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COMPUTATION OF CURVATURES OVER DISCRETE GEOMETRY USING BIHARMONIC SURFACES

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ABSTRACT

The computation of curvature quantities over discrete geometry is often required when processing geometry composed of meshes. Curvature information is often important for the purpose of shape analysis, feature recognition and geometry segmentation. In this paper we present a method for accurate estimation of curvature on discrete geometry especially those composed of meshes. We utilise a method based on fitting a continuous surface arising from the solution of the Biharmonic equation subject to suitable boundary conditions over a 1-ring neighbourhood of the mesh geometry model. This enables us to accurately determine the curvature distribution of the local area. We show how the curvature can be computed efficiently by means of utilising an analytic solution representation of the chosen Biharmonic equation. In order to demonstrate the method we present a series of examples whereby we show how the curvature can be efficiently computed over complex geometry which are represented discretely by means of mesh models.

KEY WORDS
Curvature computation, discrete geometry, Biharmonic surfaces.

1. Introduction

The efficient computation of curvature over geometry is an important step in many of the geometric design related tasks. With the advent of range scanning systems, complex objects defined through discrete geometry, for example in form of 3D meshes, are increasing becoming popular in the area of geometric modelling and computer graphics. Therefore, the need for efficient and accurate computation of curvature on discrete geometry has also become a key task. Some of the geometric applications where curvature of the discrete geometry concerned is required include surface registration, geometry smoothing and simplification. Examples of areas of application include reverse engineering [1,15], segmentation [7,16] and recognition [3,6].

Methods for computation of curvature over discrete geometry (in particular for triangular meshes) can be divided into two categories, namely discrete and continuous. Methods based on discrete formulation utilise a closed form of differential geometry operators which are applied to the discrete representation (e.g. on a 1-ring neighbourhood of the triangular mesh) of the underlying geometry. Methods based on continuous formulation involve fitting a surface locally and computing curvatures using the definition of the local surface. Popular examples of discrete methods include that of Meyer et al. [9] and that of Taubin [13], while examples of continuous methods include the fitting methods suggested by Goldfeather et al. [5] and that proposed by Rusinkiewicz [10]. The method described in this paper fall under the continuous category where we fit a Biharmonic surface over a 1-ring neighbourhood of the mesh geometry in order to accurately estimate the curvatures.

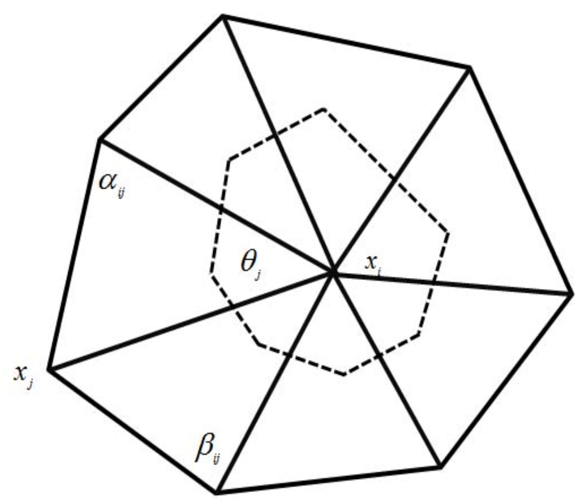

Figure 1. 1-ring neighborhood of vertex used for curvature computation using the method of Meyer et.al.

The most common discrete method widely utilised for computation of curvatures over triangular meshes is that of Meyer et. al. and is formulated through the discretization of the Laplace-Beltrami operator applied to the 1-ring neighbourhood. Given a patch of triangles surrounding a point x_i as shown in Figure 1, the estimates for the Gaussian curvature, K_i and the mean curvature H_i, at x_i is given by,

$$K_i = \frac{2\pi - \sum_j \theta_j}{\frac{1}{3}A}, \tag{1}$$

$$2H_i n_i = \frac{\sum_j (\cot\alpha_{ij} + \cot\beta_{ij})(x_i - x_j)}{\frac{2}{3}A}, \tag{2}$$

where $A = \sum_j f_j$ is the sum of triangle areas, x_i, θ_j, α_{ij} and β_{ij} are as shown in Figure 1.

