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QUIVER VARIETIES AND BEILINSON-DRINFELD GRASSMANNIANS OF TYPE A

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Abstract. We construct Nakajima's quiver varieties of type A in terms of conjugacy classes of matrices and (non-Slodowy's) transverse slices naturally arising from affine Grassmannians. In full generality quiver varieties are embedded into Beilinson-Drinfeld Grassmannians of type A. Our construction provides a compactification of Nakajima's quiver varieties and a decomposition of an affine Grassmannian into a disjoint union of quiver varieties. As an application we provide a geometric version of skew and symmetric $(GL(m), GL(n))$ duality.

1. Introduction

In type A we relate Nakajima's quiver varieties, conjugacy classes of matrices, and Beilinson-Drinfeld Grassmannians. In particular, we embed quiver varieties into Beilinson-Drinfeld Grassmannians. From the point of view of Nakajima's quiver varieties our construction provides a compactification of quiver varieties. From the point of view of nilpotent orbits we construct new transverse slices to nilpotent orbits naturally arising from affine Grassmannians. From the point of view of affine Grassmannians we get a decomposition of an affine Grassmannian into a disjoint union of quiver varieties. As an application we provide a geometric version of both the skew and the symmetric version of the $(GL(m), GL(n))$ duality.

The relationship between quiver varieties and nilpotent orbits was conjectured by Nakajima [N1] and proved by Maffei [Maf]. What we do here is close to (and in part motivated by) Maffei's work, however while he uses Slodowy's normal slices to nilpotent orbits we use different slices suggested by the relation to the affine Grassmannians, and this makes the construction explicit while Maffei's approach is based on an existence result.

These observations do not literally extended beyond type A. For instance, the closures of orbits in the affine Grassmannian are normal and this is not true for the nilpotent orbits.

1.1. **The setup.** We work over the field of complex numbers $\mathbb{C}$. By $G_m = \mathbb{C}^*$ we sometimes denote the multiplicative group of this field.

Given two $(n-1)$-tuples of integers $d = (d_1, \ldots, d_{n-1})$ and $v = (v_1, \ldots, v_{n-1})$ and a central element $c = (c_1, \ldots, c_{n-1})$ of the Lie algebra $\prod_{i=1}^{n-1} \mathfrak{gl}(v_i, \mathbb{C})$, Nakajima [N1, N2] constructs quiver varieties $\mathfrak{M}_0(v, d)$ and $\mathfrak{M}(v, d)$.

From the quiver data one can produce $GL(m)$-(co)weights (partitions) λ and μ of N (cf. subsection 5.1.1), where $m = d_1 + \cdots + d_{n-1}$, and $N = \sum_{j=1}^{n-1} jd_j$. We will also consider the affine Grassmannian $\mathcal{G}$ associated to the group $G = GL(m)$, and a "convolution" Grassmannian $\widetilde{\mathcal{G}}$ equipped with a resolution map $\pi : \widetilde{\mathcal{G}} \to \mathcal{G}$.

The following theorem is a common generalization of (some of) the results of Kraft-Procesi [KP], Lusztig [L1], and Nakajima [N1]. For simplicity we will only write down here the statement in the case $c = 0$. In this paper we provide a complete proof of the Theorem below announced in [MVy].

1.2. **Theorem.** There exist algebraic isomorphisms $\phi, \widetilde{\phi}, \psi, \widetilde{\psi}$ such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathfrak{M}(v,d) & \xrightarrow[\simeq]{\widetilde{\phi}} & \mathbf{m}^{-1}(T_\lambda \cap \overline{\mathcal{O}}_\mu) & \xrightarrow[\simeq]{\widetilde{\psi}} & \pi^{-1}(L^{<0}G \cdot \lambda \cap \overline{L^{\geq 0}G \cdot \mu}) \subset \widetilde{\mathcal{G}} \\
{\scriptstyle p}\downarrow & & {\scriptstyle \mathbf{m}}\downarrow & & {\scriptstyle \pi}\downarrow \\
\mathfrak{M}_0(v,d) & \xrightarrow[\simeq]{\phi} & T_\lambda \cap \overline{\mathcal{O}}_\mu & \xrightarrow[\simeq]{\psi} & L^{<0}G \cdot \lambda \cap \overline{L^{\geq 0}G \cdot \mu} \quad \subset \mathcal{G},
\end{array}$$

where T_λ is our new transverse slice to the nilpotent orbit $\mathcal{O}_\lambda \subseteq \mathcal{N}$ of type λ in the nilpotent cone $\mathcal{N}$ of the $\mathfrak{gl}(N, \mathbb{C})$, $\overline{\mathcal{O}}_\mu$ is the closure of the nilpotent orbit of type μ in $\mathcal{N}$, $\mathbf{m} : \widetilde{\mathcal{O}}_\mu \to \overline{\mathcal{O}}_\mu$ is its Springer resolution, and $L^{\geq 0}G$ and $L^{<0}G$ are the subgroups of non-negative and negative loops respectively in the loop group $GL(m, \mathbb{C}((z)))$.

1.3. **The deformation.** For arbitrary c, the nilpotent orbits deform to general conjugacy classes, and the affine Grassmannian deforms to the Beilinson-Drinfeld Grassmannian $\mathfrak{G}_{\mathbb{A}^{(n)}}$ on the n-th symmetric power of the curve $\mathbb{A}^1$, or more precisely its fiber over the point $(0, c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_{n-1}) \in \mathbb{A}^{(n)}$. The general statement is formulated as Theorem 5.3.

1.4. **A transverse slice different from Slodowy's.** Our isomorphisms ϕ and $\widetilde{\phi}$ are similar to those conjectured and constructed in [N1, Maf]. However, in our case T_λ is *not* the Slodowy's transverse slice but rather a *different transverse slice* naturally arising from the affine Grassmannian via the isomorphism ψ. In order to illustrate the difference, let us give an example for $N = 5$ and a nilpotent element x with Jordan blocks of sizes 3 and 2. If we fix the basis in which the matrix of x has the Jordan canonical form, i.e.,

$$x = \left(\begin{array}{ccc|cc}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{array}\right).$$

In the Jordan basis the two transverse slices in questions are described by matrices of the form

$$\text{Slodowy's slice} = \left(\begin{array}{ccc|cc} a_1 & 1 & 0 & 0 & 0 \\ a_2 & a_1 & 1 & b_1 & 0 \\ a_3 & a_2 & a_1 & b_2 & b_1 \\ \hline c_1 & 0 & 0 & d_1 & 1 \\ c_2 & c_1 & 0 & d_2 & d_1 \end{array}\right), \qquad \text{our slice} = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & b_2 & b_1 \\ \hline 0 & 0 & 0 & 0 & 1 \\ c_2 & c_1 & 0 & d_2 & d_1 \end{array}\right).$$

Let $\{x, h, y\}$ be a Jacobson-Morozov $sl(2)$-triple associated with x. Recall that Slodowy's slice is $x + Z_{\mathfrak{gl}(N)}(y)$. Our slice also arises from $\{x, h, y\}$, it can be described as $x + C \subseteq \mathfrak{gl}(N)$, where h acts on C with non-positive integral eigenvalues, C is complementary to $[\mathfrak{gl}(N), x]$ in $\mathfrak{gl}(N)$, and the action of y on C is *"as close to regular nilpotent as possible"*, cf. 3.2.7. In Slodowy's case the slice is $x + C = x + Z_{\mathfrak{gl}(N)}(y)$, so h acts on C with non-positive integral eigenvalues and C is complementary to $[\mathfrak{gl}(N), x]$ in $\mathfrak{gl}(N)$, but by contrast y acts on C by zero.

Our transverse slice is advantageous in the context of this work for three reasons. First, the isomorphism ϕ is given by simple explicit formulas, at least when $c = 0$, cf. 8.1.2 and [MVy, 3.2], as opposed to an inductive procedure used in [Maf]. Second, we are able to decompose an affine Grassmannian into a disjoint union of quiver varieties, cf. 5.4.4. Finally, our construction provides a natural environment for geometric $(GL(m), GL(n))$ duality, cf. Section 9.

Remark. Slodowy's slice was discovered by Kostant, Peterson and Slodowy, cf. [Sl, CG] and references therein.

1.5. The paper is organized as follows. In Section 2 we recall some facts on the quiver varieties of type A. In Section 3 we recall Grothendieck-Springer-Ginzburg theory and discuss transverse slices to nilpotent orbits. In Section 4 we recall some facts on Beilinson-Drinfeld Grassmannians and discuss the appearance of our transverse slice in this setting. Section 5 contains the statement of the Main Theorem and its corollaries. In Section 6 we describe a particular case providing a construction of the conjugacy classes of matrices via quiver varieties. Section 7 contains the proof of the main technical lemma. Section 8 finishes the proof of the Main Theorem. Finally, in Section 9 we discuss applications to representation theory.

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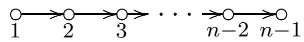
2. Quiver varieties of type A

2.1. **Definitions.**

2.1.1. Let us consider the Dynkin graph of type A_{n-1} with the following orientation Ω:



Let $I = \{1, \ldots, n-1\}$ be the set of vertices and $H = \Omega \sqcup \overline{\Omega}$ be the set of arrows of our quiver. For an arrow $h \in H$ we denote by $h' \in I$ its initial vertex and by $h'' \in I$ its terminal vertex.

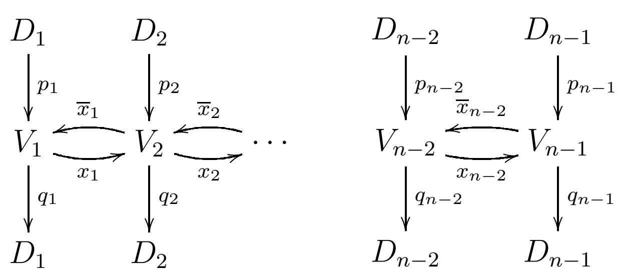
2.1.2. Following Nakajima we attach vector spaces V_i and D_i of dimensions $\dim V_i = v_i$ and $\dim D_i = d_i$, $i \in I$ to the vertices of our quiver i.e. we consider the I-graded vector spaces $V = \oplus_{i\in I} V_i$ and $D = \oplus_{i\in I} D_i$. Let $v = (v_1, \ldots, v_{n-1})$ and $d = (d_1, \ldots, d_{n-1})$ and let $M(v, d)$ be the following affine space:

$$M(v,w) = \bigoplus_{h\in H} \mathrm{Hom}(V_{h'}, V_{h''}) \oplus \bigoplus_{i\in I} \mathrm{Hom}(D_i, V_i) \oplus \bigoplus_{i\in I} \mathrm{Hom}(V_i, D_i).$$

Following Lusztig [L4] we denote an element in $M(v, w)$ as a triple (x, p, q), where

$$\begin{aligned} x &= (x_h)_{h\in H} \in \bigoplus_{h\in H} \mathrm{Hom}(V_{h'}, V_{h''}), \\ p &= (p_i)_{i\in I} \in \bigoplus_{i\in I} \mathrm{Hom}(D_i, V_i), \qquad (1) \\ q &= (q_i)_{i\in I} \in \bigoplus_{i\in I} \mathrm{Hom}(V_i, D_i). \end{aligned}$$

2.1.3. In the A_{n-1} case under consideration it is more convenient to use a different notation. Following Lusztig and Maffei we will consider an element in $M(v, w)$ as a quadruple $(x, \overline{x}, p, q)$. The notation is summarized in the following diagram:



Also denote $p_{j\to i} = \overline{x}_i \ldots \overline{x}_{j-1} p_j$ and $q_{j\to i} = q_i x_{i-1} \ldots x_j$.

The group $G(V) = \prod_{i\in I} GL(V_i)$ acts on $M(v, w)$ in the following way. If $g = (g_i)_{i\in I}$ then

$$g(x, \overline{x}, p, q) = (g_{i+1} x_i g_i^{-1}, g_i \overline{x}_i g_{i+1}^{-1}, g_i p_i, q_i g_i^{-1}). \qquad (2)$$

2.1.4. Let us denote by $\mu : M(v,d) \to \mathfrak{g}(V)$ the moment map associated to this action of $G(V)$. Here $\mathfrak{g}(V)$ is the Lie algebra of $G(V)$. A quadruple $(x, \overline{x}, p, q)$ is in $\mu^{-1}(c)$, $c = (c_1, \dots, c_{n-1})$ if and only if the following relations are satisfied:

$$\begin{aligned} c_1 + \overline{x}_1 x_1 &= p_1 q_1, \\ c_i + \overline{x}_i x_i &= x_{i-1}\overline{x}_{i-1} + p_i q_i \quad 2 \le i \le n-2, \\ c_{n-1} &= x_{n-2}\overline{x}_{n-2} + p_{n-1}q_{n-1}. \end{aligned} \tag{3}$$

We denote the set of all such quadruples by $\Lambda^c(v,d)$.

2.2. **A result on invariant polynomials.** Following [L4] let $\mathcal{R}$ be the algebra of regular functions $M(v,d) \to \mathbb{C}$ and let $\mathcal{R}(\Lambda)$ be the algebra of regular functions $\Lambda^c(v,d) \to \mathbb{C}$. The action of $G(V)$ on $M(v,d)$ (resp. $\Lambda^c(v,d)$) induces an action of $G(V)$ on $\mathcal{R}$ (resp. $\mathcal{R}(\Lambda)$). Following Lusztig [L4, 1.2] we describe two groups of invariant polynomials in $\mathcal{R}^{G(V)}$.

(a) Let $h_1, h_2 \dots, h_r$ be a cycle in our graph, that is a sequence in H such that $h_1'' = h_2', h_2'' = h_3', \dots, h_r'' = h_1'$. This cycle defines a $G(V)$-invariant polynomial in $\mathcal{R}^{G(V)}$ given by $(x,p,q) \mapsto \mathrm{Tr}(x_{h_r} x_{h_{r-1}} \dots x_{h_1}) : V_{h_1'} \to V_{h_1'}$.

(b) Let $h_1, h_2 \dots, h_r$ be a path in our graph, that is a sequence in H such that $h_1'' = h_2', h_2'' = h_3', \dots, h_{r-1}'' = h_r'$. This path together with a linear form χ on $\mathrm{Hom}(D_{h_1'}, D_{h_r''})$ defines a $G(V)$-invariant polynomial in $\mathcal{R}^{G(V)}$ given by $(x,p,q) \mapsto \chi(q_{h_r''} x_{h_r} x_{h_{r-1}} \dots x_{h_1} p_{h_1'})$.

2.2.1. **Theorem**. [L4, Theorem 1.3, 5.8] The algebra $\mathcal{R}(\Lambda)^{G(V)}$ is generated by the invariant polynomials of types (a) and (b) above for $(x,p,q) \in \Lambda^c(v,d)$.

