empty space can be studied in a rigorous manner in $\mathcal{R}$ and the results carried over into the physical space $\tilde{\mathcal{R}}$. In this way the "peeling theorem", the ten conserved quantities, the B.M.S. group and other aspects of gravitational field behaviour were investigated in a straightforward manner. [54, 55, 56] Our present work will apply Penrose's technique to the study of conformal Killing vector fields in the asymptotic regions of asymptotically-flat space-times, making use of the conform-invariance of the conformal Killing equations. We find only the "far-asymptotic" terms in this analysis, and in that respect the calculations of the present chapter do not supersede those of Chapter 5. However, the methods presented here can be readily taken to higher orders if one is prepared either to solve the Einstein field equations for asymptotically flat metrics using the Penrose conformal technique, or to transform the standard N-U results into the unphysical conformal space $\mathcal{R}$. The first of these alternatives is preferable, since it is not open to the criticism that might be levelled at asymptotic expansion techniques.

The thinking behind the conformal technique has been described many times by Penrose; e.g. [57], so only a brief introduction will be given here. The metric structure of space-time does not allow us to introduce points at infinity in a sensible way, since such a point would be infinitely distant from its neighbours. On the other hand, the conformal structure of space-time, since it involves only ratios of neighbouring infinitesimal distances, allows us to treat infinity as though it were an ordinary three-dimensional boundary $\mathscr{I}$ to a four-dimensional conformal region $\mathcal{R}$. "Asymptotic behaviour" in the physical space-time $\tilde{\mathcal{R}}$ becomes "behaviour at the hypersurface $\mathscr{I}$" in the conformal space $\mathcal{R}$. The manifold $\mathcal{R}$ has a metric tensor $g_{\mu\nu}$ which is a "conformal rescaling" (Penrose's terminology; see [12]) of the physical metric $\tilde{g}_{\mu\nu}$, given by

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu} . \qquad (1.1)$$

Throughout the following work we shall use a tilde to denote a quantity associated with physical space $\tilde{\mathcal{R}}$, the untilded quantities being associated with the conformal space $\mathcal{R}$.

In (1.1) Ω is a scalar field chosen so that $\Omega = 0$ defines $\mathscr{I}$, $\Omega > 0$ defines the interior of $\mathcal{R}$, and $\nabla_\mu \Omega \neq 0$ on $\mathscr{I}$. Then $\nabla_\mu \Omega$ defines the normal to the hypersurface $\mathscr{I}$. For a large class of metrics (including all those that have been suggested as representing gravitational radiation in asymptotically flat space-times) the hypersurface $\mathscr{I}$ separates into five disjoint parts; namely, three points I^-, I^0, I^+, representing respectively the past, spatial and future infinities, and two null hypersurfaces $\mathscr{I}^-$, $\mathscr{I}^+$, representing the past and future null infinities. Any null geodesic of $\mathcal{R}$, not on $\mathscr{I}$, originates at a point of $\mathscr{I}^-$ and terminates at a point of $\mathscr{I}^+$. Asymptotic problems concerning future null infinity in the physical space $\tilde{\mathcal{R}}$ thus reappear as problems concerning neighbourhoods bounded by $\mathscr{I}^+$ in the conformal space $\mathcal{R}$. The virtue of the conformal technique lies in the fact that quantities in $\tilde{\mathcal{R}}$ expressed as asymptotic expansions in the radial parameter $\tilde{r}$ appear in the conformal space $\mathcal{R}$ as expansions about $\mathcal{J}^+$ in terms of the small parameter r, and may be handled there in a rigorous fashion.

6.2 Conformal transformation formulae

We consider the conformal rescaling of the metric tensors of $\tilde{\mathcal{R}}$ and $\mathcal{R}$ given by

$$\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu} \quad ; \quad \tilde{g}^{\mu\nu} = \Omega^{2} g^{\mu\nu} . \tag{2.1}$$

We shall suppose that there has been no transformation of coordinates between $\mathcal{R}$ and $\tilde{\mathcal{R}}$.

The Christoffel symbols of $\tilde{\mathcal{R}}$ and $\mathcal{R}$ are related by

$$\tilde{\Gamma}^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu} - \Omega^{-1}[\delta^{\sigma}_{\mu}\Omega_{,\nu} + \delta^{\sigma}_{\nu}\Omega_{,\mu} - g^{\sigma\tau}g_{\mu\nu}\Omega_{,\tau}] . \tag{2.2}$$

Denoting covariant derivations with respect to $g_{\mu\nu}$, $\tilde{g}_{\mu\nu}$ by ∇_μ, $\tilde{\nabla}_\mu$ respectively, we have, for a scalar ψ

$$\tilde{\nabla}_\mu \psi = \nabla_\mu \psi , \tag{2.3}$$

and for a covariant vector k_μ

$$\tilde{\nabla}_\mu k_\nu = \nabla_\mu k_\nu + \Omega^{-1}[k_\mu \nabla_\nu\Omega + k_\nu \nabla_\mu \Omega - g_{\mu\nu} k^\sigma \nabla_\sigma \Omega] . \tag{2.4}$$

We shall also need the transformation rules for spinors under conformal rescaling. The relation

$$g_{\mu\nu}\, \sigma^{\mu}_{A\dot{A}}\, \sigma^{\nu}_{B\dot{B}} = \epsilon_{AB}\epsilon_{\dot{A}\dot{B}}$$

is invariant under (2.1) if we put

$$\tilde{\epsilon}_{AB} = \epsilon_{AB} \tag{2.5}$$

and

$$\tilde{\sigma}_\mu{}^{A\dot{A}} = \Omega^{-1}\sigma_\mu{}^{A\dot{A}} \quad ; \quad \tilde{\sigma}^{\mu}_{A\dot{A}} = \Omega\, \sigma^{\mu}_{A\dot{A}} . \tag{2.6}$$

The transformation for covariant derivative of a spinor is:

$$\begin{aligned}\tilde{\nabla}_{X\dot{X}}\, \tilde{\xi}_{A\cdots L\dot{A}\cdots\dot{L}} = {} & \Omega^{m+1}\nabla_{X\dot{X}}\,\xi_{A\cdots L\dot{A}\cdots\dot{L}} \\ & + \Omega^{m}\{(m-\tfrac{1}{2}r)\,\xi_{A\cdots L\dot{A}\cdots\dot{L}}\nabla_{X\dot{X}}\Omega + \xi_{X\cdots L\dot{A}\cdots\dot{L}}\nabla_{A\dot{X}}\Omega \\ & + \cdots + \xi_{A\cdots X\dot{A}\cdots\dot{L}}\nabla_{L\dot{X}}\Omega + \xi_{A\cdots L\dot{X}\cdots\dot{L}}\nabla_{X\dot{A}}\Omega \\ & + \cdots + \xi_{A\cdots L\dot{A}\cdots\dot{X}}\nabla_{X\dot{L}}\Omega\} ,\end{aligned} \tag{2.7}$$

where r is the total number of spinor indices of $\xi_{A\cdots L\dot{A}\cdots\dot{L}}$ and m is the conformal weight of $\xi_{A\cdots L\dot{A}\cdots\dot{L}}$ appearing in its transformation rule

$$\tilde{\xi}_{A\cdots L\dot{A}\cdots\dot{L}} = \Omega^{m}\,\xi_{A\cdots L\dot{A}\cdots\dot{L}} \quad . \tag{2.8}$$

The Riemann tensor decomposes in spinor form as given in Chapter 2 and [54] the spinor quantities $\tilde{\Psi}_{ABCD}$, $\tilde{\Phi}^{\dot{A}\dot{B}}_{AB}$, $\tilde{\Lambda}$ transform according to

$$\tilde{\Psi}_{ABCD} = \Omega^{2}\,\Psi_{ABCD} \quad , \tag{2.9}$$

$$\tilde{\Phi}^{\dot{A}\dot{B}}_{AB} = \Omega^{2}\,\Phi^{\dot{A}\dot{B}}_{AB} + \Omega\,\nabla_{(A}{}^{\dot{A}}\nabla_{B)}{}^{\dot{B}}\Omega \quad , \tag{2.10}$$

$$\tilde{\Lambda} = \Omega^{2}\Lambda - \tfrac{1}{4}\Omega\,\nabla_{A\dot{A}}\nabla^{A\dot{A}}\Omega + \tfrac{1}{2}\nabla_{A\dot{A}}\Omega\,\nabla^{A\dot{A}}\Omega \,. \tag{2.11}$$

6.3 Conformal Killing vectors in $\mathcal{R}$ and $\tilde{\mathcal{R}}$.

With $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ related as in (2.1) it is straightforward to show that if the vector field ξ^{μ} satisfies

$$\nabla_{(\nu}\xi_{\mu)} = \tfrac{1}{4}\,\phi\,g_{\mu\nu} \quad , \tag{3.1}$$

then $\tilde{\xi}_{\mu}$ given by

$$\tilde{\xi}_{\mu} = \Omega^{-2}\,\xi_{\mu} \tag{3.2}$$

satisfies

$$\tilde{\nabla}_{(\nu}\tilde{\xi}_{\mu)} = \tfrac{1}{4}\,\tilde{\phi}\,\tilde{g}_{\mu\nu} \tag{3.3}$$

with

$$\tilde{\phi} = \phi - 4\Omega^{-1}\xi^{\rho}\Omega_{,\rho} \quad . \tag{3.4}$$

That is, if ξ^{μ} is a conformal Killing vector of $\mathcal{R}$ then $\tilde{\xi}^{\mu}$ given by

$$\tilde{\xi}^{\mu} = \xi^{\mu} \tag{3.2a}$$

is a conformal Killing vector of $\tilde{\mathcal{R}}$.

We note however that Killing vectors ($\tilde{\phi} = 0$) of $\tilde{\mathcal{R}}$ correspond in general to <u>conformal</u> Killing vectors of $\mathcal{R}$. Thus, if we wish to carry out calculations in $\mathcal{R}$ to find the Killing vectors of $\tilde{\mathcal{R}}$, we must solve (3.1) subject to the restriction

$$\phi = 4\Omega^{-1}\xi^{\rho}\Omega_{,\rho} \,. \tag{3.5}$$

This technique has been used by Winicour and Tamburino [56] to find the asymptotic Killing symmetries of asymptotically flat spaces. In their work the equations (3.1) and (3.5) are solved on $\mathcal{J}^{+}$, giving the B.M.S. group described previously by Sachs [58] .

6.4 Properties of $\mathcal{J}^+$

Taking the case of empty space $\tilde{\mathcal{R}}$ we have from (2.11)

$$\nabla_{A\dot{A}}\Omega\ \nabla^{A\dot{A}}\Omega = 0 \qquad \text{on } \mathcal{J}^+, \qquad (4.1a)$$

and from (2.10)

$$\nabla_{(A}{}^{\dot{A}}\ \nabla_{B)}{}^{\dot{B}}\Omega = 0 \qquad \text{on } \mathcal{J}^+. \qquad (4.2a)$$

The tensor equivalent of (4.1a) is

$$\nabla_\mu\Omega\,\nabla^\mu\Omega = 0 \qquad \text{on } \mathcal{J}^+. \qquad (4.1b)$$

i.e. $\mathcal{J}^+$ is a null hypersurface.

The tensor equivalent of (4.2a) arises simply by considering

$$\begin{aligned}\nabla_A{}^{\dot{A}}\nabla_B{}^{\dot{B}}\Omega &= \nabla_{(A}{}^{\dot{A}}\nabla_{B)}{}^{\dot{B}}\Omega + \tfrac{1}{2}\epsilon_{AB}\nabla_C{}^{\dot{A}}\nabla^{C\dot{B}}\Omega \\ &= \nabla_{(A}{}^{\dot{A}}\nabla_{B)}{}^{\dot{B}}\Omega - \tfrac{1}{2}\epsilon_{AB}\left\{\nabla^{C(\dot{A}}\nabla_C{}^{\dot{B})}\Omega + \tfrac{1}{2}\epsilon^{\dot{A}\dot{B}}\nabla^C{}_{\dot{D}}\nabla_C{}^{\dot{D}}\Omega\right\},\end{aligned}$$

from which it follows that

$$\nabla_{A\dot{A}}\nabla_{B\dot{B}}\Omega - \tfrac{1}{4}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\nabla_{C\dot{D}}\nabla^{C\dot{D}}\Omega = 0 .$$

Then the required tensor equivalent is

$$\nabla_\mu\nabla_\nu\Omega - \tfrac{1}{4}g_{\mu\nu}\nabla_\rho\nabla^\rho\Omega = 0. \qquad (4.2b)$$

This last equation indicates that the normal to $\mathcal{J}^+$ constitutes a conformal Killing vector field.

We show now that a specialisation of Ω enables us to construct the rays $\Omega_{,\mu}$ so that they form a shear-free, divergence-free congruence. It is clear that the physical space $\tilde{\mathcal{R}}$ may be related by different choices of Ω to different (conformally related) unphysical spaces. Suppose two such choices Ω, Ω' relate $\tilde{\mathcal{R}}$ to $\mathcal{R}$ and $\mathcal{R}'$ respectively. Then from (2.1)

$$\tilde{g}^{\mu\nu} = \Omega^2 g^{\mu\nu} = \Omega'^2 g'^{\mu\nu} ,$$

so that we must relate Ω and Ω' according to

$$\Omega = f\,\Omega' \qquad (4.3)$$

to preserve the conformal link between $\mathcal{R}'$ and $\mathcal{R}$. f must be finite, non-zero and suitably smooth on $\mathcal{J}^+$. Alternatively, we may think of $\tilde{\mathcal{R}}$ as determining the conformal structure of $\mathcal{R}$, with

$$g'^{\mu\nu} = f^2 g^{\mu\nu} \qquad (4.4)$$

expressing the conformal freedom in choice of metric on $\mathcal{R}$. We exploit this freedom in choice of Ω to set $\nabla_\rho\nabla^\rho\Omega = 0$ on $\mathcal{J}^+$, which has the effect (see 4.2b) of making $\nabla_\mu\nabla_\nu\Omega = 0$ on $\mathcal{J}^+$.

Making use of modified forms of (2.3) and (2.4), where f replaces Ω and $'$ replaces $\sim$, we find that

$$\nabla'_\nu \Omega = \nabla_\nu \Omega \quad ,$$

and
$$\nabla'_\mu \nabla'_\nu \Omega' = f^{-1}\nabla_\mu\nabla_\nu\Omega - f^{-2}g_{\mu\nu}\nabla_\sigma f\,\nabla^\sigma\Omega - \Omega f^{-2}[\nabla_\mu\nabla_\nu f + 2f^{-1}\nabla_\mu f\,\nabla_\nu f]$$
$$+ 2\Omega f^{-3}[\nabla_\mu f\,\nabla_\nu\Omega + \tfrac{1}{2}g_{\mu\nu}\nabla^\sigma f\,\nabla_\sigma f]\ .$$

Taken on $\mathcal{J}^+$, this last equation becomes

$$\nabla'_\mu \nabla'_\nu \Omega' = f^{-1}\nabla_\mu\nabla_\nu\Omega - f^{-2}g_{\mu\nu}\nabla_\sigma f\,\nabla^\sigma\Omega \ , \quad (4.5)$$

from which

$$\nabla'_\mu \nabla'^\mu \Omega' = f\,\nabla_\mu\nabla^\mu\Omega - 4\nabla_\mu f\,\nabla^\mu\Omega\ . \quad (4.6)$$

Making a suitable choice for f enables us to put

$$\nabla'_\mu \nabla'^\mu \Omega' = 0 \quad \text{on } \mathcal{J}^+, \quad (4.7)$$

so that we have from (4.2b)

$$\nabla'_\mu \nabla'_\nu \Omega' = 0 \quad \text{on } \mathcal{J}^+. \quad (4.8)$$

Any further transformation $\Omega' \rightarrow \Omega'' = h^{-1}\Omega'$ must be such that h satisfies

$$\nabla'_\mu h\,\nabla'^\mu\Omega' = 0 \quad \text{on } \mathcal{J}^+. \quad (4.9)$$

In summary, the geometry of $\mathcal{J}^+$ at this stage is that of a null hypersurface $\Omega = 0$ with the null rays $\Omega_{,\mu}$ forming a shear- and divergence-free congruence. We stress that these are not global properties of $\mathcal{R}$, but they are of great assistance in coordinatizing a neighbourhood of $\mathcal{R}$ in the vicinity of $\mathcal{J}^+$.

6.5 Tetrad vectors in $\mathcal{R}$ and $\tilde{\mathcal{R}}$.

We see from the results of 6.4 that at any point of $\mathcal{R}$, sufficiently near $\mathcal{J}^+$, the null geodesic with tangent vector $\ell_\mu = \Omega_{,\mu}$ can be used as one member of a null tetrad. If we introduce ξ_A as the spinor corresponding to ℓ_μ then, since ℓ_μ undergoes parallel propagation along the geodesic, we have

$$\xi^B \bar{\xi}^{\dot{B}} \nabla_{B\dot{B}}\, \xi_A = 0 \quad . \quad (5.1)$$

In order to preserve (5.1) under conformal transformation into $\tilde{\mathcal{R}}$ we put

$$\tilde{\xi}_A = \Omega^{\frac{1}{2}}\, \xi_A \quad . \quad (5.2)$$

Then $\tilde{\xi}_A$ is tangent to a null geodesic in $\tilde{\mathcal{R}}$. The affine parameter r of the null geodesic in $\mathcal{R}$ is scaled according to

$$\xi^B \bar{\xi}^{\dot{B}} \nabla_{B\dot{B}} r = 1 \quad , \tag{5.3}$$

and to preserve this in $\tilde{\mathcal{R}}$ we must take

$$\tilde{r} = \int \Omega^{-2} dr \ . \tag{5.4}$$

A second spinor ι_A satisfying

$$\xi_A \iota^A = 1 \tag{5.5}$$

is now introduced and used to generate the second tetrad vector n_μ. The demand that ι_A undergoes parallel propagation along the geodesic tangent gives

$$\xi^B \bar{\xi}^{\dot{B}} \nabla_{B\dot{B}} \iota_A = 0 \quad , \tag{5.6}$$

and to preserve this relation in $\tilde{\mathcal{R}}$ we must take

$$\tilde{\iota}_A = \Omega^{-\frac{1}{2}} \iota_A + a \Omega^{\frac{1}{2}} \xi_A \quad , \tag{5.7}$$

where the complex function a satisfies

$$\xi^B \bar{\xi}^{\dot{B}} \nabla_{B\dot{B}} a = \Omega^{-2} \iota^B \bar{\xi}^{\dot{B}} \nabla_{B\dot{B}} \Omega \ . \tag{5.8}$$

(A derivation of (5.8) is given in Appendix 3)

With the spinor basis for $\mathcal{R}$ selected in this way we generate a tetrad of null vectors $l_\mu, n_\mu, m_\mu, \bar{m}_\mu$ given by

$$\left.\begin{aligned} l^\mu &= \sigma^\mu_{A\dot{B}} \xi^A \bar{\xi}^{\dot{B}} \quad , & n^\mu &= \sigma^\mu_{A\dot{B}} \iota^A \bar{\iota}^{\dot{B}} \ , \\ m^\mu &= \sigma^\mu_{A\dot{B}} \xi^A \bar{\iota}^{\dot{B}} \quad , & \bar{m}^\mu &= \sigma^\mu_{A\dot{B}} \iota^A \bar{\xi}^{\dot{B}} \ . \end{aligned}\right\} \tag{5.9}$$

which satisfy

$$\left.\begin{aligned} l^\mu n_\mu &= -m^\mu \bar{m}_\mu = 1 \\ \text{and} \quad l_\mu m^\mu &= l_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0 \ . \end{aligned}\right\} \tag{5.10}$$

We define the differential operators D, Δ, δ and $\bar{\delta}$ by

$$D \equiv \xi^A \bar{\xi}^{\dot{A}} \nabla_{A\dot{A}} \ , \quad \Delta \equiv \iota^A \bar{\iota}^{\dot{A}} \nabla_{A\dot{A}} \ , \quad \delta \equiv \xi^A \bar{\iota}^{\dot{A}} \nabla_{A\dot{A}} \ , \quad \bar{\delta} \equiv \iota^A \bar{\xi}^{\dot{A}} \nabla_{A\dot{A}} \ , \tag{5.11}$$

and then the spin coefficients are given by

$$\begin{aligned} \kappa &= \xi^A D \xi_A \quad , & \epsilon &= \iota^A D \xi_A \ , & \pi &= \iota^A D \iota_A \ , \\ \rho &= \xi^A \bar{\delta} \xi_A \quad , & \alpha &= \iota^A \bar{\delta} \xi_A \ , & \lambda &= \iota^A \bar{\delta} \iota_A \ , \\ \sigma &= \xi^A \delta \xi_A \quad , & \beta &= \iota^A \delta \xi_A \ , & \mu &= \iota^A \delta \iota_A \ , \\ \tau &= \xi^A \Delta \xi_A \quad , & \gamma &= \iota^A \Delta \xi_A \ , & \nu &= \iota^A \Delta \iota_A \ . \end{aligned} \tag{5.12}$$

We have made the same choice of tetrad as in [31] , and Chapter 2 of the present work, which gives

$$\kappa = \epsilon = \pi = 0 , \tag{5.13}$$

so that in spinor formalism we have

$$D\xi_A = D\iota_A = 0 . \tag{5.14}$$

The scaling of the affine parameter r, (5.3) leaves freedom $r \rightarrow r + \text{const.}$ In the present notation (5.3) becomes

$$Dr = 1, \tag{5.15}$$

and we note that any further conformal transformation

$$\Omega \rightarrow \Omega' = h^{-1}\Omega \tag{5.16}$$

must therefore be accompanied by the transformation

$$r \rightarrow r' = \int h^{-2}\, dr . \tag{5.17}$$

The remaining freedom in choice of affine parameter along the null geodesics is used to put r = 0 on $\mathscr{I}^+$. We know that $\mathscr{I}^+$ is given by $\Omega = 0$ so we may take

$$\Omega = -r + O(r^2) , \tag{5.18}$$

and (5.3) is now expressed as

$$\ell^\mu \nabla_\mu \Omega = -1 . \tag{5.19}$$

The conformal freedom in selection of a metric on $\mathcal{R}$, developed in 6.4, may be presented as a freedom in choice of tetrad for $\mathcal{R}$. Consider the transformation of the spinor dyad (ξ_A, ι_A) given by

$$\xi'^A = f^{\frac{1}{2}} \xi^A \tag{5.20}$$

$$\iota'^A = f^{-\frac{1}{2}} \iota^A + b f^{\frac{1}{2}} \xi^A , \tag{5.21}$$

where f is real, and complex b satisfies

$$D b = f^{-2} \bar{\delta} f , \tag{5.22}$$

combined with

$$\sigma'_\mu{}^{A\dot{A}} = f^{-1} \sigma_\mu{}^{A\dot{A}} , \quad \sigma'^\mu{}_{A\dot{A}} = f \sigma^\mu{}_{A\dot{A}} , \tag{5.23}$$

$$\epsilon'_{AB} = \epsilon_{AB} , \quad \epsilon'^{AB} = \epsilon^{AB} . \tag{5.24}$$

The induced transformation on the tetrad vectors is given by

$$\begin{aligned} \ell'^\mu &= f^2 \ell^\mu \\ n'^\mu &= n^\mu + f b m^\mu + f \bar{b} \bar{m}^\mu + f^2 b\bar{b} \ell^\mu \\ m'^\mu &= f m^\mu + f^2 \bar{b} \ell^\mu \end{aligned} \tag{5.25}$$

or, alternatively,

$$\begin{aligned} l'_\mu &= l_\mu \\ n'_\mu &= f^{-2} n_\mu + f^{-1} b\, m_\mu + f^{-1} \bar{b}\, \bar{m}_\mu + b\bar{b}\, l_\mu \\ m'_\mu &= f^{-1} m_\mu + \bar{b}\, l_\mu \quad . \end{aligned} \tag{5.26}$$

We note that the condition (5.22), which ensures that $D' \iota'_A = 0$ (see 5.14) with the new choice of tetrad, is derived in Appendix 3 . From (5.25) and (5.26) it is straightforward to show that

$$g'^{\mu\nu} = f^2 g^{\mu\nu}$$

$$\text{and} \qquad g'_{\mu\nu} = f^{-2} g_{\mu\nu} \quad ,$$

which is just the conformal freedom in choice of metric on $\mathcal{R}$, expressed previously in (4.4). To preserve (5.15), the above transformation of spinor dyad must be accompanied by the transformation

$$r' = \int f^{-2} dr$$

on the affine parameter r. Finally, the function Ω that relates the metric of $\mathcal{R}$ to the metric of the physical space $\tilde{\mathcal{R}}$ must be modified according to

$$\Omega' = f^{-1} \Omega$$

so that $\mathcal{R}$ continues to relate to the same physical space (see 4.3). From 6.1 we see that the tetrad vectors of $\tilde{\mathcal{R}}$ are related to those of $\mathcal{R}$ by

$$\begin{aligned} \tilde{l}^\mu &= \Omega^2 l^\mu \\ \tilde{n}^\mu &= n^\mu + \Omega a\, m^\mu + \Omega \bar{a}\, \bar{m}^\mu + \Omega^2 a\bar{a}\, l^\mu \\ \tilde{m}^\mu &= \Omega\, m^\mu + \Omega^2 \bar{a}\, l^\mu \quad , \end{aligned} \tag{5.27}$$

or, alternatively,

$$\begin{aligned} \tilde{l}_\mu &= l_\mu \\ \tilde{n}_\mu &= \Omega^{-2} n_\mu + \Omega^{-1} a\, m_\mu + \Omega^{-1} \bar{a}\, \bar{m}_\mu + a\bar{a}\, l_\mu \\ \tilde{m}_\mu &= \Omega^{-1} m_\mu + \bar{a}\, l_\mu . \end{aligned} \tag{5.28}$$

where a satisfies

$$D a = \Omega^{-2} \bar{\delta} \Omega . \tag{5.28a}$$

The most general transformation of tetrad in $\tilde{\mathcal{R}}$ that leaves the direction of $\tilde{l}^\mu$ unchanged is given by

$$\begin{aligned} \tilde{l}'^\mu &= A\, \tilde{l}^\mu \\ \tilde{n}'^\mu &= A^{-1} \tilde{n}^\mu + \bar{B}\, \tilde{m}^\mu + B\, \tilde{\bar{m}}^\mu + A B \bar{B}\, \tilde{l}^\mu \\ \tilde{m}'^\mu &= e^{ic} (\tilde{m}^\mu + A B\, \tilde{l}^\mu) \quad , \end{aligned} \tag{5.29}$$

where A and C are real, with $A > 0$, and B is complex.

