Cohesive Zone Models in History Dependent Materials

Layal Hakim*, Sergey Mikhailov

Dept. of Mathematics, Brunel University London, UK

Abstract

Cohesive zone model is a well known concept in nonlinear fracture mechanics of elasto-plastic materials. In contrast to that, we discuss a development of the cohesive zone model to linear, but time and history dependent, materials. The stress distribution over the cohesive zone satisfies a history dependent rupture criterion for the normalised equivalent stress, represented by a nonlinear Abel-type integral operator. The cohesive zone length at each time step is determined from the condition of zero stress intensity factor at the cohesive zone tip. It appeared that the crack starts propagating after some delay time elapses since a constant load is applied to the body. This happens when the crack tip opening displacement reaches a prescribed critical value. A numerical algorithm to compute the cohesive zone and crack length with respect to time will be discussed and graphs showing the results will be given.

Keywords: Cohesive zone, Time dependent fracture, Abel integral equation

1. Introduction

The cohesive zone in a material is the region between 2 surfaces ahead of the crack tip that are separating due to external loading but are pulled together by cohesive stresses σ, while the crack faces are traction-free, see Figure 1. Eventually, the crack will propagate through the cohesive zone points, where the criterion of the cohesive zone breakage is reached.

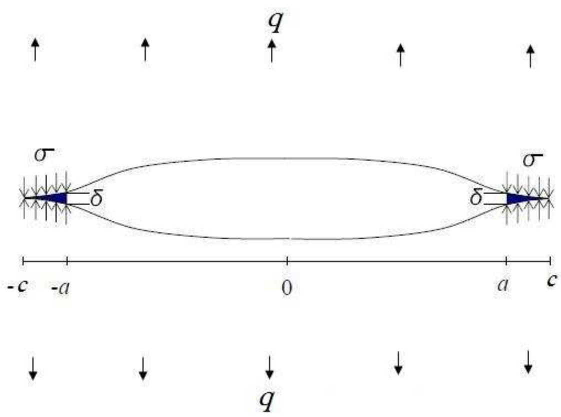

Figure 1: Cohesive zone

The Dugdale-Leonov-Panasyuk (1959-1960) (DLP) model was the first model introducing perfectly plastic cohesive zones, with constant cohesive stresses, $\sigma = \sigma_y$. This model has been modified and used in many applications in fracture mechanics. Another popular cohesive model is the Barenblatt (1962) model. The 3 main components needed to study cohesive zone models of the DLP type are: (i) the constitutive equations for the bulk of the material; (ii) the criterion on the stress (history) on the cohesive zone for the cohesive zone to appear and propagate; (iii) the criterion for the cohesive zone to break and the crack to propagate.

*Corresponding author
Email address: layal.hakim@brunel.ac.uk (Layal Hakim)

2. Problem Formulation

The following problem is an extension of the DLP model to linear, but time and history dependent, materials. To this end, we will replace the DLP cohesive zone stress condition, $\sigma = \sigma_y$, with the condition

$$\underline{\Lambda}(\boldsymbol{\sigma}; t) = 1, \tag{1}$$

where

$$\underline{\Lambda}(\boldsymbol{\sigma}; t) = \left(\frac{\beta}{b\sigma_0^\beta} \int_0^t |\boldsymbol{\sigma}(\tau)|^\beta (t-\tau)^{\frac{\beta}{b}-1} d\tau \right)^{\frac{1}{\beta}} \tag{2}$$

is the normalised equivalent stress, $|\boldsymbol{\sigma}|$ is the maximum of the principal stresses, and t denotes time. The parameters σ_0 and b are material constants in the assumed power-type relation $t_\infty(\sigma) = (\sigma/\sigma_0)^{-b}$ between the rupture time $t_\infty(\sigma)$ and the constant uniaxial tensile stress applied to a body without cracks. The parameter β is a material constant in the nonlinear accumulation rule for durability under variable load, see [1].

Note that relations (1)-(2) were implemented in [2] and [3] to solve the corresponding crack propagation problem but without a cohesive zone, i.e. it was assumed that when condition (1) is reached at a point, this point becomes part of the crack. However, such an approach appeared to be inapplicable for $b \geq 2$. In this paper, a cohesive zone approach is developed instead, in order to cover the larger range of b values relevant to structural materials. In this approach, when condition (1) is reached at a point, this point becomes part of the cohesive zone.

Let the problem geometry be as in Figure 1, i.e, the crack occupies the interval $[-a(t), a(t)]$ and the cohesive zone occupies the intervals $[-c(t), -a(t)]$ and $[a(t), c(t)]$ in an infinite plane loaded at infinity by traction q in the direction normal to the crack, which is constant in x, applied at the time $t = 0$ and kept constant in time thereafter.