Following [Maf], in the A_{n-1} case we can improve the above theorem as follows. We switch back to Maffei's notation.

Lemma. Let $h_1, h_2 \dots, h_r$ be a cycle in our quiver. Then

$$\mathrm{Tr}(x_{h_r} x_{h_{r-1}} \dots x_{h_1}) = \mathrm{Tr}(P),$$

where P is some polynomial of $q_{l\to j} p_{i \to l}$, $i, j \in \{1, \dots, n-1\}$ (necessarily $l \le \min(i,j)$).

Proof. Easily follows from relations (3). □

Lemma. Let $h_1, h_2 \dots, h_r$ be a path in our graph and let χ be a linear form on $\mathrm{Hom}(D_{h_1'}, D_{h_r''})$. Then

$$\chi(q_{h_r''} x_{h_r} x_{h_{r-1}} \dots x_{h_1} p_{h_1'}), = \chi(P),$$

where P is some polynomial of $q_{l\to j} p_{i \to l}$, $i, j \in \{1, \dots, n-1\}$ (necessarily $l \le \min(i,j)$).

Proof. Easily follows from relations (3). □

Notice that $\mathrm{Tr} : D_i \to D_i$ is a linear form on $\mathrm{Hom}(D_i, D_i)$. Now the Lusztig's theorem 2.2.1 and the lemmas above imply the following.

2.2.2. **Theorem**. The algebra of invariant functions $\mathcal{R}(\Lambda)^{G(V)}$ is generated by the invariant polynomials $\chi(q_{l\to j}p_{i\to l})$, where $i, j \in \{1, \ldots, n-1\}$, $1 \le l \le \min(i,j)$, and χ is a linear form on $\mathrm{Hom}(D_i, D_j)$.

2.2.3. Following Nakajima [N2] and Lusztig [L4, 2.11] we say that a quadruple $(x, \overline{x}, p, q)$ is stable if for any I-graded subspace U of V containing $\mathrm{Im}p$ and preserved by x and $\overline{x}$, we have $U = V$. The set of all stable quadruples in $\Lambda^c(v,d)$ is denoted by $\Lambda^c_s(v,d)$.

The following easy lemma is lifted from Maffei, [Maf, Lemma 14].

Lemma. If $(x, \overline{x}, p, q) \in \Lambda^c(v,d)$ then $(x, \overline{x}, p, q)$ is stable if and only if for all $1 \le i \le n-1$

$$\mathrm{Im}\ x_{i-1} + \sum_{j=i}^{n-1} \mathrm{Im}\ p_{j\to i} = V_i. \tag{4}$$

2.3. **Nakajima's quiver variety** [N2, 3.12]**.** The quiver variety $\mathfrak{M}(v,d)$ is the geometric quotient of $\Lambda^c_s(v,d)$ by $G(V)$. In particular the set of geometric points of $\mathfrak{M}$ is $\Lambda^c_s(v,d)/G(V)$. Below we only consider such (v,d) that $\mathfrak{M}(v,d)$ is nonempty, see [N2, 10], [Maf, Lemma 7] for explicit conditions on (v,d).

We can also consider the affine algebro-geometric quotient of $\Lambda^c(v,d)$ by $G(V)$, which we denote by

$$\mathfrak{M}_0 = \Lambda^c(v,d)//G(V) = \mathrm{Spec}\,\mathcal{R}(\Lambda^c(v,d))^{G(V)}. \tag{5}$$

We have a natural map $p : \mathfrak{M}(v,d) \to \mathfrak{M}_0(v,d)$. Following Maffei we denote

$$\mathrm{Im}\ p = \mathfrak{M}_1(v,d) \subset \mathfrak{M}_0(v,d).$$

Finally, let $\mathfrak{L}(v,d) := p^{-1}(0) \subseteq \mathfrak{M}(v,d)$ and denote by $\mathcal{H}(\mathfrak{L}(v,d))$ its top-dimensional Borel-Moore homology.

2.4. $SL(n)$**-modules.** In this subsection $c = 0$.

Theorem. [N2, *10.ii*] The space $\oplus_v \mathcal{H}(\mathfrak{L}(v,d))$ has the structure of a simple $SL(n)$-module W_d with the highest weight d (i.e., $\sum_I d_i\omega_i$ for the fundamental weights ω_i). The summand $\mathcal{H}(\mathfrak{L}(v,d))$ is the weight space for the weight $d - Cv$, where C is the Cartan matrix of type A_{n-1}.

In particular, the module W_d has a basis arising from the irreducible components of $p^{-1}(0)$, or more precisely the weight space $W_d(d - Cv)$ has a basis indexed by $\mathrm{Irr}\,\mathfrak{L}(v,d)$. Following Lusztig [L5], we call this basis *semicanonical*.

2.4.1. *From $SL(n)$ to $GL(n)$.* We may consider $\oplus_v \mathcal{H}(\mathfrak{L}(v,d))$ as a representation $W_{\check\lambda}$ of $GL(n)$ with highest weight $\check\lambda$, where $\check\lambda = \check\lambda(d) = (\check\lambda_1, \check\lambda_2, \dots, \check\lambda_n)$ is a partition of $N = \sum_{j=1}^{n-1} jd_j$ defined as follows: $\check\lambda_i = \sum_{j=i}^{n} d_j$ (here $d_n = 0$). Then $\mathcal{H}(\mathfrak{L}(v,d))$ is the weight space $W_{\check\lambda}(a)$, where $a_i = v_{n-1} + \sum_{j=i}^{n}(d - Cv)_j$ (here $(d-Cv)_n = 0$), cf. [N1, 8.3].

3. Grothendieck-Springer-Ginzburg theory and conjugacy classes of matrices

In this section we fix a vector space D of dimension N.

3.1. **Definitions of bases.**

3.1.1. Let $\mathcal{N} = \mathcal{N}(D)$ be the nilpotent cone in $\mathrm{End}(D)$. Let $a = (a_1, \dots, a_n)$ be a n-tuple of integers such that $N = \sum_{i=1}^{n} a_i$. We denote the variety if n-step flags in D and its connected components as follows:

$$\begin{aligned} \mathcal{F}^n &= \{0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = D\}, \\ \mathcal{F}^{n,a} &= \{0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = D \mid \dim F_i - \dim F_{i-1} = a_i\}. \end{aligned} \tag{6}$$

It is well known that we have the following description of the cotangent bundle $\widetilde{\mathcal{N}}^n = T^*\mathcal{F}^n$ to this flag variety and its connected components

$$\widetilde{\mathcal{N}}^{n,a} = T^*\mathcal{F}^{n,a} = \{(x,F) \in \mathcal{N} \times \mathcal{F}^{n,a} \mid x(F_i) \subseteq F_{i-1}\}. \tag{7}$$

Denote by $\mathbf{m} : \widetilde{\mathcal{N}}^n \to \mathcal{N}$ the projection onto the first factor, and by $\mathbf{m}_a$ the restriction of $\mathbf{m}$ to $\widetilde{\mathcal{N}}^{n,a}$.

3.1.2. Let $\check\lambda = \check\lambda_1 \geq \cdots \geq \check\lambda_n$, $N = \sum_{i=1}^{n} \check\lambda_i$ be a partition of N and let $\lambda = (\lambda_1, \dots, \lambda_m)$, be the dual partition. Let $x \in \mathcal{N}$ be a nilpotent element of type λ, that is, x has Jordan blocks of sizes $\lambda_1, \dots, \lambda_m$. We will denote the fiber $\mathbf{m}^{-1}(x)$ by $\mathcal{F}^n_x$ and its connected components $\mathbf{m}^{-1}(x) \cap \mathcal{F}^{n,a}$ by $\mathcal{F}^{n,a}_x$.

3.1.3. Let us extend the picture above as follows. Consider the following subbundle of the trivial vector bundle
$gl(D) \times \mathcal{F}^n$ (resp. $\mathfrak{gl}(D) \times \mathcal{F}^{n,a}$):

$$\begin{aligned} \widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}^n &= \{(x,F) \in \mathfrak{gl}(D) \times \mathcal{F}^n \mid x(F_i) \subseteq F_i\}, \\ \widetilde{\mathfrak{g}}^{n,a} &= \{(x,F) \in \mathfrak{gl}(D) \times \mathcal{F}^{n,a} \mid x(F_i) \subseteq F_i\}. \end{aligned} \tag{8}$$

We will denote the projection to the first factor by $\widetilde{\mathbf{m}} : \widetilde{\mathfrak{g}} \to \mathfrak{g} = \mathfrak{gl}(D)$. More notation: $\widetilde{\mathfrak{g}}_x := \widetilde{\mathbf{m}}^{-1}(x)$ and $\widetilde{\mathfrak{g}}^{n,a}_x := \widetilde{\mathbf{m}}^{-1}(x) \cap \widetilde{\mathfrak{g}}^{n,a}$.

3.1.4. Let us fix $x \in \mathrm{End}(D)$ with the spectrum (=set of eigenvalues) $E \subseteq \mathbb{A}^1$ such that $|E| \leq n$. For $e \in E$ let the restriction of $(x - e\,\mathrm{Id}_D)$ to the generalized e-eigenspace of x be a nilpotent of type $\mu(e)$ where $\mu(e) = (\mu_1(e) \geq \mu_2(e) \geq \cdots \geq \mu_{m(e)}(e))$ is a partition and $|\mu(e)| = \sum_{i=1}^{m(e)} \mu_i(e) = l(e)$, so $l(e)$ is the multiplicity of e. For every partition $\mu(e)$ consider its dual $\check{\mu}(e) = (\check{\mu}_1(e) \geq \check{\mu}_2(e) \geq \cdots \geq \check{\mu}_{n(e)}(e))$, so $n(e)$ is the size of the largest Jordan block associated with e. Let $\widetilde{\mu} = \{\mu(e)\}_{e \in E}$ be the collection of partitions for all eigenvalues of x.

The data $E, \widetilde{\mu}$ define the conjugacy class of x (Jordan canonical form). Let us denote this conjugacy class by $\mathcal{O}_{E,\widetilde{\mu}}$.

Let us assume now that $\sum_{e \in E} n(e) = n$. Then the set of pairs

$$M = \{(e, \check{\mu}_i(e)) \mid e \in E, 1 \leq i \leq n(e)\} \subset E \times \mathbb{Z}^n \tag{9}$$

is an n-element subset of $E \times \mathbb{Z}^n$. Let us take an arbitrary bijection $\beta : [1, n] \to M$, where $[1, n]$ is the set of integers from 1 to n. Let $\beta_1 : [1, n] \xrightarrow{\beta} M \to E$ be the composition of β with the projection of M to the first factor, and let $\beta_2 : [1, n] \xrightarrow{\beta} M \hookrightarrow E \times \mathbb{Z}^n \to \mathbb{Z}^n$ be the composition of β with the inclusion of M into $E \times \mathbb{Z}^n$ and the projection to the second factor. Denote $a = (a_1, \ldots, a_n) = (\beta_2(1), \ldots, \beta_2(n))$.

Now we can consider

$$\widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}} = \{(x, F) \in \overline{\mathcal{O}}_{E,\widetilde{\mu}} \times \mathcal{F}^{n,a} \mid x(F_i) \subseteq F_i \text{ and } x \text{ acts on } F_l/F_{l-1} \text{ as } \beta_1(l)\,\mathrm{Id}\}. \tag{10}$$

We will still denote the projection to the first factor by $\widetilde{\mathbf{m}} : \widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}} \to \overline{\mathcal{O}}_{E,\widetilde{\mu}}$. Now we need the following.

Lemma. The variety $\widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}}$ is smooth and connected, the map $\widetilde{\mathbf{m}}$ is projective and

$$\dim \widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}} = \dim \mathcal{O}_{E,\widetilde{\mu}} = N^2 - \sum_{e \in E} \sum_{i \in [1, n(e)]} \check{\mu}_i^2(e).$$

Proof. Actually $\widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}}$ is a vector bundle over $\mathcal{F}^{n,a}$ with the fiber over a particular flag F being $P(F)/L(F)$ where $P(F)$ is the parabolic preserving F and $L(F)$ its Levi factor. Also, if $x \in \mathcal{O}_{E,\widetilde{\mu}}$, then $\widetilde{\mathbf{m}}^{-1}(x) \cap \widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}}$ is a point. In fact, the conjugacy class $\mathcal{O}_{E,\widetilde{\mu}}$ is a deformation of the nilpotent class $\mathcal{O}_\mu$ where $\check{\mu}$ is the partition obtained from the n-tuple $a = (a_1, \ldots, a_n)$ as above by ordering the elements in the non-increasing order. The variety $\widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}}$ is isomorphic to $\widetilde{\mathcal{N}}^{n,a}$. In particular, $\dim \mathcal{O}_{E,\widetilde{\mu}} = \dim \mathcal{O}_\mu$. □

3.1.5. For a finite dimensional algebraic variety X we denote by $H(X)$ its top-dimensional Borel-Moore homology $H^{\mathrm{BM}}_{\dim X}(X)$. In particular, we denote

$$H(\mathcal{F}^n_x) := \bigoplus_a H^{\mathrm{BM}}_{\dim \mathcal{F}^{n,a}_x}(\mathcal{F}^{n,a}_x),$$

$$H(\widetilde{\mathfrak{g}}_x) := \bigoplus_a H^{\mathrm{BM}}_{\dim \widetilde{\mathfrak{g}}^{n,a}_x}(\widetilde{\mathfrak{g}}^{n,a}_x).$$

The following theorem is due to Ginzburg and Braverman-Gaitsgory.

3.1.6. **Theorem**.

(1) [CG, 4.2] Let $\check{x}$ be a nilpotent of type $\check{\lambda}$. The space $H(\mathcal{F}^n_{\check{x}})$ has the structure of a $\mathfrak{gl}(n)$-module W_λ with the highest weight λ.
(2) [BG] Let x be a nilpotent of type λ. The space $H(\widetilde{\mathfrak{g}}_x)$ has the structure of a $\mathfrak{gl}(n)$-module W_λ with the highest weight λ.

In particular, the module W_λ has two bases:

(1) A basis indexed by $\operatorname{Irr}\mathcal{F}^n_{\check{x}}$. More precisely, the weight space $W_\lambda(a)$ has a basis indexed by $\operatorname{Irr}\mathcal{F}^{n,a}_{\check{x}}$. It was shown in [Sav] that this basis coincides with the semicanonical basis defined in 2.4.
(2) A basis indexed by $\operatorname{Irr}\widetilde{\mathfrak{g}}_x$ (relevant irreducible components). More precisely, the weight space $W_\lambda(a)$ has a basis indexed by $\operatorname{Irr}\widetilde{\mathfrak{g}}^{n,a}_x$. We call this the *Spaltenstein basis.*

3.1.7. *Remark.* It was established in [BGV] that the Spaltenstein basis as above coincides with the Mirković-Vilonen basis of [MVi1]. As far as we know the question about the relationship between the semicanonical (as well as Lusztig's canonical [L2, L3]) and Mirković-Vilonen bases remains open.