We take $DA = DB = DC = 0$, and note that (5.29) is induced by the transformation of spinor dyad given by

$$\left.\begin{aligned} \tilde{\xi}'^{A} &= A^{\frac{1}{2}} e^{\frac{1}{2}iC}\, \tilde{\xi}^{A} \\ \tilde{\iota}'^{A} &= A^{-\frac{1}{2}} e^{-\frac{1}{2}iC}(\tilde{\iota}^{A} + A\bar{B}\, \tilde{\xi}^{A}). \end{aligned}\right\} \qquad (5.30)$$

(5.29) or (5.30) is combined with a transformation

$$\tilde{r} \longrightarrow \tilde{r}' = A^{-1}\tilde{r}\ , \qquad (5.31)$$

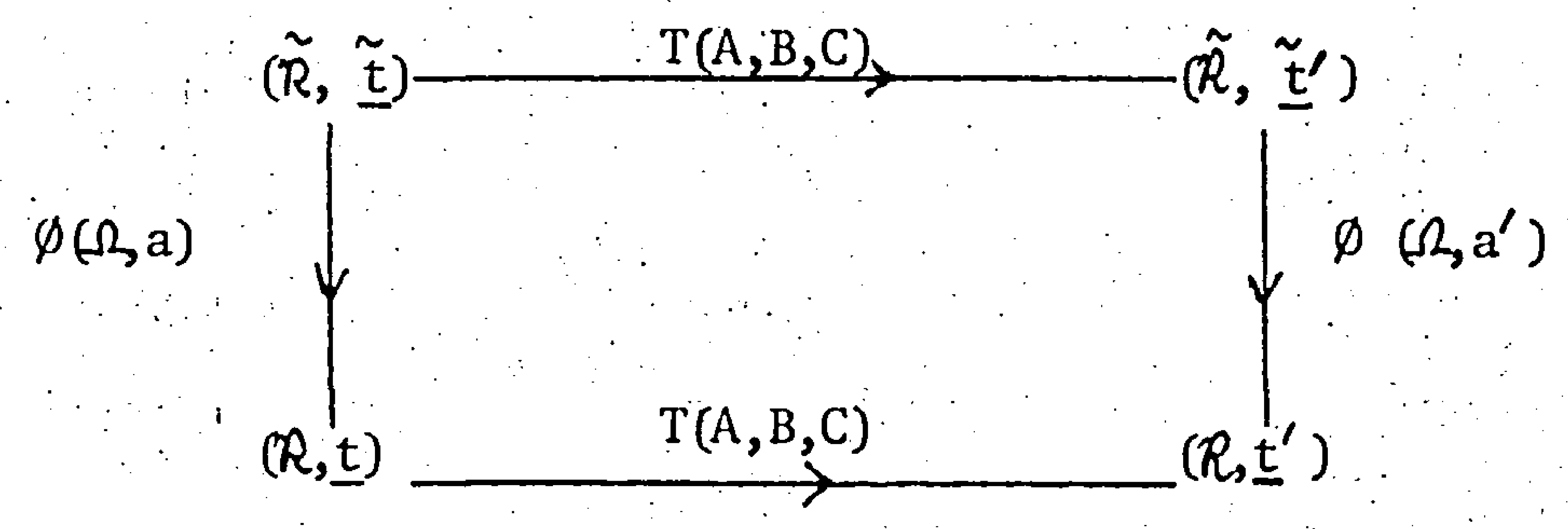
so that the metric, given by

$$\tilde{g}^{\mu\nu} = 2\tilde{l}^{(\mu}\tilde{n}^{\nu)} - 2\tilde{m}^{(\mu}\tilde{\bar{m}}^{\nu)}$$

is preserved, together with the relationships

$$\tilde{\xi}_{A}\tilde{\iota}^{A} = 1\ , \quad \tilde{D}\tilde{\xi}_{A} = \tilde{D}\tilde{\iota}_{A} = 0\ , \quad \tilde{D}\tilde{r} = 1\ .$$

If the tetrad transformation (5.29) (with parameters A,B,C) is carried out in physical space $\tilde{\mathcal{R}}$, the corresponding transformation of tetrad in $\mathcal{R}$ can be made identical in form (i.e. made to assume the form (5.29) without tildes but with unchanged parameters A,B,C), if we exploit the conformal freedom in choice of metric on $\mathcal{R}$. In fact, it will be shown that $\tilde{\mathcal{R}}$ relates to the same unphysical $\mathcal{R}$ (with no conformal change of metric on $\mathcal{R}$), since only a change of parameter a in (5.27) is involved; there is no change in choice of Ω. We recall that the parameter a in (5.27) is free up to the restriction $Da = \Omega^{-2}\bar{\delta}\Omega$. The following sketch makes the situation clear.



$\underline{t}$, $\underline{\tilde{t}}$ etc. are here used as generic symbols for tetrad vectors. T(A,B,C) denotes the transformation (5.29) with or without tildes, while $\emptyset(\Omega, a)$ denotes the inverse of transformation (5.27), given by

$$\begin{aligned} l^{\mu} &= \Omega^{-2}\tilde{l}^{\mu} \\ n^{\mu} &= a\bar{a}\tilde{l}^{\mu} + \tilde{n}^{\mu} - a\tilde{m}^{\mu} - \bar{a}\tilde{\bar{m}}^{\mu} \\ m^{\mu} &= -\Omega^{-1}\bar{a}\tilde{l}^{\mu} + \Omega^{-1}\tilde{m}^{\mu}\ . \end{aligned} \qquad (5.32)$$

The demand that

$$T(A,B,C)\,\underline{t}(\Omega,a) = \phi(\Omega,a')\,\underline{\tilde{t}}'(A,B,C)$$

leads to a functional dependence of a' upon a,A,B,C;

$$a' = e^{-iC}\left[aA^{-1} + \bar{B}(1-\Omega^{-1})\right] \qquad (5.33)$$

It is straightforward to check that a' satisfies

$$l'^{\mu} a'_{,\mu} = \Omega^{-1}\,\bar{m}'^{\mu}\,\Omega_{,\mu} \quad ,$$

which is analogous to the condition (5.28a) on a.

6.6 The spin coefficients and spinor decomposition of the Riemann tensor in $\mathcal{R}$ and $\tilde{\mathcal{R}}$.

In this section we shall set down the transformations of the spin coefficients between $\mathcal{R}$ and $\tilde{\mathcal{R}}$ and examine the form of the Riemann tensor spinor decomposition in $\mathcal{R}$ when the physical space $\tilde{\mathcal{R}}$ is empty.

It is helpful to derive first the transformations for $l_{\mu;\nu}$, $n_{\mu;\nu}$, $m_{\mu;\nu}$.

They are (using (5.19) as the transformation between $\mathcal{R}$ and $\tilde{\mathcal{R}}$)

$$\tilde{\nabla}_\nu \tilde{l}_\mu = \nabla_\nu l_\mu + l_\mu(\Omega^{-1}\Omega_{,\nu}) + l_\nu(\Omega^{-1}\Omega_{,\mu}) - g_{\mu\nu}\,\Omega^{-1}D\Omega \,, \qquad (6.1)$$

$$\begin{aligned}\tilde{\nabla}_\nu \tilde{n}_\mu = {} & a\bar{a}\nabla_\nu l_\mu + \Omega^{-2}\nabla_\nu n_\mu + \Omega^{-1}a\nabla_\nu m_\mu + \Omega^{-1}\bar{b}\,\nabla_\nu \bar{m}_\mu \\ & + l_\mu(a\bar{a}_{,\nu} + \bar{a}a_{,\nu} + \Omega^{-1}a\bar{a}\Omega_{,\nu}) + l_\nu\,\Omega^{-1}a\bar{a}\,\Omega_{,\mu} \\ & - n_\mu(\Omega^{-3}\Omega_{,\nu}) + n_\nu(\Omega^{-3}\Omega_{,\mu}) + m_\mu(\Omega^{-1}a_{,\nu}) + m_\nu(\Omega^{-2}a\,\Omega_{,\mu}) \\ & + \bar{m}_\mu(\Omega^{-1}\bar{a}_{,\nu}) + \bar{m}_\nu(\Omega^{-2}\bar{a}\,\Omega_{,\mu}) \\ & - g_{\mu\nu}(\Omega^{-1}a\bar{a}\,D\Omega + \Omega^{-3}\Delta\Omega + \Omega^{-2}a\,\delta\Omega + \Omega^{-2}\bar{a}\,\bar{\delta}\Omega)\,, \end{aligned} \qquad (6.2)$$

$$\begin{aligned}\tilde{\nabla}_\nu \tilde{m}_\mu = {} & \bar{a}\,\nabla_\nu l_\mu + \Omega^{-1}\nabla_\nu m_\mu \\ & + l_\mu(\bar{a}_{,\nu} + \Omega^{-1}\bar{a}\,\Omega_{,\nu}) + l_\nu(\Omega^{-1}\bar{a}\,\Omega_{,\mu}) \\ & + m_\nu(\Omega^{-2}\Omega_{,\mu}) - g_{\mu\nu}(\Omega^{-1}\bar{a}\,D\Omega + \Omega^{-2}\delta\Omega)\,. \end{aligned} \qquad (6.3)$$

With the spin coefficients defined in (5.4) and the usual choice of tetrad system giving

$$\tilde{\kappa} = \kappa = 0 = \tilde{\epsilon} = \epsilon = \tilde{\pi} = \pi \quad , \qquad (6.4)$$

the remaining spin coefficients transform according to

$$\tilde{\rho} = \Omega^2\rho + \Omega D\Omega$$
$$\tilde{\lambda} = \lambda + \Omega^2 a^2\rho + 2\Omega a\alpha + \bar{\delta}(\Omega a) + \tfrac{1}{2}D(\Omega^2 a^2)$$
$$\tilde{\alpha} = \Omega\alpha + \Omega^2 a\rho + \Omega D(\Omega a)$$
$$\tilde{\sigma} = \Omega^2\sigma$$
$$\tilde{\mu} = \mu + 2\Omega a\beta + \Omega^2 a^2\sigma - \Omega^{-1}\Delta\Omega + \Omega\delta a$$
$$\tilde{\beta} = \Omega\beta + \Omega^2 a\sigma$$
$$\begin{aligned}\tilde{\nu} &= \Omega^{-1}\nu + 2a\gamma + a\mu + \bar{a}\lambda + \Omega^2 a^3\sigma + 2\Omega a^2\beta + 2\Omega a\bar{a}\alpha \\ &+ \Omega a^2\tau + \Omega^2 a^2\bar{a}\rho + \Delta a + \Omega a\delta a + \Omega\bar{a}\bar{\delta}a + \Omega^{-1}a\Delta\Omega \\ &+ \Omega\bar{a}a^2 D\Omega + a^2\delta\Omega + 2a\bar{a}\bar{\delta}\Omega\end{aligned}$$
$$\tilde{\tau} = \Omega\tau + \Omega^2 a\sigma + \Omega^2\bar{a}\rho + \delta\Omega + \Omega\bar{a}D\Omega$$
$$\tilde{\gamma} = \gamma + \Omega a\beta + \Omega\bar{a}\alpha + \Omega a\tau + \Omega^2 a^2\sigma + \Omega^2 a\bar{a}\rho + \Omega^{-1}\Delta\Omega + \Omega D(a\bar{a}\Omega). \tag{6.5}$$

We note that taking $\tilde{\rho} = \bar{\tilde{\rho}}$ in $\tilde{\mathcal{R}}$ (thus making $\tilde{\ell}_\mu$ hypersurface-orthogonal) results in $\rho = \bar{\rho}$ in $\mathcal{R}$, so ℓ_μ is also hypersurface-orthogonal. If $\tilde{\ell}_\mu$ is furthermore equal to a gradient field (i.e. if $\bar{\tilde{\alpha}} + \tilde{\beta} = \tilde{\tau}$) then again ℓ_μ is also a gradient field ($\bar{\alpha} + \beta = \tau$) in $\mathcal{R}$.

To obtain the transformations of the spinor decomposition of theRiemann tensor, introduce the spinor dyad $\tilde{\zeta}_a^{\ A}$ in $\tilde{\mathcal{R}}$ by

$$\tilde{\zeta}_0^{\ A} = \tilde{\xi}^A \quad , \quad \tilde{\zeta}_1^{\ A} = \tilde{\iota}^A \quad ,$$

and then the dyad components of $\tilde{\Psi}_{ABCD}$ are given by

$$\tilde{\Psi}_{abcd} = \tilde{\Psi}_{ABCD}\,\tilde{\zeta}_a^{\ A}\tilde{\zeta}_b^{\ B}\tilde{\zeta}_c^{\ C}\tilde{\zeta}_d^{\ D} \ . \tag{6.6}$$

We have, using (2.9)

$$\tilde{\Psi}_{0000} = \Omega^4\,\Psi_{0000} \quad ,$$

which becomes, on the adoption of the Newman-Penrose definitions

$$\tilde{\Psi}_0 = \Omega^4\,\Psi_0 \ . \tag{6.7}$$

In a similar manner the remaining components of the Weyl spinor may be shown to transform according to

$$\left.\begin{aligned}\tilde{\Psi}_1 &= \Omega^3\Psi_1 + a\Omega^4\Psi_0 \\ \tilde{\Psi}_2 &= \Omega^2\Psi_2 + 2a\Omega^3\Psi_1 + a^2\Omega^4\Psi_0 \\ \tilde{\Psi}_3 &= \Omega\Psi_3 + 3a\Omega^2\Psi_2 + 3a^2\Omega^3\Psi_1 + a^3\Omega^4\Psi_0 \\ \tilde{\Psi}_4 &= \Psi_4 + 4a\Omega\Psi_3 + 6a^2\Omega^2\Psi_2 + 4a^3\Omega^3\Psi_1 + a^4\Omega^4\Psi_0\end{aligned}\right\} \tag{6.8}$$

To derive the transformations of $\Phi_{AB\dot{C}\dot{D}}$ we use (2.10), which gives, on lowering suffices,

$$\tilde{\Phi}_{AB\dot{C}\dot{D}} = \Omega^2 \Phi_{AB\dot{C}\dot{D}} + \tfrac{1}{2}\Omega\left[\nabla_{A\dot{C}}\nabla_{B\dot{D}}\Omega + \nabla_{B\dot{C}}\nabla_{A\dot{D}}\Omega\right].$$

Putting $\tilde{\Phi}_{AB\dot{C}\dot{D}} = 0$ restricts our results to the case where the physical space $\tilde{\mathcal{R}}$ is empty.

(The more general situation where $\tilde{\mathcal{R}}$ is non-empty is in any case very straightforward). Translating the above equation into spinor dyad form, we have (with $\tilde{\Phi}_{AB\dot{C}\dot{D}} = 0$)

$$\begin{aligned}\Phi_{ab\dot{c}\dot{d}} &= \Phi_{AB\dot{C}\dot{D}}\, \zeta^A_a \zeta^B_b \bar{\zeta}^{\dot{C}}_{\dot{c}} \bar{\zeta}^{\dot{D}}_{\dot{d}} \\ &= -\tfrac{1}{2}\Omega^{-1}\left\{\zeta^A_a \zeta^B_b \bar{\zeta}^{\dot{C}}_{\dot{c}} \bar{\zeta}^{\dot{D}}_{\dot{d}} \nabla_{A\dot{C}}\nabla_{B\dot{D}}\Omega + \zeta^A_a \zeta^B_b \bar{\zeta}^{\dot{C}}_{\dot{c}} \bar{\zeta}^{\dot{D}}_{\dot{d}} \nabla_{B\dot{C}}\nabla_{A\dot{D}}\Omega\right\}.\end{aligned} \tag{6.9}$$

Now, we can show that

$$\zeta^A_a \zeta^B_b \bar{\zeta}^{\dot{C}}_{\dot{c}} \bar{\zeta}^{\dot{D}}_{\dot{d}} \nabla_{A\dot{C}}\nabla_{B\dot{D}}\Omega = \partial_{a\dot{c}}\partial_{b\dot{d}}\Omega + \epsilon^{pq}\Gamma_{bpa\dot{c}}\,\partial_{q\dot{d}}\Omega + \epsilon^{\dot{r}\dot{s}}\bar{\Gamma}_{\dot{d}\dot{r}\dot{c}a}\,\partial_{b\dot{s}}\Omega .$$

Therefore

$$\begin{aligned}\Phi_{ab\dot{c}\dot{d}} = -\tfrac{1}{2}\Omega^{-1}\Big\{&\partial_{a\dot{c}}\partial_{b\dot{d}}\Omega + \partial_{b\dot{c}}\partial_{a\dot{d}}\Omega + \epsilon^{pq}\Gamma_{bpa\dot{c}}\,\partial_{q\dot{d}}\Omega \\ &+ \epsilon^{pq}\Gamma_{apb\dot{c}}\,\partial_{q\dot{d}}\Omega + \epsilon^{\dot{r}\dot{s}}\bar{\Gamma}_{\dot{d}\dot{r}\dot{c}a}\,\partial_{b\dot{s}}\Omega + \epsilon^{\dot{r}\dot{s}}\bar{\Gamma}_{\dot{d}\dot{r}\dot{c}b}\,\partial_{a\dot{s}}\Omega\Big\}.\end{aligned} \tag{6.10}$$

Then, using $\kappa = \epsilon = \pi = 0$ we find

$$\begin{aligned}
\Phi_{00} &= -\Omega^{-1} D^2\Omega \\
\Phi_{01} &= -\Omega^{-1} D\delta\Omega \\
\Phi_{11} &= -\tfrac{1}{2}\Omega^{-1}\left\{D\Delta\Omega + \bar{\delta}\delta\Omega + \rho\Delta\Omega - \alpha\delta\Omega + \bar{\beta}(\Delta\Omega + \delta\Omega) - \bar{\mu}(\bar{\delta}\Omega + D\Omega)\right\} \\
\Phi_{12} &= -\tfrac{1}{2}\Omega^{-1}\left\{\delta\Delta\Omega + \Delta\delta\Omega + (\bar{\alpha}+\beta+\tau)\Delta\Omega - (\gamma-\bar{\gamma})\delta\Omega - \bar{\lambda}\bar{\delta}\Omega - \bar{\nu}D\Omega\right\} \\
\Phi_{10} &= -\tfrac{1}{2}\Omega^{-1}\left\{D\bar{\delta}\Omega + \bar{\delta}D\Omega + \rho\bar{\delta}\Omega + \bar{\sigma}\delta\Omega - (\alpha+\bar{\beta})D\Omega\right\} \\
\Phi_{21} &= -\Omega^{-1}\left\{\bar{\delta}\Delta\Omega + (\alpha+\bar{\beta})\Delta\Omega - \lambda\delta\Omega - \bar{\mu}\bar{\delta}\Omega\right\} \\
\Phi_{02} &= -\Omega^{-1}\left\{\delta^2\Omega + \sigma\Delta\Omega - \bar{\lambda}D\Omega + (\bar{\alpha}-\beta)\delta\Omega\right\} \\
\Phi_{22} &= -\Omega^{-1}\left\{\Delta^2\Omega + (\gamma+\bar{\gamma})\Delta\Omega - \nu\delta\Omega - \bar{\nu}\bar{\delta}\Omega\right\} \\
\Phi_{20} &= -\Omega^{-1}\left\{\bar{\delta}^2\Omega + (\alpha+\bar{\beta})\bar{\delta}\Omega - \lambda D\Omega\right\} .
\end{aligned} \tag{6.11}$$

Finally, translating (2.11) into spinor dyad form gives (once again for the case when $\tilde{\mathcal{R}}$ is empty),

$$\Lambda = \tfrac{1}{2}\Omega^{-1}\{D\Delta\Omega - \delta\bar{\delta}\Omega - \bar{\rho}\Delta\Omega + \mu D\Omega + (\bar{\alpha}-\beta)\bar{\delta}\Omega\} + \Omega^{-4}\{(D\Omega)(\Delta\Omega) - (\delta\Omega)(\bar{\delta}\Omega)\} . \tag{6.12}$$

6.7 Choice of coordinates for $\mathcal{R}$

As a prelude to assigning coordinates to $\mathcal{R}$ we use the tetrad freedom of 6.5 to specialise the tetrad vectors. We use (5.29) with A = 1, C = 0, (B $\neq$ 0), to set

$$\Omega_{,\mu} = -n_\mu \tag{7.1}$$

on $\mathcal{J}^+$. Since $\ell^\mu \Omega_{,\mu} = -1$ (5.19), we can write

$$\Omega_{,\mu} = a_1 \ell_\mu - n_\mu + a_3 m_\mu + \bar{a}_3 \bar{m}_\mu \quad , \tag{7.2}$$

and from $\Omega_{,\mu}\Omega^{,\mu} = 0$ (4.1b) we have

$$a_1 + a_3 \bar{a}_3 = 0 \qquad \text{on } \mathcal{J}^+.$$

Therefore

$$\Omega_{,\mu} = -a_3\bar{a}_3 \ell_\mu - n_\mu + a_3 m_\mu + \bar{a}_3 \bar{m}_\mu \ .$$

The transformation (5.29) with A = 1, B = $-\bar{a}_3$, C = 0 gives

$$\Omega_{,\mu} = -n'_\mu \qquad \text{on } \mathcal{J}^+. \tag{7.3}$$

Further tetrad freedom is limited to

$$\ell'^\mu = A\ell^\mu \ , \quad n'^\mu = A^{-1} n^\mu \ , \quad m'^\mu = e^{iC} m^\mu , \tag{7.4}$$

with DA = DC = 0.

Making the choice $\Omega_{,\mu} = -n_\mu$ and using $\nabla_\mu \nabla_\nu \Omega = 0$ on $\mathcal{J}^+$ gives the following simplifications of spin coefficients on $\mathcal{J}^+$:-

$$\lambda = \mu = \nu = 0 \quad , \tag{7.5}$$

$$\bar{\gamma} + \gamma = 0 \quad , \quad \bar{\alpha} + \beta = 0. \tag{7.6}$$

By exploiting the freedom in choice of the $m^\mu, \bar{m}^\mu$ vectors of the tetrad we can further set

$$\gamma - \bar{\gamma} = 0 , \tag{7.7}$$

which gives finally

$$\gamma = 0 \quad \text{on } \mathscr{I}^+. \tag{7.8}$$

(7.7) is achieved as follows;

Under $\ell'^\mu = A\ell^\mu$, $n'^\mu = A^{-1}n^\mu$, $m'^\mu = e^{iC}m^\mu$ we have

$$\begin{aligned} m'_{\mu;\nu}\,\bar{m}'^\mu &= \left(e^{iC}m_{\mu;\nu} + iC_{,\nu}e^{iC}m_\mu\right)e^{-iC}\bar{m}^\mu \\ &= m_{\mu;\nu}\,\bar{m}^\mu - iC_{,\nu} . \end{aligned}$$

Hence $\gamma' - \bar{\gamma}' = -m'_{\mu;\nu}\bar{m}'^\mu n'^\nu = A^{-1}(\gamma - \bar{\gamma}) + iA^{-1}n^\nu C_{,\nu}$

so that we can choose C to make $\gamma' - \bar{\gamma}' = 0.$

The same tetrad transformation gives

$$\gamma' + \bar{\gamma}' = A^{-1}(\gamma + \bar{\gamma}) + A^{-2}\Delta A$$

and $$\bar{\alpha}' + \beta' = e^{iC}(\bar{\alpha} + \beta) + A^{-1}e^{iC}\delta A ,$$

so that (7.6) are preserved provided the additional restrictions

$$\Delta A = 0 = \delta A \tag{7.9}$$

are imposed. The tetrad freedom that remains is

$$\ell'^\mu = A\ell^\mu , \quad n'^\mu = A^{-1}n^\mu , \quad m'^\mu = e^{iC}m^\mu , \tag{7.10}$$

with A = constant, $DC = 0 = \Delta C$.

The geometrical significance of (7.5) is to indicate that there exists a congruence of null geodesics with tangent vector n_μ which is divergence-free and shear-free on $\mathscr{I}^+$. We utilise this congruence of null geodesics in $\mathscr{I}^+$ to set up a coordinate system valid in some region of $\mathcal{R}$ containing the null hypersurface $\mathscr{I}^+$.

Consider the null congruence given by

$$\frac{dx^\mu}{du} = n^\mu \tag{7.11}$$

where

$$n_\mu = -\Omega_{,\mu} . \tag{7.12}$$

This is a congruence of null geodesics lying in $\mathscr{I}^+$, with u an affine parameter for these geodesics. Select some space-like slice Σ_0 of $\mathscr{I}^+$ and label the geodesics of the above congruence by the coordinates x^A of points in the Σ_0 2-space where the geodesics cut Σ_0. Assign to each point on the same null geodesic the same labels x^A. Then u, x^A serve as coordinates for $\mathscr{I}^+$. Next consider the family of space-like slices Σ of $\mathscr{I}^+$ given by u = constant. Let N denote the unique outgoing null hypersurface intersecting $\mathscr{I}^+$ in a particular Σ 2-space.

Emanating from each point x^A of this slice there is exactly one null geodesic orthogonal to Σ and lying in N. Assign the labels x^A to all points on this null geodesic and the same label u to all points of N. Applying this procedure to the entire family of u = constant slices defines the coordinates u, x^A off $\mathscr{I}^+$.

Define r to be an affine parameter for the null geodesics emanating from the Σ-slices. $\mathscr{I}^+$ is given by r = 0. Choose

$$\ell^{\mu} = \frac{dx^{\mu}}{dr} \tag{7.13}$$

to be the second real null vector of the tetrad at each point. For the region of $\mathcal{R}$ in which this coordinate system is well-defined we have

$$g_{\mu\nu} = \begin{pmatrix} g_{11} & 1 & g_{1A} \\ 1 & 0 & 0 \\ g_{1A} & 0 & g_{AB} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & g^{22} & g^{2A} \\ 0 & g^{2A} & g^{AB} \end{pmatrix}. \tag{7.14}$$

Using the fact that u is an affine parameter for null geodesics in $\mathscr{I}^+$ we have

$$g_{11} = g_{1A} = 0 \qquad \text{on } \mathscr{I}^+, \tag{7.15}$$

(or, equivalently, $g^{22} = g^{2A} = 0$ on $\mathscr{I}^+$)

$$\text{and} \qquad \frac{\partial}{\partial r} g_{11} = 0 \qquad \text{on } \mathscr{I}^+. \tag{7.16}$$

If q_{AB} denotes the metric of the Σ 2-surfaces in $\mathscr{I}^+$, then

$$g_{AB} = q_{AB} \quad \text{on } \mathscr{I}^+. \tag{7.17}$$

The null tetrad for the region of $\mathcal{R}$ described by our coordinate system is given by

$$\left.\begin{aligned} l^\mu &= \delta^\mu_2 \\ n^\mu &= \delta^\mu_1 + U\,\delta^\mu_2 + X^A\,\delta^\mu_A \\ m^\mu &= \omega\,\delta^\mu_2 + \xi^A\,\delta^\mu_A \ , \end{aligned}\right\} \qquad (7.18)$$

where

$$\begin{aligned} g^{22} &= 2(U - \omega\bar{\omega}) \\ g^{2A} &= X^A - (\bar{\omega}\,\xi^A + \omega\,\bar{\xi}^A) \\ g^{AB} &= -(\xi^A\bar{\xi}^B + \bar{\xi}^A\xi^B) \ . \end{aligned} \qquad (7.19)$$

On $\mathcal{J}^+$, $U = X^A = \omega = 0$, so that these quantities are $O(r)$ near $\mathcal{J}^+$. The notation used in (7.18), (7.19) is in conformity with that of Newman and Penrose [17]. The tetrad vectors thus take the following form on $\mathcal{J}^+$;

$$\begin{aligned} l^\mu &= \delta^\mu_2 \ , & l_\mu &= \delta^1_\mu \ , \\ n^\mu &= \delta^\mu_1 \ , & n_\mu &= \delta^2_\mu \ , \\ m^\mu &= \xi^A\delta^\mu_A \ , & m_\mu &= \xi_A\,\delta^A_\mu \ , \\ \bar{m}^\mu &= \bar{\xi}^A\,\delta^\mu_A \ , & \bar{m}_\mu &= \bar{\xi}_A\,\delta^A_\mu \ , \end{aligned} \qquad (7.20)$$

where $\xi^A\xi_A = 0$ and $\xi^A\bar{\xi}_A = -1$. (7.21)

An important aspect of the geometry of $\mathcal{J}^+$ appears if we apply the commutation relation (7.31) of Chapter 2 to $\phi = x^A$, for this gives

$$\frac{\partial}{\partial u}\xi^A = 0 \quad \text{on } \mathcal{J}^+, \qquad (7.22)$$

from which

$$\frac{\partial}{\partial u} g_{AB} = 0 = \frac{\partial}{\partial u} g^{AB} \quad \text{on } \mathcal{J}^+ . \qquad (7.23)$$

Thus, we may make a choice of metric for the initial 2-space slice Σ_0 with the assurance that successive space-like slices Σ of $\mathcal{J}^+$ retain the same intrinsic metric. It is convenient to give the Σ 2-spaces the metric of a unit sphere and to choose coordinates x^A in one of the following ways:

(i) take (x^3, x^4) to be the coordinates employed in Chapter 3, giving

$$\left.\begin{aligned} g_{AB} &= -\tfrac{1}{2}\,P^{-2}\,\delta_{AB} \ , \\ \text{where} \quad P &= \frac{1}{\sqrt{2}}\cosh x^3 , \end{aligned}\right\} \qquad (7.24)$$

or, equivalently,

$$\left.\begin{aligned} \xi^3 &= P \quad , \quad \xi^4 = i\,P\,, \\ \xi_3 &= -\frac{1}{2P} \quad , \quad \xi_4 = -\frac{i}{2P} \quad . \end{aligned}\right\} \qquad (7.25)$$

(ii) take spherical polars (θ,ϕ) or stereographic coordinates $(\zeta,\bar{\zeta})$ to establish contact with the $\eth$ operator as developed in [38].

For our purposes the (x^3,x^4) coordinates in (i) above will be chosen, so that immediate comparison of our present results with those of Chapter 5 can be made. However, the $\eth$ operator formalism and the spin-s spherical harmonics ${}_sY_{\ell m}$ play a natural role in the analysis, and with that in mind we have already developed the required material in (x^3,x^4) coordinates in Chapter 2.

