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REPRESENTATIONS OF SEMI-SIMPLE LIE ALGEBRAS IN PRIME CHARACTERISTIC AND NONCOMMUTATIVE SPRINGER RESOLUTION

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To Joseph Bernstein with admiration and gratitude

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0. **Introduction**

Abstract. We prove most of Lusztig's conjectures from [Lu], including the existence of a canonical basis in the Grothendieck group of a Springer fiber. The conjectures also predict that this basis controls numerics of representations of the Lie algebra of a semi-simple algebraic group over an algebraically closed field of positive characteristic. We check this for almost all characteristics.

To this end we construct a non-commutative resolution of the nilpotent cone which is derived equivalent to the Springer resolution. On the one hand, this noncommutative resolution is shown to be compatible with the positive characteristic localization equivalences of [BMR1]. On the other hand, it is compatible with the t-structure arising from an equivalence with the derived category of perverse sheaves on the affine flag variety of the Langlands dual group [ArkB] inspired by local geometric Langlands duality. This allows one to apply Frobenius purity theorem of [BBD] to deduce the desired properties of the basis.

We expect the noncommutative counterpart of the Springer resolution to be of independent interest from the perspectives of algebraic geometry and geometric Langlands duality.

Let G be a reductive group over an algebraically closed field $\Bbbk$ of characteristic $p > h$ and $\mathfrak{g}$ be its Lie algebra (here h denotes the Coxeter number of G). Let $\mathcal{P}$ be a partial flag variety and consider the space $\tilde{\mathfrak{g}}_{\mathcal{P}}$ of pairs of a parabolic subalgebra $\mathfrak{p} \in \mathcal{P}$ and an element in it. For the full flag variety $\mathcal{B}$ we usually denote $\tilde{\mathfrak{g}}_{\mathcal{B}}$ simply by $\tilde{\mathfrak{g}}$. We have a map $\mu_{\mathcal{P}} : \tilde{\mathfrak{g}}_{\mathcal{P}} \to \mathfrak{g}$.

In [BMR1], [BMR2] we have shown that the derived category of $\mathfrak{g}$-modules with a fixed generalized central character is equivalent to the derived category of coherent sheaves on $\tilde{\mathfrak{g}}^*_{\mathcal{P}}$ set-theoretically supported on $\mathcal{P}_e = \mu_{\mathcal{P}}^{-1}(e)$; here the partial flag variety $\mathcal{P}$ and $e \in \mathfrak{g}^*$ depend on the central character. A numerical consequence of this equivalence is an isomorphism between the Grothendieck groups of the abelian categories $mod^{fg}(U(\mathfrak{g})_\sigma)$ and $Coh(\mathcal{P}_e)$, where $U(\mathfrak{g})_\sigma$ is the quotient of the enveloping algebra by a central character σ and mod^{fg} denotes the category of finite dimensional (equivalently, finitely generated) modules. This implies, in particular, that the number of irreducible representations with a fixed central character σ equals the rank of the Grothendieck group of $Coh(\mathcal{P}_e)$, which is known to coincide with the sum of Betti numbers of $\mathcal{P}_e$.

To derive more precise information about numerical invariants of $\mathfrak{g}$-modules one needs a characterization of the elements in $K^0(Coh(\mathcal{P}_e))$ which correspond to irreducible $\mathfrak{g}$-modules and their projective covers. Such a characterization is suggested by the work of Lusztig [Lu]. In *loc. cit.* he describes certain properties of a basis in the Grothendieck

group of a Springer fiber and conjectures that a basis with such properties exists and controls (in a certain precise sense) numerical invariants of irreducible $U(\mathfrak{g})_\sigma$ modules. (He also shows that a basis with such properties is essentially unique). The properties of a basis are similar to those enjoyed by Kazhdan-Lusztig bases of a Hecke algebra and canonical bases in modules over a quantum group, for this reason we will refer to a basis satisfying Lusztig's axioms as a *canonical basis*.

In the present paper we prove most of the conjectures from [Lu]. The first step is the construction of a non-commutative counterpart of the Springer resolution as a lift of modular representation categories to characteristic zero. By this we mean a certain noncommutative algebra A^0 defined canonically up to a Morita equivalence whose center is identified with the algebra $\mathcal{O}(\mathcal{N})$ of regular functions on the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$; here $\mathfrak{g}$ is taken over $R = \mathbb{Z}[\frac{1}{h!}]$. This noncommutative resolution is canonically derived equivalent to the ordinary Springer resolution, i.e. it comes with a canonical equivalence of triangulated categories $D^b(mod^{fg}(A^0)) \cong D^b(Coh(\widetilde{\mathcal{N}}))$ where $\widetilde{\mathcal{N}}$ is the cotangent bundle to the flag variety. Furthermore, for $\Bbbk$ as above and any $e \in \mathcal{N}(\Bbbk)$, the base change $A^0 \otimes_{\mathcal{O}(\mathcal{N})} \Bbbk_e$ is canonically Morita equivalent to a central reduction of $U(\mathfrak{g}_\Bbbk)$.

The above properties of A^0 imply that the numerics of non-restricted modular representation categories is independent of (sufficiently large) characteristic and show that A^0 provides a lifting of such representation categories to characteristic zero. As a side comment recall that a similar construction for representations of the algebraic group $G_\Bbbk$ (this setting is very close to restricted representations of the Lie algebra $\mathfrak{g}_\Bbbk$) was obtained in [AJS]. In that case the resulting category in characteristic zero turns out to be equivalent to representations of a quantum group at a root of unity. We expect that a similar statement holds for non-restricted Lie algebra modules considered in the present work, see Conjecture 1.7.1 below. Apart from that Conjecture, we avoid quantum groups in this paper.

The method of construction of the noncommutative resolution A^0 is based on an action of the affine braid group $\mathbb{B}_{\text{aff}}$ on the derived categories $D^b(Coh(\widetilde{\mathcal{N}}))$, $D^b(Coh(\tilde{\mathfrak{g}}_{\mathcal{B}}))$. Here the action of the generators of $\mathbb{B}_{\text{aff}}$ is described by certain simple correspondences. The fact that the corresponding functors obey the relations of $\mathbb{B}_{\text{aff}}$ is proven in [BR]. The algebra A^0 is determined (uniquely up to a Morita equivalence) by the t-structure on $D^b(Coh(\widetilde{\mathcal{N}}))$ corresponding to the tautological one under the equivalence with $D^b(mod^{fg}(A^0))$. This

t-structure is characterized in terms of the action of $\mathbb{B}_{\text{aff}}$. The comparison with modular localization and the proof of existence of a t-structure with required properties is based on compatibility of the $\mathbb{B}_{\text{aff}}$ action with *intertwining* (or *shuffling*) functors on the derived categories of modular representations. Notice that the latter are closely connected with "translation through the wall" functors, thus translation functors play a prominent role in our argument. The use of translation functors to establish independence of the category of modular representations on characteristic goes back (at least) to [AJS].

From the arguments alluded to above one can derive that the basis in the Grothendieck group of a Springer fiber corresponding to irreducible $\mathfrak{g}_{\Bbbk}$ modules satisfies all the axioms of a canonical basis except for one, the so-called *asymptotic orthogonality* property. The latter is reduced to certain compatibility between the above t-structures and the multiplicative group action on Slodowy slices. It says that the grading on the slice algebras, i.e. the algebras "controlling" the derived category of coherent sheaves on the resolution of a Slodowy slice, can be arranged to be positive. By this we mean that components of negative degrees in the algebra vanish, while the degree zero component is semi-simple. An analogous reformulation of Kazhdan-Lusztig conjectures is due to Soergel. Another feature parallel to Kazhdan-Lusztig theory is Koszul property of the slice algebras, see [BGS] for the corresponding facts about category O.

Properties of this type are usually deduced from a Theorem of [BBD] about weights of Frobenius acting on the stalks of l-adic intersection cohomology sheaves. Our proof also follows this strategy. The l-adic sheaves are brought into the picture by the result of [ArkB] which provides an equivalence between the derived category of G-equivariant sheaves on $\widetilde{\mathcal{N}}$ (over a field of characteristic zero) and a certain subcategory of the derived category of constructible sheaves on the affine flag variety $\mathcal{F}l$ of the Langlands dual group. This result is a categorical counterpart of one of the key ingredients in the proof of the tamely ramified local Langlands conjecture.

Below we show that the t-structure of perverse sheaves of $\mathcal{F}l$ is compatible with the t-structure coming from the equivalence with $D^b(mod^{fg}(A^0))$. This is achieved by interpreting the $\mathbb{B}_{\text{aff}}$ action on the perverse sheaves side as the geometric counterpart of the action of elements of the standard basis of the affine Hecke algebra on the anti-spherical module.

Thus the key step in our argument is compatibility between the two t-structures on $D^b(Coh(\widetilde{\mathcal{N}}))$, one coming from modular representations via the equivalence of [BMR1] and another from perverse sheaves on $\mathcal{F}l$ via the equivalence of [ArkB]. An indication of such a compatibility can be (and has been) found by unraveling logical connections between the works of G. Lusztig. However, we do not claim to have arrived at a conceptual explanation of this coincidence.

A possible conceptual approach to the material presented in this paper is via the local geometric Langlands duality formalism. Recall [FG, Fr] that the latter theory seeks to attach to a (geometric) local Langlands parameter a certain triangulated category, a categorification of a representation of a p-adic group attached to the Langlands parameter by the classical local Langlands conjectures. According to [FG] this triangulated category should arise as the derived category of an abelian category. That abelian category can conjecturally be identified with the category of modules over an affine Lie algebra at the critical level with a fixed central character. We propose the category of modules over the above algebra A^0 with a fixed central character as another construction for the so-called category of *Iwahori equivariant objects in a local Langlands category*, see Conjecture 1.7.2 for a concrete statement arising from comparing of our results with that of [FG, FG1].

We also hope that the t-structures on the derived categories of coherent sheaves (in particular, those on derived categories of coherent sheaves on varieties over $\mathbb{C}$) constructed below are of interest from the algebro-geometric point of view. We expect that the construction generalizes to other symplectic resolutions of singularities (cf. [BeKa], [Ka]) and is related to Bridgeland stability conditions; the latter topic is the subject of the forthcoming paper [AnnoB].

The paper is organized as follows. In section 1 we describe the affine braid group action on the derived categories of coherent sheaves and state existence and uniqueness of a t-structure characterized in terms of this action. We refer to [BR] for construction of the $\mathbb{B}_{\mathrm{aff}}$ actions and a proof of its properties.

Section 2 presents a proof of the facts about the t-structures. Uniqueness is deduced directly from a categorical counterpart of the quadratic relations satisfied by the action of a simple reflection $\widetilde{s_\alpha} \in \mathbb{B}_{\mathrm{aff}}$ (the action of $\widetilde{s_\alpha}$ on the corresponding Grothendieck groups satisfies quadratic relations because this action of $\mathbb{Z}[\mathbb{B}_{\mathrm{aff}}]$ factors through the affine Hecke

algebra). Existence is shown by reduction to positive characteristic, where the statement is deduced from localization in positive characteristic [BMR1], [BMR2].

Sections 3 and 4 present parabolic versions of the construction of t-structures. (They are not needed for the proof of Lusztig's conjectures: sections 5 and 6 are logically independent of sections 3 and 4).

Section 5 recalls Lusztig's conjectures [Lu] and reduces them to a positivity property of a grading on the slice algebras, stated in detail in 5.3.2. We finish the section by showing that positivity of the grading implies Koszul property of the graded algebras.

Section 6 proves this compatibility by relating the t-structure to perverse sheaves on affine flag variety of the Langlands dual group, which are related to our categories of coherent sheaves via the result of [ArkB]. Once the relationship between our graded abelian categories and perverse sheaves is established, the desired property of the grading follows from the purity theorem, similarly to the proof of Kazhdan-Lusztig conjecture.

Appendix A contains a proof of a technical statement about compatibility of the Springer representation of the Weyl group on cohomology of a Springer fiber with a certain involution of the cohomology space. This is needed in analysis of the involution of the (equivariant) cohomology space appearing in Lusztig's formulation of his conjectures. Appendix B by Eric Sommers establishes a property of the central element in an $SL(2)$ subgroup of G, which also enters comparison of our categorical picture with the formulas from [Lu].

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0.1. **Notations and conventions.** Let $G_{\mathbb{Z}}$ be a split reductive group over $\mathbb{Z}$. We work over the base ring $R = \mathbb{Z}[\frac{1}{h!}]$ where h is the maximum of Coxeter numbers of simple factors. So we denote by $G = G_R$ the base change of $G_{\mathbb{Z}}$ to R and its Lie algebra by $\mathfrak{g} = \mathfrak{g}_R$.

We will use the notation $\Bbbk$ for *geometric points* of R, i.e., maps $R \to \Bbbk$ where $\Bbbk$ is an algebraically closed field. We will use an abbreviation FGP for the set of geometric points of R that have finite characteristic. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the nilpotent cone and $\mathcal{B}$ the flag variety. We denote by $\widetilde{\mathcal{N}} = T^*\mathcal{B} \to \mathcal{N}$ the Springer resolution and by $\tilde{\mathfrak{g}} \to \mathfrak{g}$ the Grothendieck map. For convenience, we fix a nondegenerate invariant quadratic form on $\mathfrak{g}$ and use it to identify $\mathfrak{g}$ and $\mathfrak{g}^*$, hence also $\tilde{\mathfrak{g}}^*$ and $\tilde{\mathfrak{g}}$.

Let H be the abstract Cartan group of G with Lie algebra $\mathfrak{h}$. Let $\Lambda = X^*(H)$ be the weight lattice of G, $Q \subset \Lambda$ be the root lattice and W the Weyl group. Our choice of positive roots is such that for a Borel subalgebra $\mathfrak{b}$ with a Cartan subalgebra $\mathfrak{t}$, the isomorphism $\mathfrak{t} \cong \mathfrak{h}$ determined by $\mathfrak{b}$ carries roots in $\mathfrak{b}$ into negative roots. Let $I \subseteq I_{\mathrm{aff}}$ be the vertices of the Dynkin diagram for the Langlands dual group $\check{G}$ and of the affine Dynkin diagram for $\check{G}$, we consider them as affine-linear functionals on $\mathfrak{h}^*$.

Set $W_{\mathrm{aff}} = W \ltimes \Lambda$, $W_{\mathrm{aff}}^{Cox} = W \ltimes Q$. Then W_{aff}^{Cox} is a Coxeter group corresponding to the affine Dynkin graph of the Langlands dual group $\check{G}$, also $W_{\mathrm{aff}}^{Cox} \subset W_{\mathrm{aff}}$ is a normal subgroup with an abelian quotient $W_{\mathrm{aff}}/W_{\mathrm{aff}}^{Cox} \cong \Lambda/Q \cong \pi_1(\check{G})$. Thus W_{aff} is the extended affine Weyl group for $\check{G}$. Let $\mathbb{B} \subset B_{\mathrm{aff}}^{Cox} \subseteq \mathbb{B}_{\mathrm{aff}}$ denote the braid groups attached to W, W_{aff}^{Cox}, W_{aff} respectively. Let $W_{\mathrm{aff}}^{sc} \supseteq W_{\mathrm{aff}}^{Cox}$ and $\mathbb{B}_{\mathrm{aff}}^{sc} \supseteq B_{\mathrm{aff}}^{Cox}$ correspond to the simply connected cover of the derived subgroup of G.

Thus $\mathbb{B}_{\mathrm{aff}}$ contains reduced expressions $\widetilde{w}$, $w \in W_{\mathrm{aff}}$, and also a subgroup isomorphic to Λ consisting of the elements θ_λ, $\lambda \in \Lambda$, such that $\theta_\lambda = \widetilde{\lambda}$ when λ is a dominant weight. Denote by $\mathbb{B}_{\mathrm{aff}}^{+} \subseteq B_{\mathrm{aff}}^{Cox}$ the semigroup generated by lifts $\widetilde{s_\alpha}$ of all simple reflections s_α in B_{aff}^{Cox}.

$\mathcal{C}oh(X)\subseteq q\mathcal{C}oh(X)$ are the categories of coherent and quasicoherent sheaves on X. For a noetherian scheme Y we sometimes denote $\mathrm{RHom}_Y \stackrel{\mathrm{def}}{=} \mathrm{RHom}_{D^b(\mathcal{C}oh(Y))}$. The fiber products in this paper are taken in the category of schemes (as opposed to fiber product of varieties with the reduced scheme structure), unless stated otherwise. For a closed subscheme $\mathcal{Y}\subseteq\mathcal{X}$ we denote by $\mathcal{C}oh_{\mathcal{Y}}(\mathcal{X})$ the category of coherent sheaves on $\mathcal{X}$ supported set theoretically on $\mathcal{Y}$. In this paper we will consider formal neighborhood of $\mathcal{Y}$ in $\mathcal{X}$ only in the case when $\mathcal{Y}\subseteq\mathcal{X}$ is a base change of an affine closed embedding. In this situation, by the formal neighborhood $\widehat{\mathcal{Y}}$ we will understand the *completion* of $\mathcal{X}$ along $\mathcal{Y}$; this is locally the spectrum of the completion of algebras.[1]

For any abelian category $\mathcal{C}$ we denote its Grothendieck group by $K^0(\mathcal{C})$ and in a particular case of coherent sheaves on a scheme X or finitely generated modules over an algebra A we denote $K(X) = K^0[\mathcal{C}oh(X)]$ and $K(A) = K^0[mod^{fg}(A)]$.

Pull-back or push-forward functors on sheaves are understood to be the derived functors, and $Hom^i(x, y)$ means $Hom(x, y[i])$.

The base changes of $\tilde{\mathfrak{g}}$ and $\widetilde{\mathcal{N}}$ with respect to a $\mathfrak{g}$-scheme $S \to \mathfrak{g}$ will be denoted by $\widetilde{S} = S \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ and $\widetilde{S}' = S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$. For a complex of coherent sheaves $\mathcal{E}$ on $\tilde{\mathfrak{g}}$ (respectively, $\widetilde{\mathcal{N}}$) we let $\mathcal{E}_S$ (respectively, $\mathcal{E}'_S$) denote its pull-back to $\widetilde{S}$ (respectively, $\widetilde{S}'$).

1. t-structures on cotangent bundles of flag varieties: statements and preliminaries

As stated above, our basic object is the base change $G = G_R$ of a split reductive group $G_{\mathbb{Z}}$ over $\mathbb{Z}$ to the base ring $R = \mathbb{Z}[\frac{1}{h!}]$ (where h is the maximum of Coxeter numbers of simple factors).

Our main goal in the first two sections is to construct a certain t-structure $\mathcal{T}^{ex}$ on $D^b(\mathcal{C}oh(\widetilde{\mathfrak{g}}))$, called the *exotic* t-structure. The induced t-structure on $D^b(Coh(\tilde{\mathfrak{g}}_{\Bbbk}))$ for a field $\Bbbk$ of positive characteristic is related to representations of the Lie algebra $\mathfrak{g}_{\Bbbk}$. In this section we state the results on $\mathcal{T}^{ex}$ after recalling the key ingredients: the action of the affine braid group on $D^b(\mathcal{C}oh(\widetilde{\mathfrak{g}}))$, tilting generators in $D^b(\mathcal{C}oh(\widetilde{\mathfrak{g}}))$ and representation theoretic t-structures. Some proofs are postponed to later sections.

[1] Notice that we ignore the topology on completions of algebras. This somewhat unnatural setting makes base changes to a formal neighborhood fit into the framework of Theorem 1.3.2.

The next three subsections are devoted to a certain action of $\mathbb{B}_{\text{aff}}$ on the derived categories of (equivariant) coherent sheaves. In 1.1 we explain a basic formalism of convolutions on derived categories of coherent sheaves available under certain flatness assumptions, use it to define geometric action of a group on the derived categories of coherent sheaves and state the existence of a certain geometric action of $\mathbb{B}_{\text{aff}}$ on $\tilde{\mathfrak{g}}$, $\widetilde{\mathcal{N}}$.

A strengthened version of this result is presented in 1.3 where existence of a compatible collection of geometric actions of $\mathbb{B}_{\text{aff}}$ on the fiber product spaces $\tilde{\mathfrak{g}}_S$, $\widetilde{\mathcal{N}}_S$ (under some conditions on S) is stated. In fact, such compatible collections of actions arise naturally from a more transparent structure which is a direct generalization of the notion of a geometric action to the case when the space is not necessarily flat over the base. This generalization involves basics of DG-schemes theory. In an attempt to make the statements more transparent we present an informal discussion of this more general construction in 1.2.

Thus from the formal point of view subsection 1.2 and theorem 1.1.1 are not needed. We have included them in an attempt to make the exposition more transparent.

1.1. **Geometric action of the affine braid group. Definition.** By a *weak homomorphism* from a group to a monoidal category we will mean a homomorphism from the group to the group of isomorphism classes of invertible objects. A *weak action* of a group on a category $\mathcal{C}$ is a weak homomorphism from the group to the monoidal category of endo-functors of $\mathcal{C}$.

Let X be a finite type flat scheme over a Noetherian base S. Then the category $D^-(q\mathcal{C}oh(X \times_S X))$ is a monoidal category where the monoidal structure comes from convolution: $\mathcal{F}_1 * \mathcal{F}_2 = pr_{13*}(pr_{12}^*(\mathcal{F}_1) \overset{\mathrm{L}}{\otimes} pr_{23}^*(\mathcal{F}_2))$ where pr_{12}, pr_{23}, pr_{13} are the three projections $X \times_S X \times_S X \to X \times_S X$. This monoidal category acts on $D^-(q\mathcal{C}oh(X))$ by $\mathcal{F} : \mathcal{G} \mapsto pr_{1*}(\mathcal{F} \overset{\mathrm{L}}{\otimes} pr_2^*(\mathcal{G}))$.

By a weak *geometric action* of a group on X over S, we will understand a weak homomorphism from the group to $D^-(q\mathcal{C}oh(X \times_S X))$.

We will say that the action is *finite* if its image is contained in the full subcategory $D^b(\mathcal{C}oh(X \times_S X))$ and the corresponding action on the derived category of sheaves on X preserves $D^b(\mathcal{C}oh(X)) \subset D^-(q\mathcal{C}oh(X))$.

For a map $S' \to S$ we can base change the above structure in a straightforward way. Namely, the pull-back functor $D^-(q\mathcal{C}oh(X \times_S X)) \to D^-(q\mathcal{C}oh(X_{S'} \times_{S'} X_{S'})$ is monoidal and the pull-back functor $D^-(q\mathcal{C}oh(X)) \to D^-(q\mathcal{C}oh(X_{S'}))$ is compatible with the action of the monoidal categories. Thus a weak geometric action of a group on X over S induces a weak geometric action of the same group on the fiber product space $X_{S'}$ over S'.

1.1.1. *Action of $\mathbb{B}_{aff}$ on $\tilde{\mathfrak{g}}$, $\widetilde{\mathcal{N}}$ over R.* The Weyl group W acts on $\tilde{\mathfrak{g}}^{reg} \stackrel{\text{def}}{=} \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \mathfrak{g}^{reg}$. Let $\Gamma_w \subset \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ be the closure of the graph of the action of $w \in W$ and set $\Gamma'_w = \Gamma_w \cap \widetilde{\mathcal{N}}^2$.

Theorem. There exists a unique finite weak geometric action of $\mathbb{B}_{\text{aff}}$ on $\tilde{\mathfrak{g}}$ (respectively, on $\widetilde{\mathcal{N}}$) over R, such that:

i) for $\lambda \in \Lambda$, θ_λ corresponds to the direct image of the line bundle $\mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)$ (respectively, $\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)$) under the diagonal embedding.

ii) for a finite simple reflection $s_\alpha \in W$, $\widetilde{s_\alpha} \in \mathbb{B}$ corresponds to the structure sheaf $\mathcal{O}_{\Gamma_{s_\alpha}}$ (respectively, $\widetilde{s_\alpha} \mapsto \mathcal{O}_{\Gamma'_{s_\alpha}}$).

The proof appears in [BR]. We denote the weak geometric action on $\tilde{\mathfrak{g}}$ by $\mathbb{B}_{\text{aff}} \ni b \mapsto K_b \in D^b[\mathcal{C}oh(\tilde{\mathfrak{g}} \times_R \tilde{\mathfrak{g}})]$.

