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METHOD OF BOUNDARY BASED SMOOTH SHAPE DESIGN

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ABSTRACT

The discussion in this paper focuses on how boundary based smooth shape design can be carried out. For this we treat surface generation as a mathematical boundary-value problem. In particular, we utilize elliptic Partial Differential Equations (PDEs) of arbitrary order. Using the methodology outlined here a designer can therefore generate the geometry of shapes satisfying an arbitrary set of boundary conditions. The boundary conditions for the chosen PDE can be specified as curves in 3-space defining the profile geometry of the shape.

We show how a compact analytic solution for the chosen arbitrary order PDE can be formulated enabling complex shapes to be designed and manipulated in real time. This solution scheme, although analytic, satisfies exactly, even in the case of general boundary conditions, where the resulting surface has a closed form representation allowing real time shape manipulation. In order to enable users to appreciate the powerful shape design and manipulation capability of the method, we present a set of practical examples.

KEY WORDS
Smooth shape design, boundary-value problems, PDEs

1 Introduction

An important requirement of any three-dimensional (3D) geometric design system is that it should facilitate easy and accurate creation of free-form surfaces [22]. Numerous techniques for geometric design have been proposed. These include implicit surfaces [8], the B-rep based on polygonal meshes [15], spline based modeling schemes such as Bezier and Non-Uniform Rational B-Splines (NURBS) [14] and subdivision techniques [9]. In particular, both spline and subdivision based techniques tend to be very popular and can be found in most 3D modeling software packages.

In contrast to the above geometric design techniques, the method discussed here uses solutions to elliptic Partial Differential Equations (PDEs). Although traditionally elliptic PDEs have been mainly utilized in solving engineering related problems such as electromagnetism, stress/strain in physical structures and fluid flows, nowadays PDE based techniques are increasingly becoming popular in many applications of geometric design. These include computer simulation of natural phenomena and animation [13], variational fairing [18], and image inpainting [1].

The work discussed in this paper is based around the pioneering work of Bloor and Wilson on the so called PDE method [2]. The particular approach we use adopts a boundary-value approach whereby a surface is characterized by defining a number of space curves (with associated derivative information) so as to form the surface's edges, and then the surface is generated between these curves by solving a PDE. The chosen PDE is usually lower order such as the Biharmonic equation.

The PDE method for geometric design is originally developed by Bloor and Wilson as a mechanism of blend shape generation [2]. Since this initial developments, the applications of the method have broadened (e.g. [16, 12, 10, 11, 19]). Thus, apart from the method being utilized in blend and free form shape design, it has been successfully utilized for automatic design for function in various design scenarios. This is achieved by means of incorporating engineering design criteria such as functional constraints into the geometric design of PDE surfaces. Examples of automatic design using the PDE method include automatic design of ship hulls [17], propeller blades [7], engineering components [5, 17] and thin-walled structures [21, 20]. Furthermore, several numerical algorithms have also been developed to approximate PDE surfaces using standard B-splines [4, 16]. This work was intended to demonstrate that PDE surfaces are virtually compatible with other mature and well established spline-based techniques for surface design and hence PDE surfaces can be readily incorporated into existing commercial design systems.

Recent contributions of Du and Qin and others on the PDE method is also notable. In particular, these recent work include the development of interactive tools for sculpting PDE shapes by incorporating physics-based modeling techniques [12, 10]. Furthermore, such recent work also includes the integration of implicit functions with PDEs in order to demonstrate that solid modeling based on the PDE method can potentially unify both geometric constraints and functional requirements within a single design framework [12].

This paper presents a methodology which describes how the original Biharmonic based PDE model of Bloor and Wilson can be extended. In particular, we show how the governing partial differential equation is generalized to

arbitrary order allowing complex shapes to be designed as single patch PDE surfaces. Using this technique a designer has the flexibility of creating and manipulating the geometry of a shape that satisfies an arbitrary set of boundary conditions.

