Synchronization of Coupled Neutral-Type Neural Networks with Jumping-Mode-dependent Discrete and Unbounded Distributed Delays

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***Abstract*—In this paper, the synchronization problem is studied for an array of N identical delayed neutral-type neural networks with Markovian jumping parameters. The coupled networks involve both the mode-dependent discrete time-delays and the mode-dependent unbounded distributed time-delays. All the network parameters including the coupling matrix are also dependent on the Markovian jumping mode. By introducing novel Lyapunov-Krasovskii functionals and using some analytical techniques, sufficient conditions are derived to guarantee that the coupled networks are asymptotically synchronized in mean square. The derived sufficient conditions are closely related with the discrete time-delays, distributed time-delays, mode transition probability and coupling structure of the networks. The obtained criteria are given in terms of matrix inequalities that can be solved efficiently by employing the semi-definite programme method. Numerical simulations are presented to further demonstrate the effectiveness of the proposed approach.**

***Index Terms*—Synchronization; neutral-type neural networks; Markovian jumping systems; discrete time-delay; unbounded distributed time-delay; Kronecker product.**

I. Introduction

In the last decade, recurrent neural networks (RNNs) have drawn noticeable attention from many researchers working in a variety of areas such as signal and image processing, associative memories, combinatorial optimization and automatic control [1], [12], [18], [24], [30]. While traditional neural networks have been successfully applied in static data-based classification and prediction problems for various engineering systems, the dynamical behaviors of the RNNs have recently gained a lot of research interests due to their capabilities of using dynamical temporal behavior to process arbitrary sequences of inputs. Motivated from both the basic science and the technological practice, the study of synchronization problems among an array of neural networks has been an active topic of research in the past few years, see [13], [15], [20], [22], [23] for some recent publications. Note that the original notion of synchronization dates back to the 1980s after the theory of deterministic chaos has been developed. Since then, the synchronization research has been extended to the case of more complex systems, for example, the large-scale and complex networks of chaotic oscillators [14], [34], the coupled systems exhibiting spatio-temporal chaos and autowaves [28], [41], and the array of coupled neural networks with or without delays [7], [27], [37].

In practice, due to the finite speeds of the switching and transmitting signals, time delays exist in various RNNs [1], [16], [17]. It is well known that time delays may result in oscillatory behaviors or network instability (periodic oscillation and chaos). So far, most of the existing results related to the synchronization analysis for RNNs have been concerned with the discrete delay (point delay) case. Recently, the distributed delay has received an increasing research interest due to the presence of an amount of parallel pathways with a variety of axon sizes and lengths. Furthermore, as a combination of both discrete and distributed delays, the so-called mixed time-delays have gained much research attention and many relevant results have been reported in the literature, see e.g. [35], [38], [39] and the references therein. It should be pointed out that, rather than occurring in the system states (or outputs), time-delays can also appear in the derivatives of system states [4], [25], [26], [40]. This kind of time-delays is referred to as the neutral time-delays that can find a variety of applications in practice such as chemical reactors, transmission lines, partial element equivalent circuits in VLSI systems, and Lotka-Volterra systems [8]. Because of possible presence of neutral delays in implementing RNNs in VLSI circuits, the RNNs with neutral terms have stirred some attention in the past few years, see e.g. [8], [9], [19].

During the course of implementation, the RNNs often encounter the information latching problems [5], that is, the network states have finite representations (also called clusters, patterns, or modes) where the switching among the finite states is sometimes governed by a *Markovian chain*. Such kind of random mode switches may result from abrupt phenomena such as stochastic failures and repairs of the network components, changes in the interconnections of network nodes, or sudden environment switching. As such, the so-called Markovian jumping recurrent neural networks (MJRNNs) have attracted a great deal of research interest [31], [36], [42] in the past decade. For example, in [36], the exponential stability problem has been first addressed for a class of delayed recurrent neural networks with Markovian jumping parameters. In [42], the problem of exponential stability has

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been investigated for a class of stochastic neural networks with both Markovian jump parameters and mixed time delays. In [31], a noise-induced stabilization method has been proposed for RNNs with mixed time-varying delays and Markovian switching parameters. In [38], the passivity analysis has been conducted for discrete-time stochastic neural networks with both Markovian jumping parameters and mixed time delays.

Summarizing the discussion made so far, the RNNs often exhibit the phenomena of signal transmission delays and possess Markovian mode jumping behavior, where the delays could be of discrete, distributed and neutral types. As such, it should be of both theoretical and practical significance to consider the synchronization problem of such RNNs. Unfortunately, the synchronization issue for *Markovian jumping neutral-type neural networks with mode-dependent mixed time-delays* has received very little research effort due primarily to the mathematical complexity. It is, therefore, the motivation of our current investigation to shorten such a gap by launching a study on the synchronization problem for Markovian jumping neural networks of neutral type where *all discrete, distributed and neutral delays are mode-dependent and the distributed delays are allowed to be unbounded*. It is noticeable that, in two recent papers [2], [3], the passivity and stability analysis problems have been addressed for neural networks of neutral type with Markovian jumping parameters and time delays, where the time-delays are not mode-dependant.

In this paper, we are concerned with the synchronization problem for a new class of continuous-time neural networks of neutral-type with Markovian jumping parameters as well as mode-dependent mixed time-delays. Note that the mixed time-delays comprise both the discrete and distributed delays that are all dependent on the Markovian jumping mode. *The main contributions of this paper can be highlighted as follows: 1) some novel analysis techniques are developed to tackle the mathematical difficulty resulting from the presence of the mode-dependent neutral delays; 2) a new Lyapunov functional is proposed to reflect the Markovian jumps of the delay bounds; and 3) a unified framework is established to handle the Markovian jumping parameters, neutral terms and mixed time-delays*. We derive sufficient conditions to guarantee that the coupled networks are asymptotically synchronized in mean square. Note that the derived sufficient conditions are expressed by means of the system parameters, discrete time-delays, distributed time-delays, mode transition probability and coupling structure of the array of neural networks. Such conditions are in the form of LMIs, which could be easily checked by utilizing the recently developed interior-point methods available in Matlab toolbox, and no turning of parameters will be needed. Numerical simulations are presented to further demonstrate the effectiveness of the proposed approach.

Notations: The notations are quite standard. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes matrix transposition and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). I_n is the $n \times n$ identity matrix. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^TA)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. The Kronecker product of an $n\times m$ matrix X and a $p\times q$ matrix Y is defined by an $np\times mq$ matrix $X \otimes Y$ as follows

$$X \otimes Y = \begin{bmatrix} x_{11}Y & \cdots & x_{1m}Y \\ \vdots & & \vdots \\ x_{n1}Y & \cdots & x_{nm}Y \end{bmatrix}.$$

The asterisk $*$ in a matrix is used to denote term that is induced by symmetry. $\mathbb{E}[x]$ and $\mathbb{E}[x|y]$ will, respectively, mean the expectation of x and the expectation of x conditional on y. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem formulation

Let $r(t)$ $(t \geq 0)$ be a right-continuous Markov chain on a probability space taking values in a finite state space $\mathscr{N} = \{1, 2, ..., n_0\}$ with generator $\Pi = \{\pi_{ij}\}$ given by

$$P\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \pi_{ij}\Delta + o(\Delta), & \text{if } i = j. \end{cases}$$

Here $\Delta > 0$, and $\pi_{ij} \geq 0$ is the transition rate from i to j if $j \neq i$ while

$$\pi_{ii} = -\sum_{j\neq i} \pi_{ij}.$$

For a given array of N identical neutral-type neural networks, we assume that each single neural network consists of n neurons and the dynamics of kth neutral-type neural network is governed by

$$\begin{aligned} \dot{x}_k(t) &= E(r(t))\dot{x}_k(t-\tau_{1,r(t)}) - A(r(t))x_k(t) \\ &\quad + B(r(t))f(x_k(t)) + C(r(t))g(x_k(t-\tau_{2,r(t)})) \\ &\quad + D(r(t))\int_{-\infty}^{t-\tau_{3,r(t)}} \varphi(t-s)h(x_k(s))ds + u(t), \end{aligned} \tag{1}$$

where $x_k(t) = [x_{k1}(t), x_{k2}(t), \cdots, x_{kn}(t)]^T$ is the state vector of the kth delayed neural network; $A(r(t)) = \text{diag}\{a_1(r(t)), a_2(r(t)), ..., a_n(r(t))\} > 0$ is a diagonal matrix with a_j representing the rate with which the jth neuron will reset its potential to the resting state in isolation; $B(r(t)) = (b_{ij}(r(t)))_{n\times n}$, $C(r(t)) = (c_{ij}(r(t)))_{n\times n}$, $D(r(t)) = (d_{ij}(r(t)))_{n\times n}$ and $E(r(t)) = (e_{ij}(r(t)))_{n\times n}$ denote connection weight matrices of the neurons; $u(t) = [u_1(t), ..., u_n(t)]^T$ is the input vector function; and $f(\cdot) = (f_1(\cdot), f_2(\cdot), ..., f_n(\cdot))^T$, $g(\cdot) = (g_1(\cdot), g_2(\cdot), ..., g_n(\cdot))^T$, $h(\cdot) = (h_1(\cdot), h_2(\cdot), ..., h_n(\cdot))^T$ denote the activation function vectors; $\tau_{1,r(t)}$ and $\tau_{2,r(t)}$ denote the mode-dependent discrete time delays while $\tau_{3,r(t)}$ characterizes the mode-dependent upper bound of the distributed time-delay.

