Optimal Design of Thin-Walled Structures by means of Efficient Parameterization

Hassan Ugail*
University of Leeds, Leeds LS2 9JT, United Kingdom

The design and functional optimization of thin-walled structures made from plastics which are used for food packaging is considered. Such objects are created in vast numbers that an important task in the design of these objects is to minimize the amount of material used subject to a given set of functional constraints, such as the strength and volume, to reduce the cost of production and to conserve raw material.

Here a procedure is described which enables to create the optimal shapes of such containers by means of an automatic optimization where a series of possible shapes are analyzed so as to minimize the weight of such containers subject to a given set of functional constraints. The shape parameterization is carried out using the PDE method which enables complex shapes of the thin-walled structures to be efficiently defined in terms of a small set of design variables

Nomenclature

E^3	euclidean 3-space
x, y, z	cartesian coordinates
Ω	domain of solution in $\Re^2$
$\underline{X}(u, v)$	vector of Cartesian coordinates of the surface points
$\underline{X}_u$	derivative vector
a	smoothing parameter
p_i	ith positional boundary curve
d_i	ith derivative boundary curve
c_{kPi}	($k = 1, 2$ and $i = x, y, z$), parameterization P of a boundary curve c
σ^i	shear stress at a point i of the surface

Introduction

The aim of this paper is to describe a strategy for automatic design optimization, by means of physical analysis, of thin-walled structures made from polymeric materials. In particular, the optimal design of such structures by means of creating a parameterized CAD model within in an interactive environment is discussed. By making use of a standard optimization routine, automatic variations of the parameters associated with the CAD model enables to create a rich variety of possible designs that are then subjected to physical analysis. The process is terminated when the parameters defining a design which fits our requirement is found. Thus, here it is shown how the process of automatic design optimization is made feasible and, furthermore, it is shown that the key for this is the efficient and consistent parameterization of the geometry model.

For the purpose of demonstrating the techniques discussed here, the task considered here is to minimize the amount of material used in packaging subject to a given set of functional constraints such as the strength and the volume, in order to reduce the cost of production and to conserve the raw materials used. The potential impact of this work comes from the vast range of products used in the food industry composed of thin-walled polymers. For example, the types of products that make use of such containers are dairy (yogurt and ice-cream containers, mayonnaise and cooking sauces), convenience foods (ready meals and snack foods), clear packs (sandwich packaging, wines and beers). Thus, one can readily appreciate that one of the challenges facing the relevant manufacturing industries at present is to reduce the amount of material within a container, whilst at the same time maintaining adequate strength and volume. Therefore, it is clear that given the cost of raw materials, the ability to reduce the amount of material used to fabricate a particular object, whilst maintaining its required physical properties such as strength, would have a significant economic and environmental impact. For example, a saving of 1gm of material from a typical margarine tub might, for one factory over the space of a year, save 48 tons of raw material or about £33,000 ($46,760).

In the food packaging sector, the nature of product development often leads to very short time scales. Many food products undergo some changes to their packaging on an annual basis, and some much more frequently than this. These may be purely cosmetic (for increased customer impact, re-branding etc) or be more fundamental in nature. e.g. reduction of weight. A typical request of

*Research Fellow, Department of Applied Mathematics.

this latter type would be '*make pack X 10% lighter but maintain rigidity*'. For a typical manufacturer of food containers the target for producing samples that demonstrate the new concept is usually within 4 working days. To achieve this they must design/redesign the product, have a sample mould made and then produce the sample containers. It is clear that during this process, the length of time for radical product design and more importantly optimization using conventional approaches is currently either small or non-existent. Thus, it is worthwhile to address the issue of automatic design optimization within the context of functional design of thin-walled structures used for food packaging.

Applying the idea of parameterization schemes to structural optimization in packaging is not entirely new. For example, Bhakuni *et al*[1] performed optimization on beverage cans where the main objective was to minimize the weight of the can. The weight of the can in this case was measured as the diameter of the blank sheet used to create the can. More recently, Dijk *et al*[6] describe optimal design of bottles with ribs by trying to minimize the mean thickness subject to the shape of the ribs. In this case the design parameters were those introduced on the ribs allowing the shape of the ribs to be changed. However, a major drawback in these optimization problems is the lack of an efficient and coherent mechanism to parameterize the entire geometry of the object under consideration.

