HIGHER DERIVATIVE D-BRANE COUPLINGS

A Dissertation

by

GUANGYU GUO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2011

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Approved by:

Chair of Committee,	Katrin Becker
Committee Members,	Melanie Becker
	Stephen A. Fulling
	Teruki Kamon
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ABSTRACT

Higher Derivative D-brane Couplings. (August 2011)

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Chair of Advisory Committee: Dr. Katrin Becker

This dissertation covers two different but related topics: the construction of consistent models in type IIB and heterotic string theories, and the higher derivative couplings for D-brane action, which will enable us to relate some models of type IIB to the heterotic side through duality chain.

In the first part, we describe an alternative to the KKLT scenario, in which one can achieve de-Sitter space after fixing all moduli. We fix complex structure moduli and the axio-dilaton by deriving the stability conditions for the critical points of the no-scale scalar potential that governs the dynamics of the complex structure moduli and the axio-dilaton in compactifications of type IIB string theory on Calabi-Yau three-folds.

In the second part, we show the existence of a class of flux backgrounds in heterotic string theory. The background metric we will consider is a T^2 fibration over a K3 base times four-dimensional Minkowski space. Unbroken space-time supersymmetry determines all background fields except one scalar function which is related to the dilaton. The heterotic Bianchi identity gives the same differential equation for the dilaton, and we will discuss in detail the solvability of this equation for backgrounds preserving an N=2 supersymmetry.

In the third part, we obtain the higher derivative D-brane action by using both linearized T-duality and string disc amplitude computation. We evaluate disc amplitude of one R-R field $C^{(p-3)}$ and two NS-NS fields in the presence of a single Dp-brane in type II string theory. We obtain the action for the higher derivative brane interactions among one R-R field $C^{(p-3)}$ and two NS-NS B-fields after carefully comparing the supergravity amplitudes with the corresponding string amplitude up to α'^2 order. We also show that these higher derivative brane couplings are invariant under both R-R and NS-NS B-field gauge transformations, and compatible with linear T-duality.

To my family

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TABLE OF CONTENTS

CHAPTER Page

LIST OF FIGURES

CHAPTER I

INTRODUCTION

The quest for a theory of everything, or a framework that can accommodate every known interaction, has attracted physicists for generations since the great dream of Albert Einstein. In the early 1970s, Quantum field theory proved to be an appropriate framework to organize our knowledge at low energy scale, and sometimes people even argue that it is the only way to satisfy the principles of both quantum mechanics and special relativity (aside from theories like string theory that have an infinite number of particle types)[1]. Both electroweak and strong interactions can be described by standard model, which is the quantum field theory with the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The theory has been compared with a wild range of experiments to very high precision for a broad range distance from $10^{-15}m$ to $10^{8}m$.

Despite the great success of standard model, there are still big problems ahead [2]. 1) Hierarchy problem: in standard model all fermions are chiral, so their masses are protected by gauge symmetry. However, no such symmetry can prevent scalars, like Higgs particle, to receive huge mass correction with quadratical divergence. To avoid the ridiculous fine-tuning one need new physics at TeV scale. 2) Dark matter and dark energy: the familiar particles in the standard model only account for about four percents of total energy of the universe. 3) Quantization of gravity: gravity, i.e. Einstein's general relativity, is still a classical theory. There exist a few proposals, like supersymmetry and large extra dimensions, to address the first two problems, but the last problem is more challenging in the QFT framework. Consistent quantum

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This dissertation follows the style of Nuclear Physics B.

field theory seems to exist only for particles with spin no bigger than 1, but graviton has spin 2. To quantize the gravity, we need something very different.

String theory[1] arises from dual models as a candidate to describe the strong interaction in the 1960s. Despite the partial success of dual models, they are replaced by QCD (Quantum Chromodynamics) as the leading candidate for strong interaction in the early 1970s. Even though string theory does not fit for strong force, it turns out to be an appealing framework to address quantum gravity, as every consistent string theory includes a massless spin 2 particle, which has the same properties as graviton, governed by general relativity.

String theories can reconcile general relativity and quantum mechanics without the annoying UV divergence. But it came at a price. There are five different string theories in 10-dimension and one M theory in 11 dimension, which conflict with the daily experience that the world around us is only 4-dimensional. To get in tough with the everyday physics, we need to compactify the extra six dimensions.

To build models of particle phenomenonlogy from string theory, we can start with four dimensional vacua with $\mathcal{N} = 1$ supersymmetry. One can obtain such models, for example, by compactifying M-theory on G_2-holonomy manifolds, F-theory on Calabi-Yau four-folds or type II theories on Calabi-Yau orientifolds, see [7, 8] for review. All these models have a moduli space of vacuum states, and concrete predictions can not be made until one can identifies the mechanism that picks the vacuum state of string theory. By including fluxes as background fields the continuous ambiguity associated with the vacuum expectation values of the moduli fields is replaced by a discrete freedom associated with the choice of flux numbers. However, the number of possible vacuum states is still enormous and it has been argued to built a whole landscape of

[1]Interested readers can see [3, 4, 5, 6] for introduction.

solutions [9]. However, most of these string theory backgrounds have flat directions and there exists very few solutions with all moduli fixed.

In the dissertation, we will explore a few flux backgrounds in both type IIB [10] and heterotic string theory [11]. One can employ U-duality to connect the models in type IIB side to corresponding models in heterotic string theory, but the complete understanding of these duality chains requests better knowledge regarding the higher derivative brane couplings [12, 13, 14], which will be the main topic of this dissertation.

A. Flux background of type IIB string

Stabilizing all the scalar fields associated with a Calabi-Yau compactification of string theory at weak coupling is a particularly hard problem. In the context of compactifications of type IIB string theory on a Calabi-Yau orientifold, one of the fields which is conventionally unstabilized using fluxes is radial modulus ρ. In KKLT model [15], complex structure moduli and the axio-dilaton acquire an expectation value due to perturbative fluxes while preserving an $\mathcal{N}=1$ supersymmetry. The non-perturbative correction to the superpotential cause the radial modulus ρ to become heavy compared to the AdS scale. However, the masses of the complex structure moduli will generically be of the order to the inverse AdS length which means that for all practical purposes they can be considered stabilized [16]. This situation changes once these vacua are lifted to dS spaces. In [15] this has been achieved by assuming the presence of an anti-D3 brane which contributes a factor

$$\Delta V \sim \frac{1}{(\mathrm{Im}\rho)^3}, \tag{1.1}$$

to the scalar potential. Once this contribution is taken into account the potential for the radial modulus displays a metastable minimum at which the scalar potential takes a positive value and as a result corresponds to a dS space.

An alternative [17] to uplift the potential to positive value is to obtain a potential contribution resembling the one resulting from anti-D3 branes by considering flux configurations for which $\mathcal{D}_I W \neq 0$ for some I and superpotential W [18]. From the no-scale form of the potential it follows that such a contribution is positive and it's dependence on ρ is precisely equal to the one originating from anti-D3 branes. Since $\mathcal{D}_I W \neq 0$ the flux can no longer be imaginary self-dual (ISD) but will acquire an imaginary anti-self dual (IASD) component.

In Chapter II, we will analyze the stability conditions of fluxes derived by requiring that the scalar potential is critical in the complex structure and axio-dilaton directions, and also show these critical points are metastable. We then consider the four-dimensional theory obtained from compactifications of type IIB string theory on backgrounds which are mirror to rigid Calabi-Yau manifolds, *i.e.* non-geometric backgrounds with no Kähler structure. In this case case the flux induced superpotential does depend explicitly on all scalar fields, *i.e.* the complex structure moduli and the axio-dilaton. Mirror symmetry implies that on the type IIB side the Kähler potential for the axio-dilaton differs from the conventional one obtained from dimensional reduction [16]. This fact enables us to find a scalar potential which stabilizes all the complex structure moduli in terms of RR fluxes only while requiring no orientifold charge. However the axio-dilaton is not fixed and slides off to weak coupling. The axio-dilaton could be stabilized if H_{NS} is taken into account and supersymmetry is broken to render the scalar fields heavy enough. Another possibility is to take perturbative corrections to the Kähler potential and non-perturbative corrections to the superpotential into account [16].

B. Flux background of heterotic string

Even though moduli stabilization and model building in type II string theories have been intensively studied, much less is known about the heterotic string compactification with flux. The background geometry of supersymmetric heterotic compactifications with non-zero H-flux are topologically different from the zero-flux Calabi-Yau, and the geometry is non-Kähler [19, 20]. The excitations of the low-energy effective action are no longer the same as those in the no-flux case. That is, due to the lack of a Kähler structure, there is no longer a one-to-one correspondence between harmonic forms and massless modes, so the distinction between light and heavy modes on non-Kähler manifolds is not as clear as it is for Calabi-Yau manifolds [21]. From the mathematical point of view algebraic geometry techniques are still missing even though some progress has been made in describing these spaces with an explicit metric [22].

Aside from intellectual curiosity, non-Kähler compactifications of the heterotic string possess some appealing features of a physical value. In particular, non-trivial background fluxes admit a possible mechanism for spontaneous supersymmetry breaking. Such manifolds admit a globally defined spinor, however, the connection under which that spinor is covariantly constant is no longer the Levi-Civita connection, but rather, a connection with non-zero torsion. The flux as well as the torsion induce a superpotential and hence provide the possibility of fixing at least some of the moduli. A complete understanding of either of these mechanisms depends upon computation of the four-dimensional effective action.

In Chapter III, we will consider torsional heterotic backgrounds which are a T^2 fibered over a K3 base, which has been considered in [23, 24, 25]. This heterotic background is dual to a type IIB background. The duality chain has been described explicitly in ref. [26] based on earlier work by Sen [27, 28]. We are interested in analyzing α' corrections to the heterotic SUGRA background. Even though the heterotic vacua is related to type IIB backgrounds by duality, we will not use duality to obtain the α' corrected heterotic background. Rather we will follow a different route and construct the α' corrected background directly on the heterotic side, in which the action and supersymmetry transformations are known to all relevant orders. The low-energy effective action of the heterotic string to $O(\alpha'^3)$ has been constructed by Bergshoeff and de Roo by supersymmetrizing the Chern-Simons term [29]. Our goal is to construct the background which solves the α' corrected equations of motion.

Depending on the choice of flux different amounts of four-dimensional supersymmetry are preserved. While solutions preserving an N=2,1 supersymmetry have been discussed before in the literature, starting with ref.[26] (see in particular [30, 31]), the supersymmetry breaking solutions are new. We explicitly check that the backgrounds solve the equations of motion. For solutions preserving an N=1,0 supersymmetry we check this at the SUGRA level. While for solutions preserving an N=2 supersymmetry we show how to solve the equations of motion including the first α' correction. The spinor equations determine the background except one scalar function related to the dilaton. The Bianchi identity for $\mathcal{H}$ gives rise to a differential equation for this scalar function which is of Laplace type, so the existence of solution is guaranteed.

C. Higher derivative D-brane couplings

In the previous section, the reason that we have to construct heterotic vacua directly rather than by duality chain from type IIB vacua, is that the present knowledge about the relevant interactions on the world-volume of Dp-branes and O-planes is insufficient. For example, the anomalous couplings

$$\frac{\pi^2\alpha'^2}{24}Ce^{B+2\pi\alpha' F}|_{(\mathrm{p}-3)-\mathrm{form}} \wedge (\mathrm{Tr}R_T \wedge R_T - \mathrm{Tr}R_N \wedge R_N) + \mathcal{O}((\alpha')^4), \tag{1.2}$$

described in [32, 33, 34, 35, 36, 37, 38, 39] are not compatible with T-duality and additional dependence on NS-NS and R-R fields are required to obtain world-volume actions compatible with T-duality.

The duality between type IIB and heterotic flux backgrounds of the previous section can be explicitly checked at the level of SUGRA but beyond leading order the duality map makes predictions about higher derivative corrections to the world-volume action describing Dp-branes in type IIB theories. Higher derivative D-brane couplings are very important in finding consistent compactifications, as they can sometimes be needed to satisfy an important class of consistency conditions known as tadpole equations.

For instance in type IIB, the equation of motion for $C^{(4)}$ wrapping the directions of Minkowski space is an internal closed six-form which gets contributions both from fluxes (terms proportional to $F^{(3)} \wedge H_3$) and from delta-function forms corresponding to localized sources such as D3-branes and O3-planes, and can also receive contributions from higher-derivative corrections to the action. If the six-form is not exact, then there can be a topological obstruction to solving the tadpole equation, and the compactification would be inconsistent. In fact, it turns out that in some examples of this sort (as well as in some other contexts), there may be no way to solve the tadpole constraint at leading order in a momentum expansion. Higher derivative corrections must then be included that often change the global structure drastically - either by allowing the existence of solutions, or perhaps by spoiling the consistency of solutions that otherwise appeared to be fine. For this reason, it is crucial to understand these corrections and their global properties.

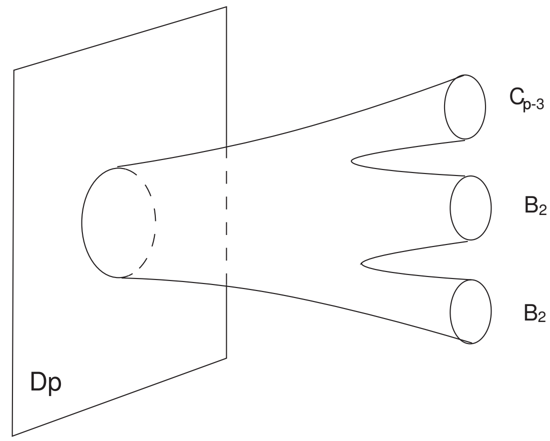

Fig. 1. String theory amplitude for a Dp-brane to absorb two B-fields and emit a (p-3) form R-R potential.

In Chapter IV, we will first use T-duality to deduce some more couplings which involve derivatives of B-fields, or will involve R-R fields of different degree, etc. It is not clear that these couplings will necessarily lead to new topological restrictions, but in some contexts they might, and they will certainly modify the local tadpole equation. Similar couplings have been obtained via U-duality in M-theory and string theory in [40, 41], where they have been used to avoid no-go theorems in IIA and M-theory flux compactifications. Clearly, these issues need to be examined more closely than they have been. But T-duality alone can not fix all the higher derivative brane couplings, so we also employ the string amplitude computation (see Figure 1) to get these brane couplings. At the low energy limit, this string amplitude can be substituted by six supergravity Feynman diagrams shown in Figure 2.

What really interests us is the amplitude for Figure 2f), which represents the contact interaction among one R-R field and two B-fields on D-brane. Once we evaluate the amplitude of first five Feynman diagrams of the Figure 2, we can obtain the amplitude of Figure 2f) by subtracting the amplitudes of the first five diagrams in

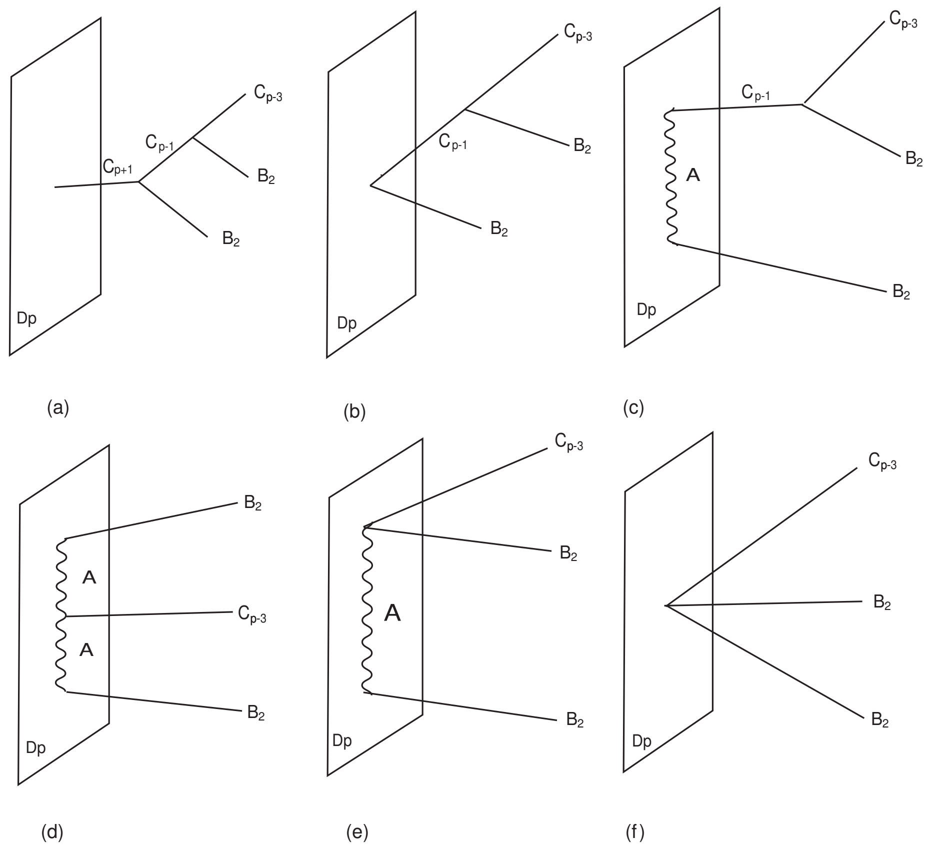

Fig. 2. Six supergravity Feynman diagrams that replace string amplitude at low energy

Figure 2 from the string amplitude. In Chapter IV, we will see that the final higher derivative couplings we obtain are invariant under both R-R and NS-NS B-field gauge transformations, and compatible with linear T-duality.

CHAPTER II

METASTABLE DE SITTER FLUX VACUA IN TYPE IIB THEORY*

In this chapter, we will consider geometric compactifications of type IIB string theory on Calabi-Yau three-folds. In section A, we derive the conditions imposed on the flux configurations to lead to stable critical points of the scalar potential in the complex structure and axio-dilaton directions. We explicitly show that the critical points do correspond to minima of the potential by computing the Hessian matrix. We illustrate the idea in the example of a torus orientifold. In section B, we consider the four-dimensional theory obtained from compactifications of type IIB strings on mirrors of rigid Calabi-Yau manifolds. We find a scalar potential which stabilizes all the complex structure moduli in terms of RR fluxes only while requiring no orientifold charge. We discuss several possibilities to stabilize the axio-dilaton at weak coupling.

A. Type IIB string theory compactified on Calabi-Yau three-folds

In this section, we start deriving the form of the scalar potential following closely [42]. Then we derive the conditions to obtain a critical point of the potential and explicitly check that the critical points correspond to minima by computing the Hessian matrix. At the end, we present a concrete example of T^6 orientifold.

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1. The scalar potential

We start with the low-energy effective action of type IIB string in the ten-dimensional Einstein frame

$$\begin{aligned} S_{IIB} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left[R - \frac{\partial_M\tau\partial^M\bar{\tau}}{2(\mathrm{Im}\tau)^2} - \frac{G\cdot\overline{G}}{12\mathrm{Im}\tau} - \frac{\tilde{F}_{(5)}^2}{4\cdot 5!}\right] \\ &\quad -\frac{1}{8i\kappa_{10}^2}\int\frac{C_{(4)}\wedge G\wedge\overline{G}}{\mathrm{Im}\tau} + S_{loc}. \end{aligned} \tag{2.1}$$

Here the axio-dilaton τ is written in terms of the RR scalar $C_{(0)}$ and the dilaton ϕ according to

$$\tau = C_{(0)} + ie^{-\phi}, \tag{2.2}$$

and the self-dual condition for five form

$$\tilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)}\wedge H_{NS} + \frac{1}{2}B_{(2)}\wedge H_{RR}, \tag{2.3}$$

should be imposed at the equation of motion level. Here H_{RR} and H_{NS} are the field strengths for field potentials $C_{(2)}$ and $B_{(2)}$ respectively and $G \equiv H_{RR} - \tau H_{NS}$. The Bianchi identity for the five-form field can be written as

$$d\tilde{F}_{(5)} = H_{NS}\wedge H_{RR} + 2\kappa_{10}^2 T_3\rho_3^{loc}. \tag{2.4}$$

After integrating both sides of the Bianchi identity over the internal manifold $\mathcal{M}_6$, we get

$$\frac{1}{(2\pi)^4\alpha'^2}\int_{\mathcal{M}_6} H_{NS}\wedge H_{RR} + Q_3^{loc} = 0, \tag{2.5}$$

where we have used the relation $2\kappa_{10}^2 T_3 = (2\pi)^4\alpha'^2$. This identity means the sum of the D3 charges from background fields and localized sources vanishes. After dimension reduction of the action Eq. (2.1), one obtain the four-dimensional scalar potential

$$V = \frac{1}{24\kappa_{10}^2(\mathrm{Im}\rho)^3}\int_{\mathcal{M}_6} d^6y\sqrt{g}\frac{G\cdot\overline{G}}{\mathrm{Im}\tau} - \frac{i}{4\kappa_{10}^2(\mathrm{Im}\rho)^3}\int_{\mathcal{M}_6}\frac{G\wedge\overline{G}}{\mathrm{Im}\tau} \tag{2.6}$$

This scalar potential can be written in terms of the flux induced superpotential [18]

$$W = \int_{\mathcal{M}_6} G\wedge\Omega, \tag{2.7}$$

and the Kähler potential

$$\mathcal{K} = -3\log[-i(\rho-\bar{\rho})] - \log[-i(\tau-\bar{\tau})] - \log[-i\int_{\mathcal{M}_6}\Omega\wedge\overline{\Omega}], \tag{2.8}$$

where ρ is the radial modulus, as the standard $\mathcal{N}=1$ supergravity form

$$V = e^{\mathcal{K}}\left(g^{a\bar{b}}\mathcal{D}_aW\mathcal{D}_{\bar{b}}\overline{W} - 3|W|^2\right) \tag{2.9}$$

where a and b label all moduli and the axio-dilaton. Even though the scalar potential (2.6) take the explicit $N=1$ supersymmetric form, the background preserves maximal $N=2$ supersymmetry. Because the superpotential is independent of ρ the scalar potential takes the no-scale form

$$V = e^{\mathcal{K}}F_I\bar{F}^I, \tag{2.10}$$

where I and J label the complex structure moduli and the axio-dilaton. Here and in the following we will be using the notation of [43]

$$F_I = \mathcal{D}_IW, \qquad Z_{IJ} = \mathcal{D}_I\mathcal{D}_JW, \qquad U_{IJK} = \mathcal{D}_I\mathcal{D}_J\mathcal{D}_KW, \tag{2.11}$$

and indices are raised using the inverse of the Kähler metric $g_{I\bar{J}} = \partial_I\partial_{\bar{J}}\mathcal{K}$.

2. Critical points of the scalar potential

The local minimum of the scalar potential (2.10) in the complex structure and axio-dilaton directions can be achieved after imposing the condition that the first derivatives vanish, *i.e.*

$$\partial_I V = e^{\mathcal{K}} \left(Z_{IJ} \bar{F}^J + F_I \overline{W} \right) = 0. \tag{2.12}$$

There exist non-trivial solution for the above condition. For example, one obvious solution of this condition is given by flux configurations satisfying $F_I = 0$. Using the explicit expression for the superpotential we have

$$F_i = \int_{\mathcal{M}_6} G \wedge \chi_i \qquad \text{and} \qquad F_\tau = -\frac{1}{\tau - \bar{\tau}} \int_{\mathcal{M}_6} \overline{G} \wedge \Omega, \tag{2.13}$$

where χ_i is the basis of harmonic $(2,1)$ forms and with lower case indices i, j we label the complex structure moduli only. This implies that the non-vanishing components of G can lie in the $(0,3)$ or $(2,1)$ directions. In other words, G is imaginary self-dual, $\star G = iG$. Moreover, this critical point is stable because the scalar potential (2.10) is positive semi-definite and at the critical points the potential vanishes.

In the following we would like to find the most general solution of Eq.(2.12). We start by rewriting Eq. (2.12) in the form

$$\begin{aligned} Z_{\tau\tau} \bar{F}^\tau + Z_{\tau j} \bar{F}^j + F_\tau \overline{W} &= 0, \\ Z_{i\tau} \bar{F}^\tau + Z_{ij} \bar{F}^j + F_i \overline{W} &= 0. \end{aligned} \tag{2.14}$$

Note that

$$Z_{ij} = \kappa_{ijk} \frac{\int_{\mathcal{M}_6} G \wedge \overline{\chi}^k}{\int_{\mathcal{M}_6} \Omega \wedge \overline{\Omega}}, \quad Z_{\tau i} = -\frac{1}{\tau - \bar{\tau}} \int_{\mathcal{M}_6} \bar{G} \wedge \chi_i, \quad Z_{\tau\tau} = 0. \tag{2.15}$$

A simple computation (we include the details in appendix A) shows that the first condition in Eq. (2.14) is equivalent to

$$\int_{\mathcal{M}_6} G \wedge \star G = 0, \tag{2.16}$$

while the second condition leads to

$$(B\bar{B}_k + A\bar{A}_k) \int_{\mathcal{M}_6} \Omega \wedge \overline{\Omega} + \kappa_{ijk} A^i B^j = 0. \tag{2.17}$$

Here we introduced the Hodge decomposition

$$G = A\Omega + A^i \chi_i + \bar{B}^{\bar{i}} \bar{\chi}_{\bar{i}} + \bar{B}\overline{\Omega} \tag{2.18}$$

and κ_{ijk} are the Yukawa couplings. The scalars (2.10) does not always have local minimum for an arbitrary choice of flux. Only if Eq. (2.16) and Eq. (2.17) are satisfied can we find a critical point in all directions except the size. This is not always possible. If $H_{NS} = 0$, for example, then the dilaton cannot be stabilized since the only non-vanishing contribution to the dilaton potential comes from the overall factor $e^{\mathcal{K}}$. As a result no critical point exists since Eq.(2.16) is violated.

It is not difficult to see that all flux combinations can lead to critical points of the potential except if G is given by a combination of the following components

$$G_{(3,0)} + G_{(0,3)}, \qquad G_{(3,0)} + G_{(2,1)}, \qquad G_{(3,0)} + G_{(0,3)} + G_{(2,1)}, \tag{2.19}$$

or their complex conjugates. A flux of the form $G_{(3,0)} + G_{(0,3)}$, for example, is easily seen to violate the condition (2.16).

Among the possible flux combinations leading to critical points of the scalar potential only a flux lying in the $(2,1)$ or $(1,2)$ directions preserves supersymmetry. The $(2,1)$ component obviously preserves supersymmetry, as it satisfies

$$\mathcal{D}_I W = \mathcal{D}_\rho W = 0. \tag{2.20}$$

However a flux in the $(1,2)$ direction also preserves supersymmetry if accompanied by a change in the sign of the tadpole due to fluxes. The reason for this is that type IIB supergravity in ten dimensions is invariant under the change of sign of all RR fluxes. Changing the signs of RR fields replaces G by $-\bar{G}$ and as a result a flux lying in the (2,1) direction should lead to the same physics as a flux in the (1,2) direction. The (1,2) component does satisfy the conventional supergravity constraint $\mathcal{D}_I\widetilde{W} = \mathcal{D}_\rho\widetilde{W} = 0$, but with a superpotential given by

$$\widetilde{W} = \int_{M_6} \overline{G} \wedge \Omega. \tag{2.21}$$

The derivation of this superpotential will be discussed in appendix B. The two superpotentials W and $\widetilde{W}$ are related to each other by a CPT transformation. Any other flux components satisfying Eq. (2.16) and (2.17) will not preserve supersymmetry and lead to a positive cosmological constant or vanishing cosmological constant if only a $(3,0)$ (or $(0,3)$) component is turned on. On the other hand, due to the no-scale structure of the potential the radial modulus cannot be stabilized.

3. The Hessian matrix

The no-scale potential (2.10) is positive definite, so the solutions which lead to a vanishing potential at the critical point $V_\star$ are necessarily stable. However, we are interested in solutions for which $V_\star > 0$ and as a result we have to check the stability of the solutions[2]. In order to determine if the critical points are stable we compute the Hessian matrix H. It turns out that it only has positive eigenvalues which means that the critical points are minima in the complex structure and axio-dilaton directions.

[2]Stability conditions for flux compactifications and the corresponding uplift has been considered before in ref. [44].

The second derivatives of the scalar potential are given by

$$\partial_I \partial_J V = e^{\mathcal{K}} \left(U_{IJK} \bar{F}^K + 2 Z_{IJ} \bar{W} \right) \tag{2.22}$$
$$\partial_I \partial_{\bar{J}} V = e^{\mathcal{K}} (g_{I\bar{J}} F_K \bar{F}^K - R_{I\bar{J}K}{}^L F_L \bar{F}^K + 2 F_I \bar{F}_{\bar{J}} + Z_{IL} \bar{Z}_{\bar{J}\bar{K}} g^{L\bar{K}} + g_{I\bar{J}} |W|^2)$$

The critical points will be stable if

$$d\Sigma^2 = H_{\alpha\beta} dw^\alpha dw^\beta \geq 0, \tag{2.23}$$

where w^α labels all coordinates, *i.e.* α and β label the axio-dilaton, complex structure moduli and their complex conjugates. Using formulas which are explicitly presented in appendix A we obtain

$$d\Sigma^2 = e^{\mathcal{K}} g^{\gamma\sigma} \left(Z_{\alpha\gamma} \overline{Z}_{\beta\sigma} dw^\alpha dw^\beta + g^{\tau\bar{\tau}} U_{\alpha\gamma\tau} \overline{U}_{\beta\sigma\bar{\tau}} dw^\alpha dw^\beta \right) \tag{2.24}$$

where $U_{\alpha\gamma\sigma} = \mathcal{D}_\alpha \mathcal{D}_\gamma \mathcal{D}_\sigma W$ and $Z_{\alpha\gamma} = \mathcal{D}_\alpha \mathcal{D}_\gamma W$ are the generalization of U_{IJK} and Z_{IJ}. As a result the Hessian matrix is positive semi-definite and the critical points correspond to minima.

4. An example

In this section we describe a concrete example in terms of a type IIB orientifold compactification. This example is closely related to examples discussed in [17, 45]. We will be following their notation. Let x^i and y^i, for $i = 1, 2, 3$ be the six real coordinates on T^6. These coordinates are subjected to the periodic identifications $x^i \equiv x^i + 1$ and $y^i \equiv y^i + 1$. The complex structure is parameterized by complex parameters τ^{ij}, and

$$z^i = x^i + \tau^{ij} y^j, \tag{2.25}$$

are global holomorphic coordinates. The explicit orientifold is $T^6/\Omega R(-1)^{F_L}$, where R is the involution which changes the sign of all torus coordinates, $R : (x^i, y^i) \to -(x^i, y^i)$. The holomorphic three-form is

$$\Omega = dz^1 \wedge dz^2 \wedge dz^3, \tag{2.26}$$

and the metric is

$$ds^2 = dz^i d\bar{z}^{\bar{i}}. \tag{2.27}$$

We choose the following orientation

$$\int_{T^6} dx^1 \wedge dx^2 \wedge dx^3 \wedge dy^1 \wedge dy^2 \wedge dy^3 = 1, \tag{2.28}$$

and the basis of $H^3(T^6, \mathbb{Z})$:

$$\begin{aligned}
\alpha_0 &= dx^1 \wedge dx^2 \wedge dx^3 \\
\alpha_{ij} &= \frac{1}{2}\varepsilon_{ilm} dx^l \wedge dx^m \wedge dy^j, \quad 1 \leq i, j \leq 3 \\
\beta^{ij} &= -\frac{1}{2}\varepsilon_{jlm} dy^l \wedge dy^m \wedge dx^i, \quad 1 \leq i, j \leq 3 \\
\beta^0 &= dy^1 \wedge dy^2 \wedge dy^3
\end{aligned} \tag{2.29}$$

which satisfies $\int_{T^6} \alpha_I \wedge \beta^J = \delta_I^J$. The fluxes can be expanded in this basis

$$\begin{aligned}
\frac{1}{(2\pi)^2\alpha'} H_{RR} &= a^0\alpha_0 + a^{ij}\alpha_{ij} + b_{ij}\beta^{ij} + b_0\beta^0 \\
\frac{1}{(2\pi)^2\alpha'} H_{NS} &= c^0\alpha_0 + c^{ij}\alpha_{ij} + d_{ij}\beta^{ij} + d_0\beta^0.
\end{aligned} \tag{2.30}$$

Here we take $a^0, a^{ij}, b_0, b_{ij}, c^0, c^{ij}, d_0, d_{ij}$ to be even integers, so that all the O3-planes are of the standard type and the issues regarding flux quantization discussed in ref. [46] can be avoided. In this case, the total number of O3-planes is 64 and each plane has D3-brane charge $-1/4$. For simplicity we only turn on the diagonal components of the flux, so that we can set the off-diagonal components of τ^{ij} equal to zero at the critical points. This condition can be imposed by restricting to an enhanced symmetry locus on the moduli space of the T^6 [17]. For example, we will consider configurations which are symmetric under

$$\begin{aligned} R_1 : (x^1, x^2, x^3, y^1, y^2, y^3) &\rightarrow (-x^1, -x^2, x^3, -y^1, -y^2, y^3) \\ R_2 : (x^1, x^2, x^3, y^1, y^2, y^3) &\rightarrow (x^1, -x^2, -x^3, y^1, -y^2, -y^3) \end{aligned} \tag{2.31}$$

Only the diagonal components of the complex structure τ^{ij}, and the three forms $\alpha_0, \alpha_{ii}, \beta^0, \beta^{ii}$ are preserved under these symmetries, so that the only non-vanishing flux components are a^0, a^{ii}, b_0, b_{ii} and c^0, c^{ii}, d_0, d_{ii}. We are left with 3 non-vanishing complex moduli and the axio-dilaton.