Remark. By the discussion preceding the Theorem, we also get geometric actions of $\mathbb{B}_{\text{aff}}$ on, say, $\tilde{\mathfrak{g}}_{\Bbbk}, \widetilde{\mathcal{N}}_{\Bbbk}$ where k is a field mapping to R. For applications below we need to consider more general base changes, these are dealt with in 1.3 below.

1.1.2. *Remark.* It is possible to deduce the Theorem from the results of [BMR1] which provide an action of $\mathbb{B}_{\text{aff}}$ on the derived category of modular representations using the reduction to prime characteristic techniques of section 2 below. This would make the series of [BMR1], [BMR2] and the present paper self-contained. However, this would further increase the amount of technical details without adding new conceptual features to the picture. For this reason we opted for a reference to a more satisfactory proof in [BR], (see also [Ri1] for a partial result in this direction).

1.2. **Digression: convolution operation via DG-schemes.** This subsection serves the purpose of motivating the formulation in the next Theorem 1.3.2 which is a strengthening of Theorem 1.1.1. It relies on some basic elements of the formalism of DG-schemes.

Neither that formalism nor the statements of the present subsection will be used in the rest of the paper (except for Remark 1.5.4). See [BR] for details.

Let $X \to S$ be again a morphism of finite type with S Noetherian, but let us no longer assume that it is flat. Then one can consider the *derived fiber product*[2] $X^2_S = X \overset{\mathrm{L}}{\times}_S X$, this is a differential graded scheme whose structure sheaf is the derived tensor product $\mathcal{O}(X) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}(S)} \mathcal{O}(X)$.

Definitions similar to the ones presented in 1.1 work also in this context providing the triangulated category $DG\mathcal{C}oh(X^2_S)$ of coherent $\mathcal{O}_{X^2_S}$-modules with the convolution monoidal structure. This monoidal category acts on the category $D^b(Coh(X))$.

The example relevant for us is when $S = \mathfrak{g}_R$ and $X = \tilde{\mathfrak{g}}_R$ or $X = \widetilde{\mathcal{N}}_R$. Notice that in the first case one can show that $Tor^{\mathcal{O}_{\mathfrak{g}}}_{>0}(\mathcal{O}_{\tilde{\mathfrak{g}}}, \mathcal{O}_{\tilde{\mathfrak{g}}}) = 0$ which implies that the derived fiber product reduces to the ordinary fiber product and $DG\mathcal{C}oh(\tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \tilde{\mathfrak{g}}) \cong D^b(\mathcal{C}oh(\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}))$. However, even in this case the definition of monoidal structure can not (to our knowledge) be given without using derived schemes, as it involves the triple fiber product $\tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \tilde{\mathfrak{g}}$ where higher Tor vanishing does not hold.

Given a pair of morphisms $X \to S \to U$ we get a natural morphism $i^S_U : X^2_S \to X^2_U$. It turns out that the functor of direct image $(i^S_U)_*$ can be equipped with a natural monoidal structure and the action of $DG\mathcal{C}oh(X^2_S)$ on $D^b(\mathcal{C}oh(X))$ factors through $DG\mathcal{C}oh(X^2_U)$.

For example, we can take $X = \tilde{\mathfrak{g}}$ or $\widetilde{\mathcal{N}}$, $S = \mathfrak{g}$, $U = R$. The composed map $X \to U$ is flat, so the construction of the monoidal structure and the action in this case reduces to the more elementary case described in Theorem 1.1.1.

The advantage of considering the finer structure of a geometric action on X over S rather than the weaker structure of a geometric action on X over U (which in our example happens to be more elementary) is the possibility to perform the base change construction for the base S.

Namely, given a morphism $S' \to S$ consider $X_{S'} = S' \overset{\mathrm{L}}{\times}_S X$ and $X^2_{S'} := X_{S'} \overset{\mathrm{L}}{\times}_{S'} X_{S'} \cong (X \overset{\mathrm{L}}{\times}_S X) \overset{\mathrm{L}}{\times}_S S'$. Then $D^b[\mathcal{C}oh(X^2_{S'})]$ is a monoidal category acting on $D^b[\mathcal{C}oh(X_{S'})]$.

[2]It may be more logical to denote the fiber product by $\overset{\mathrm{R}}{\times}$ as it can be thought of as a right derived functor in the category of schemes, corresponding to the left derived functor $\overset{\mathrm{L}}{\otimes}$ in the category of rings, opposite to the category of affine schemes.

The functor of pull-back under the morphism $X^2_{S'} \to X^2_S$ turns out to be monoidal while the pull-back and push-forward functors for the morphism $X_{S'} \to X_S$ are compatible with the module category structure. In particular a (weak) geometric action of a group Γ on X over S yields weak actions of Γ on $DG\mathcal{C}oh(X_{S'})$ for any $S' \to S$. This way our action of $\mathbb{B}_{\mathrm{aff}}$ yields some actions considered by other authors, see Remark 1.3.3.

The geometric actions of $\mathbb{B}_{\mathrm{aff}}$ on $\tilde{\mathfrak{g}}$, $\widetilde{\mathcal{N}}$ from Theorem 1.1.1, actually lift to geometric actions over $\mathfrak{g}$ and this provides a rich supply of interesting base changes of the action. Rather than spelling out the details on geometric actions over base $\mathfrak{g}$ we will here record a collection of actions of $\mathbb{B}_{\mathrm{aff}}$ on the derived categories of a class of *exact* base change varieties and the compatibilities they enjoy. The exactness condition on the base change $S \to \mathfrak{g}$ guarantees that $\tilde{\mathfrak{g}}_S$ (or $\widetilde{\mathcal{N}}_S$) is an ordinary scheme rather than a DG-scheme. It excludes some examples natural from representation-theoretic perspective, see Remark 1.5.4, but is still sufficient for our present purposes.

1.3. $\mathbb{B}_{\mathbf{aff}}$ **actions on exact base changes.** We say that a fiber product $X_1 \times_Y X_2$ is *exact* if

$$Tor^{\mathcal{O}_Y}_{>0}(\mathcal{O}_{X_1}, \mathcal{O}_{X_2}) = 0. \tag{1}$$

(We also say that the base change $X_1 \to X$ of X_2 is exact.) We let $\mathcal{BC}$ (respectively, $\mathcal{BC}'$) denote the category of affine Noetherian $\mathfrak{g}$ schemes $S \to \mathfrak{g}$ such that the base change of $\tilde{\mathfrak{g}}$ (respectively, $\widetilde{\mathcal{N}}$) to S is exact. We set $\widetilde{S} = S \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ and $\widetilde{S}' = S \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$. [3]

1.3.1. *Lemma.* Base changes $\widetilde{S}$, $\widetilde{S}'$, $\Gamma_{s_\alpha} \times_{\mathfrak{g}} S$ and $\Gamma'_{s_\alpha} \times_{\mathfrak{g}} S$ are exact for the following maps $S \to \mathfrak{g}$: (i) $\mathfrak{g}_{\mathcal{R}}$ for any Noetherian R-scheme $\mathcal{R}$; (ii) the spectrum $\widehat{\mathfrak{X}}$ of a completion of $\mathcal{O}_{\mathfrak{g}_{\mathcal{R}}}$ at any closed $\mathfrak{X} \subseteq \mathfrak{g}_{\mathcal{R}}$, (iii) any normal slice $S \subset \mathfrak{g}_{\mathcal{R}}$ to a nilpotent orbit in $\mathfrak{g}_{\mathcal{R}}$.

Proof. (iii) The conjugation map $G \times S \to \mathfrak{g}$ is smooth since the differential in the direction of G produces orbital directions and the one in direction of S produces the normal directions. □

1.3.2. *Action of braid group on base changes.* We will use the action of $G \times \mathbb{G}_m$ on $\mathfrak{g}$ (and all related objects), where $\mathbb{G}_m$ acts on $\mathfrak{g}$ by dilations.

[3] It may be possible to treat the two cases uniformly by considering also base changes with respect to the morphism $\tilde{\mathfrak{g}} \to \mathfrak{h}$. We do not develop this approach here.

Theorem. Let $\mathcal{G}$ be a group with a fixed homomorphism to $G\times\mathbb{G}_m$.

a) Let S be a scheme with a $\mathcal{G}$ action and $S\to\mathfrak{g}$ be a $\mathcal{G}$-equivariant affine map. If it is in $\mathcal{BC}$ (respectively, in $\mathcal{BC}'$), then the category $D^b(\mathcal{C}oh^{\mathcal{G}}(\widetilde{S}))$ (respectively, $D^b(\mathcal{C}oh^{\mathcal{G}}(\widetilde{S}'))$), carries a canonical weak action of $\mathbb{B}_{\mathrm{aff}}$ such that

- (i) For a finite simple reflection $s_\alpha\in W$ the generator $\widetilde{s_\alpha}$ acts by convolution with $\mathcal{O}_{\Gamma_{s_\alpha}\times_{\mathfrak{g}}S}$, respectively $\mathcal{O}_{\Gamma'_{s_\alpha}\times_{\mathfrak{g}}S}$, *provided* that the fiber product $\Gamma_{s_\alpha}\times_{\mathfrak{g}}S$, respectively $\Gamma'_{s_\alpha}\times_{\mathfrak{g}}S$, is exact.
- (ii) The generators θ_λ, $\lambda\in\Lambda$ act by tensoring with the line bundle $\mathcal{O}(\lambda)$.

For a $\mathcal{G}$-morphism $S_1\to S_2$ in $\mathcal{BC}$ (respectively, $\mathcal{BC}'$), the pull-back and push-forward functors are compatible with the $\mathbb{B}_{\mathrm{aff}}$ action. The change of equivariance functors for $\mathcal{G}'\to\mathcal{G}$ commute with the $\mathbb{B}_{\mathrm{aff}}$ action.

b) Let $\Bbbk$ be an algebraically closed field of characteristic zero or $p>h$ and $e\in\mathfrak{g}^*_{\Bbbk}$ be a nilpotent element. If the group $\mathcal{G}$ fixes e, then the induced action of $\mathbb{B}_{\mathrm{aff}}$ on $K^0(Coh^{\mathcal{G}}_{\mathcal{B}_{\Bbbk,e}}(\tilde{\mathfrak{g}}_{\Bbbk}))=K^0(Coh^{\mathcal{G}}_{\mathcal{B}_{\Bbbk,e}}(\widetilde{\mathcal{N}}_{\Bbbk}))=K^{\mathcal{G}}(\mathcal{B}_{\Bbbk,e})$ factors through the standard action of the affine Hecke algebra [Lu] in the following way.

- (i) For a finite simple reflection s_α, the action of $\widetilde{s_\alpha}$ on the K-group of $\mathcal{B}_{\Bbbk,e}$ is by

$$\widetilde{s_\alpha} \;=\; (-v)^{-1}T_{s_\alpha}, \tag{2}$$

  where T_{s_α} is the action (from [Lu]), of the Hecke algebra on the K-group.
- (ii) For $\lambda\in\Lambda$ the action of $\theta_\lambda\in\mathbb{B}_{\mathrm{aff}}$ is compatible with the action of θ_λ in the affine Hecke algebra defined in [Lu].

In particular, under the Chern character map[4] $K^0(Coh_{\mathcal{B}_{\Bbbk,e}}(\tilde{\mathfrak{g}}_{\Bbbk}))\to H_*(\mathcal{B}_{\Bbbk,e})$, the action of $\mathbb{B}\subseteq\mathbb{B}_{\mathrm{aff}}$ on the source factors through the Springer representation of W on the target.

Remark. Notice that the statement involving $\widetilde{\mathcal{N}}$ is *not* a particular case of the statement about $\tilde{\mathfrak{g}}$, because (in particular), the fiber product $\mathcal{N}\times_{\mathfrak{g}}\tilde{\mathfrak{g}}$ is not reduced, so is not isomorphic to $\widetilde{\mathcal{N}}=(\mathcal{N}\times_{\mathfrak{g}}\tilde{\mathfrak{g}})^{red}$ as a scheme.

[4]Here by homology we mean l-adic homology ($l\neq\mathrm{char}(\Bbbk)$), or the classical homology with rational coefficients if $\Bbbk=\mathbb{C}$.

1.3.3. *Examples.* (1) When $S \subset \mathfrak{g}_{\mathbb{C}}$ is the slice to the subregular orbit, then $\widetilde{S}'$ is the minimal resolution of a Kleinian singularity. The $\mathbb{B}_{\text{aff}}$ action in this case is generated by reflections at spherical objects, (see [Br] or references therein).

(2) Let us notice a relation to an action on coherent sheaves on affine Grassmannians.

Let S be a normal slice to a nilpotent e_n in $sl(2n)$, with two equal Jordan blocks. Then by the result of [Anno] the restriction to $\mathbb{B}\subseteq\mathbb{B}_{\text{aff}}$ of the above action on $\widetilde{S}'$, coincides with the action constructed by Cautis and Kamnitzer [CK] (up to a possible change of normalization).

1.3.4. *Some properties of the action.* Let $b \to \Pi(b)$ denote the composed map $\mathbb{B}_{\text{aff}} \to W_{\text{aff}} \to W_{\text{aff}}/\Lambda = W$. Let $i^{\Delta} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ and $pr : \tilde{\mathfrak{g}} \to \mathfrak{h}$ be the diagonal embedding and the projection.

Lemma. a) For $\mathcal{F} \in D^b(Coh(\mathfrak{h}))$ and $b \in \mathbb{B}_{\text{aff}}$ we have

$$K_b * i^{\Delta}_* pr^*(\mathcal{F}) * K_{b^{-1}} \cong i^{\Delta}_* pr^*(\Pi(b)_*\mathcal{F}).$$

b) For $\alpha \in I$, $K_{\widetilde{s_\alpha}^{-1}} \cong \Omega^{top}_{\Gamma_{s_\alpha}} \cong \mathcal{O}_{\Gamma_{s_\alpha}}\langle -\rho, -\alpha + \rho\rangle$.

c) $\widetilde{w}(\mathcal{O}) \cong \mathcal{O}$ for $w \in W$.

Proof. a) It suffices to construct the isomorphism for the generators of $\mathbb{B}_{\text{aff}}$. These isomorphisms come from the fact that K_{θ_λ} and $K_{\widetilde{s_\alpha}}$ are supported on the preimage under the map $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \to \mathfrak{h} \times_{\mathfrak{h}/W} \mathfrak{h}$ of, respectively, the diagonal and the graph of s_α.

b) is proved in [Ri1].

c) This reduces immediately to the case of SL_2 where it follows from the description of Γ_{s_α} as the blow up of $\tilde{\mathfrak{g}}$ along the zero section $\mathcal{B}$. □

1.4. **Certain classes of t-structures on coherent sheaves.**

1.4.1. *Braid positive and exotic t-structures on $T^*(G/B)$.* A t-structure on $D^b(\mathcal{C}oh(\widetilde{S}))$, is called *braid positive* if for any vertex α of the affine Dynkin graph of the dual group the action of $\widetilde{s_\alpha} \in \mathbb{B}_{\text{aff}}$ is right exact. It is called *exotic* if it is braid positive and also the functor of direct image to S is exact with respect to this t-structure on $D^b(\mathcal{C}oh(\widetilde{S}))$ and ordinary t-structure on $D^b(\mathcal{C}oh(S))$.

1.4.2. *Locally free t-structures.* Here we isolate a class of t-structures which admit certain simple construction. One advantage is that such t-structures can be pulled-back under reasonable base changes (say, base changes that are affine and exact, see lemma 2.5.2.a).

For a map of Noetherian schemes $f : X \to S$ we will say that a coherent sheaf $\mathcal{E}$ on X is a (relative) *tilting generator* if the functor from $D^b(\mathcal{C}oh(X))$ to $D^b(\mathcal{C}oh(f_*\mathcal{E}nd(\mathcal{E})^{op}))$ given by $\mathcal{F} \mapsto \mathrm{R}f_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ is an equivalence. This in particular implies that $f_*\mathcal{E}nd(\mathcal{E})$ is a coherent sheaf of rings, that the functor lands in the bounded derived category of coherent modules, and that $\mathrm{R}f_*\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{E}) = f_*\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{E})$.

If $\mathcal{E} \in D^b(\mathcal{C}oh(X))$ is a relative tilting generator, then the tautological t-structure on the derived category $D^b[\mathcal{C}oh(f_*(\mathcal{E}nd(\mathcal{E})^{op}))]$ induces a t-structure $\mathcal{T}_\mathcal{E}$ on $D^b(\mathcal{C}oh(X))$. We call it the $\mathcal{E}$ t-structure. We say that a t-structure is *locally free over* S if it is of the form $\mathcal{T}_\mathcal{E}$ where the relative tilting generator $\mathcal{E}$ is a vector bundle. Then $\mathcal{T}_\mathcal{E}$ is given by

$$\mathcal{F} \in D^{\geq 0} \iff \mathrm{R}f_*\mathrm{R}\mathcal{H}\mathrm{om}^{<0}(\mathcal{E}, \mathcal{F}) = 0 \quad \text{and} \quad \mathcal{F} \in D^{\leq 0} \iff \mathrm{R}f_*\mathrm{R}\mathcal{H}\mathrm{om}^{>0}(\mathcal{E}, \mathcal{F}) = 0.$$

If S is affine we omit "relative" and say that $\mathcal{E}$ is a tilting generator of $D^b[\mathcal{C}oh(X)]$. Then $\mathcal{E}$ is a projective generator for the heart of $\mathcal{T}_\mathcal{E}$. In particular, two tilting generators $\mathcal{E}$, $\mathcal{E}'$ define the same t-structure iff they are *equiconstituted*, where two objects M_1, M_2 of an additive category are called *equiconstituted* if for $k = 1, 2$ we have $M_k \cong \bigoplus N_i^{\oplus d_k^i}$ for some N_i and $d_k^i > 0$.

1.4.3. *Weak generators and tilting generators.* We say that an object X of a triangulated category $\mathcal{D}$ is a *weak generator* if $X^\perp = 0$, i.e. if $\mathrm{Hom}^\bullet(X, S) = 0 \Rightarrow S = 0$.

For future reference we recall the following

Theorem. [HvdB, Thm 7.6][5] Assume that the scheme X is projective over an affine Noetherian scheme. Then a coherent sheaf $\mathcal{E}$ is a tilting generator if and only if $\mathcal{E}$ is a weak generator for $D(q\mathcal{C}oh(X))$ and it is a quasi-exceptional object, i.e., $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$ for $i \neq 0$.

1.5. **Exotic t-structures and noncommutative Springer resolution.**

[5] In *loc. cit.* this statement is stated under the running assumption that the scheme is of finite type over $\mathbb{C}$. However, the same proof works in the present generality.

1.5.1. *Theorem.* Let $S \to \mathfrak{g}$ be an exact base change of $\widetilde{\mathfrak{g}}$ (resp. of $\widetilde{\mathcal{N}}$) with affine Noetherian S.

a) There exists a unique exotic t-structure $\mathcal{T}^{ex}_S$ on the derived category of coherent sheaves on $\widetilde{S}$ (resp $\widetilde{S}'$). It is given by:

$$D^{\geq 0,ex} = \{\mathcal{F} \ ; \ \mathrm{R}\,pr_*(b^{-1}\mathcal{F}) \in D^{\geq 0}(\mathcal{C}oh(S)) \ \forall b \in \mathbb{B}^+_{\mathrm{aff}}\};$$

$$D^{\leq 0,ex} = \{\mathcal{F} \ ; \ \mathrm{R}\,pr_*(b\mathcal{F}) \in D^{\leq 0}(\mathcal{C}oh(S)) \ \forall b \in \mathbb{B}^+_{\mathrm{aff}}\}.$$

b) This t-structure is locally free over S. In fact, there exists a $G \times \mathbb{G}_m$-equivariant vector bundle $\mathcal{E}$ on $\tilde{\mathfrak{g}}$ such that for any S as above, its pull-back $\mathcal{E}_S$ to $\widetilde{S}$ (resp. $\mathcal{E}'_S$ to $\widetilde{S}'$), is a tilting generator over S, and the corresponding t-structure is the exotic structure $\mathcal{T}^{ex}$. In particular, the pull-back $\mathcal{E}_S$ is a projective generator of the heart of $\mathcal{T}^{ex}_S$.

Proof. In proposition 2.2.1 we check that, for S as above, any exotic t-structure satisfies the description from (a). This proves uniqueness. The existence of a vector bundle $\mathcal{E}$ on $\tilde{\mathfrak{g}}$ whose pull backs produce exotic t-structures for any S as above is proved in 2.5.5. □

We denote the heart of the exotic t-structure $\mathcal{T}^{ex}_S$ by $\mathcal{E}coh(\widetilde{S})$ (resp. $\mathcal{E}coh(\widetilde{S}')$).

1.5.2. *Remark.* While the definition of an exotic t-structure involves only the *non-extended* affine Weyl group W^{Cox}_{aff}, the Theorem shows that the same property – the right exactness of the canonical lifts $\widetilde{w} \in \mathbb{B}_{\mathrm{aff}}$ – also holds for all w in the extended affine Weyl group W_{aff}. In particular, the stabilizer Ω of the fundamental alcove in W_{aff} acts by t-exact automorphisms of $D^b(\mathcal{C}oh(\widetilde{S}))$ (this is an abelian group $\Omega \cong \Lambda/Q \cong W_{\mathrm{aff}}/W^{Cox}_{\mathrm{aff}} \cong \pi_1(\check{G})$).

1.5.3. *Algebras A and A^0.* The exotic t-structures described in Theorem 1.5.1 can also be recorded as follows. For any S as in the Theorem we get an associative algebra $A_S \stackrel{\mathrm{def}}{=} End(\mathcal{E}_S)^{op}$ (respectively, $A^0_S \stackrel{\mathrm{def}}{=} End(\mathcal{E}'_S)^{op}$), together with an equivalence $D^b[Coh(\tilde{\mathfrak{g}}_S)] \cong D^b[mod^{fg}(A_S)]$ (respectively, $D^b[Coh(\widetilde{\mathcal{N}}_S)] \cong D^b[mod^{fg}(A^0_S)]$), sending $\mathcal{E}_S$ to the free rank one module. It is clear that the algebra together with the equivalence of derived categories determines the t-structure, while the t-structure determines the algebra uniquely up to a Morita equivalence.

According to the terminology of, say, [BoOr], the noncommutative $\mathcal{O}(\mathcal{N})$-algebra $A^0 = \mathrm{End}(\mathcal{E}|_{\widetilde{\mathcal{N}}})$ is a *noncommutative resolution* of singularities of the singular affine algebraic

variety $\mathcal{N}$, while $A = \mathrm{End}(\mathcal{E})$ is a noncommutative resolution of the affinization $\mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h}$ of $\tilde{\mathfrak{g}}$. In view of its close relation to Springer resolution, we call A^0 a *noncommutative Springer resolution*, while A will be called a *noncommutative Grothendieck resolution* of $\mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h}$, cf. [B3].

For future reference we record some properties of the algebra A that directly follow from the Theorem.

Lemma. (a) A is a vector bundle and a Frobenius algebra over $\mathfrak{g}$.

(b) $A^0 \stackrel{\mathrm{def}}{=} \mathrm{End}(\mathcal{E}|_{\widetilde{\mathcal{N}}})$ is a base change $A^0 \cong A\overset{L}{\otimes}_{\mathcal{O}(\mathfrak{h})}\mathcal{O}_0$ of $A = \mathrm{End}(\mathcal{E})$ in the direction of $\mathfrak{h}$.

(c) Algebras associated to base changes $S \to \mathfrak{g}$ are themselves base changes in the direction of $\mathfrak{g}$: $A_S \cong A \otimes_{\mathcal{O}(\mathfrak{g})} \mathcal{O}(S)$ (resp. $A^0_S \cong A^0 \otimes_{\mathcal{O}(\mathfrak{g})} \mathcal{O}(S)$).