This paper demonstrates how the automatic design optimization of thin-walled structures can be carried out by efficiently defining and parameterzing the shape of these structures. For this purpose the PDE method[2] is used which adopts a boundary-value approach in which the shape of the object in question is decomposed into a series of surface patches bounded by 'character-lines', where the number of patches are being kept as low as possible. Using the boundary data appropriately defined along the character-lines the PDE method enables to produce smooth surfaces between them. More importantly, the shape of the surface produced is efficiently parameterized with minimal number of shape parameters thus enabling automatic optimization to be carried out.

The PDE geometry definition of the shape of the food container is used to generate a valid finite element mesh with which to analyze its physical properties, in particular its load-bearing characteristics. The results of the finite element computation provide us with a measure for the strength of the container. The aim here is to identify a shape of a container that uses as little material as possible whilst possessing a predefined level of strength. In order to carry out a realistic optimization a volume constraint in the form of an equality condition is also introduced within the optimization procedure. Then using standard optimization procedures, the shape parameters, defining the geometry of the container, are iteratively modified to improve the object's load-bearing characteristics for a fixed volume.

The PDE Method and Geometry Generation

In geometric design, it is common practice to define curves and surfaces using some form which represents the surface parametrically. Thus, surfaces are defined in terms of two parameters u and v so that any point on the surface $\underline{X}$ is given by an expression of the form:

$$\underline{X} = \underline{X}(u, v). \qquad (1)$$

Equation (1) can be viewed as a mapping from a domain Ω in the (u, v) parameter space to Euclidean 3-space. In the case of the PDE method this mapping is defined as a partial differential operator:

$$L^m_{uv}(\underline{X}) = \underline{F}(u, v), \qquad (2)$$

where the partial differential operator L is of degree m. Thus, here, surface design is treated as an appropriately posed boundary-value problem with boundary conditions imposed on $\partial\Omega$, the boundary of Ω. The partial differential operator L is usually taken to be that of elliptic type and the degree m of this operator depends on the level of surface control and continuity required at the boundaries of the surface. The function $\underline{F}(u, v)$ is included for completeness and is generally taken to be zero.

The PDE method has been discussed before by a number of different references, e.g. Vida *et al*,[16] Bloor and Wilson[2] and Bloor and Wilson.[3] It has been shown how surfaces satisfying a wide range of functional requirements can be created by a suitable choice of the boundary conditions and appropriate values for the various design parameters associated with the method.[5,12,13]

Interactive Design

For the work described here, and for the majority of previous work carried out using the PDE method described elsewhere, the PDE chosen is of the form:

$$\left(\frac{\partial^2}{\partial u^2} + a^2\frac{\partial^2}{\partial v^2}\right)^2 \underline{X}(u, v) = 0, \qquad (3)$$

where the condition on the function $\underline{X}(u, v)$ and its normal derivatives $\frac{\partial \underline{X}}{\partial n}$ can be imposed at the edges of the surface patch. The parameter a is a

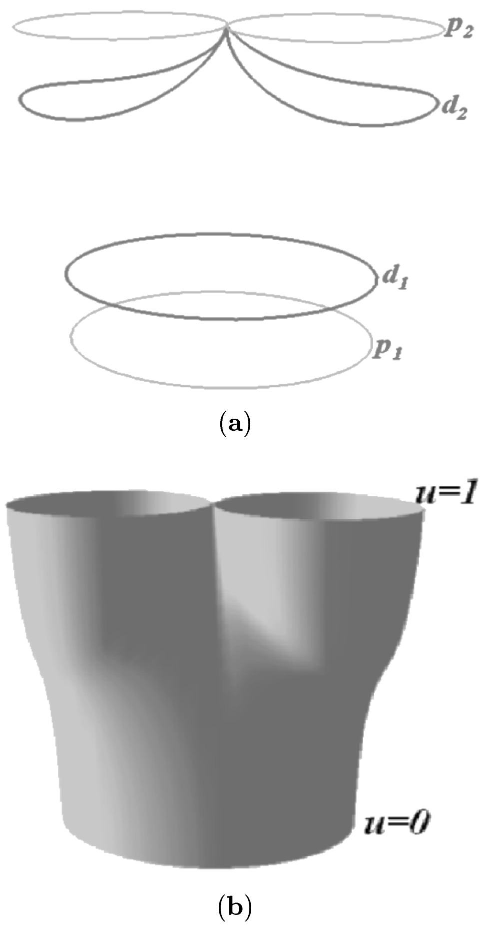


Fig. 1 Typical PDE surface. (a) The boundary curves. (b) The corresponding PDE surface patch

special design parameter which controls the relative smoothing of the surface in the u and v directions.[3] For periodic boundary conditions (e.g. $0 \le u \le 1$, $0 \le v \le 2\pi$), a pseudo-spectral method has been developed for the solution of equation (3) which allows $\underline{X}(u, v)$ to be expressed in closed form.[4]