Using Newman-Penrose definitions we have now

$$\alpha-\bar{\beta} = \tfrac{1}{2}\left\{\xi^A\bar{\xi}^B\bar{\xi}_{A;B} - \bar{\xi}^A\bar{\xi}^B\xi_{A;B}\right\} = \frac{1}{\sqrt{2}}\sinh x^3, \qquad (7.26)$$

and making use of (7.6) gives, on $\mathscr{I}^+$,

$$\alpha \doteq \frac{1}{2\sqrt{2}}\sinh x^3 \quad , \quad \beta \doteq \frac{-1}{2\sqrt{2}}\sinh x^3. \qquad (7.27)$$

It is straightforward to show that on $\mathscr{I}^+$ the $\eth$ operator acts according to

$$\frac{1}{\sqrt{2}}\,\eth\eta \doteq -\delta\eta - s(\alpha-\bar{\beta})\eta \quad , \qquad (7.28)$$

and

$$\frac{1}{\sqrt{2}}\,\bar{\eth}\eta \doteq -\bar{\delta}\eta + s(\alpha-\bar{\beta})\eta \quad , \qquad (7.29)$$

where η is a function of spin-weight s, suitably defined on $\mathscr{I}^+$.

The above choice of coordinates and tetrad gives a simplification of the spin coefficients in some neighbourhood of $\mathcal{R}$ containing $\mathscr{I}^+$, and will enable us to present the conformal Killing equations in a particularly simple form in this region. To be precise, we have

$$\kappa = \pi = \epsilon = 0, \quad \tau = \bar{\alpha}+\beta \quad , \quad \rho = \bar{\rho}\,, \qquad (7.30)$$

$$\alpha = -\beta = \frac{1}{2\sqrt{2}}\sinh x^3 + O(r)\,, \qquad (7.31)$$

$$\sigma = \sigma_0 + \sigma_1 r + O(r^2)\,, \qquad (7.32)$$

$$\rho = \rho_1 r + O(r^2)\,, \qquad (7.33)$$

sufficiently near to $\mathscr{I}^+$, and all other spin coefficients are $O(r)$. To account for (7.32), (7.33) we note first that

$$D\Omega = -1 \quad , \qquad DD\Omega = 0 \qquad \text{on } \mathcal{J}^+ ,$$

the second of these conditions following from (4.8) because

$$\begin{aligned} l^\mu \nabla_\mu (l^\nu \nabla_\nu \Omega) &= l^\mu \{ l^\nu \nabla_\mu \nabla_\nu \Omega + \nabla_\nu \Omega \nabla_\mu l^\nu \} \\ &= (l^\mu \nabla_\mu l^\nu) \nabla_\nu \Omega \\ &= 0 , \text{ since } l^\mu \text{ is tangent} \end{aligned}$$

to geodesics. Then

$$\Omega = - r + O(r^3) \tag{7.34}$$

and so (5.4) leads to

$$\tilde{r}^{-1} = - r + O(r^3). \tag{7.35}$$

The results

$$\begin{aligned} \tilde{\rho} &= -\tilde{r}^{-1} + O(\tilde{r}^{-3}) , \\ \tilde{\sigma} &= \sigma_0 \tilde{r}^{-2} + O(\tilde{r}^{-4}) , \end{aligned}$$

given in [31], lead immediately to expansions for ρ, σ in the form (7.32), (7.33) when use is made of (7.34), (7.35) and the appropriate relations from (6.5).

6.8 The conformal Killing equations in $\mathcal{R}$.

The simplifications arising from a judicious choice of tetrad and coordinates in the conformal space $\mathcal{R}$ are now employed in setting down the conformal Killing equations in $\mathcal{R}$. In addition to (7.30-33) we shall use the fact that the other spin coefficients are $O(r)$ sufficiently near $\mathcal{J}^+$. We adopt the null tetrad (7.18), noting again that U, X^A, ω are $O(r)$ and that our choice of coordinates (x^3, x^4) gives

$$\xi^3 = \frac{1}{\sqrt{2}} \cosh x^3 + O(r) \quad , \tag{8.1}$$

$$\xi^4 = \frac{i}{\sqrt{2}} \cosh x^3 + O(r) \quad , \tag{8.2}$$

sufficiently near $\mathcal{J}^+$.

We follow the work of Chapter 2 by introducing tetrad components for the conformal Killing vector field $\mathcal{S}^\mu$ according to the following scheme;

$$V_1 = \xi^\mu l_\mu = \xi^1, \tag{8.3}$$

$$V_2 = \xi^\mu n_\mu = \xi^2 + O(r), \tag{8.4}$$

$$V_3 = \xi^\mu m_\mu = -\frac{1}{\sqrt{2}}\operatorname{sech} x^3\, \xi^3 - \frac{i}{\sqrt{2}}\operatorname{sech} x^3\, \xi^4 + O(r) \tag{8.5}$$

$$V_4 = \xi^\mu \bar{m}_\mu = -\frac{1}{\sqrt{2}}\operatorname{sech} x^3\, \xi^3 + \frac{i}{\sqrt{2}}\operatorname{sech} x^3\, \xi^4 + O(r) \tag{8.6}$$

In tetrad form the conformal Killing equations are

$$D V_1 = 0 \tag{8.7}$$

$$\Delta V_2 = -(\gamma+\bar{\gamma})V_2 + \nu V_3 + \bar{\nu} V_4 \tag{8.8}$$

$$\Delta V_1 + D V_2 - \tfrac{1}{2}\phi = (\gamma+\bar{\gamma})V_1 - \bar{\tau}V_3 - \tau V_4 \tag{8.9}$$

$$\delta V_3 = \bar{\lambda} V_1 - \sigma V_2 - (\bar{\alpha}-\beta)V_3 \tag{8.10}$$

$$\delta V_1 + D V_3 = \tau V_1 - \rho V_3 - \sigma V_4 \tag{8.11}$$

$$\delta V_2 + \Delta V_3 = \bar{\nu} V_1 - 2\tau V_2 + (\mu+\gamma-\bar{\gamma})V_3 + \bar{\lambda} V_4 \tag{8.12}$$

$$\bar{\delta} V_3 + \delta V_4 + \tfrac{1}{2}\phi = (\mu+\bar{\mu})V_1 - 2\rho V_2 + (\alpha-\bar{\beta})V_3 + (\bar{\alpha}-\beta)V_4 \;. \tag{8.13}$$

The integrability conditions for these equations have been given by Collinson and French in [36], and in Chapter 2 of the present work.

The equations (8.7 - 8.13) admit solutions in the form of power series expansions in the parameter r. Our object here is to find the leading terms in these expansions, giving us the "far-asymptotic" form of the conformal Killing vectors in $\mathcal{R}$.

We consider first the three "radial" equations of the above set. (8.7) gives immediately

$$V_1 = \overset{\circ}{V}_1(u, x^A) \;, \tag{8.14}$$

where $\overset{\circ}{V}_1$ is an arbitrary function (suitably well-behaved) in $\mathcal{R}$.

Writing (8.11) in the form

$$D V_3 = -\delta V_1 + \tau V_1 - \rho V_3 - \sigma V_4$$

and putting

$$V_3 = \overset{\circ}{V}_3 + \overset{1}{V}_3 r + \overset{2}{V}_3 r^2 + \cdots$$

$$V_4 = \overset{\circ}{V}_4 + \overset{1}{V}_4 r + \overset{2}{V}_4 r^2 + \cdots$$

shows that $\overset{1}{V}_3, \overset{1}{V}_4, \overset{2}{V}_3, \overset{2}{V}_4, \ldots$ etc. are determined in terms of $\overset{\circ}{V}_3, \overset{\circ}{V}_4$ and known functions but $\overset{\circ}{V}_3$ and $\overset{\circ}{V}_4$ themselves are not determined. We have, for example,

$$\overset{1}{V}_3 = \frac{1}{\sqrt{2}} \eth V_1 - \sigma_0 \overset{0}{V}_4 , \qquad (8.15)$$

$$\overset{1}{V}_4 = \frac{1}{\sqrt{2}} \bar{\eth} V_1 - \bar{\sigma}_0 \overset{0}{V}_3 , \qquad (8.16)$$

where we have used the fact that V_1, given in (8.14), is a spin-weight zero function on $\mathscr{I}^+$. The final radial equation (8.9) takes the form

$$D V_2 = -\Delta V_1 + \tfrac{1}{2}\phi + (\gamma+\bar{\gamma}) V_1 - \bar{\tau} V_3 - \tau V_4 . \qquad (8.17)$$

Making use of (8.13) and the relationship $\tau = \bar{\alpha} + \beta$ we can write (8.17) as

$$DV_2 + 2\rho V_2 = -\Delta V_1 - \bar{\delta} V_3 - \delta V_4 - 2\bar{\beta} V_3 - 2\beta V_4 + (\gamma + \bar{\gamma} + \mu + \bar{\mu}) V_1 . (8.18)$$

Assuming a solution for V_2 in the form

$$V_2 = \overset{0}{V}_2 + \overset{1}{V}_2 r + \overset{2}{V}_2 r^2 + \cdots$$

leads to a situation in which $\overset{1}{V}_2$, $\overset{2}{V}_2$, ... etc. are determined in terms of $\overset{0}{V}_2$, $\overset{0}{V}_3$, $\overset{0}{V}_4$ and known functions but $\overset{0}{V}_2$ is not determined. We find, for example, that the O(1) terms in (8.18) give

$$\overset{1}{V}_2 = -V_{1,1} + \frac{1}{\sqrt{2}} \bar{\eth} \overset{0}{V}_3 + \frac{1}{\sqrt{2}} \eth \overset{0}{V}_4 , \qquad (8.19)$$

where we have used the fact that $\overset{0}{V}_2$, $\overset{0}{V}_3$, $\overset{0}{V}_4$ have spin weights 0,1,-1 respectively. We would point out here that the conformal factor ϕ takes the form

$$\phi = \sqrt{2} \left(\bar{\eth} \overset{0}{V}_3 + \eth \overset{0}{V}_4 \right) \quad \text{on } \mathscr{I}^+, \qquad (8.20)$$

which follows at once from (8.13).

The "non-radial" equations (8.8), (8.10), (8.12) impose conditions on the leading terms $\overset{0}{V}_2$, $\overset{0}{V}_3$, $\overset{0}{V}_4$ introduced in expansions above. The O(1) terms in (8.8) give

$$\overset{0}{V}_{2,1} = 0 , \qquad (8.21)$$

from which

$$\overset{0}{V}_2 = \overset{0}{V}_2 (x^A) . \qquad (8.22)$$

There are of course other conditions arising from terms containing higher powers of r but since we require only a "first-order" solution they are not relevant to our purpose. These higher-order conditions would assume an important role if we wished to develop the Killing vector field away from $\mathscr{I}^+$.

The leading terms of (8.12) give

$$\eth \mathring{V}_2 = \sqrt{2}\, \mathring{V}_{3,1} \quad , \tag{8.23}$$

and finally, from the leading terms of (8.10) we obtain

$$\eth \mathring{V}_3 = \sqrt{2}\, \sigma_0 \mathring{V}_2 \ . \tag{8.24}$$

We comment that σ_0 has spin weight 2, a fact which follows from the definition of σ in terms of the tetrad vectors.

Before examining the asymptotic conformal symmetries in some specific cases, we take the opportunity here of summarising the pertinent equations from the foregoing analysis:-

$$V_1 = \mathring{V}_1(u, x^A) \quad , \tag{8.25a}$$

$$\mathring{V}_{2,1} = 0 \quad , \tag{8.25b}$$

$$\eth \mathring{V}_2 = \sqrt{2}\, \mathring{V}_{3,1} \quad , \tag{8.25c}$$

$$\eth \mathring{V}_3 = \sqrt{2}\, \sigma_0 \mathring{V}_2 \ . \tag{8.25d}$$

As an immediate consequence of these equations we see that a necessary condition for the existence of asymptotic conformal symmetries is that σ_0 should be linear in u, a result which was also obtained in §4 of Chapter 5. Differentiating (8.25d) with respect to u and making use of (8.25c) gives

$$\eth \eth \mathring{V}_2 = 2 \mathring{V}_2 \frac{\partial \sigma_0}{\partial u}$$

Differentiating with respect to u again, and making use of 8.25b) gives

$$\frac{\partial^2}{\partial u^2} \sigma_0 = 0 \ .$$

6.9 Asymptotic C-K vectors in asymptotically shear-free space-times.

Putting $\sigma_0 = 0$ in the equations (8.25) gives

$$V_1 = \mathring{V}_1(u, x^A) \quad , \tag{9.1a}$$

$$\mathring{V}_{2,1} = 0 \quad , \tag{9.1b}$$

$$\eth \mathring{V}_2 = \sqrt{2}\, \mathring{V}_{3,1} \quad , \tag{9.1c}$$

$$\eth \mathring{V}_3 = 0 \ . \tag{9.1d}$$

$\mathring{V}_3$ is of spin weight 1 and (9.1d) shows that it consists only of $\ell = 1$ harmonics since $\eth$ operates on spin-1 spherical harmonics according to

$$\eth_1 Y_{\ell m} = [(\ell-1)(\ell+2)]^{\frac{1}{2}} {}_2Y_{\ell m} .$$

Therefore, we can expand $\overset{\circ}{V}_3$ in the form

$$\overset{\circ}{V}_3 = \sum_{-1}^{+1} a_m(u) \, {}_1Y_{1m} , \tag{9.2}$$

where the $a_m(u)$ are functions of u only. From (9.1b) and (9.1c) it follows that

$$\overset{\circ}{V}_{3,1,1} = 0$$

so that

$$\frac{\partial^2}{\partial u^2} a_m(u) = 0 . \tag{9.3}$$

$\overset{\circ}{V}_2$ is of spin weight zero and may be expanded in the form

$$\overset{\circ}{V}_2 = \sum_{\ell,m} b_{\ell m}(u) \, {}_0Y_{\ell m} . \tag{9.4}$$

Substitution of (9.2) and (9.4) into (9.1c) gives

$$\sum_{\ell,m} b_{\ell m} [\ell(\ell+1)]^{\frac{1}{2}} {}_1Y_{\ell m} = \sqrt{2} \sum_m \frac{\partial a_m}{\partial u} \, {}_1Y_{1m} .$$

Making use of the orthogonality relations of the ${}_sY_{\ell m}$ [(8.30) of Chapter 2], we see that $\overset{\circ}{V}_2$ has the expansion

$$\overset{\circ}{V}_2 = \sum_{-1}^{+1} \frac{\partial a_m}{\partial u} \, {}_0Y_{1m} + \alpha \, {}_0Y_{00} , \tag{9.5}$$

where α is an arbitrary constant, and the $a_m(u)$ are coefficients in the expansion (9.2) of $\overset{\circ}{V}_3$.

Finally, we remark that $\overset{\circ}{V}_1$ can be expanded in the form

$$\overset{\circ}{V}_1 = \sum_{\ell,m} c_{\ell m}(u) \, {}_0Y_{\ell m} , \tag{9.6}$$

where the $c_{\ell m}$ are arbitrary functions of u.

Using the asymptotic relations

$$\tilde{V}_1 \sim V_1 ,$$
$$\tilde{V}_2 \sim \tilde{r}^2 V_2 ,$$
$$\tilde{V}_3 \sim \tilde{r} V_3 ,$$

which follow from (5.28), we can transform the tetrad components of the C.K. vectors from the unphysical space $\mathcal{R}$ to the physical space $\tilde{\mathcal{R}}$. We thus obtain the far-asymptotic terms of the generators of the asymptotic conformal symmetry group set down in §7 of the previous Chapter of this work. It is perhaps important to stress once more that although the Penrose conformal technique developed here has led to the far-asymptotic terms of the C-K vector field with relative ease, if our wish is to find the higher order terms then it has no virtues from a computational point of view over the asymptotic expansion technique that we used in Chapter 5. However, the Penrose method is extremely valuable in its provision of a rigorous foundation for the analysis of asymptotic problems , since it avoids the objections which might be levelled at the asymptotic expansion approach.

6.10 Asymptotic C-K vectors in space-times which are not asymptotically shear-free

In this final section we briefly reconsider the equations (8.25);

$$V_1 = \overset{\circ}{V}_1(u, x^A) \quad , \qquad (10.1a)$$

$$\overset{\circ}{V}_{2,1} = 0 \quad , \qquad (10.1b)$$

$$\eth \overset{\circ}{V}_2 = \sqrt{2}\, \overset{\circ}{V}_{3,1} \quad , \qquad (10.1c)$$

$$\eth \overset{\circ}{V}_3 = \sqrt{2}\, \sigma_o \overset{\circ}{V}_2 \quad , \qquad (10.1d)$$

in the case when

$$\sigma_o \neq 0 . \qquad (10.2)$$

The only conditions on σ_0 now are that it should be linear in u, a property which was noted at the end of §8 of the present chapter, and secondly, that it has spin weight 2. We shall show how, in principle, the above equations may be solved when the form of σ_0 is prescribed.

Suppose the expansion of σ_0 is given by

$$\sigma_o = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{+\ell} \sigma_{\ell m}(u) \; {}_2Y_{\ell m}(x^A) \quad , \qquad (10.3)$$

where the $\sigma_{\ell m}(u)$ with $\frac{d^2}{du^2}\sigma_{\ell m} = 0$, are known functions.

Then, assuming that $\overset{\circ}{V}_2$ is capable of expansion in the form

$$\overset{\circ}{V}_2 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \underset{(2)}{V}_{\ell m} \; {}_0Y_{\ell m}(x^A) \quad , \qquad (10.4)$$

where the ${}_{(2)}V_{\ell m}$ are constants, we know that $\sigma_0 \overset{\circ}{V}_2$ will have an expansion of the form

$$\sigma_0 \overset{\circ}{V}_2 = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{+\ell} \alpha_{\ell m}(u)\, {}_2Y_{\ell m}(x^A) , \qquad (10.5)$$

where $\frac{d^2}{du^2}\alpha_{\ell m} = 0$, and the $\alpha_{\ell m}$ contain combinations of products of the unknowns ${}_{(2)}V_{\ell m}$ and the known quantities $\sigma_{\ell m}(u)$. The explicit form of (10.5) may be generated from (10.3) and (10.4) in a case where the expansion of σ_0 is known by making use of the relation [59],

$$ {}_{s_1}Y_{\ell_1 m_1} \times {}_{s_2}Y_{\ell_2 m_2} = \sum_{\ell} \left[\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)} \right] \langle \ell_1, \ell_2 ; m_1, m_2 | \ell, m \rangle \langle \ell_1, \ell_2 ; -s_1, -s_2 | \ell, -s \rangle {}_sY_{\ell m} , \quad (10.6)$$

which expresses the product of two spin-s harmonics as a linear combination of other spin-weighted harmonics.

In (10.6)

$$m = m_1 + m_2$$

$$s = s_1 + s_2$$

$$|\ell_1 - \ell_2| \leq \ell \leq |\ell_1 + \ell_2| ,$$

and $\langle \ell_1, \ell_2 ; m_1, m_2 | \ell, m \rangle$ is a Clebsch-Gordan coefficient of the rotation group [60]. Expanding $\overset{\circ}{V}_3$ in the form

$$\overset{\circ}{V}_3 = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{+\ell} {}_{(3)}V_{\ell m}(u)\ {}_1Y_{\ell m}(x^A) , \qquad (10.7)$$

where the ${}_{(3)}V_{\ell m}(u)$ are linear functions of u, followed by substitution of (10.3,4,5,7) into the equations (10.1c,d) gives finally

$$\frac{d}{du}\, {}_{(3)}V_{\ell m} = \left[\frac{\ell(\ell+1)}{2} \right]^{\frac{1}{2}} {}_{(2)}V_{\ell m} , \qquad (10.8)$$

and

$$\alpha_{\ell m}(u) = \left[\frac{(\ell-1)(\ell+2)}{2} \right]^{\frac{1}{2}} {}_{(3)}V_{\ell m} , \qquad (10.9)$$

where the usual orthogonality relation on spin-s spherical harmonics of the same spin weight has been employed. (For this orthogonality property see for example [38] or Chapter 2 of the present work). The quantities $\alpha_{\ell m}$ in

(10.9) involve the unknowns ${}_{(2)}V_{\ell m}$ and the (known) functions $\sigma_{\ell m}(u)$, so in any particular case (10.8) and (10.9) can be solved to give the ${}_{(2)}V_{\ell m}$ and the ${}_{(3)}V_{\ell m}$ in terms of known functions. This esentially completes the solution scheme; one need only add that from (10.1a) it follows that V_1 is of the form

$$V_1 = \overset{0}{V}_1 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} {}_{(1)}V_{\ell m}(u)\ {}_{0}Y_{\ell m} \quad (10.10).$$

CHAPTER 7

PHYSICAL INTERPRETATION OF THE CONFORMAL TRANSFORMATIONS IN MINKOWSKI SPACE-TIME

7.1 Introduction

Since the pioneer work of Bateman and Cunningham [7,8] in 1910, the Minkowski space conformal group has, until recent years, made only occasional reappearances in the literature. We give below a survey of the early work in this area, tracing the conceptual development from the conform-invariance of Maxwell's equations, through Page's "new relativity", and onwards to the more recent interpretations proposed by particle-physicists, notably Kastrup [40,61] and Rosen [62]. We shall adhere throughout to the terminology and notation of Chapter 4. In particular, the special conformal transformations, which are the most interesting part of the group as far as interpretation is concerned, will be denoted by $\delta(a^{\mu})$: $x^{\mu} \rightarrow \bar{x}^{\mu}$ with

$$\bar{x}^{\mu} = \frac{x^{\mu} - a^{\mu}(\underline{x}.\underline{x})}{1 - 2\,\underline{a}.\underline{x} + (\underline{a}.\underline{a})\,(\underline{x}.\underline{x})} \quad . \tag{1.1}$$

The paper of Cunningham referred to above falls into two sections. The material of the first part is of no concern to us here, but the second part presents what the author calls an extension of the relativity principle of Einstein. At that time the usual foundation of special relativity was the assumption, well supported by observation, that no experiment of an electromagnetic nature could be used to detect the uniform motion of an observer relative to the aether. The mathematical content of the principle lay in the fact that the Maxwell field equations are invariant under the Lorentz transformations. Cunningham's work consisted in showing that the Maxwell equations are also invariant under the group of conformal transformations in Minkowski space. He saw that these transformations can be built up of inversions in the hyperspheres of the space, and the results in his paper come from consideration of the simplest conformal transformation of this type; viz, inversion with the origin as centre. He acknowledges that the inversions, no matter how simple they may be algebraically, are not easy to describe in geometrical terms. However, it is pointed out that no conformal transformation can be found to connect frames of reference in uniform relative rotation, and, in consequence, the equations of electrodynamics will not be invariant under transformation to a rotating frame. We shall have reason to mention this point again in section 7.3.

7.2 Conformal transformations as transformations of coordinates

The early work on the Minkowski space conformal group took the "passive" view-point of transformations as changes of coordinates in flat space-time. It is probable that the influence of Einstein's original work on special relativity and the Lorentz transformations prompted this interpretation of the more general conformal transformations. The outcome of this approach is to give a new description of flat space-time using certain systems of curvilinear coordinates. The transformations between coordinate systems of this type were widely interpreted as transformations between frames of reference in constant relative acceleration, in contradistinction to the preferred frames of reference in special relativity which have constant relative velocities. This sort of physical interpretation is made in the so-called "non-relativistic" limit as $c \to \infty$.

We shall look at the argument leading to this interpretation by taking a special conformal transformation (1·1) with

$$a^{\mu} = (0,\ a,\ 0,\ 0). \qquad (2.1)$$

This particular choice of parameters will simplify the algebra without destroying the generality of the conclusions. Since it is necessary at a later stage of the calculation to pick out terms of specific orders in c, the speed of light, we will take coordinates

$$x^{\mu} = (ct,\ x,\ y,\ z) \qquad (2.2)$$

instead of making the usual choice in which $c = 1$. An additional advantage in making this choice of coordinates is the consequent possibility of using dimensional considerations as a check in some of the unwieldy algebra. The conformal transformation that we are considering takes the form

$$t' = \frac{t}{1 + 2ax - a^2(x^\alpha x_\alpha)} , \qquad (2.3a)$$

$$x' = \frac{x - a(x^\alpha x_\alpha)}{1 + 2ax - a^2(x^\alpha x_\alpha)} , \qquad (2.3b)$$

$$y' = \frac{y}{1 + 2ax - a^2(x^\alpha x_\alpha)} , \qquad (2.3c)$$

$$z' = \frac{z}{1 + 2ax - a^2(x^\alpha x_\alpha)} , \qquad (2.3d)$$

where $x^\alpha x_\alpha = c^2t^2 - x^2 - y^2 - z^2$. It is a straightforward, if tedious, matter to calculate the expressions connecting the 3-velocities $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ and $\left(\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'}\right)$. They are

$$\frac{dx'}{dt'} = \frac{1}{\Sigma}\left\{ -2ac^2t(1+ax) + \left[1+2ax+a^2(c^2t^2+x^2-y^2-z^2)\right]\frac{dx}{dt} + 2a(1+ax)y\frac{dy}{dt} + 2a(1+ax)z\frac{dz}{dt}\right\} , \qquad (2.4a)$$

$$\frac{dy'}{dt'} = \frac{1}{\Sigma}\left\{ 2a^2c^2ty - 2ay(1+ax)\frac{dx}{dt} + \left[1+2ax-a^2(c^2t^2-x^2+y^2-z^2)\right]\frac{dy}{dt} - 2a^2yz\frac{dz}{dt}\right\} , \qquad (2.4b)$$

$$\frac{dz'}{dt'} = \frac{1}{\Sigma}\left\{ 2a^2c^2tz - 2az(1+ax)\frac{dx}{dt} - 2a^2yz\frac{dy}{dt} + \left[1+2ax-a^2(c^2t^2-x^2-y^2+z^2)\right]\frac{dz}{dt}\right\} , \qquad (2.4c)$$

where

$$\Sigma = \left[1+2ax+a^2(x^\alpha x_\alpha)\right] - 2at(1+ax)\frac{dx}{dt} - 2a^2yt\frac{dy}{dt} - 2a^2zt\frac{dz}{dt}. \qquad (2.4d)$$

We shall put $\frac{dx'}{dt'} = 0 = \frac{dy'}{dt'} = \frac{dz'}{dt'}$ in (2.4a - d), let $c \rightarrow \infty$

to give the non-relativistic limit, and solve the resulting differential equations for the trajectory in the x^μ coordinates of a particle which is at rest in the $x^{\mu'}$ coordinates.

(2.4a) reduces to

$$\frac{dx}{dt} = \frac{2}{at}(1 + ax) \qquad (2.5)$$

from which

$$x = \frac{At^2}{a} - \frac{1}{a}, \qquad (2.6)$$

where A is an arbitrary constant. This is evidently a motion in the x-direction with constant acceleration $\frac{2A}{a}$. The other equations (2.4 b,c) lead to

$$\frac{dy}{dt} = \frac{2y}{t} \quad \text{and} \quad \frac{dz}{dt} = \frac{2z}{t},$$

with solutions

$$y = Bt^2, \quad z = Ct^2,$$

where B, C are arbitrary constants. We see then that, in the limit as $c \to \infty$, a particle at rest in the frame coordinatised by the $x^{\mu'}$ is observed to move with constant acceleration in the frame coordinatised by the x^μ. We therefore interpret the change of coordinates here as a transformation between two frames of reference in constant relative acceleration.

An immediate objection can be raised against this particular physical interpretation of the conformal transformations when it is noted that a relative velocity greater than the velocity of light is apparently allowed by the transformations linking $\frac{dx}{dt}$ and $\frac{dx'}{dt'}$. If, in the equations (2.4 a-d) we put $x = y = z = 0$ and $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$ it follows readily that

$$\frac{dx'}{dt'} = -2ac^2 t',$$

where we have made use of (2.3a). Evidently then, for $t' > \frac{1}{2ac}$ we have $\left|\frac{dx'}{dt'}\right| > c$. This result suggests that a particle at rest at the origin in one frame might be seen to travel with a speed exceeding the speed of light in a second frame. This is in contradiction to observation and would therefore seem to remove all physical importance from the "passive" interpretation of the transformations.

7.3 Page's "New Relativity"

In 1936 Page [63] and Page and Adams [64] described a new relativity that was subsequently seen to depend upon the flat-space conformal group in the same sense that Einstein's special relativity of 1905 depended upon the Lorentz group. The Page theory was based on two primary assumptions;

(1) there exists at least one Euclidean reference system with constant light velocity;

(2) all equivalent Euclidean reference systems with constant light velocity are physically indistinguishable as regards the formulation of the laws of nature.
("equivalent" here amounts to the property of being related one to another by transformations of the conformal group viewed in the passive sense.)

Page developed his work from the belief that only the properties of light should be used in the operational methodology of space-time theories; the Einsteinian concepts of the rigid measuring rod and the ideal clock do not appear. The consequences of Page's assumptions depend upon the fact that in his theory the metric interval

$$c^2dt^2 - dx^2 - dy^2 - dz^2$$

between neighbouring events is no longer invariant. Instead, the interval in two "Page-equivalent" reference frames transforms according to

$$c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = K(x^\mu)\left[c^2dt^2 - dx^2 - dy^2 - dz^2\right].$$

The invariant quantity in Page's theory is the null interval

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0,$$

so that it is only the null-cone and null-line structures in flat space-time that are preserved under his transformations.