3.2. **On normal (transverse) slices.** Let $\mathfrak{g} = \mathfrak{gl}(D)$ and $G = GL(D)$.

3.2.1. *Normal slices to nilpotent orbits.* We will say that a normal slice (in $\mathfrak{g}$) to a nilpotent orbit α at $e \in \alpha$, is a submanifold S of $\mathfrak{g}$ such that

(1) (Infinitesimal normality.) $T_e\alpha \oplus T_eS = \mathfrak{g}$, (cf. [CG, 3.2.19]) and
(2) (Contraction.) There is an action of G_m on S which contracts it to e and preserves intersections with the Lusztig strata in $\mathfrak{g}$. (For the definition of Lusztig strata cf. [Mir, 5.5] and references therein.)

We will use the terminology "normal slice" and "transverse slice" interchangeably.

3.2.2. *Lemma.* For a normal slice S

(1) $S \cap \alpha = \{e\}$.
(2) S meets Lusztig stratum β iff $\alpha \subseteq \overline{\beta}$.
(3) S meets Lusztig strata transversally.

3.2.3. *Lemma.* A sufficient data for a normal slice to the orbit ${}^{G}e$ at e is given by a pair (h, C) where $h \in \mathfrak{g}$ is semisimple integral (i.e., eigenvalues of ad h are integral), and $[h, e] = 2e$; while $C \subseteq \mathfrak{g}$ is an h-invariant vector subspace complementary to $T_e(\alpha) = [\mathfrak{g}, e]$, such that the eigenvalues of h in C are ≤ 1. Then $S = e + C$ is a normal slice.

Proof. Such h lifts to a homomorphism $\iota : G_m \to G$ and we can construct an action of G_m on the vector space $\mathfrak{g}$ by $s * x = s^{-2} \cdot {}^{\iota(s)}x, \ s \in G_m, \ x \in \mathfrak{g}$ which fixes e and preserves $e + C$. □

3.2.4. For a nilpotent e let $\{e, h, f\}$ be a Jacobson-Morozov $sl(2)$-triple. We can build normal slices to the nilpotent orbit ${}^{G}e$ at e using h and f.

3.2.5. *Example: Slodowy's slice.* Take h, f from a Jacobson-Morozov $sl(2)$-triple, and let $C = Z_{\mathfrak{g}}(f)$. Clearly, the conditions of Lemma 3.2.3 are satisfied, and $S = e + Z_{\mathfrak{g}}(f)$ is the best-known example of a normal slice.

3.2.6. *Another slice.* We will consider another slice arising from a Jacobson-Morozov $sl(2)$-triple $\{e, h, f\}$. First, let ${}^{h}\mathfrak{g}_{\leq 0} \subseteq \mathfrak{g}$ be the $\{h, f\}$-invariant subspace such that the eigenvalues of h in ${}^{h}\mathfrak{g}_{\leq 0}$ are ≤ 0. Then f will act as a nilpotent in ${}^{h}\mathfrak{g}_{\leq 0}$ and to build a normal slice $S = e + C$ if suffices to choose $C \subseteq {}^{h}\mathfrak{g}_{\leq 0}$ complementary to $T_e(\alpha) = [\mathfrak{g}, e]$.

If we choose $C = \ker_{\mathfrak{g}}(f) \subseteq {}^{h}\mathfrak{g}_{\leq 0}$ we recover the Slodowy's slice.

We would like to consider a $C \subseteq {}^{h}\mathfrak{g}_{\leq 0}$ with the property that f restricted to C is *"as close to regular nilpotent as possible"*, cf. 3.2.7 for more details. In particular, if e is regular, then f restricted to our C will be regular.

More precisely, the vector space D considered as an $sl(2)$-module decomposes as:

$$D = \bigoplus_i M_i \otimes L_i, \qquad \text{and} \qquad \mathrm{End}(D) = D^* \otimes D \simeq \bigoplus_{i,j} \mathrm{Hom}(M_j, M_i) \otimes L_j^* \otimes L_i, \tag{11}$$

where L_i is a simple $sl(2)$-module of highest weight i, $\dim L_i = i + 1$, and M_i is its multiplicity in the decomposition above.

Now consider C to be a subspace

$$C = \bigoplus_{i,j} \mathrm{Hom}(M_j, M_i) \otimes \ker_{L_j^*}(f^{i+1}) \otimes \ker_{L_i}(f) \subseteq \mathrm{End}(D), \tag{12}$$

where $\ker_{L_j^*}(f^{i+1})$ (resp. $\ker_{L_i}(f)$) is the kernel of the natural action of f^{i+1} (resp. f) on L_j^* (resp. L_i.). Notice that $\dim \ker_{L_j^*}(f^{i+1}) = i + 1$, and $\dim \ker_{L_i}(f) = 1$, and also $\dim C = \dim Z_{\mathfrak{g}}(f)$.

It is elementary to see that h, C satisfy the conditions of Lemma 3.2.3, and thus $S = e + C$ is a normal slice.

3.2.7. *"As close to regular nilpotent as possible"*. Let e be acting on D as a nilpotent of type $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$. Then we could say that

$$D = \bigoplus_{i=1}^{m} L_{\lambda_i - 1}, \qquad \text{and} \qquad \mathrm{End}(D) = \bigoplus_{i,j=1}^{m} L^*_{\lambda_j - 1} \otimes L_{\lambda_i - 1}, \tag{13}$$

where L_{λ_i-1} is a simple $sl(2)$-module of highest weight $\lambda_i - 1$, $\dim L_i = \lambda_i$. If $\lambda_i \geq \lambda_j$ we have

$$L^*_{\lambda_j-1} \otimes L_{\lambda_i-1} = L_{\lambda_i+\lambda_j-2} \oplus L_{\lambda_i+\lambda_j-4} \oplus \cdots \oplus L_{\lambda_i-\lambda_j},$$

λ_j summands in all. Let f be the element of a Jacobson-Morozov $sl(2)$-triple. Observe that f restricted to $C \cap (L^*_{\lambda_j-1} \otimes L_{\lambda_i-1})$ acts as a regular nilpotent. It is easy to see that f acts on C defined as above as a nilpotent of type

$$\lambda_f = (\lambda_1 \geq \lambda_2 \geq \lambda_2 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq \cdots \geq \lambda_m), \tag{14}$$

where λ_k repeats with multiplicity $2k-1$, $1 \leq k \leq m$. Then λ_f is a partition of $\sum_{i=1}^{m} (\check{\lambda}_i)^2 = N^2 - \dim \mathcal{O}_\lambda$ and the largest such partition possible for $C \subseteq {}^h\mathfrak{g}_{\leq 0}$ and C being complementary to $T_e(\alpha) = [\mathfrak{g}, e]$. By contrast in the Slodowy's situation f acts on $C = \ker_{\mathfrak{g}}(f)$ as 0 and so its type $(1, \ldots, 1)$ is the smallest possible partition of $\sum_{i=1}^{m} (\check{\lambda}_i)^2$.

3.3. **Our slice in Jordan basis.** We will adjust the notation a bit here: the nilpotent e will be denoted x in this subsection.

3.3.1. Again, let D be a vector space, $\dim D = N$, and $\mathcal{N}$ be the nilpotent cone in $\mathrm{End}(D)$. Let x be a nilpotent operator of type $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$. Moreover, $e_{k,i}$, $1 \leq k \leq \lambda_i$ be a basis in D in which x is exactly the direct sum of nilpotent blocks and x restricted to the span of $\{e_{k,i} \mid 1 \leq k \leq \lambda_i\}$ is the Jordan block of size λ_i, that is $x : e_{k,i} \mapsto e_{k-1,i},\ e_{1,i} \mapsto 0$.

Now define:

$$T_x := \{x + f,\ f \in \mathrm{End}(D) \mid f^{l,j}_{k,i} = 0,\ \text{if } k \neq \lambda_i,\ \text{and } f^{l,j}_{\lambda_i,i} = 0,\ \text{if } l > \lambda_i\}, \tag{15}$$

where $f^{l,j}_{k,i} : \mathbb{C}e_{l,j} \to \mathbb{C}e_{k,i}$ are the matrix elements of f in our basis. For example, if $\lambda = (\lambda_1 \geq \lambda_2) = (3,2)$ the matrices in T_x in the basis $e_{k,i}$, $1 \leq k \leq \lambda_i$ will have the form

$$\left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ f^{1,1}_{3,1} & f^{2,1}_{3,1} & f^{3,1}_{3,1} & f^{1,2}_{3,1} & f^{2,2}_{3,1} \\ \hline 0 & 0 & 0 & 0 & 1 \\ f^{1,1}_{2,2} & f^{2,1}_{2,2} & 0 & f^{1,2}_{2,2} & f^{2,2}_{1,2} \end{array}\right). \tag{16}$$

The set T_x (denoted by $e + C$ above) will sometimes be denoted by T_λ.

3.3.2. For μ such that $\mathcal{O}_\lambda \subseteq \overline{\mathcal{O}}_\mu$ define

$$T_{x,\mu} := T_x \cap \overline{\mathcal{O}}_\mu.$$

We have seen in 3.2.6 that

Lemma. T_x is a transverse slice to the orbit of x. In particular,

$$\dim T_{x,\mu} = \dim \mathcal{O}_\mu - \dim \mathcal{O}_\lambda = \sum_{i=1}^{\lambda_1} (\check{\lambda}_i)^2 - \sum_{i=1}^{\mu_1} (\check{\mu}_i)^2. \tag{17}$$

3.3.3. Let $\lambda \le \mu$ and take any permutation $a = (a_1, \dots, a_n)$ of the dual partition $\check{\mu}$. We will restrict the resolution $\mathbf{m}$ to the slice $T_{x,\mu}$:

$$\widetilde{T}^a_x := \mathbf{m}_a^{-1}(T_{x,\mu}) \subset \widetilde{\mathcal{N}}^{n,a}.$$

Lemma. The variety $\widetilde{T}^a_x$ is smooth and connected of dimension $\sum_{i=1}^{\lambda_1} (\check{\lambda}_i)^2 - \sum_{i=1}^{\mu_1} (\check{\mu}_i)^2$. It is nonempty if and only if $x \in \overline{\mathcal{O}}_\mu$.

The map $\mathbf{m}_a : \widetilde{T}^a_x \to T_x \cap \overline{\mathcal{O}}_\mu$ is projective.

Proof. $\widetilde{T}^a_x$ is smooth because $G{\cdot}T_{x,\mu}$ is open in $\mathfrak{g}$ and near $T_{x,\mu}$ it is a product of $T_{x,\mu}$ and the orbit $G{\cdot}x$. The dimension counts follow from

$$\dim \mathcal{O}_\lambda = N^2 - \sum_{i=1}^{\lambda_1} (\check{\lambda}_i)^2 = N^2 - \sum_{i=1}^{m} (2i-1)\lambda_i.$$

□

3.3.4. We also need to study the intersection $T_x \cap \mathcal{O}_{E,\widetilde{\mu}}$, where $\mathcal{O}_{E,\widetilde{\mu}} \subseteq \operatorname{End}(D)$ is a conjugacy class defined in 3.1.4.

Lemma. We have

$$\dim T_x \cap \mathcal{O}_{E,\widetilde{\mu}} = \sum_{i=1}^{\lambda_1} (\check{\lambda}_i)^2 - \sum_{e \in E} \sum_{i \in [1,n(e)]} \check{\mu}_i^2(e).$$

Moreover, $T_x \cap \mathcal{O}_{E,\widetilde{\mu}}$ is nonempty if and only of $x \in \overline{\mathcal{O}}_\mu$, where μ is obtained from $\widetilde{\mu}$ as in 3.1.4.

Proof. Follows from general smoothness results. □

Lemma. The variety $\widetilde{\mathbf{m}}(T_x \cap \mathcal{O}_{E,\widetilde{\mu}}) \cap \widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}}$ is smooth and connected of dimension equal to $\dim T_x \cap \mathcal{O}_{E,\widetilde{\mu}}$.

Proof. The proof is the same as above, for connectedness cf. [Sp]. □

4. Beilinson-Drinfeld Grassmannians of type A

We recall some standard facts about the affine Grassmannians of type A. In this section $G = GL(m)$ unless indicated otherwise.

4.1. **Local picture.**

4.1.1. Let m be a positive natural number, and V a vector space of dimension m. Let us fix a direct sum decomposition of V

$$V = V_1 \oplus \cdots \oplus V_m, \tag{18}$$

where $\dim V_i = 1$, $1 \le i \le m$. Let us fix nonzero elements $\mathbf{e}_i \in V_i$. The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ is a basis in V.

Let $O := \mathbb{C}[[z]]$ be the ring of formal power series in z and $K := \mathbb{C}((z))$ be its field of fractions. Let $V(K) = V \otimes K$ and let $L_0 = V \otimes O$. A *lattice* L in $V((z))$ is an O-submodule of $V(K)$ such that $L \otimes_O K = V(K)$.

The affine Grassmannian $\mathcal{G}_G$ is a (reduced) ind-scheme whose $\mathbb{C}$-points can be described as all lattices in $V(K)$ or as $G(K)/G(O)$. Its connected components $\mathcal{G}_{(N)}$ are indexed by integers $N \in \mathbb{Z}$. If $N \ge 0$ then $\mathcal{G}_{(N)}$ contains the finite dimensional subscheme

$$\mathcal{G}_N = \{\text{lattices } L \text{ in } V((z)) \text{ such that } L_0 \subseteq L, \dim L/L_0 = N\}. \tag{19}$$

To a dominant coweight $\lambda \in \mathbb{Z}^m$ of G, one attaches the lattice $L_\lambda = \oplus_1^m \mathbb{C}[[z]]{\cdot}z^{-\lambda_i} e_i$.

The $G(O)$-orbits $\mathcal{G}_\lambda$ in $\mathcal{G}_G$ are parameterized by the dominant coweights (partitions) λ via $\mathcal{G}_\lambda = G(O){\cdot}L_\lambda$.

The $G(O)$-orbits in $\mathcal{G}_N$ correspond to partitions $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_m)$ of N into at most m parts. These orbits can be explicitly described as follows:

$$\mathcal{G}_\mu = \{L \in \mathcal{G}_N \mid z \text{ restricted to } L/L_0 \text{ has Jordan blocks of sizes } \mu_1, \mu_2, \ldots, \mu_m\}. \tag{20}$$

4.1.2. Let $G = PGL(m)$. Then the points of $\mathcal{G}_G$ can be thought of as lattices in $V((z))$ only up to a shift by z, or as $PGL(m, K)/PGL(m, O)$. Set theoretically $\mathcal{G}_{PGL(m)}$ is a union of m connected components of $\mathcal{G}_{GL(m)}$.