For the purpose of interactive design, the boundary conditions are usually defined in terms of curves in 3-space. For example, Fig. 1 shows a typical set of boundary curves and the corresponding PDE surface showing the port of a bifurcated transfer port of a 2-stroke engine. Here the value of a was taken to be 1.0100. Note that the curves marked p_1 and p_2 correspond to the boundary conditions on the function $\underline{X}(u, v)$. A vector field corresponding to the difference between the points on the curves marked p_1 and p_2 and those marked d_1 and d_2 respectively, corresponds to the conditions on the function $\frac{\partial \underline{X}}{\partial n}$ such that

$$\frac{\partial \underline{X}}{\partial n} = s \left[\underline{p}(v) - \underline{d}(v) \right], \quad (4)$$

parameter	value
d_{1T_x}	0.000
d_{1T_y}	-0.850
d_{1T_z}	0.000
d_{1D_x}	0.771
d_{1D_y}	0.792
d_{1D_z}	0.000
d_{1R_x}	3.138
d_{1R_y}	0.000
d_{1R_z}	0.000

Table 1 Values for the design parameters for the boundary $d = 1$ of the surface shown in Fig. 1

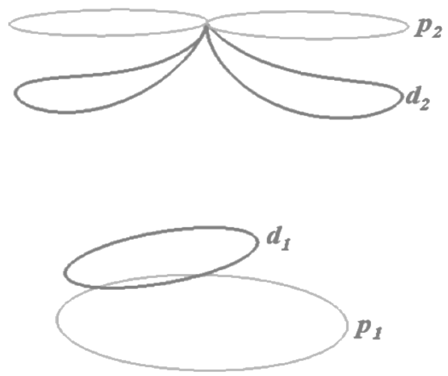
where s is a scalar. The conditions defined by p_1, p_2 and d_1, d_2 are known as the 'positional boundary conditions' and 'derivative boundary conditions' respectively.[13] Note that the surface patch will not necessarily pass through the curves which define the derivative boundary conditions.

Interactive Parameterization of Geometry Model

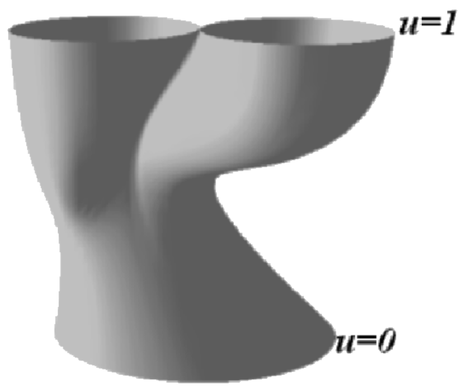
In this work the definition of the shape geometry is carried out using the PDE parameter model discussed in Ugail *et al*,[13] where the parameterized boundary curves are used to define the shape of the surface. Essentially, this parameterization is defined in such a way that linear transformations, such as translation, rotation and dilation, of the boundary curves can be carried out interactively. The result of this is that the designer is presented with tools which enable him/her to create and modify the geometry an intuitive manner.

For convenience, the parameterization on the boundary curves is denoted using the notation c_{kP_i} $(k = 1, 2)$, $(i = x, y, z)$. Here c indicates the type of curve, with the letter p denoting the position curves and the letter d denoting the derivative curves. The index k ranges from 1 to 2 corresponding to the $u = 0$ and $u = 1$ boundary edges (respectively) of the surface. The letter P denotes the type of parameter: T for a translation, R for a rotation and D for a dilation. Finally the letter i denotes the coordinate directions relevant to a particular type of parameter. Adjustments to the values of these parameters along with the value of a in equation (3) can be used to create and manipulate complex geometries.

As mentioned earlier, the effect of these parameters on the surface shape is easy to appreciate. Table 1 shows the values of the chosen parameters for $d = 1$ for the surface shown in Fig. 1.