In his original paper Page expresses the view that there are, in all probability, transformations other than those of his theory that relate reference frames in which the paths of light rays are invariant. In particular, he and Adams [64] mention the possibility of equivalent Euclidean frames of reference in relative rotation.

These aspirations must seem, to modern commentators, more than a little surprising since the early work of Lie [65] on the theory of groups of transformations could easily be used to demonstrate that the 15-parameter conformal group is the largest group with the property of leaving

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0$$

invariant. This work preceded that of Page by more than forty years, but was evidently unknown to him. Nor could Page have been aware of Cunningham's work of 1910, in which the possibility of a relativity principle based on frames of reference in relative rotation was dismissed. Needless to say, Page's theory faded from the literature at the first mention [66] of Lie's definitive treatment of the problem, and remains nowadays as little more than a mathematical curiosity without physical importance.

7.4 Conformal transformations as gauge transformations in Minkowski space-time

The mathematical framework for this discussion has already been developed in § 3 of Chapter 4, where the local isomorphism $C \approx SO(2,4)$ was demonstrated by mapping Minkowski space $\mathcal{M}$ onto a particular hyperquadric Q in the five-dimensional projective space P_5. The interpretation to be given here presents the dilatations $\mathcal{D}$ and the special conformal transformations $\mathcal{S}$ as geometrical gauge transformations in Minkowski space-time. In the case of the $\mathcal{D}$ transformations the gauge factor is a constant, but for the $\mathcal{S}$ transformations it is coordinate-dependent.

Since some small notational changes have been made in the present section we will briefly recapitulate on the results of § 3, Chapter 4 before proceeding.

(ξ^μ, ξ^4, ξ^5), $\mu = 0,1,2,3$, are taken as homogeneous coordinates in P_5 and then the mapping of P_5 into $\mathcal{M}$ has the form

$$x^\mu = \frac{\xi^\mu}{\xi^5} \quad , \quad \frac{\xi^4}{\xi^5} = x^\mu x_\mu \tag{4.1}$$

The transformations of the conformal group C are induced in $\mathcal{M}$ by certain linear transformations of homogeneous coordinates in P_5. In particular, the Poincaré transformations are induced by

$$\xi^{\mu'} = \Lambda^{\mu'}_{\ \mu}\, \xi^\mu + \xi^5\, \delta^{\mu'}_{\mu}\, b^\mu \ , \tag{4.2a}$$

$$\xi^{4'} = \xi^4 + 2\Lambda_{\mu'\nu}\, b^{\mu'} \xi^\nu + \xi^5\, b^\mu b_\mu \ , \tag{4.2b}$$

$$\xi^{5'} = \xi^5 \tag{4.2c}$$

where

$$\Lambda^{\mu'}_{\ \mu}\, g_{\mu'\nu'}\, \Lambda^{\nu'}_{\ \nu} = g_{\mu\nu} \ .$$

The translation subgroup $\mathcal{J}(b^\mu)$ is given when $\Lambda^{\mu'}_{\ \mu} = \delta^{\mu'}_{\mu}$.

The dilatations $\mathcal{D}$: $x^{\mu'} = \rho x^\mu \delta^{\mu'}_{\mu}$ are induced by

$$\xi^{\mu'} = \delta^{\mu'}_{\mu}\, \xi^\mu \quad , \tag{4.3a}$$

$$\xi^{4'} = \rho\, \xi^4 \quad , \tag{4.3b}$$

$$\xi^{5'} = \rho^{-1}\, \xi^5 \quad , \tag{4.3c}$$

where ρ is a real constant with $\rho > 0$.

The inversion transformation $\mathcal{I}$ is induced by

$$\xi^{0'} = -\xi^0 \tag{4.4a}$$

$$\xi^{i'} = \xi^i \ , \quad i = 1,2,3, \tag{4.4b}$$

$$\xi^{4'} = -\xi^5 \tag{4.4c}$$

$$\xi^{5'} = -\xi^4 \tag{4.4d}$$

Using the identity (2.6) of Chapter 4 it is straightforward to show that the special conformal transformation $\mathcal{S}(a^\mu)$ in $\mathcal{M}$ is induced by the transformation

$$\xi^{\mu'} = \delta^{\mu'}_{\mu}(\xi^{\mu} - \xi^4 a^{\mu}) \quad , \tag{4.5a}$$
$$\xi^{4'} = \xi^4 \quad , \tag{4.5b}$$
$$\xi^{5'} = \xi^5 - 2\xi^{\mu}a_{\mu} + \xi^4 a^{\mu}a_{\mu} \, , \tag{4.5c}$$

in P_5. (4.2) - (4.5) are all transformations that leave the quadratic form

$$\xi^{\mu}\xi_{\mu} - \xi^4\xi^5$$

invariant, and this is essentially the invariant quadratic form of SO(2,4).

In what follows we adopt the "active" view of transformations in space-time; the coordinate system is fixed, the transformation maps one region point-wise into another. We note that ξ^5 is invariant under transformations of the Poincaré group. Following the interpretation of Kastrup [61] we use ξ^5 to fix a standard of length. The coordinates ξ^{μ} are taken as coordinates of position in space-time, and are to be thought of here as dimensionless numbers. ξ^5 has dimension $[L]^{-1}$. As far as Poincaré transformations are concerned, ξ^{μ} and the usual coordinates of position x^{μ} are interchangeable; it is only where $\mathcal{D}$ and $\mathcal{S}$ transformations are involved that Kastrup's interpretation of the ξ^{μ} is significant.

The Minkowski metric interval between points x^{μ}, $x^{\mu} + dx^{\mu}$ is given in $\mathcal{M}$ by

$$ds^2 = dx^{\mu}dx_{\mu} = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 . \tag{4.6}$$

In terms of quantities in P_5 we find easily from (4.1) that

$$dx^{\mu} = (\xi^5)^{-2}\left[\xi^5 d\xi^{\mu} - \xi^{\mu} d\xi^5\right] , \tag{4.7}$$

and so

$$ds^2 = (\xi^5)^{-2}\left[d\xi^{\mu}d\xi_{\mu} - d\xi^4 d\xi^5\right] . \tag{4.8}$$

Again from (4.1), the hyperquadric $Q \subset P_5$ has equation

$$\xi_{\mu}\xi^{\mu} - \xi^4\xi^5 = 0 \tag{4.9}$$

so the differentials in (4.8) are subject to the restriction

$$2\xi^{\mu}d\xi_{\mu} - \xi^4 d\xi^5 - \xi^5 d\xi^4 = 0 . \tag{4.10}$$

We may add here that the finite displacement form of (4.8) is easily shown to be

$$\left(\underset{(2)}{x}^{\mu}-\underset{(1)}{x}^{\mu}\right)\left(\underset{(2)}{x}_{\mu}-\underset{(1)}{x}_{\mu}\right)=\left(\underset{(1)}{\xi}^{5}\underset{(2)}{\xi}^{5}\right)^{-1}\left[\left(\underset{(2)}{\xi}^{\mu}-\underset{(1)}{\xi}^{\mu}\right)\left(\underset{(2)}{\xi}_{\mu}-\underset{(1)}{\xi}_{\mu}\right)-\left(\underset{(2)}{\xi}^{4}-\underset{(1)}{\xi}^{4}\right)\left(\underset{(2)}{\xi}^{5}-\underset{(1)}{\xi}^{5}\right)\right], \quad (4.11)$$

where the notation is obvious.

It is easy to verify directly that (4.8) is invariant under the Poincaré transformations (4.2). Under dilatations (4.3) we find that

$$ds'^2 = \rho^2 ds^2 , \quad (4.12)$$

and under the special conformal transformations (4.5) we have

$$ds'^2 = \left[1 - 2 x^{\mu} a_{\mu} + (x^{\mu} x_{\mu}) (a^{\kappa} a_{\kappa})\right]^{-2} ds^2. \quad (4.13)$$

These results are identical in form with those obtained from the conventional "active" view-point, as adopted, for example, in Chapters 3 and 4 of the present work.

On the other hand, however, the interpretation of the δ transformations as gauge transformations gives a physical content to results like (4.13) different from that of the "active" interpretation. Referring back to (1·1) we note that the transformation

$$x^{\mu} \rightarrow \frac{x^{\mu} - a^{\mu}(x^{\alpha} x_{\alpha})}{1 - 2a_{\mu} x^{\mu} + (a_{\mu} a^{\mu})(x^{\kappa} x_{\kappa})} \quad (4.14)$$

viewed in the "active" sense maps to infinity the points on the hypersurface

$$1 - 2a_{\mu} x^{\mu} + (a_{\mu} a^{\mu})(x^{\kappa} x_{\alpha}) = 0. \quad (4.15)$$

Writing this equation in the form

$$(a^{\alpha} a_{\alpha}) \left[x^{\mu} - \frac{a^{\mu}}{(a^{\kappa} a_{\alpha})}\right] \left[x_{\mu} - \frac{a_{\mu}}{(a^{\alpha} a_{\alpha})}\right] = 0 , \quad (4.16)$$

and assuming that a^{μ} is neither null nor zero, we see that the hypersurface that maps to infinity under δ(a) is the null-cone with vertex at $\frac{a^{\mu}}{a^{\alpha} a_{\alpha}}$.

In the interpretation of Kastrup things are rather different; the transformation to be considered now is (4.5), under which space-time points (coordinatised here by the ξ^μ) suffer a translation

$$\xi^\mu \longrightarrow \xi^{\mu\prime} = \xi^\mu - (\xi^5)^{-1}(\xi^\alpha \xi_\alpha)\, a^\mu \quad ,$$

whilst there is a simultaneous gauge transformation

$$\xi^5 \longrightarrow \xi^{5\prime} = \xi^5 - 2\,\xi^\mu a_\mu + (a_\mu a^\mu)(\xi^\alpha \xi_\alpha)(\xi^5)^{-1} .$$

The points on the hypersurface (4.15) take part in the general translation but it is easy to see also that $\xi^{5\prime}$ becomes zero on this hypersurface, so that ds^2 given by (4.8) becomes infinite for separations in the hypersurface.

The advantage of looking at the conformal transformations in P_5 and employing Kastrup's physical interpretation lies in the one-to-one correspondence between points and their images which obtains in that space. The disadvantage is the appearance of a strange gauge on the hypersurface (4.15) which leads to infinite measures of separation between neighbouring points in the hypersurface. The usual "active" interpretation sends the hypersurface (4.15) to infinity, thus losing the identity of individual points in the hypersurface, whilst also, via (4.13), allocating infinite separations to pairs of image points at infinity.

7.5 The conformal group and causality - the "active" view-point

A pair of events with coordinates $x^\mu_{(1)}$, $x^\mu_{(2)}$ in space-time are causally related if the interval $x^\mu_{(2)} - x^\mu_{(1)}$ is time-like or null. Physically, this means that an influence propagating with a velocity less than or equal to the velocity of light can link the two events. The temporal ordering of the events $x^\mu_{(1)}$, $x^\mu_{(2)}$ is governed by the sign of the quantity $x^0_{(2)} - x^0_{(1)}$; if this is positive then event $x_{(1)}$ precedes event $x_{(2)}$; if it is negative then event $x_{(2)}$ precedes event $x_{(1)}$. It is well known that the Poincaré transformations and the dilatations preserve the causal structure of space-time, and it has been shown [67] that this is the largest connected group of transformations with this property. It follows then that the $\mathcal{S}$ transformations do not preserve the causal relations between pairs of events, and the work of this section and the next explores this violation of causality in some detail.

Let us look at the problem first of all from the "active" viewpoint using information from section 7.4. We have seen that

$$dx^{\mu}\, dx_{\mu} = (\xi^5)^{-1}\left[d\xi^{\mu} d\xi_{\mu} - d\xi^4 d\xi^5 \right] , \qquad (5.1)$$

from which it follows that $dx^{\mu}dx_{\mu}$ and $d\xi^{\mu}d\xi_{\mu} - d\xi^4\, d\xi^5$ always have the same sign, or vanish together, provided $\xi^5 \neq 0$. Thus, we may use the infinitesimal interval in P_5 to decide whether the corresponding interval in Minkowski space $\mathcal{M}$ is space-like, time-like or null. But $d\xi^{\mu}d\xi_{\mu} - d\xi^4 d\xi^5$ is invariant under the conformal group C, and so the concepts "time-like", "space-like" and "null", applied to infinitesimal intervals are invariant under C.

To investigate the infinitesimal temporal interval dt we use

$$\bar{t} = \frac{t - a^0(x^{\alpha}x_{\alpha})}{1 - 2a_{\mu}x^{\mu} + (a^{\alpha}a_{\alpha})(x^{\mu}x_{\mu})} ,$$

from which it follows that

$$d\bar{t} = \frac{1}{\Sigma^2}\Big\{ dt\left[1-(a^{\alpha}a_{\alpha})(t^2-x^ix_i)+(2a^0)^2(t^2-x^ix_i)-2a_{\mu}x^{\mu}+4a^0t(x^ia_i)\right]$$
$$+ dx^i\left[4a^0x_i(a^{\alpha}x_{\alpha})+2ta_i-2a^0a_i(x^{\mu}x_{\mu})-2a^0x_i-2(a^{\alpha}a_{\alpha})tx_i\right]\Big\} , \qquad (5.2)$$

where $\Sigma = 1-2a_{\mu}x^{\mu}+(a^{\alpha}a_{\alpha})(x^{\mu}x_{\mu})$.

There is no loss of generality in choosing $x^i = 0$, since any infinitesimal time-like interval can be brought to the form (dt, 0,0,0) by a suitable Lorentz transformation. Then (5.2) gives

$$d\bar{t} = \frac{dt}{\Sigma^2}\left\{ (1-a^0t)^2 + t^2\left[(a^1)^2 + (a^2)^2 + (a^3)^2\right] \right\} ,$$

from which it is clear that the sign of $d\bar{t}$ is the same as that of dt.

Thus we see that, provided we restrict ourselves to infinitesimal intervals, Einsteinian causality in Minkowski space-time is not affected by conformal transformations.

Turning our attention to finite intervals now, we must replace (5.1) by

$$(\underset{(2)}{x}^{\mu}-\underset{(1)}{x}^{\mu})(\underset{(2)}{x}_{\mu}-\underset{(1)}{x}_{\mu}) = \left(\underset{(1)}{\xi}^{5}\,\underset{(2)}{\xi}^{5}\right)^{-1}\left[(\underset{(2)}{\xi}^{\mu}-\underset{(1)}{\xi}^{\mu})(\underset{(2)}{\xi}_{\mu}-\underset{(1)}{\xi}_{\mu}) - (\underset{(2)}{\xi}^{4}-\underset{(1)}{\xi}^{4})(\underset{(2)}{\xi}^{5}-\underset{(1)}{\xi}^{5})\right], \tag{5.3}$$

and note that under the transformation $\mathcal{J}(a^{\mu})$ we have

$$(\underset{(2)}{x}^{\mu\prime}-\underset{(1)}{x}^{\mu\prime})(\underset{(2)}{x}_{\mu}{}'-\underset{(1)}{x}_{\mu}{}') = \left(\underset{(1)}{\xi}^{5}/\underset{(1)}{\xi}^{5\prime}\right)\left(\underset{(2)}{\xi}^{5}/\underset{(2)}{\xi}^{5\prime}\right)(\underset{(2)}{x}^{\mu}-\underset{(1)}{x}^{\mu})(\underset{(2)}{x}_{\mu}-\underset{(1)}{x}_{\mu}). \tag{5.4}$$

It can be shown that provided $\underset{(2)}{x}^{\mu}-\underset{(1)}{x}^{\mu}$ is not null it is possible to find parameters a^{μ} such that $\underset{(1)}{\xi}^{5}/\underset{(1)}{\xi}^{5\prime}$ and $\underset{(2)}{\xi}^{5}/\underset{(2)}{\xi}^{5\prime}$ have opposite signs. Then from (5.4) it follows that time-like intervals map into space-like and vice versa. There are a number of different cases to examine in the proof but the problem has been exhaustively treated in a paper [62] by Rosen. In physical terms the result means that any pair of causally related events with a finite non-null separation will, under certain transformations of the conformal group, map into a pair of events that cannot be causally related.

This behaviour of finite intervals under conformal transformations has led many writers (e.g. [68]) to claim that the conformal group cannot have any relevance for space-time physics. However, there are others, Kastrup among them, who maintain that because the special conformal transformations are non-linear it is only to the infinitesimal intervals that we can attach physical significance, and therefore it is the behaviour of infinitesimal elements under conformal transformation that is important to the question of physical content. This type of thinking is of course inspired by our knowledge of general relativity, coupled with the fact that if we adopt the "active" view of conformal transformations then the transformed space-time region has non-zero curvature and thus properly requires a general-relativistic treatment.

7.6 The conformal group and causality - a particle physicist's view-point

In his interesting paper [62], Rosen points out that since experiments of physics are interpreted "flatly", any useful interpretation of the conformal transformations should be consistent with the adoption of rectilinear coordinates in a flat space-time. He suggests an alternative to the "active" and "passive" view-points by proposing that space-time transformations be interpreted as mapping only the events and world-lines of the physical process under consideration, whilst leaving the space-time manifold and the coordinate system unaffected. A single observer then sees a different process taking place but in the same background of flat space-time. If every physically valid process transforms in this sense into a second physically valid process then the transformation group concerned constitutes a symmetry of physics.

In the present work we shall generalise some results of Rosen by examining the effect of the special conformal transformation (4.14) on a general linear trajectory

$$x^\mu = z^\mu + \lambda \ell^\mu, \quad \lambda \text{ real}, \tag{6.1}$$

in the flat background space-time. (6.1) is a straight line through the event z^μ with direction given by ℓ^μ. If ℓ^μ is time-like then (6.1) gives the world-line of a massive particle in free space; if ℓ^μ is null we have the world-line of a zero rest-mass particle or a photon. The image of x^μ under (4.14) is

$$\bar{x}^\mu = \frac{[z^\mu + \lambda \ell^\mu] - a^\mu[\underline{z}.\underline{z} + 2\lambda \underline{z}.\underline{\ell} + \lambda^2 \underline{\ell}.\underline{\ell}]}{[1 - 2\underline{a}.\underline{z} - 2\lambda \underline{a}.\underline{\ell}] + (\underline{a}.\underline{a})[\underline{z}.\underline{z} + 2\lambda(\underline{z}.\underline{\ell}) + \lambda^2(\underline{\ell}.\underline{\ell})]} \tag{6.2}$$

where, as usual, $\underline{u}.\underline{v}$ denotes the Minkowskian scalar product $u^\alpha v_\alpha$. Letting $\lambda \to \infty$ in (6.2) reveals that the point $\bar{x}^\mu = -\frac{a^\mu}{\underline{a}.\underline{a}}$ lies on the image trajectory. Then we can write (6.2) as

$$\bar{x}^\mu = \frac{-a^\mu}{\underline{a}.\underline{a}} + \lambda N\left[\ell^\mu - \frac{2(\underline{a}.\underline{\ell})}{\underline{a}.\underline{a}} a^\mu\right] + N\left[z^\mu + \left(\frac{1-2\underline{a}.\underline{z}}{\underline{a}.\underline{a}}\right) a^\mu\right], \tag{6.3}$$

where $N = \left\{\left[1 - 2\underline{a}.\underline{z} - 2\lambda(\underline{a}.\underline{\ell})\right] + (\underline{a}.\underline{a})\left[\underline{z}.\underline{z} + 2\lambda(\underline{z}.\underline{\ell}) + \lambda^2(\underline{\ell}\underline{\ell})\right]\right\}^{-1}$,

showing that the image trajectory lies in the plane through $-\frac{a^\mu}{\underline{a}.\underline{a}}$ spanned by the vectors $\ell^\mu - \frac{2(\underline{a}.\underline{\ell})}{\underline{a}.\underline{a}} a^\mu$ and $z^\mu + \left(\frac{1-2\underline{a}.\underline{z}}{\underline{a}.\underline{a}}\right) a^\mu$.

Secondly, we can show that $\bar{x}^\mu$ lies on a certain hypersphere as λ varies, and then deduce finally that the image trajectory of (6.1) is the intersection of this hypersurface with the plane defined above.

The hypersphere in question has equation

$$\left[\bar{x}^\mu + \frac{b^\mu}{2\underline{b}.\underline{b}}\right]\left[\bar{x}_\mu + \frac{b_\mu}{2\underline{b}.\underline{b}}\right] = \frac{1}{4\underline{b}.\underline{b}}, \qquad (6.4)$$

where
$$b^\mu = a^\mu + \frac{\ell^\mu[(\underline{z}.\underline{z})(\underline{a}.\underline{\ell})-(\underline{a}.\underline{z})(\underline{z}.\underline{\ell})] + z^\mu[(\underline{\ell}.\underline{\ell})(\underline{a}.\underline{z})-(\underline{a}.\underline{\ell})(\underline{z}.\underline{\ell})]}{(\underline{z}.\underline{\ell})^2-(\underline{z}.\underline{z})(\underline{\ell}.\underline{\ell})}. \qquad (6.5)$$

It is a straightforward, but tedious, matter to show that $\bar{x}^\mu$ defined in (6.2) lies on the hypersphere (6.4) for all λ. This completes the geometrical analysis of the transformed trajectory.

To develop further insight into the properties of the transformed trajectories, let us make some specialisations in the above analysis. Suppose now that the original trajectory is a straight line passing through the origin. It is immediately evident that the origin maps into itself, and that the image trajectory is a curve lying in the plane defined by ℓ^μ and a^μ. There are two degenerate cases that might be noted here:

(i) when ℓ^μ is null the image trajectory is just the original null line;

(ii) when a^μ is in the direction of ℓ^μ the image is just the original straight line in the direction of ℓ^μ.

In the non-degenerate case it is easy to show also that the image trajectory lies in the hypersphere

$$\left[\bar{x}^\mu + \frac{b^\mu}{2\underline{b}.\underline{b}}\right]\left[x_\mu + \frac{b_\mu}{2\underline{b}.\underline{b}}\right] = \frac{1}{4\underline{b}.\underline{b}}, \qquad (6.6)$$

where $b^\mu = a^\mu - \frac{a\cdot\ell}{\ell\cdot\ell}\ell^\mu$, so that we may define the image trajectory as the intersection of (6.6) with the plane through the origin spanned by ℓ^μ and a^μ. As an illustrative example take $\ell^\mu = \delta^\mu_0$ and $a^\mu = a_x\,\delta^\mu_1$ with $a_x > 0$, so that we are finding the image trajectory for a particle at rest at the spatial origin, transformed under a "boost" in the direction of the x^1-axis. Explicitly, we have

$$\bar{x}^0 = \frac{\lambda}{1-a_x^2\lambda^2}, \qquad \bar{x}^1 = -\frac{a_x\lambda^2}{1-a_x^2\lambda^2}, \tag{6.7}$$

from which

$$\bar{x}^{0\,2} - \left(\bar{x}^1 - \frac{1}{2a_x}\right)^2 = -\frac{1}{4a_x^2}, \tag{6.8}$$

indicating that the image trajectory is an hyperbola. The 3-velocity of this world-line is easily shown to be

$$\bar{v}_x = -\frac{2a\lambda}{1+a_x^2\lambda^2}, \tag{6.9}$$

which satisfies

$$\bar{v}_x\,(1-\bar{v}_x^2)^{-\frac{1}{2}} = \pm\, 2a_x\bar{x}^0. \tag{6.10}$$

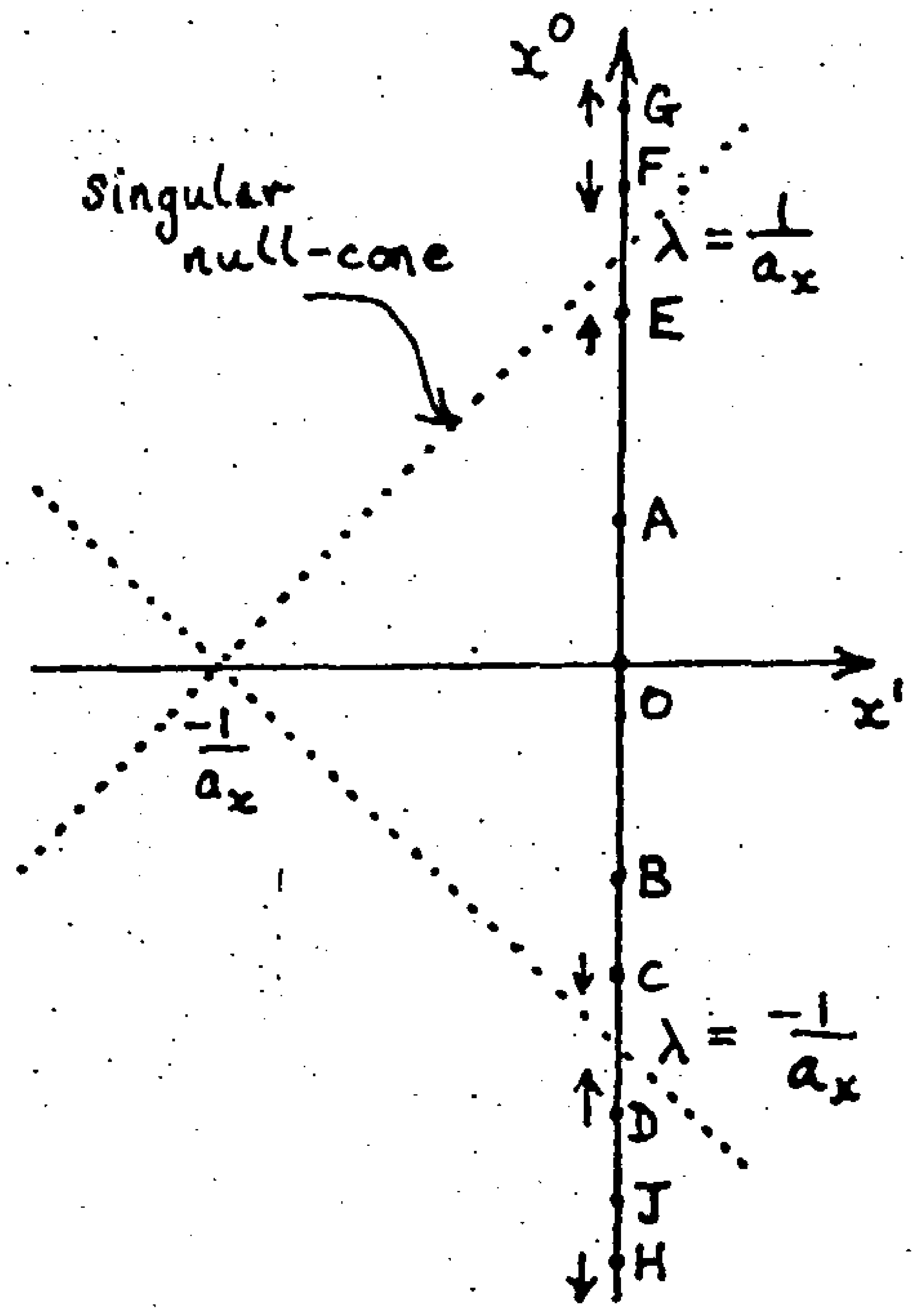
To relate events on the original particle world-line with their images we note the following results:

(i) as $\lambda \to \pm\infty$, $\bar{x}^0 \to 0\mp$ and $\bar{x}^1 \to \frac{1}{a_x}$;

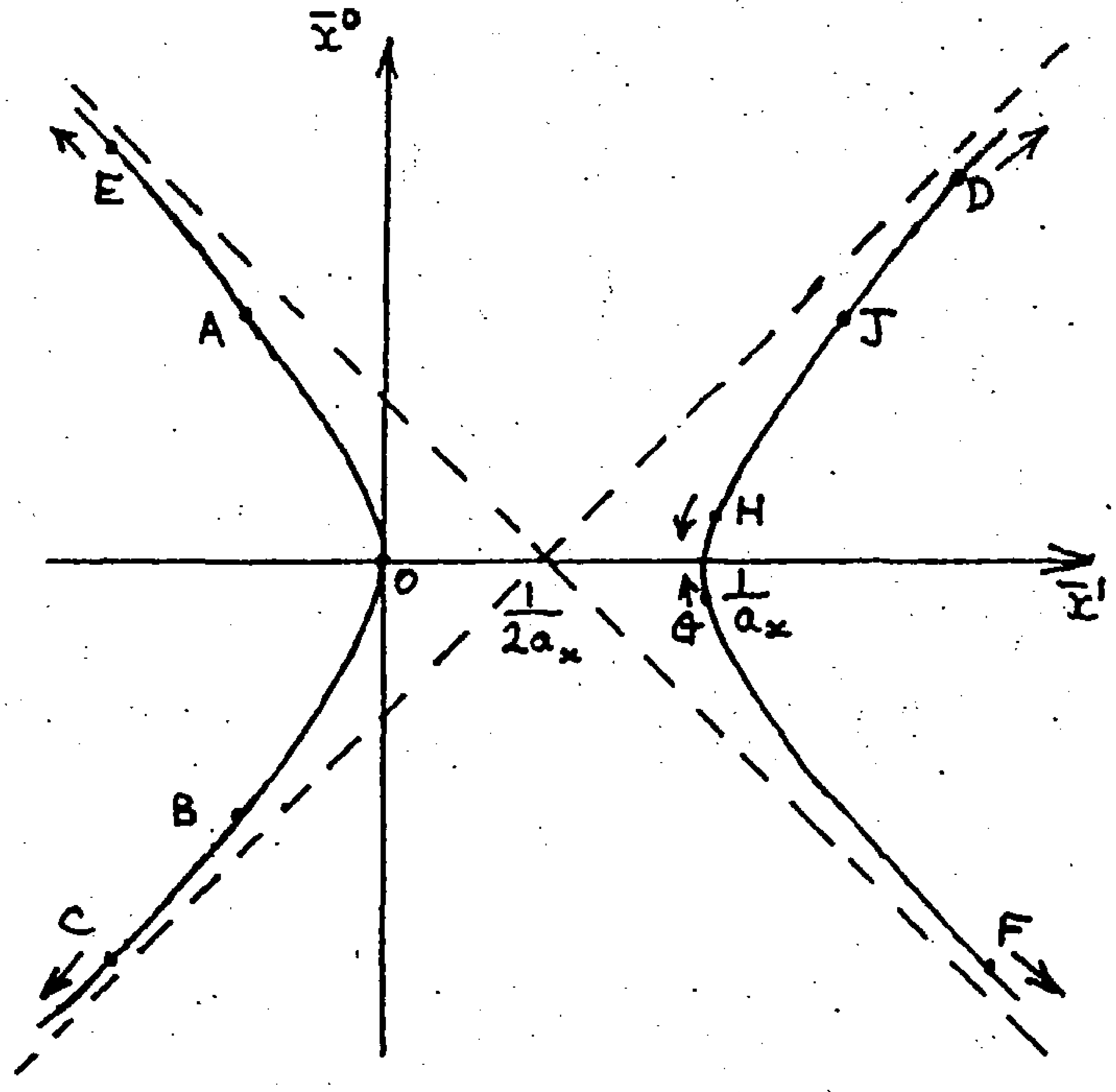
(ii) as $\lambda \to \frac{1}{a_x}\pm$, $\bar{x}^0 \to \mp\infty$ and $\bar{x}^1 \to \pm\infty$;

(iii) as $\lambda \to -\frac{1}{a_x}\pm$, $\bar{x}^0 \to \mp\infty$ and $\bar{x}^1 \to \mp\infty$.