The rest of the paper is organized as follows. In Section (2) we describe details of the local surface fitting methodology based on the boundary-value problem to the Biharmonic equation. In particular, we describe how the chosen Biharmonic equation is solved explicitly over a 1-ring neighborhood of the discrete surface mesh. In this section we also discuss a test example where we compare our proposed method with the discrete method of Meyer et al. In Section (3) we present some further results and examples whereby we show the curvature computation on complex discrete geometry. Finally in Section (4) we conclude the paper.

2. Method of Biharmonic Surfaces

The method discussed here for computing estimation to the curvatures is based on fitting a continuous surface patch based on the solution to a suitably posed boundary value problem whereby the Biharmonic Equation is solved over a 1-ring-neighbourhood of the mesh geometry. The problem of generating a continuous surface through solving the Biharmonic equation is based on defining the boundary-value problem within a parametric domain subject to a given set of boundary conditions.

Thus, we define a parametric surface patch $\mathbf{X}(u,v)$ as a function of two parameters u and v on a finite domain $\Omega \subset R^2$, by specifying boundary data around the edge region of $\partial\Omega$. The boundary data here are specified in the form of $\mathbf{X}(u,v)$ and its normal $\frac{\partial \mathbf{X}(u,v)}{\partial n}$ at the boundary $\partial\Omega$ of the domain Ω. Hence we assume that the coordinate of a point u and v in the parametric domain is mapped from that point in Ω to a point in the physical space within which the continuous surface patch lies. To satisfy these requirements the surface $\mathbf{X}(u,v)$ is regarded as a solution of the standard Biharmonic equation,

$$\frac{\partial^4 \mathbf{X}}{\partial u^4} + 2\frac{\partial^4 \mathbf{X}}{\partial u^2 \partial v^2} + \frac{\partial^4 \mathbf{X}}{\partial v^4} = 0 \tag{3}$$

Note that the motivation for using the above formulation for surface representation is that the partial differential operator associated with the Biharmonic equation (3) is an elliptic operator which posses smoothing properties. Hence, with this formulation one can see that the Biharmonic operator in Equation (3) represents a smoothing process in which the value of the function at any point on the surface is, in some sense, a weighted average of the surrounding values. In this way a surface is obtained as a smooth transition between the chosen set of boundary conditions. It is noteworthy to point out that similar methods based on the Biharmonic equation or its variations have been utilized for geometric design. e.g. [2,4,12,14,17]

2.1 Explicit Solution of the Biharmonic Equation

Our motivation here is to develop a method that can estimate the discrete geometry with a collection of continuous surface patches which has an analytic representation. Analytic representation of the surface patch has the advantage that the curvature quantities can be directly computed since the derivative information required for computation of the curvatures can be directly computed using the mathematical functions representing the surface patch.

For this purpose we seek an exact solution of Equation (3). To do this we assume the boundary conditions for a continuous patch over a 1-ring neighborhood of the mesh is continuous and periodic.

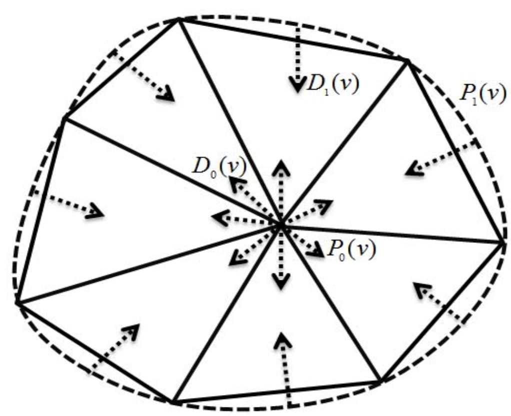

Figure 2. Description of the boundary conditions for the Biharmonic Equation over a 1-ring neighborhood of the mesh geometry.

Figure 2 shows a typical 1-ring neighbourhood of the mesh geometry over which we can define a continuous surface patch by means defining appropriate boundary conditions.