(a)



(b)

Fig. 2 The effect on the shape of the surface by changing the design parameters corresponding to the boundary $d = 1$. (a) The boundary curves. (b) The corresponding PDE surface patch

In order to show the effect of the design parameters a different set of values for the parameters for the boundary $d = 1$ of the surface shown in Fig. 1 is chosen. The new values chosen for the parameters are shown in Table 3 and the resulting surface is shown in Fig. 2. Note the value of the a for the surface shown in Fig. 2 is the same as that shown in Fig. 1, i.e. a=1.0100. Essentially, the new values of the parameters produced a dilation followed by a translation which is followed by a rotation of the boundary curve $d = 1$. The parameters introduced on the boundary curves are varied using a graphical interface where the corresponding surface is visualized simultaneously. The spectral approximation method for solving the PDE, mentioned earlier, is fast enough for the surfaces to be created and manipulated in real time.

In order to create the geometry corresponding to complex shapes, more than one surface patch often needs to be joined together along common boundaries thereby enabling one to form a composite surface.[11,12] The parametric model discussed above has been extended to cater for such composite bodies. For example, Fig. 3(**a**) shows the shape of a yogurt pot created using two surface patches. The ridges at the base of the container are created so as to obtain a more realistic shape for the container. This is done by means of the corresponding derivative curve which in this case is defined by means of a cubic B-Spline.[17] Details of how this is done are discussed later in this paper. Similarly, Fig. 3(**b**) shows the shape of a yogurt container with dual compartments created using 4 separate surface patches. In each case the composite shapes are parameterized by means of the boundary curves corresponding to the surface patches which make up the shape.

parameter	value
d_{1T_x}	-0.310
d_{1T_y}	-0.850
d_{1T_z}	0.000
d_{1D_x}	0.371
d_{1D_y}	0.792
d_{1D_z}	0.000
d_{1R_x}	3.138
d_{1R_y}	0.000
d_{1R_z}	0.220

Table 2 Values for the design parameters for the boundary $d = 1$ of the surface shown in Fig. 2

Design Optimization

In this section we show how automatic design optimization of thin-walled containers can be carried out using the PDE parametric model discussed above. In particular, an example taken from a practical setting, i.e., the optimization for strength of a yogurt container shown in Fig. 3(**a**) is discussed.

Essentially this work is an extension of previous work[14] where it has been demonstrated how complex shapes can be defined and parameterized efficiently using the PDE method. Furthermore, in that work it has been demonstrated how such techniques can be used for efficient design optimization in a generic setting. Thus, the discussion here is focused on a more realistic application resulting from the previous work where the aim is to design practical objects of thin-walled structures for food packaging.

A typical design optimization problem can be thought of as maximizing or minimizing an objective function without violating a set of constraints.

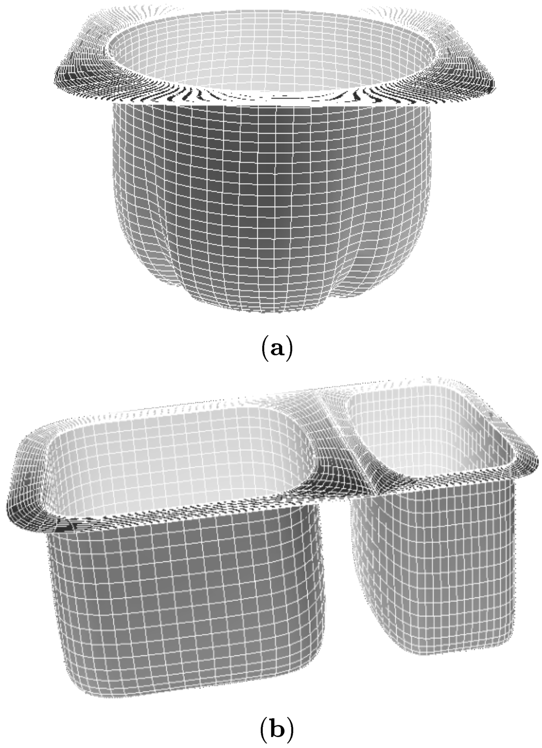


Fig. 3 Composite shapes build from multiple surface patches. (a) Shape of a yogurt container with a single compartment. (b) Shape of a yogurt container with dual compartments.

There exist a wide variety of methods for numerical optimization. The choice of a particular method is problem specific and involves considerations such as the computational cost of evaluating the function to be optimized, and also the behavior of the function within the design space.