The paper is organized as follows. Section (2) discusses the PDE method within the context of the Biharmonic equation based PDE model of Bloor and Wilson. Section (3) then presents the methodology by which the PDE method can be generalized to arbitrary order PDEs. This section also presents an analytic solution technique that can be utilized to create PDE surfaces of arbitrary order very efficiently in real time. In Section (4) we present a set of examples to demonstrate how the method of generalized PDE surfaces of arbitrary order can be utilized to generate complex geometry.

2   PDE Surfaces

A PDE surface is a parametric surface patch $\mathbf{X}(u, v)$, defined as a function of two parameters u and v on a finite domain $\Omega \subset R^2$, by specifying boundary data around the edge region of $\partial\Omega$. Typically the boundary data are specified in the form of $\mathbf{X}(u, v)$ and a number of its derivatives on $\partial\Omega$. Here one should note that the coordinates of a point u and v is mapped from that point in Ω to a point in the physical space. To satisfy these requirements the surface $\mathbf{X}(u, v)$ is regarded as a solution of a PDE of the form,

$$D_{u,v}^m \mathbf{X}(u, v) = \mathbf{F}(u, v), \qquad (1)$$

where $D_{u,v}^m \underline{X}(u, v)$ is a partial differential operator of order m in the independent variables u and v, while $\underline{F}(u, v)$ is vector valued function of u and v. Since boundary-value problems are of concern here, it is natural to choose $D_{u,v}^m \mathbf{X}(u, v)$ to be elliptic.

The most widely used PDE is based on the Biharmonic equation namely,

$$\left(\frac{\partial^2}{\partial u^2} + a^2 \frac{\partial^2}{\partial v^2} \right)^2 \mathbf{X}(u, v) = 0. \qquad (2)$$

Here the boundary conditions on the function $\mathbf{X}(u, v)$ and its normal derivatives $\frac{\partial \mathbf{X}}{\partial n}$ are imposed at the edges of the surface patch.

With this formulation one can see that the elliptic partial differential operator in Equation (2) represents a smoothing process in which the value of the function at any point on the surface is, in some sense, a weighted average of the surrounding values. In this way a surface is obtained as a smooth transition between the chosen set of boundary conditions. Note that the parameter a is a special design parameter which controls the relative smoothing of the surface in the u and v directions [3].

3   Generalized PDEs of Arbitrary Order

As described above, we assume that the surface we are dealing with is described by a parametric function $\mathbf{X}(u, v)$. In order to enable us to utilise the analytic solution method described later in the paper we further assume that parametric region Ω to be bounded by $\{u, v : 0 \leq u \leq 1; 0 \leq v \leq 2\pi\}$, so that the resulting surface patch is periodic.

With the above formulations we now seek a generalization to Equation (2) such that the arbitrary elliptic PDE satisfies a given number of $2N$ boundary conditions. Here N is an arbitrary integer such that $N \geq 2$. The general $2N$ boundary conditions can be written in the form,

$$\mathbf{X}(0, v) = \mathbf{f}_1(v), \qquad (3)$$
$$\mathbf{X}(u_i, v) = \mathbf{g}_i(v), \ i = 2 \ldots 2N - 1, \qquad (4)$$
$$\mathbf{X}(1, v) = \mathbf{f}_{2N}(v), \qquad (5)$$

where $\mathbf{f}_1(v)$ in Equation (3) and $\mathbf{f}_{2N}(v)$ in Equation (5) are function boundary conditions specified at $u = 0$ and $u = 1$ respectively. The conditions $\mathbf{X}(u_i, v) = \mathbf{g}_i(v)$ in Equation (4) can take the form either

$$\mathbf{X}(u_i, v) = \mathbf{f}_i(v) \text{ for } 0 < u_i < 1, \ i = 2 \ldots 2N - 1, \quad (6)$$

or

$$\frac{\partial \mathbf{X}}{\partial u}, \frac{\partial^2 \mathbf{X}}{\partial u^2}, \frac{\partial^3 \mathbf{X}}{\partial u^3}, \cdots, \frac{\partial^{2N-2} \mathbf{X}}{\partial u^{2N-2}}$$
$$\text{for } 0 \leq u_i \leq 1, \ i = 2 \ldots 2N - 1. \qquad (7)$$