Consider the following linearly coupled dynamical system comprising the above N identical neutral-type neural networks:

$$\begin{aligned}\dot{x}_k(t) =\ & \dot{x}_k(t-\tau_{1,r(t)}) - A(r(t))x_k(t) + B(r(t))f(x_k(t)) \\ & + C(r(t))g(x_k(t-\tau_{2,r(t)})) \\ & + D(r(t))\int_{-\infty}^{t-\tau_{3,r(t)}}\varphi(t-s)h(x_k(s))ds + u(t) \\ & + \sum_{j=1}^{N} w_{kj}(r(t))\Gamma(r(t))x_j(t), k=1,2,...,N, \quad (2)\end{aligned}$$

where $\Gamma(r(t)) = \mathrm{diag}(\gamma_1(r(t)), \gamma_2(r(t)), ..., \gamma_n(r(t))) \geq 0$ is a diagonal matrix linking the jth state variable of each neural network if $\gamma_j(r(t)) \neq 0$; $W = (w_{ij}(r(t))) \in \mathbb{R}^{N\times N}$ is the coupling configuration matrix of the system with $w_{ij}(r(t)) \geq 0$ $(i \neq j)$ but not all zero.

Remark 1: In the array of coupled neural networks (2), the distributed delay $\int_{-\infty}^{t-\tau_{3,r(t)}} \varphi(t-s)h(x_k(s))ds$ is included with the upper bound dependent on the Markov chain. Note that the time-delay s can vary from $-\infty$ to $t-\tau_{3,r(t)}$ in a distributed way. As such, the unboundedness and the mode-dependence of such a distributed time-delay would have a great impact on the stability analysis on the overall coupled system. For the practical applications of such unbounded distributed delays, we refer the authors to [10], [11], [21]. It is worth mentioning that the finite distributed delays, which are another type of distributed delays whose lower and upper bounds are both limited, have been intensively investigated in [35], [38], [39].

Throughout this paper, we make the following assumptions.

Assumption 1: The coupling configuration matrix $W(r(t)) = (w_{ij}(r(t)))$ is symmetric (i.e., $W(r(t)) = W^T(r(t))$) and satisfies

$$\sum_{j=1}^{N} w_{ij} = \sum_{j=1}^{N} w_{ji} = 0,\ i=1,2,...,N. \quad (3)$$

Assumption 2: As in [21], for $j \in \{1,2,...,n\}, \forall s_1, s_2 \in \mathbb{R}, s_1 \neq s_2$, the neuron activation functions satisfy

$$l_j^- \leq \frac{f_j(s_1)-f_j(s_2)}{s_1-s_2} \leq l_j^+, \quad (4)$$

$$\sigma_j^- \leq \frac{g_j(s_1)-g_j(s_2)}{s_1-s_2} \leq \sigma_j^+, \quad (5)$$

$$\upsilon_j^- \leq \frac{h_j(s_1)-h_j(s_2)}{s_1-s_2} \leq \upsilon_j^+, \quad (6)$$

where l_j^-, l_j^+, σ_j^-, σ_j^+, υ_j^-, υ_j^+ are some constants.

Remark 2: As discussed in [21], the constants l_j^-, l_j^+, σ_j^-, σ_j^+, υ_j^-, υ_j^+ in Assumption 2 are allowed to be positive, negative or zero. Hence, the resulting activation functions could be non-monotonic, and more general than the usual sigmoid functions. In addition, when using Lyapunov stability theory to analyze the stability, such a description is particularly suitable since it quantifies the lower and upper bounds of the activation functions that offer the possibility of reducing the induced conservatism.

Assumption 3: The delay kernel $\varphi(\cdot) : [0,+\infty) \to [0,+\infty)$ is continuous and integrable, and also satisfies

$$\int_0^{+\infty}\varphi(s)ds < +\infty, \quad \int_0^{+\infty} s\varphi(s)ds < +\infty. \quad (7)$$

Let

$$\begin{aligned} x(t) &= (x_1^T(t), x_2^T(t), ..., x_N^T(t))^T, \\ \mathbf{f}(x(t)) &= (f^T(x_1(t)), f^T(x_2(t)), ..., f^T(x_N(t)))^T, \\ \mathbf{g}(x(t)) &= (g^T(x_1(t)), g^T(x_2(t)), ..., g^T(x_N(t)))^T, \\ \mathbf{h}(x(t)) &= (h^T(x_1(t)), h^T(x_2(t)), ..., h^T(x_N(t)))^T, \\ \mathbf{u}(t) &= (u^T(t), u^T(t), ..., u^T(t))^T. \end{aligned}$$

With the above symbols and the Kronecker product of matrices, we rewrite the system (2) in the following compact form:

$$\begin{aligned}\dot{x}(t) =\ & (I_N \otimes E(r(t)))\dot{x}(t-\tau_{1,r(t)}) - (I_N \otimes A(r(t)))x(t) \\ & + (I_N \otimes B(r(t)))\mathbf{f}(x(t)) + (I_N \otimes C(r(t))) \\ & \times \mathbf{g}(x(t-\tau_{2,i})) + (I_N \otimes D(r(t)))\int_{-\infty}^{t-\tau_{3,r(t)}}\varphi(t-s)\mathbf{h}(x(s))ds \\ & + \mathbf{u}(t) + W(r(t)) \otimes \Gamma(r(t))x(t). \quad (8)\end{aligned}$$

Definition 1: The coupled system (2) or (8) is said to be globally asymptotically synchronized in mean square if

$$\lim_{t\to\infty} \mathbb{E}|x_k(t) - x_l(t)|^2 = 0$$

holds for any $k, l \in \{1,2,...,N\}$.

In this paper, we aim to deal with the synchronization problem of the system (8) coupled by an array of N identical delayed neutral-type neural networks with Markovian jumping parameters. The coupled networks involve both mode-dependent discrete time-delays and distributed time-delays with the mode-dependent upper bound. The coupled matrices are allowed to be mode-dependent as well. By constructing novel Lyapunov-Krasovskii functionals and using some analytical techniques, we shall derive easy-to-verify sufficient conditions to guarantee the coupled system to be asymptotically synchronized in mean square. The obtained criteria are given in terms of matrix inequalities that can be solved efficiently by employing the semi-definite programme method.

III. Main results and proofs

Before stating our main results, we introduce the following lemmas.

Lemma 1 ([20]): Let $\mathcal{U} = (\alpha_{ij})_{N\times N}$, $P \in \mathbb{R}^{n\times n}$, $x = (x_1^T, x_2^T, ..., x_N^T)^T$, and $y = (y_1^T, y_2^T, ..., y_N^T)^T$ with x_i, $y_i \in \mathbb{R}^n$. If $\mathcal{U} = \mathcal{U}^T$ and each row sum of $\mathcal{U}$ is zero, then

$$x^T(\mathcal{U}\otimes P)y = -\sum_{1\leq i<j\leq N}\alpha_{ij}(x_i-x_j)P(y_i-y_j).$$

Lemma 2 ([20]): Suppose that $\mathcal{B} = \mathrm{diag}\{\beta_1, \beta_2, ..., \beta_n\}$ is a positive semi-definite diagonal matrix. Let $y = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n$, and $\mathcal{H}(y) = (\hbar_1(y_1), \hbar_2(y_2), ..., \hbar_n(y_n))^T$ be a continuous nonlinear function satisfying

$$l_i^- \leq \frac{\hbar_i(s)}{s} \leq l_i^+,\ s\neq 0,\ s\in\mathbb{R},\quad i=1,2,...,n \quad (9)$$

with l_i^- and l_i^+ being constant scalars. Then

$$y^T\mathcal{B}L_1y - 2y^T\mathcal{B}L_2\mathcal{H}(y) + \mathcal{H}^T(y)\mathcal{B}\mathcal{H}(y) \leq 0$$

where $L_1 = \mathrm{diag}\{l_1^+l_1^-, l_2^+l_2^-, ..., l_n^+l_n^-\}$ and $L_2 = \mathrm{diag}\{\frac{l_1^++l_1^-}{2}, \frac{l_2^++l_2^-}{2}, ..., \frac{l_n^++l_n^-}{2}\}$.