4.1.3. The orbits of $PGL(m, O)$ on $\mathcal{G}_{PGL(m)}$ are parametrized by the dominant weights of the Langlands dual group ${}^L PGL(m) = SL(m)$. If we consider $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_m)$ defined up to simultaneous shift of by an integer as a dominant weight of $SL(m)$ then the $PGL(m, O)$-orbit $\mathcal{G}_\mu$ is described as follows:

$$\mathcal{G}_\mu = \{L \in \mathcal{G}_N \mid z \text{ restricted to } L/L_0 \text{ has Jordan blocks of sizes } \mu\}. \tag{21}$$

This is well defined since the lattice L is considered up to a shift by z.

4.2. **Global picture.** Let X be a curve, which in our case will always be $\mathbb{A}^1$. Let $\mathbb{A}^{(n)} = \mathbb{A}^1 \times \cdots \times \mathbb{A}^1 // \mathfrak{S}_n$ be the symmetric n-fold product of $\mathbb{A}^1$

Beilinson-Drinfeld Grassmannian [BD, MVi1, MVi2] is a (reduced) ind-scheme $\mathfrak{G}_{\mathbb{A}^{(n)}}$ whose $\mathbb{C}$-points are described as follows:

$$\mathfrak{G}_{\mathbb{A}^{(n)}}(\mathbb{C}) = \{(b, \mathcal{V}, t) \mid t : \mathcal{V}_{X-E} \to (X \times V)|_{X-E} \text{ is an isomorphism } \}, \tag{22}$$

where $b = (b_1, \dots, b_n) \in \mathbb{A}^{(n)}$, $E = \{b_1, \dots, b_n\} \subseteq \mathbb{A}^1$, $\mathcal{V}$ is a vector bundle of rank m, and t is the trivialization of $\mathcal{V}$ off E. The pairs $(\mathcal{V}, t)$ are considered up to an isomorphism. If we fix $b = (b_1, \dots, b_n)$ (and therefore $E = \{b_1, \dots, b_n\}$) then the corresponding ind-subscheme of $\mathfrak{G}_{\mathbb{A}^{(n)}}$ is called the fiber of $\mathfrak{G}_{\mathbb{A}^{(n)}}$ at b and is denoted by $\mathcal{G}_b^{BD}$. If $n = 1$ we will also write $\mathcal{G}_e$ for $e \in \mathbb{A}^1$. It is well known [BD, MVi1] that

$$\mathfrak{G}_b = \prod_{e \in E} \mathcal{G}_e. \tag{23}$$

4.2.1. Let $\mathbb{C}[z]$ be the ring of polynomials in z and $\mathbb{C}(z)$ be its field of fractions i.e. rational functions. Let $V(z) = V \otimes \mathbb{C}(z)$ and let $\mathcal{L}_0 = V \otimes O$. A *lattice* in $V(z)$ is an $\mathbb{C}[z]$-submodule $\mathcal{L}$ of $V(z)$ such that $L \otimes_{\mathbb{C}[z]} \mathbb{C}(z) = V(z)$.

The points of $\mathcal{G}_b$ can be described as lattices $\mathcal{L}$ in $V(z) = V \otimes \mathbb{C}(z)$ such that their localizations $L(e)$ at $e \in \mathbb{A}^1 - E$ are isomorphic to $L_0(e) = V \otimes \mathbb{C}[[z - e]]$. Define:

$$\mathfrak{G}_N = \{ \text{ lattices } \mathcal{L} \supseteq \mathcal{L}_0 \mid \dim \mathcal{L}/\mathcal{L}_0 = N \}.$$

Slightly generalizing the exposition [Ngo, Partie I], we fix a polynomial P of degree n, where $n \le N \le mn$. Define

$$\mathfrak{G}_N(P) = \{ \text{ lattices } \mathcal{L} \supseteq \mathcal{L}_0 \mid \dim \mathcal{L}/\mathcal{L}_0 = N \text{ and } P(z|_{\mathcal{L}/\mathcal{L}_0}) = 0\},$$

where $z|_{\mathcal{L}/\mathcal{L}_0}$ is the linear operator on $\mathcal{L}/\mathcal{L}_0$ obtained by the restriction of z.

Let $P = \prod_{e \in E}(z - e)^{n(e)}$. Then a version of (23) is

$$\mathfrak{G}_N(P) = \bigsqcup_{\substack{l(e) \ge n(e) \\ \sum_{e \in E} l(e) = N}} \prod_{e \in E} (\mathcal{G}_e)_{l(e)}, \tag{24}$$

where the finite dimensional subscheme $(\mathcal{G}_e)_{l(e)}$ of the affine Grassmannian $\mathcal{G}_e$ is defined as in (19).

Finally, if $(b_1, \dots, b_n) \in \mathbb{A}^{(n)}$ and $E = \{b_1, \dots, b_n\} \subset \mathbb{A}^1$, then we set $\mathfrak{G}_{N,b}(P) := \mathfrak{G}_b \cap \mathfrak{G}_N(P)$.

4.2.2. Let $a = (a_1, \dots, a_n)$ such that $\sum_{i=1}^n a_i = N$. Let us introduce a convolution Grassmannian $\widetilde{\mathfrak{G}}_N^{n,a}$ as the (reduced) scheme whose $\mathbb{C}$-points are n-step flags of lattices in $V(z)$:

$$\widetilde{\mathfrak{G}}_N^{n,a} = \{\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \ldots \subseteq \mathcal{L}_n = \mathcal{L} \mid \dim \mathcal{L}_i/\mathcal{L}_{i-1} = a_i \text{ for } 1 \le i \le n\},$$

where $\mathcal{L}_0 = V \otimes \mathbb{C}[z]$. We have a map $\pi_N^{n,a} = \pi : \widetilde{\mathfrak{G}}_N^{n,a} \to \mathfrak{G}_N$ such that $\pi : (\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \ldots \subseteq \mathcal{L}_n) \mapsto \mathcal{L} = \mathcal{L}_n$.

Let $(b_1, \ldots, b_n) \in \mathbb{A}^n$. Let us also introduce a subscheme in the fiber of $\widetilde{\mathfrak{G}}^{n,a}$ over the point $(b_1, \ldots, b_n) \in \mathbb{A}^{(n)}$.

$$\widetilde{\mathfrak{G}}_b^{n,a} = \{(\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \ldots \subseteq \mathcal{L}_n) \in \widetilde{\mathfrak{G}}_N^{n,a} \mid \mathcal{L}_n \in \mathfrak{G}_b, \text{ and } z \text{ acts on } \mathcal{L}_i/\mathcal{L}_{i-1} \text{ as } b_i\},$$

Finally, if $\{b_1, \ldots, b_n\} = E \subset \mathbb{A}^1$, and P is a polynomial as in (24), then we define $\widetilde{\mathfrak{G}}_b^{n,a}(P) = \widetilde{\mathfrak{G}}_b^{n,a} \cap \pi^{-1}(\mathfrak{G}_N(P))$.

4.2.3. Let us also consider the local version of the convolution Grassmannian. Let μ be a partition of N into at most m parts and let $\mathcal{G}_\mu \subseteq \mathcal{G}_N$ be a $G(O)$-orbit in $\mathcal{G}_N$. Let $a = (a_1, \ldots, a_n)$ be a permutation of the dual partition $\check{\mu}$. Consider the (reduced) ind-scheme $\widetilde{\mathcal{G}}_\mu^a = \mathcal{G}_{\omega_{a_1}} * \cdots * \mathcal{G}_{\omega_{a_n}}$ (here ω_k is the k-th fundamental coweight of $GL(m)$) whose $\mathbb{C}$-points are n-step flags of lattices in $V(O)$:

$$\widetilde{\mathcal{G}}_\mu^a = \{L_0 = \ L_0 \subseteq \ L_1 \subseteq \cdots \subseteq L_n = \ L \mid L \in \mathcal{G}_\mu, \dim L_i/L_{i-1} = a_i, z(L_i) \subseteq L_{i-1}\},$$

where $L_0 = V(O)$. It is known that $\pi_\mu^a = \pi : \widetilde{\mathcal{G}}_\mu^a \to \overline{\mathcal{G}}_\mu$ is a resolution of singularities [MVi1].

Consider $L \in \mathcal{G}_\lambda \subseteq \overline{\mathcal{G}}_\mu$. Observe that $(\pi_\mu^a)^{-1}(L) = \mathcal{F}_x^{n,a}$, where $\mathcal{F}_x^{n,a}$ is the Springer-Ginzburg fiber defined in 3.1.2.

4.3. **Perverse sheaves on affine Grassmannians.** In this subsection G denotes $GL(m)$ or $PGL(m)$.

Let $\mathrm{Perv}_{G(O)}(\mathcal{G}_G)$ be the category of $G(O)$-equivariant perverse sheaves on $\mathcal{G}_G$. We will denote by $\mathrm{IC}_\mu = \mathrm{IC}(\overline{\mathcal{G}}_\mu)$ the intersection cohomology complex on the closure of the orbit $\mathcal{G}_\mu$. There is a tensor product (convolution) construction [MVi1, MVi2] which makes the category $\mathrm{Perv}_{G(O)}(\mathcal{G}_G)$ into a tensor category.

Theorem: geometric Satake correspondence [MVi1, 7.1]**.** The semisimple tensor category $\mathrm{Perv}_G(\mathcal{G})$ is equivalent to the category $\mathrm{Rep}\, G^L$ of rational representations of the Langlands dual group G^L. Under this equivalence the sheaf IC_μ corresponds to the highest weight representation V_μ of G^L.

Under the equivalence above the convolution $\mathrm{IC}_{a_1} * \cdots * \mathrm{IC}_{a_n}$ corresponds to the tensor product $V_{a_1} \otimes \cdots \otimes V_{a_n}$. By Gabber's decomposition theorem,

$$\mathrm{IC}_{a_1} * \cdots * \mathrm{IC}_{a_n} = \bigoplus_\lambda L_\lambda \otimes \mathrm{IC}_\lambda \,. \tag{25}$$

On the level of representation theory, we have a decomposition

$$V_{a_1} \otimes \cdots \otimes V_{a_n} = \bigoplus_{\lambda} \mathrm{Mlt}_\lambda \otimes V_\lambda, \tag{26}$$

where the sum is over all partitions $\lambda \leq \mu$, and Mlt_λ are the multiplicity vector spaces.

Taking the hypercohomology in the left and right hand side of the equation (25) and comparing it to the equation (26) we see that

$$\mathrm{Hom}_{GL_m}(V_{a_1} \otimes \cdots \otimes V_{a_n}, V_\lambda) = \mathrm{Mlt}_\lambda = H(\pi^{-1}(L_\lambda)). \tag{27}$$

4.4. **Transverse slices arising from affine Grassmannians.**

4.4.1. Let us recall the setup of 3.3.1: x is a nilpotent operator of type λ in $\mathrm{End}(D)$, $\dim D = N$. Let $b = (b_1, \ldots, b_m)$ be a permutation of λ (notice that $b_i \geq 1$). Consider b as a coweight of $GL(m)$ and consider the lattice L_b generated by the elements $z^{-b_i}\mathbf{e}_i$, $1 \leq i \leq m$. Clearly, $L_b \in \mathcal{G}_\lambda$.

Let $D = L_b/L_0$. Then $\dim D = N$. Define

$$D_j = \mathrm{span}\{e_i \mid b_i = j\}, \qquad \text{and} \qquad d_j = \dim D_j.$$

We have a decomposition of D as follows:

$$D = \bigoplus_{1 \leq k \leq j \leq n-1} z^{-k} D_j. \tag{28}$$

4.4.2. Let us consider the group ind-scheme $G(\mathbb{C}[z^{-1}])$, and let $L^{<0}G(K)$ be subgroup of $G(\mathbb{C}[z^{-1}])$ which is the kernel of the map $G(\mathbb{C}[z^{-1}]) \to G$ defined by $z^{-1} \mapsto 0$. Denote the $L^{<0}G(K)$-orbit of the lattice L_b in $\mathcal{G}_G$ by T_b.

4.4.3. We can choose a complement L_b^- to L_b such that $V(K) = L_b \oplus L_b^-$. We define L_b^- as the subspace of $V(K)$ spanned by $z^{-j}e_i$, $j > b_i$. Denote the projection of $V(K)$ to L_b along L_b^- by π_b.

We can describe an open neighborhood $\mathcal{U}_b^N$ of L_b in $\mathcal{G}_N$ as follows:

$$\mathcal{U}_b^N = \{L \in \mathcal{G}_N \mid \text{the projection } \pi_b : L \to L_b \text{ is an isomorphism }\}.$$

4.4.4. We can describe the set $\mathcal{U}_b^N$ in terms of certain maps, generalizing a construction of [L1].

Any lattice $L \in \mathcal{U}_b^N$ is of the form $(1+f)L_b$ where $f : L_b \to L_b^-$ is a linear map such that $L_0 \subseteq \ker f$. We can decompose f as follows. Let us consider the m-dimensional vector space $V_b = \{z^{-b_i}\mathbf{e}_i \mid 1 \leq i \leq m\}$. Then

$$f = \sum_{k=1}^{\infty} z^{-k} f_k,$$

where $f_k : L_b/L_0 \to V_b$ are linear maps. It is easy to see that since $(1+f)L_b$ is a lattice, we have $f_k = f_1(z+f_1)^{k-1}$ and the operator $z+f_1 : L_b/L_0 \to L_b/L_0$ is nilpotent. Altogether:

$$f = \sum_{k=1}^{\infty} z^{-k} f_1(z+f_1)^{k-1}. \tag{29}$$

Observe that if $L = (1+f)L_b$ then the isomorphism π_b intertwines the action of z on L/L_0 with the action of $z + f_1$ on L_b/L_0.

4.4.5. Now we consider the action on $\mathcal{G}_G$ of the group of "loop rotations" isomorphic to the multiplicative group $\mathbb{C}^*$: $z \mapsto sz, \quad s \in \mathbb{C}^*$, which acts on $V((z)) = V(K)$ by sending $z^k \mathbf{e}_i$ to $(sz)^k \mathbf{e}_i$. Denote by $s \circ L$ the result of this action of $s \in \mathbb{C}^*$ on a lattice $L \in \mathcal{G}_G$.

Consider this action on the lattices in $\mathcal{U}_b^N$ i.e., lattices of the form $L = (1+f)L_b$. Our $\mathbb{C}^*$-action on $V(K)$ restricts to the action on L_b, L_b^-, L_b/L_0 and V_b and we denote $s \circ f = s \cdot f \cdot s^{-1}$ and $s \circ f_1 = s \cdot f \cdot s^{-1}$.