Proof. (a) The sheaf of algebras $\mathcal{A} = \mathcal{E}nd(\mathcal{E})$ is Frobenius for the trace functional tr. Since $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ are Calabi-Yau and have the same dimension, Grothendieck duality implies that the sheaf $A = (\tilde{\mathfrak{g}} \stackrel{\pi}{\to} \mathfrak{g})_*\mathcal{A}$ is self-dual. In particular, it is a Cohen-Macaulay sheaf, so since $\mathfrak{g}$ is smooth this implies that it is a vector bundle. Moreover, since $\tilde{\mathfrak{g}}$ is finite and flat over the regular locus $\mathfrak{g}_r \subseteq \mathfrak{g}$, Frobenius structure tr on $\mathcal{A}$ induces a Frobenius structure on $A|_{\mathfrak{g}_r}$. Now, since the complement is of codimension three, this extends to a Frobenius structure on the algebra bundle A.

For (b) we have

$$A^0 \stackrel{(1)}{=} \mathrm{R}\Gamma(\mathcal{A}|_{\widetilde{\mathcal{N}}}) \stackrel{(2)}{=} \mathrm{R}\Gamma(\mathcal{A}\overset{L}{\otimes}_{\mathcal{O}(\mathfrak{h})} \mathcal{O}_0) = \mathrm{R}\Gamma(\mathcal{A})\overset{L}{\otimes}_{\mathcal{O}(\mathfrak{h})} \mathcal{O}_0 \stackrel{(3)}{=} A\overset{L}{\otimes}_{\mathcal{O}(\mathfrak{h})} \mathcal{O}_0,$$

here vanishing statements (1) and (3) come from $\mathcal{E}$ and $\mathcal{E}|_{\widetilde{\mathcal{N}}}$ being tilting generators, while (2) follows from $\widetilde{\mathcal{N}} = \tilde{\mathfrak{g}}\times_{\mathfrak{h}} 0$ and flatness of $\tilde{\mathfrak{g}} \to \mathfrak{h}$.

(c) follows from base change isomorphisms $\mathrm{R}\Gamma(\mathcal{E}nd(\mathcal{E}_S)) \cong \mathcal{O}(S) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}(\mathfrak{g})} A$ (and similarly for S'), which follow from the exactness assumption on base change to S. Since the space in the right hand side belongs to $D^{\leq 0}$ while the space in the left hand side lies in $D^{\geq 0}$, both in fact lie in homological degree zero and the above isomorphisms hold.

1.5.4. *Remark on DG-version of the theorem.* We have required exactness of base change to avoid dealing with DG-schemes. Using some basic elements of that formalism one can derive the following generalization of Theorems 1.3.2 and 1.5.1. Let S be an arbitrary

affine Noetherian scheme equipped with a morphism $S \to \mathfrak{g}$. First, $\mathbb{B}_{\text{aff}}$ acts naturally on the triangulated category $DG\mathcal{C}oh(\tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} S)$ of differential graded coherent sheaves on the derived fiber product $DG\mathcal{C}oh(\tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} S)$. Then this allows us to extend the definition of an exotic t-structure to that context.

Finally, let $\mathcal{E}$ be as in the theorem and $A = End(\mathcal{E})$ as above, then we have an equivalence of triangulated categories (the tensor product $A \otimes_{\mathcal{O}_{\mathfrak{g}}} \mathcal{O}(S)$ does not have to be derived since A is flat over $\mathcal{O}(\mathfrak{g})$ by Lemma 1.5.3) :

$$DG\mathcal{C}oh(\tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} S) \;\cong\; D^b(mod^{fg}[A \otimes_{\mathcal{O}_{\mathfrak{g}}} \mathcal{O}(S)]).$$

The t-structure on $DG\mathcal{C}oh(\tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} S)$ corresponding to the tautological t-structure on $D^b(mod^{fg}[A \otimes_{\mathcal{O}_{\mathfrak{g}}} \mathcal{O}(S)])$ is the unique exotic t-structure.

In particular, when $S = \{e\}$ is a $\mathbb{k}$ point of $\mathfrak{g}$ where $\mathbb{k}$ is an algebraically closed field of characteristic $p > h$, then the category $mod^{fg}[A \otimes_{\mathcal{O}_{\mathfrak{g}}} \mathcal{O}_e]$ is identified with a regular block in the category of $U_{\mathbb{k},e}$-modules, where the subscript denotes reduction of $U_{\mathbb{k}}$ by the corresponding maximal ideal in the Frobenius center. The category $DG\mathcal{C}oh(\tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} e)$ of coherent sheaves on the DG Springer fiber is studied in [Ri2].

1.6. **Representation theoretic t-structures on derived categories of coherent sheaves.** We now record a particular case of Theorem 1.5.1 that follows from the results of [BMR1], [BMR2]. In the next section we will deduce the general case from this particular case.

Fix $\mathbb{k} \in \mathrm{FGP}$ and a nilpotent $e \in \mathcal{N}(\mathbb{k})$.

1.6.1. *The center of* $U\mathfrak{g}_{\mathbb{k}}$. The description of the center of enveloping algebra in characteristic $p > h$ is $Z(U\mathfrak{g}_{\mathbb{k}}) \cong \mathcal{O}_{\mathfrak{g}_{\mathbb{k}}^{*(1)} \times_{\mathfrak{h}_{\mathbb{k}}^{*}/W^{(1)}} \mathfrak{h}_{\mathbb{k}}^{*}/W}$, where $X^{(1)}$ denotes the Frobenius twist of a $\mathbb{k}$-scheme X and the map $\mathfrak{h}_{\mathbb{k}}^{*} \to \mathfrak{h}_{\mathbb{k}}^{*(1)}$ is Artin-Schreier map [BMR1]. When X is one of $\mathfrak{g}_{\mathbb{k}}, \mathfrak{h}_{\mathbb{k}}$ we can use the canonical $\mathbb{F}_p$-rational structure to identify $\mathbb{k}$-scheme $X^{(1)}$ with X. This gives isomorphism $\mathfrak{g}_{\mathbb{k}}^{*(1)} \times_{\mathfrak{h}_{\mathbb{k}}^{*}/W^{(1)}} \mathfrak{h}_{\mathbb{k}}^{*}/W \cong \mathfrak{g}_{\mathbb{k}}^{*} \times_{\mathfrak{h}_{\mathbb{k}}^{*}/W} \mathfrak{h}_{\mathbb{k}}^{*}/W \cong \mathfrak{g}_{\mathbb{k}} \times_{\mathfrak{h}_{\mathbb{k}}/W} \mathfrak{h}_{\mathbb{k}}/W$.

A compatible pair of $e \in \mathfrak{g}_{\mathbb{k}}$ and $\lambda \in \mathfrak{h}_{\mathbb{k}}$, gives a central character of $U_{\mathbb{k}}$ which we can then (partially) impose on $U_{\mathbb{k}}$ or $mod(U_{\mathbb{k}})$. We denote $U_{\mathbb{k},e}^{\lambda} = U_{\mathbb{k}} \otimes_{Z(U_{\mathbb{k}})} \mathbb{k}_{\lambda,e}$, while $U_{\mathbb{k},\widehat{e}}^{\widehat{\lambda}}$ is the completion of $\mathrm{U}_{\mathbb{k}}$ at (λ, e) and $mod_e^{\lambda}(U_{\mathbb{k}})$ is the category of modules with generalized character (λ, e). Similarly, we get $U_{\mathbb{k}}^{\lambda}$ or $U_{\mathbb{k},e}$ by imposing a central character

in one direction only and we may combine these ideas in various ways to get objects like $mod^{\lambda}(U_{\Bbbk,\widehat{e}})$.

1.6.2. *Representation theoretic t-structures.* By the main result of [BMR1], for integral regular λ we have canonical equivalence of categories of $\mathfrak{g}$-modules and coherent sheaves[6]

$$\begin{array}{ccc} D^b(\mathcal{C}oh(\widehat{\mathcal{B}_{\Bbbk,e}})) & \xrightarrow[\cong]{} & D^b[mod^{fg}(\mathrm{U}^{\widehat{\lambda}}_{\Bbbk,\widehat{e}})] \\ \subseteq\uparrow & & \subseteq\uparrow \\ D^b(\mathcal{C}oh_{\mathcal{B}_{\Bbbk,e}}(\widetilde{\mathfrak{g}}_{\Bbbk})) & \xrightarrow[\cong]{} & D^b[mod^{fg,\lambda}_{e}(\mathrm{U}_{\Bbbk})] \end{array} \quad \text{and} \quad \begin{array}{ccc} D^b(\mathcal{C}oh(\widehat{\mathcal{B}_{\Bbbk,e}}')) & \xrightarrow[\cong]{} & D^b[mod^{fg}(\mathrm{U}^{\lambda}_{\Bbbk,\widehat{e}})] \\ \subseteq\uparrow & & \subseteq\uparrow \\ D^b(\mathcal{C}oh_{\mathcal{B}_{\Bbbk,e}}(\widetilde{\mathcal{N}}_{\Bbbk})) & \xrightarrow[\cong]{} & D^b[mod^{fg}_{e}(\mathrm{U}^{\lambda}_{\Bbbk})] \end{array}.$$

(Recall that index $\mathcal{B}_{\Bbbk,e}$ means sheaves set-theoretically supported on $\mathcal{B}_{\Bbbk,e}$.) Here, the second line is Theorem 5.4.1 in [BMR1], the first line is stated in the footnote on the same page.

These equivalences provide each of the derived categories of coherent sheaves with a t-structure – the image of the tautological t-structure on the derived category of modules. According to lemma 6.1.2.a in [BMR1], this t-structure depends only on the alcove to which $\frac{\lambda+\rho}{p}$ belongs, not on λ itself.

We call the t-structure obtained from λ such that $\frac{\lambda+\rho}{p}$ is in the fundamental alcove (e.g. $\lambda = 0$), the *representation theoretic t-structure* on the derived category of coherent sheaves (RT t-structure for short).

Here by an *alcove* we mean a connected component of the complement to the affine coroot hyperplanes $H_{\check{\alpha},n} = \{\lambda | \langle \check{\alpha}, \lambda \rangle = n\}$, in the dual space $\mathfrak{h}^*_{\mathbb{R}}$ to the real Cartan algebra $\mathfrak{h}_{\mathbb{R}}$; here $\check{\alpha}$ runs over the set of coroots and $n \in \mathbb{Z}$. The *fundamental alcove* $\mathfrak{A}_0$ is the locus of points where all positive coroots take values between zero and one. Let Alc be the set of alcoves.

1.6.3. *Theorem. For any $\Bbbk \in FGP$ and $e \in \mathcal{N}(\Bbbk)$ the RT t-structure on $D^b[\mathcal{C}oh_{\mathcal{B}_{\Bbbk,e}}(\tilde{\mathfrak{g}}_{\Bbbk})]$, $D^b[\mathcal{C}oh_{\mathcal{B}_{\Bbbk,e}}(\widetilde{\mathcal{N}}_{\Bbbk})]$ is exotic. Therefore, for $\lambda \in \Lambda$ such that $\frac{\lambda+\rho}{p} \in \mathfrak{A}_0$, there are canonical equivalences of categories*

$$mod^{fg}(U^{\widehat{\lambda}}_{\widehat{e}}) \cong \mathcal{E}coh(\widehat{\mathcal{B}_{\Bbbk,e}}) \quad \textit{and} \quad mod^{fg}(U^{\lambda}_{\widehat{e}}) \cong \mathcal{E}coh(\widehat{\mathcal{B}_{\Bbbk,e}}').$$

[6] A priori such equivalences require a choice of a splitting bundle for certain Azumaya algebra, by "canonical equivalences" we mean that we use the *standard* splitting bundle from [BMR1, Remark 5.2.2.2]. Also, we are suppressing Frobenius twist $X^{(1)}$ from the notation using identifications $X^{(1)} \cong X$ that are available when X is defined over the prime subfield.

The proof is based on

1.6.4. *Proposition. The equivalence of Theorem 5.4.1 of* [BMR1] *is compatible with the* $\mathbb{B}_{aff}$ *action.*

Proof follows from [Ri1, Section 5]. □

1.6.5. *Proof of Theorem 1.6.3.* In [BMR2, 2.2.1] the action $\mathbf{I}^*_\alpha$ of Coxeter generators $\widetilde{s_\alpha}$ of $\mathbb{B}_{\text{aff}}$ is defined through a canonical distinguished triangle

$$M \to R_\alpha(M) \to \widetilde{s_\alpha}(M), \quad M \in D^b(mod^{0,fg}(\mathrm{U}_\Bbbk)),$$

where R_α is the so called *reflection functor*. Thus exactness of R_α implies that $\widetilde{s_\alpha}$ acts by right exact functors. Also, we have a commutative diagram [BMR2, Lemma 2.2.5]:

$$\begin{array}{ccc} D^b(mod_e^{fg}(U_\Bbbk^0) & \longrightarrow & D^b[\mathcal{C}oh_{\mathcal{B}_{\Bbbk,e}}(\widetilde{\mathcal{N}}_\Bbbk)] \\ {\scriptstyle T_0^{-\rho}}\downarrow & & \downarrow{\scriptstyle \mathrm{R}\Gamma} \\ D^b(mod_e^{fg}(U_\Bbbk^{-\rho})) & \longrightarrow & D^b[\mathcal{C}oh_e(\mathcal{N}_\Bbbk)] \end{array}$$

Here the horizontal arrows are localization equivalences, and $T_0^{-\rho}$ is the translation functor. Thus exactness of $T_0^{-\rho}$ implies that the RT t-structure satisfies the normalization requirement in the definition of an exotic t-structure. □

1.6.6. *Equivariant version of representation theoretic t-structures.* We will also need an equivariant version of localization Theorem of [BMR1] and its relation to exotic t-structures.

Let $\Bbbk, e, \lambda$ be as in Theorem 1.6.3 and let C be a torus with a fixed map to the centralizer of e in G.

Recall a traditional enhancement of $mod(U_e^\lambda)$. Since e vanishes on the image of $\mathfrak{c} = Lie(C)$ in $\mathfrak{g}$, the action of $\mathfrak{c}$ on any object of $mod(U_e^\lambda)$ has zero p-character. The category of restricted $\mathfrak{c}$-modules is semi-simple with simple objects indexed by $\mathfrak{c}^*(\mathbb{F}_p) = X^*(C)/p$; thus every $M \in mod(U_e^\lambda)$ carries a canonical grading by $X^*(C)/p$. One considers the category $mod_{gr}(U_e^\lambda)$ whose object is an U_e^λ module together with a grading by $X^*(C)$. The grading should be compatible with the natural $X^*(C)$ grading on U_e^λ and the induced $X^*(C)/p$ grading should coincide with the above canonical one.

The goal of this subsection is to describe a geometric realization for $mod_{gr}(U_e^\lambda)$. To simplify the statement of the derived equivalences we need to enlarge the category (without changing the set of irreducible objects nor the Grothendieck group).

Define the categories $mod^{fg}(U_{\widehat{e}}^{\widehat{\lambda}}, C)$, $mod^{fg}(U_{\widehat{e}}^{\lambda}, C)$ as follows. An object M of $mod^{fg}(U_{\widehat{e}}^{\widehat{\lambda}}, C)$ (respectively, $mod^{fg}(U_{\widehat{e}}^{\lambda}, C)$) is an object of $mod^{fg}(U_{\widehat{e}}^{\widehat{\lambda}})$ (respectively, $mod^{fg}(U_{\widehat{e}}^{\lambda})$) together with a C action (equivalently, an $X^*(C)$ grading) such that:

(i) The action map $U \to End(M)$ is C-equivariant.

(ii) Consider the two actions of $\mathfrak{c}$ on M: the derivative α_C of the C action and the composition $\alpha_{\mathfrak{g}}$ of the maps $\mathfrak{c} \to \mathfrak{g} \to End(M)$. We require that the operator $\alpha_{\mathfrak{g}}(x) - \alpha_C(x)$ is nilpotent for all $x \in \mathfrak{c}$.

Notice that the actions $\alpha_{\mathfrak{g}}$ and α_C commute; moreover, condition (i) implies that the difference $\alpha_{\mathfrak{g}}(x) - \alpha_C(x)$ commutes with the action of $\mathfrak{g}$.

Also, if $M \in mod^{fg}(U_e^{\widehat{\lambda}})$ then the action $\alpha_{\mathfrak{g}}$ is semi-simple, thus in this case conditions (i,ii) above imply that $\alpha_{\mathfrak{g}} = \alpha_C$ and $M \in mod_{gr}^{fg}(U_{\widehat{e}}^{\widehat{\lambda}})$. This applies in particular when M is irreducible.

For future reference we mention also that one can consider the categories $mod^{C,fg}(U_{\widehat{e}}^{\widehat{\lambda}})$, $mod^{C,fg}(U_{\widehat{e}}^{\lambda})$ of modules equipped with a C action subject to the condition (i) above only. A finite dimensional module M in one of these categories splits as a direct sum $M = \bigoplus_{\eta \in \mathfrak{c}^*} M_\eta$ of generalized eigenspaces of operators $\alpha_C(x) - \alpha_{\mathfrak{g}}(x)$, $x \in \mathfrak{c}$; moreover, $M_\eta = 0$ unless $\eta \in \mathfrak{c}^*(\mathbb{F}_p)$.

For a general $M \in mod^{C,fg}(U_{\widehat{e}}^{\widehat{\lambda}})$ the quotient M_n of M by the n-th power of the maximal ideal in Z_{Fr} corresponding to e is finite dimensional. It is easy to see that the above decompositions for M_n for different n are compatible, thus we get a decomposition of the category

$$mod^{C,fg}(U_{\widehat{e}}^{\widehat{\lambda}}) = \bigoplus_{\eta \in \mathfrak{c}^*(\mathbb{F}_p)} mod_\eta^{C,fg}(U_{\widehat{e}}^{\widehat{\lambda}}), \tag{3}$$

and similarly for $mod^{C,fg}(U_{\widehat{e}}^{\lambda})$. Notice that $mod_0^{C,fg}(U_{\widehat{e}}^{\widehat{\lambda}}) = mod^{fg}(U_{\widehat{e}}^{\widehat{\lambda}}, C)$ and for $\widetilde{\eta} \in X^*(C)$ twisting the C-action by η gives an equivalence $mod_\theta^{C,fg}(U_{\widehat{e}}^{\widehat{\lambda}}) \cong mod_{\theta+\eta}^{C,fg}(U_{\widehat{e}}^{\widehat{\lambda}})$ where $\eta = \widetilde{\eta} \bmod p \cdot X^*(C)$.

1.6.7. *Theorem.* a) There exist compatible equivalences of triangulated categories

$$D^b[\mathcal{C}oh^C(\widehat{\mathcal{B}_{\Bbbk,e}})] \cong D^b[mod^{fg}(U_{\hat{e}}^{\hat{\lambda}}, C)] \quad \text{and} \quad D^b[\mathcal{C}oh^C(\widehat{\mathcal{B}_{\Bbbk,e}}')] \cong D^b[mod^{fg}(U_{\hat{e}}^{\lambda}, C)].$$

b) Under the functor of forgetting the equivariant structure, these equivariant equivalences are compatible with the equivalences of [BMR1] from 1.6.2.

c) The *representation theoretic t-structures* that these equivalences define on categories of coherent sheaves coincide with the exotic t-structures, so we have induced equivalences

$$mod^{fg}(U_{\hat{e}}^{\hat{\lambda}}, C) \cong \mathcal{E}coh^C(\widehat{\mathcal{B}_{\Bbbk,e}}) \quad \text{and} \quad mod^{fg}(U_{\hat{e}}^{\lambda}, C) \cong \mathcal{E}coh^C(\widehat{\mathcal{B}_{\Bbbk,e}}').$$

The equivalences will be constructed in 5.2.4. Compatibility with forgetting the equivariance will be clear from the construction, while compatibility with t-structures follows from compatibility with forgetting the equivariance.

1.6.8. *Splitting vector bundles.* Theorem 1.6.3 has a geometric consequence. Recall that for $\Bbbk \in \mathrm{FGP}$, $e \in \mathcal{N}_{\Bbbk}$ and $\lambda \in \Lambda$ the Azumaya algebra coming from λ-twisted differential operators splits on the formal neighborhood $\widehat{\mathcal{B}_{\Bbbk,e}}'$ of the Springer fiber $\mathcal{B}'_{\Bbbk,e}$ in $\widetilde{\mathcal{N}}_{\Bbbk}$. Let $\mathcal{E}_e^{spl}(\lambda)$ be the splitting vector bundle constructed in [BMR1, Remark 5.2.2.2] (the unramified shift of λ that we use is $-\rho$).

Corollary. When $\frac{\lambda+\rho}{p}$ lies in the fundamental alcove,[7] the splitting vector bundle $\mathcal{E}_e^{spl}(\lambda)$ does not depend on p up to equiconstitutedness. More precisely, there exists a vector bundle $\mathcal{V}$ on $\tilde{\mathfrak{g}}$, defined over R, whose base change to $\widehat{\mathcal{B}_{\Bbbk,e}}'$ is equiconstituted with $\mathcal{E}_e^{spl}(\lambda)$ for every $\Bbbk$, e, λ as above.

Proof. Take $\mathcal{E}$ as in Theorem 1.5.1 and $\mathcal{V} = \mathcal{E}^*$. Then, in view of Proposition 5.1.4, both the base change $\mathcal{E}_{\widehat{\mathcal{B}_{\Bbbk,e}}}$ and the dual of the splitting bundle $\mathcal{E}_e^{spl}(\lambda)^*$ are projective generators for $\mathcal{E}coh(\widehat{\mathcal{B}_{\Bbbk,e}})$, thus they are equiconstituted.[8] □

[7] The result can be generalized for an arbitrary alcove, see 1.8 below.

[8] The reason that $\mathcal{E}$ gets dualized is a difference of conventions. For a tilting generator $\mathcal{E}$ it is standard to use $\mathrm{RHom}(\mathcal{E}, -)$ to get to modules over an algebra, while for a splitting bundle V one uses $V \overset{L}{\otimes} -$. The first functor produces a rank one free module over the algebra when applied to $\mathcal{E}$ and the second when applied to V^*.

Example. In particular, $Fr_*(\mathcal{O}_{\mathcal{B}_{\Bbbk}})$ does not depend on $p > h$ up to equiconstitutedness. The reason is that for $e = 0$ and $\lambda = 0$ the splitting bundle for the restriction of $\mathcal{D}_{\mathcal{B}_{\Bbbk}}$ to $\widehat{\mathcal{B}_{\Bbbk,e}}'$ can be chosen so that its restriction to the zero section $\mathcal{B}_{\Bbbk}$ is $Fr_*(\mathcal{O}_{\mathcal{B}_{\Bbbk}})$ (cf. [BMR1]).

1.7. **Quantum groups, affine Lie algebras and exotic sheaves.** We finish the subsection by stating two conjectures on other appearances of the noncommutative Springer resolution (and therefore also of exotic sheaves) in representation theory. Let $e \in \mathcal{N}_{\mathbb{C}}$ and denote $A_e^0 \stackrel{\text{def}}{=} A_{\mathbb{C}}^0 \otimes_{\mathcal{O}(\mathcal{N}_{\mathbb{C}})} \mathcal{O}_e$.

Let U_ζ^{DK} be the *De Concini-Kac* form of the quantum enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ at a root of unity ζ of odd order $l > h$ [DK]. Recall that the center of U_ζ^{DK} contains a subalgebra Z_l, the so-called *l-center*. The spectrum of Z_l contains the intersection of the variety of unipotent elements in $G_{\mathbb{C}}$ with the big Bruhat cell B^+B^-. We identify the varieties of unipotent elements in $G_{\mathbb{C}}$ and nilpotent elements in $\mathfrak{g}_{\mathbb{C}}$. Fix $e \in \mathcal{N}(\mathbb{C})$ such that the corresponding unipotent element lies in the big cell.

1.7.1. *Conjecture.* The category $mod^{fg}(A_e^0)$ is equivalent to a regular block in the category of U_ζ^{DK}-modules with central character corresponding to e.

A possible way to approach this conjecture is by extending the techniques of the present paper to the framework of Backelin and Kremnizer [BK].

The next conjecture is motivated by the conjectures and results of [FG], see the Introduction.