$$\Phi_{kl}(i) = \begin{pmatrix} \Xi_{11}(i) & \Xi_{12}(i) & \Theta_i\Sigma_2 & \Xi_{14}(i) & \Omega_i\Upsilon_2 & \Xi_{16}(i) & \Xi_{17} & -\sqrt{\kappa_1}A(i)Q \\ * & -\Lambda_i & 0 & 0 & 0 & 0 & 0 & \sqrt{\kappa_1}B^T(i)Q \\ * & * & \Xi_{33}(i) & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -R & 0 & 0 & 0 & \sqrt{\kappa_1}C^T(i)Q \\ * & * & * & * & \Xi_{55}(i) & 0 & 0 & 0 \\ * & * & * & * & * & -\frac{1}{\alpha_i}S & 0 & \sqrt{\kappa_1}D^T(i)Q \\ * & * & * & * & * & * & -Q & \sqrt{\kappa_1}E^T(i)Q \\ * & * & * & * & * & * & * & -Q \end{pmatrix} < 0, \quad (11)$$

Lemma 3 ([22]): Let M be a positive semi-definite matrix, $\alpha(\cdot) : (-\infty, a] \to [0, +\infty)$ be a scalar function and $\mathcal{F}(\cdot) : (-\infty, a] \to \mathbb{R}^n$ be a vector function. If the integrations concerned are well defined, the following inequality holds:

$$\begin{aligned} &\left(\int_{-\infty}^{a} \alpha(s)\mathcal{F}(s)ds\right)^T M \left(\int_{-\infty}^{a} \alpha(s)\mathcal{F}(s)ds\right) \\ \leq\ & \int_{-\infty}^{a} \alpha(s)ds \left(\int_{-\infty}^{a} \alpha(s)\mathcal{F}^T(s)M\mathcal{F}(s)ds\right). \end{aligned} \quad (10)$$

Lemma 4 (Schur Complement [6]): Given constant matrices $\Omega_1, \Omega_2, \Omega_3$ where $\Omega_1 = \Omega_1^T$ and $\Omega_2 > 0$, then

$$\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0.$$

Lemma 5 (Barbalat's Lemma [32]): Let f be a nonnegative function defined on $[0, +\infty)$. If f is Lebesgue integrable on $[0, +\infty)$ and is uniformly continuous on $[0, +\infty)$, then $\lim_{t\to+\infty} f(t) = 0$.

For presentation convenience, in the following, we denote

$$\begin{aligned} L_1 &= \text{diag}\{l_1^+ l_1^-, ..., l_n^+ l_n^-\}, \\ L_2 &= \text{diag}\left\{\frac{l_1^+ + l_1^-}{2}, ..., \frac{l_n^+ + l_n^-}{2}\right\}, \\ \Sigma_1 &= \text{diag}\{\sigma_1^+ \sigma_1^-, ..., \sigma_n^+ \sigma_n^-\}, \\ \Sigma_2 &= \text{diag}\left\{\frac{\sigma_1^+ + \sigma_1^-}{2}, ..., \frac{\sigma_n^+ + \sigma_n^-}{2}\right\}, \\ \Upsilon_1 &= \text{diag}\{\upsilon_1^+ \upsilon_1^-, ..., \upsilon_n^+ \upsilon_n^-\}, \\ \Upsilon_2 &= \text{diag}\left\{\frac{\upsilon_1^+ + \upsilon_1^-}{2}, ..., \frac{\upsilon_n^+ + \upsilon_n^-}{2}\right\}, \\ \overline{\tau}_1 &= \max_{1\leq j\leq n_0}\{\tau_{1,j}\},\ \overline{\tau}_2 = \max_{1\leq j\leq n_0}\{\tau_{2,j}\},\ \overline{\tau}_3 = \max_{1\leq j\leq n_0}\{\tau_{3,j}\}, \\ \underline{\tau}_1 &= \min_{1\leq j\leq n_0}\{\tau_{1,j}\},\ \underline{\tau}_2 = \min_{1\leq j\leq n_0}\{\tau_{2,j}\},\ \underline{\tau}_3 = \min_{1\leq j\leq n_0}\{\tau_{3,j}\}, \\ \bar{\pi} &= \max_{1\leq i\leq n_0}\{|\pi_{ii}|\}. \end{aligned}$$

The main results of this paper are given in the following theorem.

Theorem 1: Under Assumptions 1-3, the system (8) is globally asymptotically synchronized in mean square if there exist six positive definite matrices P_1, P_2, P_3, Q, R and S, and three sets of positive definite diagonal matrices Λ_i, Θ_i and $\Omega_i (1 \leq i \leq n_0)$ such that the following LMIs (11) shown at the top of the page hold for $1 \leq k < l \leq N,\ 1 \leq i \leq n_0$, where

$$\bar{\varphi} = \bar{\pi} \sup_{\underline{\tau}_3 \leq s \leq \overline{\tau}_3} \varphi(s),\ \alpha_i = \int_{\tau_{3,i}}^{+\infty} \varphi(s)ds, \quad (12)$$

$$\hat{\alpha}_i = \alpha_i + \frac{1}{2}\bar{\varphi}(\overline{\tau}_3^2 - \underline{\tau}_3^2),\ \kappa_1 = \bar{\pi}(\overline{\tau}_1 - \underline{\tau}_1) + 1, \quad (13)$$

$$\begin{aligned} \Xi_{11}(i) = &-P_iA(i) - A(i)P_i + \overline{P}_i - Nw_{kl}(i)\big(P_i\Gamma(i) \\ &+ \Gamma(i)P_i\big) - \big(\Lambda_iL_1 + \Theta_i\Sigma_1 + \Omega_i\Upsilon_1\big) \\ &+ \kappa_1 Nw_{kl}(i)\big(\Gamma(i)QA(i) + A(i)Q\Gamma(i)\big) \\ &- \kappa_1 Nw_{kl}^{(2)}(i)\Gamma(i)Q\Gamma(i), \end{aligned} \quad (14)$$

$$\Xi_{12}(i) = P_iB(i) + \Lambda_iL_2 - \kappa_1 Nw_{kl}(i)\Gamma(i)QB(i), \quad (15)$$

$$\Xi_{33}(i) = [\bar{\pi}(\overline{\tau}_2 - \underline{\tau}_2) + 1]R - \Theta_i, \quad (16)$$

$$\Xi_{14}(i) = P_iC(i) - \kappa_1 Nw_{kl}(i)\Gamma(i)QC(i), \quad (17)$$

$$\Xi_{55}(i) = \hat{\alpha}_iS - \Omega_i, \quad (18)$$

$$\Xi_{16}(i) = P_iD(i) - \kappa_1 Nw_{kl}(i)\Gamma(i)QD(i), \quad (19)$$

$$\Xi_{17}(i) = P_iE(i) - \kappa_1 Nw_{kl}(i)\Gamma(i)QE(i), \quad (20)$$

and $w_{kl}^{(2)}(i)$ is the (k,l)-th entry of matrix $[W(i)]^2$.

Proof: Define $x_t(\cdot)$ by $x_t(s) = x(t+s)$ $(-\infty < s \leq 0)$ and denote

$$U = \begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & N-1 \end{bmatrix}_{N\times N}.$$

In order to tackle the synchronization problem of (8), we introduce the following Lyapunov-Krasovskii functional candidate:

$$V(x_t, t, r(t)) := \sum_{k=1}^{6} V_k(x_t, t, r(t)) \quad (21)$$

where

$$\begin{aligned} V_1(x_t, t, r(t)) &= x^T(t)(U \otimes P_{r(t)})x(t), \\ V_2(x_t, t, r(t)) &= \int_{t-\tau_{1,r(t)}}^{t} \dot{x}^T(s)(U \otimes Q)\dot{x}(s)ds, \\ V_3(x_t, t, r(t)) &= \int_{t-\tau_{2,r(t)}}^{t} \mathbf{g}^T(x(s))(U \otimes R)\mathbf{g}(x(s))ds, \\ V_4(x_t, t, r(t)) &= \bar{\pi}\int_{\underline{\tau}_1}^{\overline{\tau}_1}\int_{t-s}^{t} \dot{x}^T(\theta)(U \otimes Q)\dot{x}(\theta)d\theta ds \\ &\quad + \bar{\pi}\int_{\underline{\tau}_2}^{\overline{\tau}_2}\int_{t-s}^{t} \mathbf{g}^T(x(\theta))(U \otimes R)\mathbf{g}(x(\theta))d\theta ds, \end{aligned}$$

$$V_5(x_t,t,r(t)) = \int_{\tau_{3,r(t)}}^{+\infty}\varphi(s)\int_{t-s}^{t}\mathbf{h}^T(x(\eta)) \times (U\otimes S)\mathbf{h}(x(\eta))d\eta ds,$$
$$V_6(x_t,t,r(t)) = \bar\varphi\int_{\underline{\tau}_3}^{\overline{\tau}_3}\int_0^u\int_{t-s}^{t}\mathbf{h}^T(x(\eta)) \times (U\otimes S)\mathbf{h}(x(\eta))d\eta ds du$$

with $\bar\varphi$ defined in (12).