We have:

$$s \circ L = s \circ (1+f)L_b = (1 + s\circ f) s \circ L_b = (1 + s \circ f)L_b$$

since L_b is a T-invariant point in $\mathcal{G}_G$. Now,

$$s \circ f = \sum_{k=1}^{\infty} (sz)^{-k} (s\circ f_1)(sz + (s \circ f_1))^{k-1} = \sum_{k=1}^{\infty} z^{-k} s^{-1} (s\circ f_1)(z + s^{-1}(s\circ f_1))^{k-1},$$

where $s^{-1}(s \circ f_1)$ is the composition of $(s \circ f_1)$ and the operator $s^{-1}\,\mathrm{Id}_{V_b}$ on V_b.

4.4.6. Now the following lemma is clear:

Lemma. $\lim_{s\to\infty} s \circ f = 0$ if and only if $\lim_{s\to\infty} s^{-1}(s\circ f_1) = 0$.

4.4.7. Let us now study $s \circ f_1$. We will consider here f_1 as a map from L_b/L_0 to itself equipped with the basis $\{z^{-k_i}\mathbf{e}_i \mid 1 \le i \le m,\ 1 \le k_i \le b_i\}$. If $u \in L_b/L_0$ is a vector then

$$u = \sum_{k,i} u_{k,i} z^{-k} \mathbf{e}_i.$$

Denote the matrix elements of f_1 is this basis by $f_{k,i}^{l,j}$ where $f_{k,i}^{l,j} : \mathbb{C}z^{-l}\mathbf{e}_j \to \mathbb{C}z^{-k}\mathbf{e}_i$. Now recall that by construction we have

$$f_{k,i}^{l,j} = 0, \quad \text{if } k \neq \lambda_i.$$

Then

$$\begin{aligned} f(u)_{k,i} &= 0, \ \text{if } k \neq \lambda_i, \\ f(u)_{\lambda_i,i} &= \sum_{l,j} f_{k,i}^{l,j} u_{l,j}. \end{aligned} \tag{30}$$

Now for $s \circ f_1 = s \cdot f_1 \cdot s^{-1}$ we have:

$$(s \circ f_1)(u)_{k,i} = 0, \quad \text{if } k \neq c_i,$$
$$(s \circ f_1)(u)_{\lambda_i,i} = \sum_{l,j} s^{-\lambda_i} f^{l,j}_{\lambda_i,i} u_{l,j} s^l = \sum_{l,j} s^{l-\lambda_i} f^{l,j}_{\lambda_i,i} u_{l,j}. \tag{31}$$

4.4.8. *Lemma.* The following are equivalent:

(1) $\lim_{s\to\infty} s^{-1}(s \circ f_1) = 0$.
(2) $f^{l,j}_{\lambda_i,i} = 0$ if $l > \lambda_i$.

Proof. Follows immediately from (31). □

4.4.9. *Lemma.* A lattice $L \in \mathcal{G}_G$ is in the $L^{<0}G(K)$-orbit of L_b if and only if $\lim_{s\to\infty} s \circ L = L_b$.

Proof. We have the following decomposition [F, Corollary 2.2]:

$$G(K) = G([z^{-1}])X_*(T)G(O).$$

Then

$$\mathcal{G} = G(K)/G(O) = \bigcup_{\lambda \in X_*(T)} G([z^{-1}])(\lambda \cdot G(O)).$$

Now $G([z^{-1}]) = L^{<0}G(K)G$ is a semidirect product. So,

$$\mathcal{G} = \bigcup_{\lambda \in X_*(T)} L^{<0}G(K)G(\lambda \cdot G(O)).$$

The orbits of $L^{<0}G(K)$ intersect the orbits of $G(O)$ transversally, [F, Section 2. Remark]. This means in particular that if $p \in G \cdot \lambda$, then $(L^{<0}G(K) \cdot p) \cap G \cdot \lambda = p$. Then we have

$$\mathcal{G} = \bigsqcup_{\substack{\lambda \in X^+_*(T) \\ p \in G\cdot\lambda}} L^{<0}G(K) \cdot p, \tag{32}$$

where $X^+_*(T)$ is the set of dominant coweights of G.

Since for $g \in L^{<0}G(K)$ we have $\lim_{s\to\infty}(s \circ g) = 1$, it is clear that

$$L^{<0}G(K) \cdot p \subseteq \{L \in \mathcal{G} \mid \lim_{s\to\infty} s \circ L = p\}.$$

Since we have the disjoint decomposition (32), we actually have

$$L^{<0}G(K) \cdot p = \{L \in \mathcal{G} \mid \lim_{s\to\infty} s \circ L = p\}.$$

□

4.4.10. *Lemma.* If $T_b := L^{<0}G(K) \cdot L_b$ then $T_b \cap \mathcal{G}_N \subseteq \mathcal{U}^N_b$.

Proof. Clear. □

4.4.11. Again, recall the setup of 3.3.1 and the definition of the variety T_x. Let $x + f_1 \in T_x$. Construct a map

$$\begin{aligned} &\psi : T_x \cap \mathcal{N} \to \mathcal{U}_b^N, \\ &\psi : x + f_1 \mapsto (1 + \sum_{k=1}^{\infty} z^{-k} f_1 (z + f_1)^{k-1}) L_b. \end{aligned} \tag{33}$$

4.4.12. *Lemma.* The image of ψ defined above is contained in $T_b \cap \mathcal{G}_N$. Moreover, the map $\psi : T_x \cap \mathcal{N} \xrightarrow{\simeq} T_b \cap \mathcal{G}_N$ is an isomorphism of algebraic varieties.

Proof. By definition of T_x, Lemma 4.4.8, and Lemma 4.4.9

$$\psi(T_x \cap \mathcal{N}) = \{L \in \mathcal{U}_b^N \mid \lim_{s \to \infty} s \circ L = L_b\} = T_b \cap \mathcal{U}_b^N.$$

Since by Lemma 4.4.10 $T_b \cap \mathcal{G}_N \subseteq \mathcal{U}_b^N$, and $\mathcal{U}_b^N \subseteq \mathcal{G}_N$ we have $T_b \cap \mathcal{U}_b^N = T_b \cap \mathcal{G}_N$. □

4.4.13. Recall the setup of 3.3.3. Also, let $\pi = \pi_\mu^a : \widetilde{\mathcal{G}}_\mu^a \to \overline{\mathcal{G}}_\mu$ is a resolution of singularities, cf. 4.2.3.

We can lift the map ψ to the map

$$\widetilde{\psi} : \widetilde{T}_x^a \to \pi^{-1}(T_b \cap \overline{\mathcal{G}}_\mu) \subseteq \widetilde{\mathcal{G}}_\mu^a$$

since a $(x + f_1)$-invariant n-step flag in D will give rise to a n-step flag of lattices in $L = \psi(x + f_1)$.

Lemma. The map $\widetilde{\psi}$ is an isomorphism of algebraic varieties. Moreover, the following diagram of morphisms

$$\begin{array}{ccc} \widetilde{T}_x^a & \xrightarrow{\widetilde{\psi}} & \pi^{-1}(T_b \cap \overline{\mathcal{G}}_\mu) \\ {\scriptstyle \mathbf{m}_a}\downarrow & & \downarrow{\scriptstyle \pi} \\ T_{x,a} & \xrightarrow{\psi} & T_b \cap \overline{\mathcal{G}}_\mu \end{array} \tag{34}$$

commutes.

4.5. **Global version of the map ψ.**

4.5.1. Recall the setup of 4.4.1. Let us consider the scheme $\mathfrak{G}_N$ and let $\mathcal{L}_b$ be the lattice in $V(z)$ generated by the elements $z^{-b_i}\mathbf{e}_i$, $1 \le i \le m$.

Just as in the local case consider m-dimensional vector subspace $V_b = \{z^{-b_i}\mathbf{e}_i \mid 1 \le i \le m\}$. of $V(z)$, and consider a linear map $f_1 : \mathcal{L}_b/\mathcal{L}_0 \to V_b$. (Notice that $D = \mathcal{L}_b/\mathcal{L}_0 \simeq L_b/L_0$ where L_b and L_0 are the analogous local lattices.)

4.5.2. *Lemma.* For any $u \in \mathcal{L}_b/\mathcal{L}_0$ and any $e \in \mathbb{A}^1$, we have

$$(1+\sum_{k=1}^{\infty} z^{-k} f_1(z+f_1)^{k-1})(u) = (1+\sum_{k=1}^{\infty}(z-e)^{-k} f_1(z-e+f_1)^{k-1})(u).$$

Proof. Binomial formula. □

4.5.3. Now consider $z+f_1$ as an operator on $D=\mathcal{L}_b/\mathcal{L}_0$, let E be its spectrum and let $\mathrm{pr}_e : D \to D_e$, for $e \in E$, be the projection to the generalized e-eigenspace.

Once again, recall the setup of 3.3.1 and the definition of the variety T_x. For $x+f_1 \in T_x \subseteq \mathrm{End}(D)$ define the subspace $\psi(x+f_1)$ in $V(z)$ as follows

$$\psi(x+f_1) = (\sum_{e\in E}(1+\sum_{k=1}^{\infty}(z-e)^{-k} f_1(z-e+f_1)^{k-1})\,\mathrm{pr}_e)L_b. \tag{35}$$

Lemma. The subspace $\psi(x+f_1)$ is a lattice in $V(z)$ and therefore an element in $\mathfrak{G}_N$.

Proof. The same as in the local case. □

Summarizing, we have constructed an embedding

$$\psi : T_x \hookrightarrow \mathfrak{G}_N$$

As in the local case, this embedding lifts to an embedding $\widetilde{\psi} : \widetilde{\mathbf{m}}^{-1}(T_x) \cap \widetilde{\mathfrak{g}}^{n,a} \hookrightarrow \widetilde{\mathfrak{G}}_N^{n,a}$ in such a way that the diagram

$$\begin{array}{ccc} \widetilde{\mathbf{m}}^{-1}(T_x)\cap\widetilde{\mathfrak{g}}^{n,a} & \xrightarrow[\subset]{\widetilde{\psi}} & \widetilde{\mathfrak{G}}_N^{n,a} \\ {\scriptstyle \mathbf{m}_a}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ T_x & \xrightarrow[\subset]{\psi} & \mathfrak{G}_N \end{array} \tag{36}$$

commutes.

5. Main Results

5.1. **Combinatorial data.**

5.1.1. *From quiver data to $GL(n)$-data.* Let $d = (d_1, \dots, d_{n-1})$ and $v = (v_1, \dots, v_{n-1})$ be two $(n-1)$-tuples of non-negative integers. We will transform this "quiver data" into some $GL(n)$ weights.

(1) Let C be the Cartan matrix of type A_{n-1}. By $(d - Cv)_j$ we will denote the j-th component of the $(n-1)$-tuple $d - Cv$.
(2) Let $N = \sum_{j=1}^{n-1} j d_j$ and let $m = \sum_{j=1}^{n-1} d_j$.
(3) Let $\check{\lambda} = (\check{\lambda}_1, \check{\lambda}_2, \dots, \check{\lambda}_n)$ be a partition of N defined as follows (here $d_n = 0$):

$$\check{\lambda}_i = \sum_{j=i}^{n} d_j.$$

(4) Let λ be the dual partition.
(5) Let $a = (a_1, \dots, a_n)$ be defined as follows, cf. [N1, 8.3], (here $(d - Cv)_n = 0$):

$$a_i = v_{n-1} + \sum_{j=i}^{n} (d - Cv)_j. \tag{37}$$

(6) Let $\check{\mu}$ be the partition obtained from a by permutation and let μ be the dual partition.

We can view $\check{\lambda}$ as a highest weight of $GL(n)$ and a as a weight in the highest weight $GL(n)$-module $W_{\check{\lambda}}$, cf. 2.4.1.

5.1.2. *From $GL(n)$-data to a conjugacy class.* Let $c = (c_1, \dots, c_{n-1})$ be in the center of $\mathfrak{g}(V)$, where $\mathfrak{g}(V) = \prod_{i=1}^{n-1} \mathfrak{gl}(V_i)$ and $\dim V_i = v_i$.

First of all, denote

$$b_1 = 0 \quad \text{and} \quad b_i = c_1 + \dots + c_{i-1} \quad \text{for} \quad 2 \le i \le n. \tag{38}$$

Let $b = (b_1, \dots, b_n) \subset \mathbb{A}^n$. Let P be the polynomial $P(t) = \prod_{i=1}^{n} (t - b_i)$.

Consider $E = E(c) = \{b_1, \dots, b_n\}$ as a subset of $\mathbb{A}^1$ and consider b as a map $[1, n] \to E$ defined by $b(i) = b_i$.

For every $e \in E$ denote $I(e) := b^{-1}(e) = \{i \in [1, n] \mid b_i = e\}$. Now take a as in (37) and let $a(e) = (a_i)_{i \in I(e)}$ and let $\check{\mu}(e)$ be the partition obtained from $a(e)$ by permutation. Let $\mu(e)$ be the dual partition, and let $\widetilde{\mu} = \{\mu(e)\}_{e \in E}$ be the collection of all partitions attached to eigenvalues. Let $\mathcal{O}_{E, \widetilde{\mu}}$ be the conjugacy class in $\mathrm{End}(D)$, $\dim D = N$ attached to the data $E, \widetilde{\mu}$ as in 3.1.4.

5.2. Now we can formulate our main theorem. For notation on quiver varieties see 2.3, on Springer-Ginzburg resolutions see 3.1, on transverse slices see 3.3, and finally on Beilinson-Drinfeld Grassmannians see 4.2.

5.3. **Theorem.** Let $N, m, v, d, a, c, b, E, \lambda, \widetilde{\mu}$ be as above. There exist algebraic isomorphisms $\phi, \widetilde{\phi}$ and algebraic immersions $\psi, \widetilde{\psi}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathfrak{M}(v,d) & \xrightarrow[\simeq]{\widetilde{\phi}} & \widetilde{\mathbf{m}}^{-1}(T_\lambda) \cap \widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}} & \xrightarrow[\subset]{\widetilde{\psi}} & \widetilde{\mathfrak{G}}_b^{n,a}(P) \\ {\scriptstyle p}\downarrow & & {\scriptstyle \widetilde{\mathbf{m}}}\downarrow & & {\scriptstyle \pi}\downarrow \\ \mathfrak{M}_1(v,d) & \xrightarrow[\simeq]{\phi} & T_\lambda \cap \overline{\mathcal{O}}_{E,\widetilde{\mu}} & \xrightarrow[\subset]{\psi} & \mathfrak{G}_{N,b}(P). \end{array} \tag{39}$$

5.4. **Remarks and Corollaries.**

5.4.1. *Remark.* When $c = 0$ we can describe the images of the maps ψ and $\widetilde{\psi}$ and obtain a more precise result stated in the introduction and [MVy]. In particular, $(\psi \circ \phi)(0) = L_\lambda \in \mathcal{G}_0$, and $\widetilde{\psi} \circ \widetilde{\phi}$ restricts to an isomorphism

$$\widetilde{\psi} \circ \widetilde{\phi} : \mathfrak{L}(v,d) \simeq \pi^{-1}(L_\lambda). \tag{40}$$

We believe that one should be able to generalize these statements for arbitrary c.