Here the optimization is performed by solving a constrained optimization problem formulated by means of the objective function f, the design parameters associated with the geometry of the shape and the constraints imposed. i.e. strength and volume. This is carried out using an augmented Lagrange multiplier method[7] along with the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method.[9,15] The BFGS method is a Quasi-Newton which performs a series of local minimizations of the objective function f along a straight line in the parameter space. The method is iterative where at each successive iteration we obtain a vector $\underline{x}^k = ({x_1}^k, {x_2}^k, ..., x_n^k)$ of the n independent design variables which is computed from the previous iterations using the expression:

$$\underline{x}^{k+1} = \underline{x}^k + \alpha^k \underline{s}^k. \qquad (5)$$

Here $\underline{s}^k$ is a direction of search and α^k is a scalar that minimizes the one-dimensional function $F(\alpha) \equiv f(\underline{x}^k + \alpha^k \underline{s}^k)$. Thus, given a starting point, the algorithm moves in a series of steps through points in the parameter space giving a lower value of the objective function than previously, until it finds a local minimal value of the objective function. The augmented Lagrange multiplier method enables one to solve the optimization problem with equality conditions.

Calculation of the Objective Function

Here we are concerned in the design of containers possessing the minimum amount of material subject to a required strength. Moreover, the problem of stacking the containers on top of each other for the purpose of transportation and display on the supermarket shelves is considered. Assuming the container at the bottom of a stack is considered, such a container experiences a stress (due to the weight of the rest of the containers in the stack) and hence becomes slightly deformed. It is the excessive shear stress which can cause most damage to the material under consideration. Thus, a measure for the required strength of the container can be computed by calculating the maximum shear stress within the container. This is done by means of thin-shell finite element analysis where a force is applied around the rim of the container which translates to the tension in the top seal of the container due the weight of the other containers in the stack. It is also assumed that the base of the container is fixed.

Given the PDE geometry corresponding to the shape of a container, the geometry is discretized, by means of the 2-dimensional (u, v) parameter space, to obtain a valid finite element mesh. In particular, the task of node numbering is relatively easy to deal with using the discretized (u, v) parameter space. To create the appropriate shell elements a thickness is generated by means of calculating normals to the surface points defining the finite element mesh. For the yogurt container discussed here a mean thickness distribution of 0.8mm throughout the container is assumed. This assumption is taken to simplify the problem, since industrial forming processes do not often produce containers with a constant thickness distribution. As far as the material properties are concerned, assuming the material under consideration is isotropic and linearly elastic, the Young's modulus and Poisson's ratio was taken to be that of polystyrene, i.e. 2.4GPa and 0.33 respectively. Fig. 4 shows the displacement profile predicted by the finite element analysis for a typical loading for the stacking problem discussed above, where a tension of 15N was applied to the top seal of the container.

As mentioned above, the design objective here is the minimization of the mass of the container sub-

Fig. 4 The displacement profile of the container due to stacking, predicted by the finite element analysis.

ject to a given maximum shear stress. Hence the process of optimization requires the calculation of the maximum shear stress that occurs in the solution for every design to be analyzed. The principal components of shear stresses σ_{xy}, σ_{xz} and σ_{yz} are obtained by performing a finite element analysis of the structure. In the example presented, the analysis was carried by assuming the structure to be an isotopic thin shell with constant thickness distribution. Details of how the finite element analysis for a shell structure can be formulated is found in Niordson[8] and Zienkiewicz.[18]

The maximum shear stress $\sigma^p_{\max}$ occurring on any plane through a point p is given by:

$$\sigma^p_{\max} = \max\{ \mid \sigma_{xy} \mid , \mid \sigma_{xz} \mid , \mid \sigma_{yz} \mid \}. \quad (6)$$

So the measure for the strength of the container to be the maximum shear stress occurring in the whole structure, i.e.

$$f = \underbrace{\max}_{\text{(all points)}} \{\sigma^p_{\max}\}. \quad (7)$$

Optimization of the shape of a Yogurt Container

The automatic design optimization of the container shown in Fig. 3(**a**) is considered here. This particular container is created using two PDE surface patches. We are interested in determining the optimal shape corresponding to the bowl part of the container that is parameterized by means of the shape defining boundary curves discussed earlier. Since the interest here is in determining the internal shape of the container, the changes in the parameters introduced on the derivative boundary curves of the bowl part of the container is only considered. In particular the translation in y direction and dilations in the xy plane of these two curves within defined limits to obtain a favorable range of shapes is considered. The ridges introduced at the base are created so as to obtain a realistic shape for the container, where such ridges are commonly found in food containers for a variety of reasons.