A diagram will help to emphasise the important properties of the transformation.



(fig 1)



(fig 2)

We can now see how causality is violated under the special conformal transformations. It has already been mentioned [see (4.16)] that if $a^{\mu}(\neq 0)$ is not null, then the null-cone with vertex at $\frac{a^{\mu}}{a^{\alpha}a_{\alpha}}$ maps to infinity under $\mathcal{S}(a^{\mu})$. In the above example this singular null-cone has vertex at $\frac{-1}{a_x}$ and has been shown in (fig. 1). Note first that the time-like interval AE remains time-like under transformation whereas the time-like interval AF becomes space-like under transformation. This latter property is a peculiarity of intervals on the initial trajectory that contain a point of the cone of singularity. It is a general fact that any finite time-like interval may be transformed to a space-like interval by a suitable choice of transformation; we need only choose a^{μ} in $\mathcal{S}(a^{\mu})$ so that the initial time-like interval intersects the cone of singularity of the transformation.

We note secondly that the temporal ordering of events such as A and F is reversed under transformation and that it is possible for temporally distinct events such as A and J to become simultaneous events under the transformation.

The view-point adopted by Rosen permits him to argue that the violations of causality apparent from the above work do not deprive the conformal group of physical significance. With reference to the last example, in which a one-particle process transforms into a two-particle process, his interpretation requires for its application only that the initial and final trajectories in space-time should be those of a physically possible process. The initial trajectory is just that of a massive particle permanently at rest at the spatial origin; the final trajectory represents the motion of two non-interacting particles with equal and opposite charge-to-mass ratios in a constant electric field. Both of these processes are physically realisable. Thus, we have here, in Rosen's view, a possible symmetry of physics.

To demonstrate incontrovertibly that the conformal group is a symmetry of physics it must be shown that every physically valid process transforms under operations of the group only into other physically valid processes (no matter how different these may be from the original process). It is to be expected that particle-creation processes may transform into particle-annihilation processes, that particle number may not be preserved under transformation (a circumstance already revealed in our simple example above), and that conventional notions of causality will be violated. It is clear that a program to investigate the conformal group in this manner is a task for the experimenters; it does not lie within the scope of theoretical physics.

7.7 Conservation laws and laws of balance

We have already, in Chapter 3, set down 15 vectors $\{\xi_A^{\ \mu}\}$, A = 0,1,---,14, as a basis for the vector space of C-K vectors in Minkowski space-time. This basis was chosen to reflect the physical aspects of the conformal transformations. In particular, the Killing vectors $\{\xi_A\}$, A = 0,1,---,9 were seen to generate the Poincaré group (the isometry group of Minkowski space-time), while $\xi_{10}^{\ \mu}$, $\xi_{11}^{\ \mu}$, ---, $\xi_{14}^{\ \mu}$ generated true conformal transformations. It is well-known that if we consider a conservative medium in a background Minkowski space-time, then to each Killing vector of the space-time there corresponds a particular law of conservation relating to properties of that medium. Let $T^{\mu\nu}$ denote the energy-momentum tensor characterising the distribution of the medium in space-time. Then $T^{\mu\nu}$ is symmetric;

$$T^{[\mu\nu]} = 0 \quad ,$$

and has zero divergence;

$$T^{\mu\nu}{}_{,\nu} = 0 \ .$$

Consider the 10 vector quantities defined by

$$V_A^{\ \mu} = \xi_{A\nu} \ T^{\mu\nu} \quad , A = 0,1,---,9. \tag{7.1}$$

A straightforward calculation gives

$$\begin{aligned} V_A^{\ \mu}{}_{,\mu} &= \xi_{A\nu,\mu} T^{\mu\nu} + \xi_{A\nu} \ T^{\nu\mu}{}_{,\mu} \\ &= \xi_{A\nu,\mu} T^{\mu\nu} \\ &= \frac{1}{2} T^{\mu\nu} (\xi_{A\mu,\nu} + \xi_{A\nu,\mu}) \quad , \end{aligned} \tag{7.2}$$

and since the $\xi_{A\mu}$ are solutions of Killing's equation it follows that

$$V_A^{\ \mu}{}_{,\mu} = 0, \quad A = 0,1,---,9. \tag{7.3}$$

From this "differential conservation law" (7.3), we can, by the usual arguments [51], construct an integral conservation law as follows. Suppose S_3 is any closed 3-surface in space-time, enclosing the 4-volume R_4. Then one form of Stokes' Theorem gives

$$\int_{S_3} V_A^{\mu} n_{\mu}\, d^{(3)}v = \int_{R_4} V_A{}^{\mu}{}_{,\mu}\, d^{(4)}v$$

$$= 0, \text{ by virtue of (7.3),}$$

and n_{μ} is the unit outward normal to S_3. The reference mentioned above shows how this form of integral law may be related to more conventional ideas.

Suppose now that ξ_A^{μ} is a <u>conformal</u> Killing vector of $\mathcal{M}$, so that it satisfies

$$\xi_{A\mu;\nu} + \xi_{A\nu;\mu} = \frac{1}{2}\xi_A{}^{\alpha}{}_{,\alpha}\, g_{\mu\nu},$$

where $g_{\mu\nu} = \text{diag}\,(1,-1,-1,-1)$ is the Minkowskian metric tensor. (The ten Killing vectors emerge as C-K vectors having $\xi^{\alpha}{}_{,\alpha} = 0$.) Then (7.2) becomes

$$V_A{}^{\mu}{}_{,\mu} = \frac{1}{4} T^{\mu}{}_{\mu}\, \xi_A{}^{\alpha}{}_{,\alpha}. \qquad (7.4)$$

Consider the 5 basis vectors ξ_B^{μ}, B = 10,11,---,14, for which

$$\xi_A{}^{\mu}{}_{,\mu} \neq 0 .$$

Then it is clear from (7.4) that the quantities

$$V_B^{\mu} = \xi_{B\nu} T^{\mu\nu}, \quad B = 10,11,\text{---},14,$$

satisfy differential conservation laws if and only if the energy-momentum tensor $T^{\mu\nu}$ is trace-free;

$$T^{\mu}{}_{\mu} = 0 .$$

It is worth noting that this is indeed the case if $T^{\mu\nu}$ is the energy-momentum tensor of a free Maxwell field. In general, however, (7.4) does not take the form of a differential conservation law but gives what Edelen [69] has called a "law of balance".

CHAPTER 8

MULTIPOLE MOMENTS AND THE CONFORMAL GROUP

8.1 Introduction

The invariance of Maxwell's equations under the conformal group formed a starting point for the discussion of physical interpretations given in the previous chapter. However, this conform-invariance is not peculiar to the field equations of electromagnetism, since, as many writers have noted, e.g.[54, 70], the field equations of any zero rest-mass field are invariant under conformal rescalings. Thus, for each of the familiar fields of spin, 0, $\frac{1}{2}$, 1 and 2 we have field equations exhibiting this invariance. For details concerning the spin - 1 (Maxwell) field and the spin - 2 (linearised gravitation) field one may refer to Pirani's contribution in the Trautmann-Pirani-Bondi volume [23]. In each of these theories it is usual to describe the potential function as a series of multipole contributions which may be pictured as arising from certain source terms that assume the form of delta-functions and derivatives of delta-functions concentrated on some world-line in space-time. This conception of a point source with structure is particularly convenient in relativistic considerations, since it avoids the difficulties that are presented by distributions of charge or mass having finite extent.

The intimate connection between multipole moments and the conformal group is suppressed in the usual treatments, but as Geroch has pointed out [16, 71], a full recognition of this connection, and in particular the recognition that multipole moment fields are also C-K tensor fields, provides a method that it is hoped may be applicable to the study of multipole moments in general relativity.

In the present chapter we investigate the algebraic structures carried by C-K vectors and C-K tensors, viewed both as objects in Minkowski space-time and as objects over the 15-dimensional space of conformal Killing vectors. The link between C-K tensors in 3-space and the multipole moments of Poisson's equation is established in §4, and the remaining section develops a formal extension of this work to the study of multipole moments of the wave equation in Minkowski space-time. It is sufficient to restrict attention to the scalar wave equation since the equations for the potential in the Maxwell and linearised gravitation cases reduce essentially to this form when an appropriate choice of gauge conditions is made.

8.2 The algebra of C-K vectors in Minkowski space-time

In Chapter 3 we found the structure constants of the Lie algebra of the Minkowski-space conformal group, the operators of which were defined there by

$$X_A{}^{op} = \xi_A^\mu \frac{\partial}{\partial x^\mu} \quad , \; A = 0,1,\ldots,14, \tag{2.1}$$

where the $\{\underline{\xi}_A\}$ form a basis for the vector space of C-K vectors.

The structure constants, C^D_{AB} , defined from

$$\left[X^{op}_A, X^{op}_B\right] = C^D_{AB}\, X^{op}_D \tag{2.2}$$

are tabulated for reference on page 58 .

It is easy to demonstrate that the C-K vectors of $\mathcal{M}$ form a vector space over the reals. We shall denote this space by $\mathcal{C}$ in the following notes. The set $\mathcal{C}$ is also closed under the Lie bracket binary operation defined on the basis vectors $\{\underline{\xi}_A\}$ by

$$\left[\underline{\xi}_A , \underline{\xi}_B\right]^\mu = \xi_A^{\;\nu}\, \xi^\mu_{B,\nu} - \xi^\nu_B\, \xi^\mu_{A,\nu} \quad , \tag{2.3}$$

since it follows readily from this definition that

$$\left[\underline{\xi}_A, \underline{\xi}_B\right]^\mu = C^D_{AB}\, \xi^\mu_D \quad , \tag{2.4}$$

where the C^D_{AB} are just those that appear in (2.2).

More generally, if

$$\underline{X} = X^A\, \underline{\xi}_A$$

and

$$\underline{Y} = Y^A\, \underline{\xi}_A$$

are any two elements of $\mathcal{C}$, the components $[\underline{X}, \underline{Y}]^A$ of their Lie bracket are given by

$$\left[\underline{X}, \underline{Y}\right]^A = X^B\, C^A_{BD}\, Y^D. \tag{2.5}$$

From the structure constants C^A_{BD} we construct the Cartan metric G_{AB} defined by

$$G_{AB} = C^E_{AD}\, C^D_{BE} = G_{BA}. \tag{2.6}$$

The properties of G_{AB} are important in the classification of Lie groups. For example, if G_{AB} is non-singular then the associated Lie group is semi-simple. Secondly, a knowledge of the signature of G_{AB} enables one to determine whether the Lie group is compact (in which case G_{AB} is negative definite) or non-compact (in which case G_{AB} is indefinite).
These results may be applied to show that the Lie group from which the present algebra derives is semi-simple and non-compact. The detailed results from which these conclusions are drawn have been incorporated in Appendix 4 .
Within the Lie algebra of C-K vectors the matrix G_{AB} and its inverse (denoted by G^{AB}) may be used for raising and lowering suffices in the usual way.

8.3 The algebra of C-K tensors in Minkowski space-time

In this section we shall first of all treat C-K tensors as objects in $\mathcal{M}$, and then give a reappraisal of their algebraic structure in the space $\mathcal{C}$.
A necessary and sufficient condition for the completely symmetric tensor $T^{\mu_1\mu_2\cdots\mu_n}$ to be a C-K tensor of $\mathcal{M}$ is that, for some $V^{\mu_2\mu_3\cdots\mu_n}$,

$$T^{(\mu_1\mu_2\cdots\mu_n,\nu)} = g^{(\nu\mu_1}V^{\mu_2\mu_3\cdots\mu_n)} . \qquad (3.1)$$

For example, both $g^{\mu\nu}$ and $\xi_{A_1}^{(\mu_1}\xi_{A_2}^{\mu_2}\cdots\xi_{A_n}^{\mu_n)}$, where $\underline{\xi}_{A_1}, \underline{\xi}_{A_2}, \cdots \underline{\xi}_{A_n}$ are C-K vectors, are C-K tensors. We define three operations in the algebra of C-K tensors as follows:

(i) the tensor sum of two C-K tensors is defined in the usual way whenever the two tensors have the same rank;

(ii) the product of C-K tensors $P^{\mu_1\mu_2\cdots\mu_n}$, $Q^{\mu_1\mu_2\cdots\mu_m}$ is given by

$$P\cap Q \equiv P^{(\mu_1\mu_2\cdots\mu_n}Q^{\mu_{n+1}\cdots\mu_{n+m})} = Q\cap P \qquad (3.2)$$

(iii) the generalised Lie bracket of C-K tensors $P^{\mu_1\mu_2\cdots\mu_n}$, $Q^{\mu_1\mu_2\cdots\mu_m}$ is given by

$$[P,Q] = [P,Q]^{\mu_1\mu_2\cdots\mu_{n+m-1}} \equiv n\, P^{\nu(\mu_1\mu_2\cdots\mu_{n-1}}Q^{\mu_n\cdots\mu_{n+m-1})}{}_{,\nu}$$
$$- m\, Q^{\nu(\mu_1\mu_2\cdots\mu_{m-1}}P^{\mu_m\cdots\mu_{m+n-1})}{}_{,\nu} . \qquad (3.3)$$

We have here followed Geroch [16] in choice of notation.

It is easy to show that the sum and product are associative and commutative, the bracket is anti-commutative,

$$[P,Q] = -[Q,P] \tag{3.4}$$

and linear, and that $\wedge$ is distributive over $+$.

Some less trivial properties of the operations are embodied in

$$[P, Q_\wedge R] = [P,Q]_\wedge R + [P,R]_\wedge Q \tag{3.5}$$

and

$$[P, [Q,R]] + [Q, [R,P]] + [R, [P,Q]] = 0. \tag{3.6}$$

(3.6) is recognised as a generalisation of the Jacobi identity.

Using the formalism developed here, the condition (3.1) for T to be a C-K tensor can be reframed as

$$\frac{1}{2} [g, T] = g_\wedge V. \tag{3.7}$$

It is an elementary exercise now to show that $\mathcal{C}$ is closed under the operations (i), (ii) and (iii) defined above.

The remaining part of this section will be devoted to a study of the algebra of C-K tensors formulated in terms of tensors over $\mathcal{C}$ rather than over $\mathcal{M}$. Tensor products of $\mathcal{C}$ can be set up in the usual way; see for example [48, 72], and we shall assume that $\{\underline{\xi}_A\}$, A = 0,1,...14, has been chosen to be a natural basis for $\mathcal{C}$. The basis $\{\underline{e}_\mu\}$ for $\mathcal{M}$ is $\underline{e}_0 = (1,0,0,0)$, $\underline{e}_1 = (0,1,0,0)$, $\underline{e}_2 = (0,0,1,0)$, $\underline{e}_3 = (0,0,0,1)$. The components $T^{A_1A_2\ldots A_n}$ of a completely symmetric tensor over $\mathcal{C}$ can be expressed as

$$T^{A_1A_2\ldots A_n} = \sum \left\{ \lambda^{A_1}\lambda^{A_2}\ldots\lambda^{A_n} + \ldots + \eta^{A_1}\eta^{A_2}\ldots\eta^{A_n} \right\}, \tag{3.8}$$

where the summation extends over at most 15^{n-1} terms and the $\lambda^A,\ldots,\eta^A$ are components of vectors $\underline{\lambda}, \underline{\eta} \in \mathcal{C}$.

We define a linear mapping $f: \mathcal{C} \to \mathcal{M}$ such that

$$f(\mathfrak{J}_A) = \mathfrak{J}^\alpha_A \underline{e}_\alpha ,$$

where $\mathfrak{J}^\alpha_A$ are the components in $\mathcal{M}$ of $\underline{\mathfrak{J}}_A \in \mathcal{C}$. It follows then that f maps the $\mathcal{C}$-vector with components λ^A to the $\mathcal{M}$-vector with components

$$\lambda^\alpha = \lambda^A \mathfrak{J}^\alpha_A$$

where $\{\mathfrak{J}^\alpha_A\}$, A = 0,1,...14, are just the components in $\mathcal{M}$ of the basis of C-K vectors adopted in Chapter 3. More generally, the tensor (3.8) over $\mathcal{C}$ maps to a tensor over $\mathcal{M}$ having the components

$$T^{\alpha_1 \alpha_2 \cdots \alpha_n} = T^{A_1 A_2 \ldots A_n} \mathfrak{J}^{\alpha_1}_{A_1} \mathfrak{J}^{\alpha_2}_{A_2} \cdots \mathfrak{J}^{\alpha_n}_{A_n} . \qquad (3.9)$$

It is clear that $T^{\alpha_1 \alpha_2 \cdots \alpha_n}$, defined in this way is a C-K tensor of $\mathcal{M}$.

The operations $+$, , and $[\ ,\ ]$ in $\mathcal{M}$ have their analogues in $\mathcal{C}$ as follows:

(i) the sum of two $\mathcal{C}$-tensors of the same rank is defined in the usual way,

$$P^{A_1 A_2 \ldots A_n} + Q^{A_1 A_2 \ldots A_n} = (P + Q)^{A_1 A_2 \ldots A_n} ; \qquad (3.10)$$

(ii) the product of $\mathcal{C}$-tensors $P^{A_1 A_2 \ldots A_n}$ and $Q^{A_1 A_2 \ldots A_m}$ is given by

$$P \cap Q = P^{(A_1 A_2 \ldots A_n} Q^{A_{n+1} \ldots A_{n+m})} ; \qquad (3.11)$$

(iii) if $P^{A_1 A_2 \ldots A_n}$, $Q^{A_1 A_2 \ldots A_m}$ are $\mathcal{C}$-tensors of ranks n, m respectively, their Lie bracket is a $\mathcal{C}$-tensor of rank n + m - 1 defined by

$$[P,Q]^{A_2 A_3 \ldots A_n B A_{n+1} \cdots A_{n+m-1}} = nm\, P^{A_1 (A_2 \ldots A_n}$$

$$\times\ C^B_{A_1 A_{n+m}} Q^{A_{n+1} \ldots A_{n+m-1}) A_{n+m}} \qquad (3.12)$$

Under (3.9) each of the $\mathcal{C}$-tensors defined in (3.10, 11, 12) maps to a C-K tensor of $\mathcal{M}$. In particular, (3.12) defines $[P,Q]^{A_2 A_3 \ldots A_n B A_{n+1} \cdots A_{n+m-1}}$ in such a way that it maps to the Lie bracket of $P^{\alpha_1 \alpha_2 \cdots \alpha_n}$, $Q^{\alpha_1 \alpha_2 \cdots \alpha_m}$ in $\mathcal{M}$. This can be verified directly (at least for small values of n and m) by substitution of

$$P^{\alpha_1 \alpha_2 \cdots \alpha_n} = P^{A_1 A_2 \ldots A_n} \mathfrak{J}^{\alpha_1}_{A_1} \mathfrak{J}^{\alpha_2}_{A_2} \cdots \mathfrak{J}^{\alpha_n}_{A_n}$$

and $$Q^{\alpha_1\alpha_2\cdots\alpha_m} = Q^{A_1A_2\ldots A_m}\,\gamma^{\alpha_1}_{A_1}\,\gamma^{\alpha_2}_{A_2}\cdots\gamma^{\alpha_m}_{A_m} ,$$

into (3.3), but it is rather easier to proceed by using induction on n and m.

In conclusion, we may make some remarks concerning the correspondence (3.9) that have importance in the relation of the $\mathscr{C}$-tensor formalism to multipole moments of the wave equation. Whilst it is clear from (3.9) that a specification of $T^{A_1A_2\ldots A_n}$ in $\mathscr{C}$ defines a unique C-K tensor of $\mathcal{M}$, it is to be noted also that two distinct $\mathscr{C}$-tensors may define the same C-K tensor in $\mathcal{M}$; the correspondence is not one-to-one. Furthermore, there are C-K tensors of $\mathcal{M}$ which cannot be generated via (3.9) from any tensor over $\mathscr{C}$. An example of this is provided by $\psi g^{\mu\nu}$, where ψ is any scalar field. This is certainly a C-K tensor of $\mathcal{M}$, but it cannot, in general, be constructed from $\mathscr{C}$-tensors since the space of rank 2 tensors over $\mathscr{C}$ has finite dimension.

The real importance of (3.9) is embodied in the result of Geroch [16] who showed that every symmetric trace-free C-K tensor of $\mathcal{M}$ can be obtained via this mapping from some (totally-symmetric) tensor over $\mathscr{C}$. The trace-free C-K tensors are precisely those required for the discussion of multipole moments that now follows.

8.4 Multipole moments of Poisson's equation in 3-space.

In this section alone we shall use $\underline{x}$ to denote the 3-vector with components x^i, i = 1,2,3, and

$$\eta_{ij} = \mathrm{diag}\,(1,1,1) \tag{4.1}$$

to denote the 3-space metric tensor.

We shall take

$$\Phi(\underline{x}) = \int \frac{\rho(\underline{x}')}{|\underline{x} - \underline{x}'|}\, dv'^{(3)} , \tag{4.2}$$

where $dv^{(3)}$ is the 3-dimensional volume element, as the solution of Poisson's equation

$$\nabla^2 \Phi = -4\pi\rho(\underline{x}) \qquad (4.3)$$

with source distribution $\rho(\underline{x})$.

Then $\dfrac{1}{|\underline{x} - \underline{x}'|}$ can be expanded in the form

$$\frac{1}{|\underline{x} - \underline{x}'|} = \frac{1}{r} - x'^j \partial_j \left(\frac{1}{r}\right) + \frac{1}{2!} x'^j x'^k \partial_j \partial_k \left(\frac{1}{r}\right) - \frac{1}{3!} x'^j x'^k x'^\ell \partial_j \partial_k \partial_\ell \left(\frac{1}{r}\right) + \cdots \qquad (4.4)$$

where $r = |\underline{x}|$

and $$\partial_{j_1} \partial_{j_2} \cdots \partial_{j_n} \left(\frac{1}{r}\right) = \left[\frac{\partial^n}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{jn}} \left(\frac{1}{|\underline{x} - \underline{x}'|}\right)\right]_{\underline{x}' = \underline{0}} . \qquad (4.5)$$

It can be shown (proof by induction) that

$$\frac{\partial^n}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{jn}} \left(\frac{1}{|\underline{x} - \underline{x}'|}\right) = \frac{(-1)^n (2n)!}{2^n n!} \left[(x^i - x'^i)(x_i - x'_i)\right]^{-(n+\frac{1}{2})}$$

$$\times \mathcal{J}\left\{(x_{j_1} - x'_{j_1}) \cdots (x_{jn} - x'_{jn})\right\}, \qquad (4.6)$$

where $\mathcal{J}\left\{V_{j_1 j_2 \cdots j_n}\right\}$ indicates the trace-free part of the completely symmetric tensor $V_{j_1 j_2 \cdots j_n}$. From (4.6) we have then, in the notation of (4.5),

$$\partial_{j_1} \partial_{j_2} \cdots \partial_{j_n} \left(\frac{1}{r}\right) = \frac{(-1)^n (2n)!}{2^n n!} r^{-(2n+1)} \mathcal{J}\left\{x_{j_1} x_{j_2} \cdots x_{j_n}\right\} . \qquad (4.7)$$

Explicitly, (4.7) gives

$$\partial_{j_1} \left(\frac{1}{r}\right) = -\frac{x_{j_1}}{r}$$

$$\partial_{j_1} \partial_{j_2} \left(\frac{1}{r}\right) = \frac{3}{r^5} \left\{ x_{j_1} x_{j_2} - \frac{1}{3} r^2 \eta_{j_1 j_1} \right\}$$

$$\partial_{j_1} \partial_{j_2} \partial_{j_3} \left(\frac{1}{r}\right) = \frac{-15}{r^7} \left\{ x_{j_1} x_{j_2} x_{j_3} - \frac{1}{5} r^2 (x_{j_1} \eta_{j_2 j_3} + x_{j_2} \eta_{j_1 j_3} + x_{j_3} \eta_{j_1 j_2}) \right\}$$

......

(4.4) becomes

$$\frac{1}{|\underline{x} - \underline{x}'|} = \frac{1}{r} + \sum_{n=1}^{\infty} \frac{(2n)!}{2^n\ n!} \frac{x'^{j_1} x'^{j_2} \dots x'^{j_n}}{r^{2n+1}} \mathcal{J}\left\{x_{j_1} x_{j_2} \dots x_{j_n}\right\} . \tag{4.8}$$

We introduce integrals over the source distribution, defined by

$$I = \int \rho(\underline{x}')\, dv'^{(3)} ,$$

$$I^{j_1} = \int \rho(\underline{x}')\, x'^{j_1}\, dv'^{(3)} ,$$

$$I^{j_1 j_2} = 3 \int \rho(\underline{x}')\, x'^{j_1} x'^{j_2}\, dv'^{(3)} ,$$

$$I^{j_1 j_2 j_3} = 15 \int \rho(\underline{x}')\, x'^{j_1} x'^{j_2} x'^{j_3}\, dv'^{(3)}, \dots$$

with the general form

$$I^{j_1 j_2 \dots j_n} = \frac{(2n)!}{2^n\ n!} \int \rho(\underline{x}')\, x'^{j_1} x'^{j_2} \dots x'^{j_n}\, dv'^{(3)} . \tag{4.9}$$

Then, using (4.8) and (4.9) we can write (4.2) in the form

$$\Phi(\underline{x}) = \frac{I}{r} + \sum_{n=1}^{\infty} \frac{I^{j_1 j_2 \dots j_n} \mathcal{J}\left\{x_{j_1} x_{j_2} \dots x_{j_n}\right\}}{n!\ \ r^{2n+1}} . \tag{4.10}$$

It is usual to define the multipole moments of the source distribution as the family of completely symmetric trace-free tensors given by

$$Q^{j_1 j_2 \dots j_n} = \mathcal{J}\left\{I^{j_1 j_2 \dots j_n}\right\} , \tag{4.11}$$

with the $I^{j_1 j_2 \dots j_n}$ as in (4.9). The first few multipole moments are

$$Q = \int \rho(\underline{x}')\, dv'^{(3)}$$

$$Q^{j_1} = \int \rho(\underline{x}')\, x'^{j_1}\, dv'^{(3)}$$

$$Q^{j_1 j_2} = \int \rho(\underline{x}') \left[3x'^{j_1} x'^{j_2} - \eta^{j_1 j_2} (x'^{i} x'_{i})\right] dv'^{(3)}$$

$$Q^{j_1 j_2 j_3} = \int \rho(\underline{x}') \left[15 x'^{j_1} x'^{j_2} x'^{j_3} - 3(\eta^{j_1 j_2} x'^{j_3} + \eta^{j_2 j_3} x'^{j_1} + \eta^{j_3 j_1} x'^{j_2})\ (x'^{i} x'_{i})\right] dv'^{(3)}$$

......