1.7.2. *Conjecture.* [9] Fix a nilpotent $G_{\mathbb{C}}$-oper O on the formal punctured disc with residue $e \in \mathcal{N}(\mathbb{C})$. The category $\mathcal{A}_e$ of Iwahori-integrable modules over the affine Lie algebra $\widehat{\mathfrak{g}}_{\mathbb{C}}$ at the critical level with central character corresponding to O, is equivalent to the category of A_e^0-modules:

$$\mathcal{A}_e \;\cong\; mod(A_e^0).$$

1.8. **t-structures assigned to alcoves.** In 1.6.2 we used representation theory to attach to each alcove a collection of t-structures on formal neighborhoods of Springer fibers in characteristic $p > h$. The particular case of the fundamental alcove yields the exotic

[9] This Conjecture is proved in a forthcoming paper by Qian Lin.

t-structure (Theorem 1.6.3). The following generalization of Theorem 1.5.1 shows that all these t-structures can also be lifted to $\tilde{\mathfrak{g}}_R$ and $\widetilde{\mathcal{N}}_R$ and hence to zero characteristic.

For $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathrm{Alc}$ we will say that $\mathfrak{A}_1$ *lies above* $\mathfrak{A}_2$ if for any positive coroot $\check{\alpha}$ and $n \in \mathbb{Z}$, such that the affine hyperplane $H_{\check{\alpha},n} = \{\lambda, \ | \langle \check{\alpha}, \lambda \rangle = n\}$ separates $\mathfrak{A}_1$ and $\mathfrak{A}_2$, alcove $\mathfrak{A}_1$ lies above $H_{\alpha,n}$, while $\mathfrak{A}_2$ lies below $H_{\check{\alpha},n}$, in the sense that for $\lambda_i \in \mathfrak{A}_i$ we have $\langle \check{\alpha}, \lambda_2 \rangle < n < \langle \check{\alpha}, \lambda_1 \rangle$.

Recall the right action of W_{aff} on the set of alcoves, it will be denoted by $w : \mathfrak{A} \mapsto \mathfrak{A}^w$.

1.8.1. *Lemma.* a) There exists a unique map $\mathrm{Alc} \times \mathrm{Alc} \to B_{\mathrm{aff}}^{Cox} \subseteq \mathbb{B}_{\mathrm{aff}}$, $(\mathfrak{A}_1, \mathfrak{A}_2) \mapsto b_{\mathfrak{A}_1,\mathfrak{A}_2}$, such that

- i) $b_{\mathfrak{A}_1,\mathfrak{A}_2} b_{\mathfrak{A}_2,\mathfrak{A}_3} = b_{\mathfrak{A}_1\mathfrak{A}_3}$ for any $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \in \mathrm{Alc}$.
- ii) $b_{\mathfrak{A},\mathfrak{A}^w} = \widetilde{w}$ for $w \in W_{\mathrm{aff}}^{Cox}$ and $\mathfrak{A} \in \mathrm{Alc}$, provided that $\mathfrak{A}^w$ lies above $\mathfrak{A}$.

b) This map satisfies:

$$b_{\lambda+\mathfrak{A}_1,\lambda+\mathfrak{A}_2} = b_{\mathfrak{A}_1,\mathfrak{A}_2} \quad \text{for} \quad \lambda \in \Lambda,$$
$$b_{\mathfrak{A}_0,\lambda+\mathfrak{A}_0^{w^{-1}}} = \theta_\lambda \widetilde{w}^{-1} \quad \text{for} \quad w \in W,\ \lambda \in Q,$$

where $\lambda + \mathfrak{A}$ denotes the λ-shift of $\mathfrak{A}$.

c) The element $b_{\mathfrak{A}_1,\mathfrak{A}_2}$ admits the following topological description. Let $\mathfrak{h}^*_{\mathbb{C},reg} = \mathfrak{h}^*_{\mathbb{C}} \setminus \cup_{\check{\alpha},n} (H_{\check{\alpha},n})_{\mathbb{C}}$. Notice that a homotopy class of a path in $\mathfrak{h}^*_{\mathbb{C},reg}$ connecting two alcoves $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in $\mathfrak{h}^*_{\mathbb{R}}$, determines an element in $B_{\mathrm{aff}}^{Cox} = \pi_1(\mathfrak{h}^*_{\mathbb{C},reg}/W_{\mathrm{aff}}^{Cox}, \bullet)$, because each alcove is contractible, and alcoves are permuted transitively by W_{aff}^{Cox}, so they all give the same base point $\bullet$ in $\mathfrak{h}^*_{\mathbb{C},reg}/W_{\mathrm{aff}}^{Cox}$.

Let $\lambda \in \mathfrak{h}^*$ be a regular dominant weight. Then the subspace $-i\lambda + \mathfrak{h}^*_{\mathbb{R}}$ does not intersect any of the affine coroot hyperplanes in $\mathfrak{h}^*_{\mathbb{C}}$. For two points $x \in \mathfrak{A}_1$, $y \in \mathfrak{A}_2$ consider the path from x to y which is a composition of the following three paths: $t \mapsto x - it\lambda$ $(0 \leq t \leq 1)$, any path from $x - i\lambda$ to $y - i\lambda$ in $-i\lambda + \mathfrak{h}^*_{\mathbb{R}}$ and the path $t \mapsto y - i(1-t)\lambda$ $(0 \leq t \leq 1)$. Then $b_{\mathfrak{A}_1,\mathfrak{A}_2}$ is represented by this path.

Proof. Uniqueness in (a) is clear since for any two alcoves there exists an alcove which is above both of them. To check existence define $b_{\mathfrak{A}_1,\mathfrak{A}_2}$ as in part c). Then property (i) is clear. To see property (ii) it suffices to consider the case when $w = s_\alpha$ is a simple reflection. Then $\widetilde{s}_\alpha$ is represented by the loop which starts at the fundamental alcove $\mathfrak{A}_0$ and runs a half-circle (in a complex line given by the direction α) around the hyperplane of the affine

coroot α, in the positive (counterclockwise) direction and ending at $s_\alpha(\mathfrak{A}_0) = \mathfrak{A}_0^{s_\alpha}$. The element z of W_{aff}^{Cox} such that $z\mathfrak{A}_0 = \mathfrak{A}$ sends $\mathfrak{A}_0^{s_\alpha}$ to $\mathfrak{A}^{s_\alpha}$ while the simple affine coroot is sent to an affine linear functional taking positive values on $\mathfrak{A}$ and negative values on $\mathfrak{A}^{s_\alpha}$. Thus the two loops are manifestly homotopic.

b) The first property in (b) follows from uniqueness in (a), as translation by λ commutes with the right action of W_{aff} and preserves the partial order on alcoves. To check the second one, let us first consider the case when either $\lambda = 0$ or $w = 1$. If $\lambda = 0$ the statement follows from (ii) as $\mathfrak{A}_0$ lies above $w(\mathfrak{A}_0) = \mathfrak{A}_0^w$ for $w \in W$. When $w = 1$ and λ is dominant, then $\theta_\lambda = \widetilde{\lambda}$ and $\lambda + \mathfrak{A}_0$ lies above $\mathfrak{A}_0$, so the claim follows from (a,ii). Then the case $w = 1$ and arbitrary λ follows from the first property in b). Finally, the general case follows from (a,i) since $b_{\lambda+\mathfrak{A}_0,\lambda+\mathfrak{A}_0^{w^{-1}}} = b_{\mathfrak{A}_0,\mathfrak{A}_0^{w^{-1}}} = \widetilde{w}^{-1}$ by the first property in (b). □

Example. $b_{\mathfrak{A}_0,-\mathfrak{A}_0} = b_{\mathfrak{A}_0,\mathfrak{A}_0^{w_0{}^{-1}}} = \widetilde{w_0}^{-1}$, hence $b_{-\mathfrak{A}_0,\mathfrak{A}_0} = (b_{\mathfrak{A}_0,\mathfrak{A}_0^{w_0{}^{-1}}})^{-1} = \widetilde{w_0}$.

1.8.2. The first part of the next Theorem is a reformulation of Theorem 1.5.1.

Theorem. (cf. [B3, 2.1.5]) a) Let $X = \tilde{\mathfrak{g}}$ and $S \in \mathcal{BC}$ or let $X = \widetilde{\mathcal{N}}$ and $S \in \mathcal{BC}'$.

There is a unique map from Alc to the set of t-structures on $D^b[\mathcal{C}oh(S\times_{\mathfrak{g}} X)]$, $\mathfrak{A} \mapsto \mathcal{T}_{\mathfrak{A}}^{S,X}$ such that

(1) (Normalization) The derived global sections functor RΓ is t-exact with respect to the t-structure $\mathcal{T}_{\mathfrak{A}_0}^{S,X}$ corresponding to the fundamental alcove $\mathfrak{A}_0$.
(2) (Compatibility with the braid action) The action of the element $b_{\mathfrak{A}_1,\mathfrak{A}_2}$ sends $\mathcal{T}_{\mathfrak{A}_1}^{S,X}$ to $\mathcal{T}_{\mathfrak{A}_2}^{S,X}$.
(3) (Monotonicity) If $\mathfrak{A}_1$ lies above $\mathfrak{A}_2$, then $D_{\mathfrak{A}_1}^{\geq 0}(X) \supseteq D_{\mathfrak{A}_2}^{\geq 0}(X)$.

b) For a fixed alcove $\mathfrak{A}$, the t-structures $\mathcal{T}_{\mathfrak{A}}^{S,X}$ are compatible with base change $S_1 \to S$ in the sense that the direct image functor is t-exact.

c) For each S the t-structure $\mathcal{T}_{\mathfrak{A}}^{S,X}$ is locally free and one can choose the corresponding tilting generators $\mathcal{E}_{\mathfrak{A}}^S$, $S \in \mathcal{BC}_X$, as pull-backs of a $G \times \mathbb{G}_m$-equivariant locally free tilting generator $\mathcal{E}_{\mathfrak{A}}$ on $\tilde{\mathfrak{g}}$.

d) When $\mathfrak{A} = \mathfrak{A}_0$ is the fundamental alcove then $\mathcal{T}_{\mathfrak{A}_0}^{S,X}$ is the exotic t-structure of Theorem 1.5.1.

e) If vector bundle $\mathcal{E}^S_{\mathfrak{A}}$ is a tilting generator for $\mathcal{T}^{S,X}_{\mathfrak{A}}$ then the dual $(\mathcal{E}^S_{\mathfrak{A}})^*$ is a tilting generator for $\mathcal{T}^{S,X}_{-\mathfrak{A}}$.

The proof will be given in section 2.6.

2. Construction of exotic t-structures

2.1. **Action of simple reflections $\widetilde{s_\alpha}^{\pm 1}$ on coherent sheaves.** We will need an additional property of the action, which can be viewed as a geometric version of the quadratic relation in the affine Hecke algebra.

We start with an elementary preliminary Lemma.

2.1.1. *Lemma.* For any $\alpha \in I_{\text{aff}}$, $\widetilde{s_\alpha}$ is conjugate in the *extended* affine braid group $\mathbb{B}^{sc}_{\text{aff}}$ to some $\widetilde{s_\beta},\ \beta \in I$.

Proof. Consider first the case when $\alpha \in I_{\text{aff}}$ is connected in the affine Dynkin diagram to some root $\beta \in I$ of the same length. Then $u = s_\alpha s_\beta$ has length two and $us_\alpha = s_\alpha s_\beta s_\alpha = s_\alpha s_\beta s_\alpha = s_\beta u$ has length three, so $\widetilde{u}\widetilde{s_\alpha} = \widetilde{s_\beta}\widetilde{u}$ in $\mathbb{B}_{\text{aff}}$. Therefore, $\widetilde{s_\alpha} = {}^{\widetilde{u}^{-1}}\widetilde{s_\beta}$. This observation suffices in all cases but C_n.

For C_n the affine Dynkin diagram is a line with two roots of equal length $\alpha \in I_{\text{aff}} - I$ and $\beta \in I$ at the ends. The stabilizer Ω of the fundamental alcove in the extended affine Weyl group W^{sc}_{aff} acts on the affine Dynkin diagram and an element $\omega \in \Omega$ realizes the symmetry that exchanges α and β. Since the length function vanishes on Ω we find that $s_\alpha = {}^{\omega}s_\beta$ in W^{sc}_{aff} lifts to $\widetilde{s_\alpha} = {}^{\widetilde{\omega}}\widetilde{s_\beta}$ in $\mathbb{B}^{sc}_{\text{aff}}$. □

We are ready to deduce the desired property of the action. To state it we need some notations. As before, we denote the above weak geometric action by $\mathbb{B}_{\text{aff}} \ni b \mapsto K_b \in D^b[\mathcal{C}oh(\tilde{\mathfrak{g}}\times_{\mathfrak{g}}\tilde{\mathfrak{g}})]$. For a root $\alpha \in I_{\text{aff}}$ let $H_\alpha \subset \mathfrak{h}^*$ be the hyperplane passing through $0 \in \mathfrak{h}^*$ and parallel to the affine-linear hyperplane of α.

2.1.2. *Proposition. a) For every simple root $\alpha \in I_{aff}$ we have an exact triangle in $D^b[\mathcal{C}oh(\tilde{\mathfrak{g}}\times_{\mathfrak{g}}\tilde{\mathfrak{g}})]$:*

$$K_{\widetilde{s_\alpha}^{-1}} \xrightarrow{a_\alpha} K_{\widetilde{s_\alpha}} \xrightarrow{b_\alpha} \Delta_*(\mathcal{O}_{\tilde{\mathfrak{g}}\times_{\mathfrak{h}}H_\alpha}), \tag{4}$$

where $\Delta : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^2$ is the diagonal embedding.

b) For every $\alpha \in I_{aff}$, and every $\mathcal{F} \in D = D^b(\mathcal{C}oh(\tilde{\mathfrak{g}}_S))$ we have a (canonical) isomorphism in the quotient category $D/\langle \mathcal{F} \rangle$

$$\widetilde{s_\alpha}(\mathcal{F}) \cong \widetilde{s_\alpha}^{-1}(\mathcal{F}) \quad mod\ \langle \mathcal{F} \rangle.$$

Here $\langle \mathcal{F} \rangle$ denotes the thick triangulated subcategory generated by $\mathcal{F}$, i.e. the smallest full triangulated subcategory closed under direct summands and containing $\mathcal{F}$.

Proof. a) By Lemma 1.3.4(a) validity of the claim for a given $\alpha \in I_{\mathrm{aff}}$ implies its validity for any $\beta \in I_{\mathrm{aff}}$ such that $\widetilde{s_\beta}$ is conjugate to $\widetilde{s_\alpha}$ in $\mathbb{B}^{sc}_{\mathrm{aff}}$. Thus in view of Lemma 2.1.1 it suffices to prove the claim for $\alpha \in I$. By Lemma 1.3.4(b) we have only to check that the divisor $D_\alpha \stackrel{\mathrm{def}}{=} \Delta(\tilde{\mathfrak{g}} \times_{\mathfrak{h}} H_\alpha)$ in Γ_{s_α} satisfies

$$\mathcal{O}_{\Gamma_{s_\alpha}}(-D_\alpha) \cong \mathcal{O}_{\Gamma_{s_\alpha}}\langle -\rho, -\alpha + \rho \rangle. \tag{5}$$

It is easy to see that D_α is the scheme-theoretic intersection of Γ_{s_α} with the diagonal $\Delta_{\tilde{\mathfrak{g}}} = \Gamma_e$. Set $Z_\alpha = \tilde{\mathfrak{g}} \times_{\tilde{\mathfrak{g}}_\alpha} \tilde{\mathfrak{g}}$. Then Γ_{s_α} and $\Delta_{\tilde{\mathfrak{g}}}$ are irreducible components of Z_α and (5) follows from the isomorphism of line bundles on Z_α: $\mathcal{J}_{\Delta_{\tilde{\mathfrak{g}}}} \cong \mathcal{O}_{Z_\alpha}\langle -\rho, -\alpha + \rho \rangle$, where $\mathcal{J}$ stands for the ideal sheaf. It suffices to check that the two line bundles have isomorphic restrictions to $\Delta_{\tilde{\mathfrak{g}}}$ and to a fiber of the projection $pr_2 : Z_\alpha \to \tilde{\mathfrak{g}}$ (which is isomorphic to $\mathbb{P}^1$). It is easy to see that in both cases these restrictions are isomorphic to $\mathcal{O}_{Z_\alpha}(-\alpha)$, $\mathcal{O}_{\mathbb{P}^1}(-1)$ respectively. Thus (5) is verified.

The distinguished triangle in (a) implies that for $\mathcal{F} \in D$ we have canonical distinguished triangle $\widetilde{s_\alpha}^{-1}(\mathcal{F}) \to \widetilde{s_\alpha}(\mathcal{F}) \to \mathcal{F}'$ where $\mathcal{F}' = \mathcal{O}_{D_\alpha} * \mathcal{F}$. On the other hand, the obvious exact sequence of coherent sheaves $0 \to \mathcal{O}_{\Delta_{\tilde{\mathfrak{g}}}} \xrightarrow{\alpha} \mathcal{O}_{\Delta_{\tilde{\mathfrak{g}}}} \to \mathcal{O}_{D_\alpha} \to 0$ yields a distinguished triangle $\mathcal{F} \to \mathcal{F} \to \mathcal{F}'$ which shows that $\mathcal{F}' \in \langle \mathcal{F} \rangle$. This implies (b). □

2.2. **Uniqueness.** Here we prove the following description of the braid positive t-structure.

2.2.1. *Proposition.* Let $S \to \mathfrak{g}$ be an exact affine base change of $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ (resp. of $\widetilde{\mathcal{N}} \to \mathfrak{g}$). If an exotic t-structure on $\widetilde{S}$ (resp. $\widetilde{S}'$) exists then it is unique and given by

$$D^{\geq 0} = \{\mathcal{F}\ ;\ \mathrm{R}\Gamma(b^{-1}(\mathcal{F})) \in D^{\geq 0}(\mathcal{A}b)\ \ \forall b \in \mathbb{B}^+_{\mathrm{aff}}\};$$

$$D^{\leq 0} = \{\mathcal{F}\ ;\ \mathrm{R}\Gamma(b(\mathcal{F})) \in D^{\leq 0}(\mathcal{A}b)\ \ \forall b \in \mathbb{B}^+_{\mathrm{aff}}\}.$$

2.2.2. Let $\mathcal{A} = D^{\geq 0} \cap D^{\leq 0}$ be the heart of an exotic t-structure $\mathcal{T}$ and let $H^i_{\mathcal{T}} : D \to \mathcal{A}$ be the corresponding cohomology functors. Recall that we denote for $w \in W_{\mathrm{aff}}$ by $\widetilde{w} \in \mathbb{B}_{\mathrm{aff}}$ its canonical lift.

Lemma. For $M \in \mathcal{A}$ let $\mathcal{A}_M \subset \mathcal{A}$ be the Serre subcategory generated by M. Then for any $\alpha \in I_{\mathrm{aff}}$

$$H^i_{\mathcal{T}}(\widetilde{s_\alpha} M),\ H^i_{\mathcal{T}}(\widetilde{s_\alpha}^{-1} M) \in \mathcal{A}_M \ \text{ for } \ i \neq 0, \quad \text{and} \quad H^0_{\mathcal{T}}(\widetilde{s_\alpha} M) \cong H^0_{\mathcal{T}}(\widetilde{s_\alpha}^{-1} M) \bmod \mathcal{A}_M$$

Proof. Proposition 2.1.2(b) implies that $H^i_{\mathcal{T}}(\widetilde{s_\alpha} M) \cong H^i_{\mathcal{T}}(\widetilde{s_\alpha}^{-1} M) \bmod \mathcal{A}_M$ for all i. On the other hand, by the definition of braid positivity we have $H^i_{\mathcal{T}}(\widetilde{s_\alpha} M) = 0$ for $i > 0$ while $H^i_{\mathcal{T}}(\widetilde{s_\alpha}^{-1} M) = 0$ for $i < 0$. □

2.2.3. *Corollary.* Set $D_{-1} = D$ and let $D_0 \subset D$ be the full subcategory of objects $\mathcal{F}$ such that $\mathrm{R}\Gamma(\mathcal{F}) = 0$. For $i > 0$ define inductively a full triangulated subcategory $D_i \subset D$ by:

$$D_i = \{\mathcal{F} \in D_{i-1}\ ;\ \widetilde{s_\alpha}(\mathcal{F}) \in D_{i-1}\ \forall \alpha \in I_{\mathrm{aff}}\}.$$

Set $\mathcal{A}_i = D_i \cap \mathcal{A},\ i \geq 0$, then we have

a) $\mathcal{A}_i$ is a Serre abelian subcategory in $\mathcal{A}$.

b) Any exotic t-structure $\mathcal{T}$ induces a bounded t-structure on D_i, whose heart is $\mathcal{A}_i$.

c) For $i > 0$ and any $\alpha \in I_{\mathrm{aff}}$, the composition of $\widetilde{s_\alpha}$ with the projection to $D/\langle \mathcal{A}_i \rangle$ sends $\mathcal{A}_i$ to $\mathcal{A}_{i-1}/\mathcal{A}_i \subset D_{i-1}/D_i$; it induces an exact functor $\mathcal{A}_i \to \mathcal{A}_{i-1}/\mathcal{A}_i$.

Proof. We prove the statements together by induction. Assume they are known for $i \leq i_0$. The statement (c) implies that for any $\alpha_1, \ldots, \alpha_{i_0+1}$ the functor

$$\mathcal{F} \mapsto \mathrm{R}^k\Gamma(\widetilde{s_{\alpha_1}} \cdots \widetilde{s_{\alpha_{i_0+1}}}(\mathcal{F}))$$

restricted to $\mathcal{A}_{i_0}$ vanishes for $k \neq 0$, and induces an exact functor $\mathcal{A}_{i_0} \to Vect$. The subcategory $\mathcal{A}_{i_0+1} \subset \mathcal{A}_{i_0}$ is, by definition, the intersection of the kernels of all such functors; this shows it is a Serre abelian subcategory in $\mathcal{A}_{i_0}$, hence in $\mathcal{A}$. Thus (a) holds for $i = i_0 + 1$.

Moreover, we see that for $\mathcal{F} \in D_{i_0}$ vanishing of $\mathrm{R}^\bullet\Gamma(\widetilde{s_{\alpha_1}} \cdots \widetilde{s_{\alpha_{i_0+1}}}(\mathcal{F}))$ implies that for an exotic t-structure $\mathcal{T}$ all $\mathrm{R}^\bullet\Gamma(\widetilde{s_{\alpha_1}} \cdots \widetilde{s_{\alpha_{i_0+1}}} H^k_{\mathcal{T}}(\mathcal{F}))$ vanish for $k \in \mathbb{Z}$. Thus for $\mathcal{F} \in D_{i_0+1}$ we have $H^k_{\mathcal{T}}(\mathcal{F}) \in D_{i_0+1}$ for all k. This shows that the truncation functors preserves D_{i_0+1}, i.e. (b) holds for $i = i_0 + 1$.

Finally statement (c) for $i = i_0 + 1$ follows from Lemma 2.2.2. □

2.2.4. *Proof of Proposition 2.2.1.* Let $\mathcal{T}$ be an exotic t-structure. Assume that $\mathcal{F} \in D$, $\mathcal{F} \notin D^{\leq 0}$, and let $i > 0$ be the largest integer such that $H^i_{\mathcal{T}}(\mathcal{F}) = M \neq 0$. It suffices to show that $\mathrm{R}^i\Gamma(b(\mathcal{F})) \neq 0$ for some $b \in \mathbb{B}^+_{\mathrm{aff}}$.

Lemma 2.4.1(a) implies that $\bigcap_i \mathcal{A}_i = \{0\}$, so $M \notin \mathcal{A}_d$ for some d; let d be smallest integer with this property. If $d = 0$ then $\mathrm{R}^i\Gamma(\mathcal{F}) \neq 0$, so we are done. Otherwise let $b = \widetilde{s_{\alpha_1}} \cdots \widetilde{s_{\alpha_d}}$ be an element such that $b(M) \notin D_0$. Then by Corollary 2.2.3 we have $\mathrm{R}^0\Gamma(b(M)) \neq 0$. Consider the exact triangle

$$\tau^{\mathcal{T}}_{<i}(\mathcal{F}) \to \mathcal{F} \to M[-i],$$

and apply b to it. Since $\widetilde{b}$ is $\mathcal{T}$-right exact, and $\mathrm{R}\Gamma$ is $\mathcal{T}$-exact, we see that $\mathrm{R}\Gamma(b(\tau_{<i}(\mathcal{F}))) \in D^{<i}(Vect)$, thus we see that $\mathrm{R}^i\Gamma(b(\mathcal{F})) \underset{\cong}{\to} \mathrm{R}^i\Gamma(b(M)[-i]) \neq 0$.