Let $\mathscr{L}$ be the weak infinitesimal generator of the random process $\{(x_t,r(t)),\ t\ge 0\}$ along the network (8) defined by

$$\mathscr{L}V(x_t,t,i) = \lim_{\Delta\to 0^+}\frac{1}{\Delta}\Big[\mathbb{E}\big[V\big(x_{t+\Delta},t+\Delta,r(t+\Delta)\big)\ \Big|\ x_t, r(t)=i\big] - V(x_t,t,i)\Big].$$

Then, we have

$$\mathscr{L}V(x_t,t,i) = \sum_{k=1}^{6}\mathscr{L}V_k(x_t,t,i) \tag{22}$$

where $\mathscr{L}V_k(x_t,t,i)$ $(k = 1,2,\cdots,6)$ are calculated as follows.

First of all, it follows that

$$\begin{aligned}
&\mathscr{L}V_1(x_t,t,i)\\
&= 2x^T(t)(U\otimes P_i)\Big[(I_N\otimes E(i))\dot x(t-\tau_{1,r(t)})\\
&\quad -(I_N\otimes A(i))x(t)+(I_N\otimes B(i))\mathbf{f}(x(t))\\
&\quad +(I_N\otimes C(i))\mathbf{g}(x(t-\tau_{2,i}))+(I_N\otimes D(i))\\
&\quad \times\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\mathbf{h}(x(s))ds+\mathbf{u}(t)\\
&\quad + W(i)\otimes\Gamma(i)x(t)\Big]+\sum_{j=1}^{n_0}\pi_{ij}x^T(t)(U\otimes P_j)x(t)\\
&= 2x^T(t)\Big[(U\otimes(P_iE(i)))\dot x(t-\tau_{1,r(t)})\\
&\quad -(U\otimes(P_iA(i)))x(t)+(U\otimes(P_iB(i)))\mathbf{f}(x(t))\\
&\quad +(U\otimes(P_iC(i)))\mathbf{g}(x(t-\tau_{2,i}))+(U\otimes(P_iD(i)))\\
&\quad \int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\mathbf{h}(x(s))ds+NW(i)\otimes(P_i\Gamma(i))x(t)\Big]\\
&\quad + x^T(t)(U\otimes\overline{P}_i)x(t),
\end{aligned}\tag{23}$$

where we have used the facts that $UW(i) = NW(i)$ and $U\otimes\mathbf{u}(t)=0$, which are not difficult to verify.

Next, it can be obtained that

$$\begin{aligned}
&\mathscr{L}V_2(x_t,t,i)\\
=\ & \dot x^T(t)(U\otimes Q)\dot x(t)-\dot x^T(t-\tau_{1,i})(U\otimes Q)\dot x(t-\tau_{1,i})\\
&+\sum_{j=1}^{n_0}\pi_{ij}\int_{t-\tau_{1,j}}^{t}\dot x^T(s)(U\otimes Q)\dot x(s)ds\\
=\ & \dot x^T(t)(U\otimes Q)\dot x(t)-\dot x^T(t-\tau_{1,i})(U\otimes Q)\dot x(t-\tau_{1,i})\\
&+\sum_{j\neq i}^{n_0}\pi_{ij}\left[\int_{t-\tau_{1,i}}^{t}+\int_{t-\tau_{1,j}}^{t-\tau_{1,i}}\right]\dot x^T(s)(U\otimes Q)\dot x(s)ds\\
&+\pi_{ii}\int_{t-\tau_{1,i}}^{t}\dot x^T(s)(U\otimes Q)\dot x(s)ds\\
\le\ & \dot x^T(t)(U\otimes Q)\dot x(t)-\dot x^T(t-\tau_{1,i})(U\otimes Q)\dot x(t-\tau_{1,i})\\
&+\bar\pi\int_{t-\overline{\tau}_1}^{t-\underline{\tau}_1}\dot x^T(s)(U\otimes Q)\dot x(s)ds.
\end{aligned}\tag{24}$$

Similar to (24), it follows that

$$\begin{aligned}
\mathscr{L}V_3(x_t,t,i) \le\ & \mathbf{g}^T(x(t))(U\otimes R)\mathbf{g}(x(t))\\
&-\mathbf{g}^T(x(t-\tau_{2,i}))(U\otimes R)\mathbf{g}(x(t-\tau_{2,i}))\\
&+\bar\pi\int_{t-\overline{\tau}_2}^{t-\underline{\tau}_2}\mathbf{g}^T(x(s))(U\otimes R)\mathbf{g}(x(s))ds.
\end{aligned}\tag{25}$$

It is easy to see

$$\begin{aligned}
\mathscr{L}V_4(x_t,t,i) =\ & \bar\pi(\overline{\tau}_1-\underline{\tau}_1)\dot x^T(t)(U\otimes Q)\dot x(t)\\
&-\bar\pi\int_{t-\overline{\tau}_1}^{t-\underline{\tau}_1}\dot x^T(s)(U\otimes Q)\dot x(s)ds\\
&+\bar\pi(\overline{\tau}_2-\underline{\tau}_2)\mathbf{g}^T(x(t))(U\otimes R)\mathbf{g}(x(t))\\
&-\bar\pi\int_{t-\overline{\tau}_2}^{t-\underline{\tau}_2}\mathbf{g}^T(x(s))(U\otimes R)\mathbf{g}(x(s))ds.
\end{aligned}\tag{26}$$

Then, it follows

$$\begin{aligned}
&\mathscr{L}V_5(x_t,t,i)\\
&=\int_{\tau_{3,i}}^{+\infty}\varphi(s)ds\mathbf{h}^T(x(t))(U\otimes S)\mathbf{h}(x(t))\\
&\quad -\int_{\tau_{3,i}}^{+\infty}\varphi(s)\mathbf{h}^T(x(t-s))(U\otimes S)\mathbf{h}(x(t-s))ds\\
&\quad +\sum_{j=1}^{n_0}\pi_{ij}\int_{\tau_{3,j}}^{+\infty}\varphi(s)\int_{t-s}^{t}\mathbf{h}^T(x(\eta))(U\otimes S)\mathbf{h}(x(\eta))d\eta ds\\
&=\alpha_i\mathbf{h}^T(x(t))(U\otimes S)\mathbf{h}(x(t))\\
&\quad -\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\mathbf{h}^T(x(s))(U\otimes S)\mathbf{h}(x(s))ds\\
&\quad +\sum_{j\neq i}\pi_{ij}\int_{\tau_{3,j}}^{+\infty}\varphi(s)\int_{t-s}^{t}\mathbf{h}^T(x(\eta))(U\otimes S)\mathbf{h}(x(\eta))d\eta ds\\
&\quad +\pi_{ii}\int_{\tau_{3,i}}^{+\infty}\varphi(s)\int_{t-s}^{t}\mathbf{h}^T(x(\eta))(U\otimes S)\mathbf{h}(x(\eta))d\eta ds\\
&=\alpha_i\mathbf{h}^T(x(t))(U\otimes S)\mathbf{h}(x(t))\\
&\quad -\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\mathbf{h}^T(x(s))(U\otimes S)\mathbf{h}(x(s))ds\\
&\quad +\sum_{j\neq i}\pi_{ij}\int_{\tau_{3,j}}^{\tau_{3,i}}\varphi(s)\int_{t-s}^{t}\mathbf{h}^T(x(\eta))(U\otimes S)\mathbf{h}(x(\eta))d\eta ds\\
&\le\alpha_i\mathbf{h}^T(x(t))(U\otimes S)\mathbf{h}(x(t))\\
&\quad -\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\mathbf{h}^T(x(s))(U\otimes S)\mathbf{h}(x(s))ds\\
&\quad +\bar\pi\max_{\underline{\tau}_3\le s\le\overline{\tau}_3}\varphi(s)\int_{\underline{\tau}_3}^{\overline{\tau}_3}\int_{t-s}^{t}\mathbf{h}^T(x(\eta))(U\otimes S)\mathbf{h}(x(\eta))d\eta ds\\
&=\alpha_i\mathbf{h}^T(x(t))(U\otimes S)\mathbf{h}(x(t))\\
&\quad -\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\mathbf{h}^T(x(s))(U\otimes S)\mathbf{h}(x(s))ds\\
&\quad +\bar\varphi\int_{\underline{\tau}_3}^{\overline{\tau}_3}\int_{t-s}^{t}\mathbf{h}^T(x(\eta))(U\otimes S)\mathbf{h}(x(\eta))d\eta ds,
\end{aligned}\tag{27}$$

where α_i is defined in (12).