To use the conditions (2.16) and (2.17) which we derived in subsection A.2, we need to transform the scalar potential (2.6) into the standard $\mathcal{N} = 1$ supergravity formula (2.9). For tori having a general complex structure the result is complicated (see for example [17, 47]). However for tori with diagonal complex structure, we can express the scalar potential in the form

$$V = e^{\mathcal{K}} \left(\sum_{i,j=1}^{3} g^{i\bar{j}} \mathcal{D}_{\tau_i} W \overline{\mathcal{D}_{\tau_j} W} + g^{\tau\bar{\tau}} \mathcal{D}_\tau W \overline{\mathcal{D}_\tau W} \right), \tag{2.32}$$

with superpotential (2.7) and “Kähler potential”,

$$\mathcal{K} = -3 \log[-i(\rho - \bar{\rho})] - \log[-i(\tau - \bar{\tau})] - \log[i(\tau_1 - \bar{\tau}_1)(\tau_2 - \bar{\tau}_2)(\tau_3 - \bar{\tau}_3)], \tag{2.33}$$

where we used τ_i to replace τ^{ii}. Before we proceed we have one more comment. Generally we can only set $\tau^{ij} = 0$ (for $i \neq j$), after computing the first derivative of the scalar potential (2.6), but on the symmetric locus, the criticality conditions $\partial_{\tau^{ij}} V = 0$ (for $i \neq j$) are automatically satisfied. As a result we can set $\tau^{ij} = 0$ (for $i \neq j$) at the beginning of the computation and only deal with the conditions

$\partial_{\tau^{ii}}V = 0$. However, when computing the second derivatives we can not set $\tau^{ij} = 0$ before we differentiate, as there are non-vanishing terms of the form $\partial^2_{\tau^{ij}}V$, which will disappear if we set $\tau^{ij} = 0$ (for $i \neq j$) at the beginning.

Next we consider a flux in the (2,1)+(1,2) direction, so the conditions (2.17) and (2.16) take the form

$$\kappa_{ijk}A^jB^k = 0 \qquad \text{and} \qquad g_{i\bar{j}}A^i\bar{B}^{\bar{j}} = 0. \tag{2.34}$$

Since we are working with a torus we set $\kappa_{123} = 1$ and one solution to the above condition is

$$A^3 = B^3 = 0, \qquad A^1B^2 = -B^1A^2, \qquad \frac{A^1\bar{B}^1}{(\mathrm{Im}\tau_1)^2} + \frac{A^2\bar{B}^2}{(\mathrm{Im}\tau_2)^2} = 0. \tag{2.35}$$

For the concrete torus orientifold we are considering the tadpole cancelation condition takes the form

$$\frac{i}{2\mathrm{Im}\tau(2\pi)^4\alpha'^2}\int_{T^6} G \wedge \overline{G} = 32. \tag{2.36}$$

In the following we will present a concrete solution of Eq. (2.35). For simplicity we redefine the parameters according to

$$A^i = -2i\mathrm{Im}\tau_i\mathrm{Im}\tau\tilde{A}^i, \qquad \text{and} \qquad \bar{B}^{\bar{i}} = 2i\mathrm{Im}\tau_i\mathrm{Im}\tau\tilde{\bar{B}}^{\bar{i}} \tag{2.37}$$

and drop the tilde in the following. The conditions (2.35) and (2.36) can be written as

$$A^1B^2 = -B^1A^2, \qquad B^1\bar{B}^1 = B^2\bar{B}^2, \qquad (A^2\bar{A}^2 - B^2\bar{B}^2)\mathrm{Im}\tau\prod_{i=1}^{3}\mathrm{Im}\tau_i = 4 \tag{2.38}$$

and the non-vanishing components of H_{RR} and H_{NS} are

$$\begin{aligned}
a^0 &= -\mathrm{Im}[\bar{\tau}(A^1 + A^2 + \bar{B}^1 + \bar{B}^2)] \\
a^{11} &= -\mathrm{Im}[\bar{\tau}(A^1\bar{\tau}_1 + A^2\tau_1 + \bar{B}^1\tau_1 + \bar{B}^2\bar{\tau}_1)] \\
a^{22} &= -\mathrm{Im}[\bar{\tau}(A^1\bar{\tau}_2 + A^2\tau_2 + \bar{B}^1\tau_2 + \bar{B}^2\bar{\tau}_2)] \\
a^{33} &= -\mathrm{Im}[\bar{\tau}(A^1\bar{\tau}_3 + A^2\tau_3 + \bar{B}^1\tau_3 + \bar{B}^2\bar{\tau}_3)] \\
b_0 &= -\mathrm{Im}[\bar{\tau}(A^1\bar{\tau}_1\tau_2\tau_3 + A^2\tau_1\bar{\tau}_2\tau_3 + \bar{B}^1\tau_1\bar{\tau}_2\bar{\tau}_3 + \bar{B}^2\bar{\tau}_1\tau_2\bar{\tau}_3)] \\
b_{11} &= \mathrm{Im}[\bar{\tau}(A^1\tau_2\tau_3 + A^2\bar{\tau}_2\tau_3 + \bar{B}^1\bar{\tau}_2\bar{\tau}_3 + \bar{B}^2\tau_2\bar{\tau}_3)] \\
b_{22} &= \mathrm{Im}[\bar{\tau}(A^1\bar{\tau}_1\tau_3 + A^2\tau_1\tau_3 + \bar{B}^1\tau_1\bar{\tau}_3 + \bar{B}^2\bar{\tau}_1\bar{\tau}_3)] \\
b_{33} &= \mathrm{Im}[\bar{\tau}(A^1\bar{\tau}_1\tau_2 + A^2\tau_1\bar{\tau}_2 + \bar{B}^1\tau_1\bar{\tau}_2 + \bar{B}^2\bar{\tau}_1\tau_2)] \\
c^0 &= -\mathrm{Im}[A^1 + A^2 + \bar{B}^1 + \bar{B}^2] \\
c^{11} &= -\mathrm{Im}[A^1\bar{\tau}_1 + A^2\tau_1 + \bar{B}^1\tau_1 + \bar{B}^2\bar{\tau}_1] \\
c^{22} &= -\mathrm{Im}[A^1\bar{\tau}_2 + A^2\tau_2 + \bar{B}^1\tau_2 + \bar{B}^2\bar{\tau}_2] \\
c^{33} &= -\mathrm{Im}[A^1\bar{\tau}_3 + A^2\tau_3 + \bar{B}^1\tau_3 + \bar{B}^2\bar{\tau}_3] \\
d_0 &= -\mathrm{Im}[A^1\bar{\tau}_1\tau_2\tau_3 + A^2\tau_1\bar{\tau}_2\tau_3 + \bar{B}^1\tau_1\bar{\tau}_2\bar{\tau}_3 + \bar{B}^2\bar{\tau}_1\tau_2\bar{\tau}_3] \\
d_{11} &= \mathrm{Im}[A^1\tau_2\tau_3 + A^2\bar{\tau}_2\tau_3 + \bar{B}^1\bar{\tau}_2\bar{\tau}_3 + \bar{B}^2\tau_2\bar{\tau}_3] \\
d_{22} &= \mathrm{Im}[A^1\bar{\tau}_1\tau_3 + A^2\tau_1\tau_3 + \bar{B}^1\tau_1\bar{\tau}_3 + \bar{B}^2\bar{\tau}_1\bar{\tau}_3] \\
d_{33} &= \mathrm{Im}[A^1\bar{\tau}_1\tau_2 + A^2\tau_1\bar{\tau}_2 + \bar{B}^1\tau_1\bar{\tau}_2 + \bar{B}^2\bar{\tau}_1\tau_2].
\end{aligned} \tag{2.39}$$

Usually one starts with certain flux numbers and then determines the values of moduli fields. Here we solve the inverse problem, namely, we start with the value of the moduli and determine the flux numbers which stabilize the moduli at the given values. To solve Eq. (2.38) using even flux numbers (2.39) we use the ansatz

$$\mathrm{Im}\tau = 4, \qquad \mathrm{Im}\tau_1 = \mathrm{Im}\tau_2 = \mathrm{Im}\tau_3 = 1 \tag{2.40}$$

So one solution of Eq. (2.38) is

$$A^1 = -3i, \qquad A^2 = 3i, \qquad \bar{B}^1 = 2 + 2i, \qquad \bar{B}^2 = 2 + 2i \tag{2.41}$$

From Eq. (2.39), we can explicitly compute the flux numbers and obtain

$$\begin{aligned}
(a^0, a^{11}, a^{22}, a^{33}) &= (16, -24, 24, 16), \\
(b_0, b_{11}, b_{22}, b_{33}) &= (16, 0, 0, -16) \\
(c^0, c^{11}, c^{22}, c^{33}) &= (-4, 0, 0, 4), \\
(d_0, d_{11}, d_{22}, d_{33}) &= (4, 6, -6, 4)
\end{aligned} \tag{2.42}$$

which are all even integrals.

B. Type IIB mirrors

In this section we would like to generalize the previous analysis to type IIB theories which arise as mirrors of type IIA models compactified on rigid Calabi-Yau threefolds, *i.e.* with $h_{2,1} = 0$. On the type IIB side these correspond to models with $h_{1,1} = 0$ and consequently are not ordinary Calabi-Yau manifolds since a Kähler form is missing but can nevertheless be described using conformal field theory techniques. Here we will be interested in the properties of the resulting four-dimensional theories which contain $h_{2,1} + 1$ four-dimensional $\mathcal{N} = 1$ chiral superfields originating from the complex structure moduli and the axio-dilaton. The number of these fields will in general be reduced if we consider an orientifold projection.

It has been shown in ref. [16] that for compactifications of type IIB strings on backgrounds with no Kähler structure the Kähler potential for the axio-dilaton and the complex structure is

$$\mathcal{K} = -4 \log\left[-i(\tau - \bar{\tau})\right] - \log\left[-i \int \Omega \wedge \bar{\Omega}\right], \tag{2.43}$$

which differs by a subtle factor 4 from the conventional Kähler potential for the axio-dilaton. This unconventional factor 4 has the consequence that supersymmetric flux configurations are no longer required to be ISD [16]. The Kähler potential (2.43) also causes the scalar potential to display new and interesting properties. In order to illustrate this imagine one considers a real three-form flux, *i.e.* a flux configuration with $H_{NS} = 0$. Then

$$W = W_{RR} = \int H_{RR} \wedge \Omega, \tag{2.44}$$

and the scalar potential can be written in the form

$$V = e^{\mathcal{K}} \left(g^{i\bar{j}} D_i W_{RR} \overline{D_j W_{RR}} + \mid W_{RR} \mid^2 \right), \tag{2.45}$$

which is positive definite and depends non-trivially on the complex structure. If

$$\partial_i V = 0 \qquad \text{for} \qquad i = 1, \ldots, h_{2,1}, \tag{2.46}$$

the potential is critical in all the complex structure directions. So for example, one solution of Eq. (2.46) is given by

$$H_{RR} = a \left(\Omega + \bar{\Omega} \right), \tag{2.47}$$

where a is some real constant. This equation determines the complex structure moduli. Indeed, it turns out that this is nothing else than the equation defining a rank 1 attractor which is well known from black hole physics. Eq. (2.47) can, for example, be explicitly solved in the large complex structure limit as has been shown by Shmakova in ref. [48] (see also ref. [49]). These critical points are stable since the only non-vanishing entries of the Hessian matrix are

$$\partial_{\bar{i}} \partial_j V = 2 e^{\mathcal{K}} g_{\bar{i}j} |W_{RR}|^2. \tag{2.48}$$

The scalar potential (2.45) has been studied before in the literature in the context of non-supersymmetric attractors (for a partial list of references on non-supersymmetric attractors see [50]). In particular, the critical points of the potential are the solutions of

$$H_{RR} = 2\mathrm{Im}\left[e^{\mathcal{K}_{cs}}\left(\Omega\bar{W} - \bar{F}^i\chi_i\right)\right], \tag{2.49}$$

subjected to the constraint

$$Z_{ij}\bar{F}^j + 2F_i\bar{W} = 0 \tag{2.50}$$

which can be written as

$$2F_i\bar{W}\int \Omega\wedge\overline{\Omega} + \kappa_{ijk}\bar{F}^j\bar{F}^k = 0. \tag{2.51}$$

Moreover, these critical points are stable since the Hessian matrix written in terms of [3]

$$d\Sigma^2 = 2e^{\mathcal{K}}\left(g^{\gamma\sigma}Z_{\alpha\gamma}\overline{Z}_{\beta\sigma}dw^\alpha dw^\beta + \bar{F}_\alpha F_\beta dw^\alpha dw^\beta\right), \tag{2.52}$$

is positive definite (the stability of non-supersymmetric black hole solutions has been analyzed in [51, 52]). In this form the critical points correspond non-supersymmetric attractor points as described in ref. [53]. This indicates that within a non-geometric model with $h_{1,1} = 0$ the proposal of ref. [17] leads to an interesting new class of backgrounds in which all the complex structure moduli can be stabilized in terms of RR fluxes only with no need of negative energy sources like orientifold planes.

Using the solution (2.47) shows that the potential at the minimum satisfies

$$V_\star > 0, \tag{2.53}$$

if $a \neq 0$ so the external space is dS. However, before we can conclude that supersym-

[3]Here the indices α, β label the complex structure moduli and their complex conjugates.

metry is spontaneously broken by the solution (2.47) we should take into account the dependence on the axio-dilaton arising from the overall factor $e^{\mathcal{K}} \sim (\mathrm{Im}\tau)^{-4}$. This factor causes the potential to slope to zero at infinity so a supersymmetric state is gained back at infinity and as it stands the theory has no ground state at all. Here (as in [16]) we will simply assume that perturbative corrections to the Kähler potential and non-perturbative corrections to the superpotential could achieve this stabilization and lead to a metastable ground state.

In order to stabilize the axio-dilaton using perturbative fluxes the only possibility is to use a non-vanishing H_{NS} flux. By including RR and NS three-form fluxes one obtains a four-dimensional superpotential which does depend non-trivially on all moduli fields. Any geometric compactification would lead to a superpotential which is independent of the Kähler moduli and consequently the radial modulus would slide off to infinity. As a result even in the absence of any type of corrections moduli stabilization may be possible within the non-geometric model by including all possible fluxes. Moreover, in order to obtain moduli fields which are heavy enough we may have to break supersymmetry [16]. But note that once the NS flux is non-vanishing the scalar potential is no longer positive definite and it is not obvious that supersymmetry breaking vacua, and in particular the phenomenologically interesting vacua leading to a positive cosmological constant, exist. As an illustrative toy example lets consider a non-geometric model with $h_{2,1} = 0$, *i.e.* a model with only one massless scalar field, the axio-dilaton, with a Kähler potential

$$\mathcal{K} = -4 \log \left[-i(\tau - \bar{\tau})\right], \tag{2.54}$$

and a superpotential

$$W = W_{RR} - \tau W_{NS}, \tag{2.55}$$

where W_{RR} and W_{NS} are constants. The condition for unbroken supersymmetry has one solution only

$$\tau = \frac{1}{W_{NS}\bar{W}_{NS}} \left[\mathrm{Re}(\bar{W}_{NS}W_{RR}) + 2i\mathrm{Im}(\bar{W}_{NS}W_{RR})\right] . \tag{2.56}$$

However, it is not difficult to see that the scalar potential is also critical if

$$\tau = \frac{1}{W_{NS}\bar{W}_{NS}} \left[\mathrm{Re}(\bar{W}_{NS}W_{RR}) - \frac{i}{2}\mathrm{Im}(\bar{W}_{NS}W_{RR})\right] , \tag{2.57}$$

which leads to $D_\tau W \neq 0$ so that supersymmetry is broken. Moreover, the scalar potential at the minimum is negative so that the external space is AdS. As a result supersymmetry breaking critical points of the potential do exist even though in this case they lead to an AdS space. However, it is interesting that a single four-dimensional chiral field with a Kähler potential of the form (2.54) avoids the no-go theorem of ref. [54] according to which dS or Minkowski space vacua with a broken supersymmetry are never possible in a theory with a single chiral field for any superpotential if the Kähler potential is $\mathcal{K} = -n \log\left[-i(\tau - \bar{\tau})\right]$ with $1 \leq n \leq 3$. As a result stable dS vacua are no longer excluded. It will be very interesting to see if by considering a 'realistic' model with a non-vanishing number of complex structure moduli fields stable critical points of the potential at which supersymmetry is broken can be found.

CHAPTER III

FLUX BACKGROUND IN HETEROTIC STRING*

In this chapter, we study different aspects of string theory compactifications in the presence of background flux. Our main focus is the heterotic string compactified to four dimensions with background NS three-form $\mathcal{H}$. We start by discussing, and mostly reviewing, flux compactifications of type IIB string theory on K3$\times$T^2 orientifolds (see for example refs.[26, 55, 56]). Depending on the choice of flux the solutions preserve an N=2,1,0 supersymmetry in four dimensions. The backgrounds solve the equations of motion and in the supersymmetric case the spinor equations. We check this to the leading order in α', *i.e.* in the SUGRA approximation. To set up our notation we also review the low-energy effective 'action' in section A.1 and derive the equations of motion of type IIB SUGRA in section A.2. In section A.3 we present the background which solves the equations of motion of type IIB SUGRA and check the amount of four-dimensional supersymmetry preserved by the different backgrounds in section A.4. Taking the type IIB background as a starting point we proceed in section B to construct the heterotic flux background. To set up the notation we review in section B.1 the heterotic effective action to $O(\alpha')$ and in section B.2 we derive the corresponding equations of motion. In section B.3 we present the background and show that it solves the SUGRA equations of motion. In section C, we discuss the α' corrected background. We start by presenting explicit results for $\mathrm{Tr}(R\wedge R)$ which are

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needed to solve the Bianchi identity and Einstein equation. In section C.1 we review the proof that $\mathrm{Tr}(R\wedge R)$ is a four-form of type (2, 2) to leading order in α', a condition which is needed for the solvability of the Bianchi identity. In section C.3, focusing on solutions with N=2 supersymmetry, we show how to construct the background which solves the α' corrected Bianchi identity and supersymmetry transformations.

A. Type IIB SUGRA background

In this section we review type IIB flux backgrounds in which the space-time metric is a warped product of flat 4d Minkowski space and a K3$\times$T^2 orientifold (see refs.[26, 55, 56, 57, 58]). To set up the notation we start summarizing our conventions for the type IIB SUGRA 'action' together with the corresponding equations of motion. Then we summarize the solutions preserving different amounts of four-dimensional supersymmetry. The analysis is done at the level of SUGRA, *i.e.* without taking actions describing brane sources into account.

1. The action

The bosonic part of the type IIB supergravity 'action' in the 10d string frame is

$$S = S_{\mathrm{NS}} + S_{\mathrm{R}} + S_{\mathrm{CS}}. \tag{3.1}$$

Here S_{NS} is

$$S_{\mathrm{NS}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\, e^{-2\phi_{\mathrm{B}}}\left[R + 4(\partial\phi_{\mathrm{B}})^2 - \frac{1}{2}|H_3|^2\right], \tag{3.2}$$

while the parts of the action describing the massless R-R sector fields are given by

$$S_{\mathrm{R}} = -\frac{1}{4\kappa^2}\int d^{10}x\sqrt{-g}\left(|F_1|^2 + |\widetilde{F}_3|^2 + \frac{1}{2}|\widetilde{F}_5|^2\right), \tag{3.3}$$

$$S_{\mathrm{CS}} = \frac{1}{4\kappa^2} \int C_4 \wedge H_3 \wedge F_3. \tag{3.4}$$

In these formulas $F_{n+1} = dC_n$, $H_3 = dB_2$ and $\widetilde{F}_3 = F_3 - C_0\, H_3$.

2. Equations of motion

The equations of motion are as follows

$$\begin{aligned} d \star F_1 &= \star\tilde{F}_3 \wedge H_3, \\ d \star \tilde{F}_3 &= \tilde{F}_5 \wedge H_3, \\ d \star \tilde{F}_5 &= -F_3 \wedge H_3, \end{aligned} \tag{3.5}$$

from the R-R fields, and

$$\begin{aligned} & R - 4(\partial\phi_{\mathrm{B}})^2 + 4\nabla^2\phi_{\mathrm{B}} - \frac{1}{2} \mid H_3 \mid^2 = 0, \\ & d\left(e^{-2\phi_{\mathrm{B}}} \star H_3\right) = F_1 \wedge \star\tilde{F}_3 - \tilde{F}_5 \wedge \tilde{F}_3, \end{aligned} \tag{3.6}$$

in the NS-NS sector. The variation of the action with respect to the metric leads to

$$G_{MN} + e^{2\phi_{\mathrm{B}}} \left(g_{MN}\nabla^2 e^{-2\phi_{\mathrm{B}}} - \nabla_M \nabla_N e^{-2\phi_{\mathrm{B}}}\right) = -\frac{2\kappa^2}{\sqrt{-g}} \frac{\delta S_{\mathrm{tensor}}}{\delta g^{MN}} e^{2\phi_{\mathrm{B}}}, \tag{3.7}$$

where G_{MN} is the Einstein tensor and S_{tensor} is the action for all the tensor fields including the dilaton. The left hand side arises from the variation of the Einstein-Hilbert action with the dilaton contribution arising from the non-canonically normalized curvature term. Moreover, the tensor fields satisfy the Bianchi identities

$$\begin{aligned} dH_3 &= 0, \\ dF_1 &= 0, \\ d\tilde{F}_3 &= H_3 \wedge F_1, \\ d\tilde{F}_5 &= H_3 \wedge F_3. \end{aligned} \tag{3.8}$$

3. The SUGRA background

We are interested in a solution of the 10d equations of motion in which the space-time contains four non-compact dimensions and six compact dimension. We require maximal symmetry in the non-compact dimensions which means all tensor fields except F_5 have components along the internal directions only, while F_5 is required to take the form

$$\tilde{F}_5 = (1+\star)d\alpha(y) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{3.9}$$

where x, y denote the 4d and 6d coordinates respectively. Moreover, we would like to consider a background which arises as the orientifold limit of a flux background of M-theory compactified on K3×K3. In this case the RR axion vanishes and the type IIB dilaton $\phi_{\rm B}$ is constant. The space-time metric is of the form

$$ds^2 = e^{2A(y)+\phi_{\rm B}/2}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)+\phi_{\rm B}/2}\left(g_{ij}dy^i dy^j + dw_1^2 + dw_2^2\right), \tag{3.10}$$

where g_{ij} is the metric of K3 and the factor involving the dilaton arises since this is the metric in the 10d string frame and $e^{-2A(y)}$ is the warp factor depending on the coordinates of the internal space only. The function α in (3.9) is related to the warp factor according to

$$\alpha(y) = e^{4A(y)}. \tag{3.11}$$

The complex three-form $G_3 = \tilde{F}_3 - ie^{-\phi_{\rm B}}H_3$ is imaginary self-dual in the internal dimensions, *i.e.*

$$\star G_3 = iG_3. \tag{3.12}$$

Moreover, the warp factor satisfies the Poisson equation

$$\nabla^2 e^{-4A(y)} + e^{-\phi_{\rm B}}|H_3|^2 = 0. \tag{3.13}$$

Away from the orientifold points this is a solution of the equations of motion as can be explicitly verified.

Note that the three-form tensor fields H_3 and $\tilde{F}_3$ are harmonic forms on the internal part of the space (3.10). It turns out that the Hodge numbers of K3 are

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{3.14}$$

and in particular there are no harmonic one-forms or three-forms on K3. As a result H_3 and $\tilde{F}_3$ have to be the product of harmonic two-forms on K3, which we will denote by $(h_3)_i$ and $(\tilde{f}_3)_i$ and a one-form in the fiber directions, dw^i, *i.e.*

$$H_3 = (h_3)_i \wedge dw^i \qquad \text{and} \qquad \tilde{F}_3 = (\tilde{f}_3)_i \wedge dw^i, \qquad i = 1, 2, \tag{3.15}$$

where $w_i \sim w_i + 1$ and

$$(\tilde{f}_3)_i,\ (h_3)_i \in H^2(K3, \mathbb{Z}). \tag{3.16}$$

Moreover, the condition that G_3 is imaginary self-dual requires the complex three-form to be

$$G_3 = g_+ \wedge d\bar{w} + g_- \wedge dw, \tag{3.17}$$

where

$$dw = dw_1 + i dw_2, \tag{3.18}$$

and $g_\pm$ can be expanded in (anti)-self dual harmonic two-forms on K3

$$g_+ \in H^{2,0}(K3) \oplus H^{0,2}(K3) \oplus H^{1,1}_+(K3) \qquad \text{and} \qquad g_- \in H^{1,1}_-(K3). \tag{3.19}$$

There are 3 self-dual two-forms and 19 anti-self dual two-forms which are of type $(1,1)$ and primitive. In the following we will see that the different solutions of the equations of motion preserve different amounts of supersymmetry. In particular, the amount of unbroken supersymmetry will depend on the choices of two-forms on K3.

4. Supersymmetry

Let us represent the dilatino and gravitino fields by Weyl spinors λ and Ψ_μ, respectively. Similarly, the infinitesimal supersymmetry parameter is represented by a Weyl spinor ε. The supersymmetry transformations of the fermi fields of type IIB supergravity (to leading order in fermi fields) are

$$\delta\lambda = \frac{1}{2}\left(\partial\!\!\!/\phi_{\rm B} - ie^{\phi_{\rm B}}\partial\!\!\!/ C_0\right)\varepsilon + \frac{1}{4}\left(ie^{\phi_{\rm B}}\widetilde{F\!\!\!/}_3 - H\!\!\!\!/_3\right)\varepsilon^\star, \tag{3.20}$$

and

$$\delta\Psi_M = \left(\nabla_M + \frac{i}{8}e^{\phi_{\rm B}}F\!\!\!/_1\,\Gamma_M + \frac{i}{16}e^{\phi_{\rm B}}\widetilde{F\!\!\!/}_5\,\Gamma_M\right)\varepsilon - \frac{1}{8}\left(2(H\!\!\!\!/_3)_M + ie^{\phi_{\rm B}}\widetilde{F\!\!\!/}_3\,\Gamma_M\right)\varepsilon^\star. \tag{3.21}$$

Upon reducing to 4d the Lorentz algebra decomposes according to

$$SO(9,1)\to SO(3,1)\times SO(6). \tag{3.22}$$

The Weyl spinor ε then decomposes as

$$\mathbf{16}\to(\mathbf{2},\mathbf{4})+(\mathbf{2}',\mathbf{4}'). \tag{3.23}$$

Under the further decomposition $SO(6)\to SO(4)\times SO(2)$

$$\begin{aligned}\mathbf{4}&\to(\mathbf{2},\mathbf{1})+(\mathbf{2}',\mathbf{1}')\\ \mathbf{4}'&\to(\mathbf{2},\mathbf{1}')+(\mathbf{2}',\mathbf{1})\end{aligned} \tag{3.24}$$

The holonomy of K3 is $SU(2)$ and under the reduction $SO(4) \to SU(2)$

$$\begin{aligned} \mathbf{2} &\to \mathbf{1} + \mathbf{1} \\ \mathbf{2}' &\to \mathbf{2}. \end{aligned} \tag{3.25}$$

This means that either $\mathbf{4}$ or $\mathbf{4}'$ of $SO(6)$ gives rise to two $SU(2)$ singlets leading to an N=4 supersymmetry in 4d. Next we analyze the constraints imposed by the orientifold projection $\mathbb{Z}_2 = \Omega(-1)^{F_L}\mathcal{I}$. Writing $\varepsilon = \varepsilon_1 + i\varepsilon_2$ the different parity transformations act according to

$$\varepsilon = \varepsilon_1 + i\varepsilon_2 \xrightarrow{\Omega} \varepsilon_2 + i\varepsilon_1 \xrightarrow{(-1)^{F_L}} -\varepsilon_2 + i\varepsilon_1 \xrightarrow{\mathcal{I}} i\Gamma_\star(-\varepsilon_2 + i\varepsilon_1), \tag{3.26}$$

where $\Gamma_\star$ is the chirality operator of $SO(2)$. Combining these operations and requiring the spinor to be left invariant by the orientifold action imposes

$$\varepsilon = -\Gamma_\star \varepsilon. \tag{3.27}$$

Before we proceed, lets determine how the spinor projection relates to the one in the type I string. After two T-dualities on torus, the left moving spinor ε_1 is unaffected, however the right moving spinor ε_2, transforms as

$$\varepsilon_2 \to \Gamma^8\Gamma^9\varepsilon_2, \tag{3.28}$$

from which we get the transformation of Eq.(3.27),

$$(1 + \Gamma_\star)(\varepsilon_1 - \varepsilon_2) = 0 \tag{3.29}$$

Because the gamma matrix $\Gamma_\star$ is pure imaginary in our representation, this condition leads to $\varepsilon_1 = \varepsilon_2$, the spinor that survives the world sheet projection of type IIB string, *i.e.* type I string. This is an alternative way to see how type I string emerges after performing T-dualities of type IIB orientifold.

Eqn.(3.27) means that spinor has a definite chirality on the torus, which we choose to be $\mathbf{1}$ in eqn. (3.24), while $\mathbf{1'}$ is projected out. As a result the $SU(2)$ singlets which are not projected out by the orientifold arise from the $\mathbf{4}$ in eqn. (3.24). The orientifold breaks the 4d supersymmetry from N=4 to N=2. Moreover, the two 4d spinors are in the $\mathbf{2}$ of $SO(3,1)$ so have the same chirality. We denote the resulting spinors by η_i, and by an $SO(4)$ transformation we can choose them to satisfy

$$\Gamma_i \eta_1 = \Gamma_w \eta_1 = 0 \qquad \text{and} \qquad \Gamma_{\bar{i}} \eta_2 = \Gamma_w \eta_2 = 0, \tag{3.30}$$

where $(y^i, y^{\bar{i}})$ and $(w, \bar{w})$ are complex coordinates on K3 and the torus respectively.

Using these supersymmetry transformations the unbroken supersymmetries are those that satisfy $\delta(\text{fermi}) = 0$. Evaluated in the background metric (3.10), using the relation between the warp factor $A(y)$ and $\alpha(y)$ and the fact that the spinors have definite 4d chirality the supersymmetry conditions become

$$\nabla_i \left(e^{-A/2} \varepsilon \right) = 0, \tag{3.31}$$

which is satisfied with a spinor proportional to the covariantly constant spinors on K3$\times$T^2 and

$$\mathbf{G}_m \varepsilon^\star = 0 \qquad \text{and} \qquad \mathbf{G} \varepsilon = \mathbf{0}. \tag{3.32}$$

Next we solve the constraints (3.32) and we will check that depending on the choice of flux different amounts of supersymmetry are preserved. Lets analyze the amount of unbroken supersymmetry

- ▷ if $G = g_- \wedge dw$, then

$$G_{wi\bar{j}} \Gamma^{i\bar{j}} \eta_k^\star = G_{iw\bar{j}} \Gamma^{w\bar{j}} \eta_k^\star = G_{\bar{j}iw} \Gamma^{iw} \eta_k^\star = G_{wi\bar{j}} \Gamma^{wi\bar{j}} \eta_k = 0, \tag{3.33}$$

for $k = 1, 2$. This is solved by requiring G to be primitive with respect to the base, *i.e.*

$$G_{wi\bar{j}} g^{i\bar{j}} = 0, \tag{3.34}$$

while both spinors η_k for $k = 1, 2$ are non-vanishing. Since g_- are expanded in a basis of anti-self dual (1,1) forms eqn. (3.34) is always satisfied. This leads to an N=2 supersymmetry in 4d.

- ▷ if $G = g_+^{2,0} \wedge d\bar{w}$, eqn. (3.32) requires

$$G_{\bar{w}ij} \Gamma^{ij} \eta_2^\star = 0, \tag{3.35}$$

which is solved by $\eta_2 = 0$, while the conditions on η_1 are

$$G_{\bar{w}ij} \Gamma^{ij} \eta_1^\star = G_{i\bar{w}j} \Gamma^{\bar{w}j} \eta_1^\star = G_{ij\bar{w}} \Gamma^{ij\bar{w}} \eta_1. \tag{3.36}$$

These conditions are always satisfied which implies that the 4d supersymmetry arising from η_1 is unbroken. This flux configuration leads to an N=1 supersymmetry in 4d.

- ▷ if $G = g_+^{0,2} \wedge d\bar{w}$, eqn. (3.32) requires

$$G_{\bar{w}\bar{i}\bar{j}} \Gamma^{\bar{i}\bar{j}} \eta_k^\star = G_{\bar{i}\bar{w}\bar{j}} \Gamma^{\bar{w}\bar{j}} \eta_k^\star = G_{\bar{j}\bar{i}\bar{w}} \Gamma^{\bar{i}\bar{w}} \eta_k^\star = G_{\bar{w}\bar{i}\bar{j}} \Gamma^{\bar{w}\bar{i}\bar{j}} \eta_k = 0, \tag{3.37}$$

for $k = 1, 2$. These conditions are solved by requiring $\eta_1 = 0$ while $\eta_2 \neq 0$ and as a result there is an N=1' unbroken supersymmetry in 4d. We label this supersymmetry with N=1' since it preserves a different subgroup of the supersymmetry than the $G_{\bar{w}ij}$ component.

- ▷ if $G = g_+^{1,1} \wedge d\bar{w}$, eqn. (3.32) requires $\eta_1 = \eta_2 = 0$ and supersymmetry is completely broken.