The specific form of the boundary conditions for generating a given surface patch over a 1-ring neighborhood is given as,

$$\mathbf{X}(0,v) = P_0(v)\,, \tag{4}$$
$$\mathbf{X}_u(0,v) = D_0(v)\,, \tag{5}$$
$$\mathbf{X}(1,v) = P_1(v)\,, \tag{6}$$
$$\mathbf{X}_u(0,v) = D_1(v)\,, \tag{7}$$

where $P_0(v)$ and $P_1(v)$ are the position boundary conditions and $D_0(v)$ and $D_1(v)$ are the normal boundary conditions at the edges of the surfaces patch. Referring to Figure 2, $P_0(v)$ corresponds to the common vertex of the 1-ring neighborhood and $P_1(v)$ corresponds to a Fourier curve representing the other vertices of the 1-ring neighborhood. $D_0(v)$ and $D_1(v)$ are the derivative vectors which are represented by arrows as shown in Figure 2.

Thus, assuming our region of interest for the PDE solution to be $0 \le u \le 1$and $0 \le v \le 2\pi$, and also assuming that the conditions required to solve Equation (3), are periodic functions, we can use the method of separation of variables to write down the explicit solution of Equation (3) as,

$$\mathbf{X}(u,v) = \mathbf{X}(u)\cos(nv) + \mathbf{X}(u)\sin(nv), \tag{8}$$

where n is an integer. The form of $\mathbf{X}(u)$ subject to the general boundary conditions given in (4)-(7) is given as,

$$\mathbf{X}(u) = c_1 e^{nu} + c_2 u e^{nu} + c_3 e^{-nu} + c_4 u e^{-nu} \quad , \tag{9}$$

where c_1, c_2, c_3 and c_4 are given by,

$$\begin{aligned} c_1 &= [-P_0\left(2n^2e^{2n} + 2ne^{2n} + e^{2n} - 1\right) \\ &+ P_1\left(ne^{3n} + e^{3n} + ne^n - e^n\right) \\ &- 2P_s ne^{2n} - P_t\left(e^{3n} - e^n\right)] / d, \end{aligned} \tag{10}$$

$$\begin{aligned} c_2 &= [P_0\left(2n^2e^{2n} + ne^{2n} - n\right) \\ &- P_1\left(ne^{3n} + 2n^2e^n - ne^n\right) - P_s\left(2ne^{2n} - e^{2n} + 1\right) \\ &+ P_t\left(e^{3n} - 2ne^n - e^n\right)] / d, \end{aligned} \tag{11}$$

$$\begin{aligned} c_3 &= [P_0\left(e^{4n} - 2n^2e^{2n} + 2ne^{2n} - e^{2n}\right) \\ &- P_1\left(ne^{3n} + e^{3n} + ne^n - e^n\right) + 2P_s ne^{2n} \\ &+ P_t\left(e^{3n} - e^n\right)] / d, \end{aligned} \tag{12}$$

$$\begin{aligned} c_4 &= [P_0\left(ne^{4n} + 2n^2e^{2n} - ne^{2n}\right) \\ &- P_1\left(2n^2e^{3n} + ne^{3n} - ne^n\right) + P_s\left(e^{4n} - 2ne^{2n} - e^{2n}\right) \\ &+ P_t\left(2ne^{3n} - e^{3n} + e^n\right)] / d, \end{aligned} \tag{13}$$

where $d = e^{4n} - 4n^2e^{2n} - 2e^{2n} + 1.$

Given the above explicit solution scheme the unknowns c_1, c_2, c_3 and c_4 can be determined by the imposed conditions at the edges of the surface patch. The boundary conditions imposed (both position and derivative) can be represented by continuous functions, which are also periodic in v. We can then write down the Fourier series representation for each boundary condition such that,

$$P_i(v) = \mathbf{C}_0^1 + \sum_{n=1}^{\infty}\left[\mathbf{C}_n^1 \cos(nv) + \mathbf{S}_n^1(v) Sin(nv)\right] \tag{14}$$

where n is the integer defined in (8) which represents the chosen Fourier mode. Thus, for each Fourier mode representing the boundary condition, an appropriate linear system involving c_1, c_2, c_3 and c_4 can be formulated which can be solved using standard methods such as LU decomposition [11].