This proves the description of $D^{\leq 0}_{\mathcal{T}}$, the description of $D^{\geq 0}_{\mathcal{T}}$ is proved similarly. □

2.3. **Reflection functors $\mathcal{R}_\alpha$ for coherent sheaves.** Reflection functors can be considered as a categorical counterpart of the idempotent of the sign representation in a Levi subalgebra of a Hecke algebra (or the group algebra of the affine Weyl group). Reflection functors on representation categories are usually defined using translation functors which are direct summands of the functor of tensoring by a finite dimensional representation. In this subsection we define geometric reflection functors and show some favorable properties they share with reflection functors in representation theory. In fact, using the results of [BMR2] it is not hard to check that these functors are compatible with the usual reflection functors for modules over the Lie algebra in positive characteristic. We neither check this in detail nor use in the present paper; however, the proof of Theorem 1.6.3 above is closely related to this fact.

For $\alpha \in I_{\mathrm{aff}}$, let $\Xi_\alpha \in D^b[\mathcal{C}oh_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}(\tilde{\mathfrak{g}}^2)]$ denote the pull-back of the extension $K^{-1}_{\widetilde{s_\alpha}} \to K_{\widetilde{s_\alpha}} \to \mathcal{O}_{\Delta \times_{\mathfrak{h}} H_\alpha}$ under the surjection $\mathcal{O}_\Delta \to \mathcal{O}_{\Delta \times_{\mathfrak{h}} H_\alpha}$, so we have an extension $K^{-1}_{\widetilde{s_\alpha}} \to \Xi_\alpha \to \mathcal{O}_\Delta$. We define the reflection functor $\mathcal{R}_\alpha$ by the integral kernel Ξ_α.

2.3.1. *Adjoints of reflection functors.* We first consider finite simple roots $\alpha \in I$. Let $P_\alpha \supseteq B$ be a minimal parabolic of type α. The canonical projection $\tilde{\mathfrak{g}} = G \times_B \mathfrak{b} \xrightarrow{\pi_\alpha} \tilde{\mathfrak{g}}_\alpha$ is generically a ramified two-sheet covering.

Lemma. For $\alpha \in I$, $\Xi_\alpha = \mathcal{O}_{\tilde{\mathfrak{g}}\times_{\tilde{\mathfrak{g}}_\alpha}\tilde{\mathfrak{g}}}$ and the reflection functor $\mathcal{R}_\alpha$ is isomorphic to the functor $(\pi_\alpha)^*(\pi_\alpha)_*$.

Proof. We will only consider the case of sl_2, the general case follows by considering an associated bundle. Notice that $\tilde{\mathfrak{g}}\times_{\tilde{\mathfrak{g}}_\alpha}\tilde{\mathfrak{g}}$ has two irreducible components $\Delta_{\tilde{\mathfrak{g}}}$ and S_α which meet transversely along $\Delta_{\widetilde{\mathcal{N}}}$. Then $\mathcal{O}_{S_\alpha}(-\rho,-\rho)$ is the ideal of $\Delta_{\tilde{\mathfrak{g}}}\cap S_\alpha$ inside S_α and of $\Delta_{\tilde{\mathfrak{g}}}$ inside $\Delta_{\tilde{\mathfrak{g}}}\cup S_\alpha = \tilde{\mathfrak{g}}\times_{\tilde{\mathfrak{g}}_\alpha}\tilde{\mathfrak{g}}$. So, one has

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{S_\alpha}(-\rho,-\rho) & \longrightarrow & \mathcal{O}_{\tilde{\mathfrak{g}}\times_{\tilde{\mathfrak{g}}_\alpha}\tilde{\mathfrak{g}}} & \longrightarrow & \mathcal{O}_{\Delta_{\tilde{\mathfrak{g}}}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{S_\alpha}(-\rho,-\rho) & \longrightarrow & \mathcal{O}_{S_\alpha} & \longrightarrow & \mathcal{O}_{\Delta_{\widetilde{\mathcal{N}}}} & \longrightarrow & 0. \end{array}$$

The lower line is the construction of the exact triangle in the Proposition 2.1.2.a), and then the upper line says that Ξ_α is $\mathcal{O}_{\tilde{\mathfrak{g}}\times_{\tilde{\mathfrak{g}}_\alpha}\tilde{\mathfrak{g}}}$. The claim for $\mathcal{R}_\alpha$ follows.

Corollary. For any simple root $\alpha \in I_{\mathrm{aff}}$:

(a) We have two canonical distinguished triangles

$$K^{-1}_{\widetilde{s_\alpha}} \to \Xi_\alpha \to \mathcal{O}_{\Delta_{\tilde{\mathfrak{g}}}} \quad \text{and} \quad \mathcal{O}_{\Delta_{\tilde{\mathfrak{g}}}} \to \Xi_\alpha \to K_{\widetilde{s_\alpha}}. \tag{6}$$

(b) The left and right adjoints of $\mathcal{R}_\alpha$ are both isomorphic to $\mathcal{R}_\alpha$.

(c) $\mathcal{R}_\alpha$ is exact relative to an exotic t-structure.

Proof. The first triangle appears in the definition of Ξ_α. To get the second one it suffices, in view of Lemma 1.3.4(b), to show that

$$\mathcal{S}(\Xi_\alpha)[-\dim \mathfrak{g}] \cong \Xi_\alpha \tag{7}$$

for all $\alpha \in I_{\mathrm{aff}}$, where $\mathcal{S}$ denotes Grothendieck-Serre duality. For $\alpha \in I$ isomorphism (7) follows from Lemma 2.3.1.

Furthermore, Lemma 1.3.4(b) implies that the conjugation action of $b \in \mathbb{B}_{\mathrm{aff}}$ commutes with Serre duality. Thus (7) holds in general by Lemma 2.1.1.

This proves (a). To get (b) we use the following general fact. If X is Gorenstein and $\mathcal{F} \in D^b[\mathcal{C}oh(X\times X)]$, then it is not hard to show that the left adjoint to the functor of (left) convolution with $\mathcal{F}$ is given by (left) convolution with

$$\mathcal{F}^\vee \stackrel{\mathrm{def}}{=} \iota^*\left(\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{O}_{X^2})\otimes pr_2^*K_X\right) = \iota^*\left(\mathcal{S}_{X^2}(\mathcal{F})\otimes pr_1^*K_X^{-1}\right); \tag{8}$$

where $\iota : X^2 \to X^2$ is the involution $\iota(x,y) = (y,x)$ while $K_X = \Omega_X^{top}[\dim X]$ is the dualizing sheaf on X.

Now isomorphism (7) implies (b) in view of (8).

Statement (c) follows from (a), since left exactness of $\mathcal{R}_\alpha$ follows from the first distinguished triangle, while right exactness follows from the second one. □

Remark. Notice that unlike the generators for the affine braid group action, the geometric reflection functors do not induce a functor on the derived category of sheaves on $\widetilde{\mathcal{N}}$ (or varieties obtained from the latter by base change). This is related to the fact that the restriction of Ξ_α to the preimage of 0 under the first projection to $\mathfrak{h}$ is not supported (scheme-theoretically) on the preimage of 0 under the second projection.

2.4. **Weak generators for the derived category arising from reflection functors.** The notion of a weak generator was recalled in subsection 1.4.3. In the present subsection we construct weak generators for $D[q\mathcal{C}oh(\tilde{\mathfrak{g}})]$, which will turn out to be locally free sheaves satisfying the requirements of Theorem 1.5.1.

For a finite sequence $J = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of elements of I_{aff}, we set

$$\Xi_J \stackrel{\mathrm{def}}{=} \mathcal{R}_{\alpha_1} \cdots \mathcal{R}_{\alpha_k}(\mathcal{O}_{\tilde{\mathfrak{g}}}).$$

For a finite collection $\mathcal{J}$ of finite sequences let $\Xi_{\mathcal{J}} \stackrel{\mathrm{def}}{=} \bigoplus_{J \in \mathcal{J}} \Xi_J$. Notice that these are $G \times \mathbb{G}_m$ equivariant by construction.

2.4.1. *Lemma.* a) There exists a finite collection of elements $b_i \in \mathbb{B}_{\mathrm{aff}}^+$, such that each of the two objects $\oplus b_i(\mathcal{O}_{\tilde{\mathfrak{g}}})$ and $\oplus b_i^{-1}(\mathcal{O}_{\tilde{\mathfrak{g}}})$ is a weak generator for $D[q\mathcal{C}oh(\tilde{\mathfrak{g}})]$.

b) There exists a finite collection $\mathcal{J}$ of finite sequences of I_{aff}, such that $\Xi_{\mathcal{J}}$ is a weak generator for $D[q\mathcal{C}oh(\tilde{\mathfrak{g}})]$.

Proof. Pick a very ample line bundle $\mathcal{O}(\lambda)$ on $\tilde{\mathfrak{g}}$ with $\lambda \in \Lambda$; thus there exists a locally closed embedding $\tilde{\mathfrak{g}} \stackrel{i}{\hookrightarrow} \mathbb{P}^N$, such that $\mathcal{O}(\lambda) \cong \mathcal{O}_{\mathbb{P}^N}(1)\big|_{\tilde{\mathfrak{g}}}$. It is well-known (and follows from [Be]) that the object $\bigoplus_{i=0}^{N} \mathcal{O}(i)$, generates $D^b[\mathcal{C}oh(\mathbb{P}^n)]$ as a Karoubian triangulated category. Then so does also its $\mathcal{O}(-N)$ twist $\bigoplus_{i=0}^{N} \mathcal{O}(-i)$.

If $G \in D^b[\mathcal{C}oh(\mathbb{P}^n)]$ generates $D^b[\mathcal{C}oh(\mathbb{P}^n)]$ as a Karoubian triangulated category, we claim that i^*G is a weak generator for $D(q\mathcal{C}oh(\tilde{\mathfrak{g}}))$. To see this, let $\mathcal{F} \in D(q\mathcal{C}oh(\tilde{\mathfrak{g}}))$ be a complex such that $\mathrm{Ext}^\bullet(i^*(G), \mathcal{F}) = 0$. Since the Karoubian triangulated subcategory of $D^b[q\mathcal{C}oh(\tilde{\mathfrak{g}})]$ generated by the restriction i^*G contains all $i^*\mathcal{O}_{\mathbb{P}^n}(k),\ k \in \mathbb{Z}$, we have $\mathrm{Ext}^\bullet(i^*\mathcal{O}_{\mathbb{P}^n}(k), \mathcal{F}) = 0$. Assuming that $\mathcal{F} \neq 0$ we can find d such that the d-th cohomology sheaf of $\mathcal{F}$ does not vanish. There exists a coherent subsheaf $\mathcal{G}$ in the kernel of the differential $\partial_d : \mathcal{F}^d \to \mathcal{F}^{d+1}$ which is not contained in the image of ∂_{d+1}. The sheaf $\mathcal{G}(k)$ is generated by global sections for some $k \gg 0$, thus we get a nonzero map $i^*\mathcal{O}_{\mathbb{P}^n}(-k)[-d] \to \mathcal{F}$ contradicting the above.

Thus the collection of multiples of λ, $b_i = i{\cdot}\lambda \in \ \Lambda^+ \subset \mathbb{B}^+_{\mathrm{aff}},\ 0 \leq i \leq N$, satisfies the requirement in (a). To deduce (b) from (a) it suffices to show that for every $b \in \mathbb{B}_{\mathrm{aff}}$ there exists a finite collection of finite sequences J_i in I_{aff} such that $b(\mathcal{O}_{\tilde{\mathfrak{g}}})$ lies in the triangulated subcategory generated by Ξ_{J_i}. Let us express b as a product $b = \widetilde{s_{\alpha_1}}^{\pm 1} \cdots \widetilde{s_{\alpha_d}}^{\pm k}$ with $\alpha_j \in I_{\mathrm{aff}}$. The exact triangles (6) imply that $b(\mathcal{O}_{\tilde{\mathfrak{g}}})$ lies in the triangulated category generated by all Ξ_J where J runs over subsequences of $(\alpha_1, \ldots, \alpha_k)$. □

2.5. **Existence.** This subsection contains a construction of exotic t-structures.

For an object $\mathcal{E} \in D^b[\mathcal{C}oh(\tilde{\mathfrak{g}})]$ and $(R \to \Bbbk) \in \mathrm{FGP}$ we denote by $\mathcal{E}^{\hat{0}}_{\Bbbk}$ the pull-back of $\mathcal{E}_{\Bbbk}$ to the formal neighborhood $\widehat{\mathcal{B}_{\Bbbk}}$ of the zero section in $\tilde{\mathfrak{g}}_{\Bbbk}$.

For $S \in \mathcal{BC}$ or $S \in \mathcal{BC}'$ (see section 1.3 for the notation) we will say that a tilting generator $\mathcal{E} \in Coh(\tilde{\mathfrak{g}}_S)$ (respectively, $\mathcal{E} \in Coh(\widetilde{\mathcal{N}}_S)$) is exotic if the t-structure $\mathcal{T}_{\mathcal{E}}$ (notation of 1.4.2) is braid positive.

Theorem. Let $\mathcal{E}$ be any weak generator $\mathcal{E} = \Xi_{\mathcal{J}}$ from Lemma 2.4 with $\mathcal{J} \ni \emptyset$. Let $\mathcal{E}_S$ (respectively, $\mathcal{E}'_S$) be the sheaf on $\tilde{\mathfrak{g}}_S$ (respectively, $\widetilde{\mathcal{N}}_S$) obtained from $\mathcal{E}$ by pull-back.

Then for any $S \in \mathcal{BC}$ (respectively, $S \in \mathcal{BC}'$) $\mathcal{E}_S$ (respectively, $\mathcal{E}'_S$) is a locally free exotic tilting generator.

2.5.1. *Reduction to the formal neighborhood of zero section in finite characteristic.* The proof of the Theorem in subsection 2.5.5 will proceed by reduction to the case of positive characteristic which is done by invoking the representation theoretic picture. The reduction is achieved in the following Proposition.

Proposition. Let $\mathcal{E}$ be an object of $D^b(\mathcal{C}oh(\tilde{\mathfrak{g}}))$ containing $\mathcal{O}$ as a direct summand.

a) If $\mathcal{E}$ is an exotic locally free tilting generator then for any $S \in \mathcal{BC}$ (respectively, $S \in \mathcal{BC}'$) the sheaf $\mathcal{E}_S$ on $\tilde{\mathfrak{g}}_S$ (respectively, $\mathcal{E}'_S$ on $\widetilde{\mathcal{N}}_S$) obtained by pull-back from $\mathcal{E}$ is a locally free exotic tilting generator.

b) Assume that $\mathcal{E}$ is $\mathbb{G}_m$ equivariant and $\mathcal{E}^{\widehat{0}}_{\Bbbk}$ is an exotic locally free tilting generator for any $(R \to \Bbbk) \in \mathrm{FGP}$. Then $\mathcal{E}$ itself is an exotic locally free tilting generator.

2.5.2. *Lemma.* a) If $\mathcal{E} \in D^b[\mathcal{C}oh^{\mathbb{G}_m}(\tilde{\mathfrak{g}})]$ is such that for any $(R \to \Bbbk) \in \mathrm{FGP}$ the object $\mathcal{E}^{\widehat{0}}_{\Bbbk}$ is a locally free sheaf, then $\mathcal{E}$ is a locally free sheaf.

b) If $\mathcal{E}, \mathcal{F} \in D^b[\mathcal{C}oh^{\mathbb{G}_m}(\tilde{\mathfrak{g}})]$ are such that $\mathrm{Ext}^{>0}_{\mathcal{C}oh(\widehat{\mathcal{B}}_{\Bbbk})}(\mathcal{E}^{\widehat{0}}_{\Bbbk}, \mathcal{F}^{\widehat{0}}_{\Bbbk}) = 0$ for any $(R \to \Bbbk) \in \mathrm{FGP}$, then $\mathrm{Ext}^{>0}_{\mathcal{C}oh(\tilde{\mathfrak{g}})}(\mathcal{E}, \mathcal{F}) = 0$.

Proof. (a) Let U be the maximal open subset such that $H^i(\mathcal{E})\big|_U$ is a locally free sheaf and $H^i(\mathcal{E})\big|_U = 0$ for $i \neq 0$. Then U is $\mathbb{G}_m$-invariant, and the condition on $\mathcal{E}$ shows that this set contains all closed points of the zero fiber. Hence $U = \tilde{\mathfrak{g}}$.

Statement (b) is obtained by applying similar considerations to the object $\pi_*(\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F}))$ where π is the Grothendieck-Springer map. Here we use the formal function theorem which shows that $\mathrm{Ext}^i(\mathcal{E}^{\widehat{0}}_{\Bbbk}, \mathcal{F}^{\widehat{0}}_{\Bbbk})$ is the space of sections of the pull-back of $\mathrm{R}^i\pi_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{E}_{\Bbbk}, \mathcal{F}_{\Bbbk})$ to the formal neighborhood of zero. □

2.5.3. *Lemma.* Let $\mathcal{E}$ be a tilting generator for $D^b(\mathcal{C}oh(X))$ where $X = \widetilde{S}$ or $X = \widetilde{S}'$ for some S in $\mathcal{BC}$ or in $\mathcal{BC}'$. Then $\mathcal{E}$ t-structure on $D^b(\mathcal{C}oh(X))$ is exotic iff $\mathrm{Hom}^i(\mathcal{E}, \widetilde{s_\alpha}(\mathcal{E})) = 0$ for $i > 0$ and $\mathcal{E}$ contains $\mathcal{O}$ as a direct summand.

Proof. Assume that $\mathcal{E}$ t-structure is exotic. Since $\mathcal{O}$ represents the functor $\mathrm{R}\Gamma$ which is exact with respect to this t-structure, it is a direct summand of any projective generator of the heart, in particular of $\mathcal{E}$ (a tilting generator is a projective generator of the heart of the corresponding t-structure). Also, right exactness of $\widetilde{s_\alpha}$ and projectivity of $\mathcal{E}$ show that $\mathrm{Hom}^i(\mathcal{E}, \widetilde{s_\alpha}(\mathcal{E})) = 0$ for $i > 0$. Conversely, if $\mathcal{O}$ is a direct summand of $\mathcal{E}$, then the derived global sections functor is t-exact. Also, if $\mathrm{Hom}^{>0}(\mathcal{E}, \widetilde{s_\alpha}(\mathcal{E})) = 0$, then $\widetilde{s_\alpha}(\mathcal{E})$ lies in $D^{\leq 0}$ with respect to this t-structure, which implies that $\widetilde{s_\alpha}$ sends $D^{\leq 0}$ to itself. □

2.5.4. *Proof of Proposition 2.5.1.* a) It is obvious that the pull-back functor sends a locally free sheaf containing $\mathcal{O}$ as a direct summand to a locally free sheaf containing $\mathcal{O}$ as a direct

summand, while pull-back under an affine morphism sends a weak generator of $D(q\mathcal{C}oh)$ to a weak generator of $D(q\mathcal{C}oh)$. In view of the characterization of tilting generators quoted in Theorem 1.4.3 and criterion for a t-structure to be exotic from Lemma 2.5.3, it remains to see that $\mathrm{Ext}^i(\mathcal{E}_S, \mathcal{E}_S) = 0$, $\mathrm{Ext}^i(\mathcal{E}_S, \widetilde{s_\alpha}(\mathcal{E}_S)) = 0$ for $i > 0$ (or the similar equalities for $\mathcal{E}_{S'}$). The required equalities follow by the base change theorem which is applicable due to the Tor vanishing condition. This proves part (a).

Assume now that $\mathcal{E}$ satisfies the conditions of (b). Then $\mathcal{E}$ is locally free by Lemma 2.5.2(a). By Lemma 2.5.2(b) it satisfies the above Ext vanishing conditions. Thus $\mathcal{E}$ is a tilting generator by Theorem 1.4.3, and it is an exotic tilting generator in view of Lemma 2.5.3. □

2.5.5. *Proof of theorem 2.5.* Let $\mathcal{J}$ be a finite collection of finite sequences in I_{aff} such that $\Xi_{\mathcal{J}}$ is a weak generator for $D^b(\mathcal{C}oh(\tilde{\mathfrak{g}}))$ and $\mathcal{J} \ni \emptyset$. We have to check that $\mathcal{E} = \Xi_{\mathcal{J}}$ satisfies the properties from the theorem 2.5. We will first reduce the verification to formal neighborhoods of zero sections over closed geometric points of positive characteristic, then the claim will follow from translation to $\mathfrak{g}$-modules.

Recall from 2.4 that $\mathcal{E}$ is $G \times \mathbb{G}_m$-equivariant by construction and $\mathcal{O}$ is a direct summand of $\mathcal{E}$ since $\emptyset \in \mathcal{J}$. Therefore, by the Proposition 2.5.1 it suffices to check that for any $R \to \Bbbk$ in FGP the restriction $\mathcal{E}^{\hat{0}}_{\Bbbk}$ of $\mathcal{E}$ to the formal neighborhood of the zero section in $\tilde{\mathfrak{g}}_{\Bbbk}$ or $\widetilde{\mathcal{N}}_{\Bbbk}$ is a locally free tilting generator for an exotic t-structure.

This will follow once we show that $\mathcal{E}^{\hat{0}}_{\Bbbk}$ is a projective generator for the heart of the RT t-structure. Indeed, then $\mathcal{E}^{\hat{0}}_{\Bbbk}$ is locally free because another projective generator for the same heart is locally free (any splitting vector bundle for the Azumaya algebra of differential operators on $\mathcal{B}_{\Bbbk}$). Also the t-structure given by $\mathcal{E}^{\hat{0}}_{\Bbbk}$ is the RT t-structure, but we know that it is exotic from Theorem 1.6.3.

To check that $\mathcal{E}^{\hat{0}}_{\Bbbk}$ is a projective generator for the heart of the RT t-structure it is enough to check that it is (1) a projective object of the heart, and (2) is a weak generator.

Statement (2) is clear since $\mathcal{E}$ is a weak generator for $D^b(q\mathcal{C}oh(\tilde{\mathfrak{g}}))$. To check (1) it is enough to treat the case of $\tilde{\mathfrak{g}}$, the case of $\widetilde{\mathcal{N}}$ follows because direct image under closed embeddings $\widetilde{\mathcal{N}} \hookrightarrow \tilde{\mathfrak{g}}$ is exact relative to the RT t-structures, this is clear since it corresponds to the full embedding of the categories of modular representations $mod(U^\lambda) \to mod(U^{\widehat{\lambda}})$. We now consider the case of $\tilde{\mathfrak{g}}$.

The pull-back functor under an exact base change preserves convolutions, so each summand of $\mathcal{E}^{\widehat{0}} = \Xi^{\widehat{0}}_{\mathcal{J}}$ is of the form $\mathcal{R}_{\alpha_1,\Bbbk}\cdots\mathcal{R}_{\alpha_p,\Bbbk}\widehat{\mathcal{O}}_{\Bbbk}$ where $\widehat{\mathcal{O}}_{\Bbbk}$ is the structure sheaf of the formal neighborhood of the zero section in $\tilde{\mathfrak{g}}_{\Bbbk}$, and the functor $\mathcal{R}_{\alpha,\Bbbk}$ is the convolution with the base change of Ξ_α under $\mathfrak{g}_{\Bbbk}\to\mathfrak{g}$.

The structure sheaf $\widehat{\mathcal{O}}_{\Bbbk}$ is projective for the RT structure because RT structure is exotic by Theorem 1.6.3, so the functor $\mathrm{RHom}(\widehat{\mathcal{O}}_{\Bbbk},-)$ is exact (it can be identified with the direct image to $\mathfrak{g}$). Thus we will be done if we show that functors $\mathcal{R}_{\alpha,\Bbbk}$ preserve the subcategory of projective objects in the RT heart.