Finally, we have

$$\begin{aligned}
&\mathscr{L}V_6(x_t,t,i)\\
=\;& \bar{\varphi}\int_{\underline{\tau}_3}^{\overline{\tau}_3}\int_0^u \mathbf{h}^T(x(t))(U\otimes S)\mathbf{h}(x(t))d\eta ds du\\
&-\bar{\varphi}\int_{\underline{\tau}_3}^{\overline{\tau}_3}\int_0^u \mathbf{h}^T(x(t-s))(U\otimes S)\mathbf{h}(x(t-s))ds du\\
=\;& \frac{1}{2}\bar{\varphi}(\overline{\tau}_3^2-\underline{\tau}_3^2)\mathbf{h}^T(x(t))(U\otimes S)\mathbf{h}(x(t))\\
&-\bar{\varphi}\int_{\underline{\tau}_3}^{\overline{\tau}_3}\int_{t-s}^t \mathbf{h}^T(x(\eta))(U\otimes S)\mathbf{h}(x(\eta))d\eta ds. \quad (28)
\end{aligned}$$

Substituting (23)-(28) into (22) yields that

$$\begin{aligned}
&\mathscr{L}V(x_t,t,i)\\
\le\;& 2x^T(t)\Big[(U\otimes(P_iE(i)))\dot{x}(t-\tau_{1,r(t)})\\
&-(U\otimes(P_iA(i)))x(t)+(U\otimes(P_iB(i)))\mathbf{f}(x(t))\\
&+(U\otimes(P_iC(i)))\mathbf{g}(x(t-\tau_{2,i}))+(U\otimes(P_iD(i)))\\
&\times\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\mathbf{h}(x(s))ds+NW(i)\otimes(P_i\Gamma(i))x(t)\Big]\\
&+x^T(t)(U\otimes\overline{P}_i)x(t)+[\bar{\pi}(\overline{\tau}_1-\underline{\tau}_1)+1]\dot{x}^T(t)\\
&\times(U\otimes Q)\dot{x}(t)-\dot{x}^T(t-\tau_{1,i})(U\otimes Q)\dot{x}(t-\tau_{1,i})\\
&+[\bar{\pi}(\overline{\tau}_2-\underline{\tau}_2)+1]\mathbf{g}^T(x(t))(U\otimes R)\mathbf{g}(x(t))\\
&-\mathbf{g}^T(x(t-\tau_{2,i}))(U\otimes R)\mathbf{g}(x(t-\tau_{2,i}))\\
&+\hat{\alpha}_i\mathbf{h}^T(x(t))(U\otimes S)\mathbf{h}(x(t))\\
&-\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\mathbf{h}^T(x(s))(U\otimes S)\mathbf{h}(x(s))ds, \quad (29)
\end{aligned}$$

where $\hat{\alpha}_i$ is defined in (13).

For the sake of the presentation simplicity, we also denote

$$\mathbf{x}_{kl}(t)=x_k(t)-x_l(t),\quad \bar{\mathbf{f}}_{kl}(t)=f(x_k(t))-f(x_l(t)),$$
$$\bar{\mathbf{g}}_{kl}(t)=g(x_k(t))-g(x_l(t)),\quad \bar{\mathbf{h}}_{kl}(t)=h(x_k(t))-h(x_l(t)).$$

By applying Lemma 1 to (29), we have

$$\begin{aligned}
&\mathscr{L}V(x_t,t,i)\\
=&\sum_{1\le k<l\le N}2\mathbf{x}_{kl}^T(t)\Big[P_iE(i)\dot{\mathbf{x}}_{kl}(t-\tau_{1,i})-P_iA(i)\mathbf{x}_{kl}(t)\\
&+P_iB(i)\bar{\mathbf{f}}_{kl}(t)+P_iC(i)\bar{\mathbf{g}}_{kl}(t-\tau_{2,i})+P_iD(i)\\
&\times\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\bar{\mathbf{h}}_{kl}(s)ds-Nw_{kl}(i)P_i\Gamma(i)\mathbf{x}_{kl}(t)\Big]\\
&+\sum_{1\le k<l\le N}\Big[\mathbf{x}_{kl}^T(t)\overline{P}_i\mathbf{x}_{kl}(t)-\dot{\mathbf{x}}_{kl}^T(t-\tau_{1,i})Q\dot{\mathbf{x}}_{kl}(t-\tau_{1,i})\\
&+[\bar{\pi}(\overline{\tau}_2-\underline{\tau}_2)+1]\bar{\mathbf{g}}_{kl}^T(t)R\bar{\mathbf{g}}_{kl}(t)\\
&-\bar{\mathbf{g}}_{kl}^T(t-\tau_{2,i})R\bar{\mathbf{g}}_{kl}(t-\tau_{2,i})+\hat{\alpha}_i\bar{\mathbf{h}}_{kl}^T(t)S\bar{\mathbf{h}}_{kl}(t)\\
&-\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\bar{\mathbf{h}}_{kl}^T(s)S\bar{\mathbf{h}}_{kl}(s)ds\Big]\\
&+[\bar{\pi}(\overline{\tau}_1-\underline{\tau}_1)+1]\dot{x}^T(t)(U\otimes Q)\dot{x}(t). \quad (30)
\end{aligned}$$

From Assumption 2 and Lemma 2, we can deduce that

$$\begin{aligned}
&\mathbf{x}_{kl}^T(t)\Lambda_iL_1\mathbf{x}_{kl}(t)-2\mathbf{x}_{kl}^T(t)\Lambda_iL_2\bar{\mathbf{f}}_{kl}(t)\\
&+\bar{\mathbf{f}}_{kl}^T(t)\Lambda_i\bar{\mathbf{f}}_{kl}(t)\le 0, \quad (31)\\
&\mathbf{x}_{kl}^T(t)\Theta_i\Sigma_1\mathbf{x}_{kl}(t)-2\mathbf{x}_{kl}^T(t)\Theta_i\Sigma_2\bar{\mathbf{g}}_{kl}(t)\\
&+\bar{\mathbf{g}}_{kl}^T(t)\Theta_i\bar{\mathbf{g}}_{kl}(t)\le 0, \quad (32)\\
&\mathbf{x}_{kl}^T(t)\Omega_i\Upsilon_1\mathbf{x}_{kl}(t)-2\mathbf{x}_{kl}^T(t)\Omega_i\Upsilon_2\bar{\mathbf{h}}_{kl}(t)\\
&+\bar{\mathbf{h}}_{kl}^T(t)\Omega_i\bar{\mathbf{h}}_{kl}(t)\le 0. \quad (33)
\end{aligned}$$

Also, in terms of Lemma 3, it is easy to see that

$$\begin{aligned}
&\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\bar{\mathbf{h}}_{kl}^T(s)S\bar{\mathbf{h}}_{kl}(s)ds\\
\ge\;&\frac{1}{\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)ds}\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\bar{\mathbf{h}}_{kl}^T(s)dsS\\
&\times\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\bar{\mathbf{h}}_{kl}(s)ds\\
=\;&\frac{1}{\alpha_i}\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\bar{\mathbf{h}}_{kl}^T(s)dsS\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\bar{\mathbf{h}}_{kl}(s)ds. \quad (34)
\end{aligned}$$

From (30)-(34), it follows that

$$\begin{aligned}
&\mathscr{L}V(x_t,t,i)\\
\le&\sum_{1\le k<l\le N}2\mathbf{x}_{kl}^T(t)\Big[P_iE(i)\dot{\mathbf{x}}_{kl}(t-\tau_{1,i})-P_iA(i)\mathbf{x}_{kl}(t)\\
&+P_iB(i)\bar{\mathbf{f}}_{kl}(t)+P_iC(i)\bar{\mathbf{g}}_{kl}(t-\tau_{2,i})+P_iD(i)\\
&\times\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\bar{\mathbf{h}}_{kl}(s)ds-Nw_{kl}(i)P_i\Gamma(i)\mathbf{x}_{kl}(t)\Big]\\
&+\sum_{1\le k<l\le N}\Big[\mathbf{x}_{kl}^T(t)\overline{P}_i\mathbf{x}_{kl}(t)-\dot{\mathbf{x}}_{kl}^T(t-\tau_{1,i})Q\dot{\mathbf{x}}_{kl}(t-\tau_{1,i})\\
&+[\bar{\pi}(\overline{\tau}_2-\underline{\tau}_2)+1]\bar{\mathbf{g}}_{kl}^T(t)R\bar{\mathbf{g}}_{kl}(t)-\bar{\mathbf{g}}_{kl}^T(t-\tau_{2,i})R\\
&\times\bar{\mathbf{g}}_{kl}(t-\tau_{2,i})+\hat{\alpha}_i\bar{\mathbf{h}}_{kl}^T(t)S\bar{\mathbf{h}}_{kl}(t)\\
&-\frac{1}{\alpha_i}\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\bar{\mathbf{h}}_{kl}^T(s)dsS\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\bar{\mathbf{h}}_{kl}(s)ds\Big]\\
&-\sum_{1\le k<l\le N}\Big[\mathbf{x}_{kl}^T(t)\big(\Lambda_iL_1+\Theta_i\Sigma_1+\Omega_i\Upsilon_1\big)\mathbf{x}_{kl}(t)\\
&-2\mathbf{x}_{kl}^T(t)\Lambda_iL_2\bar{\mathbf{f}}_{kl}(t)+\bar{\mathbf{f}}_{kl}^T(t)\Lambda_i\bar{\mathbf{f}}_{kl}(t)-2\mathbf{x}_{kl}^T(t)\Theta_i\Sigma_2\bar{\mathbf{g}}_{kl}(t)\\
&+\bar{\mathbf{g}}_{kl}^T(t)\Theta_i\bar{\mathbf{g}}_{kl}(t)-2\mathbf{x}_{kl}^T(t)\Omega_i\Upsilon_2\bar{\mathbf{h}}_{kl}(t)+\bar{\mathbf{h}}_{kl}^T(t)\Omega_i\bar{\mathbf{h}}_{kl}(t)\Big]\\
&+[\bar{\pi}(\overline{\tau}_1-\underline{\tau}_1)+1]\dot{x}^T(t)(U\otimes Q)\dot{x}(t), \quad (35)
\end{aligned}$$

where κ_1 is defined in (13).