2.1 Computation of the surface curvatures

Using the above method we can fit an analytic and continuous surface patch for a given 1-ring neighbourhood of the mesh in question. This surface patch has the analytic form given in Equation (8). Given this, the computation of the relevant curvature quantities is a straightforward procedure involving the computation of the derivatives and normal for the surface. For example, given the unit normal to the surface $\mathbf{X}_n = \dfrac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}$, the mean curvature K and the Gaussian curvature H is given by,

$$K = \frac{Eg - 2fF + eG}{2(EG - F^2)} \quad , \tag{15}$$

and

$$H = \frac{eg - f^2}{EG - F^2} \quad , \tag{16}$$

where, $E = \mathbf{X}_u \bullet \mathbf{X}_u$, $F = \mathbf{X}_u \bullet \mathbf{X}_v$, $G = \mathbf{X}_v \bullet \mathbf{X}_v$, $e = \mathbf{X}_{uu} \bullet \mathbf{X}_n$, $f = \mathbf{X}_{uv} \bullet \mathbf{X}_n$ and $g = \mathbf{X}_{vv} \bullet \mathbf{X}_n$.

2.1 Test Example

Here we take a test example in order to compare the results of the curvature for the Biharmonic surface fitting technique discussed above. For comparison purposes we use the cartenoid surface for which the exact curvature in the form of an analytic function is available. Thus, we compute the curvature distribution over the catenoid using the explicit method. We also compute the curvature distribution using the method of Meyer et. al. These are then compared with the continuous Biharmonic surface fitting method discussed above.

Figure 3 shows the surface of a cartenoid which can be analytically defined as, $x(u,v)=u$, $y(u,v)=\cosh u \cos v$, $z(u,v)=\cosh u \sin v$.

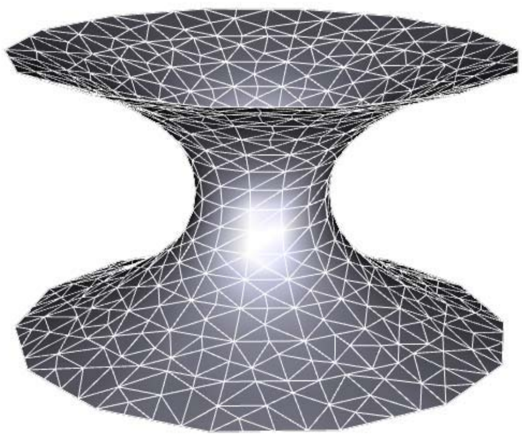

Figure 3. Geometry of the cartenoid shape.

Using Equation (16) the explicit description of the Gaussian curvature $H_{cartenoid}$ is given as,

$$H_{cartenoid} = -\frac{1}{\cosh u^4}. \tag{17}$$

Figure 4 shown the distribution of the curvature using the discrete method of Meyer et. al. Here the solid line indicates the exact curvature distribution while the broken line describes the curvature computed using the discrete method of Meyers et. al.

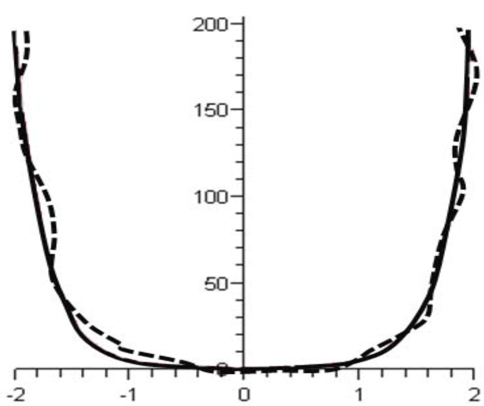

Figure 4. Comparison of the actual Gaussian curvature with that produced by the method of Meyers et al.

Figure 5 shows the variation of the Gaussian curvature for the cartenoid along the u direction (i.e. along a vertical profile for the cartenoid surface shown in Figure 3. Again the solid line indicates the exact curvature distribution while the broken line indicates the discrete curvature using the Biharmonic method.