B. Heterotic SUGRA background

In this section we analyze the heterotic flux backgrounds. To set up the notation we review the heterotic low-energy effective action to $O(\alpha'^2)$ in section 3.1. In section 3.2 we summarize the equations of motion. In section 3.3 we present the backgrounds solving the SUGRA equations to leading order in α'. In section 3.4 we analyze the amount of unbroken four-dimensional supersymmetry. This section is confined to solutions solving the SUGRA equations to leading order in α' and the corrected background is discussed in section 4.

1. The action

The bosonic part of the heterotic supergravity action to $O(\alpha'^2)$ in the 10d string frame is [29, 59, 60, 61, 62]

$$S_{het} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{2}|\mathcal{H}|^2 - \frac{\alpha'}{4}\mathrm{tr}(\mathcal{F}^2 - R_+^2) \right], \tag{3.38}$$

where

$$\mathcal{H} = d\mathcal{B} + \frac{\alpha'}{4}\omega_3, \tag{3.39}$$

is the NS three-form and $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the gauge field strength. Moreover, $\omega_3 = \omega_{\mathrm{L}} - \omega_{\mathrm{YM}}$ is given in terms of the Lorentz and Yang-Mills Chern-Simons three-forms

$$\omega_{\mathrm{L}} = \mathrm{tr}\left(\Omega_+ \wedge d\Omega_+ + \frac{2}{3}\Omega_+ \wedge \Omega_+ \wedge \Omega_+ \right) \quad \text{and} \quad \omega_{\mathrm{YM}} = \mathrm{tr}\left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right). \tag{3.40}$$

The contribution to the action which is quadratic in the Riemann tensor is

$$\mathrm{tr}R_+^2 = \frac{1}{2}R_{MNAB}(\Omega_+)R^{MNAB}(\Omega_+), \tag{3.41}$$

while the quadratic term in $\mathcal{F}$ is the standard gauge field kinetic term. Note that the Einstein-Hilbert action is formulated in terms of the spin connection while the quadratic term in the Riemann tensor is expressed in terms of a connection involving the NS three-form which explicitly is defined by

$$\Omega_{\pm}^{AB}{}_{M} = \Omega^{AB}{}_{M} \pm \frac{1}{2}\mathcal{H}^{AB}{}_{M}. \tag{3.42}$$

Also, we will follow ref. [29] according to which the action involves the Ω_+ connection while the supersymmetry transformations involve the Ω_- connection. The supersymmetry tranformations will be described in more detail below.

2. Equations of motion

The equations of motion arising from the action presented in the previous section are

▷ for the dilaton

$$R - 4(\nabla\phi)^2 + 4\nabla^2\phi - \frac{1}{2}|\mathcal{H}|^2 - \frac{\alpha'}{4}\mathrm{tr}(\mathcal{F}^2 - R_+^2) = 0, \tag{3.43}$$

▷ for $\mathcal{B}$

$$d(e^{-2\phi} \star_{10} \mathcal{H}) = 0, \tag{3.44}$$

▷ for the metric

$$\begin{aligned} &R_{MN} + 2\nabla_M\nabla_N\phi - \frac{1}{4}\mathcal{H}_{MPQ}\mathcal{H}_N{}^{PQ} + \\ &\frac{\alpha'}{4}[R_{MPQR}(\Omega_+)R_N{}^{PQR}(\Omega_+) - \mathcal{F}_{MP}\mathcal{F}_N{}^P] = 0, \end{aligned} \tag{3.45}$$

▷ for the Yang-Mills field

$$e^{2\phi}d(e^{-2\phi} \star_{10} \mathcal{F}) + \mathcal{A} \wedge \star_{10}\mathcal{F} - \star_{10}\mathcal{F} \wedge \mathcal{A} + \mathcal{F} \wedge \star_{10}\mathcal{H} = 0. \tag{3.46}$$

The Bianchi identities are

$$d\mathcal{H} = \frac{\alpha'}{4}[\mathrm{tr}(R_+ \wedge R_+) - \mathrm{Tr}(\mathcal{F} \wedge \mathcal{F})] \qquad \text{and} \qquad d\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0. \tag{3.47}$$

3. The SUGRA background

In the following, we present the background that solves the SUGRA equations of motion to leading order in α' (see ref.[26, 30, 31] for supersymmetric backgrounds). As we will see non-trivial solutions of the Bianchi identity exist only for non-compact backgrounds. This conclusion is modified once α' corrections are taken into account. The background metric is

$$ds^2_{het} = \eta_{\mu\nu}dx^\mu dx^\nu + e^{-4A(y)}g_{ij}dy^i dy^j + E_{w_1}E_{w_1} + E_{w_2}E_{w_2}, \tag{3.48}$$

where

$$E_{w_1} = dw_1 + B_{iw_1}dy^i \qquad \text{and} \qquad E_{w_2} = dw_2 + B_{iw_2}dy^i, \tag{3.49}$$

and $B_{w_k} = B_{iw_k}dy^i$, for $k = 1, 2$ are one-forms on the base. These one-forms are constrained by the condition that

$$H_{w_1} = dB_{jw_1}dy^j \qquad \text{and} \qquad H_{w_2} = dB_{jw_2}dy^j, \tag{3.50}$$

are harmonic non-trivial two-forms on K3. Note that E_{w_k} have to be globally defined since otherwise the metric is not be globally defined. As a result on the 6d space $H_{w_k} = dE_{w_k}$ become exact even though these forms are non-trivial on K3. We will expand H_{w_k} in harmonic non-trivial two-forms on K3. Depending on the choice of flux different amounts of 4d supersymmetry will preserved as we will see in the next section. The three-form is

$$\mathcal{H} = e^{2\phi} \star_6 d\left(e^{-2\phi}E^{w_1} \wedge E^{w_2}\right) = \star_b de^{-4A(y)} - \star_b H_{w_1} \wedge E^{w_1} - \star_b H_{w_2} \wedge E^{w_2}, \tag{3.51}$$

where $\star_6$ denotes the Hodge dual with respect to the 6d internal space and $\star_b$ denotes the Hodge dual with respect to the unwarped base. The dilaton is given by

$$\phi = -2A(y). \tag{3.52}$$

The Yang-Mills field is assumed to be a two-form on K3 only and to satisfy the hermitian Yang-Mills equations, *i.e.*

$$\mathcal{F}_{i\bar{j}} J^{i\bar{j}} = 0 \qquad \text{and} \qquad \mathcal{F}_{ij} = \mathcal{F}_{\bar{i}\bar{j}} = 0. \tag{3.53}$$

Here J is the Kähler form of K3. Moreover, $A(y)$ is a scalar function depending on the coordinates of the base only. To leading order it is required to solve the differential equation

$$\nabla^2 e^{-4A(y)} + \mid H_{w_1} \mid^2 + \mid H_{w_2} \mid^2 = 0. \tag{3.54}$$

Next we show that this background satisfies the equations of motion to leading order in α'. The equation of motion of $\mathcal{B}$ is satisfied since (3.51) implies

$$\star_{10}\mathcal{H} = -e^{2\phi} d\left(e^{-2\phi} E^{w_1} \wedge E^{w_2}\right) \wedge dx^{0123}. \tag{3.55}$$

The equation of motion for the metric has several components

$$(\mu,\nu),\ (i,j),\ (w_1,i),\ (w_2,i),\ (w_1,w_2). \tag{3.56}$$

The (i,j) component, with two indices on K3, is satisfied assuming $A(y)$ solves (3.54). Moreover, it is easy to see that all other components vanish to this order in α'. Next we consider the dilaton equation of motion. Using the metric $(B.4)$ to compute the scalar curvature R, the dilaton equation of motion is solved assuming $A(y)$ solves eqn. (3.54). On the other hand the Bianchi identity leads to

$$d\mathcal{H} = -\left(\nabla^2 e^{-4A(y)} + \mid H_{w_1} \mid^2 + \mid H_{w_2} \mid^2\right) \star_b 1 = 0, \tag{3.57}$$

which again is solved after imposing eqn. (3.54). Note that eqn. (3.54) only has non-trivial solutions if the internal space is non-compact. Below we will describe in detail how to construct compact solutions by going beyond the leading order in α'.

4. Supersymmetry

Next let us analyze the supersymmetry of the solutions of the equation of motion. The supersymmetry transformations leaving the 10d heterotic string frame effective action invariant are

$$\begin{aligned}
\delta\Psi_M &= \nabla_M\varepsilon - \frac{1}{4}\not{\mathcal{H}}_M\varepsilon, \\
\delta\lambda &= \not{\partial}\phi_h\varepsilon - \frac{1}{2}\not{\mathcal{H}}\,\varepsilon, \\
\delta\chi &= 2\not{\mathcal{F}}\,\varepsilon,
\end{aligned}$$

where Ψ_M is the gravitino, λ the dilatino and χ the gaugino. All spinors are Majorana-Weyl. The covariant derivative of a spinor is defined according to

$$\nabla_M\epsilon = \partial_M\varepsilon + \frac{1}{4}\Omega^{AB}{}_M\Gamma_{AB}\varepsilon, \tag{3.58}$$

where Ω is the spin connection. Note that the gravitino variation can then be written in the form

$$\delta\Psi_M = \partial_M\varepsilon + \frac{1}{4}\Omega_{-}^{AB}{}_M\Gamma_{AB}\varepsilon, \tag{3.59}$$

where

$$\Omega_{\pm}^{AB}{}_M = \Omega^{AB}{}_M \pm \frac{1}{2}H^{AB}{}_M. \tag{3.60}$$

Explicitly the components of the spin connection are

$$\begin{aligned}
\Omega_{\pm\ a}^{w_1} =& \frac{1}{2}e^{2A}(H_{w_1} \mp \star_b H_{w_1})_{ab}e^b, \\
\Omega_{\pm\ a}^{w_2} =& \frac{1}{2}e^{2A}(H_{w_2} \mp \star_b H_{w_2})_{ab}e^b, \\
\Omega_{\pm b}^{a} =& 2\left[\partial^a A e_b - \partial_b A e^a \mp (\star_b dA)^a{}_{bc}e^c\right] + \omega^a{}_b \\
& - \frac{1}{2}e^{4A}(H_{w_1} \pm \star_b H_{w_1})^a{}_b E^{w_1} - \frac{1}{2}e^{4A}(H_{w_2} \pm \star_b H_{w_2})^a{}_b E^{w_2}.
\end{aligned} \tag{3.61}$$

Note the sign differences between the first two components of the spin connection and the last one. These sign differences will play a crucial role in the supersymmetry analysis. Under the decomposition $SO(9,1) \to SO(3,1) \times SO(6)$ a 10d Weyl spinor decomposes as $\mathbf{16} \to (\mathbf{2}, \mathbf{4}) + (\mathbf{2}', \mathbf{4}')$. Imposing the Majorana condition we set

$$\epsilon = \zeta \otimes \eta + \zeta^\star \otimes \eta^\star, \tag{3.62}$$

where $\zeta \otimes \eta$ transforms as $(\mathbf{2}, \mathbf{4})$. Since the complex conjugate is not an independent spinor each 6d Weyl spinor gives rise to one minimal 4d supersymmetry.

Lets solve the supersymmetry constraints. The gravitino condition with the index in the external space-time is satisfied if the spinor does not depend on the coordinates of the external space-time. Projecting onto spinors with definite 4d chirality the supersymmetry conditions become

$$\begin{aligned}
&\nabla_M \eta - \frac{1}{4}\not{\mathcal{H}}_M \eta = 0, \\
&\not{\partial}\phi\eta - \frac{1}{2}\not{\mathcal{H}}\,\eta = 0, \\
&\not{\mathcal{F}}\,\eta = 0,
\end{aligned}$$

which are equations constraining the 6d spinor η. To solve this supersymmetry conditions the spinor η has to satisfy

$$\partial_{w_i}\eta = 0 \qquad \text{and} \qquad \partial_i \eta + \frac{1}{4}\omega^{ab}{}_i \gamma_{ab}\eta = 0, \tag{3.63}$$

i.e. η is a covariantly constant spinor on the base. We denote the two covariantly constant spinors of K3 by η_k, $k = 1, 2$. Moreover, we require η_k to solve

$$\begin{aligned}&(H_{w_1} - \star_b H_{w_1})_{ab}\gamma^{ab}\eta_k = (H_{w_2} - \star_b H_{w_2})_{ab}\gamma^{ab}\eta_k = 0,\\&(H_{w_1} + \star_b H_{w_1})_{ab}\gamma^{w_1 a}\eta_k + (H_{w_2} + \star_b H_{w_2})_{ab}\gamma^{w_2 a}\eta_k = 0,\end{aligned} \tag{3.64}$$

which after introducing complex coordinates $w = w_1 + iw_2$, so that

$$H_w = \frac{1}{2}(H_{w_1} - iH_{w_2}) \qquad \text{and} \qquad H_{\bar{w}} = \frac{1}{2}(H_{w_1} + iH_{w_2}), \tag{3.65}$$

take the form

$$\begin{aligned}&\left[(1 - \star_b)H_w\right]_{ab}\gamma^{ab}\eta_k = 0,\\&\left[(1 - \star_b)H_{\bar{w}}\right]_{ab}\gamma^{ab}\eta_k = 0,\\&[(1 + \star_b)H_w]_{ab}\gamma^{wa}\eta_k + [(1 + \star_b)H_{\bar{w}}]_{ab}\gamma^{\bar{w}a}\eta_k = 0.\end{aligned} \tag{3.66}$$

Note that the contributions involving the warp factor arising from the spin connection components $\Omega_{\pm\ c}^{ab}$ and contributing to the component of the gravitino variation along the base cancel since the two spinors η_k have positive chirality on the base *i.e.*

$$-\gamma_{1234}\eta_k = \eta_k \qquad k = 1, 2. \tag{3.67}$$

Now depending on the choice of flux different amounts of supersymmetry are preserved [31]. The different cases are

- ▷ if H_w is proportional to an anti-self dual (1,1) form on the K3 base, the conditions $(B.18)$ are satisfied for both spinors η_k, $k = 1, 2$. An N=2 supersymmetry is preserved in 4d. Indeed, the third condition is trivially satisfied and the first two conditions are satisfied since the anti-self dual (1,1) forms are primitive with respect to the base.

- ▷ if H_w is proportional to the self-dual (0,2) form on the base the supersymmetry generated by η_1 is preserved while $\eta_2 = 0$. There is an N=1 unbroken supersymmetry in 4d.

- ▷ if H_w is proportional to the self-dual (2,0) form on the base the supersymmetry generated by η_2 is unbroken while $\eta_1 = 0$. There is an N=1' unbroken supersymmetry in 4d.

- ▷ if H_w is proportional to the self-dual (1,1) form on the base (B.18) requires the two spinors to vanish. So N=0 in 4d.

C. The α' corrected torsional heterotic geometry

In this section we will consider α' corrections to the torsional heterotic geometries. We will see that these α' corrections to the background are required since otherwise the α' corrected equations of motion are not satisfied. Once the background is corrected in α' compact solutions become possible. As a first step to solve the Bianchi identity we need to compute $\mathrm{tr}(R_+ \wedge R_+)$, which appears on the right hand side of the Bianchi identity.

1. tr$(R_+ \wedge R_+)$

In general, the curvature two-form is defined by

$$R^A{}_B = d\Omega^A{}_B + \Omega^A{}_C \wedge \Omega^C{}_B, \tag{3.68}$$

for some connection Ω. According to Bergshoeff and de Roo [29] the connection used in the supersymmtry transformations is Ω_- while in the Bianchi identity the Ω_+

connection is used. The connection coefficients are

$$\begin{aligned}\Omega_{+\ \,a}^{w_k} =& \frac{1}{2}e^{2A}(H_{w_k} - \star_b H_{w_k})_{ij} e_a^i dy^j, \qquad k=1,2 \\ \Omega_{+b}^{a} =& \sigma^a{}_b + \omega^a{}_b - \frac{1}{2}(H_{w_k} + \star_b H_{w_k})_{ij} E_c^i E_b^j \eta^{ac} E^{w_k},\end{aligned} \tag{3.69}$$

where, the last term involves a sum over $k = 1, 2$. We denote with E^a the vielbeine of the warped base while e^a are those of the unwarped K3. Moreover,

$$\sigma_{ab} = 2\left[\partial_a A e_b - \partial_b A e_a - (\star_b dA)_{abc} e^c\right]. \tag{3.70}$$

Note that σ_{ab} is self-dual in its indices, *i.e.* it satisfies

$$\sigma_{ab} = \frac{1}{2}\varepsilon_{abcd}\sigma^{cd}. \tag{3.71}$$

We are denoting the spin connection coefficients and curvature two-form of the K3 base by $\omega^a{}_b$ and $r^a{}_b$.

Before describing in detail the results for the curvature two-form and $\mathrm{Tr}(R_+ \wedge R_+)$, where R_+ is computed with respect to the Ω_+ connection, we will first establish that the curvature two-form of the torsional space is of type (1,1) to leading order in α' if computed with respect to the Ω_+ connection. This implies that $\mathrm{Tr}(R_+ \wedge R_+)$ is a (2,2) form which is a necessary condition for the Bianchi identity to admit a non-trivial solution. Indeed, up to terms of $O(\alpha'^2)$ unbroken supersymmetry requires the flux and the fundamental (1,1) form to be related according to $\mathcal{H} = i(\partial - \bar{\partial})J$. As a result $d\mathcal{H} = -2i\partial\bar{\partial}J$ is a (2,2) form. This is the left hand side of the Bianchi identity. The right hand side of the Bianchi identity is $\mathrm{Tr}(R_+ \wedge R_+)$, which is required to be a four-form of type (2,2) since otherwise the background is over-constrained.

Here we follow the presentation of ref. [63]. By definition

$$\Omega_{+\ \ \,M}^{AB} = \Omega^{AB}{}_M + \frac{1}{2}H^{AB}{}_M, \tag{3.72}$$

which implies that the connection in the coordinate basis is modified to

$$\Gamma^J_{+IK} = G^{JL}(E^A_L \partial_I E^A_K + \Omega_+^{AB}{}_I E^A_L E^B_K) = \Gamma^J_{IK} - \frac{1}{2}\mathcal{H}_{IK}{}^J. \tag{3.73}$$

By definition

$$\Gamma^J_{IK} = \frac{1}{2} g^{JN} \left(\partial_I g_{NK} + \partial_K g_{NI} - \partial_N g_{IK}\right). \tag{3.74}$$

Supersymmetry requires $\mathcal{H}$ to be related to the derivative of the metric according to

$$\mathcal{H}_{MN\bar{P}} = -\partial_M g_{N\bar{P}} + \partial_N g_{M\bar{P}}, \tag{3.75}$$

and the complex conjugate. Here we have introduced complex coordinates. Using the fact that the metric of the torsional space is hermitian eqn. (3.74) implies that the non-vanishing connection coefficients are

$$\Gamma^I_{+JK} = g^{I\bar{N}} \partial_J g_{K\bar{N}} \qquad \text{and} \qquad \Gamma^I_{+J\bar{K}} = g^{I\bar{N}} \partial_{\bar{K}} g_{J\bar{N}} - g^{I\bar{N}} \partial_{\bar{N}} g_{J\bar{K}}. \tag{3.76}$$

So in contrast to Kähler geometry there are connection coefficients with mixed indices. The Riemann tensor is obtained from the connection coefficients according to

$$R_{MN}{}^K{}_L = \partial_M \Gamma^K_{NL} - \partial_N \Gamma^K_{ML} + \Gamma^K_{MR}\Gamma^R_{NL} - \Gamma^K_{NR}\Gamma^R_{ML}, \tag{3.77}$$

and the curvature two-form is related to the Riemann tensor according to

$$R^A{}_B = \frac{1}{2} R_{CD}{}^A{}_B E^C E^D. \tag{3.78}$$

Introducing complex coordinates it is not difficult to see that

$$R_{+MN}{}^K{}_L = R_{+MN}{}^{\bar{K}}{}_L = R_{+MN}{}^{\bar{K}}{}_{\bar{L}} = 0. \tag{3.79}$$

Moreover,

$$R_{+\bar{M}\bar{N}}{}^{\bar{K}}{}_L = g^{P\bar{K}} \left(g_{P[\bar{N},\bar{M}]L} - g_{L[\bar{N},\bar{M}]P}\right) = O(\alpha'). \tag{3.80}$$

This quantity is subleading since the right hand side is the (2,2) component of $d\mathcal{H}$ which is $O(\alpha')$ after using the Bianchi identity. Therefore we conclude that to leading order in α', $\mathrm{Tr}(R_+ \wedge R_+)$ is of type (2,2).

Next we present the explicit results for the curvature two-forms and $\mathrm{Tr}(R_+ \wedge R_+)$ and show how to solve the Bianchi identity. We will focus on solutions with N=2 supersymmetry.

2. N=2 background at $O(\alpha')$

In this case the forms H_{w_i} are proportional to anti-self dual (1,1) forms on the K3 base. From (3.69) we see that the only non-vanishing components of the spin connection are

$$\begin{aligned} \Omega_{+\ a}^{w_k} &= e^{2A}(H_{w_k})_{ij} e_a^i dy^j, \qquad k = 1, 2 \\ \Omega_{+b}^{a} &= \sigma^a{}_b + \omega^a{}_b. \end{aligned} \tag{3.81}$$

In this case the curvature two-form computed with respect to the Ω_+ connection is a two-from on K3 explicitly given by

$$\begin{aligned} R^{w_1}{}_{w_2} &= -e^{4A}(H_{w_1})_a (H_{w_2})^a \\ R^a{}_{w_k} &= -\nabla[e^{2A}(H_{w_k})_a] - e^{2A}(H_{w_k})_b \sigma^b{}_a, \qquad k = 1, 2 \\ R^a{}_b &= r^a{}_b + \nabla\sigma^a{}_c + \sigma^a{}_c \sigma^c{}_b - e^{4A}(H_{w_k})_a (H_{w_k})_b, \end{aligned} \tag{3.82}$$

where $r^a{}_b$ is the curvature two-form of K3 and ∇ is the covariant derivative with respect to the $\omega^a{}_b$ connection. Explicitly

$$\nabla\sigma^a{}_b = d\sigma^a{}_b + \omega^a{}_c \sigma^c{}_b + \sigma^a{}_c \omega^c{}_b. \tag{3.83}$$

A convenient way to compute $\mathrm{Tr}(R_+ \wedge R_+)$ is to use the Chern-Simons formula which relates the results for $\mathrm{Tr}(R \wedge R)$ computed with two connections Γ and $\tilde{\Gamma}$ according to

$$\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(\tilde{R} \wedge \tilde{R}) = dQ(\Gamma, \tilde{\Gamma}), \tag{3.84}$$

where

$$Q(\Gamma, \tilde{\Gamma}) = 2\alpha \wedge R - \alpha \wedge d\alpha - 2\alpha \wedge \Gamma \wedge \alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha \tag{3.85}$$

where $\alpha = \Gamma - \tilde{\Gamma}$. Setting

$$\begin{aligned} \tilde{\Gamma}^a{}_b &= \Omega^a_{+b} \qquad \text{and} \qquad \tilde{\Gamma}^{w_k}{}_a = \Omega_+{}^{w_k}{}_a \\ \Gamma^a{}_b &= \Omega^a_{+b} \qquad \text{and} \qquad \Gamma^{w_k}{}_a = 0, \qquad k = 1, 2, \end{aligned} \tag{3.86}$$

or in other words choosing

$$\alpha^a{}_b = 0 \qquad \text{and} \qquad \alpha^{w_k}{}_a = -e^{2A}(H_{w_k})_{ij} e^i_a dy^j, \tag{3.87}$$

we obtain

$$\mathrm{Tr}(R_+ \wedge R_+) = \mathrm{Tr}[R(\Gamma) \wedge R(\Gamma)] + 2d\left\{e^{2A}(H_{w_k})_b \nabla[e^{2A}(H_{w_k})_b] + e^{4A}(H_{w_k})_b \sigma^b{}_c (H_{w_k})_c\right\}, \tag{3.88}$$

where

$$\mathrm{Tr}[R(\Gamma) \wedge R(\Gamma)] = -(\nabla\sigma^a{}_b + r^a{}_b + \sigma^a{}_c\sigma^c{}_b)(\nabla\sigma^b{}_a + r^b{}_a + \sigma^b{}_c\sigma^c{}_a) \tag{3.89}$$

This result can be further simplified by using the Chern-Simons formula again, this time with

$$\tilde{\Gamma}^a{}_b = \omega^a{}_b \qquad \text{and} \qquad \Gamma^a{}_b = \omega^a{}_b + \sigma^a{}_b. \tag{3.90}$$

The result is

$$\mathrm{Tr}[R(\Gamma) \wedge R(\Gamma)] = \mathrm{Tr}(r \wedge r) - 2^4 d\left[2(\nabla^2 A) \star dA - \star d(\nabla A)^2 - 8(\nabla A)^2 \star dA\right]. \tag{3.91}$$

A straightforward but tedious computation then shows

$$\begin{aligned}\mathrm{Tr}(R_+ \wedge R_+) =& \mathrm{Tr}(r \wedge r) + 4d \star_b d \left(\nabla^2 A\right) + \\ & d \star_b d \left[(\nabla^2 e^{-4A} + |H|^2) e^{4A}\right] + \\ & 2d \left[(\nabla^2 e^{-4A} + |H|^2) \star_b de^{4A}\right],\end{aligned} \tag{3.92}$$

where

$$|H|^2 = |H_{w_1}|^2 + |H_{w_2}|^2. \tag{3.93}$$

Note that the last two lines in eqn. (3.92) involve the leading order equation of motion (3.54). Thus we establish that for solutions preserving an N=2 supersymmetry in four dimensions $\mathrm{Tr}(R_+ \wedge R_+)$ is a (2,2) form with components along the K3 base only. Note that this fact is a consequence of having used the Ω_+ connection to compute $\mathrm{Tr}(R_+ \wedge R_+)$. Since $\mathrm{Tr}(R_+ \wedge R_+)$ has components along the base only the fiber is not required to be of $O(\alpha')$ and can be chosen to be large.

Next we will use this result and solve the Bianchi identity

$$d\mathcal{H} = \frac{\alpha'}{4}[\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(\mathcal{F} \wedge \mathcal{F})] \tag{3.94}$$

to $O(\alpha')$. First we note that the second and third line on the right hand side of Eq.(3.92) are proportional to the dual of $d\mathcal{H}$ and are therefore $O(\alpha')$. As a result they contribute to the Bianchi identity only to $O(\alpha'^2)$. Keeping all terms up to $O(\alpha')$ the Bianchi identity becomes

$$d \star_b de^{-4A} - \star_b H_{w_k} \wedge H_{w_k} + O(\alpha') = \frac{\alpha'}{4}[\mathrm{Tr}(r \wedge r) - \mathrm{Tr}(\mathcal{F} \wedge \mathcal{F})] + \alpha' d \star_b d(\nabla^2 A). \tag{3.95}$$

Here we have allowed a correction to $O(\alpha')$ on the left hand side. Since the supersymmtry transformations receive only corrections at $O(\alpha'^2)$ any corrections to the left hand side of eqn.(3.95) have to solve the leading order supersymmetry conditions. Since the supersymmtry conditions do not determine $A(y)$ we can redefine the warp factor and still obtain a supersymmetric situation. In particular if we define

$$e^{-4A'} = e^{-4A} + \alpha' \nabla^2 A. \tag{3.96}$$

and allow the background to receive an $O(\alpha')$ correction according to

$$\begin{aligned} &\phi = -2A'(y), \\ &\mathcal{H} = \star_b de^{-4A'(y)} - \star_b H_{w_1} \wedge E_{w_1} - \star_b H_{w_2} \wedge E_{w_2} \\ &ds^2_{het} = \eta_{\mu\nu} dx^\mu dx^\nu + e^{-4A'(y)} g_{ij} dy^i dy^j + E_{w_1} E_{w_1} + E_{w_2} E_{w_2} \end{aligned} \tag{3.97}$$

supersymmtry will still be preserved. To this order in α' the Bianchi identity becomes an equation of Laplace type, namely

$$d \star_b de^{-4A} - \star_b H_{w_k} \wedge H_{w_k} = \frac{\alpha'}{4} [\mathrm{Tr}(r \wedge r) - \mathrm{Tr}(\mathcal{F} \wedge \mathcal{F})]. \tag{3.98}$$

Note that we have obtained a linear differential for the dilaton even though the Bianchi identity could, in principle, lead to a highly non-linear differential equation. This fact depends crucially on choosing the Ω_+ connection to construct $\mathrm{Tr}(R_+ \wedge R_+)$. There is a preferred set of fields for which this connection is required by space-time supersymmetry as shown by Bergshoeff and de Roo [29]. A different choice of connection is always possible but it leads to a different choice of fields for which in general the supersymmetry transformations will receive corrections at $O(\alpha')$. We have found a differential equation of Laplace type using the Ω_+ connection and the solvability of the equation is immediate if the integrated equation is satified. Choosing the hermitian connection, on the other hand, will lead to a highly non-linear differential equation of Monge-Ampere type as shown in refs. [23, 24] .

In the following we will show that the α' corrected background solves the equations of motion presented in section 3.2. First we note that the equation of motion of $\mathcal{B}$ is satisfied since in the background (3.97)

$$\star_{10}\mathcal{H} = -e^{2\phi} d\left(e^{-2\phi} E^{w_1} \wedge E^{w_2}\right) \wedge dx^{0123}. \tag{3.99}$$

The Bianchi identity for $\mathcal{H}$ is solved by construction. To solve the equations of motion for the metric we first establish some properties of the Riemann tensor. First, the Ricci tensor of the torsional metric is

$$R_{ij} = 4\nabla_i\partial_j A' + 8\partial_i A'\partial_j A' - \frac{1}{2}e^{4A'} H_{w_k ai} H^{w_k a}{}_j + g_{ij}\left[2\nabla^2 A' - 8(\partial A')^2\right], \tag{3.100}$$

where (i, j) are indices on the base and ∇_i involves connections on the base only. Note that this derivative is not identical to $\nabla_i^{(6)}$, which is the covariant derivative constructed with respect to the connections on the six-dimensional torsional space. So for example

$$\nabla_i^{(6)}\partial_j\phi = \nabla_i\partial_j\phi - 8\partial_i A'\partial_j A' + 4g_{ij}(\partial A')^2. \tag{3.101}$$

Up to terms of $O(\alpha')$ the curvature two-form constructed from the Ω_+ connection R^A_{+B} satisfies

$$\star_b R_+{}^A{}_B = -R_+{}^A{}_B + O(\alpha') \tag{3.102}$$

This condition can be derived using the integrability condition of the supersymmetry constrain on the gravitino

$$\left[\nabla_{-M}, \nabla_{-N}\right]\varepsilon = \frac{1}{4}R_{-MNPQ}\Gamma^{PQ}\varepsilon = 0, \tag{3.103}$$

which implies

$$R_{-MNPQ}J^{PQ} = 0. \tag{3.104}$$

Moreover, one has

$$R_{-PQMN} = R_{+MNPQ} - 2\nabla_{[P}\mathcal{H}_{MNQ]} = R_{+MNPQ} + O(\alpha'), \tag{3.105}$$

which implies

$$R_{+PQMN}J^{PQ} + O(\alpha') = 0. \tag{3.106}$$

From here we obtain the following identity

$$R_{+mPAB}R_{+n}{}^{PAB} = \frac{1}{4}R_{+PQAB}R_{+}{}^{PQAB}g_{mn} + O(\alpha') \tag{3.107}$$

where now (m,n) are indices on the K3 base only, while if these indices are along the fiber the result vanishes. Also,

$$\mathrm{Tr}(R_+ \wedge R_+) = -\frac{1}{2}R_{+PQAB}R_{+}{}^{PQAB} \star_b 1 + O(\alpha'). \tag{3.108}$$

Using the above result for the curvature we can now verify the equation of motion for the metric and the dilaton. The only non-trivial component of the Einstein equation is the $(M,N) = (m,n)$ component with both indices along the base. All terms, except the ones proportional to the base metric g_{mn} cancel. The coefficient of g_{mn}, on the other hand, turns out to be the Hodge dual of the Bianchi identity (A), as can be verified with a bit of patience. As a result the Einstein equation, Bianchi identity and equation of motion for $\mathcal{B}$ are satisfied. Explicit computation shows that also the dilaton equation of motion is solved.

We end by describing torsional spaces with an N=2 supersymmetry in which the twist of the fiber is 'exchanged' by vacuum expectation values of abelian gauge fields. This type of solutions were suggested in refs. [25, 64]. In this case the torus fiber is not twisted and the background fields are

$$\begin{aligned}
ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + e^{-4A'(y)}g_{ij}dy^i dy^j + dw_1^2 + dw_2^2,\\
\mathcal{H} &= \star_b de^{-4A'(y)},\\
\mathcal{F} &= \mathcal{F}_{i\bar{j}}dy^i dy^{\bar{j}},\\
\phi &= -2A'(y),
\end{aligned} \tag{3.109}$$

where now an abelian gauge field is included as part of the background and $\mathcal{F}$ is an anti-self dual form on K3. This background solves the supersymmetry constraints preserving an N=2 supersymmtry. Moreover, it is not difficult to see that the Bianchi identity reduces to the differential equation

$$\begin{aligned} -\nabla^2 e^{-4A(y)} \star_b 1 = & \frac{\alpha'}{4} \left[\mathrm{Tr}(r \wedge r) - \mathrm{Tr}(\mathcal{F} \wedge \mathcal{F}) \right] \\ & + \frac{3\alpha'}{4} d(\nabla^2 e^{-4A} \star_b de^{4A}) \\ & + \frac{\alpha'}{4} d(e^{4A} \star_b d\nabla^2 e^{-4A}). \end{aligned} \tag{3.110}$$

The computation of $\mathrm{Tr}(R_+ \wedge R_+)$ for these solutions is greatly simplified since the fiber is not twisted. In this case the second and third lines on the right hand side of eqn. (3.110) are again corrections of order $O(\alpha'^2)$ or higher and can only be consistently taken into account once the supersymmetry transformations are corrected to $O(\alpha'^2)$. Therefore to $O(\alpha')$ the differential equation is again of Laplace type and solvability is guaranteed. The form of the $O(\alpha'^2)$ corrections to the supersymmetry transformations has been described in ref. [29]. It would be interesting to analysis to $O(\alpha'^2)$ of solutions preserving an N=2 supersymmetry and show the solvability of the Bianchi identity for backgrounds preserving an N=1 supersymmetry.