For the last term in the above inequality, we have

$$\begin{aligned}
&\dot{x}^T(t)(U\otimes Q)\dot{x}(t)\\
=\;&\Big[(I_N\otimes E(i))\dot{x}(t-\tau_{1,i})-(I_N\otimes A(i))x(t)\\
&+(I_N\otimes B(i))\mathbf{f}(x(t))+(I_N\otimes C(i))\mathbf{g}(x(t-\tau_{2,i}))\\
&+(I_N\otimes D(i))\int_{-\infty}^{t-\tau_{3,i}}\varphi(t-s)\mathbf{h}(x(s))ds+\mathbf{u}(t)\\
&+W(i)\otimes\Gamma(i)x(t)\Big]^T(U\otimes Q)\Big[(I_N\otimes E(i))\\
&\times\dot{x}(t-\tau_{1,i})-(I_N\otimes A(i))x(t)
\end{aligned}$$

$$
\begin{aligned}
&+ (I_N \otimes B(i))\mathbf{f}(x(t)) + (I_N \otimes C(i))\mathbf{g}(x(t-\tau_{2,i})) \\
&+ (I_N \otimes D(i)) \int_{-\infty}^{t-\tau_{3,i}} \varphi(t-s)\mathbf{h}(x(s))ds + \mathbf{u}(t) \\
&+ W(i)\otimes\Gamma(i)x(t)\Big] \\
= \quad & \Big[(I_N \otimes E(i))\dot{x}(t-\tau_{1,i}) - (I_N \otimes A(i))x(t) \\
&+ (I_N \otimes B(i))\mathbf{f}(x(t)) + (I_N \otimes C(i))\mathbf{g}(x(t-\tau_{2,i})) \\
&+ (I_N \otimes D(i)) \int_{-\infty}^{t-\tau_{3,i}} \varphi(t-s)\mathbf{h}(x(s))ds\Big]^T \\
&\times (U\otimes Q)\Big[(I_N \otimes E(i))\dot{x}(t-\tau_{1,i}) \\
&- (I_N \otimes A(i))x(t) + (I_N \otimes B(i))\mathbf{f}(x(t)) \\
&+ (I_N \otimes C(i))\mathbf{g}(x(t-\tau_{2,i})) + (I_N \otimes D(i)) \\
&\times \int_{-\infty}^{t-\tau_{3,i}} \varphi(t-s)\mathbf{h}(x(s))ds\Big] + 2x^T(t) \\
&\times \big(W(i)\otimes\Gamma(i)\big)(U\otimes Q)\Big[(I_N \otimes E(i))\dot{x}(t-\tau_{1,i}) \\
&- (I_N \otimes A(i))x(t) + (I_N \otimes B(i))\mathbf{f}(x(t)) \\
&+ (I_N \otimes C(i))\mathbf{g}(x(t-\tau_{2,i})) + (I_N \otimes D(i)) \\
&\times \int_{-\infty}^{t-\tau_{3,i}} \varphi(t-s)\mathbf{h}(x(s))ds\Big] + x^T(t)\big(W(i)\otimes\Gamma(i)\big) \\
&\times (U\otimes Q)\big(W(i)\otimes\Gamma(i)\big)x(t). \qquad (36)
\end{aligned}
$$

Noticing the relationships

$$
\begin{aligned}
&\big(W(i)\otimes\Gamma(i)\big)(U\otimes Q) = N\big(W(i)\otimes(\Gamma(i)Q\big), \\
&\big(W(i)\otimes\Gamma(i)\big)(U\otimes Q)\big(W(i)\otimes\Gamma(i)\big) \\
= \quad & N[W(i)]^2 \otimes \big(\Gamma(i)Q\Gamma(i)\big),
\end{aligned}
$$

we can infer from Lemma 1 that

$$
\begin{aligned}
&\dot{x}^T(t)(U\otimes Q)\dot{x}(t) \\
= \quad & \sum_{1\le k<l\le N} \Big[E(i)\dot{\mathbf{x}}_{kl}(t-\tau_{1,i}) - A(i)\mathbf{x}_{kl}(t) \\
&+ B(i)\bar{\mathbf{f}}_{kl}(x(t)) + C(i)\bar{\mathbf{g}}_{kl}(t-\tau_{2,i}) \\
&+ D(i)\int_{-\infty}^{t-\tau_{3,i}} \varphi(t-s)\bar{\mathbf{h}}_{kl}(s)ds\Big]^T Q\Big[E(i)\dot{\mathbf{x}}_{kl}(t-\tau_{1,i}) \\
&- A(i)\mathbf{x}_{kl}(t) + B(i)\bar{\mathbf{f}}_{kl}(x(t)) + C(i)\bar{\mathbf{g}}_{kl}(t-\tau_{2,i}) \\
&+ D(i)\int_{-\infty}^{t-\tau_{3,i}} \varphi(t-s)\bar{\mathbf{h}}_{kl}(s)ds\Big] - 2N\sum_{1\le k<l\le N}\mathbf{x}_{kl}^T(t) \\
&\times w_{kl}(i)\Gamma(i)Q\Big[E(i)\dot{\mathbf{x}}_{kl}(t-\tau_{1,i}) - A(i)\mathbf{x}_{kl}(t) \\
&+ B(i)\bar{\mathbf{f}}_{kl}(x(t)) + C(i)\bar{\mathbf{g}}_{kl}(t-\tau_{2,i}) \\
&+ D(i)\int_{-\infty}^{t-\tau_{3,i}} \varphi(t-s)\bar{\mathbf{h}}_{kl}(s)ds\Big] \\
&- N\sum_{1\le k<l\le N}\mathbf{x}_{kl}^T(t)w_{kl}^{(2)}(i)\Gamma(i)Q\Gamma(i)\mathbf{x}_{kl}(t). \qquad (37)
\end{aligned}
$$

Substituting (37) into (35) leads to

$$
\begin{aligned}
&\mathscr{L}V(x_t,t,i) = \sum_{1\le k<l\le N}\chi_{kl}^T(t,i)\big[\Psi_{kl}(i) \\
+ \quad & \mathcal{A}^T(i)\kappa_1 Q\mathcal{A}(i)\big]\chi_{kl}(t,i), \qquad (38)
\end{aligned}
$$

where $\Psi_{kl}(i)$ is defined as

$$
\begin{pmatrix}
\Xi_{11}(i) & \Xi_{12}(i) & \Theta_i\Sigma_2 & \Xi_{14}(i) & \Omega_i\Upsilon_2 & \Xi_{16}(i) & Xi_{17} \\
* & -\Lambda_i & 0 & 0 & 0 & 0 & 0 \\
* & * & \Xi_{33}(i) & 0 & 0 & 0 & 0 \\
* & * & * & -R & 0 & 0 & 0 \\
* & * & * & * & \Xi_{55}(i) & 0 & 0 \\
* & * & * & * & * & -\frac{1}{\alpha_i}S & 0 \\
* & * & * & * & * & * & -Q
\end{pmatrix},
$$

and

$$
\begin{aligned}
&\mathcal{A}(i) = \big[-A(i) \;\; B(i) \;\; 0 \;\; C(i) \;\; 0 \;\; D(i) \;\; E(i)\big], \\
&\chi_{kl}(t,i) = \Big[\mathbf{x}_{kl}^T(t) \;\; \bar{\mathbf{f}}_{kl}^T(s) \;\; \bar{\mathbf{g}}_{kl}^T(t) \;\; \bar{\mathbf{g}}_{kl}^T(t-\tau_{2,i}) \;\; \bar{\mathbf{h}}_{kl}^T(t) \\
&\qquad \int_{-\infty}^{t-\tau_{3,i}} \varphi(t-s)\bar{\mathbf{h}}_{kl}(s)ds \;\; \dot{\mathbf{x}}_{kl}(t-\tau_{1,i})\Big]^T.
\end{aligned}
$$

In terms of Lemma 4, (11) is equivalent to

$$
\begin{aligned}
&\Psi_{kl}(i) + \mathcal{A}^T(i)\kappa_1 Q\mathcal{A}(i) < 0, \qquad (39) \\
&(1\le k<l\le N,\ 1\le i\le n_0).
\end{aligned}
$$