In simpler terms the above boundary condition implies that for a PDE surface patch of order $2N$ we can specify two function boundary conditions, as given in Equations (3) and (5), that should be satisfied at the edges (at $u = 0$ and $u = 1$) of the surface patch, and a number of function or derivative conditions, as given in Equation (4), amounting to $2N - 2$ boundary conditions which the PDE should also satisfy.

With the above formulation we take standard Laplace operator, $\nabla = 0$, as a base PDE and generalize it to the Nth order such that,

$$\left(\frac{\partial^2}{\partial u^2} + a^2 \frac{\partial^2}{\partial v^2} \right)^N \underline{X}(u, v) = 0. \qquad (8)$$

As one can easily observe the above equation is a generalization of the usual 4th order elliptic PDE where the corresponding Biharmonic equation can be derived by choosing N to be 2.

3.1   Analytic Solution of PDEs of Arbitrary Order

Given a set of $2N$ boundary conditions as defined in Equations (3), (4) and (5), we take the (u, v) parameter space Ω to be the region $\{u, v : 0 \leq u \leq 1; 0 \leq v \leq 2\pi\}$. Thus, we assume that all the boundary conditions are periodic in v in the sense $\mathbf{f}_1(0) = \mathbf{f}_1(2\pi)$, $\mathbf{f}_{2N}(0) = \mathbf{f}_{2N}(2\pi)$

and $\mathbf{g}_i(0) = \mathbf{g}_i(2\pi)$. We further assume that all the boundary conditions are continuous functions within the domain of Ω.

With the above assumptions on the boundary conditions, we can utilize the method of separation of variables to write the analytic solution of Equation (8) as,

$$\mathbf{X}(u,v) = \mathbf{A}_0(u) + \sum_{n=1}^{\infty}[\mathbf{A}_n(u)\cos(nv) + \mathbf{B}_n(u)\sin(nv)], \tag{9}$$

where

$$\mathbf{A}_0 = \mathbf{a}_{00} + \mathbf{a}_{01}u + \mathbf{a}_{02}u^2 + \ldots + \mathbf{a}_{(2N-1)}u^{2N-1}, \tag{10}$$

$$\begin{aligned}\mathbf{A}_n = {} & \mathbf{a}_{n1}e^{anu} + \mathbf{a}_{n2}ue^{anu} + \mathbf{a}_{n3}e^{-anu} + \mathbf{a}_{n4}ue^{-anu} \\ & + \ldots + \mathbf{a}_{n(2N-3)}u^{N-2}e^{anu} + \mathbf{a}_{n(2N-2)}u^{N-1}e^{anu} \\ & + \mathbf{a}_{n(2N-1)}u^{N-2}e^{-anu} + \mathbf{a}_{n2N}u^{N-1}e^{-anu}, \tag{11}\end{aligned}$$

$$\begin{aligned}\mathbf{B}_n = {} & \mathbf{b}_{n1}e^{anu} + \mathbf{b}_{n2}ue^{anu} + \mathbf{b}_{n3}e^{-anu} + \mathbf{b}_{n4}ue^{-anu} \\ & + \ldots + \mathbf{b}_{n(2N-3)}u^{N-2}e^{anu} + \mathbf{b}_{n(2N-2)}u^{N-1}e^{anu} \\ & + \mathbf{b}_{n(2N-1)}u^{N-2}e^{-anu} + \mathbf{b}_{n2N}u^{N-1}e^{-anu}, \tag{12}\end{aligned}$$