CHAPTER IV

HIGHER DERIVATIVE D-BRANE COUPLINGS*

As we mentioned in the introduction, one needs more complete knowledge regarding the higher derivative D-brane couplings to connect the flux backgrounds we described in Chapter III to vacua in type IIB side at the α' order. In this chapter, we will compute the higher derivative D-brane couplings by using both T-duality rules and string disc amplitude approaches. In section A, we use spacetime T-duality to argue that there should be additional higher derivative terms to the well known anomaly couplings at Eq.(1.2), and we will in fact use the Buscher rules to compute several terms which must be present, eventually arriving at (4.28), which is the key result of this section. In section B we evaluate disc amplitudes with insertions of three vertex operators for one R-R field $C^{(p-3)}$ and two NS-NS fields. We will focus on the case that both NS-NS fields are anti-symmetric B-fields, and only briefly summarize the results for other situations. In section C, we present the supergravity diagrams that replace the string amplitude at low energy limit. Using all known low energy effective action of type II string, we are able to evaluate the amplitudes for most of these diagrams, except the one with only one vertex, representing the contact interaction among one R-R field and two B-fields on D-brane. After subtracting all known supergravity amplitudes from the string amplitude we get the amplitude arising from the brane

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couplings including both leading and higher derivative terms. In section D, we write down the action that reproduce the higher derivative amplitude in section C, in terms of either field strength H or $B + 2\alpha' F$, so the action is manifestly invariant under B-field gauge transformation. We show that the require of R-R gauge invariance impose the corrections of our action. we also show that the modified higher derivative action is compatible with linear T-duality. Finally, we will discuss how to fix the arbitrary terms we left behind.

A. Predictions from T-duality

1. Buscher rules

In backgrounds which include a $U(1)$ isometry, type II string theories appear to enjoy a duality,called T-duality, relating one background which solves the equations of motion to another. Pick coordinates such that the isometry corresponds to translation in one coordinate, y, and let the remaining coordinates be labeled by indices μ, ν, etc. Then the explicit T-duality transformations for the NS-NS fields are given by [65]

$$g'_{yy} = \frac{1}{g_{yy}}, \qquad g'_{\mu y} = \frac{B_{\mu y}}{g_{yy}}, \qquad g'_{\mu\nu} = g_{\mu\nu} - \frac{g_{\mu y} g_{\nu y} - B_{\mu y} B_{\nu y}}{g_{yy}},$$

$$B'_{\mu y} = \frac{g_{\mu y}}{g_{yy}}, \qquad B'_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu y} g_{\nu y} - g_{\mu y} B_{\nu y}}{g_{yy}}, \qquad \Phi' = \Phi - \frac{1}{2}\ln g_{yy}, \tag{4.1}$$

and for the R-R potentials we have [66]

$$\begin{aligned} C^{(p)\prime}_{\mu_1\cdots\mu_{p-1}y} &= C^{(p-1)}_{\mu_1\cdots\mu_{p-1}} - (p-1)\,\frac{C^{(p-1)}_{[\mu_1\cdots\mu_{p-2}|y|}g_{\mu_{p-1}]y}}{g_{yy}}, \\ C^{(p)\prime}_{\mu_1\cdots\mu_p} &= C^{(p+1)}_{\mu_1\cdots\mu_p y} + pC^{(p-1)}_{[\mu_1\cdots\mu_{p-1}}B_{\mu_p]y} + p\,(p-1)\,\frac{C^{(p-1)}_{[\mu_1\cdots\mu_{p-2}|y|}B_{\mu_{p-1}|y|}g_{\mu_p]y}}{g_{yy}}. \end{aligned} \tag{4.2}$$

Under this duality, the type IIA and type IIB supergravity actions are mapped into each other, and in fact the action for the NS-NS sector fields is invariant under T-duality.

2. Using T-duality to construct or constrain actions

Suppose that we didn't actually know the two-derivative action for NS-NS sector fields, but knew only that it was invariant under diffeomorphisms and B-field gauge transformations. In this case there are four possible terms we could write down in the Lagrangian,

$$f_1(\Phi)\sqrt{-g}R, \qquad f_2(\Phi)\sqrt{-g}H^2, \qquad f_3(\Phi)\sqrt{-g}\nabla^2\Phi, \qquad f_4(\Phi)\sqrt{-g}\left(\nabla\Phi\right)^2, \quad (4.3)$$

where the f_i are arbitrary functions of Φ. Note that one combination of these would be a total derivative, but if we continue to work at the level of Lagrangians, we can keep all four terms. If we also know that the Lagrangian was invariant under the Buscher rules above, then we can actually fix the action up to an overall constant. We would do this by assuming a background with a $U(1)$ isometry, evaluating each of the terms above in that situation, and demanding that the result be invariant. One finds the invariant combination

$$\mathcal{L} \supset \mathcal{N} e^{-2\Phi}\sqrt{-g}\left(R - \frac{1}{12}H^2 + 4\nabla^2\Phi - 4\left(\nabla\Phi\right)^2\right), \qquad (4.4)$$

with $\mathcal{N}$ an arbitrary constant[4]. If we knew the coefficient of one of the terms, like the Einstein-Hilbert term, then the other terms are determined. In this way, T-duality can be used to fix the form of the action.

T-duality is also a useful guide in the presence of D-branes, converting a brane which wraps the direction of the $U(1)$ isometry into one which is localized at a point

[4] One can compare this result with equation (1.10) of [67], which is obtained by slightly different reasoning.

in the circle direction[5]. T-duality should map the actions on such dual pairs of branes into one another. In this chapter we will be focused on the Wess-Zumino part of the D-brane action, its higher derivative corrections, and terms related to it by T-duality. Formally, these terms can be written as

$$T_p \int_{Dp} \mathcal{L}_{WZ}^{(p+1)}, \tag{4.5}$$

where T_p is the tension of the D-brane and $\mathcal{L}_{WZ}^{(p+1)}$ is a $(p+1)$-form on the worldvolume of the D-brane. A naive guess for the zero-derivative piece of this action would be $\mathcal{L}_{WZ}^{(p+1)} = C^{(p+1)}$, but it turns out that this is inconsistent with T-duality. Indeed, the requirement of consistency with T-duality is equivalent to demanding (we use a prime to indicate that the expression should be transformed by the Buscher rules (4.1) and (4.2))

$$\mathcal{L}_{WZ\ \mu_1\cdots\mu_{p+1}}^{(p+1)\prime} = \mathcal{L}_{WZ\ \mu_1\cdots\mu_{p+1}y}^{(p+2)}, \qquad \mathcal{L}_{WZ\ \mu_1\cdots\mu_p y}^{(p+1)\prime} = \mathcal{L}_{WZ\ \mu_1\cdots\mu_p}^{(p)}, \tag{4.6}$$

which is not satisfied by $C^{(p+1)}$ because of the non-linear pieces in the transformation rules (4.2). Rather, we should proceed as before and write down the possible terms which can appear, evaluate them in a circle isometry ansatz, and impose T-duality. Doing so, we arrive at the T-duality completion of this naive term,

$$\mathcal{L}_{WZ}^{(p+1)} = Ce^B|_{(p+1)-\mathrm{form}}, \tag{4.7}$$

where C is a formal sum of R-R potentials and

$$e^B = 1 + B + \frac{1}{2} B \wedge B + \cdots . \tag{4.8}$$

[5]In this discussion, we are referring to probe branes, not to branes or stacks of branes that backreact on the geometry. A supergravity solution corresponding to a stack of branes wrapping a circle isometry with backreaction taken into account is converted, by T-duality, into a solution where a stack of lower-dimensional branes are smeared along the circle direction. Instead, we are typically interested in only a single brane which is localized, not smeared.

It is not hard to see that (considered as forms in the ten-dimensional spacetime) the expression (4.7) satisfies (4.6).

Thus, if one knew about T-duality, and knew that we expected at least a term in the Lagrangian like $\int_{Dp} C^{(p+1)}$, then we could deduce that it must be part of a larger "T-duality invariant", $\int_{Dp} Ce^B$, where the $(p+1)$-form integrand here is understood to be pulled back to the worldvolume of the Dp-brane. Of course, if we also considered invariance under B-field gauge transformations, then we would be lead to introduce more terms, so that the final result was

$$S_{WZ}^{(0)} = T_p \int_{Dp} Ce^{B+2\pi\alpha' F}, \tag{4.9}$$

where $F = dA$ is the field strength of the worldvolume gauge field which transforms under B-field gauge transformations $B \to B + d\Lambda$ as $A \to A - \Lambda/(2\pi\alpha')$. In most of what follows we will set the gauge field to zero, though of course the eventual task of constructing a full non-linear action will require its inclusion, along with many other terms that we have not written down, in order to satisfy B-field gauge invariance.

3. Higher derivative corrections

Now we turn to four-derivative terms. It is known that (up to field redefinitions), the type II two-derivative supergravity action gets no corrections until certain eight-derivative terms predicted from string theory appear. Thus the action receives only $(\alpha')^3$ corrections, and is uncorrected at order α' and $(\alpha')^2$. It then follows, trivially, that the Buscher rules which we wrote down before continue to be symmetries of (the NS-NS part of) the action to order $(\alpha')^2$.

We will then assume that this observation holds also in the presence of branes, where suddenly the idea that the Buscher rules remain uncorrected at order $(\alpha')^2$ becomes a powerful tool. The worldvolume actions of D-branes, and the Wess-Zumino piece in particular, is known to receive four-derivative corrections at order $(\alpha')^2$. If the original Buscher rules continue to describe T-duality at this order, then they can be used to strongly constrain these corrections to the action, since the four-derivative parts of the action will need to be T-duality covariant by themselves. On the other hand, if the Buscher rules *were* corrected to this order, then it would be much more difficult to extract any useful information, since we would have to contend with mixing between T-duality transformations of the zero-derivative and four-derivative parts of the action.

It's not completely clear that our assumption is reasonable - one could perhaps imagine corrections to the Buscher rules which were non-vanishing only in the presence of branes or other sources. However, for now we will proceed with this idea, and we will find that the result we got from string amplitude approach in section D will confirm the predictions we make here, thus justifying, to some extent, our assumptions.

Now we turn to the known α'^2 corrections to the Wess-Zumino action (1.2), which is proportional to a four-form

$$\begin{aligned} X^{(4)}_{\text{original}} &= \mathrm{Tr} R_T \wedge R_T - \mathrm{Tr} R_N \wedge R_N \\ &= \frac{1}{4}\left(-g_T^{eg} g_T^{fh} \left(R_T\right)_{abef} \left(R_T\right)_{cdgh} + \delta^{ik}\delta^{j\ell} \left(R_N\right)_{ab}{}^{ij} \left(R_N\right)_{cd}{}^{k\ell}\right) dx^a \wedge dx^b \wedge dx^c \wedge dx^d, \end{aligned} \tag{4.10}$$

where g_T is the induced metric on the brane worldvolume, R_T is the curvature tensor built from g_T, and R_N is the curvature of the normal bundle. Here and throughout this chapter we use the indices a, b, etc. to refer to the worldvolume of the D-brane, and indices i, j, etc. to refer to the normal bundle. Our notation largely follows that of [68]. We will use indices μ, ν, etc. for the ten-dimensional spacetime. If the brane positions are given by $X^\mu(x^a)$, then we have $(g_T)_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$, and we can pick an orthonormal frame ξ_i^μ for the normal bundle which satisfies $g_{\mu\nu}\xi_i^\mu \xi_j^\nu = \delta_{ij}$

and $g_{\mu\nu}\partial_a X^\mu \xi_i^\nu = 0$.

In order to relate the curvatures R_T and R_N to the ten-dimensional spacetime curvature, we must first introduce the second fundamental form [69],

$$\Omega^i_{ab} = \delta^{ij} g_{\mu\nu}\xi^\mu_j \left(\partial_a\partial_b X^\nu - (\Gamma_T)^c_{ab}\,\partial_c X^\nu + \Gamma^\nu_{\rho\sigma}\partial_a X^\rho \partial_b X^\sigma\right). \tag{4.11}$$

In this expression, $\Gamma^\nu_{\rho\sigma}$ and $(\Gamma_T)^c_{ab}$ are the Christoffel symbols constructed from the spacetime and worldvolume metrics respectively.

We then use the Gauss-Codazzi equations, which state

$$\begin{aligned}
(R_T)_{abcd} &= R_{abcd} + \delta_{ij}\left(\Omega^i_{ac}\Omega^j_{bd} - \Omega^i_{ad}\Omega^j_{bc}\right), \\
(R_N)_{ab}{}^{ij} &= -R_{ab}{}^{ij} + g_T^{cd}\left(\Omega^i_{ac}\Omega^j_{bd} - \Omega^j_{ac}\Omega^i_{bd}\right).
\end{aligned} \tag{4.12}$$

Here we raise and lower indices with $(g_T)_{ab}$ or δ_{ij}, as appropriate, and we pull back indices from spacetime using either $\partial_a X^\mu$ or ξ^μ_i, so

$$R_{abcd} = \partial_a X^\mu \partial_b X^\nu \partial_c X^\rho \partial_d X^\sigma R_{\mu\nu\rho\sigma}, \qquad R_{ab}{}^{ij} = \delta^{ik}\delta^{j\ell}\partial_a X^\mu \partial_b X^\nu \xi^\rho_k \xi^\sigma_\ell R_{\mu\nu\rho\sigma}. \tag{4.13}$$

We will work in a linearized approximation, which means that we expand all of our fields around a flat background and work to leading order in the fluctuations. We do this both to greatly simplify our calculations, and also because these are really the only results that we can realistically compare to the disc amplitudes we compute in section B. Fortunately, this does provide an enormous simplification since the second fundamental form vanishes in the flat background and so must be at least first order in fluctuations, which means that it contributes to R_T and R_N only at second order in the fields or higher. Meanwhile, the spacetime curvature does have a piece which is first order in the fluctuations,

$$R_{\mu\nu\rho\sigma} = -\partial_{\mu[\rho} h_{\sigma]\nu} + \partial_{\nu[\rho} h_{\sigma]\mu} + \mathcal{O}(h^2), \tag{4.14}$$

where we have split the metric into background plus fluctuation, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Thus, to leading order in the fluctuations,

$$\begin{aligned}&\left(X^{(4)}_{\text{original}}\right)_{abcd}\\ &= 12\left(-\partial_{[a}{}^{e}h_{b}{}^{f}\partial_{c|e|}h_{d]f} + \partial_{[a}{}^{e}h_{b}{}^{f}\partial_{c|f|}h_{d]e} + \partial_{[a}{}^{i}h_{b}{}^{j}\partial_{c|i|}h_{d]j} - \partial_{[a}{}^{i}h_{b}{}^{j}\partial_{c|j|}h_{d]i}\right) + \mathcal{O}(h^3).\end{aligned} \tag{4.15}$$

4. T-dualizing the corrections

Now we note that the action so far (to this order in α') is not consistent with T-duality, since

$$\mathcal{L}^{(p+1)}_{WZ} = \frac{\pi^2\left(\alpha'\right)^2}{24}\left(Ce^B\right)^{(p-3)} \wedge X^{(4)}_{\text{original}} \tag{4.16}$$

does not satisfy (4.6). In order to find an action that is consistent with T-duality, we make the following ansatz[6]

$$\begin{aligned}\frac{24}{\pi^2\left(\alpha'\right)^2}\mathcal{L}^{(p+1)}_{a_1\cdots a_{p+1}} \;&=\; \frac{(p+1)!}{4!\,(p-3)!}\left(Ce^B\right)^{(p-3)}_{[a_1\cdots a_{p-3}}X^{(4)}_{a_{p-2}a_{p-1}a_pa_{p+1}]}\\ &\quad+\frac{(p+1)!}{3!\,(p-2)!}\left(Ce^B\right)^{(p-1)}_{[a_1\cdots a_{p-2}|i|}X^{(3)\,i}_{a_{p-1}a_pa_{p+1}]}\\ &\quad+\frac{(p+1)!}{2^2\,(p-1)!}\left(Ce^B\right)^{(p+1)}_{[a_1\cdots a_{p-1}|i_1i_2|}X^{(2)\,i_1i_2}_{a_pa_{p+1}]}\end{aligned} \tag{4.17}$$

We assume that the objects $X^{(n)}$ are built out of NS-NS sector closed string fields[7].

---

[6] The normalizations here are chosen so as to make the T-duality rules in (4.19) simple. In principle we could also include terms with $X^{(1)\,i_1i_2i_3}_a$ and $X^{(0)\,i_1i_2i_3i_4}$, which would in turn correspond to couplings of higher degree forms $C^{(p+3)}$ and $C^{(p+5)}$ to the D-brane. However, it turns out that these couplings do not occur in the T-duality invariants built from $X^{(4)}_{\text{original}}$.

[7] Note that the Buscher rules always preserve the number of R-R fields which appear in an expression, so this Wess-Zumino term does not mix under T-duality with terms that contain no R-R fields, such as DBI, or with terms that contain more than one R-R field.

To impose consistency under T-duality, we must ensure that this ansatz satisfies (4.6), which happens iff

$$X^{(4)\prime}_{a_1a_2a_3a_4} = X^{(4)}_{a_1a_2a_3a_4}, \qquad X^{(3)\prime\, i}_{a_1a_2a_3} = X^{(3)\, i}_{a_1a_2a_3}, \qquad X^{(2)\prime\, i_1i_2}_{a_1a_2} = X^{(2)\, i_1i_2}_{a_1a_2}, \tag{4.18}$$

and[8]

$$X^{(3)\prime\, y}_{a_1a_2a_3} = X^{(4)}_{a_1a_2a_3y}, \qquad X^{(2)\prime\, iy}_{a_1a_2} = X^{(3)\, i}_{a_1a_2y}, \tag{4.19}$$

where a prime means that we have used the Buscher rules to transform the object in question. This ansatz and these consistency conditions should in fact hold even beyond the linearized approximation, though at higher orders we may also have to incorporate open string fields.

Now we would like to build an action which includes the known terms (4.10) but which is consistent with the T-duality rules expressed above. Note that all four of the terms in (4.15) have two of the four antisymmetrized free indices attached to derivatives. The Buscher rules, given our assumption that they are exact to this order in α', will preserve this fact - any terms which can mix with these four terms under T-duality must also have two of the antisymmetrized indices occupied by derivatives. One immediate consequence of this is that we need not consider terms in $X^{(n)}$ which are linear order in NS-NS fluctuations, since in that case all derivatives would be hitting the same field and antisymmetrizing any two derivatives would give zero. This is not to say that terms with only one NS-NS field will not occur (indeed they are expected, see [71]), but simply that they cannot appear in the same T-duality invariant as (4.15). Furthermore, applying the Buscher rules never reduces the number

[8]Here the T-duality transformation swaps an upper y index with a lower y index (though of course at linearized order around a flat background this is irrelevant). This is a frequent feature of T-duality transformations of NS-NS fields and fluxes, such as for example so-called generalized NS-NS fluxes [70].

of fluctuations in a term, so we see that we can restrict ourselves to terms which are quadratic in the fluctuations and we can also restrict ourselves to the linearized version of the Buscher rules,

$$h'_{yy} = -h_{yy}, \qquad h'_{\mu y} = B_{\mu y}, \qquad B'_{\mu y} = h_{\mu y}, \qquad \Phi' = \Phi - \frac{1}{2}h_{yy}, \tag{4.20}$$

with $h_{\mu\nu}$ and $B_{\mu\nu}$ left invariant.

Under these transformations, it is not hard to verify that the terms in (4.15) can only mix with certain terms, which we can enumerate,

$$\begin{aligned}
X^{(4)}_{a_1a_2a_3a_4} &= \alpha_1\partial_{[a_1}{}^b h_{a_2}{}^c\partial_{a_3|b|}h_{a_4]c} + \alpha_2\partial_{[a_1}{}^b h_{a_2}{}^c\partial_{a_3|c|}h_{a_4]b} + \alpha_3\partial_{[a_1}{}^j h_{a_2}{}^k\partial_{a_3|j|}h_{a_4]k} \\
&\quad +\alpha_4\partial_{[a_1}{}^j h_{a_2}{}^k\partial_{a_3|k|}h_{a_4]j} + \alpha_5\partial_{[a_1}{}^b B_{a_2}{}^j\partial_{a_3|b|}B_{a_4]j} + \alpha_6\partial_{[a_1}{}^j B_{a_2}{}^b\partial_{a_3|j|}B_{a_4]b}, \\
X^{(3)\,i}_{a_1a_2a_3} &= \beta_1\partial_{[a_1}{}^b h_{a_2}{}^c\partial_{a_3]b}B^i{}_c + \beta_2\partial_{[a_1}{}^b h_{a_2}{}^c\partial_{a_3]c}B^i{}_b + \beta_3\partial_{[a_1}{}^j h_{a_2}{}^k\partial_{a_3]j}B^i{}_k \\
&\quad +\beta_4\partial_{[a_1}{}^j h_{a_2}{}^k\partial_{a_3]k}B^i{}_j + \beta_5\partial_{[a_1}{}^b h^{ij}\partial_{a_2|b|}B_{a_3]j} + \beta_6\partial_{[a_1}{}^j h^{ib}\partial_{a_2|j|}B_{a_3]b}, \\
X^{(2)\,i_1i_2}_{a_1a_2} &= \gamma_1\partial_{[a_1}{}^b h^{[i_1|j|}\partial_{a_2]b}h^{i_2]}{}_j + \gamma_2\partial_{[a_1}{}^j h^{[i_1|b|}\partial_{a_2]j}h^{i_2]}{}_b + \gamma_3\partial_{[a_1}{}^b B^{[i_1|c|}\partial_{a_2]b}B^{i_2]}{}_c \\
&\quad +\gamma_4\partial_{[a_1}{}^b B^{[i_1|c|}\partial_{a_2]c}B^{i_2]}{}_b + \gamma_5\partial_{[a_1}{}^j B^{[i_1|k|}\partial_{a_2]j}B^{i_2]}{}_k + \gamma_6\partial_{[a_1}{}^j B^{[i_1|k|}\partial_{a_2]k}B^{i_2]}{}_j.
\end{aligned} \tag{4.21}$$

From (4.15) we know that $-\alpha_1 = \alpha_2 = \alpha_3 = -\alpha_4 = 12$, but we would like to use our T-duality constraints to determine the remaining fourteen constants. To proceed, we need to evaluate the expressions above in an ansatz with a circle bundle. For instance, suppose the circle bundle is along the brane, then we would evaluate $X^{(4)}$ as

$$X^{(4)}_{a_1a_2a_3a_4} = \widehat{X}^{(4)}_{a_1a_2a_3a_4} + \alpha_1\partial_{[a_1}{}^{\hat b}h_{a_2|y|}\partial_{a_3|\hat b|}h_{a_4]y} + \alpha_6\partial_{[a_1}{}^j B_{a_2|y|}\partial_{a_3|j|}B_{a_4]y}, \tag{4.22}$$

where hatted indices are summed over all directions along the brane excluding y, and where $\widehat{X}^{(4)}$ represents the expression for $X^{(4)}$ but with y excluded from all sums.

Under T-duality, this expression becomes

$$X^{(4)\prime}_{a_1a_2a_3a_4} = \widehat{X}^{(4)}_{a_1a_2a_3a_4} + \alpha_1\partial_{[a_1}{}^{b}B_{a_2|y|}\partial_{a_3|b|}B_{a_4]y} + \alpha_6\partial_{[a_1}{}^{\hat{\jmath}}h_{a_2|y|}\partial_{a_3|\hat{\jmath}|}h_{a_4]y}. \qquad (4.23)$$

Meanwhile, if the circle bundle is normal to the brane we have

$$X^{(4)}_{a_1a_2a_3a_4} = \widehat{X}^{(4)}_{a_1a_2a_3a_4} + \alpha_3\partial_{[a_1}{}^{\hat{\jmath}}h_{a_2|y|}\partial_{a_3|\hat{\jmath}|}h_{a_4]y} + \alpha_5\partial_{[a_1}{}^{b}B_{a_2|y|}\partial_{a_3|b|}B_{a_4]y}. \qquad (4.24)$$

Comparing (4.23) and (4.24) we learn that $\alpha_1 = \alpha_5$ and $\alpha_6 = \alpha_3$. Similar considerations for $X^{(3)}$ and $X^{(2)}$ show that $\beta_1 = \beta_5$, $\beta_6 = \beta_3$, $\gamma_2 = \gamma_5$, and $\gamma_3 = \gamma_1$.

Next, we also compute

$$\begin{aligned} X^{(4)\prime}_{a_1a_2a_3y} \;=\; & \frac{1}{2}\alpha_1\left(\partial_{[a_1}{}^{b}h_{a_2}{}^{c}\partial_{a_3]b}B_{cy} - \partial_{[a_1}{}^{b}h_{|yy|}\partial_{a_2|b|}B_{a_3]y}\right) + \frac{1}{2}\alpha_2\partial_{[a_1}{}^{b}h_{a_2}{}^{c}\partial_{a_3]c}B_{by} \\ & + \frac{1}{2}\alpha_3\partial_{[a_1}{}^{\hat{\jmath}}h_{a_2}{}^{\hat{k}}\partial_{a_3]\hat{\jmath}}B_{\hat{k}y} + \frac{1}{2}\alpha_4\partial_{[a_1}{}^{\hat{\jmath}}h_{a_2}{}^{\hat{k}}\partial_{a_3]\hat{k}}B_{\hat{\jmath}y} \\ & - \frac{1}{2}\alpha_5\partial_{[a_1}{}^{b}h^{\hat{\jmath}}{}_{y}\partial_{a_2|b|}B_{a_3]\hat{\jmath}} - \frac{1}{2}\alpha_6\partial_{[a_1}{}^{\hat{\jmath}}h^{b}{}_{y}\partial_{a_2|\hat{\jmath}|}B_{a_3]b}, \end{aligned} \qquad (4.25)$$

and

$$\begin{aligned} X^{(3)\,y}_{a_1a_2a_3} \;=\; & -\beta_1\partial_{[a_1}{}^{b}h_{a_2}{}^{c}\partial_{a_3]b}B_{cy} - \beta_2\partial_{[a_1}{}^{b}h_{a_2}{}^{c}\partial_{a_3]c}B_{by} - \beta_3\partial_{[a_1}{}^{\hat{\jmath}}h_{a_2}{}^{\hat{k}}\partial_{a_3]\hat{\jmath}}B_{\hat{k}y} \\ & - \beta_4\partial_{[a_1}{}^{\hat{\jmath}}h_{a_2}{}^{\hat{k}}\partial_{a_3]\hat{k}}B_{\hat{\jmath}y} + \beta_5\left(\partial_{[a_1}{}^{b}h^{\hat{\jmath}}{}_{|y|}\partial_{a_2|b|}B_{a_3]\hat{\jmath}} + \partial_{[a_1}{}^{b}h_{|yy|}\partial_{a_2|b|}B_{a_3]y}\right) \\ & + \beta_6\partial_{[a_1}{}^{\hat{\jmath}}h^{b}{}_{|y|}\partial_{a_2|\hat{\jmath}|}B_{a_3]b}, \end{aligned} \qquad (4.26)$$

from which we deduce that $\beta_1 = -\frac{1}{2}\alpha_1$, $\beta_2 = -\frac{1}{2}\alpha_2$, $\beta_3 = -\frac{1}{2}\alpha_3$, $\beta_4 = -\frac{1}{2}\alpha_4$, $\beta_5 = -\frac{1}{2}\alpha_5 = -\frac{1}{2}\alpha_1$, and $\beta_6 = -\frac{1}{2}\alpha_6 = -\frac{1}{2}\alpha_3$.

A comparison of $X^{(3)\,i\,\prime}_{a_1a_2y}$ and $X^{(2)\,iy}_{a_1a_2}$ then lead us also to $\gamma_1 = -\frac{1}{3}\beta_5 = -\frac{1}{3}\beta_1$, $\gamma_2 = -\frac{1}{3}\beta_6$, $\gamma_3 = -\frac{1}{3}\beta_1$, $\gamma_4 = -\frac{1}{3}\beta_2$, $\gamma_5 = -\frac{1}{3}\beta_3$, and $\gamma_6 = -\frac{1}{3}\beta_4$. Note that all the conditions are self-consistent, and we are left with the result,

$$
\begin{aligned}
X^{(4)}_{a_1a_2a_3a_4} &= 12\Big(-\partial_{[a_1}{}^{b}h_{a_2}{}^{c}\partial_{a_3|b|}h_{a_4]c}+\partial_{[a_1}{}^{b}h_{a_2}{}^{c}\partial_{a_3|c|}h_{a_4]b}+\partial_{[a_1}{}^{j}h_{a_2}{}^{k}\partial_{a_3|j|}h_{a_4]k}\\
&\quad -\partial_{[a_1}{}^{j}h_{a_2}{}^{k}\partial_{a_3|k|}h_{a_4]j}-\partial_{[a_1}{}^{b}B_{a_2}{}^{j}\partial_{a_3|b|}B_{a_4]j}+\partial_{[a_1}{}^{j}B_{a_2}{}^{b}\partial_{a_3|j|}B_{a_4]b}\Big)\,,\\
X^{(3)\,i}_{a_1a_2a_3} &= 6\Big(\partial_{[a_1}{}^{b}h_{a_2}{}^{c}\partial_{a_3]b}B^{i}{}_{c}-\partial_{[a_1}{}^{b}h_{a_2}{}^{c}\partial_{a_3]c}B^{i}{}_{b}-\partial_{[a_1}{}^{j}h_{a_2}{}^{k}\partial_{a_3]j}B^{i}{}_{k}\\
&\quad +\partial_{[a_1}{}^{j}h_{a_2}{}^{k}\partial_{a_3]k}B^{i}{}_{j}+\partial_{[a_1}{}^{b}h^{ij}\partial_{a_2|b|}B_{a_3]j}-\partial_{[a_1}{}^{j}h^{ib}\partial_{a_2|j|}B_{a_3]b}\Big)\,, \qquad (4.27)\\
X^{(2)\,i_1i_2}_{a_1a_2} &= 2\Big(-\partial_{[a_1}{}^{b}h^{[i_1|j|}\partial_{a_2]b}h^{i_2]}{}_{j}+\partial_{[a_1}{}^{j}h^{[i_1|b|}\partial_{a_2]j}h^{i_2]}{}_{b}-\partial_{[a_1}{}^{b}B^{[i_1|c|}\partial_{a_2]b}B^{i_2]}{}_{c}\\
&\quad +\partial_{[a_1}{}^{b}B^{[i_1|c|}\partial_{a_2]c}B^{i_2]}{}_{b}+\partial_{[a_1}{}^{j}B^{[i_1|k|}\partial_{a_2]j}B^{i_2]}{}_{k}-\partial_{[a_1}{}^{j}B^{[i_1|k|}\partial_{a_2]k}B^{i_2]}{}_{j}\Big)\,.
\end{aligned}
$$

Taking into account the factorial factors in (4.17), we see that this result can be written in the form

$$
\begin{aligned}
S_{WZ} \quad &\supset T_p\frac{\pi^2(\alpha')^2}{24}\int_{Dp}dx^{a_1}\wedge\cdots\wedge dx^{a_{p+1}}\\
&\Big\{\frac{1}{2}\frac{1}{(p-3)!}C^{(p-3)}_{a_1\cdots a_{p-3}}(-2\partial_{a_{p-2}}{}^{[b}h_{a_{p-1}}{}^{c]}\partial_{a_pb}h_{a_{p+1}c}+2\partial_{a_{p-2}}{}^{[j}h_{a_{p-1}}{}^{k]}\partial_{a_pj}h_{a_{p+1}k}\\
&-\partial_{a_{p-2}}{}^{b}B_{a_{p-1}}{}^{j}\partial_{a_pb}B_{a_{p+1}j}+\partial_{a_{p-2}}{}^{j}B_{a_{p-1}}{}^{b}\partial_{a_pj}B_{a_{p+1}b})\\
&+\frac{1}{(p-2)!}C^{(p-1)}_{a_1\cdots a_{p-2}i}(2\partial_{a_{p-1}}{}^{[b}h_{a_p}{}^{c]}\partial_{a_{p+1}b}B^{i}{}_{c}-2\partial_{a_{p-1}}{}^{[j}h_{a_p}{}^{k]}\partial_{a_{p+1}j}B^{i}{}_{k}\\
&+\partial_{a_{p-1}}{}^{b}h^{ij}\partial_{a_pb}B_{a_{p+1}j}-\partial_{a_{p-1}}{}^{j}h^{ib}\partial_{a_pj}B_{a_{p+1}b})\\
&+\frac{1}{2}\frac{1}{(p-1)!}C^{(p+1)}_{a_1\cdots a_{p-1}i_1i_2}(-\partial_{a_p}{}^{b}h^{i_1j}\partial_{a_{p+1}b}h^{i_2}{}_{j}+\partial_{a_p}{}^{j}h^{i_1b}\partial_{a_{p+1}j}h^{i_2}{}_{b}\\
&-2\partial_{a_p}{}^{b}B^{i_1c}\partial_{a_{p+1}[b}B^{i_2}{}_{c]}+2\partial_{a_p}{}^{j}B^{i_1k}\partial_{a_{p+1}[j}B^{i_2}{}_{k]})\Big\}. \qquad (4.28)
\end{aligned}
$$

Above action is compatible with linearized T-duality rules, but it is not invariant under either B-field or R-R gauge transformation, even if we restore the terms that depend on gauge field strength F. However, this does not mean action (4.28) is wrong. There could be additional terms in $X^{(4)}$, which map to themselves under the T-duality transformation, and these new terms can combine with the terms in action action (4.28) to give an action with good property. Starting from new section, we will do string disc amplitude computation to obtain the additional terms to action (4.28).

