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Parametric Surface Meshing for Design Optimization using a PDE formulation

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Abstract

The problem of parametric surface meshing for the purpose of design optimisation using finite element analysis is considered. Here the surface mesh is generated as a solution of a suitably posed boundary value problem implemented on a 2D parameter space. A robust meshing scheme is presented where an initial mesh is manipulated, with the aid of the 2D parameter space, so as to obtain a suitable surface triangulation. This meshing scheme can then be used to create suitable finite element meshes with which accurate design optimisations can be carried out.

Introduction

The problem of creating a surface triangulation suitable for engineering analysis is of great interest for many Computer-Aided Engineering (CAE) applications. There are great many references citing the significant efforts that have been made towards developing efficient and robust algorithms for automatic triangulation of complex geometric shapes. Among the existing techniques for surface triangulations are algebraic mapping methods [9], variational methods [15], Delaunay triangulation [4], Quadtrees [18] and Advancing Front method [12]. While the bulk of these existing methods are well equipped to tackle the problem of meshing complex domains, from the point of view of physical analysis, certain problems still need addressing with regard to the actual quality of the mesh [10, 5]. Furthermore the existing mesh improvement techniques can be computationally extensive.

There exist many methods whereby surface meshes are generated by means of mapping meshes produced in the parameter space onto surface. All of these methods fall into the general mesh generation techniques described earlier. For example, [19] uses an algebraic mapping technique to generate surface meshes where a triangulation update scheme (based on edge swapping and smoothing) is adopted on the parameter space in order to improve the surface mesh. Similarly, in [6] the use of Delaunay triangulation is made to generate surface meshes. Here, as a means of checking and improving the quality of the surface mesh, a mapping technique based on the circumcircle property of the triangular elements in the parameter space is used to update the surface mesh.

This paper shows how a PDE formulation can be used to generate suitable surface meshes from the point of view of design optimisation using finite element analysis. In the following sections an overview of the PDE method is discussed, which shows how a 6th order biharmonic PDE can be used to create an initial mesh over a complex parametric surface. A discussion of how an initially created PDE surface mesh can then be improved to generate a mesh, which is suitable for use in finite element analysis is presented.

Surface Design and Mesh Generation using the PDE Method

In geometric design, it is common practice to define curves and surfaces using a parametric representation. Thus, surfaces are defined in terms of two parameters u and v so that any point on the surface $\underline{X}$ is given by an expression of the form:

$$\underline{X} = \underline{X}(u,v) . \tag{1}$$

Equation (1) can be viewed as a mapping from a domain Ω in the (u,v) parameter space to Euclidean 3-space. In the case of the PDE method [2, 3] this mapping is defined by means of a partial differential operator:

$$L_{uv}^{m}(\underline{X}) = \underline{F}(u,v) , \tag{2}$$

where the elliptic partial differential operator L is of degree m. Thus, effectively, surface design is treated as an appropriately posed boundary-value problem with appropriate boundary conditions imposed on $\partial\Omega$, the boundary of Ω. The chosen PDE is solved subject to a set of boundary conditions which are usually defined at the edges of the surface patch.

The Design Approach

For the work described here, the sixth order PDE:

$$\left(\frac{\partial}{\partial u^2} + a^2 \frac{\partial}{\partial v^2}\right)^3 \underline{X}(u,v) = 0 , \tag{3}$$

is used, where the condition on the function $\underline{X}(u,v)$, its first and second derivatives can be imposed at the edges of the surface patch. The parameter a

is a special design parameter which controls the relative smoothing of the surface in the u and v directions [3]. For periodic boundary conditions (e.g. $0 \le u \le 1$, $0 \le v \le 2\pi$) a closed form solution of equation (3) which allows $\underline{X}(u,v)$ to be written as:

$$X(u,v) = A_0(u) + \sum_{n=1}^{N} A_n(u)\cos(2nv) + B_n(u)\sin(2nv), \tag{4}$$

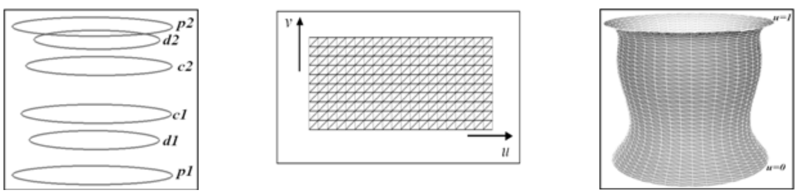
where

$$A_0 = a_{00} + a_{01}u + a_{02}u^2 + a_{03}u^3 + a_{04}u^4 + a_{05}u^5, \tag{5}$$