5.4.2. *Dimensions.* Let $c = 0$. First of all we'll check that the varieties $\mathfrak{M}(v,d)$ and $\widetilde{T}_x^a$ have the same dimension. According to Nakajima [N2, Corollary 3.12] $\mathfrak{M}(v,d)$, if nonempty, is a smooth variety of dimension ${}^t v(2d - Cv)$ where C is the Cartan matrix of type A_{n-1}. If $\check{\lambda}$ and $\check{\mu}$ are defined by v, d as in 5.1.1, then we have

$$\begin{aligned} \dim \mathfrak{M}(v,d) &= {}^t v(2d - Cv) = 2\sum_{i=1}^{n-1} v_i d_i - 2\sum_{i=1}^{n-1} v_i^2 + 2\sum_{i=1}^{n-2} v_i v_{i+1} \\ &= \sum_{i=1}^{n-1} [(\check{\lambda}_i)^2 - (\check{\mu}_i)^2] = \dim \widetilde{T}_{x,\mu}. \end{aligned} \tag{41}$$

We will list here two applications of our Main Theorem.

5.4.3. **A compactification of quiver varieties.** The closure in $\mathfrak{G}_{N,b}(P)$ of the image of $\mathfrak{M}_1(v,d)$ under the map $\psi \circ \phi$ gives us a compactification of $\mathfrak{M}_1(v,d)$. Analogously, the closure in $\widetilde{\mathfrak{G}}_b^{n,a}(P)$ of the image of $\mathfrak{M}(v,d)$ under the map $\widetilde{\psi} \circ \widetilde{\phi}$ gives us a compactification of the quiver variety $\mathfrak{M}(v,d)$.

5.4.4. **A decomposition of the affine Grassmannian.** The following is a corollary of the main theorem. Here $c = 0$.

Corollary. We can decompose $\overline{\mathcal{G}}_\mu$ into the following disjoint union:

(42)
$$\overline{\mathcal{G}}_\mu = \bigsqcup_{\substack{y \in G\cdot\lambda \\ \lambda \leq \mu}} \mathfrak{M}_0(v,d)_y,$$

where λ varies over the set of dominant coweights of G, $G \cdot \lambda$ is the G-orbit of λ in $\mathcal{G}_G$, and $\mathfrak{M}_0(v,d)_y$ is a copy of quiver variety $\mathfrak{M}_0(v,d)$ for every point $y \in G\cdot\lambda$, with v, d obtained from λ, μ by reversing the procedures of 5.1.1.

Proof. As in the proof of 4.4.9, we have:

(43)
$$\mathcal{G}_G = \bigsqcup_{\substack{\lambda \in X_*^+(T) \\ y \in G\cdot\lambda}} L^{<0}G(K)\cdot y.$$

Then:

(44)
$$\overline{\mathcal{G}}_\mu = \bigsqcup_{\substack{\lambda \in X_*^+(T) \\ y \in G\cdot\lambda}} (L^{<0}G(K)\cdot y) \cap \overline{\mathcal{G}}_\mu = \bigsqcup_{\substack{\lambda \in X_*^+(T) \\ y \in G\cdot\lambda}} \mathfrak{M}_0(v,d)_y$$

since every $(L^{<0}G(K)\cdot y) \cap \overline{\mathcal{G}}_\mu$, for $y \in G\cdot\lambda$ is isomorphic to a copy of $\mathfrak{M}_0(v,d)$. □

5.4.5. *Remarks.*

(1) An "affine analogue" of our construction has recently appeared in the paper [BF].
(2) We would also like to mention another example of a decomposition of an infinite Grassmannian into a disjoint union of quiver varieties. Generalizing a result of G. Wilson [W], V. Baranovsky, V. Ginzburg, and A. Kuznetsov [BGK] constructed a decomposition of (a part of) adelic Grassmannian into a disjoint union of *deformed* versions of quiver varieties $\mathfrak{M}(v,d)$ associated to affine quivers of type A.

6. On quiver varieties and conjugacy classes of matrices

6.1. **Definitions.** Let us consider a particular case of the Main Theorem. Let $d = (N, 0, \ldots, 0)$ and $v = (v_1, \ldots, v_{n-1})$ be the $(n-1)$-tuple of non-negative integers such that $N \geq v_1 \geq v_2 \geq \cdots \geq v_{n-1}$.

6.1.1. Define the algebraic morphisms $\widetilde{\phi} : \mathfrak{M}(v,d) \rightarrow \widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}}$ and $\phi : \mathfrak{M}_1(v,d) \rightarrow \overline{\mathcal{O}}_{E,\widetilde{\mu}}$ as follows:

(45)
$$\begin{aligned} &\widetilde{\phi} : (x, \overline{x}, p, q) \mapsto (q_1 p_1, \{0\} \subseteq \ker p_1 \subseteq \ker x_1 p_1 \subseteq \ker x_{n-1} \ldots x_1 p_1), \\ &\phi : (x, \overline{x}, p, q) \mapsto q_1 p_1. \end{aligned}$$

The following theorem is a common generalization of (some of) the results of [KP] and [N1], cf. [CB].

6.2. **Theorem.** The maps $\phi, \widetilde{\phi}$ defined above are isomorphisms of algebraic varieties and the following diagram commutes

$$\begin{array}{ccc} \mathfrak{M}(v,d) & \xrightarrow{\widetilde{\phi}} & \widetilde{\mathfrak{g}}^{n,a,E,\widetilde{\mu}} \\ p\big\downarrow & & \widetilde{\mathbf{m}}\big\downarrow \\ \mathfrak{M}_1(v,d) & \xrightarrow{\phi} & \overline{\mathcal{O}}_{E,\widetilde{\mu}} \end{array} \tag{46}$$

Proof. Following the logic of [N2, Maf], it is not hard to check that $\widetilde{\phi}$ is a bijective morphism between two smooth varieties of the same dimension and thus an isomorphism. The map ϕ is a closed immersion and it is surjective since both p and $\widetilde{\mathbf{m}}$ are surjective. □

6.2.1. In particular, if all the numbers $0, c_1, c_1 + c_2, \ldots, c_1 + c_2 + \cdots + c_{n-1}$ are pairwise distinct, then the quiver variety $\mathfrak{M}(v,d)$ is isomorphic to the conjugacy class of a semisimple element (diagonal matrix)

$$\mathrm{diag}(b_1, \ldots, b_1, b_2, \ldots, b_2, \ldots, b_n, \ldots, b_n),$$

where $b_1 = 0$ appears with multiplicity a_1, $b_2 = c_1$ appears with multiplicity a_2, and so on, and $b_n = c_1 + c_2 + \ldots, c_{n-1}$ appears with multiplicity a_n.

6.2.2. *Remark.* In fact one can also prove that the quiver variety $\mathfrak{M}_0(v,d)$ is isomorphic to a conjugacy class which is generally different from the conjugacy class considered above. The two classes coincide when the $SL(n)$ weight $d - Cv$ is dominant, i.e. when $a_1 \geq a_2 \geq \cdots \geq a_n$.

7. Proof of the Main Lemma

7.1. D'après Maffei.

7.1.1. We borrow Maffei's [Maf] notations and conventions. Let $v = (v_1, \ldots, v_{n-1})$ and $d = (d_1, \ldots, d_{n-1})$ be two $(n-1)$-tuples of integers and let us define $(n-1)$-tuples $\widetilde{v}$ and $\widetilde{d}$ as follows:

$$\begin{aligned} \widetilde{d}_1 &:= \sum_{j=1}^{n-1} j d_j, \\ \widetilde{d}_i &:= 0, \ \text{for } i > 1, \\ \widetilde{v}_i &:= v_i + \sum_{j=i+1}^{n-1} (j-i) d_j. \end{aligned} \tag{47}$$

Our goal is to construct a map from $\Lambda^c(v,d)$ to $\Lambda^c(\widetilde{v},\widetilde{d})$, that is we have to send a quadruple $(x,\overline{x},p,q)\in\Lambda^c(v,d)$ to a quadruple $(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta})\in\Lambda^c(\widetilde{v},\widetilde{d})$. First of all, the I-graded vector spaces $\widetilde{V}_i$ and $\widetilde{D}_i$ such that $\dim\widetilde{V}_i=\widetilde{v}_i$ and $\widetilde{D}_i=\widetilde{d}_i$ are constructed as follows. Let $D_j^{(k)}$ be a copy of D_j.

$$\begin{aligned}\widetilde{D}_1&=\bigoplus_{1\le k\le j\le n-1}D_j^{(k)},\\ \widetilde{D}_i&=0,\ \text{for } i>1,\\ \widetilde{V}_i&=V_i\oplus\bigoplus_{1\le k\le j-i\le n-i-1}D_j^{(k)}.\end{aligned}\tag{48}$$

We need the following subspaces of $\widetilde{V}_i$.

$$D'_i=\bigoplus_{\substack{i+1\le j\le n-1\\ 1\le k\le j-i}}D_j^{(k)},\qquad D_i^+=\bigoplus_{\substack{i+2\le j\le n-1\\ 2\le k\le j-i}}D_j^{(k)},\qquad D_i^-=\bigoplus_{\substack{i+2\le j\le n-1\\ 1\le k\le j-i-1}}D_j^{(k)}.\tag{49}$$

In order to make the notation more homogeneous we set $\widetilde{V}_0:=\widetilde{D}_1$, $\widetilde{A}_0=\widetilde{\gamma}_1$, $\widetilde{B}_0=\widetilde{\delta}_1$. We will name the blocks of the maps $\widetilde{A}_i$ and $\widetilde{B}_i$ as follows

$$\begin{aligned}\pi_{D_j^{(h)}}\widetilde{A}_i|_{D_{j'}^{(h')}}&={}^it_{j,h}^{j',h'} & \pi_{D_j^{(h)}}\widetilde{B}_i|_{D_{j'}^{(h')}}&={}^is_{j,h}^{j',h'}\\ \pi_{D_j^{(h)}}\widetilde{A}_i|_{V_i}&={}^it_{j,h}^{V} & \pi_{D_j^{(h)}}\widetilde{B}_i|_{V_{i+1}}&={}^is_{j,h}^{V}\\ \pi_{V_{i+1}}\widetilde{A}_i|_{D_{j'}^{(h')}}&={}^it_{V}^{j',h'} & \pi_{V_i}\widetilde{B}_i|_{D_{j'}^{(h')}}&={}^is_{V}^{j',h'}\end{aligned}\tag{50}$$

We define also the following operator z_i on D'_i

$$\begin{aligned}z_i|_{D_j^{(1)}}&=0,\\ z_i|_{D_j^{(h)}}&=Id_{D_j}:D_j^{(h)}\to D_j^{(h-1)}\end{aligned}\tag{51}$$

7.1.2. Following Maffei let us introduce the following degrees:

$$\begin{aligned}\deg({}^it_{j,h}^{j',h'})&=\min(h-h'+1,h-h'+1+j'-j),\\ \deg({}^is_{j,h}^{j',h'})&=\min(h-h',h-h'+j'-j).\end{aligned}\tag{52}$$

7.1.3. A quadruple $(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta})\in\Lambda^c(\widetilde{v},\widetilde{d})$ is called *transversal* if it satisfies the following two groups of relations for $0\le i\le n-2$

(1) first group (Maffei)

$$
\begin{aligned}
{}^i t_{j,h}^{j',h'} &= 0 && \text{if } \deg(t_{j,h}^{j',h'}) < 0\\
{}^i t_{j,h}^{j',h'} &= 0 && \text{if } \deg(t_{j,h}^{j',h'}) = 0 \text{ and } (j',h') \neq (j,h+1)\\
{}^i t_{j,h}^{j',h'} &= Id_{D_j} && \text{if } \deg(t_{j,h}^{j',h'}) = 0 \text{ and } (j',h') = (j,h+1)\\
{}^i t_{i,j,h}^{V} &= 0 &&\\
{}^i t_{V}^{j',h'} &= 0 && \text{if } h' \neq 1\\
{}^i s_{j,h}^{j',h'} &= 0 && \text{if } \deg(s_{j,h}^{j',h'}) < 0\\
{}^i s_{j,h}^{j',h'} &= 0 && \text{if } \deg(s_{j,h}^{j',h'}) = 0 \text{ and } (j',h') \neq (j,h)\\
{}^i s_{j,h}^{j',h'} &= Id_{D_j} && \text{if } \deg(s_{j,h}^{j',h'}) = 0 \text{ and } (j',h') = (j,h)\\
{}^i s_{j,h}^{V} &= 0 && \text{if } h \neq j-i\\
{}^i s_{V}^{j',h'} &= 0 &&
\end{aligned} \tag{53}
$$

(2) second group

$$\pi_{D_j^{(h)}} \widetilde{B}_i \widetilde{A}_i|_{D_{j'}^{(h')}} - x_i = 0 \qquad \text{unless } h = j-i \tag{54}$$

Let us denote the set of all transversal elements in $\Lambda^c(\widetilde{v}, \widetilde{d})$ by S. The set of all stable transversal elements is denoted by $S^s = S \cap \Lambda^{c,s}(\widetilde{v}, \widetilde{d})$.

7.1.4. We will need more notation. First of all denote

$$b_j^i = c_{i+2} + \cdots + c_j \qquad \text{for } -1 \leq i \leq n-3, \text{ and } i+2 \leq j \leq n-1. \tag{55}$$

Now we introduce some invariant polynomials of $q_{i\to j} p_{j\to i}$ as follows. First,

$$P(i,1,j) = q_{i+2\to j} p_{j \to i+2} \tag{56}$$

and for $2 \leq h' \leq j-i-1$

$$
\begin{aligned}
P(i,h',j) = {} & q_{i+h'+1\to j} p_{j\to i+h'+1}\\
& + \sum_{k=1}^{j-i-h'-1} (-1)^k \sigma_k(b_{i+2}^i, \ldots, b_{i+h'-1+k}^i) q_{i+h'+1+k\to j} p_{j\to i+h'+1+k}\\
& + (-1)^{j-i-h'-1} \sigma_{j-i-h'}(b_{i+2}^i, \ldots, b_{j-1}^i).
\end{aligned} \tag{57}
$$

where σ_k is the k-th elementary symmetric function.