The latter property is equivalent to the existence of a right adjoint to $\mathcal{R}_{\alpha,\Bbbk}$ which is exact relative to the RT t-structure. According to the Lemma 2.3.1 the right adjoint of $\mathcal{R}_\alpha$ is isomorphic to $\mathcal{R}_\alpha$ itself and $\mathcal{R}_\alpha$ can be written as an extension of the identity functor with either the action of $\widetilde{s_\alpha}$ or $\widetilde{s_\alpha}^{-1}$, the same then holds for $\mathcal{R}_{\alpha,\Bbbk}$. On $\mathfrak{g}$-modules $\widetilde{s_\alpha}$ is right exact and $\widetilde{s_\alpha}^{-1}$ is left exact. Since the equivalence of categories of representations and of coherent sheaves intertwines the two actions of $\mathbb{B}_{\mathrm{aff}}$ 1.6.4 we see that $\mathcal{R}_{\alpha,\Bbbk}$ is both left and right exact for the RT t-structure. □

2.5.6. *Example.* For $G = SL_2$ vector bundle $\mathcal{E}$ and therefore also the algebra A can be described explicitly. $\mathcal{E}$ is a sum of positive multiples of $\mathcal{O}_{\tilde{\mathfrak{g}}}$ and $\mathcal{O}_{\tilde{\mathfrak{g}}}(1) \stackrel{\mathrm{def}}{=} (\tilde{\mathfrak{g}}\to\mathcal{B})^*\mathcal{O}_{\mathbb{P}^1}(1)$. We know that $\mathcal{O}_{\tilde{\mathfrak{g}}}$ is a summand of $\mathcal{E}$ and since ρs_α fixes the fundamental alcove according to Remark 1.5.2, $(\rho s_\alpha)^\sim\ \mathcal{O}_{\tilde{\mathfrak{g}}} = \mathcal{O}_{\tilde{\mathfrak{g}}}(1)$ is also a summand. To see that these are all indecomposable summands, it suffices to see that $\mathcal{O}_{\tilde{\mathfrak{g}}}$ and $\mathcal{O}_{\tilde{\mathfrak{g}}}(1)$ are weak generators of $D^b\mathcal{C}oh(\tilde{\mathfrak{g}})$. This is true since $\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)$ are weak generators of $D^b\mathcal{C}oh(\mathbb{P}^1)$ and $\tilde{\mathfrak{g}}\hookrightarrow\mathbb{P}^1\times\mathfrak{g}$. For further explicit computations of this sort (over algebraically closed fields) we refer to: [BMR1] for the (sub)regular case, and the case $e=0$ for SL_3; and to [Anno] for the case when $e\in sl(2n)$ has two Jordan blocks of equal size.

2.6. **Proof of Theorem 1.8.2.** In a), the compatibility with the affine braid group action (axiom (2)) says that the choice of a t-structure $\mathcal{T}^{S,X}_{\mathfrak{A}_0}$ determines the t-structures $\mathcal{T}^{S,X}_{\mathfrak{A}} = b_{\mathfrak{A}_0,\mathfrak{A}}(\mathcal{T}^{S,X}_{\mathfrak{A}_0})$ for all $\mathfrak{A}$. We need to check that this collection of t-structures satisfies the monotonicity property (3) if and only if the t-structure $\mathcal{T}^{S,X}_{\mathfrak{A}_0}$ is exotic.

Notice that for a simple reflection we have $b_{\mathfrak{A}_0,s_\alpha\mathfrak{A}_0} = \widetilde{s}_\alpha^{\pm 1}$ where the power is $+1$ exactly when $s_\alpha\mathfrak{A}_0$ is above $\mathfrak{A}_0$, i.e., when α is not in the finite root system. So, braid positivity of $\mathcal{T}^{S,X}_{\mathfrak{A}_0}$ is implied by monotonicity – it amounts to monotonicity applied to pairs of alcoves

$\mathfrak{A}_0, s_\alpha\mathfrak{A}_0$ where s_α runs over all simple reflections. On the other hand, assume $\mathcal{T}^{S,X}_{\mathfrak{A}_0}$ is braid positive. To check property (3) for the collection $\mathcal{T}^{S,X}_{\mathfrak{A}} = b_{\mathfrak{A}_0,\mathfrak{A}}(\mathcal{T}^{S,X}_{\mathfrak{A}_0})$, it suffices to consider a pair of neighboring alcoves $\mathfrak{A}_2 = \mathfrak{A}_1^{s_\alpha}$. Then the automorphism $b^{-1}_{\mathfrak{A}_0,\mathfrak{A}_1}$ sends the pair of t-structures $(\mathcal{T}^{S,X}_{\mathfrak{A}_1}, \mathcal{T}^{S,X}_{\mathfrak{A}_2})$ to the pair $(\mathcal{T}^{S,X}_{\mathfrak{A}_0}, \widetilde{s}_\alpha(\mathcal{T}^{S,X}_{\mathfrak{A}_0}))$, thus the desired relation between $\mathcal{T}^{S,X}_{\mathfrak{A}_1}$ and $\mathcal{T}^{S,X}_{\mathfrak{A}_2}$ follows from braid positivity.

Thus part (a) – and at the same time part (d) – of the Theorem follows from existence and uniqueness of exotic t-structure (Theorem 1.5.1).

Part (b) follows from compatibility of the affine braid group action with base change (Theorem 1.3.2).

For (c) notice that for any S, X, a projective generator for the heart of $\mathcal{T}^{S,X}_{\mathfrak{A}}$ can be obtained by pull-back from a projective generator for the heart of $\mathcal{T}^{\mathfrak{g},\tilde{\mathfrak{g}}}_{\mathfrak{A}}$ which is of the form $b_{\mathfrak{A},\mathfrak{A}_0}(\Xi_{\mathcal{J}})$. So we just need to check that $\mathcal{E}_{\mathfrak{A}} \stackrel{\text{def}}{=} b_{\mathfrak{A}_0,\mathfrak{A}}(\Xi_{\mathcal{J}}) \in D^b(Coh(\tilde{\mathfrak{g}}))$ is a locally free sheaf.

As in the above arguments, it suffices to check that for any $\Bbbk \in \mathrm{FGP}$ the pull-back of $b_{\mathfrak{A}_0,\mathfrak{A}}(\Xi_J)$ to the formal neighborhood of $\mathcal{B}_\Bbbk$ in $\tilde{\mathfrak{g}}_\Bbbk$ is locally free. This follows from compatibility with localization for $\mathfrak{g}_\Bbbk$-modules at a Harish-Chandra central character λ such that $\frac{\lambda+\rho}{p}$ lies in the alcove $\mathfrak{A}$, as explained in section 1.6.

(e) It remains to check that $\mathcal{E}^*_{\mathfrak{A}}$ is a tilting generator for $\mathcal{T}^{\mathfrak{g}_R,\tilde{\mathfrak{g}}}_{-\mathfrak{A}}$. We will prove the equivalent claim that $b_{-\mathfrak{A},\mathfrak{A}_0}(\mathcal{E}^*_{\mathfrak{A}})$ is a tilting generator for the exotic t-structure.

We will check that: (i) $(\mathcal{E}^S_{\mathfrak{A}})^*$ is a tilting generator for $\mathcal{T}^{S,\tilde{\mathfrak{g}}}_{-\mathfrak{A}}$ where S is the formal neighborhood of the zero section $\mathcal{B}_\Bbbk$ in $\tilde{\mathfrak{g}}_\Bbbk$ for $\Bbbk \in \mathrm{FGP}$, and also that (ii) $\mathcal{O}$ is a direct summand in $b_{-\mathfrak{A},\mathfrak{A}_0}(\mathcal{E}^*_{\mathfrak{A}})$. Then Proposition 2.5.1(b) implies the desired statement.

Statement (i) is immediate from the standard isomorphism $(\mathcal{D}^{\mathcal{L}}_X)^{op} \cong \mathcal{D}^{\mathcal{L}^{-1}\otimes\Omega_X}_X$ where X is a smooth algebraic variety over a field, $\mathcal{L}$ is a line bundle, Ω_X is the line bundle of top degree forms, and $\mathcal{D}^{\mathcal{L}}_X$ is the sheaf of $\mathcal{L}$-twisted differential operators (cf. [BMR1]). In particular, we get $(\mathcal{D}^\lambda_{\mathcal{B}})^{op} \cong \mathcal{D}^{-2\rho-\lambda}_{\mathcal{B}}$ which shows that the dual of a splitting vector bundle for $\mathcal{D}^\lambda_{\mathcal{B}}$ on the formal neighborhood of the zero section is a splitting vector bundle for $\mathcal{D}^{-2\rho-\lambda}_{\mathcal{B}}$. It is easy to see that the choice of the splitting vector bundle for $\mathcal{D}^\lambda_{\mathcal{B}}$ as in [BMR1, Remark 5.2.2.2] leads to dual vector bundles for λ and $-2\rho-\lambda$.

To check the claim that $\mathcal{O}$ is a direct summand in $b_{-\mathfrak{A},\mathfrak{A}_0}(\mathcal{E}^*_{\mathfrak{A}})$ recall that for $w \in W$, $\lambda \in Q$ such that $\mathfrak{A} = \lambda + w^{-1}\mathfrak{A}_0$ one has $b_{\mathfrak{A}_0,\mathfrak{A}} = \theta_\lambda \widetilde{w}^{-1}$ (Lemma 1.8.iii). Since $-\mathfrak{A} = -\lambda + w^{-1}w_0\mathfrak{A}_0 = -\lambda + (w_0w)^{-1}\mathfrak{A}_0$, this also implies that $b_{\mathfrak{A}_0,-\mathfrak{A}} = \theta_{-\lambda}\widetilde{w_0w}^{-1}$ and so $b_{-\mathfrak{A},\mathfrak{A}_0} = \widetilde{w_0w}\theta_\lambda$.

Since $\mathcal{O}$ is a summand in $\mathcal{E}_{\mathfrak{A}_0}$, the sheaf $\mathcal{E}_{\mathfrak{A}} = b_{\mathfrak{A}_0,\mathfrak{A}}\mathcal{E}_{\mathfrak{A}_0} = \theta_\lambda \widetilde{w}^{-1}\mathcal{E}_{\mathfrak{A}_0}$ has a summand $\theta_\lambda \widetilde{w}^{-1}\mathcal{O} \cong \mathcal{O}(\lambda)$ (recall that $\widetilde{w}\mathcal{O} \cong \mathcal{O}$ for $w \in W$ by Lemma 1.3.4(c)) and $\theta_\lambda(\mathcal{O}) \cong \mathcal{O}(\lambda)$. Therefore, $\mathcal{O}(-\lambda)$ is a direct summand in $\mathcal{E}^*_{\mathfrak{A}}$ and $b_{-\mathfrak{A},\mathfrak{A}_0}(\mathcal{E}^*_{\mathfrak{A}}) = \widetilde{w_0w}\theta_\lambda(\mathcal{E}^*_{\mathfrak{A}})$ has a summand $\widetilde{w_0w}\theta_\lambda\ \mathcal{O}(-\lambda) \cong \widetilde{w_0w}\ \mathcal{O} \cong \mathcal{O}$. □

3. t-structures on $\tilde{\mathfrak{g}}_{\mathcal{P}}$ corresponding to alcoves on the wall

Let $\mathcal{P}$ be a partial flag variety and consider the space $\tilde{\mathfrak{g}}_{\mathcal{P}}$ of pairs of a parabolic subalgebra $\mathfrak{p} \in \mathcal{P}$ and an element in it. We have a map $\pi_{\mathcal{P}} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_{\mathcal{P}}$.

Recall that we consider the partition of $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ into alcoves, which are connected components of the complement to the hyperplanes $H_{\check{\alpha},n} := \{\lambda\ ;\ \langle \lambda, \check{\alpha}\rangle = n\}$ parametrized by all $\alpha \in \Delta$ and $n \in \mathbb{Z}$. The fundamental alcove $\mathfrak{A}_0 \subseteq \Lambda_{\mathbb{R}}$ is given by $0 < \langle \lambda, \check{\alpha}\rangle < 1$ for all positive coroots $\check{\alpha}$.

The *$\mathcal{P}$-wall* $\mathcal{W}_{\mathcal{P}} \subseteq \Lambda_{\mathbb{R}}$ is given by $\langle \lambda, \check{\alpha}\rangle = 0$ for roots α in the Levi root subsystem defined by $\mathcal{P}$. By a *$\mathcal{P}$-alcove* we mean a connected component of the complement in $\mathcal{W}_{\mathcal{P}}$ to those affine coroot hyperplanes which do not contain it.

Let $S \to \mathfrak{g}$ be an affine exact base change of $X \to \mathfrak{g}$ where $X = \tilde{\mathfrak{g}}_{\mathcal{P}}$ (in particular, X can be $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{\mathcal{B}}$). An example would be a (Slodowy) slice (the proof for $\tilde{\mathfrak{g}}$ in Lemma 1.3.1 works also for $\tilde{\mathfrak{g}}_{\mathcal{P}}$). The base change of $\pi_{\mathcal{P}}$ to S is a map $\pi^S_{\mathcal{P}} : \tilde{\mathfrak{g}}_S \to \tilde{\mathfrak{g}}_{\mathcal{P},S}$.

3.0.1. *Lemma.* Let $\varpi : X \to Y$ be a proper morphism of finite Tor dimension and assume that $\mathrm{R}\varpi_*\mathcal{O}_X \cong \mathcal{O}_Y^{\oplus N}$ for some N. Let $\mathcal{T}_X$, $\mathcal{T}_Y$ be t-structure on $D^b(\mathcal{C}oh(X))$, $D^b(Coh(Y))$ respectively.

a) If ϖ^* is *t*-exact, then $\mathcal{T}_Y$ is given by

$$\mathcal{F} \in D^{\leq 0}_{\mathcal{T}_Y} \iff \varpi^*\mathcal{F} \in\ D^{\leq 0}_{\mathcal{T}_X}; \qquad \mathcal{F} \in D^{\geq 0}_{\mathcal{T}_Y} \iff \varpi^*\mathcal{F} \in\ D^{\geq 0}_{\mathcal{T}_X}.$$

In particular, $\mathcal{T}_Y$ is then uniquely determined by $\mathcal{T}_X$.

The same applies with ϖ^* replaced by $\varpi^!$.

b) If $\mathcal{E}$ is a projective generator for the heart of $\mathcal{T}_X$ and $\varpi^!$ is t-exact then $\varpi_*(\mathcal{E})$ is a projective generator for the heart of $\mathcal{T}_Y$.

c) Assume that $\mathcal{T}_X$ corresponds to a tilting generator $\mathcal{E}_X$, and $\mathcal{T}_Y$ corresponds to a tilting generator $\mathcal{E}_Y$. Then the functor $\varpi^!$ is t-exact iff $\mathcal{E}_Y$ is equiconstituted with $\varpi_*(\mathcal{E}_X)$.

The functor ϖ_* is t-exact iff $\varpi^*(\mathcal{E}_Y)$ is a direct summand in $\mathcal{E}_X^{\oplus N}$ for some N.

Proof. The "$\Rightarrow$" implication in (a) is immediate from the t-exactness assumption. We check the converse for $D^{<0}$, the argument for $D^{>0}$ is similar. If $\mathcal{F} \notin D^{<0}$ then we have a nonzero morphism $\phi : \mathcal{F} \to \mathcal{G}$, $\mathcal{G} \in D^{\geq 0}_{\mathcal{T}_Y}$. The projection formula shows that $\varpi_* \circ \varpi^* \cong Id^{\oplus N}$, thus the map $\varpi^*(\phi) : \varpi^*\mathcal{F} \to \varpi^*\mathcal{G}$ is not zero. Since $\varpi^*(\mathcal{G}) \in D^{\geq 0}_{\mathcal{T}_X}$ this implies that $\varpi^*(\mathcal{F}) \notin D^{\leq 0}_{\mathcal{T}_X}$. This proves the statement about ϖ^*, the proof for $\varpi^!$ is parallel, using the fact that $\varpi_* \circ \varpi^! \cong Id^{\oplus N}$ (this isomorphism follows from the one for $\varpi_*\varpi^*$ by Grothendieck-Serre duality).

To prove (b) recall that a functor between abelian categories sends projective objects to projective ones provided that its right adjoint is exact. Thus ϖ_* sends projective objects in the heart of $\mathcal{T}_X$ to projective ones in the heart of $\mathcal{T}_Y$ if $\varpi^!$ is t-exact. Also it sends weak generators of $D(q\mathcal{C}oh(X))$ to weak generators of $D(q\mathcal{C}oh(Y))$ since $\varpi^!$ is conservative (kills no objects), which is clear from $\varpi_* \circ \varpi^! \cong Id^{\oplus N}$.

The "only if" direction in the first statement in (c) follows from (b), while the "if" part is clear from the definition of the t-structure corresponding to a tilting generator and adjointness between ϖ_* and $\varpi^!$. The second statement is clear from the fact that a functor between abelian categories is exact if and only if its left adjoint sends a projective generator to a projective object (equivalently, to a summand of a some power of a projective generator). □

Recall from 1.8 that there is a collection of locally free t-structures $\mathcal{T}^S_{\mathfrak{A}}$ on $\widetilde{S}$ indexed by alcoves $\mathfrak{A}$, and one can choose the corresponding tilting generator $\mathcal{E}^S_{\mathfrak{A}}$ as a pull-back of a $G \times \mathbb{G}_m$-equivariant locally free tilting generator $\mathcal{E}_{\mathfrak{A}}$ on $\tilde{\mathfrak{g}}$.

3.0.2. *Theorem.* a) There exists a unique map $\mathfrak{A}_{\mathcal{P}} \mapsto \mathcal{T}^S_{\mathfrak{A}_{\mathcal{P}}}$ from the set of $\mathcal{P}$-alcoves $\mathfrak{A}_{\mathcal{P}}$ to the set of t-structures on $D^b(Coh(\tilde{\mathfrak{g}}_{\mathcal{P},S}))$ such that:

If $\mathfrak{A}_{\mathcal{P}}$ is a $\mathcal{P}$-alcove in the closure of an alcove $\mathfrak{A}$ then both of the functors $(\pi^S_{\mathcal{P}})_*$ and $(\pi^S_{\mathcal{P}})^* \cong (\pi^S_{\mathcal{P}})^!$, between $D^b(\mathcal{C}oh(\tilde{\mathfrak{g}}_S))$ and $D^b(\mathcal{C}oh(\tilde{\mathfrak{g}}_{\mathcal{P},S}))$, are exact for t-structures $\mathcal{T}^S_{\mathfrak{A}}$ and $\mathcal{T}^S_{\mathfrak{A}_{\mathcal{P}}}$.

b) Each $\mathcal{T}^S_{\mathfrak{A}_{\mathcal{P}}}$ is bounded and locally free. Moreover, for $\mathfrak{A}$, $\mathfrak{A}_{\mathcal{P}}$ as above, any projective generator $\mathcal{E}^S_{\mathfrak{A}}$ for $\mathcal{T}^S_{\mathfrak{A}}$ produces a locally free projective generator $\mathcal{E}^S_{\mathfrak{A}_{\mathcal{P}}} = \mathrm{R}(\pi^S_{\mathcal{P}})_*(\mathcal{E}^S_{\mathfrak{A}})$ for $\mathcal{T}^S_{\mathfrak{A}_{\mathcal{P}}}$.

c) If S lies over a geometric point $\Bbbk$ of R then $(\pi^S_{\mathcal{P}})_*$ sends any irreducible object in the heart of $\mathcal{T}^S_{\mathfrak{A}}$ either to zero, or to an irreducible object in the heart of $\mathcal{T}^S_{\mathfrak{A}_{\mathcal{P}}}$. This gives a bijection of $\mathcal{T}^S_{\mathfrak{A}}$-irreducibles with non-zero images and $\mathcal{T}^S_{\mathfrak{A}_{\mathcal{P}}}$-irreducibles.

Proof. Isomorphism $(\pi^S_{\mathcal{P}})^* \cong (\pi^S_{\mathcal{P}})^!$ follows from the fact that $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{g}}_{\mathcal{P}}$ are smooth over R of the same dimension.

The direct image $(\widetilde{\mathfrak{g}} \to \mathfrak{g})_*\mathcal{O}_{\widetilde{\mathfrak{g}}}$ is $\mathcal{O}_{\mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h}}$ and this is a free module of rank $|W|$ over $\mathcal{O}_{\mathfrak{g}}$ since the same is true for $\mathcal{O}(\mathfrak{h})$ as a module for $\mathcal{O}(\mathfrak{h}/W)$. Thus Lemma 3.0.1 applies.

Uniqueness of a t-structure $\mathcal{T}^S_{\mathfrak{A}_{\mathcal{P}}}$ for which $(\pi^S_{\mathcal{P}})^*$ is t-exact follows from Lemma 3.0.1(a). The remaining part of statements (a,b) is equivalent, in view of Lemma 3.0.1(c), to the following statement:

($\bullet_S$) $\mathrm{R}(\pi^S_{\mathcal{P}})_*(\mathcal{E}^S_{\mathfrak{A}})$ is a locally free tilting generator. Moreover, $(\pi^S_{\mathcal{P}})^*\mathrm{R}(\pi^S_{\mathcal{P}})_*(\mathcal{E}^S_{\mathfrak{A}})$ is a direct summand in $(\mathcal{E}^S_{\mathfrak{A}})^{\oplus N}$ for some N.

In the special case when S is the formal neighborhood $\widehat{0}_{\Bbbk}$ of zero in $\mathfrak{g}_{\Bbbk}$, $\Bbbk \in \mathrm{FGP}$, statements (a,b) and hence ($\bullet_S$), follow from results of [BMR2]. More precisely, the t-structure arising from the singular localization theorem and compatible splitting bundle satisfies the exactness properties because direct and inverse image functors correspond to translation functors to/from the wall [BMR2, Lemma 2.2.5], which are well known to be exact. A projective generator for the heart of the t-structure can in this case be chosen to be a splitting bundle on $\widehat{0}_{\Bbbk}$ for an Azumaya algebra $\widetilde{\mathcal{D}}$ on $\tilde{\mathfrak{g}}^{(1)}_{\mathcal{P}}$ [BMR2, Remark 1.3.5]. This projective generator (and hence any) is clearly locally free.

The general case of ($\bullet_S$) follows from the above special case by the reasoning of section 2.

First, $\mathrm{R}(\pi^S_{\mathcal{P}})_*(\mathcal{E}^S_{\mathfrak{A}})$ is a weak generator because $\mathcal{E}^S_{\mathfrak{A}}$ is a weak generator and the right adjoint functor $(\pi^S_{\mathcal{P}})^!$ is conservative.

Now (b) would follow once we verify that $\mathrm{R}(\pi_{\mathcal{P}}^S)_*(\mathcal{E}_{\mathfrak{A}}^S)$ is locally free and satisfies the Ext vanishing condition in Theorem 1.4.3, while $\mathrm{Hom}(\mathcal{E}_{\mathfrak{A}}^S, (\pi_{\mathcal{P}}^S)^*\mathrm{R}(\pi_{\mathcal{P}}^S)_*(\mathcal{E}_{\mathfrak{A}}^S))$ is projective as a module over $End(\mathcal{E}_{\mathfrak{A}}^S)$ (notice that vanishing of $\mathrm{Ext}^i(\mathcal{E}_{\mathfrak{A}}^S, (\pi_{\mathcal{P}}^S)^*\mathrm{R}(\pi_{\mathcal{P}}^S)_*(\mathcal{E}_{\mathfrak{A}}^S))$ for $i \neq 0$ follows from the tilting property of $\mathrm{R}(\pi_{\mathcal{P}}^S)_*(\mathcal{E}_{\mathfrak{A}}^S)$)). It suffices to do it in the "absolute" case $S = \mathfrak{g}$. Then local freeness and Ext vanishing follow from the above special case in view of Lemma 2.5.2. Similarly, projectivity holds since it holds after base change to any algebraically closed field of positive characteristic and completing by the grading topology.