The cohesive zone condition (1)-(2) at a point x on the cohesive zone can be rewritten as

$$\int_{t_c(x)}^t \sigma^\beta(x,\tau)(t-\tau)^{\frac{\beta}{b}-1} d\tau = \frac{b\sigma_0^\beta}{\beta} - \int_0^{t_c(x)} \sigma^\beta(x,\tau)(t-\tau)^{\frac{\beta}{b}-1} d\tau, \quad t \geq t_c(x), \tag{3}$$

where $a(t) \leq |x| \leq c(t)$. Here, $t_c(x)$ denotes the time when the point x becomes part of the cohesive zone. Equation (3) is an inhomogeneous nonlinear Volterra integral equation of the Abel type with unknown function $\sigma(x,t)$, $t \geq t_c(x)$.

Assuming that the bulk of the material is linearly elastic and applying the results by Muskhelishvili (see [4], Section 120), we have for the stresses ahead of the crack,

$$\sigma(x,t) = \frac{1}{\sqrt{x - c(t)}} \frac{x}{\sqrt{x + c(t)}} \left(q - \frac{2}{\pi} \int_{a(t)}^{c(t)} \frac{\sqrt{c^2(t) - \xi^2}}{x^2 - \xi^2} \sigma(\xi, t) d\xi \right), \quad t \leq t_c(x), \tag{4}$$

where $|x| > c(t)$. A sufficient condition for the normalised equivalent stress, Λ, to be bounded at the cohesive zone tip is that the stress intensity factor, denoted by K is zero at the cohesive zone tip. Multiplying the stress in equation (4) by $\sqrt{x - c(t)}$ and taking the limit as x tend to $c(t)$ yields

$$K(t) = \sqrt{\frac{c(t)}{2}} \left(q - \frac{2}{\pi} \int_{a(t)}^{c(t)} \frac{1}{\sqrt{c^2(t) - \xi^2}} \sigma(\xi, t) d\xi \right).$$

We will normalise σ, t and x, a and c using the following transformations but keeping the same notations:

$$\sigma \to \frac{\sigma}{q}, \ t \to \frac{t}{t_\infty(q)} = t\left(\frac{q}{\sigma_0}\right)^b, \ x \to \frac{x}{a(0)}, \ a \to \frac{a}{a(0)}, \ c \to \frac{c}{a(0)}.$$

Thus, after the normalisation, we state the following principle equations for the considered problem:
(a) the cohesive zone condition on stresses using equation (1):

$$\int_{t_c(x)}^t \sigma(x,\tau)^\beta (t-\tau)^{\frac{\beta}{b}-1} d\tau = \frac{b}{\beta} - \int_0^{t_c(x)} \sigma^\beta(x,\tau)(t-\tau)^{\frac{\beta}{b}-1} d\tau \qquad a(t) \leq |x| \leq c(t); \tag{5}$$

(b) the expression for the stress ahead of the cohesive zone:

$$\sigma(x,t) = \frac{x}{\sqrt{x^2 - c^2(t)}} \left(1 - \frac{2}{\pi} \int_{a(t)}^{c(t)} \frac{\sqrt{c^2(t) - \xi^2}}{x^2 - \xi^2} \sigma(\xi, t) d\xi \right) \qquad |x| > c(t); \tag{6}$$

(c) the zero stress intensity factor at the cohesive zone tip:

$$K(t) = -\frac{\sqrt{2c(t)}}{\pi}\int_{a(t)}^{c(t)} \frac{1}{\sqrt{c^2(t)-\xi^2}}\sigma(\xi,t)d\xi + \frac{\sqrt{c(t)}}{\sqrt{2}} = 0. \tag{7}$$

3. Cohesive Zone Growth for a Stationary Crack

First, we will consider the stage, when the crack is stationary, $a(t) = 1$, and thus only the cohesive zone grows with time. This is related to the times before the crack starts propagating. Our aim is to find the cohesive zone tip position $c(t)$ and the crack opening at the crack tip $a(t) = 1$.