Let $\rho_0 = \max\big\{\lambda_{\max}\big(\Psi_{kl}(i) + \mathcal{A}^T(i)\kappa_1 Q\mathcal{A}(i)\big)\ \big|\ 1\le k< l\le N,\ 1\le i\le n_0\big\}$. Obviously, $\rho_0<0$ and it then follows from (38) that

$$
\begin{aligned}
\mathscr{L}V(x_t,t,i) \quad &\le \quad \rho_0\sum_{1\le k<l\le N}\chi_{kl}^T(t,i)\chi_{kl}(t,i) \\
&\le \quad \rho_0\sum_{1\le k<l\le N}|\mathbf{x}_{kl}(t)|^2. \qquad (40)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\mathbb{E}V(x(t),t,r(t)) \\
&= \mathbb{E}V(x(0),0,r(0)) + \mathbb{E}\int_0^t LV(x(s),s,r(s))ds \\
&\le \mathbb{E}V(x(0),0,r(0)) + \rho_0\sum_{1\le k<l\le N}\int_0^t \mathbb{E}|\mathbf{x}_{kl}(s)|^2 ds. \quad (41)
\end{aligned}
$$

Since $\rho_0<0$ and $V(x(t),t,r(t))>0$, it follows readily from (41) that

$$
\sum_{1\le k<l\le N}\int_0^t \mathbb{E}|\mathbf{x}_{kl}(s)|^2 ds \le \frac{1}{|\rho|}\mathbb{E}V(x(0),0,r(0)), \qquad (42)
$$

which implies that the integral $\sum_{1\le k<l\le N}\int_0^{+\infty}\mathbb{E}|\mathbf{x}_{kl}(s)|^2 ds < +\infty$.

By Lemma 5, we have

$$
\lim_{t\to+\infty}\sum_{1\le k<l\le N}\mathbb{E}|\mathbf{x}_{kl}(t)|^2 = 0,
$$

or $\lim_{t\to+\infty}\mathbb{E}|x_k(t)-x_l(t)|^2 = 0$ for $1\le k<l\le N$. In other words, the system (8) is globally asymptotically synchronized in mean square. This completes the proof of the theorem. ■

The system (8) is rather general. In what follows, we consider two special cases. In Case 1, we show that our main results can be specialized to the synchronization problem for coupled system without involving the derivatives of the past history (i.e., $E(r(t))=0$), which reduces to a retarded

$$\hat{\Phi}_{kl}(i)=\begin{pmatrix}\hat{\Xi}_{11}(i) & PB(i)+\Lambda_i L_2 & \Theta_i\Sigma_2 & PC(i) & \Omega_i\Upsilon_2 & PD(i)\\ * & -\Lambda_i & 0 & 0 & 0 & 0\\ * & * & \Xi_{33}(i) & 0 & 0 & 0\\ * & * & * & -R & 0 & 0\\ * & * & * & * & \Xi_{55}(i) & 0\\ * & * & * & * & * & -\frac{1}{\alpha_i}S\end{pmatrix}<0, \quad (44)$$

$$\tilde{\Phi}(i)=\begin{pmatrix}\tilde{\Xi}_{11}(i) & P_iB(i)+\Lambda_i L_2 & \Theta_i\Sigma_2 & P_iC(i) & \Omega_i\Upsilon_2 & P_iD(i) & P_iE(i) & -\sqrt{\kappa_1}A(i)Q\\ * & -\Lambda_i & 0 & 0 & 0 & 0 & 0 & \sqrt{\kappa_1}B^T(i)Q\\ * & * & \Xi_{33}(i) & 0 & 0 & 0 & 0 & 0\\ * & * & * & -R & 0 & 0 & 0 & \sqrt{\kappa_1}C^T(i)Q\\ * & * & * & * & \Xi_{55}(i) & 0 & 0 & 0\\ * & * & * & * & * & -\frac{1}{\alpha_i}S & 0 & \sqrt{\kappa_1}D^T(i)Q\\ * & * & * & * & * & * & -Q & \sqrt{\kappa_1}E^T(i)Q\\ * & * & * & * & * & * & * & -Q\end{pmatrix}<0, \quad (46)$$

functional differential equation. In Case 2, we consider the same array of neural networks with discrete time-delay only.

Case 1. In the case of $E(r(t))=0$, the system (8) reduces to

$$\begin{aligned}\dot{x}(t) \;=\; & -(I_N\otimes A(r(t)))x(t)+(I_N\otimes B(r(t)))\mathbf{f}(x(t))\\ & +(I_N\otimes C(r(t)))\mathbf{g}(x(t-\tau_{2,i}))\\ & +(I_N\otimes D(r(t)))\int_{-\infty}^{t-\tau_{3,r(t)}}\varphi(t-s)\mathbf{h}(x(s))ds\\ & +\mathbf{u}(t)+W(r(t))\otimes\Gamma(r(t))x(t). \qquad (43)\end{aligned}$$

For the system (43), the following result can be derived based on Theorem 1.

Corollary 1: Under Assumptions 1-3, the system (43) is globally asymptotically synchronized in mean square if there exist *five* positive definite matrices P_1, P_2, P_3, R and S, and three sets of positive definite diagonal matrices Λ_i,Θ_i and $\Omega_i(1\le i\le n_0)$ such that the following LMIs (44) shown at the top of the page hold for $1\le k<l\le N,\ 1\le i\le n_0$, where each symbol has its previous meaning except $\hat{\Xi}_{11}(i)=-P_iA(i)-A(i)P_i+\overline{P}_i-Nw_{kl}(i)\big(P_i\Gamma(i)+\Gamma(i)P_i\big)-\big(\Lambda_iL_1+\Theta_i\Sigma_1+\Omega_i\Upsilon_1\big)$.

Case 2. In this case, with $D(r(t))=0$, the system (8) is simplified as

$$\begin{aligned}\dot{x}(t) \;=\; & (I_N\otimes E(r(t)))\dot{x}(t-\tau_{1,r(t)})-(I_N\otimes A(r(t)))x(t)\\ & +(I_N\otimes B(r(t)))\mathbf{f}(x(t))+(I_N\otimes C(r(t)))\\ & \times\mathbf{g}(x(t-\tau_{2,i}))+\mathbf{u}(t)\\ & +W(r(t))\otimes\Gamma(r(t))x(t). \qquad (45)\end{aligned}$$

For the system (45), the following result is readily available.

Corollary 2: Under Assumptions 1-3, the system (45) is globally asymptotically synchronized in mean square if there exist *five* positive definite matrices P_1, P_2, P_3, Q, R and R, and *two* sets of positive definite diagonal matrices Λ_i and $\Theta_i(1\le i\le n_0)$ such that the following LMIs

$$\bar{\Phi}_{kl}(i)<0$$

hold for $1\le k<l\le N,\ 1\le i\le n_0$, where $\bar{\Phi}_{kl}(i)$ is defined as

$$\begin{pmatrix}\bar{\Xi}_{11}(i) & \Xi_{12}(i) & \Theta_i\Sigma_2 & \Xi_{14}(i) & \Xi_{17} & -\sqrt{\kappa_1}A(i)Q\\ * & -\Lambda_i & 0 & 0 & 0 & \sqrt{\kappa_1}B^T(i)Q\\ * & * & \Xi_{33}(i) & 0 & 0 & 0\\ * & * & * & -R & 0 & \sqrt{\kappa_1}C^T(i)Q\\ * & * & * & * & -Q & \sqrt{\kappa_1}E^T(i)Q\\ * & * & * & * & * & -Q\end{pmatrix},$$

and each symbol has its previous meaning except $\bar{\Xi}_{11}(i)=-P_iA(i)-A(i)P_i+\overline{P}_i-Nw_{kl}(i)\big(P_i\Gamma(i)+\Gamma(i)P_i\big)-\big(\Lambda_iL_1+\Theta_i\Sigma_1\big)$.

Remark 3: Notice that in the case of $W(i)=0$ or $\Gamma(i)=0$ for all i, the system (8) is uncoupled, and the dynamics of each single neutral-type network is independent of the other networks. Hence, by means of Theorem 1, a sufficient condition can be obtained to guarantee the global asymptotic stability in mean square for each single neutral-type neural network.

Corollary 3: Under Assumptions 1-3, the neutral-type neural network (1) is globally asymptotically stable in mean square if there exist six positive definite matrices P_1, P_2, P_3, Q, R and S, and three sets of positive definite diagonal matrices Λ_i,Θ_i and $\Omega_i(1\le i\le n_0)$ such that the following LMIs (46) shown at the top of the page hold for $1\le i\le n_0$, where each symbol has its previous meaning except $\tilde{\Xi}_{11}(i)=-P_iA(i)-A(i)P_i+\overline{P}_i-\big(\Lambda_iL_1+\Theta_i\Sigma_1+\Omega_i\Upsilon_1\big)$.