Finally (c) follows from the next Proposition. □

We keep the notations of the Theorem, fix (and drop from notations) the alcove $\mathfrak{A}$, and $S = \mathfrak{g}$. Set $\mathcal{A} = End(\mathcal{E})^{op}$, $\mathcal{A}_{\mathcal{P}} = End(\mathcal{E}_{\mathcal{P}})^{op}$, $\mathcal{M} = \mathrm{Hom}(\mathcal{E}, \pi_{\mathcal{P}}^*(\mathcal{E}_{\mathcal{P}}))$. Thus $\mathcal{M}$ is an $\mathcal{A} - \mathcal{A}_{\mathcal{P}}$ bimodule.

3.0.3. *Proposition.* a) Under the above equivalences $D^b(Coh(\tilde{\mathfrak{g}})) \cong D^b(mod^{fg}(\mathcal{A}))$, $D^b(Coh(\tilde{\mathfrak{g}}_{\mathcal{P}})) \cong D^b(mod^{fg}(\mathcal{A}_{\mathcal{P}}))$ the functor $\mathrm{R}\pi_*$ is identified with the functor $N \mapsto N \overset{\mathrm{L}}{\otimes} \mathcal{M}$.

b) The natural map $\mathcal{O}(\mathfrak{h}^*) \otimes_{\mathcal{O}(\mathfrak{h}^*)^{W_L}} \mathcal{A}_{\mathcal{P}}^{op} \longrightarrow End_{\mathcal{A}}(\mathcal{M})$ is an isomorphism.

Proof. a) is obvious from the definitions. It suffices to prove that (b) becomes true after base change to the formal neighborhood of zero in $\mathfrak{g}_{\Bbbk}^*$, $\Bbbk \in \mathrm{FGP}$. It is clear that validity of the statement is independent on the choice of tilting generators $\mathcal{E}$, $\mathcal{E}_{\mathcal{P}}$ for the hearts of a given t-structure. An appropriate choice of $\mathcal{E}$, $\mathcal{E}_{\mathcal{P}}$ yields $\mathcal{A} = \Gamma(\widetilde{\mathcal{D}})$, $\mathcal{A}_{\mathcal{P}} = \Gamma(\widetilde{\mathcal{D}}_{\mathcal{P}})$, while $\mathcal{M}$ is the space of sections of the bimodule $\mathcal{B}_\lambda^\mu$ providing the equivalence between the Azumaya algebras $\widetilde{\mathcal{D}}^{\hat{\lambda}}$, $\widetilde{\mathcal{D}}^{\hat{\mu}}$ on $FN(\widetilde{\mathcal{N}})_{\tilde{\mathfrak{g}}}$ [BMR2, Remark 1.3.5]. Then the statement follows from

$$End_{\mathcal{A}}(\mathcal{M}) = \Gamma(\mathcal{E}nd_{\widetilde{\mathcal{D}}^{\hat{\lambda}}}(\mathcal{B}_\lambda^\mu)) = \Gamma(\widetilde{\mathcal{D}}^{\hat{\mu}})^{op}$$
$$\cong \Gamma(\pi_{\mathcal{P}}^*(\bar{\mathcal{D}}^{\hat{\mu}})^{op}) = \mathcal{O}(\mathfrak{h}^*) \otimes_{\mathcal{O}(\mathfrak{h}^*/W_L)} \Gamma((\bar{\mathcal{D}}^{\hat{\mu}})^{op}) = \mathcal{O}(\mathfrak{h}^*) \otimes_{\mathcal{O}(\mathfrak{h}^*/W_L)} \mathcal{A}_{\mathcal{P}}.$$

4. **t-structures on $T^*(G/P)$ corresponding to alcoves on the wall**

Let $P \subset G$ be a parabolic, $\mathcal{P} = G/P$, $L \subseteq P$ be a Levi subgroup. The projection $\mathcal{B} \to \mathcal{P}$ yields the maps $T^*(\mathcal{P}) \xleftarrow{pr_1} T^*(\mathcal{P}) \times_{\mathcal{P}} \mathcal{B} \xhookrightarrow{i} T^*\mathcal{B}$.

Let $\Lambda^P \subset \Lambda$ be the sublattice of weights orthogonal to coroots in L. We have embeddings of finite index $\Lambda^P \subset Pic(\mathcal{P})$, $\Lambda \subset Pic(\mathcal{B})$ compatible with the pull-back under projection map $\mathcal{B} \to \mathcal{P}$.

(If G is simply connected then both embeddings are isomorphisms .)

4.1. **t-structures in positive characteristic.** In the next Proposition we work over a field $\Bbbk = \bar{\Bbbk}$ of characteristic $p > h$. As in [BMR2, 1.10], $\mathfrak{D}^\lambda_{\mathcal{P}}$ denotes the sheaf of crystalline differential operators on $\mathcal{P}$ with a twist $\lambda \in \Lambda^P$.

Proposition. For a weight $\lambda \in \Lambda^P$ such that the element "$(\lambda+\rho)$ mod p" of $\mathfrak{h}^*$ is regular, consider the functor

$$\pi^\star : \ D^b(\mathcal{C}oh(T^*\mathcal{P})) \to D^b(\mathcal{C}oh(T^*\mathcal{B})), \quad \pi^\star\mathcal{F} = i_* pr_1^*(\mathcal{F}) \otimes \mathcal{O}(\rho)$$

and its left adjoint $\pi_\star\mathcal{G} = (pr_1)_* i^*[\mathcal{G}(-\rho)]$.

a) If $\mathrm{R}^{>0}\Gamma(\mathfrak{D}^\lambda_{\mathcal{P}}) = 0$ then there exists a unique t-structure on $D^b(\mathcal{C}oh(T^*\mathcal{P}))$, such that the functor $\pi^\star$ is t-exact, where the target is equipped with the t-structure $\mathcal{T}_{\mathfrak{A}}$ corresponding to the alcove $\mathfrak{A}$ containing λ.

This t-structure is locally free and a $\mathbb{G}_m$-equivariant projective generator of its heart is given by $\mathcal{E}_{\mathcal{P}} \stackrel{\text{def}}{=} \pi_\star\mathcal{E}_{\mathfrak{A}}$ for any $\mathbb{G}_m$-equivariant projective generator $\mathcal{E}_{\mathfrak{A}}$ of the heart of $\mathcal{T}_{\mathfrak{A}}$.

b) If the map $U\mathfrak{g} \to \Gamma(\mathfrak{D}^\lambda_P)$ is surjective, then the t-exact functor $\pi^\star$ sends irreducible objects to irreducible ones, and it is injective on isomorphism classes of irreducibles.

Proof. The functor $\pi^\star\mathcal{F} = i_* pr_1^*(\mathcal{F}) \otimes \mathcal{O}(\rho)$ is conservative, thus its left adjoint $\pi_\star$ sends a generator to a generator. Thus for a t-structure on $D^b(Coh(T^*\mathcal{P}))$ the functor $\pi^\star$ is t-exact iff $\mathcal{E}_{\mathcal{P}} = \pi_\star[\mathcal{E}_{\mathfrak{A}}]$ is a projective generator of its heart. This shows uniqueness in (a) and reduces the rest of statement (a) to showing that $\mathcal{E}_{\mathcal{P}}$ is a tilting vector bundle.

As above, it suffices to check that the restriction of $\mathcal{E}_{\mathcal{P}}$ to the formal neighborhood of the zero section of $T^*\mathcal{P}$, is a tilting vector bundle. Reversing the argument of the previous paragraph we see that it suffices to show the existence of a locally free t-structure on $D^b(Coh_{\mathcal{P}}(T^*\mathcal{P}))$ such that $\pi^\star$ is t-exact.

It is shown in [BMR2, 1.10] that under the assumptions of (a) the derived global sections functor $\mathrm{R}\Gamma : D^b[mod^{fg}(\mathfrak{D}^\lambda_{\mathcal{P}})] \to D^b[mod^{fg}(\Gamma(\mathfrak{D}^\lambda_{\mathcal{P}}))]$ is an equivalence. Furthermore, $\mathfrak{D}^\lambda_{\mathcal{P}}$ is an Azumaya algebra over $T^*\mathcal{P}^{(1)}$ which is split on the formal neighborhood of the fibers

of the moment map $\mu_{\mathcal{P}} : T^*\mathcal{P}^{(1)} \to (\mathfrak{g}^*)^{(1)}$, in particular on the formal neighborhood of the zero fiber. Thus we get an equivalence between $D^b[Coh_{\mathcal{P}}(T^*\mathcal{P})]$ and the derived category of modules over a certain algebra [BMR2, Corollary 1.0.4]. In view of [BMR2, Proposition 1.10.7] we see that the resulting t-structure on $D^b[Coh_{\mathcal{P}}(T^*\mathcal{P})]$ satisfies the desired exactness property, thus (a) is proved.

The same Proposition also implies claim (b), since it shows that the functor between the abelian hearts induced by the functor $D^b(Coh_{\mu_{\mathcal{P}}^{-1}(e)}(T^*\mathcal{P})) \xrightarrow{\pi^\star} D^b(Coh_{\pi^{-1}(e)}(T^*\mathcal{B}))$, can be identified with the pull-back functor between the categories of modules (with a fixed generalized central character) corresponding to the ring homomorphism $U^\lambda(\mathfrak{g}) \to \Gamma(\mathfrak{D}^\lambda_{\mathcal{P}})$. If this ring homomorphism is surjective then the pull-back functor sends irreducible modules to irreducible ones and distinguishes the isomorphism classes of irreducibles. □

4.2. **Lifting to characteristic zero.** We now return to considerations over an arbitrary base. Let $S \to \mathfrak{g}$ be a base change exact for both $T^*(\mathcal{B}) \to \mathfrak{g}$ an $T^*(\mathcal{P}) \to \mathfrak{g}$. Again, any Slodowy slice is an example of such S. Making base change to S we get maps i_S, $pr_{1,S}$ and the functor $\pi_S^\star : D^b[Coh(\widetilde{S}_{\mathcal{P}})] \to D^b[Coh(\widetilde{S})]$, where $\widetilde{S}_{\mathcal{P}} = S \times_{\mathfrak{g}} T^*\mathcal{P}$.

Theorem. There exists an integer $N > 0$ (depending on the type of G only), such that the following is true provided that N is invertible on S.

(a) Fix an alcove $\mathfrak{A}$. Assume that there exists a weight $\lambda \in \Lambda^P$, such that $\frac{\lambda+\rho}{p} \in \mathfrak{A}$.

Then there exists a unique t-structure $\mathcal{T}_{\mathfrak{A}}^{\widetilde{S}_{\mathcal{P}}}$ on $D^b(\mathcal{C}oh(\widetilde{S}_P))$, such that the functor $\pi_S^\star : D^b[Coh(\widetilde{S}_{\mathcal{P}})] \to D^b[Coh(\widetilde{S})]$ is t-exact, where the target is equipped with the t-structure $\mathcal{T}_{\mathfrak{A}}^S$ corresponding to $\mathfrak{A}$.

The t-structure $\mathcal{T}_{\mathfrak{A}}^{\widetilde{S}_{\mathcal{P}}}$ is locally free over S, a projective generator of its heart is given by $\mathcal{E}_{\mathcal{P}}^S = pr_{1,S_*} i_S^*[\mathcal{E}_{\mathfrak{A}}(-\rho)]$ for any projective generator $\mathcal{E}_{\mathfrak{A}}$ of the heart of $\mathcal{T}_{\mathfrak{A}}^S$.

(b) Let $\Bbbk$ be a geometric point of R such that the pull-back map $\mathcal{O}(\mathfrak{g}_\Bbbk) \to \Gamma(\mathcal{O}(T^*(\mathcal{P}_\Bbbk)))$ is surjective. Then the t-exact functor $\pi_S^\star$ sends irreducible objects to irreducible ones and is injective on isomorphism classes of irreducibles.

Proof. It is well known [Bro] that for fields $\Bbbk$ of characteristic zero, hence also for $\Bbbk$ of a sufficiently large positive characteristic, we have $H^{>0}(T^*\mathcal{P}_\Bbbk, \mathcal{O}) = 0$. It follows that the cohomology vanishing condition of Proposition 4.1(a) holds over such a field. Thus Proposition 4.1 shows that statement (a) is true when $S = \mathfrak{g}^*_\Bbbk$ where $\Bbbk$ is an algebraically

closed field of sufficiently large positive characteristic. Then the general case follows, as in the proof of Theorem 1.5.1, by Proposition 2.5.1. This proves part (a).

The general case of statement (b) follows by a standard argument from the case when $\mathbb{k}$ has (large) positive characteristic. (Notice that if the map $\mathcal{O}(\mathfrak{g}_{\mathbb{k}}) \to \mathcal{O}(T^*\mathcal{P}_{\mathbb{k}})$ is surjective for $\mathbb{k}$ of characteristic zero, then it is surjective for $\mathbb{k}$ of large positive characteristic). In the latter case the statement follows from Proposition 4.1(b), since surjectivity of the map $\mathcal{O}(\mathfrak{g}_{\mathbb{k}}) \to \mathcal{O}(T^*\mathcal{P}_{\mathbb{k}})$ implies surjectivity of the map $U(\mathfrak{g}_{\mathbb{k}}) \to \Gamma(\mathfrak{D}^{\lambda}_{\mathcal{P}_{\mathbb{k}}})$. □

4.2.1. *Remark.* a) It is well known that conditions of part (b) hold when G is of type A_n and $\mathbb{k}$ is of characteristic zero.

b) The twist by ρ appearing in the last Theorem is caused by normalizations in the definition of the braid group action. Removing thus shift would produce a shift in the preceding Theorem.

c) Notice that control on the set of primes for which the result is valid is weaker here than in other similar results of [BMR2] and sections 1, 3. The only reason for this is that higher cohomology vanishing for the sheaf $\mathcal{O}(T^*\mathcal{P})$ has not been established in general (to our knowledge) in positive characteristic p except for indefinitely large p.

4.2.2. *Remark.* It can be deduced from the results of [Ri2] that the construction of the present subsection is related to that of the preceding one by Koszul duality (see *loc. cit.* for details).

The matching of combinatorial parameters is explained in the next subsection.

4.3. **Shifting the alcoves.** The set of t-structures constructed in this subsection is indexed by the set of alcoves $\mathfrak{A}$ such that $\frac{\lambda+\rho}{p} \in \mathfrak{A}$ for some integral $\lambda \in \Lambda^P$. Let us denote this set by $\mathrm{Alc}_{\mathcal{P}}$.

4.3.1. *Lemma.* The set $\mathrm{Alc}_{\mathcal{P}}$ is in a canonical bijection with the set of $\mathcal{P}$-alcoves. The bijection sends $\mathfrak{A} \in \mathrm{Alc}_{\mathcal{P}}$ to the interior of $\bar{\mathfrak{A}} \cap \mathcal{W}_{\mathcal{P}}$.

Proof. By a $\mathcal{P}$-*chamber* we will mean a connected component of the complement in the $\mathcal{P}$-wall $\mathcal{W}_{\mathcal{P}}$, of intersections with coroot hyperplanes not containing the wall. The $\mathcal{P}$-chambers are in bijection with parabolic subalgebras in the Langlands dual[10] algebra $\check{\mathfrak{g}}$ with a Levi $\check{\mathfrak{l}} \subseteq \check{\mathfrak{g}}$ whose semisimple part is given by coroots orthogonal to λ.

A Weyl chamber is a connected component of the complement to coroot hyperplanes in $\mathfrak{h}^*_{\mathbb{R}}$. We will say that a Weyl chamber is *near the* $\mathcal{P}$ *wall* if it contains a $\mathcal{P}$-chamber in its closure. The set of Weyl chambers is in bijection with the set of Borel subalgebras with a fixed Cartan. So, a Weyl chamber is near the $\mathcal{P}$-wall iff the corresponding Borel subalgebra $\check{\mathfrak{b}}$ is contained in a parabolic subalgebra with Levi $\check{\mathfrak{l}}$, in other words if the subspace $\check{\mathfrak{l}} + \check{\mathfrak{b}} \subset \check{\mathfrak{g}}$ is a subalgebra.

It suffices to show that for every weight $\lambda \in \Lambda^P$ such that $\lambda + \rho$ is regular, $\lambda + \rho$ lies in a Weyl chamber which is near the $\mathcal{P}$ wall. This amounts to showing that if $\check{\alpha}, \check{\beta}$ and $\check{\alpha} + \check{\beta}$ are coroots and $\langle \check{\alpha}, \lambda + \rho \rangle > 0$, $\langle \check{\beta}, \lambda \rangle = 0$, then either $\langle \check{\alpha} + \check{\beta}, \lambda + \rho \rangle > 0$ or $\langle \check{\alpha} + \check{\beta}, \lambda \rangle = 0$.

It is enough to check that $\langle \check{\alpha} + \check{\beta}, \lambda + \rho \rangle > 0$ assuming $\check{\beta}$ is a simple negative coroot in $\check{\mathfrak{l}}$. For such $\check{\beta}$ we have $\langle \check{\beta}, \lambda + \rho \rangle = \langle \check{\beta}, \rho \rangle = -1$, so $\langle \check{\alpha} + \check{\beta}, \lambda + \rho \rangle \geq 0$. However, since we have assumed that $\lambda + \rho$ is regular and $\check{\alpha} + \check{\beta}$ is a coroot, this implies that $\langle \check{\alpha} + \check{\beta}, \lambda + \rho \rangle \neq 0$. $\square$

[10] Of course, these are also in bijection with parabolic subalgebras in $\mathfrak{g}$ with the fixed Levi. However, the argument uses coroots rather than roots, hence the appearance of $\check{G}$.

5. Applications to Representation Theory

The results of section 2 imply independence of the numerics of $\mathfrak{g}_{\Bbbk}$ modules on the characteristic p of the base field $\Bbbk$ for $p \gg 0$. This is spelled out in subsection 5.1.

In 5.3, 5.4 we briefly recall Lusztig's conjectural description of the numerical structure of the theory and reduce its verification to a certain positivity property of a grading on the category of representations.

Given the results of sections 1, 2 it is easy to construct a family of gradings on the category. However, showing that this family contains a grading which satisfies the positivity property is more subtle. This is established in section 6 by making use of the results of [ArkB]. This reduction relies on G-equivariant versions of some of the above constructions. This technical variation is presented in 5.2.

To simplify notations we only treat the case of a regular block and a nilpotent Frobenius central character, generalization to any block and an arbitrary p-central character is straightforward.

5.1. **Generic independence of p.** In this section $\mathcal{E}$ is a vector bundle on $\tilde{\mathfrak{g}}$ that satisfies Theorem 1.5.1.b (so up to equiconstitutedness it is (i) unique, and (ii) of the form $\Xi_{\mathcal{J}}$ from Theorem 2.5); while A is the R-algebra $\mathrm{End}(\mathcal{E})^{op}$.

5.1.1. *Theorem.* The algebra A satisfies the following. For any $\Bbbk \in \mathrm{FGP}$ and any $e \in \mathcal{N}_{\Bbbk}$, there are canonical Morita equivalences compatible with the action of $\mathcal{O}_{\mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h}}$:

$$U^{\hat{0}}_{\Bbbk,\hat{e}} \ \sim \ A^{\hat{0}}_{\Bbbk,\hat{e}} \quad \text{and} \quad U^{0}_{\Bbbk,\hat{e}} \ \sim \ A^{0}_{\Bbbk,\hat{e}}.$$

Here, $U^{\hat{0}}_{\Bbbk,\hat{e}}$, $A^{\hat{0}}_{\Bbbk,\hat{e}}$ are completions of $U_{\Bbbk}$ and $A_{\Bbbk} = A \otimes_R \Bbbk$, at the central ideals corresponding to e and to $0 \in \mathfrak{h}$ (resp., $A^{0}_{\Bbbk,\hat{e}}$, $U^{0}_{\Bbbk,\hat{e}}$ are completions of $U^0_{\Bbbk}$ and $A^0_{\Bbbk} = A_{\Bbbk} \otimes_{\mathcal{O}(\mathfrak{h})} \Bbbk_0$ at the central ideals corresponding to e).

Proof. Recall that $A^0 \stackrel{\mathrm{def}}{=} A \otimes_{\mathcal{O}(\mathfrak{h})} \mathcal{O}_0$ is the algebra $\mathrm{End}(\mathcal{E}|_{\widetilde{\mathcal{N}}})^{op}$ (Lemma 1.5.3), and that there are canonical equivalences

$$mod^{fg}(U^{\hat{0}}_{\hat{e}}) \cong \mathcal{E}coh(\widehat{\mathcal{B}_{\Bbbk,e}}) \ \xleftarrow{\text{restriction}} \ \mathcal{E}coh(\tilde{\mathfrak{g}}) \cong mod^{fg}(A)$$

and $mod^{fg}(U^{0}_{\hat{e}}) \cong \mathcal{E}coh(\widehat{\mathcal{B}_{\Bbbk,e}}') \ \xleftarrow{\text{restriction}} \ \mathcal{E}coh(\widetilde{\mathcal{N}}) \cong mod^{fg}(A^0)$. Both times, the first equivalence is from Theorem 1.6.3, and the second one is the fact that $\mathcal{E}$ (resp. $\mathcal{E}|_{\widetilde{\mathcal{N}}}$) is a tilting generator and that the locally free t-structure it produces is the exotic t-structure.

The following induced equivalences of categories are the desired Morita equivalences of algebras:

$$mod^{fg}(U^{\widehat{0}}_{\widehat{e}}) \cong \mathcal{E}coh(\widehat{\mathcal{B}_{\Bbbk,e}}) \cong mod^{fg}(A^{\widehat{0}}_{\Bbbk,\widehat{e}}) \quad \text{and} \quad mod^{fg}(U^{0}_{\widehat{e}}) \cong \mathcal{E}coh(\widehat{\mathcal{B}_{\Bbbk,e}}') \cong mod^{fg}(A^{0}_{\Bbbk,\widehat{e}}).$$

We can define an action of $\mathcal{O}_{(\mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h}/W)_\Bbbk}$ on the leftmost terms using the center of $U_\Bbbk$. Recall that it is isomorphic to $\mathcal{O}_{\mathfrak{g}_\Bbbk\times_{\mathfrak{h}_\Bbbk/W}\mathfrak{h}_\Bbbk/W}$, where map $\mathfrak{h}_\Bbbk \to \mathfrak{h}_\Bbbk$ is the Artin-Schreier map (see 1.6.1). However, for a *regular* $\lambda \in \Lambda$ (say $\lambda = 0$) and any $e \in \mathcal{N}_\Bbbk$, the completion of this center at the point $(e, W\lambda)$ is canonically isomorphic to the completion of $\mathcal{O}_{\mathfrak{g}_\Bbbk\times_{\mathfrak{h}_\Bbbk/W}\mathfrak{h}_\Bbbk}$ (where this time $\mathfrak{h}_\Bbbk \to \mathfrak{h}_\Bbbk/W$ is just the quotient map), at the point (e, λ). □

5.1.2. *Remarks.* (1) Let $e \in \mathcal{N}(R)$. Since Morita invariance is inherited by central reductions we get also R-algebras A^0_e, $A^{\widehat{0}}_e$, $A^0_{\hat{e}}$ whose base change to any $\Bbbk \in FGP$ is Morita equivalent to the corresponding central reductions $U^0_{\Bbbk,e}, U^{\widehat{0}}_{\Bbbk,e}, U^0_{\Bbbk,\widehat{e}}$ of the enveloping algebra. Here $U^{\widehat{0}}_{\Bbbk,e}$ is the most popular version – its category of modules is the principal block of the category of $U_{\Bbbk,e}$-modules.

(2) A similar result for representations of algebraic groups (rather than Lie algebras) has been established by Andersen, Jantzen and Soergel [AJS]. This is equivalent to the case $e = 0$ of the theorem.

5.1.3. *Cartan matrices.* Recall that the set of nilpotent conjugacy classes in $\mathfrak{g}_\Bbbk$ does not depend on the algebraically closed field $\Bbbk$ provided its characteristic is a good prime.

Corollary. There exists a finite set of primes Π such that for any $\Bbbk, \Bbbk' \in$ FGP with characteristics outside Π, the following holds. If the conjugacy classes of $e \in \mathcal{N}_\Bbbk$, $e' \in \mathcal{N}_{\Bbbk'}$ correspond to each other, then the Cartan matrices[11] of $U^{\hat{0}}_e$, $U^{\hat{0}}_{e'}$ coincide.