B. String disc amplitude

In this section we compute the three-point function involving one RR field $C^{(p-3)}$ and 2 NS-NS fields in the present of one Dp-brane. When one of the NS-NS field is symmetric and the other NS-NS field is antisymmetric, the amplitude vanishes because of symmetry. This also can be checked through explicit string disc amplitude computation. When both NS-NS fields are gravitons, the disc amplitude are well known [72, 73, 74],

$$\mathcal{L}_{CGG} = T_p \frac{\pi^2(\alpha')^2}{12(p-3)!}\epsilon^{a_1\cdots a_{p+1}} C^{(p-3)}_{a_1\cdots a_{p-3}} \left[\partial_{a_{p-2}}{}^{[j} h_{a_{p-1}}{}^{k]} \partial_{a_p j} h_{a_{p+1}k} - \partial_{a_{p-2}}{}^{[b} h_{a_{p-1}}{}^{c]} \partial_{a_p b} h_{a_{p+1}c}\right] \tag{4.29}$$

Here and throughout this chapter we use the indices a, b, etc. to refer to the world-volume of the D-brane, and indices i, j, etc. to refer to the normal bundle.

What interests us most is the case that both two NS-NS fields are antisymmetric. In this section, we will put much effort to compute the complete disc amplitude. In the following subsection, we start with a short summary of the basic conventions we will use throughout this chapter.

1. Basic conventions

On the upper half-plane, the holomorphic fields have OPEs among themselves[9]

$$\begin{aligned} X^\mu(z)X^\nu(w) &\sim -\eta^{\mu\nu}\log(z-w), \\ \psi^\mu(z)\psi^\nu(w) &\sim \frac{\eta^{\mu\nu}}{z-w}, \\ \phi(z)\phi(w) &\sim -\log(z-w), \end{aligned} \tag{4.30}$$

with similar expressions for the antiholomorphic fields. Because of the boundary, representing the D-brane, there are also non-trivial OPEs between holomorphic and antiholomorphic fields,

$$\begin{aligned} X^\mu(z)\widetilde{X}^\nu(\bar{w}) &\sim -D^{\mu\nu}\log(z-\bar{w}), \\ \psi^\mu(z)\widetilde{\psi}^\nu(\bar{w}) &\sim \frac{D^{\mu\nu}}{z-\bar{w}}, \\ \phi(z)\widetilde{\phi}(\bar{w}) &\sim -\log(z-\bar{w}). \end{aligned} \tag{4.31}$$

Here the matrix $D^{\mu\nu}$ is a diagonal matrix that agrees with $\eta^{\mu\nu}$ in directions along the brane (Neumann boundary conditions) and with $-\eta^{\mu\nu}$ in directions normal to the brane (Dirichlet boundary conditions). In our previous notation, $D^{ab} = \eta^{ab}$, $D^{ij} = -\delta^{ij}$, $D^{ai} = 0$. Using $\eta_{\mu\nu}$ to raise or lower indices, then we have $D^\mu{}_\rho D^\rho{}_\nu = \delta^\mu_\nu$. One can now use a convenient trick [75, 76] when computing amplitudes. One can make the replacements

$$\widetilde{X}^\mu(\bar{z}) \to D^\mu{}_\nu X^\nu(\bar{z}), \qquad \widetilde{\psi}(\bar{z}) \to D^\mu{}_\nu \psi^\nu(\bar{z}), \qquad \widetilde{\phi}(\bar{z}) \to \phi(\bar{z}), \tag{4.32}$$

and then use only the holomorphic OPEs (4.30), but where we now regard z and $\bar{z}$ as independent insertion points.

In order to construct R-R vertex operators, we will also need spin fields $S_A(z)$

[9] In this section we will mostly work in units where $\alpha' = 2$, and the OPE for ψ^μ differs from [12] by a sign.

and $\widetilde{S}_B(\bar{z})$, where A and B are spinor indices. Rather than give the individual OPEs involving spin fields, it will suffice to quote the general fermion sector expectation values that we will need[10],

$$\begin{aligned}\Big\langle S_A(z)\widetilde{S}_B(\bar{z})\psi^{\mu_1}(z_1)\dots\psi^{\mu_n}(z_n)\Big\rangle &= \frac{1}{2^{n/2}}\frac{(z-\bar{z})^{n/2-5/4}}{\sqrt{(z_1-z)(z_1-\bar{z})\dots(z_n-z)(z_n-\bar{z})}} \\ \times\Big[(\Gamma^{\mu_n\dots\mu_1}\mathcal{C}^{-1}M^T)_{AB} &+ \psi^{\mu_1}\widehat{(z_1)\psi^{\mu_2}}(z_2)(\Gamma^{\mu_n\dots\mu_3}\mathcal{C}^{-1}M^T)_{AB}\pm\dots \\ &+\psi^{\mu_1}\widehat{(z_1)\psi^{\mu_2}}(z_2)\psi^{\mu_3}\widehat{(z_3)\psi^{\mu_4}}(z_4)(\Gamma^{\mu_n\dots\mu_5}\mathcal{C}^{-1}M^T)_{AB}\pm\dots\Big],\end{aligned} \tag{4.33}$$

where

$$\psi^{\mu_i}\widehat{(z_i)\psi^{\mu_j}}(z_j) = \eta^{\mu_i\mu_j}\frac{(z_i-z)(z_j-\bar{z})+(z_j-z)(z_i-\bar{z})}{(z_i-z_j)(z-\bar{z})}. \tag{4.34}$$

In these expressions we use real symmetric 32×32 gamma matrices $(\Gamma^\mu)_A{}^B$ which satisfy

$$\{\Gamma^\mu,\Gamma^\nu\} = 2\eta^{\mu\nu}, \tag{4.35}$$

$\mathcal{C}^{AB}$ is an antisymmetric charge conjugation matrix, and $M_A{}^B$ encodes the Neumann and Dirichlet boundary conditions as they are realized on spinor indices, so that it satisfies $\Gamma^\mu M = D^\mu{}_\nu M\Gamma^\nu$. It is explicitly given by

$$M = \begin{cases} \pm\frac{i}{(p+1)!}\,(\varepsilon^v)_{a_0\cdots a_p}\,\Gamma^{a_0}\cdots\Gamma^{a_p}, & \text{for } p \text{ even},\\ \pm\frac{1}{(p+1)!}\,(\varepsilon^v)_{a_0\cdots a_p}\,\Gamma^{a_0}\cdots\Gamma^{a_p}\Gamma_{11}, & \text{for } p \text{ odd},\end{cases} \tag{4.36}$$

where ε^v is the epsilon tensor on the brane worldvolume and where

$$\Gamma_{11} = \frac{1}{10!}\varepsilon_{\mu_0\cdots\mu_9}\Gamma^{\mu_0}\cdots\Gamma^{\mu_9} = \Gamma^0\cdots\Gamma^9. \tag{4.37}$$

We will not be attempting to compute the overall normalization of our result (as opposed to relative phases, which will of course be crucial), so we can freely ignore

[10] A similar expression appears in [77], though their result restricts to fermions on the boundary of the disc. We need the more general result shown here.

the ± 1 or $\pm i$ in the definition of M.

The tree level string amplitude (see Figure 1) is given by

$$\mathcal{A}^{string}_{CBB} =< V_C^{(-\frac{1}{2},-\frac{1}{2})}(p_1)V_B^{(-1,0)}(\varepsilon_2,p_2)V_B^{(0,0)}(\varepsilon_3,p_3) > \tag{4.38}$$

The two vertex operators for two B-fields are not in the same picture, so the above string amplitude don't enjoy the manifest symmetry under the exchange of two B-fields, and being able to write the final result symmetrically is a very useful way to control the error. The vertex operators in above amplitude are

$$\begin{aligned}
V_C^{(-\frac{1}{2},-\frac{1}{2})} &= (\mathcal{C}P_+\not{F}^{(p-2)})^{AB}\int d^2z_1 e^{-\frac{1}{2}\phi}S_A e^{ip_1X}(z_1) : e^{-\frac{1}{2}\phi}\widetilde{S}_B e^{ip_1DX}(\bar{z}_1) \\
V_B^{(-1,0)} &= (\varepsilon_2 D)_{\mu\nu}\int d^2z_2 e^{-\phi}\psi^\mu e^{ip_2X}(z_2) : (\partial X^\nu - ip_2D\psi\psi^\nu)e^{ip_2DX}(\bar{z}_2) \\
V_B^{(0,0)} &= (\varepsilon_3 D)_{\mu\nu}\int d^2z_3 (\partial X^\mu - ip_3\psi\psi^\mu)e^{ip_3X}(z_3) : (\partial X^\nu - ip_3D\psi\psi^\nu)e^{ip_3DX}(\bar{z}_3)
\end{aligned} \tag{4.39}$$

One also can use the R-R vertex operator in (-3/2,-1/2) picture [77, 78]

$$V^{(-3/2,-1/2)} = (\mathcal{C}P_-\not{C})^{AB}\int d^2z_1 e^{-\frac{3}{2}\phi}e^{ip_1X}S_A(z_1) : e^{-\frac{1}{2}\phi}e^{ip_1DX}\widetilde{S}_B(\bar{z}_1), \tag{4.40}$$

as long as the total picture charge of all three vertex operators equals to -2. Because the whole disc amplitude is complicated and it is difficult to keep track of all terms at once, which is especially true when we compare it with supergravity amplitude, we want to separate the amplitude into five pieces,

$$\mathcal{A}^{string}_{CBB} = \mathcal{A}^{string}_1 + \mathcal{A}^{string}_2 + \mathcal{A}^{string}_3 + \mathcal{A}^{string}_4 + \mathcal{A}^{string}_5 \tag{4.41}$$

according to different index structures and list these $\mathcal{A}^{string}_n$ in the following.

1. $(\varepsilon_2 \cdot p)(\varepsilon_3 \cdot p)$ and $(\varepsilon_2 \cdot \varepsilon_3)$ terms

   Sum of the terms proportional to either $(\varepsilon_2 \cdot p)(\varepsilon_3 \cdot p)$ or $(\varepsilon_2 \cdot \varepsilon_3)$ for arbitrary polarization ε_2 and ε_3 equals to

$$\begin{aligned}
\mathcal{A}_1^{string} &= \frac{i}{2\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}\times\\
&\Big[(p_2p_3)(\varepsilon_2D\varepsilon_3)_{\beta_3\beta_4}I_0-(p_2Dp_3)(\varepsilon_2\varepsilon_3)_{\beta_3\beta_4}I_0+(p_2D\varepsilon_2)_{\beta_3}(p_3D\varepsilon_3)_{\beta_4}I_3\\
&-(p_2D\varepsilon_2)_{\beta_3}(p_2D\varepsilon_3)_{\beta_4}I_7+(p_3D\varepsilon_2)_{\beta_3}(p_2\cdot\varepsilon_3)_{\beta_4}I_8\\
&-(p_2D\varepsilon_2)_{\beta_3}(p_1N\varepsilon_3)_{\beta_4}I_4-(p_3D\varepsilon_2)_{\beta_3}(p_1N\varepsilon_3)_{\beta_4}I_5\\
&-(p_2D\varepsilon_2)_{\beta_3}(p_2\cdot\varepsilon_3)_{\beta_4}I_6+(p_3\cdot\varepsilon_2)_{\beta_3}(p_1N\varepsilon_3)_{\beta_4}I_9\\
&+(p_1N\varepsilon_2)_{\beta_3}(p_1N\varepsilon_3)_{\beta_4}I_{10}\Big]+\Big[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Big]
\end{aligned}\tag{4.42}$$

In this amplitude, I_n are integrals, whose definition and value at small momentum limit can be found in the appendix A. In appendix B, we compute the integral I_{10} in much detail to illuminate the method we use to evaluate all other integrals for small momentum expansion.

2. $(p\cdot\varepsilon\cdot p)(\varepsilon)$ term

The sum of the terms proportional to $(p\cdot\varepsilon\cdot p)(\varepsilon)$ for arbitrary polarization ε_2 and ε_3 equals to

$$\begin{aligned}
\mathcal{A}_2^{string} &= \frac{i}{4\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_4a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}(\varepsilon_2)_{\mu_3\mu_4}\times\\
&\Big[(p_2\varepsilon_3Dp_3)I'_6+(p_2D\varepsilon_3Dp_3)I'_7+(p_2D\varepsilon_3Np_1)I_5-(p_2\cdot\varepsilon_3Np_1)I_9\\
&+(p_2\varepsilon_3Dp_2)(I_8-2I_0)\Big]+\Big[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Big]
\end{aligned}\tag{4.43}$$

where $I'_n=I_n(p_2\leftrightarrow p_3)$.

3. $(\varepsilon\cdot p)(\varepsilon)$ term

Depending on weather all the polarization of R-R field potential C^{p-3} is along the brane direction or not, all terms proportional to $(\varepsilon\cdot p)(\varepsilon)$ for arbitrary polarization ε_2 and ε_3 can be separated into two parts:

$$
\begin{aligned}
\mathcal{A}_3^{string} \;=\; & \frac{-i}{8\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta\mu_3\mu_4\mu_6 a_1\cdots a_{p-3}} C_{a_1\cdots a_{p-3}}(p_1)_\beta(\varepsilon_2)_{\mu_3\mu_4}\times \\
& \Big[(p_2\cdot p_3)(p_2\cdot\varepsilon_3)_{\mu_6} I_2 - (p_2Dp_3)(p_2D\varepsilon_3)_{\mu_6} I_1 + 2(p_2Dp_3)(p_2\cdot\varepsilon_3)_{\mu_6} I_0 \\
& -2(p_2\cdot p_3)(p_2D\varepsilon_3)_{\mu_6} I_0 - (p_2Dp_3)(p_2\cdot\varepsilon_3)_{\mu_6} I_8 + (p_2\cdot p_3)(p_2D\varepsilon_3)_{\mu_6} I_8 \\
& +2(p_2Dp_3)(p_3D\varepsilon_3)_{\mu_6} I_7' + 2(p_2\cdot p_3)(p_3D\varepsilon_3)_{\mu_6} I_6' \\
& -(p_3Dp_3)(p_2\cdot\varepsilon_3)_{\mu_6} I_6' - (p_3Dp_3)(p_2D\varepsilon_3)_{\mu_6} I_7' \\
& -2(p_2\cdot p_3)(p_1N\varepsilon_3)_{\mu_6} I_9 + 2(p_2Dp_3)(p_1N\varepsilon_3)_{\mu_6} I_5\Big] \\
& +\frac{-i}{8\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta\mu_4\mu_5\mu_6 a_1\cdots a_{p-3}} C_{a_1\cdots a_{p-3}}(p_1)_\beta(\varepsilon_3)_{\mu_5\mu_6}\times \\
& \Big[(p_2\cdot p_3)(p_3\cdot\varepsilon_2)_{\mu_4} I_2 - (p_2Dp_3)(p_3D\varepsilon_2)_{\mu_4} I_1 + 2(p_2Dp_3)(p_3\cdot\varepsilon_2)_{\mu_4} I_0 \\
& -2(p_2\cdot p_3)(p_3D\varepsilon_2)_{\mu_4} I_0 - (p_3Dp_3)(p_1N\varepsilon_2)_{\mu_4} I_4' - (p_3Dp_3)(p_3\cdot\varepsilon_2)_{\mu_4} I_6' \\
& -(p_3Dp_3)(p_3D\varepsilon_2)_{\mu_4} I_7' + (p_2\cdot p_3)(p_3D\varepsilon_2)_{\mu_4} I_8 \\
& -(p_2Dp_3)(p_3\cdot\varepsilon_2)_{\mu_4} I_8 + 2(p_3Dp_3)(p_2D\varepsilon_2)_{\mu_4} I_3\Big] \\
& +\frac{i}{4\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta\mu_3\mu_4\mu_6 a_1\cdots a_{p-3}} C_{a_1\cdots a_{p-3}}(p_3)_\beta\varepsilon_{\mu_3\mu_4}\times \qquad (4.44) \\
& \Big[(p_1Np_3)(p_2\cdot\varepsilon_3) I_9 - (p_1Np_3)(p_2D\varepsilon_3) I_5 + (p_1Np_2)(p_3D\varepsilon_3) I_4' \\
& +(p_1Np_2)(p_2\cdot\varepsilon_3) I_9 - 2(p_1Np_2)(p_1N\varepsilon_3) I_{10} + (p_1Np_2)(p_2D\varepsilon_3) I_5\Big] \\
& +\frac{i}{4\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta\mu_4\mu_5\mu_6 a_1\cdots a_{p-3}} C_{a_1\cdots a_{p-3}}(p_3)_\beta\varepsilon_{\mu_5\mu_6}\times \\
& \Big[(p_1Np_2)(p_3\cdot\varepsilon_2) I_9 + (p_1Np_2)(p_3D\varepsilon_2) I_5 + 2(p_1Np_3)(p_1N\varepsilon_2) I_{10} \\
& -(p_1Np_3)(p_3D\varepsilon_2) I_5 + (p_1Np_3)(p_3\cdot\varepsilon_2) I_9 - (p_1Np_3)(p_2D\varepsilon_2) I_4\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_4^{string} \;=\; & \frac{i}{4\sqrt{2}}\frac{p-3}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_4\mu_6 a_2\cdots a_{p-3}} C_{ia_2\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}(\varepsilon_2)_{\mu_3\mu_4}\times \\
& \Big[p_3^i(p_2\cdot\varepsilon_3) I_9 - p_3^i(p_2D\varepsilon_3) I_5 - 2p_2^i(p_1N\varepsilon_3) I_{10} + p_2^i(p_2\cdot\varepsilon_3) I_9
\end{aligned}
$$

$$+p_2^i(p_2D\varepsilon_3)I_5+p_2^i(p_3D\varepsilon_3)I_4'\Big]+\Big[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Big]. \qquad (4.45)$$

As we use asymmetric vertex operator for two B-fields, it is not unexpected that $\mathcal{A}_3^{string}$ appears asymmetric under the exchange of $p_2 \leftrightarrow p_3$, and $\varepsilon_2 \leftrightarrow \varepsilon_3$. However the integrals I_n in above amplitudes are not independent, and they satisfy the following identities,

$$\begin{aligned}
&(p_2\cdot p_3)I_6'+(p_2Dp_3)I_7'+(p_1Np_2)I_4'-(p_2Dp_2)I_3'=0\\
&(p_3Dp_3)I_4'-4(p_1Np_3)I_{10}+2(p_2Dp_3)I_5+2(p_2\cdot p_3)I_9=0 \qquad (4.46)\\
&2(p_2\cdot p_3)I_8-(p_3Dp_3)I_7'+(p_2Dp_2)I_7+2(p_1Np_2-p_1Np_3)I_5=0\\
&(p_3Dp_3)I_6'-2(p_1Np_3+p_1Np_2)I_9+2(p_2Dp_3)I_8-(p_2Dp_2)I_6=0
\end{aligned}$$

which can be checked using our expression of these integrals at appendix A. After using these identities, one can rewrite A_3^{string} in a symmetric form,

$$\begin{aligned}
\mathcal{A}_3^{string} \;=\;& \frac{i}{8\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta\mu_3\mu_4\mu_6a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_\beta(\varepsilon_2)_{\mu_3\mu_4}\times\\
&\Big[(p_2\cdot p_3)(p_2\cdot\varepsilon_3)_{\mu_6}I_2-(p_2Dp_3)(p_2D\varepsilon_3)_{\mu_6}I_1+2(p_2Dp_3)(p_2\cdot\varepsilon_3)_{\mu_6}I_0\\
&-2(p_2\cdot p_3)(p_2D\varepsilon_3)_{\mu_6}I_0-(p_2Dp_3)(p_2\cdot\varepsilon_3)_{\mu_6}I_8+(p_2\cdot p_3)(p_2D\varepsilon_3)_{\mu_6}I_8\\
&+2(p_2Dp_3)(p_3D\varepsilon_3)_{\mu_6}I_7'+2(p_2\cdot p_3)(p_3D\varepsilon_3)_{\mu_6}I_6'\\
&-(p_3Dp_3)(p_2\cdot\varepsilon_3)_{\mu_6}I_6'-(p_3Dp_3)(p_2D\varepsilon_3)_{\mu_6}I_7'\\
&-2(p_2\cdot p_3)(p_1N\varepsilon_3)_{\mu_6}I_9+2(p_2Dp_3)(p_1N\varepsilon_3)_{\mu_6}I_5\Big]\\
&+\frac{i}{8\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta\mu_4\mu_5\mu_6a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_\beta(\varepsilon_3)_{\mu_5\mu_6}\times\\
&\Big[(p_2\cdot p_3)(p_3\cdot\varepsilon_2)_{\mu_4}I_2-(p_2Dp_3)(p_3D\varepsilon_2)_{\mu_4}I_1+2(p_2Dp_3)(p_3\cdot\varepsilon_2)_{\mu_4}I_0\\
&-2(p_2\cdot p_3)(p_3D\varepsilon_2)_{\mu_4}I_0-(p_3Dp_3)(p_1N\varepsilon_2)_{\mu_4}I_4'-(p_3Dp_3)(p_3\cdot\varepsilon_2)_{\mu_4}I_6'\\
&-(p_3Dp_3)(p_3D\varepsilon_2)_{\mu_4}I_7'+(p_2\cdot p_3)(p_3D\varepsilon_2)_{\mu_4}I_8-(p_2Dp_3)(p_3\cdot\varepsilon_2)_{\mu_4}I_8\\
&+2(p_3Dp_3)(p_2D\varepsilon_2)_{\mu_4}I_3\Big]+\Big[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Big] \qquad (4.47)
\end{aligned}$$

4. $(\varepsilon_2)(\varepsilon_3)$ term

Sum of the terms proportional to $(\varepsilon_2)_{\beta_1\beta_2}(\varepsilon_3)_{\beta_3\beta_4}$ for arbitrary polarization ε_2 and ε_3 equals to

$$\begin{aligned}
\mathcal{A}_5^{string} &= \frac{i}{16\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\mu_3\mu_4\mu_5\mu_6 a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6}\times \\
&\Big[2(p_2Dp_3)(p_1Np_3)I_5 - 2(p_2p_3)(p_1Np_3)I_9 - (p_3Dp_3)(p_1Np_2)I_4'\Big] \\
&+\frac{i}{16\sqrt{2}}\frac{p-3}{(p-2)!}\epsilon^{\mu_3\mu_4\mu_5\mu_6\beta a_2\cdots a_{p-3}}C_{ia_2\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6}\times \\
&\Big[p_1^\beta p_2^i(p_3Dp_3)I_4' + 2p_1^\beta p_3^i(p_2\cdot p_3)I_9 - 2p_1^\beta p_3^i(p_2Dp_3)I_5 \\
&-4p_3^\beta p_3^i(p_1Np_2)I_{10} + 4p_3^\beta p_2^i(p_1Np_3)I_{10}\Big] \\
&+\frac{i}{4\sqrt{2}}\frac{(p-3)(p-4)}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_4\mu_5\mu_6 a_3\cdots a_{p-3}}C_{ija_3\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6} \\
&\times(p_2)_{\beta_1}(p_3)_{\beta_2}p_3^i p_2^j I_{10}
\end{aligned}\tag{4.48}$$

Because integrals I_n satisfy the identities,

$$2(p_2Dp_3)I_5 - 2(p_2\cdot p_3)I_9 + (p_2Dp_2)I_4 - 4(p_1Np_2)I_{10} = 0 \tag{4.49}$$

$$\begin{aligned}
&2(p_2Dp_3)(p_1Np_3)I_5 - 2(p_2p_3)(p_1Np_3)I_9 - (p_3Dp_3)(p_1Np_2)I_4' \\
&= (p_2Dp_3)^2I_1 - (p_2\cdot p_3)^2I_2 - (p_2Dp_2)(p_3Dp_3)I_3
\end{aligned}\tag{4.50}$$

one can rewrite $\mathcal{A}_5^{string}$ as

$$\begin{aligned}
\mathcal{A}_5^{string} &= \frac{i}{16\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\mu_3\mu_4\mu_5\mu_6 a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6}\times \\
&\Big[(p_2Dp_3)^2I_1 - (p_2\cdot p_3)^2I_2 - (p_2Dp_2)(p_3Dp_3)I_3\Big] \\
&+\frac{i}{16\sqrt{2}}\frac{p-3}{(p-2)!}\epsilon^{\mu_3\mu_4\mu_5\mu_6\beta a_2\cdots a_{p-3}}C_{ia_2\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6}\times \\
&\Big[2p_3^\beta p_2^i(p_2Dp_3)I_5 + 2p_3^\beta p_2^i(p_2\cdot p_3)I_9 - p_2^\beta p_2^i(p_3Dp_3)I_4'
\end{aligned}$$

$$+2p_2^\beta p_3^i(p_2Dp_3)I_5-2p_2^\beta p_3^i(p_2\cdot p_3)I_9-p_3^\beta p_3^i(p_2Dp_2)I_4\Big]$$
$$+\frac{i}{4\sqrt{2}}\frac{(p-3)(p-4)}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_4\mu_5\mu_6a_3\cdots a_{p-3}}C_{ija_3\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}$$
$$\times(\varepsilon_3)_{\mu_5\mu_6}(p_2)_{\beta_1}(p_3)_{\beta_2}p_3^ip_2^jI_{10}, \tag{4.51}$$

In the above expression, integrals I_1, I_2, I_3, and I_5 are symmetric, but I_9 is anti-symmetric under the exchange $p_2 \leftrightarrow p_3$, so $\mathcal{A}_5^{string}$ is symmetric under the exchange of two B-fields.

In the appendix D and E, we have evaluated all integrals I_n to α'^0 order, which means that we have expanded the string amplitude $\mathcal{A}_{CBB}^{string}$ to α'^2 order. In the next section, we will compute the supergravity interpretation of this string amplitude by evaluating the corresponding Feynamn diagrams to α'^2 order.

C. Supergravity interpretation

At the low energy limit, our string amplitude $\mathcal{A}_{CBB}^{string}$ (see Figure 1) can be substituted by six supergravity Feynman diagrams shown in Figure 2. What really interests us is the amplitude for Figure 2f), which represent the contact interaction among one R-R field and two B-fields on D-brane. Once we evaluate the amplitude of first five Feynman diagrams of the Figure 2, we can obtain the amplitude of Figure 2f) by subtracting the amplitudes of the first five diagrams in Figure 2 from the string amplitude.

Now the challenge is to compute the amplitude for supergravity diagrams to order α'^2. To achieve this, we first need to obtain the α'^2 corrections of all vertices that appear in these diagrams. Even though all vertices in the bulk are derived from the 10-dimension supergravity action, which has no correction at order α'^2, three vertices on the D-brane (see Figure 3) do receive correction at this order. In the

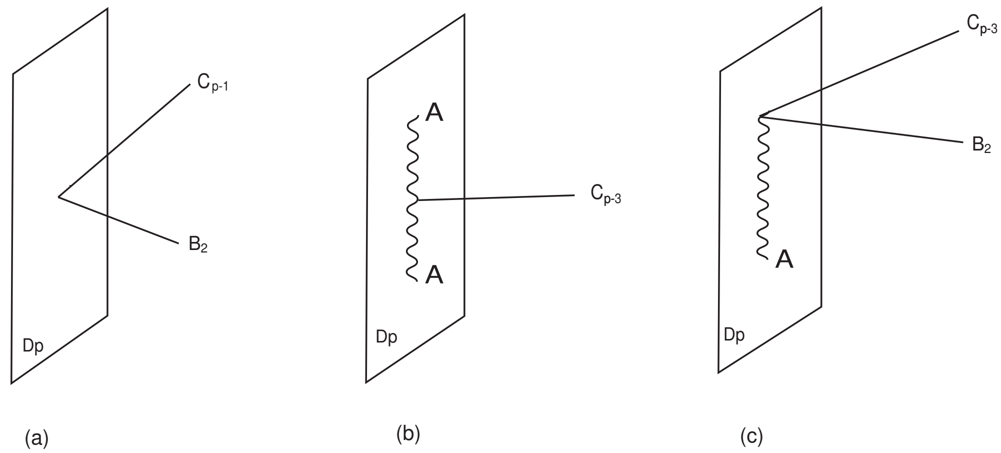

Fig. 3. Three brane vertices with higher derivative corrections

subsection 3.1, we first compute the α'^2 corrections for the vertex in Figure 3a), and then evaluate the amplitude for Figure 2a), 2b) and 2c) to order α'^2. In subsection 3.2 and 3.3, we compute the amplitude for Figure 2d) and 2e) respectively, after obtaining the higher order correction for the vertices in Figure 3b) and 3c). Finally, in subsection 3.4, we write down the amplitude for Figure 2f), so that the sum of the amplitudes of all the Feynman diagrams in Figure 2 reproduces the string amplitude $\mathcal{A}^{string}_{CBB}$.

1. Amplitude for diagram 2a), 2b), and 2c)

To compute the higher order correction of the coupling in Figure 3a), we follow the similar strategy that we want to use to compute the coupling in Figure 3f). The string disc amplitude with insertions of one R-R and one NS-NS B-field vertex operators equals to

$$\begin{aligned}
\mathcal{A}_{BC}^{string} &= < V^{(-\frac{1}{2},-\frac{1}{2})}(p_1)V_B^{(-1,0)}(p_2,\varepsilon) > \\
&= \frac{T_p}{(p-1)!\times 2}\frac{\Gamma[1+p_2Dp_2]\Gamma[1+\frac{(p_1+p_2)^2}{2}]}{\Gamma[1+p_2Dp_2+\frac{(p_1+p_2)^2}{2}]}\epsilon^{\nu_1\cdots\nu_{p-1}\mu\nu}\Big[C^{(p-1)}_{\nu_1\cdots\nu_{p-1}} \\
&\Big(\frac{2}{(p_1+p_2)^2}(p_2D\varepsilon)_\mu(p_2)_\nu+\frac{2}{p_2Dp_2}(p_2D\varepsilon)_\mu(p_2)_\nu-\frac{2}{(p_1+p_2)^2}(p_1\varepsilon)_\mu(p_2)_\nu \\
&+(1+\frac{p_2Dp_2-p_1\cdot p_2}{(p_1+p_2)^2})\varepsilon_{\mu\nu}\Big)-\frac{(p-1)}{(p_1+p_2)^2}C_{\nu_1\cdots\nu_{p-2}\beta}(p_2)_{\nu_{p-1}}(Dp_2)^\beta\varepsilon_{\mu\nu}\Big]
\end{aligned} \tag{4.52}$$

This string amplitude should be replaced by the three Feynman diagrams in Figure 4 at the low momentum limit. The supergravity amplitude for the Figure 4a) and 4b)

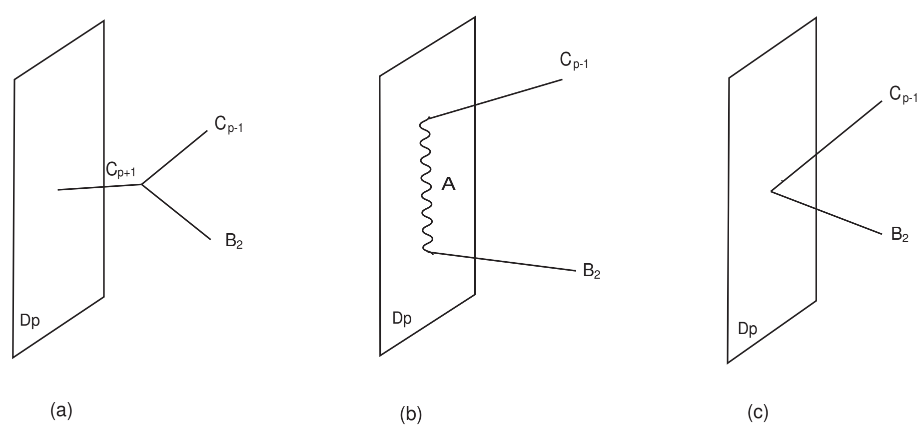

Fig. 4. Three supergravity Feynman diagrams that replace string amplitude $\mathcal{A}_{BC}^{string}$ at low energy.

are

$$\begin{aligned}
A_{BC}^{(a)} &= \frac{-T_p}{(p-1)!\times 4}\epsilon^{a_1\cdots a_{p+1}}\Big[\frac{2(p-1)}{(p_1+p_2)^2}C_{a_1\cdots a_{p-2}i}\varepsilon_{a_{p-1}a_p}p_{1\,a_{p+1}}p_2^i-C_{a_1\cdots a_{p-1}} \\
&\Big((-1+\frac{p_2Dp_2}{(p_1+p_2)^2})\varepsilon_{a_pa_{p+1}}-\frac{4}{(p_1+p_2)^2}(2\varepsilon_{a_pb}p_{1\,a_{p+1}}p_1^b+\varepsilon_{a_pi}p_{1\,a_{p+1}}p_1^i)\Big)\Big]
\end{aligned} \tag{4.53}$$

and

$$A^{(b)}_{BC} = \frac{T_p}{(p-1)!}\frac{2}{p_2Dp_2}\epsilon^{a_1\cdots a_{p+1}}C_{a_1\cdots a_{p-1}}\varepsilon_{ba_p}p_{1\,a_{p+1}}p_1^b \tag{4.54}$$

After subtracting the supergravity amplitudes $A^{(a)}_{BC}$ and $A^{(b)}_{BC}$ from the string amplitude $\mathcal{A}^{string}_{BC}$, we obtain the supergravity amplitude $A^{(c)}_{BC}$ for Figure 4c), and it can be derived from following action:

$$\begin{aligned}
\mathcal{L}_{BC} \;=\; & \frac{T_p}{(p-1)!\times 2}\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}}B_{\beta_1\beta_2}C^{(p-1)}_{\nu_1\cdots\nu_{p-1}} \\
& -\frac{T_p}{(p-1)!\times 4}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}}\nabla^a{}_aH_{\beta_1\beta_2 i}\nabla^iC^{(p-1)}_{\nu_1\cdots\nu_{p-1}} \\
& +\frac{T_p}{(p-1)!\times 2}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}}\nabla^a{}_iH_{\beta_1\beta_2 a}\nabla^iC^{(p-1)}_{\nu_1\cdots\nu_{p-1}} \\
& +\frac{T_p}{(p-2)!\times 12}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\,\nu_2\cdots\nu_{p-1}}\nabla^{ia}{}_aH_{\beta_1\beta_2\beta_3}C^{(p-1)}_{i\,\nu_2\cdots\nu_{p-1}} \\
& +\frac{T_p}{(p-1)!\times 4}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}}\nabla^aH_{\beta_1\beta_2 a}\nabla^\mu{}_\mu C^{(p-1)}_{\nu_1\cdots\nu_{p-1}}
\end{aligned} \tag{4.55}$$

to the order α'^2. This clarifies a confusion regarding the string theory amplitude computation of the $\int C\wedge B$ coupling mentioned in [72, 79]. In the above action, we have used the notation $I_0 = -\pi^4/3$, and Taylor expansion

$$\frac{\Gamma[1+p_2Dp_2]\Gamma[1+\frac{(p_1+p_2)^2}{2}]}{\Gamma[1+p_2Dp_2+\frac{(p_1+p_2)^2}{2}]} = 1-\frac{\pi^2}{12}(p_1+p_2)^2p_2Dp_2+O[\alpha'^3] \tag{4.56}$$

We would like to make a few comments before we proceed:

1) The Feynamn diagrams 2a), 2b), and 2c) can be constructed from the three diagrams of Figure 4 by adding the same $C^{(p-1)}$ field propagator and vertex from $|C^{p-1}+H\wedge C^{p-3}|^2$ term of 10d action. So we would like to compute the total amplitude for diagrams 2a), 2b), and 2c) by using $\mathcal{A}^{string}_{BC}$ directly, rather than from the low energy effective action.