Remark 4: In this paper, the synchronization problem is dealt with for a new class of continuous-time neural networks of neutral-type with Markovian jumping parameters as well as mode-dependent mixed time-delays. Note that the mixed time-delays comprise both the discrete and distributed delays that are all dependent on the Markovian jumping mode. The novelty of the main results is fourfold: 1) due to the consideration of the mode-dependent neutral delays, some novel analysis techniques are developed to tackle the resulting mathematical difficulty; 2) a new Lyapunov functional is proposed to account for the Markovian jumps of the delay bounds; 3) a unified framework is established to handle the Markovian jumping parameters, neutral terms and mixed time-delays; and 4) the main results established in Theorem 1 contain all the information of the considered coupling neural networks including physical parameters, Markovian jumping rate, the discrete time-delay as well as bounds on the distributed time-delays. In the next section, a simulation example is provided to show the usefulness of the proposed stability conditions.

IV. A Numerical example

In this section, we present a simulation example so as to illustrate the usefulness of our main results. Our aim is to examine the global asymptotic synchronization of the system (8) in mean square.

Consider a system coupled by four identical second-order neutral-type neural networks with network parameters given as follows:

$$\Pi = \begin{bmatrix} -5 & 2 & 3 \\ 4 & -5 & 1 \\ 2 & 4 & -6 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 1 & 0 \\ 0 & -0.3 \end{bmatrix},$$
$$B(1) = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & -0.1 \end{bmatrix}, \quad C(1) = \begin{bmatrix} 0.3 & -0.1 \\ -0.4 & 0.1 \end{bmatrix},$$
$$D(1) = \begin{bmatrix} 0.3 & -0.2 \\ 0.2 & 0.2 \end{bmatrix}, \quad E(1) = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0 \end{bmatrix},$$
$$A(2) = \begin{bmatrix} 1 & 0 \\ 0 & -0.4 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & -0.2 \end{bmatrix},$$
$$C(2) = \begin{bmatrix} 0.2 & -0.4 \\ 0.2 & 0.2 \end{bmatrix}, \quad D(2) = \begin{bmatrix} 0.2 & -0.4 \\ 0 & 0.2 \end{bmatrix},$$
$$E(2) = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}, \quad A(3) = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix},$$
$$B(3) = \begin{bmatrix} 0.2 & 0 \\ 0.2 & -0.2 \end{bmatrix}, \quad C(3) = \begin{bmatrix} 0.2 & 0.3 \\ -0.3 & 0 \end{bmatrix},$$
$$D(3) = \begin{bmatrix} 0.1 & -0.2 \\ 0.1 & 0.2 \end{bmatrix}, \quad E(3) = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$

$$W(1) = \begin{bmatrix} -7 & 3 & 2 & 2 \\ 3 & -8 & 2 & 3 \\ 2 & 2 & -6 & 2 \\ 2 & 3 & 2 & -7 \end{bmatrix},$$
$$W(2) = \begin{bmatrix} -7 & 2 & 3 & 2 \\ 2 & -6 & 2 & 2 \\ 3 & 2 & -7 & 2 \\ 2 & 2 & 2 & -6 \end{bmatrix},$$
$$W(3) = \begin{bmatrix} -8 & 3 & 3 & 2 \\ 3 & -7 & 2 & 2 \\ 3 & 2 & -7 & 2 \\ 2 & 2 & 2 & -6 \end{bmatrix},$$
$$\Gamma(1) = \mathrm{diag}\{4,\ 3\}, \ \Gamma(2) = \mathrm{diag}\{3,\ 3\},$$
$$\Gamma(3) = \mathrm{diag}\{4,\ 4\}, \ \tau_{1,1} = 2, \ \tau_{2,1} = 7,$$
$$\tau_{3,1} = 1.2, \ \tau_{1,2} = 1, \ \tau_{2,2} = 6, \ \tau_{3,2} = 1,$$
$$\tau_{1,3} = 3, \ \tau_{2,3} = 6.5, \ \tau_{3,3} = 0.8.$$

The activation functions are taken as follows

$$f_1(s) = g_1(s) = h_1(s) = \tanh(-0.6s),$$
$$f_2(s) = g_2(s) = h_2(s) = 0.4\tanh(s),$$

and the delayed kernel function is given by $\varphi(s) = \mathrm{e}^{-3s}$. It is not difficult to verify that

$$L_1 = \Sigma_1 = \Upsilon_1 = \mathrm{diag}\{0,\ 0\},$$
$$L_2 = \Sigma_2 = \Upsilon_2 = \mathrm{diag}\{-0.3,\ 0.2\}.$$

With the parameters given above and by using the Matlab LMI toolbox, we solve the LMI (11) and obtain the following feasible solutions:

$$P_1 = \begin{bmatrix} 0.7524 & -0.0691 \\ -0.0691 & 1.0418 \end{bmatrix}, \ P_2 = \begin{bmatrix} 1.3868 & -0.1239 \\ -0.1239 & 1.3968 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 0.7541 & -0.0940 \\ -0.0940 & 1.2698 \end{bmatrix}, \ Q = \begin{bmatrix} 0.0020 & -0.0004 \\ -0.0004 & 0.0037 \end{bmatrix},$$
$$R = \begin{bmatrix} 2.1168 & 0.0213 \\ 0.0213 & 1.5872 \end{bmatrix}, \quad S = \begin{bmatrix} 0.1123 & -0.0000 \\ -0.0000 & 0.1123 \end{bmatrix},$$
$$\Lambda_1 = \mathrm{diag}\{6.4715,\ 5.9952\}, \ \Theta_1 = \mathrm{diag}\{21.9896,\ 18.3192\},$$
$$\Omega_1 = \mathrm{diag}\{7.0265,\ 6.7963\}, \ \Lambda_2 = \mathrm{diag}\{7.4679,\ 7.5336\},$$
$$\Theta_2 = \mathrm{diag}\{22.4617,\ 18.5736\}, \ \Omega_2 = \mathrm{diag}\{6.8619,\ 6.5830\},$$
$$\Lambda_3 = \mathrm{diag}\{7.0148,\ 6.7889\}, \ \Theta_3 = \mathrm{diag}\{21.9830,\ 18.3069\},$$
$$\Omega_3 = \mathrm{diag}\{7.0304,\ 6.8070\}.$$

Therefore, it follows from Theorem 1 that the system (8) with given parameters is globally asymptotically synchronized in mean square. The numerical simulation further confirms the theoretical results. Fig. 1 and Fig. 2 display the evolution of the states of the first neutral-type neural network without coupling and with coupling, respectively. Fig. 3 shows that the synchronization error $err(t)$ approaches zero as $t \to \infty$.

Remark 5: It is worth to pointing out that the example given above is non-trivial. Note that the system matrix $A(r(t))$ of the single network is unstable, which results in the instability of either single network or coupled system. This can also be observed from Fig. 1 and Fig. 2. Nevertheless, as shown in Fig. 3, the coupled system is synchronized. The numerical simulation is in complete accord with the theoretical results.

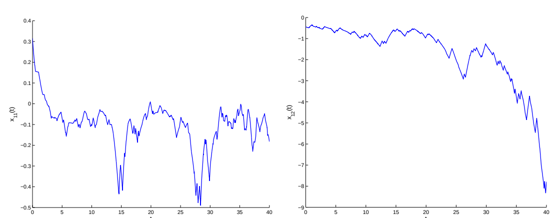

Fig. 1. State Evolution of Single Network Without Coupling.

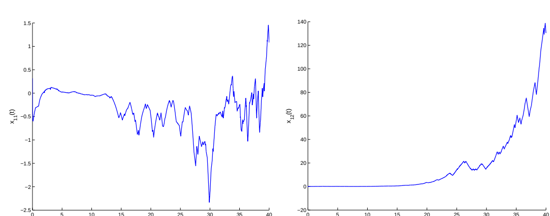

Fig. 2. State Evolution of Single Network in Coupled System.

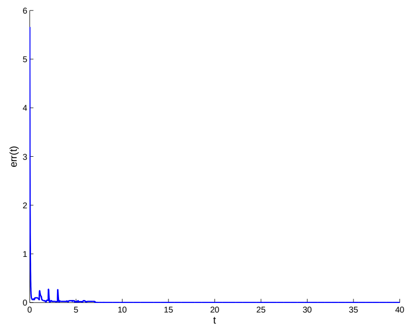

Fig. 3. State Trajectory of the Synchronization Error.

V. Conclusions

In this paper, we have investigated the synchronization problem for an array of linearly coupled neutral-type neural networks with Markovian jumping parameters and mixed time delays. The discrete time delays are mode-dependent, and distributed time delay is unbounded with mode-dependent upper bound. By utilizing a novel Lyapunov-Krasovskii functional and the Kronecker product, we have shown that the addressed synchronization problem is solvable if several linear matrix inequalities (LMIs) are feasible. A unified LMI approach has been developed to establish sufficient conditions for the coupled neural networks to be globally synchronized in mean square. A numerical example has been provided to show the usefulness of the proposed global synchronization condition.

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