We also fix the notation for binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

7.2. **Main Lemma.** We can now formulate our main lemma

Lemma.

(i) There exists a unique $G(V)$-equivariant map $\Phi : \Lambda^c(v,d) \to S$ such that

$$\pi_{V_{i+1}} \widetilde{A}_i|_{V_i} = x_i \qquad \pi_{V_i} \widetilde{B}_i|_{V_{i+1}} = \overline{x}_i \\ {}^{i}t^{i+1,1}_{V} = p_{i+1} \qquad {}^{i}s^{V}_{i+1,1} = q_{i+1} \tag{58}$$

(ii) The blocks of $\widetilde{A}_i, \widetilde{B}_i$ not defined in the equations (53) and (58) are described as follows:

$$ {}^{i}t^{j',1}_{V} = p_{j' \to i+1} \qquad {}^{i}s^{V}_{j,j-i} = q_{i+1 \to j} \tag{59}$$

When $j' \neq j$ we have

$$\begin{aligned} {}^{i}t^{j',h'}_{j,h} &= 0 \quad \text{if } (j',h') \neq (j,h+1) \\ {}^{i}s^{j',h'}_{j,h} &= 0 \quad \text{if } (j',h') \neq (j,h) \text{ and } h \neq j-i \end{aligned} \tag{60}$$

and

$$ {}^{i}s^{j',h'}_{j,j-i} = q_{i+h'+1 \to j} p_{j' \to i+h'+1} \tag{61}$$

When $j = j'$ we have

$$ {}^{i}t^{j,h'}_{j,h} = \begin{cases} 0, & \text{if } h' = 1 \\ (-1)^{h-h'+1} \binom{h-1}{h'-2} c_{i+1}^{h-h'+1}, & \text{if } 2 \leq h' \leq h+1 \end{cases} \tag{62}$$

And finally,

$$ {}^{i}s^{j,h'}_{j,h} = \begin{cases} \binom{h-1}{h'-1} c_{i+1}^{h-h'}, & \text{if } h \neq j-i \\ P(i,h',j) + \binom{h-1}{h'-1} c_{i+1}^{h-h'}, & \text{if } 1 \leq h' \leq h, \text{ and } h = j-i \end{cases} \tag{63}$$

(iii) For $x \in \Lambda^c(v,d)$ we have $\Phi(x) \in S^s$ if and only if $x \in \Lambda^c_s(v,d)$. Thus the restriction of Φ to the stable points provides the $G(V)$-equivariant map $\Phi^s : \Lambda^c_s(v,d) \to S^s$

(iv) The maps Φ and Φ^s are isomorphisms of algebraic varieties.

Proof. Following Maffei, we prove the lemma by decreasing induction on i. If $i = n-2$ the maps $\widetilde{A}_{n-2}$ and $\widetilde{B}_{n-2}$ are completely defined by the relations (58) and (53) and it is easy to see that $\widetilde{A}_{n-2}\widetilde{B}_{n-2} = c_{n-1}$.

Assume that $\widetilde{A}_k, \widetilde{B}_k$ are defined for $k > i$ by the formulas in the lemma.

We have the following equations for $\widetilde{A}_i$ and $\widetilde{B}_i$:

$$\widetilde{A}_i \widetilde{B}_i = \widetilde{B}_{i+1} \widetilde{A}_{i+1} + c_{i+1} \tag{64}$$

$$\pi_{D_j^{(h)}} \widetilde{B}_i \widetilde{A}_i|_{D_{j'}^{(h')}} - z_i = 0 \ \text{ unless } h = j-i. \tag{65}$$

Observe that

$$\pi_{V_{i+1}}\widetilde{A}_i\widetilde{B}_i|_{V_{i+1}} = A_iB_i + p_{i+1}q_{i+1} = B_{i+1}A_{i+1} + c_{i+1} = \pi_{V_{i+1}}\widetilde{B}_{i+1}\widetilde{A}_{i+1}|_{V_{i+1}} + c_{i+1}.$$

Then, in agreement with formulas (59)

$$\pi_{V_{i+1}}\widetilde{A}_i\widetilde{B}_i|_{D_j^{(h)}} = \pi_{V_{i+1}}\widetilde{B}_{i+1}\widetilde{A}_{i+1}|_{D_j^{(h)}} = K_{h,1}B_{i+1}p_{j\to i+2} = K_{h,1}p_{j\to i+1},$$
$$\pi_{D_j^{(h)}}\widetilde{A}_i\widetilde{B}_i|_{V_{i+1}} = \pi_{D_j^{(h)}}\widetilde{B}_{i+1}\widetilde{A}_{i+1}|_{V_{i+1}} = K_{h,j-i-1}q_{i+2\to j}A_{i+1} = K_{h,j-i-1}q_{i+1\to j}.$$

where

$$K_{p,q} = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases}$$

is the Kronecker symbol.

Now, in order to simplify the notation a bit we set $t^{j',h'}_{j,h} := {}^{i}t^{j',h'}_{j,h}$ and $s^{j',h'}_{j,h} := {}^{i}s^{j',h'}_{j,h}$

Case I: $j \neq j'$. In this case the equation (64) and translates into the following equations for $t^{j',h'}_{j,h}$ and $s^{j',h'}_{j,h}$:

(66)
$$s^{j',h'}_{j,h+1} + \sum_{\substack{h'<h''<h+1 \\ h'-j'<h''-j''<h+1-j}} t^{j'',h''}_{j,h}s^{j',h'}_{j'',h''} + t^{j',h'}_{j,h} = \begin{cases} 0, & \text{if } h \neq j-i-1 \\ q_{i+h'+1\to j}p_{j'\to i+h'+1}, & \text{if } h = j-i-1. \end{cases}$$

while the equation (65) translates into the following equations for $t^{j',h'}_{j,h}$ and $s^{j',h'}_{j,h}$, $h \neq j-i$:

$$t^{j',h'}_{j,h} + \sum_{\substack{h'-1<h''<h \\ h'-1-j'<h''-j''<h-j}} s^{j'',h''}_{j,h}t^{j',h'}_{j'',h''} + s^{j',h'-1}_{j,h} = 0 \tag{67}$$

We claim that the system of equations (66) and (67) has a unique solution indicated in the statement of the lemma. We will prove this claim by induction on h and h'.

First of all, observe that from the equation (67) we have $t^{j',1}_{j,1} = 0$.

We make two induction assumptions ($k \geq 1$):

(1) $t^{j',h'}_{j,h} = 0$ for all (h',h) such that $h' \leq h \leq k$ for all $j \neq j'$ at the same time.
(2) $s^{j',h'}_{j,h+1} = 0$ for all (h',h) such that $h' < h \leq k+1 \leq j-i$ for all $j \neq j'$ at the same time.

Induction Step 1. Consider the equation (67) for $h = k+1$. By assumption (2) we have $s^{j',h'-1}_{j,k+1} = 0$ and $s^{j'',h''}_{j,k+1} = 0$ for $j'' \neq j$. If $j'' = j$, then $j'' \neq j'$ and by assumption (1) $t^{j',h'}_{j'',h''} = 0$ for $h'' \leq k$. Now from equation (67) we see that $t^{j',h'}_{j,k+1} = 0$ for $h' \leq k+1$.

Induction Step 2. Consider the equation (66) for $h = k+1$. By induction step (1) $t^{j',h'}_{j,k+1} = 0$ and $t^{j'',h''}_{j,k+1} = 0$ for $j'' \neq j$. If $j'' = j$, then $j'' \neq j'$ and by assumption (2) $s^{j',h'}_{j'',h''} = 0$. Now from equation (66) we see that $s^{j',h'}_{j,k+2} = 0$ for $h' < k+2$.

Finally, if $h+1 = j-i$, then the equations (66) and the induction steps 1 and 2 yield:

$$s^{j',h'}_{j,j-i} = q_{i+h'+1\to j}\, p_{j'\to i+h'+1}. \tag{68}$$

Case II: $j = j'$. In this case we fix j and simplify the notation further a bit, by setting $t^{h'}_h := t^{j,h'}_{j,h}$ and $s^{h'}_h := s^{j,h'}_{j,h}$. Now, taking into account Case I, the equation (64) and translates into the following equations for $t^{h'}_h$ and $s^{h'}_h$:

$$s^{h'}_{h+1} + \sum_{h'<h''<h+1} t^{h''}_{j,h} s^{h'}_{h''} + t^{h'}_h = \begin{cases} 0, & \text{if } h \neq j-i-1 \text{ and } h \neq h' \\ c_{i+1} & \text{if } h \neq j-i-1 \text{ and } h = h' \\ P(i,h',j), & \text{if } h = j-i-1 \text{ and } h \neq h' \\ P(i,h',j) + c_{i+1}, & \text{if } h = j-i-1 \text{ and } h = h' \end{cases} \tag{69}$$

(In order to compute the right hand side, we need to use the following combinatorial formula

$$\sigma_a(c, c+b_1, \ldots, c+b_p) = \sum_{l=0}^{a} c^l \binom{p-a+l+1}{l} \sigma_{a-l}(b_1, \ldots, b_p)$$

for $a, p \in \mathbb{Z}$, $1 \leq a \leq p$. We assume here that $\sigma_0(b_1, \ldots, b_p) = 1$.)

The equation (65) translates into the following equations for $t^{j',h'}_{j,h}$ and $s^{j',h'}_{j,h}$, $h < j-i$:

$$t^{h'}_h + \sum_{h'-1<h''<h} s^{h''}_h t^{h'}_{h''} + s^{h'-1}_h = 0 \tag{70}$$

Again, we claim that the system of equations (69) and (70) has a unique solution indicated in the statement of the lemma. Again, we will prove this claim by induction on h and h'.

First of all, observe that from the equation (70) we have $t^1_1 = 0$.

We make two induction assumptions ($k \geq 1$):

(1) $t^{h'}_h$ is given by equations (60) for all (h', h) such that $h' \leq h \leq k$.
(2) $s^{h'}_h$ is given by equations (61) for all (h', h) such that $h' < h \leq k+1 \leq j-i$.

Proceeding by induction as in Case I and using the formula (for $b, l \in \mathbb{Z}$, $0 \leq b \leq l-2$)

$$\sum_{a=b}^{l} (-1)^{l-a} \binom{l}{a} \binom{a+1}{b+1} = 0$$

it is easy to see that all $t^{h'}_h$ and $s^{h'}_{h+1}$ are given by formulas (60) and (61) respectively.

We have proved the assertions (i) and (ii) of the lemma. The assertion (iii) follows from the construction and Lemma 2.2.3 exactly as in [Maf, Lemma 19]. The assertion (iv) follows from the construction, cf. [Maf, Lemma 19]. □

7.2.1. It is important for us to record the formula for $\widetilde{B}_0\widetilde{A}_0 = \widetilde{\delta}_1\widetilde{\gamma}_1$. To simplify notation, we set

$$b_l := b_l^{-1} = c_1 + \cdots + c_l,$$

and

$$P'(h',j) := \sum_{k=1}^{j-h'-1} (-1)^k \sigma_k(b_1,\ldots,b_{h'-2+k}) q_{h'+k\to j} p_{j\to h'+k} + (-1)^{j-h'-1}\sigma_{j-h'}(b_1,\ldots,b_{j-1}).$$

Now we have

$$(\widetilde{\delta}_1\widetilde{\gamma}_1)_{j,h}^{j',h'} = \begin{cases} \mathrm{Id}_{D_j}, & \text{if } h'=h+1,\ j'=j, \\ q_{h'\to j}p_{j'\to h'} + K_{j,j'}P'(h',j), & \text{if } h=j, \\ 0, & \text{otherwise .}\end{cases} \tag{71}$$

where $K_{p,q}$ is the Kronecker symbol, σ_k is the k-th elementary symmetric function, and we assume that the value of σ_k at the empty collection of variables is zero.

Finally, let us record the specialization of the above formula for the case $c=0$. Clearly, in this case $P'(h',j)=0$ and we have

$$(\widetilde{\delta}_1\widetilde{\gamma}_1)_{j,h}^{j',h'} = \begin{cases} \mathrm{Id}_{D_j}, & \text{if } h'=h+1,\ j'=j, \\ q_{h'\to j}p_{j'\to h'}, & \text{if } h=j, \\ 0, & \text{otherwise.}\end{cases} \tag{72}$$

8. Proof of the Main Theorem

In this section we complete the proof of the Main Theorem (Theorem 5.3.)

8.1. The isomorphisms ϕ and $\widetilde{\phi}$. The argument in this subsection is for the case $c=0$. The argument for a general c is completely analogous. In the proof we mostly follow the logic of [Maf].

Lemma. Let $(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta}) \in S$ and let $\widetilde{g} \in G(\widetilde{V})$ be such that $\widetilde{g}(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta}) \in S$. Then $\widetilde{g}_i(V_i) \subseteq V_i$ and if we denote $g_i = \widetilde{g}_i|_{V_i}$ we have

$$\widetilde{g}(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta}) = g(\widetilde{A},\widetilde{B},\widetilde{\gamma},\widetilde{\delta}). \tag{73}$$

Proof. The proof is lifted verbatim from [Maf, Lemma 22]. □

8.1.1. Let $D = \widetilde{D}_1$ as in 7.1.1. Then $\dim D = N = \widetilde{d}_1 := \sum_{j=1}^{n-1} j d_j$. Observe that $(\widetilde{v}, \widetilde{d})$ as constructed in 7.1.1 must satisfy the conditions of Section 6 in order for $\mathfrak{M}_1(\widetilde{v}, \widetilde{d})$ and $\mathfrak{M}(\widetilde{v}, \widetilde{d})$ to be nonempty, cf. [Maf, 1.4] and therefore, if nonempty, $\mathfrak{M}_1(\widetilde{v}, \widetilde{d}) \simeq \overline{\mathcal{O}}_\mu$ and $\mathfrak{M}(\widetilde{v}, \widetilde{d}) \simeq T^*\mathcal{F}^{n,a}$ (for $c = 0$), where μ, a are defined as in 5.1.1. (For a general c the nilpotent orbit $\mathcal{O}_\mu$ deforms into a general conjugacy class, cf. 3.1.4, 5.1.2.) Now recall the definition (cf. 3.3.1) of the transverse slice T_x to the orbit $\mathcal{O}_\lambda$ where λ is obtained from (v, d) as in 5.1.1. Let $T_{x,\mu} = T_x \cap \overline{\mathcal{O}}_\mu$ be as in 3.3.1 and let $\widetilde{T}_x^a$ be as in 3.3.3.