2) String amplitude is evaluated on-shell, which means it does not determine the off-shell action, where we are not allowed to set $p_1^2=0$. When we evaluate the

amplitudes of diagrams 2a), 2b), and 2c), factor p_1^2 leads to $(p_1 + p_3)^2$ which is not zero on-shell. So to keep p_1^2 or not in Eq.(4.52) will affect the amplitudes of diagrams 2a), 2b), and 2c).

3) There are also other on-shell condition like $p_2^2 = 0$, $p_2^\mu \varepsilon_{\mu\nu} = 0$, and $p_1^\mu C^{p-1}_{\mu\nu_2\cdots\nu_{p-1}} = 0$, however these conditions do not change the amplitudes of diagrams 2a), 2b), and 2c) on-shell, so we don't bother to discuss them here, as long as our purpose is to reproduce $\mathcal{A}^{string}_{CBB}$.

4) We will use the expression of $\mathcal{A}^{string}_{BC}$ in Eq.(4.52), without imposing on-shell condition $p_1^2 = 0$, to compute the supergravity amplitude of diagrams 2a), 2b), and 2c). This means we also should not impose this condition when we derive $\mathcal{L}_{BC}$, so we end up with a term proportional to p_1^2 in the expression of $\mathcal{L}_{BC}$. So at this moment, we only make a consistent choice about keeping terms with p_1^2 factor, and this does not remove the ambiguity of the terms that include a factor p_1^2. We will turn to this issue later.

5) We will see that the amplitudes of diagrams 2a), 2b), and 2c) after using $\mathcal{A}^{string}_{BC}$ in Eq.(4.52), have already reproduced all the terms with $1/p_1 \cdot p_2$, $1/p_1 \cdot p_3$, and $1/(p_1 + p_2 + p_3)^2$ poles in string amplitude $\mathcal{A}^{string}_{CBB}$, which means the arbitrary terms in comment 4) should not give rise to any of above poles, because only diagrams 2a), 2b), and 2c) have such poles. This will largely limit the number of arbitrary terms.

In the following, we compute the total amplitudes of three Figures 2a), 2b), and 2c) directly from Eq.(4.52), without imposing condition $p_1^2 = 0$. After a long, but straight forward computation we have

$$A^{(a+b+c)} = A_1^{(a+b+c)} + A_2^{(a+b+c)} + A_3^{(a+b+c)} + A_4^{(a+b+c)} + A_5^{(a+b+c)} \tag{4.57}$$

with

$$
\begin{aligned}
A_1^{(a+b+c)} &= \frac{i}{2\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_5a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}\times \\
&\Big[(p_2D\varepsilon_2)_{\mu_3}(p_3D\varepsilon_3)_{\mu_5}(I_{10}+\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3+\frac{2\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2) \\
&+(p_2D\varepsilon_2)_{\mu_3}(p_2D\varepsilon_3)_{\mu_5}(I_{10}+\frac{2\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2)+(p_3D\varepsilon_2)_{\mu_3}(p_2\cdot\varepsilon_3)_{\mu_5}I_8 \\
&+(p_2D\varepsilon_2)_{\mu_3}(p_2\cdot\varepsilon_3)_{\mu_5}(I_9+\frac{2\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2)-(p_3D\varepsilon_2)_{\mu_3}(p_1N\varepsilon_3)_{\mu_5}I_5 \\
&-(p_2D\varepsilon_2)_{\mu_3}(p_1N\varepsilon_3)_{\mu_5}(2I_{10}+\frac{4\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2)+(p_3\cdot\varepsilon_2)_{\mu_3}(p_1N\varepsilon_3)_{\mu_5}I_9 \\
&+(p_1N\varepsilon_2)_{\mu_3}(p_1N\varepsilon_3)_{\mu_5}I_{10}\Big]+\Big[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Big] \qquad (4.58)
\end{aligned}
$$

$$
\begin{aligned}
A_2^{(a+b+c)} &= \frac{i}{4\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_4a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}\times \\
&\Big[(p_2\varepsilon_3Dp_3)(I_9-\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3)+(p_2\varepsilon_3Dp_2)(I_8)-(p_2\cdot\varepsilon_3Np_1)I_9 \\
&+(p_2D\varepsilon_3Dp_3)(-I_{10}-\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3)+(p_2D\varepsilon_3Np_1)I_5\Big](\varepsilon_2)_{\mu_3\mu_4} \\
&+\Big[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Big] \qquad (4.59)
\end{aligned}
$$

$$
\begin{aligned}
A_3^{(a+b+c)} &= \frac{i}{8\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta\mu_3\mu_4\mu_6a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_\beta(\varepsilon_2)_{\mu_3\mu_4}\Big[(p_2\cdot p_3)(p_2D\varepsilon_3)_{\mu_6}I_8 \\
&-(p_3Dp_3)(p_2D\varepsilon_3)_{\mu_6}(-I_{10}-\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3-\frac{\pi^4}{3}\frac{p_2Dp_2}{p_3Dp_3}+\frac{\pi^4}{3}) \\
&-(p_3Dp_3)(p_2\cdot\varepsilon_3)_{\mu_6}(I_9-\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3)-(p_2Dp_3)(p_2D\varepsilon_3)_{\mu_6}I_{10} \\
&+2(p_2\cdot p_3)(p_3D\varepsilon_3)_{\mu_6}(I_9-\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3)+2(p_2Dp_3)(p_1N\varepsilon_3)_{\mu_6}I_5 \\
&-2(p_2Dp_3)(p_3D\varepsilon_3)_{\mu_6}(I_{10}+\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3)-2(p_2\cdot p_3)(p_1N\varepsilon_3)_{\mu_6}I_9 \\
&+(p_2\cdot p_3)(p_2\cdot\varepsilon_3)_{\mu_6}(I_{10}+\frac{2\pi^2}{p^2(p_2\cdot p_3)}(Q_2+Q_3))-(p_2Dp_3)(p_2\cdot\varepsilon_3)_{\mu_6}I_8\Big] \\
&+\frac{i}{8\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta\mu_4\mu_5\mu_6a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_\beta(\varepsilon_3)_{\mu_5\mu_6}\Big[(p_2\cdot p_3)(p_3D\varepsilon_2)_{\mu_4}I_8
\end{aligned}
$$

$$
\begin{aligned}
&+(p_2\cdot p_3)(p_3\cdot\varepsilon_2)_{\mu_4}I_2(I_{10}+\frac{2\pi^2}{p^2(p_2\cdot p_3)}(Q_2+Q_3))-(p_2Dp_3)(p_3D\varepsilon_2)_{\mu_4}I_{10}\\
&-(p_3Dp_3)(p_1N\varepsilon_2)_{\mu_4}(2I_{10}+\frac{4\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3+\frac{2}{3}\pi^4\frac{p_2Dp_2}{p_3Dp_3}-\frac{2}{3}\pi^4)\\
&-(p_3Dp_3)(p_3\cdot\varepsilon_2)_{\mu_4}(I_9-\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3)-(p_2Dp_3)(p_3\cdot\varepsilon_2)_{\mu_4}I_8\\
&-(p_3Dp_3)(p_3D\varepsilon_2)_{\mu_4}(-I_{10}-\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3-\frac{1}{3}\pi^4\frac{p_2Dp_2}{p_3Dp_3}+\frac{1}{3}\pi^4)\\
&+2(p_3Dp_3)(p_2D\varepsilon_2)_{\mu_4}(I_{10}+\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3+\frac{2\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2\\
&-\frac{4\pi^2}{(p_2Dp_2)(p_3Dp_3)}+\frac{\pi^4}{3}\frac{p_2Dp_2}{p_3Dp_3}+\frac{2}{3}\pi^4\frac{p^2}{p_3Dp_3}-\frac{\pi^4}{3})\Big]\\
&+\Big[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Big]
\end{aligned}
\tag{4.60}
$$

$$
\begin{aligned}
A_4^{(a+b+c)} \;=\;& \frac{i}{4\sqrt{2}}\frac{p-3}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_4\mu_6a_2\cdots a_{p-3}}C_{ia_2\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}(\varepsilon_2)_{\mu_3\mu_4}\times\\
&\Big[p_3^i(p_2\cdot\varepsilon_3)I_9-p_3^i(p_2D\varepsilon_3)I_5-2p_2^i(p_1N\varepsilon_3)I_{10}+p_2^i(p_2\cdot\varepsilon_3)I_9+p_2^i(p_2D\varepsilon_3)I_5\\
&+p_2^i(p_3D\varepsilon_3)(2I_{10}+\frac{4\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3)\Big]\\
&+\Big[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Big]
\end{aligned}
\tag{4.61}
$$

$$
\begin{aligned}
A_5^{(a+b+c)} \;=\;& \frac{i}{16\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\mu_3\mu_4\mu_5\mu_6a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6}\times\\
&\Big[(p_2Dp_3)^2I_{10}-(p_2\cdot p_3)^2(I_{10}+\frac{2\pi^2}{p^2(p_2\cdot p_3)}Q_3+\frac{2\pi^2}{p^2(p_2\cdot p_3)}Q_2)\\
&-(p_2Dp_2)(p_3Dp_3)(I_{10}+\frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3+\frac{2\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2\\
&+\frac{\pi^4}{3}\frac{p_2Dp_2}{p_3Dp_3}+\frac{\pi^4}{3}\frac{p_3Dp_3}{p_2Dp_2}-\frac{2\pi^4}{3}-\frac{12\pi^2}{(p_2Dp_2)(p_3Dp_3)}+\frac{2\pi^4}{3}\frac{p^2}{p_2Dp_2}\\
&+\frac{2\pi^4}{3}\frac{p^2}{p_3Dp_3}+\frac{2\pi^4}{3}\frac{p_1\cdot p_3}{p_3Dp_3}+\frac{2\pi^4}{3}\frac{p_1\cdot p_2}{p_2Dp_2}+\frac{\pi^4}{3}\frac{p_2\cdot p_3}{p_3Dp_3}+\frac{\pi^4}{3}\frac{p_2\cdot p_3}{p_2Dp_2})\Big]\\
&+\frac{i}{16\sqrt{2}}\frac{p-3}{(p-2)!}\epsilon^{\mu_3\mu_4\mu_5\mu_6\beta a_2\cdots a_{p-3}}C_{ia_2\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6}\times\\
&\Big[2p_3^\beta p_2^i(p_2Dp_3)I_5+2p_3^\beta p_2^i(p_2\cdot p_3)I_9+2p_2^\beta p_3^i(p_2Dp_3)I_5-2p_2^\beta p_3^i(p_2\cdot p_3)I_9\\
&-p_2^\beta p_2^i(p_3Dp_3)(2I_{10}+\frac{4\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3+\frac{2}{3}\pi^4\frac{p_2Dp_2}{p_3Dp_3}-\frac{2}{3}\pi^4)
\end{aligned}
$$

$$-p_3^\beta p_3^i(p_2Dp_2)(2I_{10}+\frac{4\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2+\frac{2}{3}\pi^4\frac{p_3Dp_3}{p_2Dp_2}-\frac{2}{3}\pi^4)\Big]$$
$$+\frac{i}{4\sqrt{2}}\frac{(p-3)(p-4)}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_4\mu_5\mu_6a_3\cdots a_{p-3}}C_{ija_3\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6}\times$$
$$(p_2)_{\beta_1}(p_3)_{\beta_2}p_3^ip_2^jI_{10}, \tag{4.62}$$

where Q_2 and Q_3 are defined in the Appendix A.

2. Amplitude for diagram 2d)

In this subsection, we first compute the α'^2 correction of the brane vertex in Figure 3b). The string amplitude for a Dp-brane absorbing one R-R field and emitting two open string gauge field equals to

$$\begin{aligned}\mathcal{A}^{string}_{CAA} &= < V^{(-\frac{1}{2},-\frac{1}{2})}(p_1)V^{-1}(p_2,\zeta_2)V^0(p_3,\zeta_3) > \\ &= \frac{\Gamma[1+4p_2\cdot p_3]}{\Gamma[1+2p_2\cdot p_3]^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}F_{\beta_1\beta_2}F_{\beta_3\beta_4}C_{\nu_1\cdots n_{p-3}} \\ &\sim \left[1+\frac{2\pi^2}{3}(p_2\cdot p_3)^2\right]\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}F_{\beta_1\beta_2}F_{\beta_3\beta_4}C_{\nu_1\cdots n_{p-3}}\end{aligned} \tag{4.63}$$

where $p_2\cdot p_3=p_2^ap_3^a$, as only components p_2^a and p_3^a are non-vanishing. The two vertex operators for two gauge fields are

$$\begin{aligned}V^{-1}(p_2,\zeta_2) &= (\zeta_2)_a\int dxe^{-\phi}\psi^ae^{2ip_2\cdot X}(x) \\ V^0(p_3,\zeta_3) &= (\zeta_3)_a\int dx(\partial X^a-2ip_3\cdot\psi\psi^a)e^{2ip_3\cdot X}(x)\end{aligned} \tag{4.64}$$

At the low energy limit, the above string amplitude can be replaced by the supergravity diagram 3b), which means that at α'^2 order we have the action

$$\mathcal{L}_{CAA}=\frac{T_p(2\alpha')^2}{(p-3)!\times 8}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}C_{\nu_1\cdots\nu_{p-3}}\left[F_{\beta_1\beta_2}F_{\beta_3\beta_4}+\frac{2\pi^2}{3}\nabla^a{}_bF_{\beta_1\beta_2}\nabla^b{}_aF_{\beta_3\beta_4}\right] \tag{4.65}$$

This action also has the ambiguity that bothers $\mathcal{L}_{BC}$. For example the factor p_2Dp_2 is zero on-shell, but non-vanishing when we compute the amplitude of diagram 2d).

We will handle this issue later. The supergravity amplitude of diagram 2d) equals to

$$\begin{aligned} A^{(d)} &= \frac{i}{2\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_5 a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}\times \\ & (p_2D\varepsilon_2)_{\mu_3}(p_3D\varepsilon_3)_{\mu_5}\frac{8\pi^2}{(p_2Dp_2)(p_3Dp_3)}[1+\frac{\pi^2}{6}(p_2Dp_3+p_2\cdot p_3)^2] \end{aligned} \tag{4.66}$$

3. Amplitude for diagram 2e)

To obtain the amplitude for Feynman diagram 2e),we first need to get the correction of the vertex 3c) at the order α'^2. The disc amplitude with insertions of one R-R, one antisymmetric NS-NS, and one open string vertex operators is

$$\begin{aligned} \mathcal{A}^{string}_{CAB} &= < V^{(-\frac{1}{2},-\frac{1}{2})}(p_1)V^{-1}(p_2,\zeta)V^{(0,0)}(p_3,\varepsilon) > \qquad (4.67) \\ &= \frac{i}{2^{7/2}}\frac{1}{(p-3)!}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}C_{\nu_1\cdots\nu_{p-3}}\Big[F_{\beta_1\beta_2}(p_3)_{\beta_4}(p_2\cdot\varepsilon)_{\beta_3}\frac{2\pi^2}{3}(p_2\cdot p_3) \\ & -F_{\beta_1\beta_2}(p_3)_{\beta_4}(p_3D\varepsilon)_{\beta_3}(\frac{2}{p_3Dp_3}Q+\frac{4\pi^2}{3}\frac{(p_2\cdot p_3)^2}{p_3Dp_3}+\frac{Q}{2p_1\cdot p_3}) \\ & +F_{\beta_1\beta_2}(p_3)_{\beta_4}(p_1\cdot\varepsilon)_{\beta_3}\frac{Q}{p_1\cdot p_3}+\varepsilon_{\beta_1\beta_2}(p_3)_{\beta_3}\zeta_{\beta_4}(\frac{p_2\cdot p_3}{p_1\cdot p_3}Q-\frac{2\pi^2}{3}(p_2\cdot p_3)^2) \\ & -\varepsilon_{\beta_1\beta_2}(p_3)_{\beta_3}(p_2)_{\beta_4}(p_3\cdot\zeta)(\frac{Q}{p_1\cdot p_3}-\frac{2\pi^2}{3}p_2\cdot p_3) \\ & -F_{\beta_1\beta_2}\varepsilon_{\beta_3\beta_4}(\frac{1}{2}Q+\frac{1}{4}\frac{p_3Dp_3}{p_1\cdot p_3}Q+\frac{\pi^2}{3}(p_2\cdot p_3)^2)\Big] \\ & -\frac{i}{2^{7/2}}\frac{1}{(p-4)!}\epsilon^{\beta_1\beta_2\beta_3\beta_4\beta_5\,\nu_2\cdots\nu_{p-3}}F_{\beta_1\beta_2}\varepsilon_{\beta_3\beta_4}(p_3)_{\beta_5}p_3^iC_{i\,\nu_2\cdots\nu_{p-3}}(\frac{Q}{2p_1\cdot p_3}) \end{aligned}$$

where

$$Q=\frac{\Gamma[1+p_1\cdot p_3]\Gamma[1+p_3Dp_3]}{\Gamma[1+p_1\cdot p_3+p_3Dp_3]}\approx 1-\frac{\pi^2}{6}(p_1\cdot p_3)(p_3Dp_3) \tag{4.68}$$

This string amplitude can be replaced by three supergravity Feynman diagrams in the Figure 5. The amplitudes for supergravity diagram 5a) and 5b) are

$$\begin{aligned} A^{(a)}_{CAB} &= -\frac{i}{2^{5/2}}\frac{1}{(p-3)!}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}C_{\nu_1\cdots\nu_{p-3}}F_{\beta_1\beta_2}(p_3)_{\beta_4}(p_3D\varepsilon)_{\beta_3} \\ & (\frac{1}{p_3Dp_3}+\frac{2\pi^2}{3}\frac{(p_2\cdot p_3)^2}{p_3Dp_3}) \end{aligned} \tag{4.69}$$

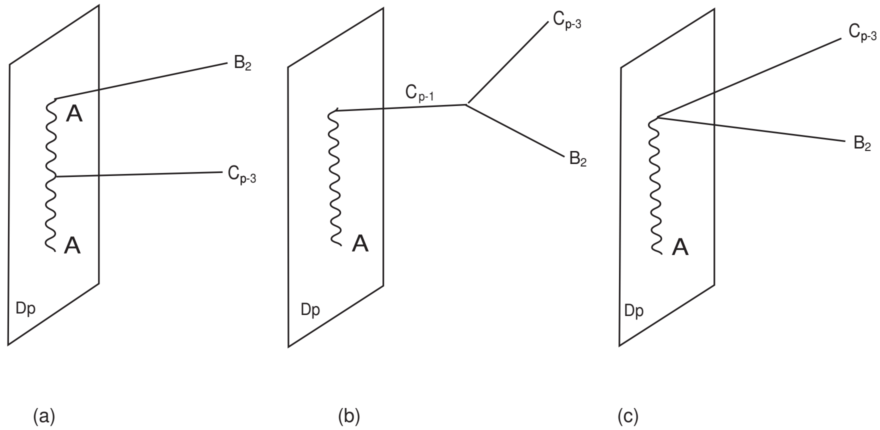

Fig. 5. Three supergravity Feynman diagrams that replace string amplitude $\mathcal{A}^{string}_{CAB}$ at low energy

and

$$
\begin{aligned}
A^{(b)}_{CAB} &= \frac{i}{2^{7/2}}\frac{1}{(p-3)!}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}C_{\nu_1\cdots\nu_{p-3}}\Big[-F_{\beta_1\beta_2}(p_3)_{\beta_4}(p_3D\varepsilon)_{\beta_3}(\frac{1}{2p_1\cdot p_3})\\
&+F_{\beta_1\beta_2}(p_3)_{\beta_4}(p_1\cdot\varepsilon)_{\beta_3}\frac{1}{p_1\cdot p_3}+\varepsilon_{\beta_1\beta_2}(p_3)_{\beta_3}\zeta_{\beta_4}(\frac{p_2\cdot p_3}{p_1\cdot p_3})\\
&-\varepsilon_{\beta_1\beta_2}(p_3)_{\beta_3}(p_2)_{\beta_4}(p_3\cdot\zeta)(\frac{1}{p_1\cdot p_3})-F_{\beta_1\beta_2}\varepsilon_{\beta_3\beta_4}(\frac{1}{4}\frac{p_3Dp_3}{p_1\cdot p_3})\Big]\\
&-\frac{i}{2^{7/2}}\frac{1}{(p-4)!}\epsilon^{\beta_1\beta_2\beta_3\beta_4\beta_5\,\nu_2\cdots\nu_{p-3}}F_{\beta_1\beta_2}\varepsilon_{\beta_3\beta_4}(p_3)_{\beta_5}p_3^iC_{i\,\nu_2\cdots\nu_{p-3}}(\frac{1}{2p_1\cdot p_3})\\
&+\frac{i}{2^{9/2}}\frac{1}{(p-3)!}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}C_{\nu_1\cdots\nu_{p-3}}F_{\beta_1\beta_2}\varepsilon_{\beta_3\beta_4} \qquad (4.70)
\end{aligned}
$$

After subtracting the supergravity amplitudes $A^{(a)}_{CAB}$ and $A^{(b)}_{CAB}$ from the string amplitude, we obtain the supergravity amplitude $A^{(c)}_{CAB}$ of Feynman diagram 5c), which can be generated by the following action:

$$
\begin{aligned}
\mathcal{L}_{CAB} &= \frac{2T_p}{(p-3)!\times 8}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}(B)_{\beta_1\beta_2}(2\alpha'F)_{\beta_3\beta_4}C_{\nu_1\cdots\nu_{p-3}}\\
&-\frac{2T_p}{(p-3)!\times 4}I_0\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}\nabla^a{}_b(B)_{\beta_1\beta_2}\nabla^b{}_a(2\alpha'F)_{\beta_3\beta_4}C_{\nu_1\cdots\nu_{p-3}}\\
&-\frac{T_p}{(p-3)!\times 8}I_0\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}\Big[-2\nabla^aH_{\beta_1\beta_2b}\nabla^b{}_a(2\alpha'F)_{\beta_3\beta_4}
\end{aligned}
$$

$$+2\nabla^a{}_b H_{\beta_1\beta_2 a}\nabla^b(2\alpha' F)_{\beta_3\beta_4} - \nabla^a{}_a H_{\beta_1\beta_2 b}\nabla^b(2\alpha' F)_{\beta_3\beta_4} \tag{4.71}$$
$$+\frac{2}{3}\nabla^{ba}{}_a H_{\beta_1\beta_2\beta_3}(2\alpha' F)_{b\beta_4} + \frac{4}{3}\nabla^a{}_b H_{\beta_1\beta_2\beta_3}\nabla^b(2\alpha' F)_{a\beta_4}\Big] C_{\nu_1\cdots\nu_{p-3}}$$
$$+\frac{T_p}{(p-3)!\times 4} I_0 \epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}\nabla^a{}_i H_{\beta_1\beta_2 a}(2\alpha' F)_{\beta_3\beta_4}\nabla^i C_{\nu_1\cdots\nu_{p-3}}$$
$$-\frac{T_p}{(p-3)!\times 8} I_0 \epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}\nabla^a{}_a H_{\beta_1\beta_2 i}(2\alpha' F)_{\beta_3\beta_4}\nabla^i C_{\nu_1\cdots\nu_{p-3}}$$
$$+\frac{T_p}{(p-4)!\times 24} I_0 \epsilon^{\beta_1\beta_2\beta_3\beta_4\beta_5\,\nu_2\cdots\nu_{p-3}}\nabla^{ia}{}_a H_{\beta_1\beta_2\beta_3}(2\alpha' F)_{\beta_4\beta_5} C_{i\nu_2\cdots\nu_{p-3}}$$

So the supergravity amplitude of diagram 2e) equals to

$$A^{(e)} = A_1^{(e)} + A_2^{(e)} + A_3^{(e)} + A_4^{(e)} \tag{4.72}$$

with

$$\begin{aligned} A_1^{(e)} \;=\; & \frac{i}{2\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_5 a_1\cdots a_{p-3}} C_{a_1\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}\times \\ & \Big[-\frac{1}{3}\pi^4(p_2D\varepsilon_2)_{\mu_3}(p_3D\varepsilon_3)_{\mu_5}\Big(\frac{p_2Dp_2}{p_3Dp_3}+\frac{p_3Dp_3}{p_2Dp_2}+2\frac{p_1\cdot p_3}{p_2Dp_2}+2\frac{p_1\cdot p_2}{p_3Dp_3}\Big) \\ & -\frac{1}{3}\pi^4(p_2D\varepsilon_2)_{\mu_3}(p_2D\varepsilon_3)_{\mu_5}\Big(\frac{p_3Dp_3}{p_2Dp_2}+2\frac{p_2\cdot p_3}{p_2Dp_2}+2\frac{p_2Dp_3}{p_2Dp_2}\Big) \\ & -\frac{1}{3}\pi^4(p_2D\varepsilon_2)_{\mu_3}(p_2\cdot\varepsilon_3)_{\mu_5}\Big(\frac{p_3Dp_3}{p_2Dp_2}+2\frac{p_2\cdot p_3}{p_2Dp_2}+2\frac{p_2Dp_3}{p_2Dp_2}\Big) \\ & +\frac{2}{3}\pi^4(p_2D\varepsilon_2)_{\mu_3}(p_1N\varepsilon_3)_{\mu_5}\frac{p_3Dp_3}{p_2Dp_2}\Big] + \Big[p_2\leftrightarrow p_3, \varepsilon_2\leftrightarrow\varepsilon_3\Big] \end{aligned} \tag{4.73}$$

$$\begin{aligned} A_2^{(e)} \;=\; & \frac{i}{4\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_4 a_1\cdots a_{p-3}} C_{a_1\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}\times \\ & \Big[\frac{1}{3}\pi^4(p_2\varepsilon_3Dp_3)\Big(\frac{p_2Dp_2}{p_3Dp_3}+2\frac{p_2\cdot p_3}{p_3Dp_3}+2\frac{p_2Dp_3}{p_3Dp_3}\Big)+\frac{1}{3}\pi^4(p_2D\varepsilon_3Dp_3)\times \\ & \Big(\frac{p_2Dp_2}{p_3Dp_3}+2\frac{p_2\cdot p_3}{p_3Dp_3}+2\frac{p_2Dp_3}{p_3Dp_3}\Big)\Big](\varepsilon_2)_{\mu_3\mu_4} + \Big[p_2\leftrightarrow p_3, \varepsilon_2\leftrightarrow\varepsilon_3\Big] \end{aligned} \tag{4.74}$$

$$\begin{aligned} A_3^{(e)} \;=\; & \frac{i}{8\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta\mu_3\mu_4\mu_6 a_1\cdots a_{p-3}} C_{a_1\cdots a_{p-3}}(p_2)_{\beta}(\varepsilon_2)_{\mu_3\mu_4}\times \\ & \Big[\frac{2}{3}\pi^4(p_2Dp_3)(p_3D\varepsilon_3)_{\mu_6}\Big(\frac{p_2Dp_2}{p_3Dp_3}+2\frac{p_2\cdot p_3}{p_3Dp_3}+2\frac{p_2Dp_3}{p_3Dp_3}\Big) \\ & +\frac{2}{3}\pi^4(p_2\cdot p_3)(p_3D\varepsilon_3)_{\mu_6}\Big(\frac{p_2Dp_2}{p_3Dp_3}+2\frac{p_2\cdot p_3}{p_3Dp_3}+2\frac{p_2Dp_3}{p_3Dp_3}\Big)\Big] \end{aligned}$$

$$
\begin{aligned}
&+\frac{i}{8\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\beta\mu_4\mu_5\mu_6 a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_\beta(\varepsilon_3)_{\mu_5\mu_6}\times \\
&\Bigg[2(p_3Dp_3)(p_2D\varepsilon_2)_{\mu_4}(\frac{8\pi^2}{(p_2Dp_2)(p_3Dp_3)}+\frac{2\pi^4}{3}\frac{(p_2Dp_3+p_2\cdot p_3)^2}{(p_2Dp_2)(p_3Dp_3)} \\
&-\frac{1}{3}\pi^4\frac{p_3Dp_3}{p_2Dp_2}-\frac{2}{3}\pi^4\frac{p_1\cdot p_3}{p_2Dp_2})\Bigg]+\Bigg[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Bigg]
\end{aligned}
\tag{4.75}
$$

$$
\begin{aligned}
A_4^{(e)} &= \frac{i}{4\sqrt{2}}\frac{p-3}{(p-2)!}\epsilon^{\beta_1\beta_2\mu_3\mu_4\mu_6 a_2\cdots a_{p-3}}C_{ia_2\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}(\varepsilon_2)_{\mu_3\mu_4}\times \\
&\quad p_2^i(p_3D\varepsilon_3)(-\frac{2}{3}\pi^4\frac{p_2Dp_2}{p_3Dp_3})+\Bigg[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Bigg]
\end{aligned}
\tag{4.76}
$$