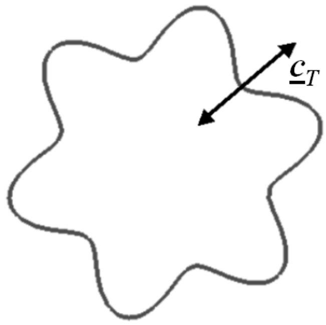


Fig. 5 Cubic B-spline curve corresponding derivative condition enabling to create the ridges at the base of the pot.

These ridges are created by means of the derivative boundary curve corresponding to the base of the pot. This curve is defined using a cubic B-spline definition such that:

$$\underline{d}_1(v) = \sum_i \underline{c}_i B_i(v), \quad (8)$$

where B_i is a cubic B-spline, and $\underline{c}_i$ are the control points.

The control points, $\underline{c}_i$, of the spline are chosen so as they initially lie on the curve which determine the value of dilation parameter. The number of control points chosen determine the number of ridges on the pot. In order to define the amplitude of the ridges the control points are translated, normal to the curve, within a confined xy planar region. Thus, the amount of translation of the control points, normal to the curve determine the prominence of the ridges. The translation of the control points, introduces an extra shape parameter which is referred as $\underline{c}_T$. Figure 5 shows the B-spline curve illustrating the parameter $\underline{c}_T$.

With the above formulation the design parameters and their initial values for the optimization of the yogurt pot are shown in Table 3. Note that the table also shows the chosen range for each parameter. The range specified for each design parameter (by means of interactively choosing a maximum and a minimum) allows the parameters to be varied within the specified ranges enabling alternative shapes to be created within the design space automatically. These ranges are chosen to ensure that the geometry of sensible shapes are fed into the optimization routine. The container was loaded so as a tension of 15N was applied to the top seal of the container. The required strength of the container is specified so as $\sigma^p_{\max} \leq 15$MPa. The design space is further restricted by choosing a volume constraint for the container. For this particular example the volume of the container was taken to to be 150ml.

Parameter	Min	Max	Initial	Optimal
d_{1T_y}	-0.400	-0.001	-0.400	-0.133
d_{1D_x}	0.100	0.800	0.450	0.298
d_{1D_y}	0.100	0.800	0.450	0.309
d_{2T_y}	0.001	0.400	0.400	0.396
d_{2D_x}	0.100	0.800	0.450	0.371
d_{2D_y}	0.100	0.800	0.450	0.379
a	1.000	7.000	1.000	1.095
$\underline{c}_T$	-0.300	0.300	0.200	0.002

Table 3 Parameter values for the yogurt pot with a single compartment.

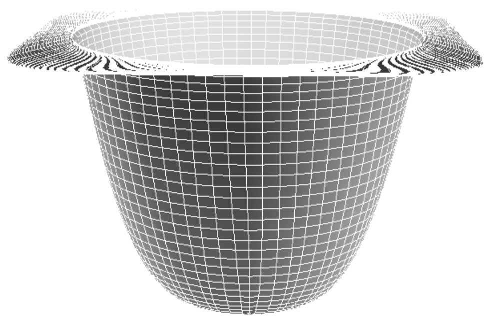


Fig. 6 Optimal design for the yogurt pot with a single compartment.

Once the geometry is parameterized, the design parameters and their ranges along with the value for the required volume of the container are fed into the optimization routine. This routine then automatically searches the design space in order to find the design with lowest possible value of the chosen merit function. Due to the extensive finite element analysis calculations, needed to calculate the value of the objective function for each iteration of the optimization, it took a little over 24 hours for the optimization to complete on a PC workstation with a 800 MHz processor.

The values of the parameters obtained for the optimal design is shown in Table 3 and the optimal shape is shown in Fig. 6. The resulting optimal shape had a relative reduction in mass of 19.5%. The maximum shear stress occurring within the resulting optimal shape was found to be 14.31MPa. As of note is that the optimal design shows a narrower base and the ridges introduced at the base have also vanished.

Conclusion

In this paper it is demonstrated how the PDE method for the description and parameterization of complex geometries can be used to set-up the shape optimization of practical thin-walled objects. In particular, it is shown how the method can be used to parameterize the geometry and more importantly how automatic optimization by means of physical analysis can be performed.