The potential (4.2) now takes the form

$$\Phi(\underset{\sim}{x}) = \frac{Q}{r} + \sum_{n=1}^{\infty} \frac{Q^{j_1 j_2 \cdots j_n} \, x_{j_1} x_{j_2} \cdots x_{j_n}}{n! \; r^{2n+1}} \quad . \qquad (4.12)$$

The above analysis is so far quite conventional, and in keeping with other similar accounts it makes no reference to the conformal tensor structure of the multipole moments. Following Geroch [16] let us attach the $Q^{j_1 j_2 \cdots j_n}$ to a particular origin. Then under changes of origin the multipole moments become tensor fields. To be precise, let the new origin $\bar{O}$ have position vector $\underset{\sim}{x}$ relative to the original origin O. Then if $\underset{\sim}{x}'$ is the position vector of a point P relative to O, and $\bar{\underset{\sim}{x}}'$ is its position vector relative to $\bar{O}$, we have

$$\bar{x}'^j = x'^j - x^j \quad . \qquad (4.13)$$

The multipole moments referred to the new origin $\bar{O}$, will be denoted by $\bar{Q}^{j_1 j_2 \cdots j_n}$, and are defined via equations of the form (4.9), (4.11) but with $\bar{x}'^j$ replacing x'^j throughout.

It is straightforward then to show that the transformations between the $Q^{j_1 j_2 \cdots j_n}$ and the $\bar{Q}^{j_1 j_2 \cdots j_n}$ are given by

$$\bar{Q} = Q \quad ,$$

$$\bar{Q}^{j_1} = Q^{j_1} - x^{j_1} Q \; ,$$

$$\bar{Q}^{j_1 j_2} = Q^{j_1 j_2} - 6 \, \mathcal{J}\{Q^{(j_1} x^{j_2)}\} + 3 \, \mathcal{J}\{Q x^{j_1} x^{j_2}\} \quad ,$$

$$\bar{Q}^{j_1 j_2 j_3} = Q^{j_1 j_2 j_3} - 15 \, \mathcal{J}\{Q^{(j_1 j_2} x^{j_3)}\} + 45 \, \mathcal{J}\{Q^{(j_1} x^{j_2} x^{j_3)}\} - 15 \, \mathcal{J}\{Q x^{j_1} x^{j_2} x^{j_3}\} \, ,$$

and in general we have

$$\begin{aligned} \bar{Q}^{j_1 j_2 \cdots j_n} = \; & Q^{j_1 j_2 \cdots j_n} + A^n_1 \, \mathcal{J}\{Q^{(j_1 j_2 \cdots j_{n-1}} x^{j_n)}\} \\ & + A^n_2 \, \mathcal{J}\{Q^{(j_1 j_2 \cdots j_{n-2}} x^{j_{n-1}} x^{j_n)}\} \\ & + \ldots + A^n_n \, \mathcal{J}\{Q x^{j_1} x^{j_2} \ldots x^{j_n}\} \quad , \end{aligned} \qquad (4.14)$$

where

$$A^n_r = (-1)^r \, ({}^nC_r) \, (2n-2r+1) \, (2n-2r+3) \ldots (2n-1) \,, \qquad (4.15)$$
$$r = 1, 2, \ldots, n.$$

Now consider the differential equations

$$Q^{j_1 j_2 \cdots j_n, k} = n(n-1) \, Q^{k(j_3 \cdots j_n} \eta^{j_1 j_2)} - n(2n-1) \, \eta^{k(j_1} Q^{j_2 \cdots j_n)}, \qquad (4.16)$$

for $n = 0,1,2, \ldots$.

Straightforward integration of the cases $n = 0,1,2,3$ leads to the solutions

$$Q = \underset{o}{Q}$$
$$Q^{j_1} = \underset{o}{Q}^{j_1} - x^{j_1} \underset{o}{Q}$$
$$Q^{j_1 j_2} = \underset{o}{Q}^{j_1 j_2} - 6 \, \mathcal{J}\left\{\underset{o}{Q}^{(j_1} x^{j_2)}\right\} + 3 \, \mathcal{J}\left\{\underset{o}{Q} \, x^{j_1} x^{j_2}\right\}$$
$$Q^{j_1 j_2 j_3} = \underset{o}{Q}^{j_1 j_2 j_3} - 15 \, \mathcal{J}\left\{\underset{o}{Q}^{(j_1 j_2} x^{j_3)}\right\} + 45 \, \mathcal{J}\left\{\underset{o}{Q}^{(j_1} x^{j_2} x^{j_3)}\right\}$$
$$- 15 \, \mathcal{J}\left\{\underset{o}{Q} \, x^{j_1} x^{j_2} x^{j_3}\right\} \,,$$

where the suffix o indicates initial data specified at the origin of the frame of reference coordinatised by the $\{x^{j_1}\}$.

It is clear also that the form of the general multipole moment $Q^{j_1 j_2 \cdots j_n}$ is given by

$$Q^{j_1 j_2 \cdots j_n} = \underset{o}{Q}^{j_1 j_2 \cdots j_n} + A^n_1 \, \mathcal{J}\left\{\underset{o}{Q}^{(j_1 j_2 \cdots j_{n-1}} x^{j_n)}\right\} + A^n_2 \, \mathcal{J}\left\{\underset{o}{Q}^{(j_1 j_2 \cdots j_{n-2}} x^{j_{n-1}} x^{j_n)}\right\}$$
$$+ \ldots + A^n_n \, \mathcal{J}\left\{\underset{o}{Q} \, x^{j_1} x^{j_2} \cdots x^{j_n}\right\} \,, \qquad (4.17)$$

where the coefficients A^n_r are defined in (4.15).

Our conclusion is then that the differential equation (4.16) is the defining equation of a set of multipole moment fields having the correct position dependence and the required dependence upon the $\underset{o}{Q}^{j_1 j_2 \cdots j_n}$ moments. It is this differential equation that we shall generalise to Minkowski space-time in the next section of this work.

From (4.16) we derive the following properties of the multipole moment fields:

(i) a knowledge of Q, Q^{j_1}, $Q^{j_1 j_2}$, ... $Q^{j_1 j_2 \cdots j_n}$ at a particular point uniquely defines the tensor field $Q^{j_1 j_2 \cdots j_n}(\underline{x})$ everywhere;

(ii) a knowledge of the tensor field $Q^{j_1 j_2 \cdots j_n}(\underline{x})$ determines all the Q-fields of rank less than n. To see this, contract (4.16) over k, j_1 to obtain

$$Q^{k j_2 \cdots j_n}{}_{,k} = -n(2n+1)\, Q^{j_2 j_3 \cdots j_n} \quad \text{, which gives} \tag{4.18}$$

$Q^{j_1 j_2 \cdots j_{n-1}}$ (up to a factor) as the divergence of $Q^{j_1 j_2 \cdots j_n}$;

(iii) each of the $Q^{j_1 j_2 \cdots j_n}$ is a C-K tensor field of $\mathcal{M}$. It is straightforward to show from (4.16) that

$$Q^{(j_1 j_2 \cdots j_n , k)} = -n^2 \eta^{(k j_1} Q^{j_2 \cdots j_n)} \quad ,$$

which is the required condition on $Q^{j_1 j_2 \cdots j_n}$;

(iv) knowing Q, Q^{j_1}, $Q^{j_1 j_2}$,... as tensor fields we may define the position vector $\bar{x}^i$ of the centre of mass of the system by

$$Q^{j_1}(\bar{x}^i) = 0.$$

Then, assuming $\underset{o}{Q} \neq 0$ we have

$$\bar{x}^i = \frac{1}{\underset{o}{Q}} \underset{o}{Q}^i \quad ,$$

where $\underset{o}{Q}^i$ is the dipole moment about the origin.

8.5 Multipole moments of the wave equation in Minkowski space-time.

We shall extend the 3-space formalism of §4 here to consider a definition of multipole moments for the equation

$$\Box \psi = \tau(x^{\mu}) \qquad (5.1)$$

in Minkowski space-time. It will be sufficient to study the scalar wave equation since the generalisation to electromagnetism and linearised gravitation just requires that each component of a vector or tensor potential satisfy a differential equation of the form (5.1).

By analogy with the properties possessed by the 3-space multipole moment fields of §4 we define the multipole moments in $\mathcal{M}$ as the family of fields Q, Q^{α_1}, $Q^{\alpha_1\alpha_2}$, ... with the following properties:

(i) each of Q, Q^{α_1}, $Q^{\alpha_1\alpha_2}$, ... is completely symmetric and trace-free;

(ii) knowledge of $Q^{\alpha_1\alpha_2\ldots\alpha_n}$ as a tensor field uniquely determines all Q-fields of lower rank;

(iii) knowledge of Q, Q^{α_1}, $Q^{\alpha_1\alpha_2}$, ... $Q^{\alpha_1\alpha_2\ldots\alpha_n}$ at a given point uniquely determines the tensor field $Q^{\alpha_1\alpha_2\ldots\alpha_n}(x^{\mu})$.

In the way that the set of differential equations (4.16) generated 3-space multipole moment fields, we propose the set

$$Q^{\alpha_1\alpha_2\ldots\alpha_n,\mu} = \tfrac{1}{2}\, n(n-1)\, Q^{\mu(\alpha_3\ldots\alpha_n}\, g^{\alpha_1\alpha_2)} - n^2\, g^{\mu(\alpha_1}Q^{\alpha_2\ldots\alpha_n)}\,, \qquad (5.2)$$

for n = 0,1,2,..., to generate multipole moment fields in Minkowski space. $g^{\mu\nu}$ indicates, as usual, the metric tensor of Minkowski space-time;

$g^{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$.

It remains to show that the solutions of (5.2) satisfy properties (i), (ii) and (iii) above. We note first that the equations (5.2) for n = 0,1,2,3,4 have been integrated explicitly, and give

$$Q = \underset{\circ}{Q} \quad ,$$

$$Q^{\alpha_1} = \underset{\circ}{Q}^{\alpha_1} - \underset{\circ}{Q}\, x^{\alpha_1} \quad ,$$

$$Q^{\alpha_1\alpha_2} = \underset{\circ}{Q}^{\alpha_1\alpha_2} - 4\,\mathcal{J}\left\{x^{(\alpha_1}\underset{\circ}{Q}^{\alpha_2)}\right\} + 2\,\mathcal{J}\left\{\underset{\circ}{Q}\, x^{\alpha_1} x^{\alpha_2}\right\} ,$$

$$Q^{\alpha_1\alpha_2\alpha_3} = \underset{\circ}{Q}^{\alpha_1\alpha_2\alpha_3} - 9\,\mathcal{J}\left\{x^{(\alpha_1}\underset{\circ}{Q}^{\alpha_2\alpha_3)}\right\} + 18\,\mathcal{J}\left\{x^{(\alpha_1}x^{\alpha_2}\underset{\circ}{Q}^{\alpha_3)}\right\}$$
$$- 6\,\mathcal{J}\left\{\underset{\circ}{Q}\, x^{\alpha_1} x^{\alpha_2} x^{\alpha_3}\right\} ,$$

$$Q^{\alpha_1\alpha_2\alpha_3\alpha_4} = \underset{\circ}{Q}^{\alpha_1\alpha_2\alpha_3\alpha_4} - 16\,\mathcal{J}\left\{x^{(\alpha_1}\underset{\circ}{Q}^{\alpha_2\alpha_3\alpha_4)}\right\} + 72\,\mathcal{J}\left\{x^{(\alpha_1}x^{\alpha_2}\underset{\circ}{Q}^{\alpha_3\alpha_4)}\right\}$$
$$- 96\,\mathcal{J}\left\{x^{(\alpha_1}x^{\alpha_2}x^{\alpha_3}\underset{\circ}{Q}^{\alpha_4)}\right\} + 24\,\mathcal{J}\left\{x^{\alpha_1}x^{\alpha_2}x^{\alpha_3}x^{\alpha_4}\underset{\circ}{Q}\right\} ,$$

where $\mathcal{J}\{\ \}$ indicates trace-free part and the suffix $\circ$ indicates initial data prescribed at the origin of coordinates.

In general, the solution for $Q^{\alpha_1\alpha_2\ldots\alpha_n}$ is

$$Q^{\alpha_1\alpha_2\ldots\alpha_n} = \underset{\circ}{Q}^{\alpha_1\alpha_2\ldots\alpha_n} + B^n_1\,\mathcal{J}\left\{x^{(\alpha_1}\underset{\circ}{Q}^{\alpha_2\ldots\alpha_n)}\right\} + B^n_2\,\mathcal{J}\left\{x^{(\alpha_1}x^{\alpha_2}\underset{\circ}{Q}^{\alpha_3\ldots\alpha_n)}\right\}$$
$$+ \ldots + B^n_n\,\mathcal{J}\left\{x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_n}\,\underset{\circ}{Q}\right\} , \qquad (5.3)$$

where

$$B^n_r = (-1)^r\,({}^nC_r)^2\, r! \quad , \quad r = 1,2,\ldots,n. \qquad (5.4)$$

Some results on the expressions for trace-free parts of tensors in Minkowski space are given in Appendix 5 .

In the following work we shall often have occasion to manipulate expressions of the form

$$A^{(\alpha_1\alpha_2\ldots\alpha_r}\,B^{\alpha_{r+1}\ldots\alpha_n)} \quad ,$$

where $A^{\alpha_1\alpha_2\ldots\alpha_r}$ and $B^{\alpha_{r+1}\ldots\alpha_n}$ are completely symmetric, and we quote here a general result that will facilitate such manipulations;

$$T^{(\alpha_1\alpha_2\ldots\alpha_n)} = ({}^nC_r)^{-1}\sum_I T^{(\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r})}\ (\text{other } \alpha_i) \quad , \qquad (5.5)$$

where the summation is taken over the set I of all integers $i_1, i_2, \ldots i_r$ with

$$1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n .$$

Let us turn our attentions now to a proof of the properties (i), (ii) and (iii) enumerated above.

It is evident from (5.3) that $Q^{\alpha_1\alpha_2\ldots\alpha_n}(x^\nu)$ is a completely symmetric trace-free tensor field provided that the initial data $\overset{\circ}{Q}{}^{\alpha_1\alpha_2\ldots\alpha_n}$ is completely symmetric and trace-free. A verification of the trace-free property is also possible using the differential equation (5.2) directly, for then we have

$$
\begin{aligned}
Q^{\alpha_1\alpha_2\ldots\alpha_n;\mu} = \tfrac{1}{2}n(n-1)\ ({}^nC_2)^{-1}\Big\{ & g^{\alpha_1\alpha_2}Q^{\alpha_3\ldots\alpha_n\mu} + g^{\alpha_1\alpha_3}Q^{\alpha_2\alpha_4\ldots\alpha_n\mu} + \ldots \\
& + g^{\alpha_2\alpha_3}Q^{\alpha_1\alpha_4\ldots\alpha_n\mu} + g^{\alpha_2\alpha_4}Q^{\alpha_1\alpha_3\ldots\alpha_n\mu} + \ldots \\
& + g^{\alpha_3\alpha_4}Q^{\alpha_1\alpha_2\alpha_5\ldots\alpha_n\mu} + \ldots\Big\} \\
- n^2({}^nC_1)^{-1}\Big\{ & g^{\mu\alpha_1}Q^{\alpha_2\alpha_3\ldots\alpha_n} + g^{\mu\alpha_2}Q^{\alpha_1\alpha_3\ldots\alpha_n} + g^{\mu\alpha_3}Q^{\alpha_1\alpha_2\ldots\alpha_n} + \ldots\Big\},
\end{aligned}
\tag{5.6}
$$

from which

$$
\begin{aligned}
Q_\beta{}^{\beta\alpha_3\ldots\alpha_n;\mu} = {} & 4\, Q^{\mu\alpha_3\ldots\alpha_n} + (n-2)\, Q^{\mu\alpha_3\ldots\alpha_n} \\
& + (n-2)\, Q^{\mu\alpha_3\ldots\alpha_n} \\
& + \text{terms in } Q_\beta{}^{\beta\alpha_4\ldots\alpha_n} \\
& - n\Big\{2\, Q^{\mu\alpha_3\ldots\alpha_n} + \text{terms in } Q_\beta{}^{\beta\alpha_4\ldots\alpha_n}\Big\} \\
= {} & 0 + \text{terms in } Q_\beta{}^{\beta\alpha_4\ldots\alpha_n},
\end{aligned}
\tag{5.7}
$$

where the Q tensor on the right hand side of this equation is of rank (n-1). We are now able to prove our result inductively, for (5.7) tells us that

$$Q^{\alpha_1\alpha_2\ldots\alpha_{n-1}} \text{ trace-free} \implies \frac{\partial}{\partial x^\mu}\, Q_\beta{}^{\beta\alpha_3\ldots\alpha_n} = 0.$$

Now $Q_\beta{}^{\beta\alpha_3\ldots\alpha_n}$ is known to be zero at the origin (part of the initial data), and therefore we have shown that if $Q^{\alpha_1\alpha_2\ldots\alpha_{n-1}}$ is everywhere trace-free then $Q^{\alpha_1\alpha_2\ldots\alpha_n}$ is everywhere trace-free. Explicit calculation has shown that $Q^{\alpha_1\alpha_2}$ is a trace-free field, so the required result follows by induction.

Using (5.6) again we can easily demonstrate property (ii), for it follows from (5.6), by contraction over α_1 and μ, that

$$Q^{\mu\alpha_2\ldots\alpha_n}{}_{,\mu} = (n-1)\, Q^{\alpha_2\ldots\alpha_n} - n\left\{ 4\, Q^{\alpha_2\ldots\alpha_n} + (n-1)\, Q^{\alpha_2\ldots\alpha_n}\right\},$$

where the complete symmetry of the Q tensors and their trace-free property have been invoked. Thus, we have

$$Q^{\mu\alpha_2\ldots\alpha_n}{}_{,\mu} = -(n+1)^2\, Q^{\alpha_2\ldots\alpha_n}, \qquad (5.8)$$

from which a knowledge of the field $Q^{\alpha_1\alpha_2\ldots\alpha_n}(x^\nu)$ enables one to determine the field $Q^{\alpha_1\alpha_2\ldots\alpha_{n-1}}(x^\nu)$. Repeated applications of (5.8) then define the Q fields of all lower ranks and property (ii) is proved.

Property (iii) is easily seen to be true since a knowledge of Q, Q^{α_1}, $Q^{\alpha_1\alpha_2}, \ldots,$ $Q^{\alpha_1\alpha_2\ldots\alpha_n}$ at a specific point leads (via equations of the form (5.3)) to a determination of the initial data $\overset{o}{Q}$, $\overset{o}{Q}{}^{\alpha_1}$, $\overset{o}{Q}{}^{\alpha_1\alpha_2}$, $\ldots$, $\overset{o}{Q}{}^{\alpha_1\alpha_2\ldots\alpha_n}$ and hence to a determination of the tensor field $Q^{\alpha_1\alpha_2\ldots\alpha_n}(x^\nu)$.

Finally, we may add that each of the fields $Q^{\alpha_1\alpha_2\ldots\alpha_n}(x^\nu)$ is a C-K tensor of Minkowski space, a result which follows readily from symmetrization over $\alpha_1, \alpha_2, \ldots, \alpha_n, \mu$ in (5.2).

The foregoing construction of multipole moment fields in Minkowski space-time must be seen as a purely formal extension of the Geroch analysis in 3-space. No attempt will be made here to establish contact with other multipole formulations in flat space-time, but it seems likely that the above work will have close links with the multipole moments defined by Pirani [23] .

CHAPTER 9

CONFORMAL SYMMETRIES IN CONFORMALLY-FLAT SPACE-TIMES

9.1 Introduction

Since C-K vectors are conform-invariant objects it is a relatively simple matter to extend the considerations made in flat and asymptotically-flat space-times to space-times which possess conformal flatness or asymptotic conformal flatness. In particular, we examine space-times of constant curvature (which are necessarily conformally flat) and a class of well-known cosmological models; those which are spatially homogeneous and isotropic. Knowing the C-K vectors of Minkowski space-time, it is easy to pick out those that become Killing vectors in the conformally related space-time, and this provides a neat approach to the derivation of the Killing symmetries.

The chapter concludes with a short discussion of asymptotically-Friedmann space-times which reproduces in a very concise manner some results of Hawking [73] on the asymptotic symmetries of such models.

9.2 Conformally flat space-times; their C-K vectors

It has already been shown in Chapter 6 that if $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are the respective metric tensors of two conformally related space-times $\mathcal{R}$ and $\bar{\mathcal{R}}$, so that

$$\bar{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu} , \tag{2.1}$$

$$\bar{g}^{\mu\nu} = \Omega^{2} g^{\mu\nu} , \tag{2.2}$$

and if ξ^{μ} is a conformal Killing vector of $\mathcal{R}$, then $\bar{\xi}^{\mu}$ given by

$$\bar{\xi}^{\mu} = \xi^{\mu} \qquad (2.3)$$

is a conformal Killing vector of $\bar{\mathcal{R}}$. The conformal factors ϕ, $\bar{\phi}$ defined by

$$\phi = \nabla_{\mu}\xi^{\mu} \text{ and } \bar{\phi} = \bar{\nabla}_{\mu}\xi^{\mu} \qquad (2.4)$$

are related according to

$$\bar{\phi} = \phi - 4\Omega^{-1}\xi^{\rho}\Omega_{,\rho} \,. \qquad (2.5)$$

If we consider conformally flat metrics with Minkowskian coordinates, so that

$$g_{\mu\nu} = \text{diag }(1,-1,-1,-1) \qquad (2.6)$$

and write

$$\bar{g}_{\mu\nu} = R^{2}(x^{\alpha})\, g_{\mu\nu} \qquad , \qquad (2.7)$$

it is straightforward to set down a basis for the 15-dimensional space of conformal Killing vectors in $\bar{\mathcal{R}}$ by modifying the results of Chapter 3. The contravariant components $\bar{\xi}^{\mu}_{A}$, A = 0,1,2,...,14, together with the associated conformal factors, are tabulated below:-

$\bar{\xi}^{\mu}_{0} = (1,0,0,0)$	$\bar{\phi}_{0} = \frac{4}{R} R_{,1}$
$\bar{\xi}^{\mu}_{1} = (0,1,0,0)$	$\bar{\phi}_{1} = \frac{4}{R} R_{,2}$
$\bar{\xi}^{\mu}_{2} = (0,0,1,0)$	$\bar{\phi}_{2} = \frac{4}{R} R_{,3}$
$\bar{\xi}^{\mu}_{3} = (0,0,0,1)$	$\bar{\phi}_{3} = \frac{4}{R} R_{,4}$
$\bar{\xi}^{\mu}_{4} = (x,t,0,0)$	$\bar{\phi}_{4} = \frac{4}{R} (xR_{,1} + tR_{,2})$
$\bar{\xi}^{\mu}_{5} = (y,0,t,0)$	$\bar{\phi}_{5} = \frac{4}{R} (yR_{,1} + tR_{,3})$
$\bar{\xi}^{\mu}_{6} = (z,0,0,t)$	$\bar{\phi}_{6} = \frac{4}{R} (zR_{,1} + tR_{,4})$
$\bar{\xi}^{\mu}_{7} = (0,y,-x,0)$	$\bar{\phi}_{7} = \frac{4}{R} (yR_{,2} - xR_{,3})$

$\bar{\xi}^{\mu}_{8} = (0,z,0,-x)$	$\bar{\phi}_{8} = \frac{4}{R}\,(zR_{,2} - xR_{,4})$
$\bar{\xi}^{\mu}_{9} = (0,0,z,-y)$	$\bar{\phi}_{9} = \frac{4}{R}\,(zR_{,3} - yR_{,4})$
$\bar{\xi}^{\mu}_{10} = (t,x,y,z)$	$\bar{\phi}_{10} = 4 + \frac{4}{R}\,\bar{\xi}^{\mu}_{10}\,R_{,\mu}$
$\bar{\xi}^{\mu}_{11} = (\tfrac{1}{2}\,[t^2+x^2+y^2+z^2],\ tx,\ ty,\ tz)$	$\bar{\phi}_{11} = 4t + \frac{4}{R}\,\bar{\xi}^{\mu}_{11}\,R_{,\mu}$
$\bar{\xi}^{\mu}_{12} = (xt,\ \tfrac{1}{2}\,[x^2+t^2-y^2-z^2],\ xy,\ xz)$	$\bar{\phi}_{12} = 4x + \frac{4}{R}\,\bar{\xi}^{\mu}_{12}\,R_{,\mu}$
$\bar{\xi}^{\mu}_{13} = (yt,\ yx,\ \tfrac{1}{2}[y^2+t^2-x^2-z^2],\ yz)$	$\bar{\phi}_{13} = 4y + \frac{4}{R}\,\bar{\xi}^{\mu}_{13}\,R_{,\mu}$
$\bar{\xi}^{\mu}_{14} = (zt,\ zx,\ zy,\ \tfrac{1}{2}\,[z^2+t^2-x^2-y^2]\ )$	$\bar{\phi}_{14} = 4z + \frac{4}{R}\,\bar{\xi}^{\mu}_{14}\,R_{,\mu}$

It is clear from (2.5) that although conformal Killing vectors of $\mathcal{R}$ become conformal Killing vectors of $\bar{\mathcal{R}}$ it is certainly not true in general that Killing vectors of $\mathcal{R}$ map into Killing vectors of $\bar{\mathcal{R}}$. However, the next section of this chapter will investigate space-times of constant curvature, and we shall find there that such spaces are not only conformally flat but also have a ten-parameter group of Killing motions. In that respect space-times of constant curvature are similar to Minkowski space-time; in each case the manifold admits the maximal dimension group of motions for a four-dimensional space.

In the work of the following sections we shall simplify the notation by dropping the bars from all quantities in $\bar{\mathcal{R}}$, except when there is a risk of ambiguity.

9.3 Space-times of constant curvature; their groups of motions

In a space-time of constant curvature it can be shown (see, for example, [15]) that the Riemann tensor $B_{\mu\nu\rho\sigma}$ takes the form

$$B_{\mu\nu\rho\sigma} = K\,(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}), \text{ K constant,} \tag{3.1}$$

from which it follows that the Ricci tensor is

$$B_{\nu\rho} = B^{\sigma}{}_{\nu\rho\sigma} = -3K\, g_{\nu\rho} \tag{3.2}$$

and the scalar curvature is

$$B = -12K. \tag{3.3}$$

Then the Weyl conformal curvature $C^{\mu}{}_{\nu\rho\sigma}$, defined by

$$C^{\mu}{}_{\nu\rho\sigma} = B^{\mu}{}_{\nu\rho\sigma} + \tfrac{1}{2}(\delta^{\mu}_{\rho} B_{\nu\sigma} - \delta^{\mu}_{\sigma} B_{\nu\rho} + g_{\nu\sigma} B^{\mu}{}_{\rho} - g_{\nu\rho} B^{\mu}{}_{\sigma}) + \frac{1}{6} B\,(\delta^{\mu}_{\sigma}\, g_{\nu\rho} - \delta^{\mu}_{\rho}\, g_{\nu\sigma}) \;, \tag{3.4}$$

is easily seen to vanish, and this is a necessary and sufficient condition for the space-time to be conformally flat.