4. Amplitude for diagram 2f)

After subtracting $A^{(a+b+c)}$, $A^{(d)}$, and $A^{(e)}$ from the string amplitude $\mathcal{A}^{string}_{CBB}$, we have the supergravity amplitude for diagram 2f)

$$
A^{(f)} = A_1^{(f)} + A_2^{(f)} + A_3^{(f)} + A_4^{(f)} + A_5^{(f)}
\tag{4.77}
$$

with

$$
\begin{aligned}
A_1^{(f)} &= \frac{i}{2\sqrt{2}}\frac{1}{(p-2)!}I_0\epsilon^{\beta_1\beta_2\mu_3\mu_5 a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}\times \\
&\quad\Bigg[(p_2p_3)(\varepsilon_2D\varepsilon_3)_{\mu_3\mu_5}-(p_2Dp_3)(\varepsilon_2\varepsilon_3)_{\mu_3\mu_5}-2(p_2D\varepsilon_2)_{\mu_3}(p_3D\varepsilon_3)_{\mu_5} \\
&\quad-(p_2D\varepsilon_2)_{\mu_3}(p_2D\varepsilon_3)_{\mu_5}+(p_2D\varepsilon_2)_{\mu_3}(p_2\cdot\varepsilon_3)_{\mu_5}+2(p_2D\varepsilon_2)_{\mu_3}(p_1N\varepsilon_3)_{\mu_5}\Bigg] \\
&\quad+\Bigg[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Bigg]
\end{aligned}
\tag{4.78}
$$

$$
\begin{aligned}
A_2^{(f)} &= \frac{i}{4\sqrt{2}}\frac{1}{(p-2)!}I_0\epsilon^{\beta_1\beta_2\mu_3\mu_4 a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}(\varepsilon_2)_{\mu_3\mu_4}\times \\
&\quad\Bigg[-(p_2\varepsilon_3Dp_3)+(p_2D\varepsilon_3Dp_3)-2(p_2\varepsilon_3Dp_2)\Bigg] \\
&\quad+\Bigg[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Bigg]
\end{aligned}
\tag{4.79}
$$

$$
\begin{aligned}
A_3^{(f)} \;=\;& \frac{i}{8\sqrt{2}}\frac{1}{(p-2)!}I_0\epsilon^{\beta\mu_3\mu_4\mu_6a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_\beta(\varepsilon_2)_{\mu_3\mu_4}\times\\
&\Big[-2(p_2\cdot p_3)(p_2D\varepsilon_3)_{\mu_6}+2(p_2Dp_3)(p_2D\varepsilon_3)_{\mu_6}+2(p_2Dp_3)(p_2\cdot\varepsilon_3)_{\mu_6}\\
&+2(p_2\cdot p_3)(p_2\cdot\varepsilon_3)_{\mu_6}+(p_2\cdot\varepsilon_3)_{\mu_6}(p_2Dp_2+p_3Dp_3+4p_2Dp_3)\\
&+(p_3Dp_3)(p_2\cdot\varepsilon_3)_{\mu_6}+(p_2\cdot\varepsilon_3)_{\mu_6}(p_2Dp_2+2p_2\cdot p_3+2p_2Dp_3)\\
&-2(p_3Dp_3)(p_2D\varepsilon_3)_{\mu_6}+2(p_2D\varepsilon_3)_{\mu_6}(p_2Dp_2+p_2\cdot p_3+p_2Dp_3)\\
&+2(p_2Dp_3)(p_3D\varepsilon_3)_{\mu_6}-2(p_2\cdot p_3)(p_3D\varepsilon_3)_{\mu_6}\Big]\\
&+\frac{i}{8\sqrt{2}}\frac{1}{(p-2)!}I_0\epsilon^{\beta\mu_4\mu_5\mu_6a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(p_2)_\beta(\varepsilon_3)_{\mu_5\mu_6}\times\\
&\Big[-2(p_2\cdot p_3)(p_3D\varepsilon_2)_{\mu_4}+2(p_2Dp_3)(p_3D\varepsilon_2)_{\mu_4}+2(p_2Dp_3)(p_3\cdot\varepsilon_2)_{\mu_4}\\
&+2(p_2\cdot p_3)(p_3\cdot\varepsilon_2)_{\mu_4}+(p_3\cdot\varepsilon_2)_{\mu_4}(p_2Dp_2+p_3Dp_3+4p_2Dp_3)\\
&+(p_3Dp_3)(p_3\cdot\varepsilon_2)_{\mu_4}+(p_3\cdot\varepsilon_2)_{\mu_4}(p_2Dp_2+2p_2\cdot p_3+2p_2Dp_3)\\
&-2(p_3Dp_3)(p_3D\varepsilon_2)_{\mu_4}+2(p_3D\varepsilon_2)_{\mu_4}(p_2Dp_2+p_2\cdot p_3+p_2Dp_3)\\
&+4(p_3Dp_3)(p_1N\varepsilon_2)_{\mu_4}-4(p_1N\varepsilon_2)_{\mu_4}(p_2Dp_2)-6(p_3Dp_3)(p_2D\varepsilon_2)_{\mu_4}\\
&+4(p_2D\varepsilon_2)_{\mu_4}(p_2Dp_2+p_1\cdot p_2+p^2)\Big]+\Big[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Big]
\end{aligned}
\tag{4.80}
$$

$$
\begin{aligned}
A_4^{(f)} \;=\;& -\frac{i}{2\sqrt{2}}\frac{p-3}{(p-2)!}I_0\epsilon^{\beta_1\beta_2\mu_3\mu_4\mu_6a_2\cdots a_{p-3}}C_{ia_2\cdots a_{p-3}}(p_2)_{\beta_1}(p_3)_{\beta_2}(\varepsilon_2)_{\mu_3\mu_4}p_2^i(p_3D\varepsilon_3)_{\mu_6}\\
&+\Big[p_2\leftrightarrow p_3,\varepsilon_2\leftrightarrow\varepsilon_3\Big]
\end{aligned}
\tag{4.81}
$$

$$
\begin{aligned}
A_5^{(f)} \;=\;& -\frac{i}{\sqrt{2}}\frac{1}{(p-2)!}\epsilon^{\mu_3\mu_4\mu_5\mu_6a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6}\\
&+\frac{i}{16\sqrt{2}}\frac{1}{(p-2)!}I_0\epsilon^{\mu_3\mu_4\mu_5\mu_6a_1\cdots a_{p-3}}C_{a_1\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6}\times\\
&\Big[4(p_2Dp_2)(p_3Dp_3)-2(p_3Dp_3)^2-2(p_2Dp_2)^2-2(p_1\cdot p_2+p_1\cdot p_3)(p_3Dp_3)\\
&-2p^2(p_3Dp_3)-(p_2\cdot p_3)(p_3Dp_3)-2p^2(p_2Dp_2)-2(p_1\cdot p_3+p_1\cdot p_2)(p_2Dp_2)\\
&-(p_2\cdot p_3)(p_2Dp_2)-(p_2\cdot p_3)(p_2Dp_2+p_3Dp_3)\Big]\\
&+\frac{i}{4\sqrt{2}}\frac{p-3}{(p-2)!}I_0\epsilon^{\mu_3\mu_4\mu_5\mu_6\beta a_2\cdots a_{p-3}}C_{ia_2\cdots a_{p-3}}(\varepsilon_2)_{\mu_3\mu_4}(\varepsilon_3)_{\mu_5\mu_6}\times
\end{aligned}
$$

$$\left[p_2^\beta p_2^i(p_3Dp_3 - p_2Dp_2) + p_3^\beta p_3^i(p_2Dp_2 - p_3Dp_3)\right] \tag{4.82}$$

D. Higher derivative brane couplings and their properties

1. Higher derivative couplings

It is straight forward to check that amplitude $A^{(f)}$ can be generated by the action,

$$\begin{aligned}
\mathcal{L}_{CBB} \;=\; & \frac{T_p}{(p-3)!\times 8}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}C_{\nu_1\cdots\nu_{p-3}}\left[B_{\beta_1\beta_2}B_{\beta_3\beta_4} - 2\frac{I_0}{\pi^2}\nabla^a{}_b B_{\beta_1\beta_2}\nabla^b{}_a B_{\beta_3\beta_4}\right] \\
& -\frac{T_p}{(p-3)!\times 8}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}C_{\nu_1\cdots\nu_{p-3}}\Big[\frac{1}{2}\nabla^i H_{\beta_1\beta_2 a}\nabla_i H_{\beta_3\beta_4}{}^a \\
& -\frac{1}{2}\nabla^a H_{\beta_1\beta_2 i}\nabla_a H_{\beta_3\beta_4}{}^i - H_{\beta_1\beta_2 i}\nabla^{ia}H_{\beta_3\beta_4 a} + \frac{2}{3}\nabla^i H_{\beta_1\beta_2\beta_3}\nabla^a H_{\beta_4 ai} \\
& -\frac{2}{3}H_{\beta_1 ai}\nabla^{ai}H_{\beta_2\beta_3\beta_4} + 2\nabla^a H_{\beta_1\beta_2 a}\nabla^b H_{\beta_3\beta_4 b} - 2\nabla^a H_{\beta_1\beta_2 b}\nabla^b{}_a B_{\beta_3\beta_4} \\
& +2\nabla^a{}_b H_{\beta_1\beta_2 a}\nabla^b B_{\beta_3\beta_4} - \nabla^a{}_a H_{\beta_1\beta_2 b}\nabla^b B_{\beta_3\beta_4} + \frac{2}{3}\nabla^{ba}{}_a H_{\beta_1\beta_2\beta_3}B_{b\beta_4} \\
& +\frac{4}{3}\nabla^a{}_b H_{\beta_1\beta_2\beta_3}\nabla^b B_{a\beta_4}\Big] - \frac{T_p}{(p-3)!\times 8}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}\times \\
& \left[-H_{\beta_1\beta_2 i}\nabla^a H_{\beta_3\beta_4 a} - 2\nabla^a{}_i H_{\beta_1\beta_2 a}B_{\beta_3\beta_4} + \nabla^a{}_a H_{\beta_1\beta_2 i}B_{\beta_3\beta_4}\right]\nabla^i C_{\nu_1\cdots\nu_{p-3}} \\
& -\frac{T_p}{(p-4)!\times 24}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\beta_5\,\nu_2\cdots\nu_{p-3}}\nabla^i H_{\beta_1\beta_2\beta_3}\nabla^b H_{\beta_4\beta_5 b}C_{i\nu_2\cdots\nu_{p-3}} \\
& +\frac{T_p}{(p-4)!\times 24}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\beta_5\,\nu_2\cdots\nu_{p-3}}\nabla^{ia}{}_a H_{\beta_1\beta_2\beta_3}B_{\beta_4\beta_5}C_{i\nu_2\cdots\nu_{p-3}}
\end{aligned} \tag{4.83}$$

The sum of this action, with $\mathcal{L}_{CAB}$ of Eq.(4.71) and $\mathcal{L}_{CAA}$ of Eq.(4.65) can be written as $\mathcal{L}_{CBB}$, after making the replacement $B \to B + 2\alpha' F$. So total action is manifestly invariant under the gauge transformation of B-field. In action $\mathcal{L}_{CBB}$, we have fixed the overall scale by fixing the coefficient of the zero derivative term. We still need to check the R-R gauge invariance of our action $\mathcal{L} = \mathcal{L}_{CB} + \mathcal{L}_{CBB} + \mathcal{L}_{CAB} + \mathcal{L}_{CAA}$, and it turns out this action does not have the desired property. We will see this problem can be solved after including a new term in $\mathcal{L}_{CB}$, and this new term vanishes on-shell, so we have the "freedom" to include it (we will fix the coefficients of these term at the last section).

2. R-R gauge invariance

In this section we focus on the variation of action $\mathcal{L}_C + \mathcal{L}$ under the R-R gauge transformations,

$$\delta C^{p+1} = d\Lambda^p + H\wedge\Lambda^{p-2}, \qquad \delta C^{p-1} = d\Lambda^{p-2} + H\wedge\Lambda^{p-4}, \qquad and \qquad \delta C^{p-3} = d\Lambda^{p-4} \tag{4.84}$$

We also should mention

$$\mathcal{L}_C = \frac{T_p}{(p+1)!}\epsilon^{\nu_1\cdots\nu_{p+1}}C^{p+1}_{\nu_1\cdots\nu_{p+1}} \tag{4.85}$$

It is easy to check that the variation of $\mathcal{L}_C + \mathcal{L}$ vanish for arbitrary Λ^p, so we only need to focus on the Λ^{p-2}, and Λ^{p-4}, ie.

$$\begin{aligned}
\delta C^{p+1}_{a_1\cdots a_{p+1}} &= \frac{(p+1)p(p-1)}{3!}H_{[a_1a_2a_3}\Lambda^{p-2}_{a_4\cdots a_{p+1}]}\\
\delta C^{p-1}_{a_1\cdots a_{p-1}} &= (p-1)\partial_{[a_1}\Lambda^{p-2}_{a_2\cdots a_{p-1}]} + \frac{(p-1)(p-2)(p-3)}{3!}H_{[a_1a_2a_3}\Lambda^{p-4}_{a_4\cdots a_{p-1}]}\\
\delta C^{p-1}_{i\,a_2\cdots a_{p-1}} &= (p-1)\partial_{[i}\Lambda^{p-2}_{a_2\cdots a_{p-1}]} + \frac{(p-2)(p-3)}{2}H_{i[a_2a_3}\Lambda^{p-4}_{a_4\cdots a_{p-1}]}\\
&\quad -\frac{(p-2)(p-3)(p-4)}{3!}H_{[a_2a_3a_4}\Lambda^{p-4}_{|i|a_5\cdots a_{p-1}]}\\
\delta C^{p-3}_{a_1\cdots a_{p-3}} &= (p-3)\nabla_{[a_1}\Lambda^{(p-4)}_{a_2\cdots a_{p-3}]}\\
\delta C^{p-3}_{i\,a_2\cdots a_{p-3}} &= \nabla_i\Lambda^{(p-4)}_{a_2\cdots a_{p-3}} - (p-4)\nabla_{[a_2}\Lambda^{(p-4)}_{|i|a_3\cdots a_{p-3}]}
\end{aligned} \tag{4.86}$$

The gauge variation of action $\mathcal{L}_{BC}$ and $\mathcal{L}_{CBB} + \mathcal{L}_{CAB} + \mathcal{L}_{CAA}$ only partly cancel,

$$\begin{aligned}
\delta(\mathcal{L}+\mathcal{L}_C) &= \delta\Bigg\{\frac{T_p}{(p-1)!\times 4}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}}\nabla^a H_{\beta_1\beta_2a}\nabla^\mu{}_\mu C^{(p-1)}_{\nu_1\cdots\nu_{p-1}}\\
&\quad -\frac{T_p}{(p-3)!\times 8}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}C_{\nu_1\cdots\nu_{p-3}}\times\\
&\quad \Bigg[+2\nabla^a H_{\beta_1\beta_2a}\nabla^b H_{\beta_3\beta_4b} + 2\nabla^a{}_b H_{\beta_1\beta_2a}\nabla^b(B+2\alpha' F)_{\beta_3\beta_4}
\end{aligned} \tag{4.87}$$

$$-\nabla^a{}_a H_{\beta_1\beta_2 b}\nabla^b(B+2\alpha' F)_{\beta_3\beta_4}+\frac{2}{3}\nabla^{ba}{}_a H_{\beta_1\beta_2\beta_3}(B+2\alpha' F)_{b\beta_4}\Big]$$
$$+\frac{T_p}{(p-3)!\times 8}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}H_{\beta_1\beta_2 i}\nabla^a H_{\beta_3\beta_4 a}\nabla^i C_{\nu_1\cdots\nu_{p-3}}$$
$$-\frac{T_p}{(p-4)!\times 24}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\beta_5\,\nu_2\cdots\nu_{p-3}}\nabla^i H_{\beta_1\beta_2\beta_3}\nabla^b H_{\beta_4\beta_5 b}C_{i\nu_2\cdots\nu_{p-3}}\Big\}$$

The obvious way to make our action invariant for arbitrary Λ^{p-2} is to introduce a similar term like the first term of r.h.s of the above equation, but with an opposite coefficient. i.e.

$$\Delta\mathcal{L}_{BC}=-\frac{T_p}{(p-1)!\times 4}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}}\nabla^a H_{\beta_1\beta_2 a}\nabla^\mu{}_\mu C^{(p-1)}_{\nu_1\cdots\nu_{p-1}} \tag{4.88}$$

which is zero on shell, and its coefficient can not fixed by two point string amplitude $\mathcal{A}^{string}_{BC}$ alone. The correction of action $\mathcal{L}_{BC}$ leads to the correction of action $\mathcal{L}_{CBB}$,

$$\begin{aligned}
\Delta\mathcal{L}_{CBB} \;=\;& -\frac{T_p}{(p-3)!\times 8}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}\nabla^a H_{\beta_1\beta_2 a}H_{\beta_3\beta_4 i}\nabla^i C_{\nu_1\cdots\nu_{p-3}}\\
&+\frac{T_p}{(p-3)!\times 8}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}\nabla^a{}_b H_{\beta_1\beta_2 a}H_{\beta_3\beta_4 b}C_{\nu_1\cdots\nu_{p-3}}\\
&+\frac{T_p}{(p-3)!\times 4}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}\nabla^a H_{\beta_1\beta_2 a}\nabla^b H_{\beta_3\beta_4 b}C_{\nu_1\cdots\nu_{p-3}}\\
&+\frac{T_p}{(p-4)!\times 24}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\beta_5\,\nu_2\cdots\nu_{p-3}}\nabla^a H_{\beta_1\beta_2 a}\nabla^i H_{\beta_3\beta_4\beta_5}C_{i\,\nu_2\cdots\nu_{p-3}}\\
&-\frac{T_p}{(p-3)!\times 12}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_2\cdots\nu_{p-3}}\nabla^a H_{\beta_1 ba}\nabla^b H_{\beta_2\beta_3\beta_4}C_{\nu_1\cdots\nu_{p-3}}
\end{aligned} \tag{4.89}$$

It is easy to check that our corrected higher derivative action

$$\begin{aligned}
\mathcal{L}'_{BC} \;=\;& \mathcal{L}_{BC}+\Delta\mathcal{L}_{BC}\\
=\;& \frac{T_p}{(p-1)!\times 2}\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}}B_{\beta_1\beta_2}C^{(p-1)}_{\nu_1\cdots\nu_{p-1}}\\
&-\frac{T_p}{(p-1)!\times 4}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}}\nabla^a{}_a H_{\beta_1\beta_2 i}\nabla^i C^{(p-1)}_{\nu_1\cdots\nu_{p-1}}\\
&+\frac{T_p}{(p-1)!\times 2}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}}\nabla^a{}_i H_{\beta_1\beta_2 a}\nabla^i C^{(p-1)}_{\nu_1\cdots\nu_{p-1}}
\end{aligned}$$

$$+\frac{T_p}{(p-2)!\times 12}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\,\nu_2\cdots\nu_{p-1}}\nabla^{ia}{}_a H_{\beta_1\beta_2\beta_3}C^{(p-1)}_{i\,\nu_2\cdots\nu_{p-1}} \tag{4.90}$$

and

$$\begin{aligned}
\mathcal{L}'_{CBB} &= \mathcal{L}_{CBB}+\Delta\mathcal{L}_{CBB}\\
&= \frac{T_p}{(p-3)!\times 8}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}C_{\nu_1\cdots\nu_{p-3}}\left[B_{\beta_1\beta_2}B_{\beta_3\beta_4}-2\frac{I_0}{\pi^2}\nabla^a{}_b B_{\beta_1\beta_2}\nabla^b{}_a B_{\beta_3\beta_4}\right]\\
&\quad -\frac{T_p}{(p-3)!\times 8}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}C_{\nu_1\cdots\nu_{p-3}}\left[\frac{1}{2}\nabla^i H_{\beta_1\beta_2 a}\nabla_i H_{\beta_3\beta_4}{}^a\right.\\
&\quad -\frac{1}{2}\nabla^a H_{\beta_1\beta_2 i}\nabla_a H_{\beta_3\beta_4}{}^i - H_{\beta_1\beta_2\mu}\nabla^{\mu a}H_{\beta_3\beta_4 a}+\frac{2}{3}\nabla^\mu H_{\beta_1\beta_2\beta_3}\nabla^a H_{\beta_4 a\mu}\\
&\quad -\frac{2}{3}H_{\beta_1 ai}\nabla^{ai}H_{\beta_2\beta_3\beta_4}-2\nabla^a H_{\beta_1\beta_2 b}\nabla^b{}_a B_{\beta_3\beta_4}+2\nabla^a{}_b H_{\beta_1\beta_2 a}\nabla^b B_{\beta_3\beta_4}\\
&\quad \left.-\nabla^a{}_a H_{\beta_1\beta_2 b}\nabla^b B_{\beta_3\beta_4}+\frac{2}{3}\nabla^{ba}{}_a H_{\beta_1\beta_2\beta_3}B_{b\beta_4}+\frac{4}{3}\nabla^a{}_b H_{\beta_1\beta_2\beta_3}\nabla^b B_{a\beta_4}\right]\\
&\quad +\frac{T_p}{(p-3)!\times 4}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}\nabla^a{}_i H_{\beta_1\beta_2 a}B_{\beta_3\beta_4}\nabla^i C_{\nu_1\cdots\nu_{p-3}}\\
&\quad -\frac{T_p}{(p-3)!\times 8}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\,\nu_1\cdots\nu_{p-3}}\nabla^a{}_a H_{\beta_1\beta_2 i}B_{\beta_3\beta_4}\nabla^i C_{\nu_1\cdots\nu_{p-3}}\\
&\quad +\frac{T_p}{(p-4)!\times 24}\frac{I_0}{\pi^2}\epsilon^{\beta_1\beta_2\beta_3\beta_4\beta_5\,\nu_2\cdots\nu_{p-3}}\nabla^{ia}{}_a H_{\beta_1\beta_2\beta_3}B_{\beta_4\beta_5}C_{i\nu_2\cdots\nu_{p-3}}
\end{aligned} \tag{4.91}$$

satisfy

$$\delta(\mathcal{L}_C+\mathcal{L}')=0 \tag{4.92}$$

for arbitrary Λ^{p-2} and Λ^{p-4}, with corrected action $\mathcal{L}'=\mathcal{L}'_{BC}+\mathcal{L}'_{CBB}+\mathcal{L}_{CAB}+\mathcal{L}_{CAA}$. The action $\mathcal{L}'$ still enjoys manifest B-field gauge invariance.

3. Linear T-duality

Formally, we can write the sum of action $\mathcal{L}'_{CBB}$ and action $\mathcal{L}_{CGG}$ of Eq.(4.29) as

$$\mathcal{L}'_{CBB}+\mathcal{L}_{CGG}=C^{p-3}\wedge X^{(4)}+C_i^{p-3}\wedge X^{(5)i} \tag{4.93}$$

Then one can read off $X^{(4)}$ and $X_i^{(5)}$ from the action $\mathcal{L}'_{CBB}$ and $\mathcal{L}_{CGG}$. Under the linear T-duality transformation, one can prove that

$$X^{(4)\prime}_{a_1a_2a_3a_4} = X^{(4)}_{a_1a_2a_3a_4}, \qquad and \qquad X^{(5)\prime i}{}_{a_1a_2a_3a_4a_5} = X^{(5)i}{}_{a_1a_2a_3a_4a_5} \tag{4.94}$$

which implies that action $\mathcal{L}'_{CBB} + \mathcal{L}_{CGG}$ is compatible with linearized T-duality.

E. Ambiguity terms

So far we have shown that Dp-brane action $\mathcal{L}_C + \mathcal{L}'$, with 10d action, can reproduce string amplitudes $\mathcal{A}^{string}$, $\mathcal{A}^{string}_{CB}$, $\mathcal{A}^{string}_{CAA}$, and $\mathcal{A}^{string}_{CAB}$. Also the action $\mathcal{L}_C + \mathcal{L}'$ is invariant under both B-field and R-R field gauge transformation, and compatible with linearized T-duality. These results will give strong constrain to the possible extra terms that we would miss for action $\mathcal{L}'$. At this moment, there are four groups of ambiguity terms,

1) On-shell vanishing terms for actions $\mathcal{L}'_{CB}$, $\mathcal{L}_{CAA}$ and $\mathcal{L}_{CAB}$.

2) On-shell non-vanishing corrections for $\mathcal{L}_{CAB}$, because of the correction in 1)

3) On-shell non-vanishing corrections for $\mathcal{L}'_{CBB}$ because of the correction 1) and 2).

4) On-shell vanishing terms for action $\mathcal{L}'_{CBB}$.

The sum of the first three groups of terms need to satisfy following conditions:

a) Give zero contribution to string amplitudes $\mathcal{A}^{string}_{CBB}$, $\mathcal{A}^{string}_{CB}$, $\mathcal{A}^{string}_{CAA}$, and $\mathcal{A}^{string}_{CAB}$ on-shell.

b) B-field and R-R gauge invariance, after combining any terms from 4).

c) Compatible with linearized T-duality, after combining any terms from 4), which need to compatible with b).

In the following, we handle the first three groups of ambiguity terms first. If the extra term include gauge fields, then at least one gauge field should appear in the combination

$$2\alpha'(F_2)_{ab} + (B_2)_{ab} - \frac{2}{p_2 D p_2}\nabla^c H_{cab}. \tag{4.95}$$

so that it give zero contribution to $\mathcal{A}^{string}_{CBB}$. Then, the request of B-field gauge invariance implies only two kinds of terms exist,

- $\left[2\alpha'(F_2)_{ab} + (B_2)_{ab} - \frac{2}{p_2Dp_2}\nabla^e H_{eab}\right]\left[2\alpha'(F_3)_{cd} + (B_3)_{cd}\right]$
- $\left[2\alpha'(F_2)_{ab} + (B_2)_{ab} - \frac{2}{p_2Dp_2}\nabla^c (H^{(2)})_{cab}\right] H^{(3)}$

As the correction of $\mathcal{L}'_{BC}$ should not introduce new poles, and should be written in terms of field strengths H_3 and $F^{(p-2)}$, we are left with three possible terms

- $\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}}\nabla^a H_{a\beta_1\beta_2}\nabla^\mu{}_{[\mu}C_{\nu_1\cdots\nu_{p-1}]}$
- $\epsilon^{\beta_1\beta_2\beta_3\,\nu_2\cdots\nu_{p-1}}\nabla^i H_{\beta_1\beta_2\beta_3}\nabla^\mu{}_{[\mu}C_{i\,\nu_2\cdots\nu_{p-1}]}$
- $\epsilon^{\beta_1\beta_2\,\nu_1\cdots\nu_{p-1}} H_{i\beta_1\beta_2}\nabla^{i\mu}{}_{[\mu}C_{\nu_1\cdots\nu_{p-1}]}$

The contribution of these terms to $\mathcal{A}^{string}_{CBB}$ need to be canceled by the corrections of action $\mathcal{L}'_{CBB}$, and actually the combination of corrected $\mathcal{L}'_{CBB}$ and $\mathcal{L}'_{BC}$ is automatically invariant R-R gauge transformation. It can be checked that the corrected $\mathcal{L}'_{BC}$ is also compatible with linearized T-duality. These arbitrary terms can not be fixed by the string amplitudes we have computed so far. It is not unexpected that the string amplitudes, which are evaluated on-shell, do not fix the action uniquely.

CHAPTER V

CONCLUSION

In Chapter IV, we have got α'^2 corrections for the D-brane couplings, which can reproduce string amplitudes $\mathcal{A}^{string}_{CBB}$, $\mathcal{A}^{string}_{CB}$, $\mathcal{A}^{string}_{CAA}$, and $\mathcal{A}^{string}_{CAB}$ up to order α'^2. The action we obtained is invariant under B-field and R-R gauge transformation, and compatible with linear T-duality. However, we can not fix it uniquely. So far, for the three point function case, we only compute the string amplitude involve R-R field with degree (p-3). It would be interesting to compute the three point function with R-R fields C^{p-1} and C^{p+1}. A lot work still need to be done to obtain the additional terms for action (4.28), to make the whole action have nice property, gauge invariance etc.

Unlike the string amplitude, which only give on-shell information, T-duality should be correct off-shell. So we expect that the arbitrary terms, we mentioned at the end of Chapter IV, can partly or all be fixed once we have finished the computation for the amplitudes involving R-R fields C^{p-1} and C^{p+1}, and request the whole action to be compatible with T-duality.

Now the α'^2 correction of D-brane action should enable us to compute the equation of motion to α'^2 order for type IIB string theory. It would be interesting to see how these equation compared with the equation of motion of heterotic string theory under the duality chain described in Chapter III.

The full collection of terms of $\mathcal{L}_{CGG}$ and $\mathcal{L}'_{CBB}$ should be expressed more elegantly. It would be interesting if one can rewrite all 3-form flux H_3 in some form of torsion [80].

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APPENDIX A

In this appendix, we present the details of some of the computations presented in this paper. To set up our notation we start by reviewing a few basic formulas regarding Calabi-Yau manifolds [81]. On a Calabi-Yau three-fold, there exists a unique harmonic (3,0) form Ω, whose first derivatives satisfy

$$\frac{\partial \Omega}{\partial z^i} = K_i \Omega + \chi_i \qquad \text{and} \qquad \frac{\partial \Omega}{\partial \bar{z}^i} = 0 \tag{A.1}$$

where χ_i is an harmonic (2,1) form. The Kähler potential on the complex structure moduli space is

$$K_{cs} = -\log[-i \int \Omega \wedge \overline{\Omega}]. \tag{A.2}$$

As is easy to check

$$\partial_i K_{cs} = -K_i \qquad \text{and} \qquad g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K_{cs} = -\frac{\int \chi_i \wedge \bar{\chi}_{\bar{j}}}{\int \Omega \wedge \overline{\Omega}}. \tag{A.3}$$

One important property of the (3,0) form Ω is that it is undefined up to multiplication by a holomorphic function $f(z)$

$$\Omega \to f(z)\Omega. \tag{A.4}$$

Under (A.4) the Kähler potential transforms as

$$K_{cs} \to K_{cs} - \log f(z) - \log \bar{f}(\bar{z}), \tag{A.5}$$

which leaves the metric on moduli space invariant. For convenience, we can define a gauge covariant derivative

$$\chi_i = \mathcal{D}_i \Omega = \partial_i \Omega + \partial_i K_{cs} \Omega, \tag{A.6}$$

and thus under the Kähler transformation, it transforms according to $\mathcal{D}_i\Omega \to f\mathcal{D}_i\Omega$, *i.e.* $\chi_i \to f\chi_i$. One can also generalize the definition of the covariant derivative to other quantities which transform like

$$\Psi^{(a,b)} \to f^a \bar{f}^b \Psi^{(a,b)} \tag{A.7}$$

under the Kähler transformation. In this case the covariant derivatives take the form

$$\begin{aligned} \mathcal{D}_i\Psi^{(a,b)} &= (\partial_i + a\partial_i K_{cs})\Psi^{(a,b)} \\ \mathcal{D}_{\bar{j}}\Psi^{(a,b)} &= (\partial_{\bar{j}} + b\partial_{\bar{j}} K_{cs})\Psi^{(a,b)}. \end{aligned} \tag{A.8}$$

The partial derivatives ∂_i and $\partial_{\bar{i}}$ are to be replaced by ordinary covariant derivatives ∇_i, $\nabla_{\bar{j}}$ when acting on tensors. It is easy to see that under Kähler transformations

$$\mathcal{D}_i\Psi^{(a,b)} \to f^a \bar{f}^b \mathcal{D}_i\Psi^{(a,b)} \qquad \text{and} \qquad \mathcal{D}_{\bar{j}}\Psi^{(a,b)} \to f^a \bar{f}^b \mathcal{D}_{\bar{j}}\Psi^{(a,b)}. \tag{A.9}$$

We also require

$$[\mathcal{D}_i, \mathcal{D}_{\bar{j}}]\Omega = -g_{i\bar{j}}\Omega, \qquad \text{and} \qquad \mathcal{D}_k g_{i\bar{j}} = 0. \tag{A.10}$$

Using the above formulas, we can get the results quoted in the table below

Derivatives of the basis	*Spans*
Ω	(3,0)
$\mathcal{D}_i\Omega = \chi_i$	(2,1)
$\mathcal{D}_i\chi_j = \frac{1}{\int \Omega\wedge\overline{\Omega}}\kappa_{ij}{}^{\bar{k}}\bar{\chi}_{\bar{k}}$	(1,2)
$\mathcal{D}_i\bar{\chi}_{\bar{j}} = g_{i\bar{j}}\overline{\Omega}$	(0,3)
$\mathcal{D}_i\overline{\Omega} = 0$	

(A.11)

where the Yukawa couplings are defined as

$$\kappa_{ijk} = \int \Omega \wedge \mathcal{D}_i\mathcal{D}_j\mathcal{D}_k\Omega. \tag{A.12}$$

The above results are the tools needed to compute the first derivative of scalar potential (2.10). Because the scalar potential is invariant under Kähler transformation, *i.e.* $a = b = 0$, we can transform the ordinary derivatives into covariant derivatives

$$\partial_I V = \mathcal{D}_I V = e^{\mathcal{K}} \left(Z_{IJ} \bar{F}^J + F_I \overline{W} \right) \tag{A.13}$$

with the notation (2.11). To obtain an explicit expression for $\partial_I V = 0$, we need to compute a few quantities,

$$\begin{aligned}
F_i &= \mathcal{D}_i W = \int_{\mathcal{M}_6} G \wedge \chi_i \\
F_\tau &= \mathcal{D}_\tau W = \partial_\tau W + \partial_\tau \mathcal{K} W = -\frac{1}{\tau - \bar{\tau}} \int_{\mathcal{M}_6} \overline{G} \wedge \Omega \\
Z_{ij} &= \mathcal{D}_i \mathcal{D}_j W = \frac{\kappa_{ijk}}{\int \Omega \wedge \overline{\Omega}} \int_{\mathcal{M}_6} G \wedge \overline{\chi}^k \\
Z_{\tau i} &= \mathcal{D}_\tau \mathcal{D}_i W = -\frac{1}{\tau - \bar{\tau}} \int_{\mathcal{M}_6} \overline{G} \wedge \chi_i \\
Z_{\tau\tau} &= \mathcal{D}_\tau \mathcal{D}_\tau W = \partial_\tau F_\tau - \Gamma^\tau_{\tau\tau} F_\tau + \partial_\tau \mathcal{K} F_\tau = 0.
\end{aligned} \tag{A.14}$$

As a result the critical condition $\partial_I V = 0$ can be explicitly written as

$$\begin{cases}
\int \overline{G} \wedge \chi_i \int \overline{G} \wedge \overline{\chi}^i + \int \overline{G} \wedge \Omega \int \overline{G} \wedge \overline{\Omega} = 0 \\
\\
\int G \wedge \overline{\chi}^k \int \overline{G} \wedge \overline{\chi}^i \left(\frac{\kappa_{ijk}}{\int \Omega \wedge \overline{\Omega}}\right) + \int G \wedge \chi_j \int \overline{G} \wedge \overline{\Omega} + \int \overline{G} \wedge \chi_j \int G \wedge \overline{\Omega} = 0
\end{cases} \tag{A.15}$$

After using the Hodge decomposition for G

$$G = A\Omega + A^i \chi_i + \bar{B}^{\bar{i}} \bar{\chi}_{\bar{i}} + \bar{B}\overline{\Omega} \tag{A.16}$$

the condition (A.15) can be further written in the form

$$\begin{cases}
\int G \wedge \star G = 0 \\
\\
(B\bar{B}_k + A\bar{A}_k) \int \Omega \wedge \overline{\Omega} + \kappa_{ijk} A^i B^j = 0
\end{cases} \tag{A.17}$$

which are Eq.(3.6) and (2.17). To derive these equations, we have used the property that the harmonic (2,1) and (0,3) forms are imaginary self-dual, and the harmonic (1,2) and (3,0) forms are imaginary anti-self-dual on Calabi-Yau three-fold.