where $\mathbf{a}_{00} + \mathbf{a}_{01}$, $\mathbf{a}_{02}$, $\ldots$, $\mathbf{a}_{2N-1}$, $\mathbf{a}_{n1} + \mathbf{a}_{n2}$, $\mathbf{a}_{n3}$, $\mathbf{a}_{n4}$, $\ldots$, $\mathbf{a}_{n(2N-3)}$, $\mathbf{a}_{n(2N-2)}$, $\mathbf{a}_{n(2N-1)}$, $\mathbf{a}_{n2N}$ and $\mathbf{b}_{n1} + \mathbf{b}_{n2}$, $\mathbf{b}_{n3}$, $\mathbf{b}_{n4}$, $\ldots$, $\mathbf{b}_{n(2N-3)}$, $\mathbf{b}_{n(2N-2)}$, $\mathbf{b}_{n(2N-1)}$, $\mathbf{b}_{n2N}$ are vector-valued constants, whose values are determined by the imposed boundary conditions at u_i where $0 \leq u_i \leq 1$.

Since the chosen boundary conditions are all continuous functions which are also periodic in v we can write down their Fourier series representation as,

$$\mathbf{f}_1(v) = \mathbf{C}_0^1 + \sum_{n=1}^{\infty}[\mathbf{C}_n^1\cos(nv) + \mathbf{S}_n^1\sin(nv)], \tag{13}$$

$$\begin{aligned}\mathbf{g}_i(v) = {} & \mathbf{C}_0^i + \sum_{n=1}^{\infty}[\mathbf{C}_n^i\cos(nv) + \\ & \mathbf{S}_n^i\sin(nv)], \ i = 2 \ldots 2N - 1, \tag{14}\end{aligned}$$

$$\mathbf{f}_{2N}(v) = \mathbf{C}_0^{2N} + \sum_{n=1}^{\infty}[\mathbf{C}_n^{2N}\cos(nv) + \mathbf{S}_n^{2N}\sin(nv)] \tag{15}$$

Let us assume for the moment that the Fourier sums in the Expressions (13), (14) and (15) have finite M modes. Then the vector constants $\mathbf{C}_0^1$, $\mathbf{C}_0^i$ for $i = 2 \ldots 2N - 1$, and $\mathbf{C}_0^{2N}$ can be obtained by directly comparing them with the constants $\mathbf{a}_{00}$, $\mathbf{a}_{01}$, $\mathbf{a}_{02}$, $\ldots$, $\mathbf{a}_{(2N-1)}$ given in Equation (10). Now for each of the Fourier modes $n = \{1, \ldots, M\}$ we can write linear systems,

$$\begin{pmatrix} \mathbf{a}_{n1} \\ \vdots \\ \mathbf{a}_{n2N} \end{pmatrix} = \mathbf{A}(a,n) \begin{pmatrix} \mathbf{C}_n^1 \\ \vdots \\ \mathbf{C}_n^{2N} \end{pmatrix} \tag{16}$$

and

$$\begin{pmatrix} \mathbf{b}_{n1} \\ \vdots \\ \mathbf{b}_{2N} \end{pmatrix} = \mathbf{B}(a,n) \begin{pmatrix} \mathbf{S}_n^1 \\ \vdots \\ \mathbf{S}_n^{2N} \end{pmatrix} \tag{17}$$

where $\mathbf{A}(a,n)$ and $\mathbf{B}(a,n)$ are $2N \times 2N$ matrices, whose coefficients can be obtained by solving the linear systems (16) and (17) subject to the Fourier coefficients corresponding to the $2N$ boundary conditions.

The above solution scheme is based on the fact that the boundary conditions can be expressed as a finite Fourier series. However, in practice this cannot be assumed. We, therefore, adopt a generalized version of the spectral approximation to the Biharmonic PDE given in [6] as described below.