3.1. Numerical Method

First, we will introduce a time mesh, with nodes t_i, for $i = 0, 1, 2, 3, ...n$. At each time step t_i, we use the secant method to find the roots, $c(t_i)$, of the equation $K(c(t_i), t_i) = 0$. For each approximation within the secant iterations, we calculate the stress intensity factor using equation (7). To compute the integral

$$\int_{a_0}^{c(t_i)} \frac{1}{\sqrt{c^2(t_i)-\xi^2}}\sigma(\xi,t_i)d\xi, \tag{8}$$

we linearly interpolate $\sigma(\xi, t_i)$ on the cohesive zone between $\xi = c(t_k)$ and $\xi = c(t_{k+1})$, where $k = 0, 1, 2, ...i-1$. On the other hand, to find $\sigma(c(t_k), t_i)$, at each $c(t_k)$ for $t_i > t_k$, we use the Abel integral equation (5). After linearly interpolating the function $\sigma^\beta(c(t_k), \tau)$ between $\tau = t_j$ and $\tau = t_{j+1}$, where $j = 0, 1, 2, ...k-1$, in the integral

$$\int_0^{t_k} \sigma^\beta(c(t_k),\tau)(t-\tau)^{\frac{\beta}{b}-1}d\tau$$

in the right hand side of equation (5), we use an analytical formula to solve the equation. To this end, in turn, we need to find $\sigma^\beta(c(t_k), t_j)$ for $t_j < t_k$. This is obtained using equation (6) (since $c(t_k) > c(t_j)$), where the integral

$$\int_{a_0}^{c(t_j)} \frac{\sqrt{c^2(t)-\xi^2}}{c(t_k)^2-\xi^2}\sigma(\xi,t_j)d\xi$$

will be calculated similar to integral (8). This means we will linearly interpolate $\sigma(\xi, t_j)$ between $\xi = c(t_m)$ and $\xi = c(t_{m+1})$ for $m = 0, 1, ...j-1$.

All programming was implemented in MATLAB.

3.2. Cohezive Zone Growth Numerical Results

Figure 2 shows the results obtained for the case $b = 4$, using various mesh sizes.

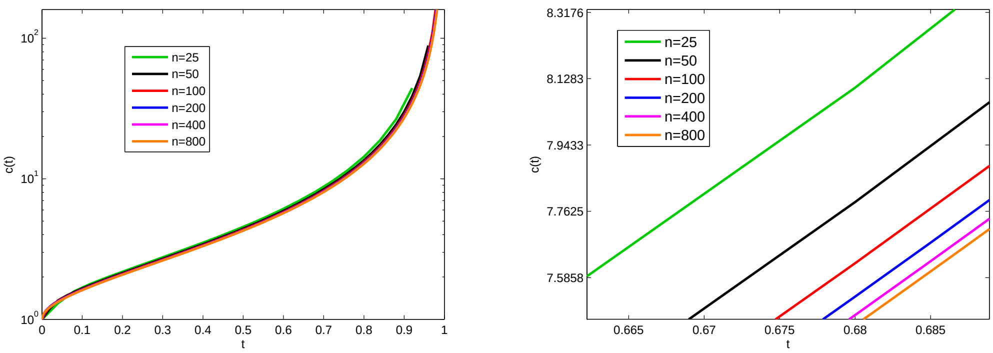

Figure 2: CZ tip position vs time for $b = 4$ and different meshes (non-propagating crack)

The graphs in Figure 3 show the results obtained when considering 3 different cases for the parameter β using $n = 500$ time steps on the interval $0 \leq t \leq 1$. The graph on the right hand side of Figure 3 is a zoomed part of the graph on the left hand side in the vicinity of the point $t = 0.56$.

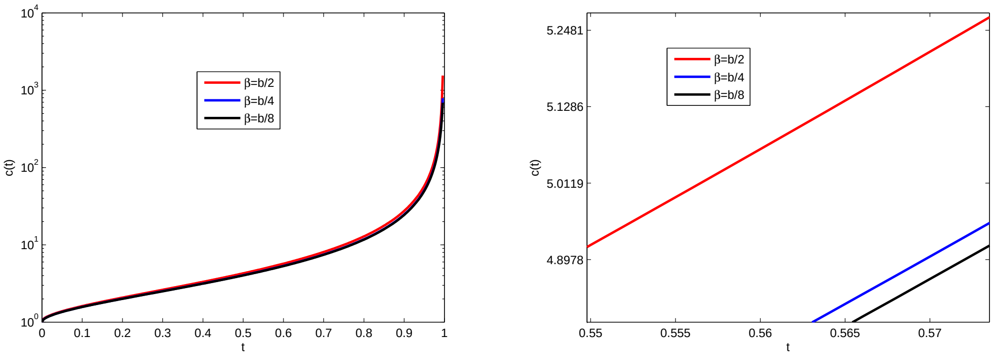

Figure 3: CZ tip position vs time for $b = 4$ and different β (non-propagating crack)

Figure 4 shows the stress behaviour with respect to time at the point $x = c(0.6)$, i.e., $t_c(x) = 0.6$.