Now we are going to compute the second derivative of scalar potential by noting that

$$\partial_I \partial_J V = \mathcal{D}_I \mathcal{D}_J V, \qquad \partial_I \partial_{\bar{J}} V = \mathcal{D}_I \mathcal{D}_{\bar{J}} V \tag{A.18}$$

at the critical point $\partial_I V = 0$. After a little algebra, the second derivatives of the scalar potential (2.10) are

$$\begin{aligned} \partial_I \partial_J V &= e^{\mathcal{K}} \left(U_{IJK} \bar{F}^K + 2 Z_{IJ} \overline{W} \right) \\ \partial_I \partial_{\bar{J}} V &= e^{\mathcal{K}} (U_{\bar{J}IK} \bar{F}^K + F_I \bar{F}_{\bar{J}} + Z_{IL} \bar{Z}_{\bar{J}\bar{K}} g^{L\bar{K}} + g_{I\bar{J}} |W|^2), \end{aligned} \tag{A.19}$$

where $U_{\bar{J}IK} = \mathcal{D}_{\bar{J}} \mathcal{D}_I \mathcal{D}_K W$. The above formula can be easily transformed to (2.22) by using the identity:

$$[\mathcal{D}_I, \mathcal{D}_{\bar{J}}] F_K = -g_{I\bar{J}} F_K + R_{I\bar{J}K}{}^L F_L \tag{A.20}$$

To get expression (2.24), we need to generalize the definition of U_{IJK} and Z_{IJ} to

$$U_{\alpha\beta\gamma} = \mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{D}_\gamma W \qquad \text{and} \qquad \overline{U}_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = \overline{U_{\alpha\beta\gamma}} \tag{A.21}$$

and

$$Z_{\alpha\beta} = \mathcal{D}_\alpha \mathcal{D}_\beta W \qquad \text{and} \qquad \overline{Z}_{\bar{\alpha}\bar{\beta}} = \overline{Z_{\alpha\beta}}, \tag{A.22}$$

where α, β, and γ label all coordinates, *i.e.* the axio-dilaton, complex structure moduli and their complex conjugates. Using the results quoted in the table (A.11), we have

$$\begin{aligned} U_{ijk} &= \mathcal{D}_i\mathcal{D}_j\mathcal{D}_k W = \frac{\int G\wedge\overline{\Omega}}{\int \Omega\wedge\overline{\Omega}}\kappa_{ijk} \\ U_{ij\tau} &= \mathcal{D}_i\mathcal{D}_j\mathcal{D}_\tau W = -\frac{\int \overline{G}\wedge\bar{\chi}^k}{\int \Omega\wedge\overline{\Omega}}\frac{\kappa_{ijk}}{\tau-\bar{\tau}} \\ U_{\bar{k}ij} &= -\frac{1}{(\int \Omega\wedge\overline{\Omega})^2}\kappa_{ij}{}^{\bar{m}}\bar{\kappa}_{\bar{k}\bar{m}\bar{n}}F^{\bar{n}} \end{aligned} \tag{A.23}$$

One consequence of Eq. (A.23) and Eq. (A.14) is

$$\begin{aligned} &\bar{F}^\tau U_{ij\tau} = \bar{F}^k U_{ijk}, \\ &Z_{\bar{J}I} = g_{I\bar{J}}W, \\ &Z_{J\bar{I}} = 0, \\ &U_{K\bar{J}I} = g_{I\bar{J}}F_K, \\ &U_{\tau\tau i} = U_{\tau\tau\tau} = U_{\bar{K}\bar{J}I} = U_{\alpha\bar{j}\tau} = 0. \end{aligned} \tag{A.24}$$

The above expressions are useful to show the equivalence of (2.24) and (A.19).

APPENDIX B

In this appendix we explicitly show the appearance of the two superpotentials

$$W = \int G \wedge \Omega, \qquad \text{and} \qquad \widetilde{W} = \int \overline{G} \wedge \Omega, \tag{B.1}$$

by dimensional reduction of ten-dimensional supergravity theories. Our convention is $\varepsilon_{01....9} = 1$, and

$$\star dx^{m_0} \wedge ... \wedge dx^{m_n} = \frac{1}{(9-n)!} \epsilon_{m_{n+1}...m_9}{}^{m_0...m_n} dx^{m_{n+1}} \wedge ... \wedge dx^{m_9} \tag{B.2}$$

We take the type IIB effective action (2.1) together with the local terms are

$$S_{loc} = -\int_{R^4 \times \Sigma} d^{p+1}\xi T_p \sqrt{-\hat{G}} + \mu_p \int_{R^4 \times \Sigma} C_{p+1} \tag{B.3}$$

To perform the dimensional reduction, we assume that the metric is independent of external coordinates

$$ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{-2A(y)} \tilde{g}_{mn}(y) dy^m dy^n \tag{B.4}$$

The Einstein equation is

$$R_{MN} = k_{10}^2 \left(T_{MN} - \frac{1}{8} g_{MN} T \right) \qquad \text{with} \qquad T_{MN} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{MN}}, \tag{B.5}$$

The non-compact components of the Einstein equation can be written as

$$R_{\mu\nu} = \left[-\frac{1}{8\mathrm{Im}\tau} |G|^2 - \frac{1}{4} e^{-8A} (\partial_m \alpha)^2 \right] g_{\mu\nu} + k_{10}^2 \left(T_{\mu\nu}^{loc} - \frac{1}{8} T^{loc} g_{\mu\nu} \right) \tag{B.6}$$

On the other hand, using the metric (B.4), we obtain

$$R_{\mu\nu} = -e^{2A} \tilde{\nabla}^2 A g_{\mu\nu} \tag{B.7}$$

which yields

$$\tilde{\nabla}^2 A = \frac{1}{8\mathrm{Im}\tau} e^{-2A}|G|^2 + \frac{1}{4} e^{-10A}|\partial_m \alpha|^2 + \frac{1}{8} k_{10}^2 e^{-2A}[T_m^m - T_\mu^\mu]^{loc}. \tag{B.8}$$

This can also be written in the form

$$\tilde{\nabla}^2 e^{4A} = \frac{1}{2\mathrm{Im}\tau} e^{2A}|G|^2 + e^{-6A}\left[(\partial_m \alpha)^2 + (\partial_m e^{4A})^2\right] + \frac{1}{2} k_{10}^2 e^{2A}\left(T_m^m - T_\mu^\mu\right)^{loc} \tag{B.9}$$

To compute the equation of motion for C_4 we only need to consider a few terms in the action namely

$$\frac{1}{8\kappa_{10}^2}\int \tilde{F}_{(5)} \wedge \star \tilde{F}_{(5)} - \frac{1}{8i\kappa_{10}^2}\int \frac{C_{(4)} \wedge G \wedge \overline{G}}{\mathrm{Im}\tau} + \frac{\mu_p}{2}\int_{R^4 \times \Sigma} C_{p+1} \tag{B.10}$$

The appearance of extra factor $\frac{1}{2}$ is a consequence of the self-duality of the five form. The Bianchi identity is

$$d \star \tilde{F}_{(5)} = -\frac{G \wedge \overline{G}}{2i\mathrm{Im}\tau} + 2k_{10}^2 T_3 \rho_3^{loc} \tag{B.11}$$

As $\tilde{F}_{(5)}$ is self-dual, we have

$$\tilde{F}_5 = (1 + \star) d\alpha \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{B.12}$$

and the Bianchi identity becomes

$$\tilde{\nabla}^2 \alpha = \frac{i}{12\mathrm{Im}\tau} e^{2A} G_{mnp} \star \overline{G}^{mnp} + 2e^{-6A}\partial_m e^{4A}\partial^m \alpha + 2k_{10}^2 T_3 \rho_3^{loc} \tag{B.13}$$

By summing or subtracting equations (B.9)and (B.13), we get

$$\begin{aligned} \tilde{\nabla}^2(e^{4A} \pm \alpha) =& \frac{1}{2\mathrm{Im}\tau} e^{2A}|G \mp i \star G|^2 + e^{-6A}|\partial_m \alpha \pm \partial_m e^{4A}|^2 \\ &+ 2k_{10}^2 e^{2A}\left(\frac{1}{4}(T_m^m - T_\mu^\mu)^{loc} \pm T_3 \rho_3^{loc}\right). \end{aligned} \tag{B.14}$$

The left hand side of the above equation vanishes when integrated over a compact manifold $\mathcal{M}_6$. As a result there are two solutions

$$\begin{aligned} \star_6\, G = -iG, \qquad & \alpha = -e^{4A}, \qquad \text{with} \qquad \overline{O}3, \overline{D}3 \\ \star_6\, G = +iG, \qquad & \alpha = +e^{4A}, \qquad \text{with} \qquad O3, D3. \end{aligned} \tag{B.15}$$

Notice that we can not have $O3$ and $\overline{D}3$ at the same time.

Using the results above we can perform the dimensional reduction

$$\int d^{10}x\sqrt{-g}\mathcal{R} = \int d^4x\sqrt{-g_4}\int d^6y\sqrt{g_6}\left[-8(\nabla A)^2 e^{4A}\right]. \tag{B.16}$$

Taking into account the fact the self-duality of the five-form we get

$$\int d^{10}x\sqrt{-g}\frac{\tilde{F}^2_{(5)}}{4} = \int d^4x\sqrt{-g_4}\int d^6y\sqrt{g_6}\frac{e^{-4A}}{2}(\partial_m\alpha)^2 \tag{B.17}$$

Since $\alpha = \mp e^{4A}$, this term gives the same contribution as the Einstein term

$$\begin{aligned} &\int d^{10}x\sqrt{-g}\left(\mathcal{R} - \frac{\tilde{F}^2_{(5)}}{4}\right) = \int d^4x\sqrt{-g_4}\int d^6y\sqrt{g_6}\left(-(\partial_m\alpha)^2 e^{4A}\right) \\ &= \int d^4x\sqrt{-g_4}\int d^6y\sqrt{g_6}\Big(\mp(\partial_m\alpha)^2 \pm 4\partial_m\alpha\partial^m A\Big) \\ &= \int d^4x\sqrt{-g_4}\int d^6y\sqrt{g_6}\Big(\pm\frac{1}{12i\mathrm{Im}\tau}e^{4A}G_{mnp}\star\overline{G}^{mnp} \mp 2e^{4A}\kappa_{10}^2 T_3\rho_3^{loc}\Big) \end{aligned} \tag{B.18}$$

Where we have used the Bianchi identity (B.13). The second term in the last equation of (B.18) will cancel the first term of S_{loc}, and the CS term cancels the second term of S_{loc}. At the end, the scalar potential is

$$S_v = \frac{1}{2\kappa_{10}^2}\int d^4x\sqrt{-g_4}\int\frac{e^{4A}}{2\mathrm{Im}\tau}G\wedge\star_6(\overline{G}\pm i\star\overline{G}) \tag{B.19}$$

From this expression, we can write the scalar potential in the standard form with

$$\widetilde{W} = \int\overline{G}\wedge\Omega, \qquad \text{or} \qquad W = \int G\wedge\Omega \tag{B.20}$$

APPENDIX C

In this appendix, we summarize our conventions and quote some useful formulas. We use indices

$$M, N, \ldots \mu, \nu, \ldots, i, j, \ldots, w_1, w_2, \quad (A, B, \ldots \alpha, \beta, \ldots, a, b, \ldots, w_1, w_2)$$

to denote the coordinate (non-coordinate) bases of any six-dimensional space, of four-dimensional Minkowski space-time, of the base and of the fiber, respectively. For coordinates on the four-dimensional base of the six-dimensional space, we use y^i while we denote the fiber coordinates by w_i, $i = 1, 2$. We define the chirality operators

$$\Gamma^{(4)} = -i\Gamma^0\Gamma^1\Gamma^2\Gamma^3, \qquad \Gamma^{(4')} = -\Gamma^4\Gamma^5\Gamma^6\Gamma^7, \qquad \Gamma_\star = -i\Gamma^8\Gamma^9 \tag{C.1}$$

where $\Gamma^{(4)}$, $\Gamma^{(4')}$, and $\Gamma_\star$ are the chirality operators for external space, K3 base and the T^2 fibre, from which we get

$$\Gamma^{(10)} = \Gamma^{(4)}\Gamma^{(4')}\Gamma_\star = \Gamma^0 \cdots \Gamma^9 \tag{C.2}$$

for the 10d space. In type the IIB theory, the 10d spinor ε satisfies

$$\Gamma^{(10)}\varepsilon = -\varepsilon \tag{C.3}$$

We also choose the orientation

$$\epsilon^{4567w_1w_2} = -1. \tag{C.4}$$

The Riemann tensor is defined by

$$R_{\mu\nu}{}^A{}_B = \partial_\mu \Omega^A{}_{B\nu} - \partial_\nu \Omega^A{}_{B\mu} + \Omega^A{}_{C\mu}\Omega^C{}_{B\nu} - \Omega^A{}_{C\nu}\Omega^C{}_{B\mu} \tag{C.5}$$

and

$$\mathrm{trR} \wedge \mathrm{R} = R^A{}_B \wedge R^A{}_B. \tag{C.6}$$

We use the notation

$$H_{w_i} = \frac{1}{2} H_{w_i ab} e^a \wedge e^b, \qquad (H_{w_i})_a = H_{w_i ab} e^b, \qquad H_{w_i ab} = H_{w_i mn} e^m_a e^n_b \tag{C.7}$$

and

$$|H|^2 = \frac{1}{2} H_{w_1 ab} H_{w_1}{}^{ab} + \frac{1}{2} H_{w_2 ab} H_{w_2}{}^{ab} \tag{C.8}$$

with e^a the vielbein for unwarped K3 base. To compute the tr$R \wedge R$, it is convenient to use the following results

$$\begin{aligned} A_{ij} A^i{}_k &= \frac{1}{4} A_{mn} A^{mn} g_{jk}, \qquad & S_{ij} S^i{}_k &= \frac{1}{4} S_{mn} S^{mn} g_{jk} \\ A_{ij} S^i{}_k &= A_{ik} S^i{}_j, & A_{ij} S^{ij} &= 0 \\ A_{ij} &= -\frac{1}{2} \epsilon_{ij}{}^{kl} A_{kl}, & S_{ij} &= \frac{1}{2} \epsilon_{ij}{}^{kl} S_{kl} \end{aligned} \tag{C.9}$$

where A_{ij} are the components of any anti-self-dual two form on the K3 base, and S_{ij} are the components of any self-dual two form on the K3 base.

APPENDIX D

We wish to evaluate the integral

$$I_0 = \int_0^{2\pi} d\theta \int_0^1 r_1 dr_1 \int_0^1 r_2 dr_2 \frac{\left(e^{i\theta} - e^{-i\theta}\right)^2}{\left|r_1 - r_2 e^{i\theta}\right|^2 \left|1 - r_1 r_2 e^{i\theta}\right|^2} \mathcal{K}, \tag{D.1}$$

at lowest order in momenta. The result is known (see, e.g. [73]), but for completeness we will present our own derivation. At this order we can set $\mathcal{K} = 1$, provided the remaining integral converges. If we split the integral up into two regions, $r_1 \leq r_2$ and $r_1 \geq r_2$, then we can expand the factors in the denominator of the integrand as Taylor series,

$$\begin{aligned} I_0 = \int_0^{2\pi} d\theta \left(e^{i\theta} - e^{-i\theta}\right)^2 \sum_{m_1,n_1,m_2,n_2=0}^{\infty} & \left\{ \int_0^1 \frac{dr_1}{r_1} \int_0^{r_1} r_2 dr_2 \left(\frac{r_2}{r_1}\right)^{n_1+n_2} (r_1 r_2)^{m_1+m_2} \right. \\ & \left. + \int_0^1 \frac{dr_2}{r_2} \int_0^{r_2} r_1 dr_1 \left(\frac{r_1}{r_2}\right)^{n_1+n_2} (r_1 r_2)^{m_1+m_2} \right\} e^{i(n_1-n_2+m_1-m_2)\theta}. \end{aligned} \tag{D.2}$$

The two regions clearly give identical contributions. Let's now rewrite the sums using $N = n_1 + n_2$, $n = (n_1 - n_2)/2$, $M = m_1 + m_2$, and $m = (m_1 - m_2)/2$,

$$\begin{aligned} I_0 &= 2 \int_0^1 dr_1 \int_0^{r_1} dr_2 \int_0^{2\pi} d\theta \sum_{N,M=0}^{\infty} r_1^{M-N-1} r_2^{M+N+1} \\ & \quad \sum_{n=-N/2}^{N/2} \sum_{m=-M/2}^{M/2} \left(e^{i\theta} - e^{-i\theta}\right)^2 e^{2i(m+n)\theta}. \end{aligned} \tag{D.3}$$

Note that the angular integral will give a non-zero result if and only if M and N have the same parity (either both even or both odd). Consider the angular integral at fixed N and M. If $N < M$, then for each allowed value of n there is precisely one allowed m satisfying each $m = -n-1$, $m = -n$, and $m = -n+1$. Thus, when we expand $(e^{i\theta} - e^{-i\theta})^2$ and perform the angular integral, the three terms precisely cancel out. Similarly, the angular integral for $N > M$ gives a vanishing result. This leaves us only with the case $N = M$,

$$\begin{aligned}
I_0 &= 2\int_0^1 \frac{dr_1}{r_1}\int_0^{r_1} r_2 dr_2 \sum_{N=0}^{\infty} r_2^{2N}\int_0^{2\pi} d\theta \sum_{n,m=-N/2}^{N/2}\left(e^{2i\theta} - 2 + e^{-2i\theta}\right) e^{2i(m+n)\theta} \\
&= 4\pi \int_0^1 \frac{dr_1}{r_1}\int_0^{r_1} r_2 dr_2 \sum_{N=0}^{\infty}\left(N - 2\left(N+1\right) + N\right) r_2^{2N} \\
&= -8\pi \int_0^1 dr_1 \sum_{N=0}^{\infty}\frac{r_1^{2N+1}}{2N+2} = -2\pi\sum_{N=0}^{\infty}\frac{1}{(N+1)^2} = -\frac{\pi^3}{3}. \qquad \text{(D.4)}
\end{aligned}$$

APPENDIX E

$$
\begin{aligned}
I_0 &= \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{(z_3\bar{z}_2 - z_2\bar{z}_3)^2}{2\mid z_2\mid^2\mid z_3\mid^2\mid 1 - z_2\bar{z}_3\mid^2\mid z_2 - z_3\mid^2}\mathcal{K} \\
I_1 &= \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{|1 + z_2\bar{z}_3|^2}{|1 - z_2\bar{z}_3|^2|z_2|^2|z_3|^2}\mathcal{K} \\
I_2 &= \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{|z_2 + z_3|^2}{|z_2 - z_3|^2|z_2|^2|z_3|^2}\mathcal{K} \\
I_3 &= \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{(1 + |z_2|^2)(1 + |z_3|^2)}{(1 - |z_2|^2)(1 - |z_3|^2)|z_2|^2|z_3|^2}\mathcal{K} \\
I_4 &= \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{2(1 + |z_2|^2)}{|z_2|^2|z_3|^2(1 - |z_2|^2)}\mathcal{K} \\
I_5 &= \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{1 - |z_2|^2|z_3|^2}{|z_2|^2|z_3|^2|1 - z_2\bar{z}_3|^2}\mathcal{K} \\
I_6 &= \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{-(1 + |z_2|^2)(|z_2|^2 - |z_3|^2)}{|z_2|^2|z_3|^2(1 - |z_2|^2)|z_2 - z_3|^2}\mathcal{K} \\
I_7 &= \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{-(1 + |z_2|^2)(1 - |z_2|^2|z_3|^2)}{|z_2|^2|z_3|^2|1 - z_2\bar{z}_3|^2(1 - |z_2|^2)}\mathcal{K} \\
I_8 &= \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{(|z_2|^2 - |z_3|^2)(1 - |z_2|^2|z_3|^2)}{|z_2|^2|z_3|^2|1 - z_2\bar{z}_3|^2|z_2 - z_3|^2}\mathcal{K} \\
I_9 &= \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{(|z_2|^2 - |z_3|^2)}{|z_2|^2|z_3|^2|z_2 - z_3|^2}\mathcal{K}
\end{aligned}
$$

$$I_{10} \quad = \quad \int_{|z_2|,|z_3|<1} d^2z_2 d^2z_3 \frac{1}{|z_2|^2|z_3|^2}\mathcal{K} \tag{E.1}$$

where

$$\mathcal{K} = (1-z_2\bar{z}_2)^{p_2\cdot Dp_2}\,(1-z_3\bar{z}_3)^{p_3\cdot Dp_3}\mid z_2\mid^{2p_1\cdot p_2}\mid z_3\mid^{2p_1\cdot p_3}\mid z_2-z_3\mid^{2p_2\cdot p_3}\mid 1-z_2\bar{z}_3\mid^{2p_2\cdot Dp_3}. \tag{E.2}$$

After a lengthy computation, these integrals equal to

$$\begin{aligned}
I_0 &= -\frac{\pi^4}{3} \\
I_1 &= I_{10} + \frac{2}{3}\pi^4 \\
I_2 &= I_{10} + \frac{2\pi^2}{p^2(p_2\cdot p_3)}Q_3 + \frac{2\pi^2}{p^2(p_2\cdot p_3)}Q_2 - \frac{2}{3}\pi^4 - \frac{1}{3}\pi^4\frac{p_2Dp_2 + p_3Dp_3 + 4p_2Dp_3}{p_2\cdot p_3} \\
I_3 &= I_{10} + \frac{2\pi^2}{(p_1\cdot p_2)(p_3Dp_3)}Q_3 + \frac{4\pi^2}{(p_2Dp_2)(p_3Dp_3)}[1 + \frac{\pi^2}{6}(p_2Dp_3 + p_2\cdot p_3)^2] \\
&\quad + \frac{2\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2 - \frac{1}{3}\pi^4[\frac{p_2Dp_2}{p_3Dp_3} + \frac{p_3Dp_3}{p_2Dp_2} + 2\frac{p_1\cdot p_3}{p_2Dp_2} + 2\frac{p_1\cdot p_2}{p_3Dp_3}] + \frac{2}{3}\pi^4 \\
I_4 &= 2I_{10} + \frac{4\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2 - \frac{2}{3}\pi^4\frac{p_3Dp_3}{p_2Dp_2} + \frac{2}{3}\pi^4 \\
I_5 &= I_{10} \\
I_6 &= -I_9 - \frac{2\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2 + \frac{\pi^4}{3}\frac{p_3Dp_3}{p_2Dp_2} + \frac{2\pi^4}{3}\frac{p_2\cdot p_3}{p_2Dp_2} + \frac{2\pi^4}{3}\frac{p_2Dp_3}{p_2Dp_2} + \frac{1}{3}\pi^4 \\
I_7 &= -I_{10} - \frac{2\pi^2}{(p_1\cdot p_3)(p_2Dp_2)}Q_2 + \frac{\pi^4}{3}\frac{p_3Dp_3}{p_2Dp_2} + \frac{2\pi^4}{3}\frac{p_2\cdot p_3}{p_2Dp_2} + \frac{2\pi^4}{3}\frac{p_2Dp_3}{p_2Dp_2} - \frac{1}{3}\pi^4
\end{aligned}$$

$$
\begin{aligned}
I_8 &= I_9 \\
I_9 &= \frac{\pi^2}{p^2(p_1 \cdot p_3)} Q_2 - \frac{\pi^2}{p^2(p_1 \cdot p_2)} Q_3 \\
I_{10} &= \frac{\pi^2}{p^2(p_1 \cdot p_3)} Q_2 + \frac{\pi^2}{p^2(p_1 \cdot p_2)} Q_3
\end{aligned}
\tag{E.3}
$$

where we have used the notation

$$
\begin{aligned}
p^2 &= p_1 \cdot p_2 + p_1 \cdot p_3 + p_2 \cdot p_3 \\
Q_2 &= [1 - \frac{\pi^2}{6} p^2 (p_2 D p_2)] \\
Q_3 &= [1 - \frac{\pi^2}{6} p^2 (p_3 D p_3)],
\end{aligned}
\tag{E.4}
$$

and we have only kept the terms to $O(p^0)$.

APPENDIX F

In this appendix, we evaluate the integration I_{10} in detail, while all other integrals in this paper can be handled similarly. In polar coordinates, I_{10} can be written as

$$\begin{aligned} I_{10} &= 2\pi \int_0^1 r_2 dr_2 \int_0^1 r_3 dr_3 \int_0^{2\pi} d\theta (r_2^2)^{p_1\cdot p_2 - 1} (r_3^2)^{p_1\cdot p_3 - 1} (1-r_2^2)^{p_2 D p_2} (1-r_3^2)^{p_3 D p_3} \\ &\quad \times (r_2 - r_3 e^{i\theta})^{p_2\cdot p_3} (r_2 - r_3 e^{-i\theta})^{p_2\cdot p_3} (1 - r_2 r_3 e^{i\theta})^{p_2 D p_3} (1 - r_2 r_3 e^{-i\theta})^{p_2 D p_3} \quad \text{(F.1)} \end{aligned}$$

after setting $z_2 = r_2 e^{i\theta_2}$ and $z_3 = r_3 e^{i\theta_3}$. The integration only depends on $\theta = \theta_2 - \theta_3$, so one angle can be integrated to get 2π factor. As we only interest in the behavior of this integral at small momentum limit, we will use binomial expansion to translate this integral into an infinite series where every single term can be integrated easily. The formula we will use frequently is

$$\frac{1}{(1-x)^s} = \sum_{n=0}^{\infty} \begin{pmatrix} s+n-1 \\ n \end{pmatrix} x^n \quad \text{(F.2)}$$

for $\mid x \mid \leq 1$. This formula is well defined for integer s, and for general s we can use the Gamma function representation of binomial coefficients, ie.

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}. \quad \text{(F.3)}$$

To apply binomial expansion to integral (F.1), we need to consider two situations $r_2 > r_3$ and $r_2 < r_3$ separately. For $r_2 > r_3$, we have

$$
\begin{aligned}
(I_{10})_{r_2>r_3} &= 2\pi \sum_{n_1,\cdots 6=0}^{\infty} \int_0^1 r_2 dr_2 \int_0^{r_2} r_3 dr_3 \int_0^{2\pi} d\theta (r_2^2)^{p_1\cdot p_2+p_2\cdot p_3-1}(r_3^2)^{p_1\cdot p_3-1} \\
&\quad \times \begin{pmatrix} -p_2Dp_2+n_1-1 \\ n_1 \end{pmatrix} \begin{pmatrix} -p_3Dp_3+n_2-1 \\ n_2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -p_2\cdot p_3+n_3-1 \\ n_3 \end{pmatrix} \begin{pmatrix} -p_2\cdot p_3+n_4-1 \\ n_4 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -p_2Dp_3+n_5-1 \\ n_5 \end{pmatrix} \begin{pmatrix} -p_2Dp_3+n_5-1 \\ n_6 \end{pmatrix} \\
&\quad \times \left(r_2^2\right)^{n_1} \left(r_3^2\right)^{n_2} \left(\frac{r_3}{r_2}e^{i\theta}\right)^{n_3} \left(\frac{r_3}{r_2}e^{-i\theta}\right)^{n_4} \left(r_2r_3e^{i\theta}\right)^{n_5} \left(r_2r_3e^{-i\theta}\right)^{n_6} \\
&= \pi^2 \sum_{n_1,\cdots 6=0}^{\infty} \frac{1}{p_1\cdot p_3+n_2+n_3+n_5}\frac{1}{p^2+n_1+n_2+n_5+n_6} \\
&\quad \times \begin{pmatrix} -p_2Dp_2+n_1-1 \\ n_1 \end{pmatrix} \begin{pmatrix} -p_3Dp_3+n_2-1 \\ n_2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -p_2\cdot p_3+n_3-1 \\ n_3 \end{pmatrix} \begin{pmatrix} -p_2\cdot p_3+n_4-1 \\ n_4 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -p_2Dp_3+n_5-1 \\ n_5 \end{pmatrix} \begin{pmatrix} -p_2Dp_3+n_5-1 \\ n_6 \end{pmatrix} \delta_{n_3+n_5-n_4-n_6,0}
\end{aligned}
\tag{F.4}
$$

where $p^2 = (p_1+p_2+p_3)^2/2 = p_1\cdot p_2 + p_1\cdot p_3 + p_2\cdot p_3$. One can exchange p_2 and p_3 in $(A_{10})_{r_2>r_3}$ to get $(A_{10})_{r_2<r_3}$. After adding these two parts of integral together, we obtain

$$\begin{aligned}
I_{10} &= (I_{10})_{r_2>r_3} + (I_{10})_{r_2<r_3} \\
&= \pi^2 \sum_{n_1,\cdots 6=0}^{\infty} \left(\frac{1}{p_1 \cdot p_3 + n_2 + n_3 + n_5} + \frac{1}{p_1 \cdot p_2 + n_1 + n_3 + n_5} \right) \\
&\quad \times \begin{pmatrix} -p_2 D p_2 + n_1 - 1 \\ n_1 \end{pmatrix} \begin{pmatrix} -p_3 D p_3 + n_2 - 1 \\ n_2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -p_2 \cdot p_3 + n_3 - 1 \\ n_3 \end{pmatrix} \begin{pmatrix} -p_2 \cdot p_3 + n_4 - 1 \\ n_4 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -p_2 D p_3 + n_5 - 1 \\ n_5 \end{pmatrix} \begin{pmatrix} -p_2 D p_3 + n_5 - 1 \\ n_6 \end{pmatrix} \\
&\quad \times \frac{1}{p^2 + n_1 + n_2 + n_5 + n_6} \delta_{n_3+n_5-n_4-n_6,0}
\end{aligned} \tag{F.5}$$

where one of the binomial coefficients can be expressed as

$$\begin{pmatrix} -p_2 D p_2 + n_1 - 1 \\ n_1 \end{pmatrix} = \frac{\Gamma[-p_2 D p_2 + n_1]}{\Gamma[-p_2 D p_2]\Gamma[n_1 + 1]} \tag{F.6}$$

which equals to 1 as $n = 0$, and behavior like $-p_2 D p_2$ for small $-p_2 D p_2$ as $n_1 \neq 0$. As we only interest in the terms up to order $O(p^0)$, for most of the time it is enough to consider only $n_i = 0$ terms. Now we separate the multiple infinite sum into several pieces,

1. $n_3 = n_4 = n_5 = n_6 = 0$

$$(I_{10})_1 = \pi^2 \sum_{n_{1,2}=0}^{\infty} \frac{\Gamma[-p_2 D p_2 + n_1]}{\Gamma[-p_2 D p_2]\Gamma[n_1 + 1]} \frac{\Gamma[-p_3 D p_3 + n_2]}{\Gamma[-p_3 D p_3]\Gamma[n_2 + 1]}$$

$$\times \frac{p^2+n_1+n_2-p_2\cdot p_3}{(p_1\cdot p_3+n_2)(p_1\cdot p_2+n_1)(p^2+n_1+n_2)}$$
$$= \frac{\pi^2}{p^2(p_1\cdot p_2)}+\frac{\pi^2}{p^2(p_1\cdot p_3)}-\frac{\pi^4}{6}\frac{p_3Dp_3}{p_1\cdot p_2}-\frac{\pi^4}{6}\frac{p_2Dp_2}{p_1\cdot p_3}+O(p^2) \quad \text{(F.7)}$$

where we have used the identity

$$\sum_{n=1}^{\infty}\frac{\Gamma[-p_2Dp_2+n]}{\Gamma[-p_2Dp_2]\Gamma[n+1]}\frac{1}{p_1\cdot p_2+n}=-\frac{\pi^2}{6}p_2Dp_2+O(p^4) \quad \text{(F.8)}$$

2. $n_3\neq 0$,$n_4\neq 0$,$n_5\neq 0$,$n_6\neq 0$

   The leading contribution to I_{10} for small momentum is order $O(p^8)$

3. $n_3=n_4=0$, $n_5=n_6\neq 0$

   The leading contribution to I_{10} for small momentum is order $O(p^4)$. Similar for following three cases

   - $n_5=n_6=0$, $n_3=n_4\neq 0$
   - $n_3=n_6=0$, $n_4=n_5\neq 0$
   - $n_4=n_5=0$, $n_3=n_6\neq 0$

4. $n_3=0$, $n_5=n_4+n_6$, $n_4\neq 0$, and $n_6\neq 0$

   The leading contribution to I_{10} for small momentum is order $O(p^6)$. Similar for following three cases

   - $n_4=0$, $n_6=n_3+n_5$, $n_3\neq 0$, and $n_5\neq 0$
   - $n_5=0$, $n_3=n_4+n_6$, $n_4\neq 0$, and $n_6\neq 0$
   - $n_6=0$, $n_4=n_3+n_5$, $n_3\neq 0$, and $n_5\neq 0$

So we have

$$I_{10}=\pi^2\frac{1}{p^2(p_1\cdot p_2)}+\pi^2\frac{1}{p^2(p_1\cdot p_3)}-\frac{\pi^4}{6}\frac{p_3Dp_3}{p_1\cdot p_2}-\frac{\pi^4}{6}\frac{p_2Dp_2}{p_1\cdot p_3}+O(p^2) \quad \text{(F.9)}$$

VITA

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