$$A_n = a_{n1}e^{anu} + a_{n2}e^{-anu} + a_{n3}ue^{anu} + a_{n4}ue^{-anu} + a_{n5}u^2e^{anu} + a_{n6}u^2e^{-anu}, \tag{6}$$

$$B_n = b_{n1}e^{anu} + b_{n2}e^{-anu} + b_{n3}ue^{anu} + b_{n4}u^2e^{-anu} + b_{n5}u^2e^{anu} + b_{n6}u^2e^{-anu}, \tag{7}$$

where the vector-valued constant coefficients in (5), (6) and (7) are determined by the imposed boundary conditions at $u = 0$ and $u = 1$.



(a) (b) (c)

Figure 1 A typical PDE surface resulting from the solution of the 6th order PDE. (a) The boundary curves. (b) A section of the (u,v) parameter space showing the triangulation. (c) The corresponding PDE surface patch.

Although the boundary conditions can be imposed by means of standard analytic functions, for the purpose of demonstration here the boundary conditions are defined in terms of curves in 3-space [16, 17]. For example, assuming v to be the periodic parameter, Figure 1(a) shows a typical set of boundary curves where the corresponding PDE surface is shown in Figure 1(c). A section of the corresponding (u,v) parameter space is also shown in Figure 1(b). The curves marked $p1$ and $p2$ correspond to the boundary conditions on the function $\underline{X}(u,v)$. A vector field corresponding to the difference between the points on the curves marked $p1$ and $p2$ and those marked $d1$ and $d2$ respectively, corresponds to the conditions on the function $\frac{\partial \underline{X}}{\partial u}$ such that:

$$\frac{\partial \underline{X}}{\partial u} = [\underline{p}(v) - \underline{d}(v)]s\ , \tag{8}$$

where s is a scalar. The condition $\frac{\partial^2 \underline{X}}{\partial u^2}$ is defined by means of the curves marked $p1$ and $p2$, $d1$ and $d2$ and those marked $c1$ and $c2$ such that:

$$\frac{\partial^2 \underline{X}}{\partial u^2} = [\underline{p}(v) - 2\underline{d}(v) + \underline{c}(v)]t\ , \tag{9}$$

where t is another scalar. As the format of the equations (8) and (9) suggests, the definitions of the boundary conditions $\frac{\partial \underline{X}}{\partial u}$ and $\frac{\partial^2 \underline{X}}{\partial u^2}$ resemble that of a finite-difference approximation scheme. Note that the surface patch does not necessarily pass through the curves defining the first and second derivative boundary conditions.

For the purpose of design optimization a parameterization is introduced via the boundary curves defining the surface. The design parameters are defined as linear transformations of these boundary curves, i.e. translations, dilations and rotation of the boundary curves corresponding to the boundary conditions of the PDE produce a wide variety of alternative shapes within a chosen design space.

Surface Meshing

As described above the PDE method produces a parametric surface patch of two parameters u and v. To produce a mesh on such a surface, a suitable discretization of the (u,v) parameter space is first carried out. The resulting surface mesh inherits the topological characteristics of the mesh resulting from a discretization based on the constant u, v lines of the parameter space. There are a number of methods by which a triangulation of a 2D region can be created, which include domain decomposition techniques such the Delaunay triangulation. For the purposes of demonstrations the following simple procedure has been used for generating a triangular mesh in the (u,v) parameter space. First a uniform discretization of the (u,v) parameter space, leading to rectangular elements, is carried out. Each rectangular element is then halved along its diagonal to obtain two triangular elements. The triangulation on the surface shown in Figure 1(c) has been implemented by using this triangulation procedure carried out on the (u,v) parameter space shown in Figure 1(b).

It is clear that as a result of the PDE mapping from parameter space to physical space a satisfactory mesh in (u,v) parameter space could result in a very poor mesh on the surface. Therefore, the aim here is to present a technique to check and improve the quality of such a surface mesh by means of direct mesh manipulation in the (u,v) parameter space.

Before the procedure for local mesh manipulation is applied, a suitable subdivision of the parameter space along u and v is sought, based on the intrinsic parameterization of the surface patch. This is an important step since the relative subdivisions along u and v directions have a direct influence on the aspect ratio of the individual triangles within the mesh.