8.1.2. Now we will construct the maps ϕ_0 and $\widetilde{\phi}$ completing the following commutative diagrams.

(74)
$$\begin{array}{ccccccc} \Lambda^c(v,d) & \xrightarrow{\Phi} & S & \quad & \Lambda^c_s(v,d) & \xrightarrow{\Phi^s} & S^s \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{M}_0(v,d) & \xrightarrow{\phi_0} & \mathfrak{M}_0(\widetilde{v},\widetilde{d}) & & \mathfrak{M}(v,d) & \xrightarrow{\widetilde{\phi}} & \mathfrak{M}(\widetilde{v},\widetilde{d}) \end{array}$$

We denote $\phi := \phi_0|_{\mathfrak{M}_1(v,d)} : \mathfrak{M}_1(v,d) \to \mathfrak{M}_0(\widetilde{v}, \widetilde{d})$. Since $\mathfrak{M}_1(\widetilde{v}, \widetilde{d}) \simeq \overline{\mathcal{O}}_\mu$ an element of $\mathfrak{M}_1(v, d)$ will be sent by ϕ to an operator $y + f \in \mathrm{End}(D)$, where y is nilpotent of type λ and f is given by the *explicit formulas* (72) (and (71) for arbitrary c). A simple inspection shows that $\mathrm{Im}\ \phi \subseteq T_{x,\mu}$, and $\mathrm{Im}\ \widetilde{\phi} \subseteq \widetilde{T}_x^a$.

8.1.3. *Lemma.* The map ϕ is a closed immersion.

Proof. It is enough to prove that ϕ_0 is closed immersion. Recall that

(75)
$$\begin{aligned} \mathfrak{M}_0(v,d) &= \Lambda^c(v,d)//G(V) = \mathrm{Spec}\,\mathcal{R}(\Lambda^c(v,d))^{G(V)}, \\ \mathfrak{M}_0(\widetilde{v},\widetilde{d}) &= \Lambda^c(\widetilde{v},\widetilde{d})//G(\widetilde{V}) = \mathrm{Spec}\,\mathcal{R}(\Lambda^c(\widetilde{v},\widetilde{d}))^{G(\widetilde{V})}. \end{aligned}$$

We will prove that the restriction map $\phi^* : \mathcal{R}(\Lambda^c(\widetilde{v}, \widetilde{d}))^{G(\widetilde{V})} \to \mathcal{R}(\Lambda^c(v, d))^{G(V)}$ is surjective.

By Theorem 2.2.2 the algebra $\mathcal{R}(\Lambda^c(\widetilde{v}, \widetilde{d}))^{G(\widetilde{V})}$ is generated by $\widetilde{\chi}(\widetilde{\delta}_1\widetilde{\gamma}_1)$ where $\widetilde{\chi}$ is a linear form on $\mathrm{Hom}(\widetilde{D}_1, \widetilde{D}_1)$. If $\widetilde{\delta}_1\widetilde{\gamma}_1$ is of the form (72) and

$$\widetilde{\chi} = \chi \in \mathrm{Hom}(D_{j'}^{(h')}, D_j^{(j)})^* \subseteq \mathrm{Hom}(\widetilde{D}_1, \widetilde{D}_1)^*,$$

then for $1 \le h' \le \min(j, j')$ we have

$$\widetilde{\chi}(\widetilde{\delta}_1\widetilde{\gamma}_1) = \chi(\pi_{D_j^{(j)}}(\widetilde{\delta}_1\widetilde{\gamma}_1)|_{D_{j'}^{(h')}}) = \chi(q_{h'\to j}p_{j'\to h'}),$$

which are all the generators of the algebra $\mathcal{R}(\Lambda^c(v, d))^{G(V)}$ according to the Theorem 2.2.2. □

8.1.4. *Lemma.* The map $\widetilde{\phi}:\mathfrak{M}(v,d)\to\widetilde{T}^a_x$ is proper and injective.

Proof. We have the following diagrams

$$\begin{array}{ccc} \mathfrak{M}(v,d) & \xrightarrow{\widetilde{\phi}} & \widetilde{T}^a_x \\ {\scriptstyle p}\downarrow & & {\scriptstyle \mathbf{m}_a}\downarrow \\ \mathfrak{M}_0(v,d) & \xrightarrow{\phi_0} & \mathfrak{M}_0(\widetilde{v},\widetilde{d}) \end{array} \qquad \begin{array}{ccc} \mathfrak{M}(v,d) & \xrightarrow{\widetilde{\phi}} & \widetilde{T}^a_x \\ {\scriptstyle p}\downarrow & & {\scriptstyle \mathbf{m}_a}\downarrow \\ \mathfrak{M}_1(v,d) & \xrightarrow{\phi} & T_{x,\mu} \end{array} \tag{76}$$

Since ϕ is a closed immersion and the morphisms p and $\mathbf{m}_a$ are projective, we see that $\widetilde{\phi}$ is proper. Since all orbits in $\Lambda^c_s(v,d)$ and $\Lambda^c_s(\widetilde{v},\widetilde{d})$ are closed, $\widetilde{\phi}$ is injective. □

8.1.5. *Lemma.* The map $\widetilde{\phi}:\mathfrak{M}(v,d)\to\widetilde{T}^a_x$ is an isomorphism of algebraic varieties.

Proof. Since $\widetilde{\phi}$ is a proper injective morphism between connected smooth varieties of the same dimension, $\widetilde{\phi}$ is an analytic isomorphism and therefore an algebraic isomorphism. □

Lemma. The map $\phi:\mathfrak{M}_1(v,d)\to T_{x,\mu}$ is an isomorphism of algebraic varieties.

Proof. Since $\mathbf{m}_a$ is surjective, from the diagram (76) we see that ϕ is surjective. Since ϕ is a surjective closed immersion, and both $\mathfrak{M}_1(v,d)$ and $T_{x,\mu}$ are reduced varieties over $\mathbb{C}$, ϕ is an algebraic isomorphism. □

8.2. **The immersions ψ and $\widetilde{\psi}$.** These immersions were constructed in section 4.5.

9. Application to representation theory: $(\mathfrak{gl}(n),\mathfrak{gl}(m))$-duality

The relationship between quiver varieties and affine Grassmannians provides a natural framework for $(GL(n),GL(m))$ duality.

9.1. **Skew $(GL(n),GL(m))$ duality.**

9.1.1. Let $V=\mathbb{C}^m$ and $W=\mathbb{C}^n$ be two vector spaces. Let us consider the $\mathfrak{gl}(m)\times\mathfrak{gl}(n)$ bimodule $V\otimes W$ and its N-th exterior power $\wedge^N(V\otimes W)$. We have the following decomposition [H, 4.1.1]:

$$\wedge^N(V\otimes W)=\bigoplus_\lambda V_\lambda\otimes W_{\check{\lambda}}, \tag{77}$$

where λ are all partitions of N which fit into the $n\times m$ box, V_λ is the highest weight representation of $\mathfrak{gl}(m)$ with highest weight λ and $W_{\check{\lambda}}$ is the highest weight representation of $\mathfrak{gl}(n)$ with highest weight $\check{\lambda}$.

9.1.2. Considering $V \otimes W$ as a $\mathfrak{gl}(m)$ module $V \otimes \mathbb{C}^n$, we have the following decomposition:

$$\wedge^N (V \otimes W) = \bigoplus_{a_1+\cdots+a_n=N} \wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V. \tag{78}$$

Considered as a representation of the torus $(\mathbb{C}^\times)^n \subseteq \mathfrak{gl}(n)$ the vector space $\wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V$ has weight $a = (a_1, \ldots, a_n)$. Thus decompositions (77) and (78) imply the following formula

$$\mathrm{Hom}_{\mathfrak{gl}(m)}(\wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V, V_\lambda) \simeq W_{\check{\lambda}}(a), \tag{79}$$

where $W_{\check{\lambda}}(a)$ is the weight space corresponding to weight a of the $\mathfrak{gl}(n)$ highest weight module $W_{\check{\lambda}}$.

9.1.3. *Geometric skew duality.* We construct a based version of the isomorphism (79), i.e., a geometric skew $(GL(n), GL(m))$ duality. More precisely, with N, v, d, a, λ as in 5.1.1, we identify the right hand side with $\mathcal{H}(\pi^{-1}(L_\lambda))$, where L_λ is a lattice in the affine Grassmannian $\mathcal{G}$, and the left hand side with $\mathcal{H}(\mathfrak{L}(v, d))$ by Theorem 2.4. The identification of irreducible components $\mathrm{Irr}\, \pi^{-1}(L_\lambda) = \mathrm{Irr}\, \mathfrak{L}(v, d)$, which follows from the isomorphism (40) matches the natural basis of the space of intertwiners $\mathrm{Hom}_{GL(m)}(\wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V, V_\lambda)$ arising from the affine Grassmannian construction (i.e., $\mathrm{Irr}\, \pi^{-1}(L_\lambda)$), and the natural basis of the weight space $W_{\check{\lambda}}(a)$ in the Nakajima construction (i.e., $\mathrm{Irr}\, \mathfrak{L}(v, d)$). Altogether:

$$\mathrm{Hom}_{GL(m)}(\wedge^{a_1} V \otimes \cdots \otimes \wedge^{a_n} V, V_\lambda) \simeq \mathcal{H}(\pi^{-1}(L_\lambda)) \simeq \mathcal{H}(\mathfrak{L}(v, d)) \simeq W_{\check{\lambda}}(a).$$

9.1.4. Dually, we have

$$\mathrm{Hom}_{\mathfrak{gl}(n)}(\wedge^{c_1} W \otimes \cdots \otimes \wedge^{c_m} W, W_{\check{\lambda}}) = V_\lambda(c), \tag{80}$$

where $V_\lambda(c)$ is the weight space corresponding to the weight $c = (c_1 \ldots, c_m)$ of the $\mathfrak{gl}(m)$ highest weight module V_λ.

9.2. **Symmetric** $(GL(m), GL(m))$ **duality.**

9.2.1. Analogously, if we consider the N-th symmetric power $\mathrm{Sym}^N(V \otimes V)$ of the $\mathfrak{gl}(m) \times \mathfrak{gl}(m)$ bimodule $V \otimes V$, we have the following decomposition (a particular case of [H, 2.1.2]):

$$\mathrm{Sym}^N(V \otimes V) = \bigoplus_\lambda V_\lambda \otimes V_\lambda, \tag{81}$$

where the sum is over all partitions λ of N with at most m parts.

Considering $V \otimes V$ as a $\mathfrak{gl}(m)$ module $V \otimes \mathbb{C}^m$, we have the following decomposition:

$$\mathrm{Sym}^N(V \otimes V) = \bigoplus_{c_1+\cdots+c_m=N} \mathrm{Sym}^{c_1} V \otimes \cdots \otimes \mathrm{Sym}^{c_m} V. \tag{82}$$

Thus decompositions (81) and (82) imply the following formula

$$\mathrm{Hom}_{\mathfrak{gl}(m)}(\mathrm{Sym}^{c_1} V \otimes \cdots \otimes \mathrm{Sym}^{c_m} V, V_\lambda) = V_\lambda(c), \tag{83}$$

where $V_\lambda(c)$ is the weight space corresponding to weight c of the $\mathfrak{gl}(m)$ highest weight module V_λ.

9.2.2. Combining the equations (80) and (83) we get

$$\mathrm{Hom}_{\mathfrak{gl}(n)}(\wedge^{c_1} W \otimes \cdots \otimes \wedge^{c_m} W, W_{\check{\lambda}}) = \mathrm{Hom}_{\mathfrak{gl}(m)}(\mathrm{Sym}^{c_1} V \otimes \cdots \otimes \mathrm{Sym}^{c_m} V, V_\lambda). \tag{84}$$

9.2.3. *Geometric symmetric duality.* Geometry allows us to find a *based* isomorphism of the left and right hand side of (84). Let N, v, d, a, λ be as in 5.1.1. First of all it follows from the quiver tensor product constructions of Malkin [Mal] and Nakajima [N4] that the relevant irreducible components $\mathrm{Irr}\, \widetilde{\mathfrak{g}}_x^{n,c}$ of the Spaltenstein fiber over a nilpotent of type λ index a natural basis in the left hand side of (84). Here

$$\widetilde{\mathfrak{g}}_x^{n,c} = \{(x, F) \in gl(D) \times \mathcal{F}^{m,c} \mid x(F_i) \subseteq F_i \text{ and } x \text{ acts on } F_i/F_{i-1} \text{ as a regular nilpotent } \}.$$

Now consider another convolution Grassmannian:

$$\begin{aligned} \widetilde{\mathcal{G}}^c &= \overline{\mathcal{G}}_{c_1\omega_1} * \cdots * \overline{\mathcal{G}}_{c_m\omega_1} \\ &= \{ \ L_0 \subseteq \ L_1 \subseteq \cdots \subseteq L_n \mid \ \dim L_i/L_{i-1} = c_i, \ z|_{L_i/L_{i-1}} \text{ is a regular nilpotent } \}, \end{aligned} \tag{85}$$

where ω_1 is the first fundamental weight of $GL(m)$. We have a map $\pi : \widetilde{\mathcal{G}}^c \to \mathcal{G}$ defined by $\pi : (L_0 \subseteq L_1 \subseteq \ldots \subseteq L_n) \mapsto L = L_n$. Consider $\pi^{-1}(L_\lambda)$ for $L_\lambda \in \mathcal{G}$. It follows from the Geometric Satake Correspondence that the set of relevant irreducible components $\mathrm{Irr}\, \pi^{-1}(L_\lambda)$ indexes a basis in the right hand side of (84).

It is clear that the varieties $\widetilde{\mathfrak{g}}_x^{n,c} \simeq \pi^{-1}(L_\lambda)$ are isomorphic. This isomorphism gives us a bijection $\mathrm{Irr}\, \widetilde{\mathfrak{g}}_x^{n,c} = \mathrm{Irr}\, \pi^{-1}(L_\lambda)$.

Summarizing:

$$\begin{aligned} \mathrm{Hom}_{GL(n)}(\wedge^{c_1} W \otimes \cdots \otimes \wedge^{c_m} W, W_{\check{\lambda}}) &\simeq \mathcal{H}(\widetilde{\mathfrak{g}}_x^{n,c}) \\ &\simeq \mathcal{H}(\pi^{-1}(L_\lambda)) \\ &\simeq \mathrm{Hom}_{GL(m)}(\mathrm{Sym}^{c_1} V \otimes \cdots \otimes \mathrm{Sym}^{c_m} V, V_\lambda). \end{aligned}$$

9.2.4. *Remark.* The second author has greatly benefited from a class taught by W. Wang at Yale [Wa1]. The "geometric symmetric duality" above has a lot in common with the construction described in [Wa2] and we believe that the "geometric skew duality" construction answers a question posed by Weiqiang Wang.

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