Although in practical terms we cannot assume that a given boundary condition can be expressed accurately using a finite Fourier series, it is reasonable to assume that the boundary conditions can be written as,

$$\begin{aligned}\mathbf{f}_1(v) = {} & \mathbf{C}_0^1 + \sum_{n=1}^{M}[\mathbf{C}_n^1\cos(nv) + \mathbf{S}_n^1\sin(nv)] \\ & + \mathbf{R}_1(v), \tag{18}\end{aligned}$$

$$\begin{aligned}\mathbf{g}_i(v) = {} & \mathbf{C}_0^i + \sum_{n=1}^{M}[\mathbf{C}_n^i\cos(nv) + \mathbf{S}_n^i\sin(nv)] \\ & + \mathbf{R}_i(v), \ i = 2 \ldots 2N - 1, \tag{19}\end{aligned}$$

$$\begin{aligned}\mathbf{f}_{2N}(v) = {} & \mathbf{C}_0^{2N} + \sum_{n=1}^{M}[\mathbf{C}_n^{2N}\cos(nv) + \mathbf{S}_n^{2N}\sin(nv)] \\ & + \mathbf{R}_{2N}(v). \tag{20}\end{aligned}$$

Thus, the basic idea here is to formulate each of the boundary conditions in terms of the sum of a finite Fourier series containing M modes and a 'remainder' term $\mathbf{R}_i(v)$, $i = 1 \ldots 2N$ which contains the higher order Fourier modes. In [6], for the case of Biharmonic equation, it is shown that the higher order Fourier modes make negligible contributions to the interior of the PDE patch and the same applies for the general case of Nth order Biharmonic PDE. Hence it is reasonable to truncate the Fourier series at some finite M, (typically $M = 6$ is adequate), and represent the contribution of the high frequency modes to the surface with a remainder function $\mathbf{R}(u,v)$. The format of this remainder function is somewhat arbitrary and for this work it is taken to be of the form,

$$\begin{aligned}\mathbf{R}(u,v) = {} & \mathbf{r}_1e^{\omega u} + \mathbf{r}_2ue^{\omega u} + \mathbf{r}_3e^{-\omega u} + \mathbf{r}_4ue^{-\omega u} \\ & + \ldots + \mathbf{r}_{(2N-3)}u^{N-2}e^{\omega u} + \mathbf{r}_{(2N-2)}u^{N-1}e^{\omega u} \\ & + \mathbf{r}_{(2N-1)}u^{N-2}e^{-\omega u} + \mathbf{r}_{2N}u^{N-1}e^{-\omega u}, \tag{21}\end{aligned}$$

where $\mathbf{r}_1$, $\mathbf{r}_2 \ldots \mathbf{r}_{(2N-1)}$, $\mathbf{r}_{2N}$ are vector-valued constants which depend on v.

Now by taking $\widetilde{\mathbf{X}}(u, v)$ to the approximate solution,

$$\widetilde{\mathbf{X}}(u, v) = \mathbf{A}_0(u) + \sum_{n=1}^{M}[\mathbf{A}_n(u)\cos(nv) + \mathbf{B}_n(u)\sin(nv)], \tag{22}$$

satisfying the boundary conditons of the finite Fourier series we define difference functions such that,

$$\mathbf{df}_1(v) = \mathbf{df}_1(v) - \widetilde{\mathbf{X}}(0, v), \tag{23}$$

$$\mathbf{dg}_i(v) = \mathbf{g}_i(v) - \widetilde{\mathbf{X}}(u_i, v),$$
$$i = 2 \ldots 2N - 1, \tag{24}$$

$$\mathbf{df}_{2N}(v) = \mathbf{df}_1(v) - \widetilde{\mathbf{X}}(1, v). \tag{25}$$

By choosing ω in the Expression (21) to be an, the vector constants $\mathbf{r}_1 \ldots \mathbf{r}_{2N}$ can be computed by means of direct comparison with the difference terms $\mathbf{df}_1(v)$, $\mathbf{df}_{2N}(v)$ and $\mathbf{dg}_i(v)$, for $i = 2 \ldots 2N - 1$ in Equations (23) - (25).