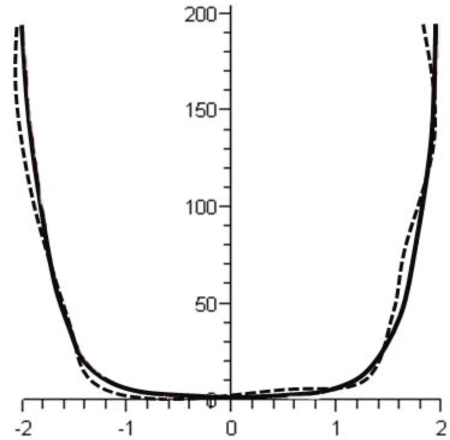

Figure 5. Comparison of the actual Gaussian curvature with that produced by the method by Biharmonic surface fitting.

Comparing the curvature results for the cartenoid shape (between Figures 4 and 5) one can clearly see that there is a more smooth distribution of the curvature when the Biharmonic method is used in comparison to that of Meyer et. al.

3. Results and Examples

In this section we discuss curvature computation on several examples of complex geometric shapes which are discretely represented using triangular meshes.

As a first example we take the geometry of the biological vesicle shown in Figure 6. Figure 7 shows the Gaussian curvature distribution over the vesicle shape.

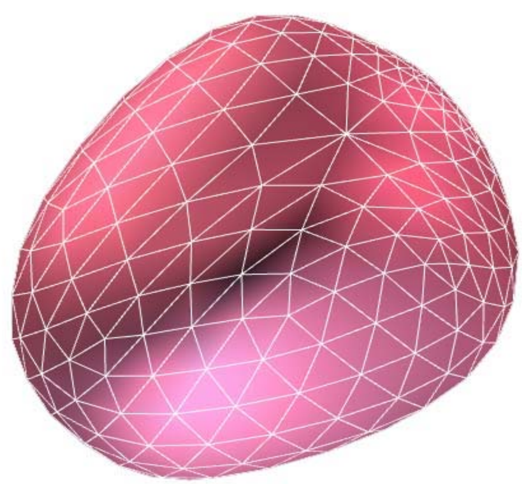

Figure 6. Mesh geometry model of a vesicle shape.

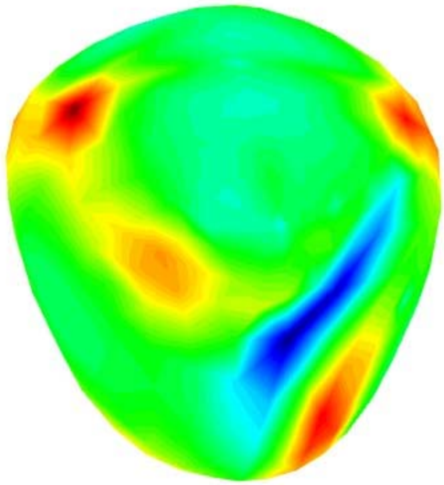

Figure 7. Gaussian curvature distribution on the surface of the vesicle shape.

As a second example in this section we discuss the computation of the curvature on the surface of a human head model shown in Figure 8. Again the head model is defined as a triangular mesh where for each 1-ring neighbourhood a continuous surface is fitted to compute the curvature explicitly. Figure 9 shows the distribution of the mean curvature over the head model.

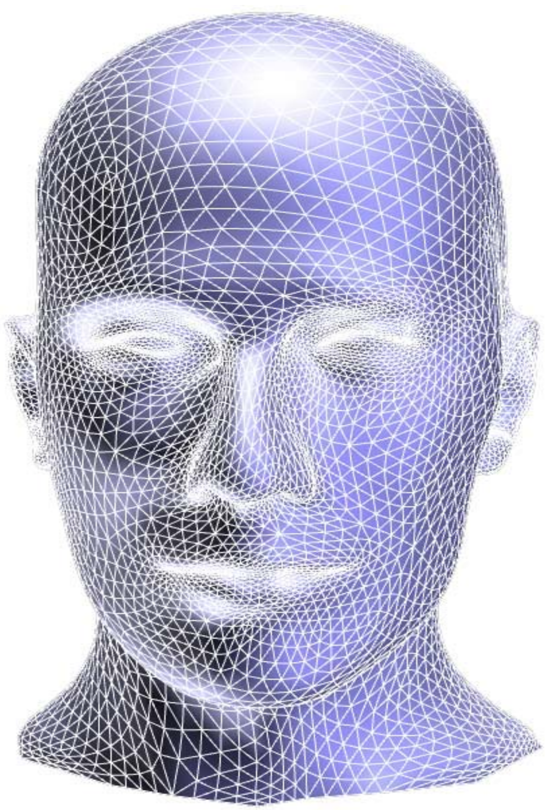

Figure 8. Mesh geometry model of a head shape.