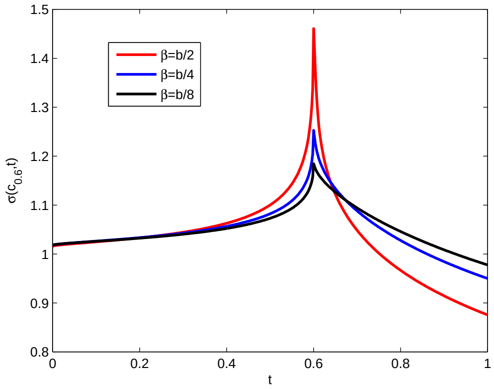

Figure 4: $\sigma(c(0.6), t)$ vs time for $b = 4$, $n = 500$

3.3. *Crack Opening*

Now, we want to find the displacement jump at the crack shores, which is also referred to as the crack opening. We denote the crack opening by $\delta(x, t)$, see Figure 1. Using the representations by Muskhelishvili (see [4], Section 120), it can be deduced that the crack opening is given by

$$\delta(x, t) = [u_2(x; c(t))] = [u_2^{(q)}(x; c(t))] + [u_2^{(\sigma)}(x; c(t), t)]$$

where

$$[u_2^{(q)}(x; c)] = \frac{4q(1 - \nu^2)}{E} \sqrt{c^2 - x^2}$$

and

$$[u_2^{(\sigma)}(x; c, t)] = \frac{4(1 - \nu^2)}{\pi E} \left(\int_a^c \sigma(\xi, t) \hat{\Gamma}(x, \xi; c) d\xi \right),$$

where

$$\hat{\Gamma}(x, \xi; c) = \ln \left[\frac{2c^2 - \xi^2 - x^2 - 2\sqrt{(c^2 - x^2)(c^2 - \xi^2)}}{2c^2 - \xi^2 - x^2 + 2\sqrt{(c^2 - x^2)(c^2 - \xi^2)}} \right].$$

In the above expressions, E and ν denote Young's modulus of elasticity and Poisson's ratio respectively. We will further normalise δ using the transformation

$$\delta \to \frac{\delta E}{aq(1-\nu^2)}. \tag{9}$$

Therefore, the normalised crack opening $\delta(x,t)$ at the crack tip, $x = a(t)$, is given as

$$\delta(a(t),t) = \frac{4}{\pi}\left(\pi\sqrt{c^2(t)-a^2(t)} + \int_{a(t)}^{c(t)} \sigma(\xi,t)\hat{\Gamma}(a(t),\xi;c(t))d\xi\right).$$

Linearly interpolating $\sigma(\xi,t_i)$ between $c(t_k)$ and $c(t_{k+1})$ for $k = 0, 1, ...i-1$, we can calculate the crack opening, $\delta(1,t_i)$, at the tip $a(t) = 1$ of the stationary crack. The graphs in Figure 5 show the results obtained.

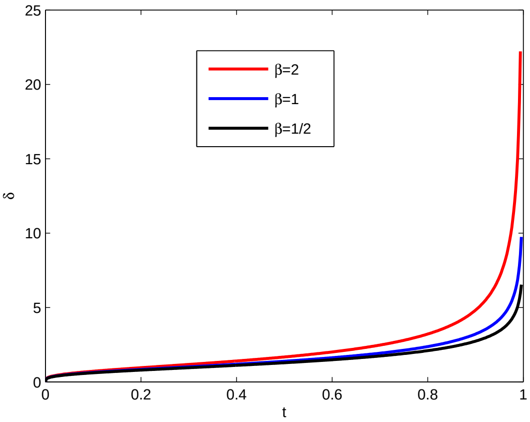

Figure 5: Crack tip opening vs time in the stationary crack for $b = 4$, $n = 500$

4. Crack Propagation

We have, so far, assumed that the crack is stationary and only the cohesive zone is growing ahead of the crack. However, at some time instant, t_d, that will be referred to as the delay time, the crack opening reaches a critical value, $\delta = \delta_c$, and the crack starts to propagate while still having the cohesive zone ahead of it. The crack and cohesive zone will not necessarily grow at the same rate. The parameter δ_c is known as the critical crack tip opening. It depends on the material and can be experimentally measured along with other material parameters. For example, for PMMA (also known as plexiglas) one can take from literature $\delta_c = 0.0016mm$, the Poisson ratio $\nu = 0.35$ and the Young modulus $E = 3.1GPa$. Thus, normalising δ_c according to (9), we will have $\delta_c = 0.113$. After determining t_d, the aim is to find the crack length and the corresponding cohesive zone length for times $t > t_d$.