In fact, it is always possible in a space of constant curvature to choose coordinates so that

$$g_{\mu\nu}\, dx^{\mu} dx^{\nu} = ds^2 = \frac{\eta_{\mu\nu} dx^{\mu} dx^{\nu}}{\left[1 + \frac{K}{4} x^{\mu} x_{\mu}\right]^2} \;, \tag{3.5}$$

where

$$\eta_{\mu\nu} = \text{diag}\,(1,-1,-1,-1) \;,$$

is the metric tensor of Minkowski space-time. This makes explicit the conformal flatness of the metric $g_{\mu\nu}$. It is a straightforward matter now to investigate the Killing motions of (3.5) by using the results of § 9.2 . We establish contact with the formalism of that section by noting that $R(x^\alpha)$ in (2.7) becomes

$$R(x^\alpha) = \left[1 + \frac{K}{4} x^\mu x_\mu\right]^{-1}. \qquad (3.6)$$

Then it follows immediately that

$$R_{,1} = \frac{-KR^2 t}{2}, \quad R_{,2} = \frac{KR^2 x}{2}, \quad R_{,3} = \frac{KR^2 y}{2}, \quad R_{,4} = \frac{KR^2 z}{2}, \qquad (3.7)$$

and so

$$\phi_4 = \phi_5 = \phi_6 = \phi_7 = \phi_8 = \phi_9 = 0, \qquad (3.8)$$

revealing that $\underline{\xi}_4, \underline{\xi}_5, \underline{\xi}_6, \underline{\xi}_7, \underline{\xi}_8, \underline{\xi}_9$ are all Killing vectors. One can show also that

$$\phi_0 = -2KRt, \quad \phi_1 = 2KRx, \quad \phi_2 = 2KRy, \quad \phi_3 = 2KRz,$$

and

$$\phi_{11} = 4Rt, \quad \phi_{12} = 4Rx, \quad \phi_{13} = 4Ry, \quad \phi_{14} = 4Rz,$$

from which it follows that

$$\underline{\xi}_{11} + \frac{2}{K}\underline{\xi}_0, \quad \underline{\xi}_{12} - \frac{2}{K}\underline{\xi}_1, \quad \underline{\xi}_{13} - \frac{2}{K}\underline{\xi}_2, \quad \underline{\xi}_{14} - \frac{2}{K}\underline{\xi}_3$$

are the four remaining Killing vectors.

9.4 Homogeneous isotropic cosmological models

Two basic assumptions concerning the large-scale nature of the universe have been made in the derivation of general-relativistic world-models, and although these principles have some support from observation, they are by no means incontrovertible. Perhaps, as Tolman [74] says, we should look upon these assumptions as an attempt "... to secure a definite and relatively simple mathematical problem, rather than to secure a correspondence with known reality".

The first assumption is that space (i.e. space-like 3-space of space-time) is homogeneous; in other words the large-scale features of the universe are the same in all places. The second assertion is that space is isotropic; an observer at rest with respect to the mean motion of matter in his vicinity records large-scale observations which show no dependence upon direction. Starting from these assumptions, Robertson [75,76] and Walker [77] were able to set down a form of metric which made it possible to find some exact solutions of the Einstein field equations. We shall take the Robertson-Walker metric in the form

$$ds^2 = (dx^0)^2 - R^2(x^0) \left[1 + \frac{\epsilon}{4} r^2\right]^{-2} \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] , \qquad (4.1)$$

where

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 .$$

It is clear from (4.1) that the 3-spaces x^0 = constant are spaces of constant curvature and therefore admit the maximal (6-parameter) group of Killing motions permitted by a space of three dimensions. The six symmetries are just the spatial translations and spatial rotations, which one would expect to arise as a consequence of the assumptions of spatial homogeneity and spatial isotropy upon which the model is based.

A specialisation of the Robertson-Walker metric made by Friedmann [78,79] gives us the class of cosmological models (now named after him), in which the matter present in the universe is assumed to be cold dust at zero pressure. With this assumption the Einstein field equations (with cosmological term included),

$$B^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} B + \Lambda g^{\mu\nu} = \varkappa T^{\mu\nu} , \qquad (4.2)$$

where $B^{\mu\nu}$ signifies the Ricci tensor, B its contraction, and $T^{\mu\nu}$ is the energy-momentum tensor associated with the cosmological dust, take an especially simple form. In fact, (4.2) then reduces to a single differential equation for R (x), the so-called Friedmann equation,

$$\left(\frac{dR}{dx^0}\right)^2 = -\frac{\kappa}{4\pi}\frac{M}{R} + \frac{\Lambda}{3}R^2 - \varepsilon, \qquad (4.3)$$

where M is a positive constant. The different types of Friedmann models arise with different choices of M, Λ, ε. We shall examine some of these solutions in subsequent sections of this chapter.

9.5 The de Sitter model

The usual form of the de Sitter metric [80] is

$$ds^2 = (1 - \frac{r^2}{R_0^2})\, dt^2 - \left[(1 - \frac{r^2}{R_0^2})^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right], \qquad (5.1)$$

where R_0 is a constant, showing clearly that the model is a static one. Using a coordinate transformation

$$\bar{r} = r(1 - \frac{r^2}{R_0^2})^{-\frac{1}{2}} \quad e^{\frac{-t}{R_0}}, \qquad (5.2)$$

$$\bar{t} = t + \tfrac{1}{2} R_0 \log (1 - \frac{r^2}{R_0^2}) ,$$

due to Robertson [81], brings (5.1) to the form

$$ds^2 = d\bar{t}^2 - e^{\frac{2\bar{t}}{R_0}} (d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta\, d\phi^2) .$$

Dropping bars and writing

$$k = \frac{1}{R_0}$$

gives, after an obvious change of coordinates,

$$ds^2 = dt^2 - e^{2kt}(dx^2 + dy^2 + dz^2) . \qquad (5.3)$$

An alternative derivation of (5.3) appears if we solve the Friedmann equation (4.3) with

$$M = 0 ,$$

corresponding to a universe in which the mean density of matter is zero, and

$$\epsilon = 0 ,$$

corresponding to space-like 3-spaces of zero curvature. It is necessary also to make the assumption

$$\Lambda > 0 ,$$

and then (4.3) gives

$$R(x^0) = Ae^{\alpha x^0} , \tag{5.4}$$

where

$$\alpha^2 = \frac{\Lambda}{3} , \tag{5.5}$$

and A is an arbitrary constant. Putting $x^0 = t$ and substituting (5.4) into (4.1) gives, after a trivial coordinate transformation,

$$ds^2 = dt^2 - e^{2\alpha t}(dx^2 + dy^2 + dz^2) , \tag{5.6}$$

which is identical with (5.3).

To investigate the symmetries of the de Sitter model we first make a further coordinate change

$$t \longrightarrow -\frac{1}{\alpha} e^{-\alpha t}$$

which puts (5.6) into the form

$$ds^2 = \frac{1}{t^2} \left[dt^2 - (dx^2 + dy^2 + dz^2) \right] , \tag{5.7}$$

which is evidently conformally flat. It is a simple matter now to utilise the work of §9.2 to pick out Killing motions of the de Sitter model. It transpires that

$$\phi_1 = \phi_2 = \phi_3 = 0 ,$$

$$\phi_7 = \phi_8 = \phi_9 = 0 ,$$

(which follow whenever $R(x^\alpha)$ in (2.7) is a function of x^0 alone), and

$$\phi_{10} = \phi_{12} = \phi_{13} = \phi_{14} = 0 .$$

Thus, the de Sitter universe admits a 10-parameter group of Killing motions. This group is not isomorphic to the Poincaré group in general, but goes over into the Poincaré group in the limit $\alpha \rightarrow 0$; [82]. It can be shown that the de Sitter group is in fact isomorphic to SO(1,4), but we shall not give a demonstration of this isomorphism in the present work.

9.6 The Friedmann models

This class of models has metrics of the Robertson-Walker type (4.1) where the function R is determined by the Friedmann equation (4.3). We shall restrict our attention to the case $\Lambda = 0$, and write

$$ds^2 = dt^2 - R^2(t) \left(1 + \frac{\varepsilon r^2}{4}\right)^{-2} (dx^2 + dy^2 + dz^2) , \qquad (6.1)$$

where

$$\left(\frac{dR}{dt}\right)^2 = -\frac{\kappa}{4\pi}\frac{M}{R} - \varepsilon . \qquad (6.2)$$

Starting from an alternative form of (6.1) ;

$$ds^2 = dt^2 - R^2(t) \left(1 + \frac{\varepsilon r^2}{4}\right)^{-2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2); \qquad (6.3)$$

we make the coordinate change

$$r \longrightarrow r \left(1 + \frac{\varepsilon r^2}{4}\right)^{-1} ,$$

which brings (6.3) to the form

$$ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1-\varepsilon r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right] . \qquad (6.4)$$

We shall, with no loss of generality, solve (6.2) in the cases $\varepsilon = 0,+1,-1$ only. Noting that

$$\chi = \frac{-8\pi G}{c^4} , \qquad (6.5)$$

where c is the speed of light and G the gravitational constant, we find in the case $\varepsilon = 0$,

$$R = \frac{GM}{2c^4} \tau^2 , \qquad (6.6)$$

where the change of coordinates

$$t = \frac{GM}{6c^4} \tau^3 \qquad (6.7)$$

has been made. It follows then that the metric is given by

$$ds^2 = \frac{G^2M^2}{4c^8} \tau^4 \left[d\tau^2 - (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2)\right] , \qquad (6.8)$$

which is clearly conformally flat.

Taking next the case when $\varepsilon = +1$, one finds that

$$R = \frac{GM}{c^4} (1 - \cos\tau) \qquad (6.9)$$

with

$$t = \frac{GM}{c^4} (\tau - \sin\tau) , \qquad (6.10)$$

leading to the metric

$$ds^2 = \left(\frac{GM}{c^4}\right)^2 (1 - \cos\tau)^2 \left[d\tau^2 - \left(\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right)\right] . \qquad (6.11)$$

The change of radial coordinate

$$r = \sin \chi$$

then gives

$$ds^2 = \left(\frac{GM}{c^4}\right)^2 (1 - \cos\tau)^2 \left[d\tau^2 - (d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta \, d\phi^2)\right]. \quad (6.12)$$

To exhibit the conformal flatness of (6.12) we make a final change of coordinates

$$2t' = \tan \tfrac{1}{2} (\tau + \chi) + \tan \tfrac{1}{2} (\tau - \chi)$$

$$2r' = \tan \tfrac{1}{2} (\tau + \chi) - \tan \tfrac{1}{2} (\tau - \chi)$$

which gives, after dropping the dashes,

$$ds^2 = \left(\frac{GM}{c^4}\right)^2 F^2(r,t) \left[dt^2 - (dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2)\right], \quad (6.13)$$

where

$$F(r,t) = \frac{2\left\{ \left[(1 + t^2 + r^2) - 4r^2t^2\right]^{\frac{1}{2}} - 1 + t^2 - r^2 \right\}}{(1 + t^2 + r^2) - 4r^2t^2} . \quad (6.14)$$

Finally, in the case when $\varepsilon = -1$, (6.2) gives

$$R = \frac{GM}{c^4} (\cosh\tau - 1) , \quad (6.15)$$

with

$$t = \frac{GM}{c^4} (\sinh\tau - \tau) .$$

Making the substitution

$$r = \sinh \chi \quad (6.16)$$

puts the metric (6.4) into the form

$$ds = \left(\frac{GM}{c^4}\right)^2 (\cosh\tau - 1)^2 \left[d\tau^2 - (d\chi^2 + \sinh^2\chi \, d\theta^2 + \sinh^2\chi \sin^2\theta \, d\phi^2)\right] , \quad (6.17)$$

which is the form analogous to (6.12) in our previous case.

Finally, the coordinate transformation

$$r' = e^{\tau} \sinh \chi$$
$$t' = e^{\tau - \chi} + e^{\tau} \sinh \chi$$

puts (6.17) into the conformally flat form

$$ds^2 = \left(\frac{GM}{c^4}\right)^2 F^2(r,t) \left[dt^2 - (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2)\right] , \qquad (6.18)$$

where

$$F(r,t) = \tfrac{1}{2} \left\{1 - (t^2 - r^2)^{-\frac{1}{2}}\right\}^2 . \qquad (6.19)$$

The physical interpretation of the Friedmann cosmological models has been considered recently by Hawking [73, 84] with special concern for the asymptotic analysis of gravitational radiation, and it has been concluded that only the last ($\varepsilon = -1$) case is of any significance in this respect.

The Killing symmetries of (6.18), (6.19) can be discovered now using the formalism of § 9.2; one finds that

$$\phi_4 = \phi_5 = \phi_6 = 0$$

and

$$\phi_7 = \phi_8 = \phi_9 = 0 .$$

Thus the symmetry group of this particular Friedmann model is isomorphic to the homogeneous Lorentz group. (Perhaps we should point out also that this in turn is isomorphic to the group of Killing motions of a 3-space of constant negative curvature).

9.7 Asymptotically-Friedmann models; their asymptotic symmetries

Some work of Hawking [73] has suggested that the assumption of asymptotic flatness is likely to be valid only if we restrict interest to regions of the universe which are small compared with the Hubble radius. For applications of a cosmological nature the curvature and expansion of the universe must be recognised and this leads to the assumption of an asymptotically-Friedmann space-time. For the reasons quoted in the previous section we shall consider only the Friedmann models with negative constant curvature on the hypersurfaces x^0 = constant, and then (6.18), (6.19) give the form of the metric in the asymptotic limit.

It is possible to extend the asymptotic analysis of C-K vectors in asymptotically flat space-times to the case in which only asymptotic conformal flatness is required. The asymptotically-Friedmann case then emerges as a special example of this. Suppose, in the terminology of § 9.2, that $\mathcal{R}$ is asymptotically flat and $\bar{\mathcal{R}}$ is asymptotically conformally flat. Then, following the technique of Chapter 5, the equations to be solved are

$$\mathcal{L}_{\underline{\xi}}\, \mathcal{G}_{\mu\nu} = O\,(r^{n(\mu,\nu)})\ , \qquad (7.1)$$

where the $n(\mu,\nu)$ are a set of integers describing the asymptotic behaviour of the tensor density $\mathcal{G}_{\mu\nu}$ according to

$$\mathcal{G}_{\mu\nu} = O\,(r^{n(\mu,\nu)})\ . \qquad (7.2)$$

Now from

$$\bar{g}_{\mu\nu} = R^2(x^\alpha)\, g_{\mu\nu} \qquad (7.3)$$

we see that

$$\det\,(\bar{g}_{\mu\nu}) = R^8 \det(g_{\mu\nu})\ ,$$

and so the tensor density $\bar{\mathcal{G}}_{\mu\nu}$, defined in $\bar{\mathcal{R}}$ by

$$\bar{\mathcal{G}}_{\mu\nu} = \left[-\det\,(\bar{g}_{\alpha\beta})\right]^{-\frac{1}{4}}\, \bar{g}_{\mu\nu}\ , \qquad (7.4)$$

satisfies

$$\bar{\mathcal{G}}_{\mu\nu} = \mathcal{G}_{\mu\nu} = O\,(r^{n(\mu,\nu)})\ . \qquad (7.5)$$

Thus, in $\bar{\mathcal{R}}$ the equations to be solved for the asymptotic C-K vectors, given by

$$\mathcal{L}_{\underline{\xi}}\, \bar{\mathcal{G}}_{\mu\nu} = O(r^{n(\mu,\nu)})\ , \qquad (7.6)$$

have the same solutions as the equations (7.1); that is, the asymptotic C-K vectors of $\bar{\mathcal{R}}$ are precisely those of $\mathcal{R}$. On the other hand, we have seen earlier that asymptotic Killing vectors in $\mathcal{R}$ do not in general go over into asymptotic Killing vectors in $\bar{\mathcal{R}}$, although they do of course go over into asymptotic C-K vectors.

Explicitly, we have in (7.3),

$$R(x^{\alpha}) = \frac{GM}{2c^4}\left[1 - (t^2 - r^2)^{-\frac{1}{2}}\right]^2, \tag{7.7}$$

and making use of the results of Chapter 5 for the asymptotic C-K vectors of $\mathcal{R}$, it is straightforward to show that only the spatial rotations and the Lorentz rotations appear as asymptotic Killing symmetries in the asymptotically Friedmann case. The remaining symmetries from the B.M.S. group appear as asymptotic conformal symmetries in the asymptotically Friedmann space. These findings are in accord with those of Hawking in his original paper on the subject.

APPENDICES

AND

BIBLIOGRAPHY

APPENDIX 1

INTEGRATION OF THE "NON-RADIAL" EQUATIONS FOR THE C-K VECTORS OF MINKOWSKI SPACE-TIME

In § 3.2 (pages 52-56) only the "radial" equations of the set (2.2 a-f) were solved explicitly. These equations revealed the dependence on r of the ξ^{μ}; the remaining equations impose restrictions on the constants of integration that have already appeared.

From (2.4 - 2.6), (2.8 - 2.10) of § 3.2 we have

$$\xi^{1} = \mathring{\xi}^{1}(u,x^{A}) \ , \qquad \text{(A1.1)}$$

$$\xi^{2} = \mathring{\xi}^{2} r^{2} - Ar - \tfrac{1}{2}B \ , \qquad \text{(A1.2)}$$

where $$A = \mathring{\xi}^{3}{}_{,3} + \mathring{\xi}^{4}{}_{,4} - \mathring{\xi}^{1}{}_{,1} - 2\,\mathring{\xi}^{3} \tanh x^{3} \qquad \text{(A1.3)}$$

and $$B = -\cosh^{2} x^{3} \left[\mathring{\xi}^{1}{}_{,3,3} + \mathring{\xi}^{1}{}_{,4,4} \right] , \qquad \text{(A1.4)}$$

$$\xi^{3} = \mathring{\xi}^{3} - r^{-1} \cosh^{2} x^{3}\, \mathring{\xi}^{1}{}_{,3} \ , \qquad \text{(A1.5)}$$

$$\xi^{4} = \mathring{\xi}^{4} - r^{-1} \cosh^{2} x^{3}\, \mathring{\xi}^{1}{}_{,4} \ . \qquad \text{(A1.6)}$$

There are thirteen equations (2.11 - 2.23) which must be solved to give the final form of the constants $\mathring{\xi}^{\mu}$, A, B appearing above;

$$\mathring{\xi}^{2}{}_{,1} = 0 \ , \qquad \text{(A1.7)}$$

$$\mathring{\xi}^{2} + A = 0 \ , \qquad \text{(A1.8)}$$

$$A + \mathring{\xi}^{1}{}_{,1} - B_{,1} = 0 \ , \qquad \text{(A1.9)}$$

$$\mathring{\xi}^{3}{}_{,3} - \mathring{\xi}^{4}{}_{,4} = 0 \ , \qquad \text{(A1.10)}$$

$$\mathring{\xi}^{3}{}_{,4} + \mathring{\xi}^{4}{}_{,3} = 0 \ , \qquad \text{(A1.11)}$$

$$\cosh x^{3} \left(\mathring{\xi}^{1}{}_{,3,3} - \mathring{\xi}^{1}{}_{,4,4} \right) + 2 \sinh x^{3}\, \mathring{\xi}^{1}{}_{,3} = 0 \ , \qquad \text{(A1.12)}$$

$$\cosh x^{3}\, \mathring{\xi}^{1}{}_{,3,4} + \sinh x^{3}\, \mathring{\xi}^{1}{}_{,4} = 0 \ , \qquad \text{(A1.13)}$$

$$2\,\mathring{\xi}^{1}{}_{,3} - B_{,3} = 0 \ , \qquad \text{(A1.14)}$$

$$2\,\mathring{\xi}^{1}{}_{,4} - B_{,4} = 0 \ , \qquad \text{(A1.15)}$$

$$\mathring{\xi}^{1}{}_{,3,1} - A_{,3} = 0 \ , \qquad \text{(A1.16)}$$

$$\mathring{\xi}^{1}{}_{,4,1} - A_{,4} = 0 \ , \qquad \text{(A1.17)}$$

$$\cosh^{2} x^{3}\, \mathring{\xi}^{2}{}_{,3} - \mathring{\xi}^{3}{}_{,1} = 0 \ , \qquad \text{(A1.18)}$$

and $$\cosh^{2} x^{3}\, \mathring{\xi}^{2}{}_{,4} - \mathring{\xi}^{4}{}_{,1} = 0 \ . \qquad \text{(A1.19)}$$

(A1.13) gives

$$\overset{\circ}{\xi}{}^{1} = \alpha(u,x^4)\ \text{sech}\ x^3 + \beta(u,x^3) \ , \qquad \text{(A1.20)}$$

where α, β are arbitrary functions of the given variables. Hence

$$B = \alpha\,\text{sech}\ x^3 - \alpha \sinh x^3 \tanh x^3 - \cosh^2 x^3\, \beta_{,3,3} - \cosh x^3\, \alpha_{,4,4}. \qquad \text{(A1.21)}$$

Substitution from (A1.20), (A1.21) into (A1.15) leads to the solution for $\alpha(u,x^4)$;

$$\alpha(u,x^4) = \gamma(u) \sin x^4 + \delta(u) \cos x^4 + \epsilon(u) \ , \qquad \text{(A1.22)}$$

where γ, δ, ϵ are arbitrary functions of u.

(A1.12) leads to the following form for $\beta(u,x^3)$;

$$\beta(u,x^3) = -\epsilon(u)\ \text{sech}\ x^3 + \chi(u) \tanh x^3 + \lambda(u) \ , \qquad \text{(A1.23)}$$

where χ, λ are arbitrary functions of u.

Combining (A1.20), (A1.22), (A1.23) we have

$$\overset{\circ}{\xi}{}^{1} = \left[\gamma(u) \sin x^4 + \delta(u) \cos x^4\right] \text{sech}\ x^3 + \kappa(u) \tanh x^3 + \lambda(u) \ , \qquad \text{(A1.24)}$$

leading to

$$B = 2\ \text{sech}\ x^3 \left[\gamma(u) \sin x^4 + \delta(u) \cos x^4\right] + 2\,\kappa(u) \tanh x^3. \qquad \text{(A1.25)}$$

Then

$$B - 2\overset{\circ}{\xi}{}^{1} = -2\,\gamma(u) \ ,$$

so that (A1.14) is identically satisfied.

Using (A1.9 - A1.11) we can solve for $\overset{\circ}{\xi}{}^{3}$, $\overset{\circ}{\xi}{}^{4}$ in the form

$$\overset{\circ}{\xi}{}^{3} = \left[\gamma_{,1} \sin x^4 + \delta_{,1} \cos x^4\right] \sinh x^3 + \left[\mu(u) \cos x^4 - \nu(u) \sin x^4\right] \cosh x^3 - \kappa_{,1} \ , \qquad \text{(A1.26)}$$

and

$$\overset{\circ}{\xi}{}^{4} = \left[-\gamma_{,1} \cos x^4 + \delta_{,1} \sin x^4\right] \cosh x^3 + \left[\mu(u) \sin x^4 + \nu(u) \cos x^4\right] \sinh x^3 + \rho(u) \ , \qquad \text{(A1.27)}$$

where all functions of u are arbitrary at the moment.

Substituting (A1.24) and (A1.26) into the expression $A - \overset{\circ}{\xi}{}^{1}_{,1}$ gives

$$A - \overset{\circ}{\xi}{}^{1}_{,1} = -2\lambda_{,1} \ , \qquad \text{(A1.28)}$$

and we see that (A1.16), (A1.17) are identically satisfied.

(A1.28) leads to

$$A = \left[\gamma_{,1} \sin x^4 + \delta_{,1} \cos x^4\right] \operatorname{sech} x^3 + \kappa_{,1} \tanh x^3 - \lambda_{,1} \ , \qquad \text{A1.29)}$$

where we have made use of (A1.24).

Now (A1.8) gives

$$\overset{\circ}{\xi}{}^{2} = -\left[\gamma_{,1,1} \sin x^4 + \delta_{,1,1} \cos x^4\right] \operatorname{sech} x^3 + \kappa_{,1,1} \tanh x^3 - \lambda_{,1,1} \ . \qquad \text{(A1.30)}$$

Substitution of (A1.30) into (A1.18) and A1.19) gives

$$\mu_{,1} = \nu_{,1} = \rho_{,1} = 0 \ ,$$

so that μ, ν, ρ are constants.

The remaining equation (A1.7) gives, on substitution of $\overset{\circ}{\xi}{}^{2}$ from (A1.30)

$$\frac{\partial^3}{\partial u^3}\gamma = \frac{\partial^3}{\partial u^3}\delta = \frac{\partial^3}{\partial u^3}\kappa = \frac{\partial^3}{\partial u^3}\lambda = 0 \ ,$$

so that $\gamma, \delta, \kappa, \lambda$ are quadratic functions of u. We set

$$\gamma(u) = au^2 + bu + c$$
$$\delta(u) = du^2 + eu + f$$
$$\kappa(u) = gu^2 + hu + j$$
$$\lambda(u) = ku^2 + \ell u + m$$
$$\mu = n$$
$$\nu = p$$
$$\rho = q$$

where a, b, c, ..., q are 15 constants.

The final solution for $\overset{\circ}{\xi}{}^{\mu}$ is

$$\overset{\circ}{\xi}{}^{1} = \left[(au^2 + bu + c)\sin x^4 + (du^2 + eu + f)\cos x^4\right]\operatorname{sech} x^3 + \left[gu^2 + hu + j\right]\tanh x^3 + (ku^2 + \ell u + m) \quad , \qquad \text{(A1.31)}$$

$$\overset{\circ}{\xi}{}^{2} = -2\left[a \sin x^4 + d\cos x^4\right]\operatorname{sech} x^3 - 2g\tanh x^3 + 2k \quad , \qquad \text{(A1.32)}$$

$$\overset{\circ}{\xi}{}^{3} = \left[(2au + b)\sin x^4 + (2du + e)\cos x^4\right]\sinh x^3 + \left[n\cos x^4 - p\sin x^4\right]\cosh x^3 - (2gu + h) \quad , \qquad \text{(A1.33)}$$

$$\overset{\circ}{\xi}{}^{4} = \left[-(2au + b)\cos x^4 + (2du + e)\sin x^4\right]\cosh x^3 + \left[n\sin x^4 + p\cos x^4\right]\sinh x^3 + q \quad , \qquad \text{(A1.34)}$$

and by substitution into the equations (A1.1), (A1.2), (A1.5), (A1.6) the results quoted in §3.2 are obtained.

APPENDIX 2

SOME COMPOSITION PROPERTIES OF THE CONFORMAL TRANSFORMATIONS IN MINKOWSKI SPACE-TIME

We give here the calculations leading to equations (2.6) and (2.16) of Chapter 4.

The translation $\mathcal{J}(\underline{a})$ is defined by

$$x'^{\mu} = x^{\mu} + a^{\mu} \quad , \tag{A2.1}$$

and the inversion $\mathcal{I}$ is given by

$$x'^{0} = \frac{x^{0}}{x^{\alpha}x_{\alpha}} \quad , \quad x'^{i} = -\frac{x^{i}}{x^{\alpha}x_{\alpha}} \quad , \quad i = 1,2,3 \; . \tag{A2.2}$$

We investigate the form of the composite mapping

$$\mathcal{I} \; \mathcal{J}(-c^{0},c^{i}) \mathcal{I} \; ,$$

which will be shown to be that of a special conformal transformation $\mathcal{S}(c^{\mu})$:

$$x'^{\mu} = \frac{x^{\mu} - c^{\mu}(x^{\alpha}x_{\alpha})}{1 - 2c_{\alpha}x^{\alpha} + (c^{\beta}c_{\beta})\;(x^{\alpha}x_{\alpha})} \; . \tag{A2.3}$$

Operating on (A2.2) with $\mathcal{J}(b^{\mu})$ gives the mapping $x^{\mu} \longrightarrow x'^{\mu}$:

$$x'^{0} = \frac{x^{0} + b^{0}\;(x^{\alpha}x_{\alpha})}{x^{\alpha}x_{\alpha}} \quad , \quad x'^{i} = \frac{-x^{i} + b^{i}(x^{\alpha}x_{\alpha})}{x^{\alpha}x_{\alpha}} \; , \tag{A2.4}$$

from which we note that

$$x'^{\mu}x'_{\mu} = \frac{1}{(x^{\alpha}x_{\alpha})}\left[\,1 + 2(x^{0}b_{0} - x^{i}b_{i}) + (b^{\mu}b_{\mu})\;(x^{\alpha}x_{\alpha})\right] .$$

Then the effect of $\mathcal{I}$ on (A2.4) gives

$$x'^{0} = \frac{x^{0} + b^{0}(x^{\alpha}x_{\alpha})}{1 + 2(x^{0}b_{0} - x^{i}b_{i}) + (b^{\mu}b_{\mu})\;(x^{\alpha}x_{\alpha})} \quad ,$$

$$x'^{i} = \frac{x^{i} - b^{i}\;(x^{\alpha}x_{\alpha})}{1 + 2(x^{0}b_{0} - x^{i}b_{i}) + (b^{\mu}b_{\mu})\;(x^{\alpha}x_{\alpha})} \quad ,$$

and putting

$$b^0 = -c^0 \ , \ b^i = c^i$$

leads to the transformation $x^\mu \to x'^\mu$ given by (A2.3).