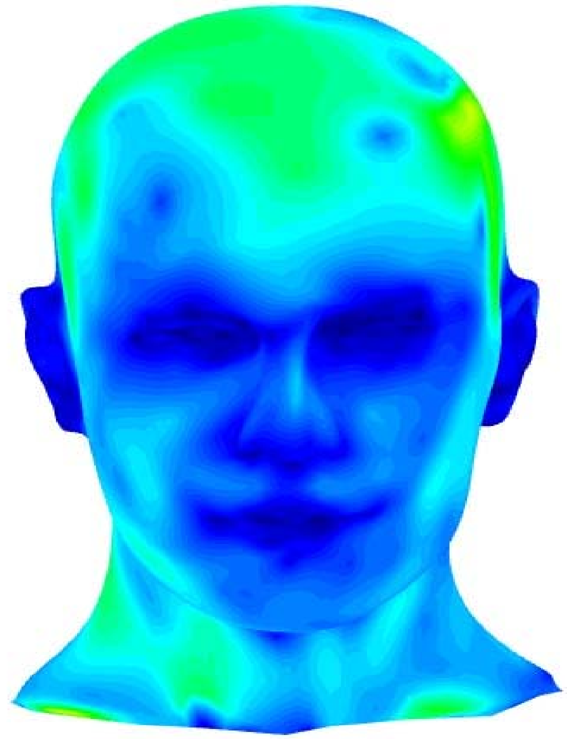

Figure 9. Mean curvature distribution on the surface of the head shape.

As a third and final example we discuss the computation of the curvature on the surface of the terrain geometry shown in Figure 10. As usual the terrain geometry is given as a triangular mesh model on which the discrete curvature has been computed. Figure 11 shows the distribution of the absolute mean curvature over the terrain geometry model.

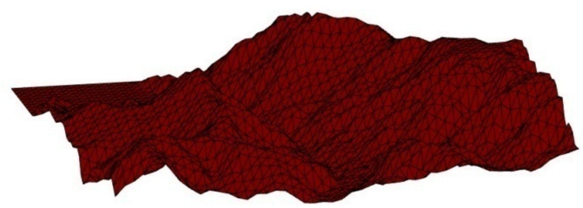

Figure 10. Mesh geometry model of a terrain surface.

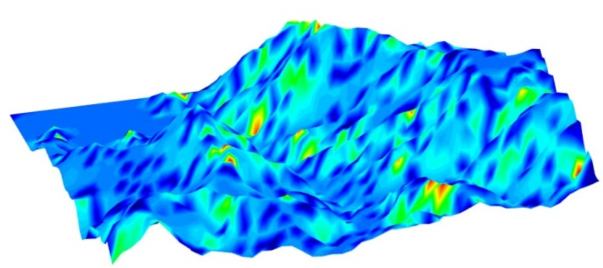

Figure 11. Absolute mean curvature distribution on the terrain surface.

4. Conclusion

In this paper we have described a method of computing curvature quantities on discrete geometry especially those that can be described by a network of mesh. We utilize a boundary value approach whereby the solution of standard Biharmonic equation is solved for appropriately defined boundary conditions over the 1-ring neighborhood of the mesh geometry. The Biharmonic equation over the chosen 1-ring neighborhood is solved explicitly which enables efficient computation of the continuous surface over which curvature quantities can be computed explicitly. The results indicate that the method produces more accurate curvature distribution when compared with the discrete method of Meyer et al. We have demonstrated the capacity of the method proposed here through several examples whereby various curvature quantities are computed on meshes representing complex geometry.

The method proposed here can be improved further. For example one can look into fitting surfaces over an extended area of the geometry such as the 2-ring neighborhood of the mesh. This may require additional treatment of the boundary conditions and may also require looking into solution of higher order elliptic partial differential equations.

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