Finally the approximate analytic solution of the PDE is given as,

$$\mathbf{X}(u, v) = \widetilde{\mathbf{X}}(u, v) + \mathbf{R}(u, v). \tag{26}$$

It is important to note that the choice of the number of Fourier terms M will affect how well $\widetilde{\mathbf{X}}(u, v)$ approximates the solutions of the general N^{th} order Biharmonic PDE. Although this may be the case, due to the choice of difference functions utilized here, i.e. by computing the difference between the original boundary conditions and the corresponding finite Fourier series, as described in Equations (23) - (25) the approximate solution satisfies the chosen set of boundary conditions exactly (to within the machine accuracy).

The above solution method can be viewed as a spectral method for the solutions of arbitrary order linear elliptic PDEs where the expansion functions are based upon the Fourier series . Thus, this solution scheme can be applied in the case of general complex periodic boundary conditions which can be seen to be suitable enough for generating a wide range of complex geometries.

4 Examples of Higher Order PDE Surfaces

In order to demonstrate the capability of complex surface generation using higher order equations we discuss several examples. As one can clearly see, there are several ways by which one could specify the boundary conditions for the Nth order Biharmonic PDE. One way to do is to specify all the boundary conditions in terms of function boundary conditions whereby the resulting surface patch will contain all the conditions. Another way is to specify two function
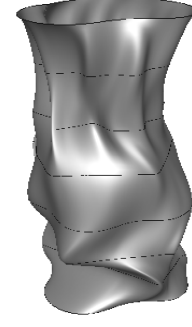

Figure 1. A typical higher order PDE patch generated as a solution to the 8th order equation.

boundary conditions corresponding to the edges of the surface patch and then specify a number of derivative conditions. Furthermore, a combination of function and derivative boundary conditions within the interior of the surface path is also possible.

Figure (1) shows a typical higher order PDE surface patch of order 8, whereby all the boundary conditions for the PDE are chosen to be function boundary conditions defined as curves in 3-space. These curves are all free-hand curves generated as cubic B-Splines within an interactive graphical environment. The surface is then generated and rendered in real time where the corresponding boundary curves are also shown on the surface.

Figure 2 shows the familiar shape of the Klein bottle generated as solution to the 40th order PDE. For the boundary conditions, we have taken cross section curves of the Klein bottle using its analytic representation as given below.

$$x = \begin{cases} \alpha \cos(u)(1 + \sin(u)) + \gamma \cos(u)\cos(v) \\ 0 \le u < \pi \\ \alpha \cos(u)(1 + \sin(u)) + \gamma \cos(v + \pi) \\ \pi \le u \le 2\pi \end{cases} \tag{27}$$

$$y = \begin{cases} \beta \sin(u) + \gamma \sin(u)\cos(v) & 0 \le u < \pi \\ \beta \sin(u) & \pi \le u \le 2\pi \end{cases} \tag{28}$$

$$z = \gamma \sin(v) \tag{29}$$

Thus, using the above analytic form, we utilized 40 cross sectional ellipses along u with $0 \le v \le 2\pi$, $\alpha = 6.0$, $\beta = 16.0$ and $\gamma = 4(1 - \cos(u)/2)$.

Figure (3)(**b**) shows a fluid membrane shape such as that corresponding to a red blood cell generated as an 8th order PDE with function and derivative conditions as shown in Figure (3)(**a**). The function boundary conditions in this case are p_1, p_2, p_3 and p_4 where p_1 and p_4 are taken to be points correspond to the edges of the surface patch. The other boundary conditions are taken to be the first and
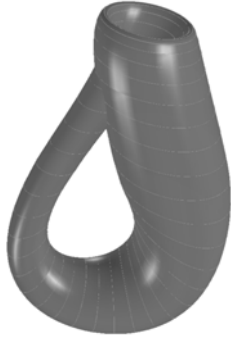
Figure 2. The Klein bottle PDE surface generated as solution to 40th order PDE.
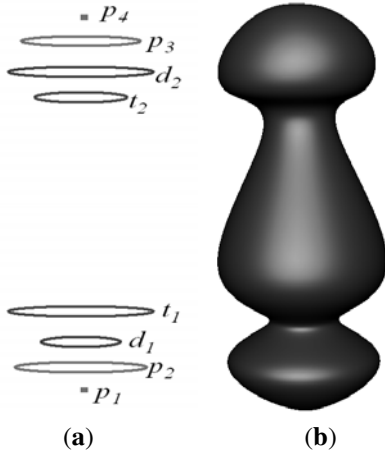