To compute a suitable value for the aspect ratio of the initial subdivision surface patch, first an arbitrary number of uniform subdivisions along the u and v directions of the (u,v) parameter space are carried out, and the corresponding $\underline{X}(u,v)$ points on the PDE surface patch are computed. Then along the u, v parameter lines on the surface patch the corresponding arc-lengths of the curves on the surface are calculated. By using these arc-lengths, a measure of the average aspect ratio, r, is obtained using:

$$r = \frac{\sum_i S(u_i)/M}{\sum_j S(v_j)/N}, \quad 1 \le i \le M \ , \ 1 \le j \le N \ , \tag{10}$$

where $S(u_i)$ and $S(v_j)$ are the arc lengths along u_i and v_j isoparametric lines respectively. The sum $\sum_i S(u_i)/M$ and $\sum_j S(v_j)/N$ are measures for the average arc-lengths along the given u and v parameter lines respectively of the surface patch. M and N are the positive integers corresponding to the arbitrary uniform subdivisions along u and v directions respectively.

Once the (u,v) parameter space is discretized according to the aspect ratio of the surface patch, mesh improvement by means of adjusting individual vertices is performed. Considering a triangular element t in the initial mesh, a shape regularity criterion [11, 1] based on the analysis of interpolation error can be defined so as to obtain a measure for the quality of individual triangles of the mesh. Given that a triangle, t, has edges d_1, d_2 and d_3 and area a, its shape regularity criterion $\alpha(t)$ can be defined as:

$$\alpha(t) = \frac{2\sqrt{3}\|a\|}{\|d_1\|^2 + \|d_2\|^2 + \|d_3\|^2} \ , \tag{11}$$

where the function $\alpha(t)$ is normalized such that $0 \le \alpha(t) \le 1$. The value of $\alpha(t)$ will attain the value 1.0 if the triangle is equilateral and approaches zero for triangles with small internal angles. For finite element analysis purposes, for a given triangle, if $\alpha(t) \ge 0.6$ then the triangle is considered to be acceptable [7].

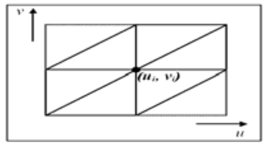
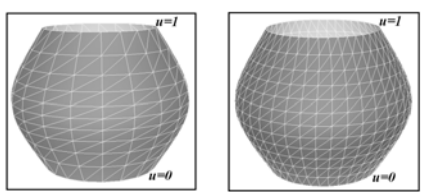


Figure 2. A section Ω_i of the (u,v) parameter space showing edges relating to a given vertex (u_i, v_i).

To describe the procedure adopted here in detail, refer to Figure 2 showing a section of (u,v) parameter space Ω_i, and consider the vertex labeled $V_i = (u_i, v_i)$, where $\Omega_i \subseteq \Omega$ contains all the elements of the mesh with the vertex V_i. For a series of triangulations T_k in Ω_i a local optimization problem can be formulated such that:

$$T_k^* = \min[\sum_i (1.0 - \alpha(t_i))^2], \; t_i \in T_k . \tag{12}$$

Referring to the elements of the mesh in Ω_i, it is clear that to effect the quality $\alpha(t)$ of the elements of the corresponding PDE surface defined by those in Ω_i we can iteratively change the position of V_i within the space of Ω_i. This local optimization problem is solved by implementing a line search algorithm [13] with appropriately imposed constraints so that the search space is restricted within a given Ω_i. These constraints are imposed to preserve the topology of the mesh structure and to prevent the unfolding of the mesh. Since this local problem is always bounded from above, the function $\sum_i (1.0 - \alpha(t_i))^2$ can only be decreased or at worst left unchanged. Thus, the unconstrained vertices (all vertices except those forming the edge) of the mesh are in turn updated iteratively improving the mesh in general. In practice one could iterate the global problem until a formal convergence criterion is met. However, it is common practice to only perform a fixed number of iterations. In fact, good quality meshes are usually obtained by using very few iterations.



(a) (b)

Figure 3. (a) An initial surface mesh in which triangles at $u = 1/2$ whose quality needs improving. (b) The improved mesh resulting from the smoothing process, by means vertex manipulation.