Thus, we have, in the formalism of Chapter 4,

$$\mathcal{J}(c^\mu) = \mathcal{I}\, \mathcal{T}(-c^0, c^i)\, \mathcal{I} \ . \qquad \text{(A2.5)}$$

The proof of (2.16) is a more lengthy business. First, it is straightforward to show that the transformation $\mathcal{I}\,\mathcal{T}(a^\mu)\,\mathcal{I} : x^\mu \to x'^\mu$ is given by

$$x'^0 = \frac{x^0 + a^0\,(x^\alpha x_\alpha)}{1 + 2(x^0 a_0 - x^i a_i) + (a^\mu a_\mu)\,(x^\alpha x_\alpha)} \ , \qquad \text{(A2.6)}$$

$$x'^i = \frac{x^i - a^i (x^\alpha x_\alpha)}{1 + 2(x^0 a_0 - x^i a_i) + (a^\mu a_\mu)\,(x^\alpha x_\alpha)} \ . \qquad \text{(A2.7)}$$

Now consider $\mathcal{T}\left(\frac{a^0}{a^\alpha a_\alpha}, \frac{-a^i}{a^\alpha a_\alpha}\right) : \quad x^\mu \to \underset{(1)}{x^\mu}$ with

$$\underset{(1)}{x^0} = x^0 + \frac{a^0}{a^\alpha a_\alpha} \ , \qquad \underset{(1)}{x^i} = x^i - \frac{a^i}{a^\alpha a_\alpha} \ ,$$

and premultiply with the inversion $\mathcal{I}$. The outcome is given by $x^\mu \to \underset{(2)}{x^\mu}$ with

$$\underset{(2)}{x^0} = \frac{x^0\,(a^\alpha a_\alpha) + a^0}{1 + 2(x^0 a_0 - x^i a_i) + (a^\alpha a_\alpha)\,(x^\mu x_\mu)}$$

$$\underset{(2)}{x^i} = \frac{-x^i (a^\alpha a_\alpha) + a^i}{1 + 2(x^0 a_0 - x^i a_i) + (a^\alpha a_\alpha)\,(x^\mu x_\mu)} \ .$$

Applying next the dilatation $\mathcal{D}\left(-\frac{1}{a^\alpha a_\alpha}\right)$ and the translation $\mathcal{T}\left(\frac{a^\mu}{a^\alpha a_\alpha}\right)$

gives the composite mapping $x^\mu \to \underset{(3)}{x^\mu}$ with

$$\underset{(3)}{x^0} = \frac{1}{\Sigma}\left\{ -x^0 + \frac{a^0}{a^\alpha a_\alpha}\left[\, 2(x^0 a_0 - x^i a_i) + (a^\mu a_\mu)\,(x^\alpha x_\alpha)\right]\right\}$$

$$\underset{(3)}{x^i} = \frac{1}{\Sigma}\left\{x^i + \frac{a^i}{a^\alpha a_\alpha}\left[2(x^o a_o - x^i a_i) + (a^\mu a_\mu)\ (x^\alpha x_\alpha)\right]\right\} ,$$

where

$$\Sigma = 1 + 2\ (x^o a_o - x^i a_i) + (a^\mu a_\mu)\ (x^\alpha x_\alpha) \ . \qquad \text{(A2.8)}$$

Finally, we apply the Lorentz transformation $\mathcal{L}(L^\mu{}_\nu)$, so that

$$\underset{(3)}{x^\mu} \longrightarrow x'^\mu = L^\mu{}_\nu \underset{(3)}{x^\nu} ,$$

where

$$L^o{}_o = \frac{2a^o a_o}{a^\alpha a_\alpha} - 1 , \qquad L^o{}_i = \frac{2a^o a_i}{a^\alpha a_\alpha}$$

$$L^i{}_o = -\frac{2a^i a_o}{a^\alpha a_\alpha} , \qquad L^i{}_j = \delta^i_j - \frac{2a^i a_j}{a^\alpha a_\alpha} .$$

(We shall show below that $L^\mu{}_\nu$ thus defined is a Lorentz transformation).

Some straightforward calculation gives the result that

$$x'^o = \frac{1}{\Sigma}\left[x^o + a^o\ (x^\alpha x_\alpha)\right] , \qquad x'^i = \frac{1}{\Sigma}\left[x^i - a^i(x^\alpha x_\alpha)\right] ,$$

where Σ is defined in (A2.8), and this is precisely the transformation given in (A2.6), (A2.7). That is, we have the following identity;

$$\mathcal{J}\,\mathfrak{I}(a^\mu)\,\mathcal{J} = \mathcal{L}(L^\mu{}_\nu)\,\mathfrak{I}\left(\frac{a^\mu}{a^\alpha a_\alpha}\right)\mathcal{D}\left(\frac{-1}{a^\alpha a_\alpha}\right)\mathcal{J}\,\mathfrak{I}\left(\frac{a^o}{a^\alpha a_\alpha} , \frac{-a^i}{a^\alpha a_\alpha}\right) ,$$

and it remains only to show that $L^\mu{}_\nu$, given by

$$L^\mu{}_\nu = \begin{pmatrix} \dfrac{2a^o a_o}{a^\alpha a_\alpha} - 1 & \dfrac{2a^o a_j}{a^\alpha a_\alpha} \\ -\dfrac{2a^i a_o}{a^\alpha a_\alpha} & \delta^i_j - \dfrac{2a^i a_j}{a^\alpha a_\alpha} \end{pmatrix} ,$$

is the matrix of a Lorentz transformation. We are required to verify that

$$L^\mu{}_\sigma \eta^{\sigma\tau} L^\nu{}_\tau = \eta^{\mu\nu} , \qquad \text{(A2.9)}$$

where $\eta^{\mu\nu}$ = diag (1,-1,-1,-1) is the metric tensor of Minkowski space-time.

Taking $(\mu,\nu) = (0,0)$ the left-hand side of (A2.9) is

$$L^o{}_o L^o{}_o + L^o{}_i \eta^{ij} L^o{}_j = \left(\frac{2a^o a_o}{a^\alpha a_\alpha} - 1\right)^2 + \frac{4(a^o)^2}{(a^\alpha a_\alpha)^2} \eta^{ij} a_i a_j$$

$$= 1 + \frac{4(a^o)^4}{(a^\alpha a_\alpha)^2} - \frac{4(a^o)^2}{(a^\alpha a_\alpha)} - \frac{4(a^o)^2}{(a^\alpha a_\alpha)^2}\left[(a^1)^2 + (a^2)^2 + (a^3)^2\right]$$

$$= \frac{4(a^o)^2}{a^\alpha a_\alpha} - \frac{4(a^o)^2}{a^\alpha a_\alpha} + 1$$

$$= 1 \text{ , which is the right-hand side of (A2.9).}$$

Taking $(\mu,\nu) = (0,j)$, we have in a similar way,

$$L^o{}_o L^j{}_o + L^o{}_i \eta^{ik} L^j{}_k = -\frac{2a^j a_o}{a^\alpha a_\alpha}\left(\frac{2a^o a_o}{a^\alpha a_\alpha} - 1\right) + \frac{2a^o a_i}{a^\alpha a_\alpha}\left(\delta^j{}_k - \frac{2a^j a_k}{a^\alpha a_\alpha}\right)\eta^{ik}$$

$$= -\frac{4a^j (a^o)^3}{(a^\alpha a_\alpha)^2} + \frac{2a^j a_o}{a^\alpha a_\alpha} + \frac{2a^o a_j}{a^\alpha a_\alpha} + \frac{2a^o a^j}{(a^\alpha a_\alpha)^2}\left[(a^1)^2 + (a^2)^2 + (a^3)^2\right]$$

$$= \frac{4a^j a^o}{a^\alpha a_\alpha} - \frac{4a^j a^o}{(a^\alpha a_\alpha)^2}\left[(a^o)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2\right]$$

$$= 0 .$$

The remaining cases $(\mu,\nu) = (i,0)$, $(\mu,\nu) = (i,j)$ follow in the same manner, and this completes the proof.

APPENDIX 3

SPIN FRAMES IN THE PENROSE CONFORMAL TECHNIQUE

In §5 of Chapter 6 we examined the transformation of the spinor dyad (ξ_A, ι_A) from the unphysical conformal manifold $\mathcal{M}$ to the physical space-time $\tilde{\mathcal{M}}$, noting that in $\mathcal{M}$ we should desire the properties

$$\xi_A \iota^A = 1 \,, \tag{A3.1}$$

$$\xi^B \bar{\xi}^{\dot{B}} \nabla_{B\dot{B}} \xi_A = 0 \,, \tag{A3.2}$$

and

$$\xi^B \bar{\xi}^{\dot{B}} \nabla_{B\dot{B}} \iota_A = 0 \tag{A3.3}$$

to hold. To preserve the form of (A3.2) in $\tilde{\mathcal{M}}$ it is sufficient to put

$$\tilde{\xi}_A = \Omega^{\frac{1}{2}} \xi_A \tag{A3.4}$$

(i.e. ξ_A transforms with conformal weight $\frac{1}{2}$ under conformal rescalings), and then the transformation $\iota_A \rightarrow \tilde{\iota}_A$ may be written in the form

$$\tilde{\iota}_A = \Omega^{-\frac{1}{2}} \iota_A + a\Omega^{\frac{1}{2}} \xi_A \,, \tag{A3.5}$$

which immediately satisfies (A3.1). The purpose of the present calculation is to find the condition that the complex function a in (A3.5) must satisfy in order to give

$$\tilde{\xi}^B \bar{\tilde{\xi}}^{\dot{B}} \tilde{\nabla}_{B\dot{B}} \tilde{\iota}_A = 0 \tag{A3.6}$$

in $\tilde{\mathcal{M}}$.

It is clear from (A3.5) that ι_A does not possess a definite conformal weight under conformal rescalings. On the other hand, it is straightforward to verify that

$$\iota_A + \frac{\Omega a}{1-\Omega} \xi_A$$

is of conformal weight $-\frac{1}{2}$, since under (A3.4), (A3.5) we have

$$\tilde{\iota}_A + \frac{\Omega a}{1-\Omega}\tilde{\xi}_A = \Omega^{-\frac{1}{2}}\left(\iota_A + \frac{\Omega a}{1-\Omega}\xi_A\right) . \qquad \text{(A3.7)}$$

Making use of equation (2.7) from Chapter 6 it follows that

$$\tilde{\nabla}_{B\dot{B}}\left(\tilde{\iota}_A + \frac{\Omega a}{1-\Omega}\tilde{\xi}_A\right) = \Omega^{\frac{1}{2}}\nabla_{B\dot{B}}\left(\iota_A + \frac{\Omega a}{1-\Omega}\xi_A\right)$$

$$+\Omega^{-\frac{1}{2}}\left[-\left(\iota_A + \frac{\Omega a}{1-\Omega}\xi_A\right)\nabla_{B\dot{B}}\Omega + \left(\iota_B + \frac{\Omega a}{1-\Omega}\xi_B\right)\nabla_{A\dot{B}}\Omega\right].$$

Multiplying this equation by $\tilde{\xi}^B\bar{\tilde{\xi}}^{\dot{B}}$ and making use of (A3.1), A3.2) and (A3.3), we get

$$\tilde{\xi}^B\bar{\tilde{\xi}}^{\dot{B}}\tilde{\nabla}_{B\dot{B}}\tilde{\iota}_A + \tilde{\xi}^A\tilde{\xi}^B\bar{\tilde{\xi}}^{\dot{B}}\tilde{\nabla}_{B\dot{B}}\left(\frac{\Omega a}{1-\Omega}\right) = \Omega^{\frac{3}{2}}\xi_A\xi^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\left(\frac{\Omega a}{1-\Omega}\right)$$

$$-\Omega^{\frac{1}{2}}\bar{\xi}^{\dot{B}}\nabla_{A\dot{B}}\Omega - \Omega^{\frac{1}{2}}\left(\iota_A + \frac{\Omega a}{1-\Omega}\xi_A\right)\xi^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\Omega .$$

Since $\frac{\Omega a}{1-\Omega}$ is a scalar we have, using (2.3) and (2.6) of Chapter 6,

$$\tilde{\nabla}_{B\dot{B}}\left(\frac{\Omega a}{1-\Omega}\right) = \Omega\nabla_{B\dot{B}}\left(\frac{\Omega a}{1-\Omega}\right),$$

and therefore

$$\tilde{\xi}^B\bar{\tilde{\xi}}^{\dot{B}}\tilde{\nabla}_{B\dot{B}}\tilde{\iota}_A = -\Omega^{\frac{5}{2}}\xi_A\xi^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\left(\frac{\Omega a}{1-\Omega}\right) + \Omega^{\frac{3}{2}}\xi_A\xi^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\left(\frac{\Omega a}{1-\Omega}\right)$$

$$-\Omega^{\frac{1}{2}}\bar{\xi}^{\dot{B}}\nabla_{A\dot{B}}\Omega - \Omega^{\frac{1}{2}}\left(\iota_A + \frac{\Omega a}{1-\Omega}\xi_A\right)\xi^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\Omega .$$

This last equation reduces to

$$\tilde{\xi}^B\bar{\tilde{\xi}}^{\dot{B}}\tilde{\nabla}_{B\dot{B}}\tilde{\iota}_A = \Omega^{\frac{5}{2}}\xi_A\xi^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}a - \Omega^{\frac{1}{2}}\bar{\xi}^{\dot{B}}\nabla_{A\dot{B}}\Omega - \Omega^{\frac{1}{2}}\iota_A\xi^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\Omega ,$$

and using the expression

$$\bar{\xi}^{\dot{B}}\nabla_{A\dot{B}}\Omega = \xi_A\left(\iota^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\Omega\right) - \iota_A\left(\xi^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\Omega\right)$$

leads to

$$\tilde{\xi}^B\bar{\tilde{\xi}}^{\dot{B}}\tilde{\nabla}_{B\dot{B}}\tilde{\iota}_A = \Omega^{\frac{1}{2}}\xi_A\left[\Omega^2\xi^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}a - \iota^B\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\Omega\right].$$

Thus the condition on a required to make

$$\tilde{\xi}^{B}\bar{\tilde{\xi}}^{\dot{B}}\tilde{\nabla}_{B\dot{B}}\tilde{\iota}_{A} = 0$$

may be written

$$\xi^{B}\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\, a = \Omega^{-2}\iota^{B}\bar{\xi}^{\dot{B}}\nabla_{B\dot{B}}\Omega,$$

as quoted (5.8) in Chapter 6.

We may add finally that, with only some slight changes of notation and interpretation, the above proof applies also to the condition (5.22) of Chapter 6.

APPENDIX 4

THE CARTAN METRIC FOR THE LIE ALGEBRA OF THE FLAT-SPACE CONFORMAL GROUP

In Chapter 8 we defined the Cartan metric G_{AB}, $A,B = 0,1,2,\ldots,14$, by

$$G_{AB} = C^{E}_{AD}\, C^{D}_{BE} = G_{BA} ,$$

where C^{E}_{AD} are the structure constants of the Lie algebra of the flat-space conformal group. The C^{E}_{AD} have been given implicitly in the commutator relations on page 58 , and lead to the following components of G_{AB} ;

$$G_{AB} = \left(\begin{array}{c|c|c}
0 & 0 & \begin{matrix} -16 & 0 & 0 & 0 \\ 0 & -16 & 0 & 0 \\ 0 & 0 & -16 & 0 \\ 0 & 0 & 0 & -16 \end{matrix} \\ \hline
0 & \begin{matrix} -8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{matrix} & 0 \\ \hline
\begin{matrix} -16 & 0 & 0 & 0 \\ 0 & -16 & 0 & 0 \\ 0 & 0 & -16 & 0 \\ 0 & 0 & 0 & -16 \end{matrix} & 0 & 0
\end{array}\right)$$

The Cartan metric has an importance in the classification of Lie groups. We state, without proof, the following results, which may be found, for example, in [30];

(i) a Lie group is semi-simple, (i.e. contains no abelian invariant subgroup), if

$$\det (G_{AB}) \neq 0 ;$$

(ii) the Cartan metric is negative definite for a compact Lie group and indefinite for a non-compact Lie group.

In the case of the flat-space conformal group we can easily verify that

$$\det G_{AB} = -16^{8} . 8^{7} ,$$

showing that the group is semi-simple, and by a similarity transformation

$$G \longrightarrow P^{-1} G P$$

with

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} I_4 & 0 & -\frac{1}{\sqrt{2}} I_4 \\ 0 & I_7 & 0 \\ \frac{1}{\sqrt{2}} I_4 & 0 & \frac{1}{\sqrt{2}} I_4 \end{pmatrix} ,$$

where I_4, I_7 are respectively the 4 x 4 and 7 x 7 identity matrices, we bring G_{AB} to the form

$$G_{AB} = 8 \operatorname{diag} (-2, -2, -2, -2, -1, -1, -1, 1, 1, 1, 1, 1, 1, 1, 1) ,$$

demonstrating that the conformal group is non-compact.

APPENDIX 5

SOME RESULTS CONCERNING TRACE-FREE PARTS OF TENSORS IN MINKOWSKI SPACE-TIME

In the calculation of multipole moment fields $Q^{\alpha_1\alpha_2\ldots\alpha_n}$ in Minkowski space-time (§5 of Chapter 8) the following results have frequently been used. We employ the same notation here as in that chapter.

$$\mathfrak{J}\{x^{(\alpha}Q^{\beta)}\} = x^{(\alpha}Q^{\beta)} - \frac{1}{4} g^{\alpha\beta}(Q^{\nu}x_{\nu})$$

$$\mathfrak{J}\{x^{\alpha}x^{\beta}\} = x^{\alpha}x^{\beta} - \frac{1}{4} g^{\alpha\beta}(x^{\nu}x_{\nu})$$

$$\mathfrak{J}\{x^{(\alpha}Q^{\beta\gamma)}\} = x^{(\alpha}Q^{\beta\gamma)} - \frac{1}{3} g^{(\alpha\beta}Q^{\gamma)\nu}x_{\nu}$$

$$\mathfrak{J}\{x^{(\alpha}x^{\beta}Q^{\gamma)}\} = x^{(\alpha}x^{\beta}Q^{\gamma)} - \frac{1}{3} g^{(\alpha\beta}x^{\gamma)}(Q^{\nu}x_{\nu}) - \frac{1}{6} g^{(\alpha\beta}Q^{\gamma)}(x^{\nu}x_{\nu})$$

$$\mathfrak{J}\{x^{\alpha}x^{\beta}x^{\gamma}\} = x^{\alpha}x^{\beta}x^{\gamma} - \frac{1}{2} g^{(\alpha\beta}x^{\gamma)}(x^{\nu}x_{\nu})$$

$$\mathfrak{J}\{x^{(\alpha}Q^{\beta\gamma\delta)}\} = x^{(\alpha}Q^{\beta\gamma\delta)} - \frac{3}{8} g^{(\alpha\beta}Q^{\gamma\delta)\nu}x_{\nu}$$

$$\mathfrak{J}\{x^{(\alpha}x^{\beta}Q^{\gamma\delta)}\} = x^{(\alpha}x^{\beta}Q^{\gamma\delta)} - \frac{1}{8} g^{(\alpha\beta}Q^{\gamma\delta)}(x^{\nu}x_{\nu}) - \frac{1}{2} g^{(\alpha\beta}x^{\gamma}Q^{\delta)\nu}x_{\nu} + \frac{1}{24} g^{(\alpha\beta}g^{\gamma\delta)}(Q^{\nu\sigma}x_{\nu}x_{\sigma})$$

$$\mathfrak{J}\{x^{(\alpha}x^{\beta}x^{\gamma}Q^{\delta)}\} = x^{(\alpha}x^{\beta}x^{\gamma}Q^{\delta)} - \frac{3}{8}(x^{\nu}x_{\nu})g^{(\alpha\beta}x^{\gamma}Q^{\delta)} - \frac{3}{8}(Q^{\nu}x_{\nu})g^{(\alpha\beta}x^{\gamma}x^{\delta)} + \frac{1}{16}(Q^{\mu}x_{\mu})(x^{\nu}x_{\nu})g^{(\alpha\beta}g^{\gamma\delta)}$$

$$\mathfrak{J}\{x^{\alpha}x^{\beta}x^{\gamma}x^{\delta}\} = x^{\alpha}x^{\beta}x^{\gamma}x^{\delta} - \frac{3}{4}(x^{\nu}x_{\nu})g^{(\alpha\beta}x^{\gamma}x^{\delta)} + \frac{1}{16}(x^{\nu}x_{\nu})^2 g^{(\alpha\beta}g^{\gamma\delta)}$$

B I B L I O G R A P H Y

[1] Anderson, J.L. Principles of Relativity Physics. Academic Press (New York & London) 1967

[2] Yano, K. The theory of Lie derivatives and its applications. North-Holland Publishing Co., (Amsterdam) 1955

[3] Wieder, S. The foundations of Quantum Theory. Academic Press (New York & London) 1973

[4] Nöther, E. Nachr. Ges. Wiss. Göttingen, 211 (1918)

[5] Rohrlich, F. Classical Charged Particles. Addison-Wesley 1965

[6] Dyson, F.J. Symmetry groups in nuclear and particle physics. W.A. Benjamin (New York) 1966

[7] Bateman, H. Proc. Lond. Math. Soc. 8, 223 (1910)

[8] Cunningham, E. Lond. Math. Soc. 8, 77 (1910)

[9] Penrose, R. Journ. Math. Phys. 8, 345 (1967)

[10] Penrose, R. Journ. Math. Phys. 10, 38 (1969)

[11] Penrose, R. Int. Journ. of Theor. Phys. 1, 61 (1968)

[12] Penrose, R. and MacCallum, M.A.H. Physics Reports, 6C, 242 (1973)

[13] Isham, C.J., Penrose, R. and Sciama, D.W. Quantum Gravity. Oxford University Press 1975

[14] Fulton, T., Rohrlich, F., Witten, L. Rev. Mod. Phys. 34, 442 (1962)

[15] Eisenhart, L.P. Riemannian Geometry. Princeton University Press (1925)

[16] Geroch, R. Journ. Math. Phys. 11, 1955 (1970)

[17] Newman, E.T. and Penrose, R. Journ. Math. Phys. 4, 998 (1962)

[18] Barut, A.O. Electrodynamics and classical theory of fields and particles. Macmillan (New York and London) 1964

[19] Bade, W.L. and Jehle, H. Rev. Mod. Phys. 25, 714 (1953)

[20] Cartan, E. Bull. Soc. Math. France. 41, 53 (1913)

[21] Van der Waerden, B.L. Nachr. Akad. Wiss. Göttingen, Math.-Physik Kl 100 (1929)

[22] Infeld, L. and van der Waerden, B.L. Sitzber-preuss. Akad. Wiss. Physik.-Math. Kl 9, 380 (1933)

[23] Trautmann, A., Pirani, F.A.E., and Bondi, H. Lectures on General Relativity, Brandeis Summer Institute in Theoretical Physics. Prentice-Hall 1964

[24] Penrose, R. Ann. Phys. 10, 171 (1960)

[25] Penrose, R. Null hypersurface initial data for classical fields of arbitrary spin and for general relativity. Report ARL 63-56, Quantization of generally-covariant field theories. Office of Aerospace Research USAF, 1963.

[26] Kobayashi, S. C.R. Acad. Sci. Paris. 238, 644 (1954)

[27] Samelson, H. Notes on Lie algebras. Van Nostrand Reinhold, New York 1969.

[28] Boerner, H. Representations of Groups. North-Holland Publishing Co. (Amsterdam) 1970

[29] Hermann, R. Lie groups for physicists. W.A. Benjamin, (New York and Amsterdam) 1966

[30] Rosen, G. Formulations of classical and quantum dynamical theory. Academic Press (New York and London) 1969

[31] Newman, E.T., and Unti, W.J. Journ. Math. Phys. 3, 891 (1962)

[32] Newman, E.T. and Tamburino, L.A. Journ. Math. Phys. 3, 902 (1962)

[33] Newman, E.T., Tamburino, L.A., and Unti, T.W.J. Journ. Math. Phys. 4, 915 (1963)

[34] Kerr, R.P. Phys. Rev. Lett. 11, 237 (1963)

[35] Talbot, C.J. Commun. Math. Phys. 13, 45 (1969)

[36] Collinson, C.D. and French, D.C. Journ. Math. Phys. 8, 701 (1967)

[37] Newman, E.T., and Penrose, R. Journ. Math. Phys. 7, 863 (1966)

[38] Goldberg, J.N., Macfarlane, A.J., Newman, E.T., Rohrlich, F. and Sudarshan, E.C.G. Journ. Math. Phys. 8, 2155 (1967)

[39] Hill, E.L. Phys. Rev. 67, 358 (1945)

[40] Kastrup, H.A. Phys. Rev. 143, 1021 (1966)

[41] Rosen, J. Ann. Phys. 47, 468 (1968)

[42] Kastrup, H.A. Ann. der Physik 9, 388 (1962)

[43] Kastrup, H.A. Phys. Rev. 150, 1183 (1966)

[44] Hall, G.G. Applied Group Theory. Longmans 1967

[45] Gürsey, F. Nuovo Cimento III, 988 (1956)

[46] Dirac, P.A.M. Ann. Math. 37, 429 (1936)

[47] Penrose, R. The light-cone at infinity. Report to the Warsaw Conference on Relativity and Gravitation 1962

[48] Willmore, T.J. An Introduction to Differential Geometry. Oxford University Press 1959

[49] Corson, E.M. Introduction to tensors, spinors, and relativistic wave-equations. Blackie 1955

[50] Murnaghan, F.D. The theory of group representations. Baltimore 1938

[51] Synge, J.L. Relativity: the Special Theory. North-Holland Publishing Co., Amsterdam 1965

[52] Unti, T.W.J. Gen. Rel. and Grav. 3, 43 (1972)

[53] Dixon, W.G. Journ. Math. Phys. 11, 1238 (1970)

[54] Penrose, R. Proc. Roy. Soc. A284, 159 (1965)

[55] Newman, E.T., and Penrose, R. Proc. Roy. Soc. A305, 175 (1968)

[56] Tamburino, L.A., and Winicour, J.H. Phys. Rev. 150, 1039 (1966)

[57] Penrose, R., in Battelle Rencontres - 1967 Lectures in mathematics and physics. Edited by De Witt, C.M., and Wheeler, J.A. W.A. Benjamin 1968

[58] Sachs, R. Phys. Rev. 128, 2851 (1962)

[59] Held, A., Newman, E.T. and Posadas, R. Journ. Math. Phys. 11, 3145 (1970)

[60] Brink, D. and Satchler, G. Angular-Momentum. Oxford University Press 1962

[61] Kastrup, H.A. Phys. Rev. 150, 1183 (1966)

[62] Rosen, J. Ann. Phys. 47, 468 (1968)

[63] Page, L. Phys. Rev. 49, 254 (1936)

[64] Page, L. and Adams, N.I. Phys. Rev. 49, 466 (1936)

[65] Lie, S. Theorie der Transformationsgruppen. Leipzig 1893

[66] Engstrom, H.T. and Zorn, M. Phys. Rev. 49, 701 (1936)

[67] Zeeman, E.C. Journ. Math. Phys. 5, 490 (1964)

[68] Gamba, A. and Luzzato, G. Nuovo Cimento 33, 1732 (1964)

[69] Edelen, D.G.B. Nuovo Cimento 67B, 29 (1970)

[70] Buchdahl, H.A. Nuovo Cimento 11,496 (1959)

[71] Geroch, R. Journ. Math. Phys. 11,2580 (1970)

[72] Lichnerowicz, A. Elements of Tensor Calculus. Methuen (London)1962

[73] Hawking, S.W. Journ. Math. Phys. 9, 598 (1968)

[74] Tolman, R.C. Relativity, Thermodynamics and Cosmology. Clarendon Press, Oxford, 1962

[75] Robertson, H.P. Astrophys. Journ. 82, 284 (1935)

[76] Robertson, H.P. Astrophys. Journ. 83, 187 (1936)

[77] Walker, A.G. Proc. Lond. Math. Soc. (2) 42, 90 (1936)

[78] Friedmann, A. Zeit. für Physik 10, 377 (1922)

[79] Friedmann, A. Zeit. für Physik 21, 326 (1924)

[80] de Sitter, W. Proc. Acad. Sci. Amsterdam, 19, 1217 (1917)

[81] Robertson, H.P. Phil. Mag. 5, 835 (1928)

[82] Gürsey, F, in Relativity, Groups and Topology. Gordon and Breach, New York 1964

[83] Hawking, S.W. and Ellis, G.F.R. The large scale structure of space-time. Cambridge University Press 1973.