        (a)   (b)

Figure 3. A membrane shape generated as solution to higher order PDE with a number of position and derivative boundary conditions. (**a**) Function and derivative boundary curves (**b**) The corresponding surface patch.

second derivative conditions at the u to be positions corresponding to the position curves p_2 and p_3 such that,

$$\frac{\partial \mathbf{X}}{\partial u}\Big|_{\mathbf{p_2}} = [\mathbf{d_1}(v) - \mathbf{p_2}(v)]\,\alpha, \qquad (30)$$

$$\frac{\partial \mathbf{X}}{\partial u}\Big|_{\mathbf{p_3}} = [\mathbf{p_3}(v) - \mathbf{d_2}(v)]\,\beta, \qquad (31)$$

and

$$\frac{\partial^2 \mathbf{X}}{\partial u^2}\Big|_{\mathbf{p_2}} = [\mathbf{p_2}(v) - 2\mathbf{d_1}(v) + \mathbf{t_1}(v)]\,\gamma, \qquad (32)$$

$$\frac{\partial^2 \mathbf{X}}{\partial u^2}\Big|_{\mathbf{p_3}} = [\mathbf{p_3}(v) - 2\mathbf{d_2}(v) + \mathbf{t_2}(v)]\,\eta, \qquad (33)$$

where α, β, γ and η are constants.

As the format of the Equations (30) - (33), as shown in Figure (3)(**a**), suggests the definitions of the derivative boundary conditions resemble that of a finite-difference approximation scheme. It is important to note that the surface patch does not necessarily pass through the curves that define the derivative boundary conditions.

Figure 4. Generic shape of a dolphin created interactively.

Finally as a practical surface design example we present the generic shape of a dolphin shown in Figure (4) created interactively using 5 surface patches. i.e. a 14th order patch for the main body of the dolphin, a 10th order patch for the tail flukes and 3 Biharmonic patches for the lateral fins and the dorsal fin. In all cases the boundary curves are taken to be function boundary conditions thus enabling the curves to pass through the surface patches. The curves themselves are cubic splines which were created and manipulated within the interactive graphical environment. The curves which form the body attachment for the lateral fins and the dorsal fin respectively, lie on the main body surface where the portions have been trimmed off from the main body surface. These trimming processes were performed via the use of the (u, v) parameter space using the techniques outlined in [19].

5 Conclusions

The discussions of paper this have been concerned with the generalization of the PDE based approach for surface generation. The governing PDE is generalized to arbitrary order where the equations are solved analytically. This solution scheme even in the case of general boundary conditions satisfies the boundary conditions exactly where the resulting surface has an analytic representation.

As one would observe, for all the examples we have discussed in this paper we have used even number of boundary conditions. This is essential if we are to utilize the analytic solution form as presented in the paper. However, this is not necessarily a restriction on the type of geometry the method is capable of generating. In fact if we are presented with odd number of boundary conditions we could easily create a fictitious position boundary condition to make the number even and utilize the analytic solution where, upon solving the even order equation, the undesired potion can be removed.

There are many possible extensions of this initial work. For example, much work has been done in the past using lower order PDEs on shape parameterization of generic designs where such parameterizations are ultimately utilized for automatic design optimizations. A shape parameterization based on higher order PDEs presents new avenues where it would be possible to parameterize complex geometry very efficiently and hence enable the possibility for automatic design optimization of very re-

alistic design scenarios.

It is also noteworthy that the method discussed here has applications in other areas of computer graphics and computer-aided geometric design, e.g., generation of higher order blend surfaces, geometric data interpolation as well smooth surface representations.

6   Acknowledgements

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