To illustrate the implementation of the above procedure to smooth a given mesh, consider the surface shown in Figure 3(a). The particular shape of this surface patch is created to exemplify a mesh in which some of the triangles have undesired geometric shapes that can be quantified in terms of the $\alpha(t)$ criterion described above. In particular, one can see that the elements towards the middle region $u = 1/2$ of the surface patch are of worst quality. The aim is to improve the quality of these elements and in general to improve the quality of triangles of the whole mesh. As a measure of the overall quality of the elements in the surface mesh the minimum value of $\alpha(t)$ taken over all the elements of the mesh is calculated here. For the initial shape shown in Figure 3(a) this minimum value of $\alpha(t)$ is 0.561.

Figure 3(b) shows the surface patch that results after the mesh smoothing process has been applied by taking into account the aspect ratio of the surface patch. As can be seen for the mesh on this surface patch there has been a significant improvement on the quality of its elements, as indicated by the minimum value of $\alpha(t)$ for the improved mesh which is now 0.738.

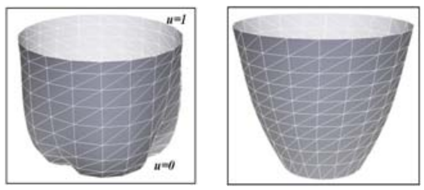
Numerical Example illustrating a Design Optimization

In this section, the PDE surface meshing scheme and the parameterization scheme is used to illustrate how a given design optimization can be carried out. The problem considered here is the minimization of the mass of a container with both a fixed volume and a prescribed level of strength.

Consider the shape shown in Figure 4 (a) describing the shape of a container suitable for packaging a food product such as yogurt. To formulate the design optimization, the problem of stacking such containers on top of each other is considered. Assuming the container to be considered is at the bottom of the stack, such a container will experience a level of stress (due to the weight of the rest of the containers in the stack) and hence becomes slightly deformed. It is the excessive shear stress that can cause most damage to the material under consideration. Thus, a measure for the required strength of the container can be computed by calculating the maximum plastic shear stress within the loaded container. Assuming the container is composed of the plastic with the material properties of that of polystyrene the strength of the container is characterized by means of the non-linear elasto-plastic thin-shell stress analysis using finite element method. A vertical force of $15Nm^{-1}$ (equivalent to the weight of about

30 yogurt containers) is applied around the rim of the container. It is also assumed that the base of the container is fixed.

With the above formulation the design objective here is set to be the minimization of the mass of the container subject to a given maximum shear stress. The optimization is performed by solving a constrained optimization problem using an augmented Lagrange multiplier method [8] along with the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method [14].



(a) (b)

Figure 4. Example illustrating the optimal design of a container for strength. (a) The initial design (b) The optimal design.

As mentioned earlier a surface parameterization is formulated via the linear transformation of the boundary curves. The objective here is to find the internal shape of the container with fixed edges. Therefore, the changes in the design parameters introduced on the first and second derivative boundary curves corresponding to the shape of the container are only varied, i.e. both the positional boundary curves are kept fixed. The required volume of the container was taken to be as 150ml. The required strength of the container is specified so as the level of stress occurring within the loaded structure is always less than 30% of the yield stress.

Once the geometry is parameterized, the design parameters and their ranges along with the value for the required volume of the container are automatically varied by the optimization routine. The routine searches the design space in order to find the design with the lowest possible value of the chosen merit function. For each iteration of the optimization a new design is generated where the corresponding triangular mesh is improved before the finite element analysis is performed.

Due to the extensive finite element analysis computations, the optimization took a little over 10 hours on a fast PC. The resulting optimal shape is shown in Figure 4 (b). This new design had a relative reduction in mass of 23.8%.

Conclusion

The problem of automatic triangulation of surfaces suitable for use in finite element analysis applications is addressed. The surface meshing is carried out by means of a sixth order elliptic PDE subject to suitable boundary conditions defined at the edges of the surface patch. A vertex manipulation procedure is adopted by means of a local optimization procedure where the mesh is iteratively updated by adjusting the individual vertices of the mesh. This local optimization problem is solved so as to optimize a geometric measure based on the shape of the triangles of mesh.

The existence of a closed form solution of the chosen PDE enables complex surfaces to be created and re-created very quickly thus making the mesh optimization scheme very fast. Furthermore, since the necessary computation for the surface meshing scheme is effectively carried out on the 2D parametric space, the process of mesh smoothing is very efficient and robust.

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