4.1. Numerical Method

Consider a time mesh with uniformly spaced time nodes, t_i. During crack growth, at each step, the crack opening $\delta(a(t_i),t_i)$ equals to the critical value δ_c, which leads to the following equation

$$\frac{4}{\pi}\left(\pi\sqrt{c(t_i)^2-a(t_i)^2} + \int_{a(t_i)}^{c(t_i)} \sigma(\xi,t_i)\hat{\Gamma}(a(t_i),\xi;c(t_i))d\xi\right) - \delta_c = 0, \quad t_i \geq t_d. \tag{10}$$

For each t_i, we apply the following algorithm to obtain $a(t_i)$ and $c(t_i)$. We use the secant method to find $a(t_i)$ by setting the crack opening equal to the critical crack opening value, i.e., by solving equation (10) for $a(t_i)$. To do this, we need to know $c(t_i)$ at each iteration. It is obtained using the secant method to set the stress intensity factor to 0; i.e., to solve the equation

$$-\frac{\sqrt{2c(t_i)}}{\pi}\int_{a(t_i)}^{c(t_i)} \frac{1}{\sqrt{c^2(t_i)-\xi^2}}\sigma(\xi,t_i)d\xi + \frac{\sqrt{c(t_i)}}{\sqrt{2}} = 0, \tag{11}$$

for $c(t_i)$. Note that we choose previous cohesive zone tip positions, $c(t_m)$, as initial approximations for $a(t_i)$ within the secant algorithm. The advantage of doing this is that we already know the stress history at these previous points. For instance, for the first step after crack growth begins, it is reasonable to take the corresponding $a(t_i)$ approximations equal to $c(t_1)$ and $c(t_2)$. At some point, we will come across the step where $a(t_i)$ will exceed $c(t_{i-1})$; this means that we can only use $c(t_{i-1})$ as one of the initial approximations for $a(t_i)$. For the steps when this occurs, we fix $a(t_i)$ to be equal to $c(t_{i-1})$ and compute the corresponding t_i and $c(t_i)$ by solving equation (10) and $K(t_i) = 0$ respectively.

4.2. Numerical results

To check the algorithm, we first used in our calculations the value $\delta_c = 1.13$ for the normalised critical crack tip opening, which is 10 times larger than the corresponding value estimated for PMMA. For the case of $b = 4$, and $\beta = 1$, we obtained $t_d = 0.0102$ and present in Figure 6 the graphs showing coordinates of the crack tip and the cohesive zone tip as well as the cohesive zone length against time.

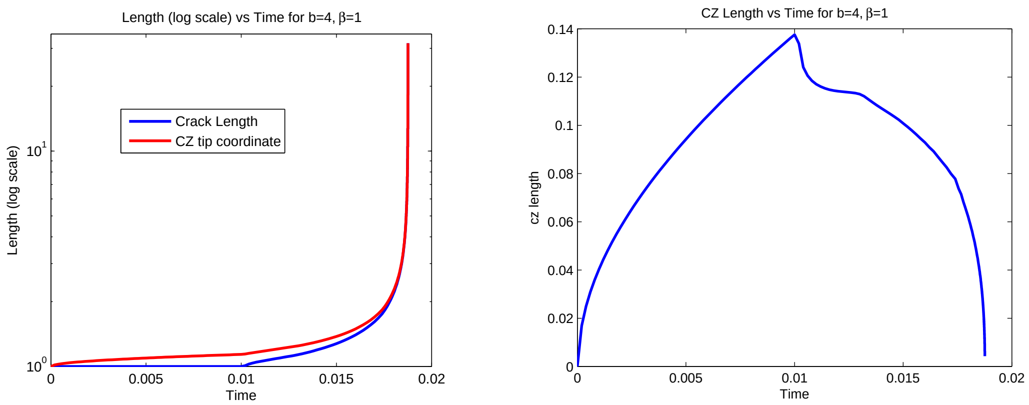

Figure 6: Length vs. time for $b = 4, \beta = 1$

5. Conclusions

The solutions for $c(t)$, $a(t)$ and $\sigma(x, t)$ converge as the mesh becomes finer. For the stationary crack stage, $t < t_d$, the rate of the cohesive zone length growth decreases when β decreases. As expected, we have an increase, with time, of the crack opening. Moreover, as β becomes smaller, the crack opening increases more slowly with time. For the growing crack stage, $t > t_d$, we can see from Figure 6, that the crack growth rate increases, while the cohesive zone length decreases with time. The time, when the cohesive zone time becomes 0 seems to coincide with the time when the crack becomes infinite and can be associated with the complete fracture of the body.

References

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