It is noteworthy that presently when considering the physical properties of new designs made from thin-walled structures, the designers usually rely upon their own knowledge and experience, and further along the design process, model testing. However, in this paper it is shown that the geometry defining and the succinct parameterization characteristics of the PDE method can be used to carry out automatic design optimization in a practical setting where the time consuming and tedious process of creating trial designs and model testing can be minimized.

Acknowledgments

The author wishes to acknowledge the support of EPSRC grant GR/M73125 ('The Optimal Design and Manufacture of Thin-walled Structures from Polymers.'). He is indebted to Professors M J Wilson and M I G Bloor of Department of Applied Mathematics of University of Leeds for useful discussions and comments on the manuscript. He also wishes to thank Dr P. Duddington of Huhtamaki Van Leer, Professor I. M. Ward and Dr A. P. Unwin of Department of Physics and Astronomy at University of Leeds for useful discussions.

References

[1]Bhakuni, N.D, Trageser, A.B., Sundaresan, S., and Ishii, K., Structural Optimisation Methods for Aluminium Beverage Can Bottoms. In *Proceedings of ASME Advances in Design Automation Conference*, Houstis, E.N., and Rice, J.R. (Eds.), ASME Publication No. **DE-Vol. 32-1**, 265–271, 1991.

[2]Bloor, M.I.G., and Wilson, M.J., "Generating blend surfaces using partial differential equations," *Computer-Aided Design*, **21**, 165–171, 1989.

[3]Bloor, M.I.G., and Wilson, M.J., "Using Partial Differential Equations to Generate Freeform Surfaces," *Computer Aided Design*, **22**, 202–212, 1990.

[4]Bloor, M.I.G., and Wilson, M.J., "Spectral Approximations to PDE Surfaces," *Computer-Aided Design*, **28**, 145–152, 1996.

[5]Dekanski, C.W., Bloor, M.I.G., and Wilson, M.J., "The Generation of Propeller Blades Using the PDE Method," *Journal of Ship Research*, **39**, 108–116, 1995.

[6]Dijk, van. R, and Keulen, van. F., Optimisation of Food Packaging; Explorative Study to the Optimal Bottle Design (Part 1). In *Proceedings of European Congress on Computational Methods in Applied Sciences and Engineering*, Oñate, E., Bugeda, G., and Suárez., B., (Eds.), CIMNE, Barcelona, CDROM, 2000.

[7]Greig, D.M., *Optimisation.* Longman, London, 1980.

[8]Niordson, F.I., *Shell Theory.* Elsevier Science Publishers, Amsterdam, 1985.

[9]Press, W.H., Teukolsky, S.A., Vetterling, W.T, and Flannery, B.P, *Numerical Recepies in C.* Cambridge University Press, UK, 1992.

[10]Struik, D., *Lectures on Classical Differential Geometry.* Addison-Wesley Pub. Co., Reading, Massachusetts, 1961.

[11]Ugail, H., Bloor, M.I.G. and Wilson, M.J., "On Interactive Design Using the PDE method," Mathematical Methods for Curves and Surfaces II, (eds) M. Dæhlen, T. Lyche, and L. L. Schumaker, Vanderbilt University Press Nashville TN, 493–500, 1998.

[12]Ugail, H., Bloor, M.I.G., and Wilson, M.J., "Techniques for Interactive Design Using the PDE Method," *ACM Transactions on Graphics*, **18**, (2), 195–212, 1999.

[13]Ugail, H., Bloor, M.I.G., and Wilson, M.J., "Manipulations of PDE Surfaces Using an Interactively Defined Parameterisation," *Computers and Graphics*, **24**, (3), 525–534, 1999.

[14]Ugail, H., Bloor, M.I.G., and Wilson, M.J., "Implementing Automatic Design Optimisation within an Interactive Environment," AIAA 2000-4858, September 2000.

[15]Vanderplaats, G.N., *Numerical Optimisation Techniques for Engineering Design: With Applications.* McGraw-Hill, New York, 1984.

[16]Vida, J., Martin, R.R., and Varady, T., "A Survey of Blending Methods that use Parametric Surfaces," *Computer-Aided Design*, **26**,(5), 341–365, 1994.

[17]Schumaker, L.L., *Spline Functions: Basic Theory.* John Wiely and Sons, New York, 1981.

[18]Zienkiewicz, O.C.Z., *The Finite Element Method.* McGraw-Hill, New York, 1977.