The same applies to U^0_e, $U^0_{e'}$.

Proof. For a finite rank R-algebra $\mathcal{A}$ independence of the Cartan matrix of $\mathcal{A}_\Bbbk$ on $\Bbbk$, for an algebraically closed field $\Bbbk$ of sufficiently large characteristic, is well known (cf. also Lemma 5.1.5 below). Thus the claim follows from part (1) of the the previous Remark. □

[11] By this we mean the matrix whose entries are multiplicities of irreducible modules in indecomposable projective ones.

5.1.4. *Lifting irreducible and projective objects.* In the remainder of the subsection we will enhance the last Corollary to a geometric statement.

A quasifinite domain over a commutative ring R is a commutative ring $\mathcal{R}$ over R such that there exists $R' \supseteq R$ such that R' is a finite domain over R and $\mathcal{R}$ is a finite localization of R'.

Proposition. For any $e \in \mathcal{N}(R)$ there exists a quasifinite domain $\mathcal{R}$ over R, such that there exist

- a finite set I_e,
- a collection $\mathcal{E}_i$, $i \in I_e$, of locally free sheaves on $\widehat{\mathcal{B}_{\mathcal{R},e}}$, and
- a collection of complexes of coherent sheaves $\mathcal{L}_i \in D^b[\mathcal{C}oh(\widehat{\mathcal{B}_{\mathcal{R},e}}')]$, $i \in I_e$;

with the following properties.

Let $\lambda \in \Lambda$ be such that $\frac{\lambda+\rho}{p}$ is in the fundamental alcove. Then for every finite characteristic geometric point $\Bbbk$ of $\mathcal{R}$, the set of isomorphism classes of irreducible $U^{\lambda}_{\Bbbk,e}$ modules is canonically parametrized by I_e, we denote this $I_e \ni i \mapsto L_{\Bbbk,i} \in Irr(U^{\lambda}_{\Bbbk,e})$. This parametrization is such that the equivalence of [BMR1] sends:

- (I) irreducible $L_{\Bbbk,i}$ to $(\mathcal{L}_i)_{\Bbbk} \stackrel{\text{def}}{=} \mathcal{L}_i \overset{L}{\otimes}_R \Bbbk \in D^b[\mathcal{C}oh(\widehat{\mathcal{B}_{\Bbbk,e}}')]$;
- (P_1) the projective cover of $L_{\Bbbk,i}$ over $U^{\hat{\lambda}}_{\Bbbk,\hat{e}}$ to $(\mathcal{E}_i)_{\Bbbk} \stackrel{\text{def}}{=} \mathcal{E}_i \otimes_R \Bbbk \in D^b[\mathcal{C}oh(\widehat{\mathcal{B}_{\Bbbk,e}})]$;
- (P_2) the projective cover of $L_{\Bbbk,i}$ over $U^{\lambda}_{\Bbbk,\hat{e}}$ to $(\mathcal{E}_i)_{\Bbbk}\big|_{\mathcal{B}'_{\Bbbk,e}} \in D^b[\mathcal{C}oh(\widehat{\mathcal{B}_{\Bbbk,e}}')]$;
- (P_3) the projective cover of $L_{\Bbbk,i}$ over $U^{\hat{\lambda}}_{\Bbbk,e}$ to $\mathcal{E}_i \overset{L}{\otimes}_{\mathcal{O}_{\mathfrak{g}}} \Bbbk_e \in D^b[\mathcal{C}oh(\widehat{\mathcal{B}_{\Bbbk,e}})]$.

Proof. The first three claims are immediate from the Theorem 5.1.1, together with the next general Lemma. Part (P_3) follows from part (P_1) because the localization equivalence is compatible with derived tensor product over the center, while the enveloping algebra U is flat over its Frobenius center. □

5.1.5. *Lemma.* a) Let R be a quasifinite domain over $\mathbb{Z}$ and $\mathcal{A}$ be a finite rank flat algebra over R. Then there exist a quasifinite domain R' over R and collections L_i, P_i of $\mathcal{A}_{\mathsf{R}'}$-modules indexed by $I = Irr(\mathcal{A}_{\overline{\mathbb{Q}}})$, such that for any geometric point $\Bbbk$ of R' the corresponding $\mathcal{A}_{\Bbbk}$ modules $(L_i)_{\Bbbk}$, $(P_i)_{\Bbbk}$, $i \in I$, provide complete nonrepeating lists of irreducible $\mathcal{A}_{\Bbbk}$-modules and their projective covers.

b) The same holds for topological algebras of the form $\varprojlim \mathcal{A}_i$ where $\mathcal{A}_i$ are as in (a) and $\mathcal{A}_i \to \mathcal{A}_{i+1}$ is a *square zero extension*, i.e., a surjective homomorphism with zero multiplication on the kernel.

Sketch of proof. Part (b) follows from (a) since irreducibles for $(\mathcal{A}_i)_{\Bbbk}$ do not depend on i and any projective module admits a lifting to a projective module over a square zero extension.

(a) Let $L_i^o,\ i \in I$, be the complete list of irreducible modules for $\mathcal{A}_{\overline{\mathbb{Q}}}$ and let P_i^o be the indecomposable projective cover of L_i^o. These modules lift to a finite extension R_1 of R, i.e., there are $\mathcal{A}_{\mathsf{R}_1}$-modules P_i, L_i such that $(P_i)_{\overline{\mathbb{Q}}} \cong P_i^o$, $(L_i)_{\overline{\mathbb{Q}}} \cong L_i^o$. (For that choose a presentation for a module as a cokernel of a map between finitely generated free modules and multiply the matrix of the map by an element in R to make its entries integral over $\mathbb{Z}$. Then R_1 is obtained by adjoining the new entries to R.)

After a finite localization R_2 of R_1 we can also achieve that each P_i is projective. The reason is that for some $d_i > 0$, the sum $\oplus_I\ (P_i)_{\mathbb{Q}}^{\oplus d_i}$ is a free $\mathcal{A}_{\mathbb{Q}}$-modules, so $\oplus_I\ P_i^{\oplus d_i}$ becomes a free module after a finite localization R_2.

After further replacing R_2 by a finite localization R_3 we can achieve that $(P_i)_{\Bbbk}$ are pairwise non-isomorphic indecomposable modules. For this consider two $\mathcal{A}_{\mathsf{R}_2}$-modules M, N and a commutative R_2 algebra $\mathfrak{R}$. The existence of (i) an isomorphism $M_{\mathfrak{R}} \cong N_{\mathfrak{R}}$, and (ii) a nontrivial idempotent in $End(M_{\mathfrak{R}})$; amounts to existence of an $\mathfrak{R}$ point of a certain affine algebraic variety over R_2. Such an algebraic variety has a $\overline{\mathbb{Q}}$ point iff it has a $\Bbbk$ point for all finite geometric points $\Bbbk$ of $Spec(\mathsf{R}_2)$ of almost any prime characteristic. Here, R_3 is obtained from R_2 by inverting finitely many primes.

Finally, $\mathrm{Hom}(P_i, L_j)$ is a finite R_3-module such that $\mathrm{Hom}(P_i, L_j) \otimes \overline{\mathbb{Q}} \cong \overline{\mathbb{Q}}^{\delta_{ij}}$. Thus after a finite localization R' we can assume that $\mathrm{Hom}(P_i, L_j) \cong (\mathsf{R}')^{\delta_{ij}}$. □

5.1.6. *Remark.* Our proof of Lusztig conjectures about numerics of modular representations gives the result for $p \gg 0$ only, because we rely on the (very general) Lemma 5.1.5, which is not constructive in the sense that it provides no information on the set of primes one needs to invert in R to get R' with needed properties. The same problem appears in [AJS] which contains results that are essentially equivalent to the case $e = 0$ of our results. Notice that Fiebig has found an explicit (and very large) bound M and showed that the argument of [AJS] works for $p > M$ [Fi].

5.1.7. *Remark.* The following is just the extra information one finds in the proofs of Theorem 5.1.1 and Proposition 5.1.4.

Corollary. Let $\mathcal{E}$ be a vector bundle on $\tilde{\mathfrak{g}}$ as in 1.5.1, then the algebra A in Theorem 5.1.1 can be chosen as $A = \mathrm{End}(\mathcal{E})^{op}$. For any $e \in \mathcal{N}(R)$, the collection $\mathcal{E}_i$ in Proposition 5.1.4, can be chosen as representatives of isomorphism classes of indecomposable constituents of the restriction of $\mathcal{E}$ to $\widehat{\mathcal{B}_{\mathcal{R},e}}$.

I_e is the set of isomorphism classes of irreducible modules of $A|_{\hat{e}} \otimes_R \mathbb{C}$ (i.e., of $A_e \otimes_R \mathbb{C}$ for $A_e = A \otimes_{\mathcal{O}(\mathfrak{g})} \mathcal{O}_e$).

5.1.8. *Numerical consequences.* For $\mathbb{k} \in FGP$ and $e \in \mathcal{N}(\mathbb{k})$ recall the isomorphisms of Grothendieck groups

$$\begin{array}{l} K(U^0_{\mathbb{k},e}) \cong K(\mathcal{B}_{\mathbb{k},e}), \\ K(\mathcal{B}_{\mathbb{k},e})_{\mathbb{Q}} \cong K(\mathcal{B}_{\overline{\mathbb{Q}},e})_{\mathbb{Q}}, \\ K(\mathcal{B}_{\overline{\mathbb{Q}},e}) \cong K((A_{\overline{\mathbb{Q}}})^0_e). \end{array} \tag{9}$$

The first isomorphism is [BMR1] Corollary 5.4.3, the second one follows from [BMR1] Lemma 7.2.1 and Proposition 7.1.7, and the last one is immediate from Theorem 1.5.1 above. By "homotopy invariance of the Grothendieck group" we have $K^0[mod^{fg}(U^0_{\mathbb{k},e})] = K^0[mod^{fg}(U^{\hat{0}}_{\mathbb{k},e})] = K^0[mod^{fg}(U^0_{\mathbb{k},\hat{e}})]$, and the same for U replaced by A.

Corollary. a) For almost all characteristics p the composed isomorphism

$$K[mod^{fg}(U^0_{\mathbb{k},e})]_{\mathbb{Q}} \cong K[mod^{fg}((A_{\overline{\mathbb{Q}}})^0_e)]_{\mathbb{Q}}$$

sends classes of irreducibles to classes of irreducibles and classes of indecomposable projectives to classes of indecomposable projectives; the same applies to $U^{\hat{0}}_{\mathbb{k},e}$ and $U^0_{\mathbb{k},\hat{e}}$.

(b) For all $p > h$ the composition sends the class of the dual of baby Verma $U_{\mathbb{k}}$-module of "highest" weight zero, to the class of a structure sheaf of a point.[12]

(c) Multiplicities of irreducible modules in baby Verma modules are independent of p for large p.

[12] A baby Verma module for $U^0_{\mathbb{k},e}$ involves data of a Borel subalgebra $\mathfrak{b} \in \mathcal{B}_{\mathbb{k},e}$ which we view geometrically as a point of $\mathcal{B}_{\mathbb{k},e}$. Notice that it is easy to see that the class $[\mathcal{O}_{\mathfrak{b}}] \in K(\mathcal{B}_{\mathbb{k},e})$ is independent of the choice of $\mathfrak{b}$.

5.2. **Equivariant versions and Slodowy slices.** From now on we denote by $\mathfrak{G}$ a copy of the group $G\times\mathbb{G}_m$, supplied with a map $\mathfrak{G}\to G\times\mathbb{G}_m$ by $(g,t)\mapsto(g,t^2)$. Since $G\times\mathbb{G}_m$ acts on $\mathfrak{g}$ by $(g,t)\ x\stackrel{\text{def}}{=}t\cdot{}^g x$ the group $\mathfrak{G}$ acts on $\mathfrak{g}$ by $(g,t):x\stackrel{\text{def}}{=}\ t^2\ {}^g x$,

5.2.1. *Equivariant tilting t-structures.*

Proposition. Let R' be a Noetherian R-algebra. Let $S\to\mathfrak{g}_{R'}$ be a map of affine schemes over R' which satisfies the Tor vanishing condition from 1.3 relative to $\tilde{\mathfrak{g}}_{R'}$.

Let H be a flat affine algebraic group over R' endowed with a morphism $\phi:H\to\mathfrak{G}_{R'}$. We assume that H acts on S and that the map $S\to\mathfrak{g}_{R'}$ (respectively $S\to\mathcal{N}_{R'}$) is H-equivariant. Then we have a natural equivalence

$$D^b(Coh^H(\widetilde{S}))\cong D^b(mod^{H,fg}(A_S)),\quad\text{respectively}\quad D^b(Coh^H(\widetilde{S}'))\cong D^b(mod^{H,fg}(A_S^0)).$$

Proof. We construct the first equivalence, the second one works the same. The tilting bundle $\mathcal{E}$ constructed in section 2 is manifestly $G\times\mathbb{G}_m$ equivariant. Thus the vector bundle $\mathcal{E}_S$ and the algebra A_S carry natural H-equivariant structures. Therefore, for $\mathcal{F}\in Coh^H(\widetilde{S})$ and $M\in mod^{H,fg}(A_S)$, the A_S-modules $\mathrm{Hom}(\mathcal{E}_S,\mathcal{F})$ and $M\overset{\mathrm{L}}{\otimes}_{A_S}\mathcal{E}_S\in\mathcal{C}oh^H(\widetilde{S})$ carry natural H-equivariant structures. Passing to the derived functors we get two adjoint functors $\mathcal{C}^H,\mathcal{I}^H$ between $D^b(Coh^H(\widetilde{S}))$ and $D^b(mod^{H,fg}(A_S))$. A standard argument shows that these functors are compatible with the pair of adjoint functors $\mathcal{C},\mathcal{I}$ between $D^b(Coh(\widetilde{S}))$ and $D^b(mod^{fg}(A_S))$, which are given by the same formulas. Moreover, this compatibility extends to the adjunction morphisms between the identity functors and compositions $\mathcal{I}^H\mathcal{C}^H,\mathcal{C}^H\mathcal{I}^H$ and $\mathcal{I}\mathcal{C},\mathcal{C}\mathcal{I}$. Since the adjunction morphisms are isomorphisms in the non-equivariant setting, they are also isomorphisms in the equivariant one. □

5.2.2. *Slodowy slices.* Let $\Bbbk$ be a geometric point of R. Fix a nilpotent $e\in\mathcal{N}(\Bbbk)$ such that there is a homomorphism $\varphi:SL(2)\to G$ such that $d\varphi\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)=e$.[13] The corresponding sl_2 triple e,h,f defines a Slodowy slice $S_{\Bbbk,e}\stackrel{\text{def}}{=}e+Z_{\mathfrak{g}_\Bbbk}(f)$ transversal to the conjugacy class of e.

Let C be a maximal torus in the centralizer of the image of φ, it is also a maximal torus in the centralizer G_e of e. Let $\phi:\mathbb{G}_m\to G$ by $\phi(t)\stackrel{\text{def}}{=}\varphi\left(\begin{smallmatrix}t&0\\0&t^{-1}\end{smallmatrix}\right)$. We denote by $\mathfrak{G}_{\mathfrak{m}}$ a

[13] This is always possible if $p>3h-3$ [Hu].

copy of the group $\mathbb{G}_m$ and by $\widetilde{C}$ the group $C\times\mathfrak{G}_{\mathfrak{m}}$ supplied with a morphism $\mathbf{i}:\widetilde{C}\to\mathfrak{G}$ by $\mathbf{i}(c,t)\stackrel{\text{def}}{=}(c\phi(t),t^{-1})$. The action of $\widetilde{C}$ on $\mathfrak{g}$ is by $(c,t)\ x=\ t^{-2\ c\phi(t)}x$, it preserves the Slodowy slice $S_{\Bbbk,e}$ and the action of $t\in\mathfrak{G}_{\mathfrak{m}}$ contracts it to e for $t\to\infty$.

Now we use the fact that C contains $\phi(-1)\boldsymbol{z}$ for some $\boldsymbol{z}\in Z(G)$, see Appendix B. Notice that the element $\boldsymbol{m}=(\phi(-1)\boldsymbol{z},-1)\in\widetilde{C}$ is sent to $(\phi(-1)\boldsymbol{z}\phi(-1),-1)=\ (\boldsymbol{z},-1)\in\ \mathfrak{G}$. So, the action of $\widetilde{C}$ on $\mathfrak{g}$ factors through the quotient by the subgroup generated by $\boldsymbol{m}$.

5.2.3. *Equivariant lifts of irreducibles and indecomposable projectives.* Let $\Bbbk$ be a geometric point of R and let $e\in\mathcal{N}(\Bbbk)$. Let H be a torus endowed with a map into the stabilizer of e in $\mathfrak{G}_{\Bbbk}$. We say that for an object $\mathcal{F}$ in the derived category of coherent sheaves its equivariant lifting is an object in the equivariant derived category whose image under forgetting the equivariance functor is isomorphic to $\mathcal{F}$.

Proposition. Let H be a $\Bbbk$-torus mapping to $\mathfrak{G}_{\Bbbk}$ and fixing e,h,f, so that in particular it preserves $S_{\Bbbk,e}$.

a) Every irreducible exotic sheaf $\mathcal{L}$ on either of the spaces: $\widetilde{S_{\Bbbk,e}}$, $\widetilde{S_{\Bbbk,e}}'$, $\widehat{\mathcal{B}_{\Bbbk,e}}$, $\widehat{\mathcal{B}_{\Bbbk,e}}'$, $\tilde{\mathfrak{g}}_{\Bbbk}$, $\widetilde{\mathcal{N}}_{\Bbbk}$, whose (set-theoretic) support is contained in $\mathcal{B}_{\Bbbk,e}$ admits an H-equivariant lift $\widetilde{\mathcal{L}}$. Any other lift is isomorphic to a twist of $\widetilde{\mathcal{L}}$ by a character of H.

b) Every projective exotic sheaf $\mathcal{W}$ on either of the spaces: $\widetilde{S_{\Bbbk,e}}$, $\widetilde{S_{\Bbbk,e}}'$, $\widehat{\mathcal{B}_{\Bbbk,e}}$, $\widehat{\mathcal{B}_{\Bbbk,e}}'$ admits an H-equivariant lifting $\widetilde{\mathcal{W}}$. If $\mathcal{W}$ is indecomposable then every equivariant lifting of $\mathcal{W}$ is isomorphic to a twist of $\widetilde{\mathcal{W}}$ by a character of H.

c) Assume that $char(\Bbbk)=0$. Then there exists a quasifinite R-domain R', such that the following holds.

- i) The nilpotent e, torus H and the homomorphism $H\to\mathfrak{G}$ are defined over R'.
- ii) For each $\mathcal{L}\in D^b(Coh_{\mathcal{B}_{\Bbbk,e}}(\widetilde{S_{\Bbbk,e}}'))$, $\mathcal{W}\in\mathcal{C}oh(\widetilde{S_{\Bbbk,e}}')$ and equivariant lifts $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{W}}$ as above there exist $\mathcal{L}_{R'}$, $\mathcal{W}_{R'}\in D^b(Coh^{H_{R'}}(\widetilde{S}'_{R',e}))$ and their equivariant lifts $\widetilde{\mathcal{L}}_{R'}$, $\widetilde{\mathcal{W}}_{R'}$, such that their base change to $\Bbbk$ is isomorphic to $\mathcal{L},\mathcal{W},\widetilde{\mathcal{L}},\widetilde{\mathcal{W}}$.
- iii) For every geometric point $\Bbbk'$ of R' the base change to $\Bbbk'$ of $\mathcal{L}_{R'}$, $\mathcal{W}_{R'}$ and $\widetilde{\mathcal{L}}_{R'}$, $\widetilde{\mathcal{W}}_{R'}$ are, respectively, irreducible and indecomposable projective equivariant exotic sheaves. Every equivariant irreducible or indecomposable projective exotic sheaf on $\widetilde{S}_{\Bbbk',e}$, $\widetilde{S}'_{\Bbbk',e}$, $\widehat{\mathcal{B}_{k',e}}$, $\widehat{\mathcal{B}_{k',e}}'$ arises in this way.

Proof. It is clear that the direct image of an irreducible exotic sheaf $\mathcal{L} \in D^b(Coh_{\mathcal{B}'_{\Bbbk,e}}(\widetilde{S_{\Bbbk,e}}'))$ under each of the closed embeddings $\widetilde{S_{\Bbbk,e}}' \hookrightarrow \widetilde{\mathcal{N}}_{\Bbbk}$, $\widetilde{S_{\Bbbk,e}}' \hookrightarrow \widetilde{S_{\Bbbk,e}} \hookrightarrow \tilde{\mathfrak{g}}_{\Bbbk}$ is again an irreducible exotic sheaf supported on $\mathcal{B}'_{\Bbbk,e}$, and all irreducible exotic sheaves on these spaces supported on $\mathcal{B}_{\Bbbk,e}$ are obtained this way. The resulting sheaves on $\tilde{\mathfrak{g}}_{\Bbbk}$, $\widetilde{\mathcal{N}}_{\Bbbk}$ can be also thought of as sheaves on $\widehat{\mathcal{B}_{\Bbbk,e}}$, $\widehat{\mathcal{B}_{\Bbbk,e}}'$ respectively. Thus in (a) it suffices to consider the case of $\widetilde{S_{\Bbbk,e}}'$ only.

Applying Proposition 5.2.1 we get an equivalence $D^b(Coh^H(\widetilde{S_{\Bbbk,e}}')) \cong D^b(mod^{H,fg}(A^0_{S_{\Bbbk,e}}))$. Thus, statement (a) reduces to showing that every irreducible A^0_S module with central character e admits an H equivariant structure. The torus H acts trivially on the finite set of irreducible A^0_S-modules with central character e. It follows that H acts projectively on such a representation. Since every cocharacter of $PGL(n)_{\Bbbk}$ admits a lifting to $GL(n)_{\Bbbk}$, we see that the representation admits an H equivariant structure. This proves (a).

Similarly, in order to check (b) it suffices to equip indecomposable projective modules over the respective algebras with an H equivariant structure. By a standard argument, projective cover of an irreducible module in the category of graded (equivalently, H-equivariant) modules is also a projective cover in the category of non-graded modules, which yields (b).

To check (c), it suffices to consider equivariant projective modules on $\widetilde{S_{\Bbbk,e}}'$ (then the rest follows as in 5.1.4). Equivariant indecomposable projectives over $\Bbbk$ are direct summands of $\mathcal{E}_{\widetilde{S_{\Bbbk,e}}'}$. We can find a quasifinite R-domain R' such that the corresponding idempotents are defined and orthogonal over R'. □

5.2.4. *Equivariant localization.* In this subsection we link C-equivariant exotic sheaves to representations graded by weights of C by proving Theorem 1.6.6. In this argument it will be important to distinguish between a variety and its Frobenius twist, so we bring the twist back into the notations. We concentrate on the first equivalence, the second one is similar.

The torus $C^{(1)}$ acts on $\widehat{\mathcal{B}_{\Bbbk,e}}^{(1)}$, composing this action with the Frobenius morphism $C \to C^{(1)}$ we get an action of C on $\widehat{\mathcal{B}_{\Bbbk,e}}^{(1)}$. Consider the category of equivariant coherent sheaves $Coh^C(\widehat{\mathcal{B}_{\Bbbk,e}}^{(1)})$. The finite group scheme $C_1 = Ker(C \xrightarrow{Fr} C^{(1)})$ maps to automorphisms of the identity functor in this category; since the category of C_1-modules is semisimple

with simple objects indexed by $\mathfrak{c}^*(\mathbb{F}_p) = X^*(C)/p$, the category splits into a direct sum

$$Coh^C(\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}) = \bigoplus_{\eta \in X^*(C)/p} Coh^C_\eta \tag{10}$$

where Coh^C_η consists of such equivariant sheaves that C_1 acts on each fiber by the character η. Notice that $Coh^C_0(\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}) \cong Coh^{C^{(1)}}(\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)})$ canonically and for every $\widetilde{\eta} \in X^*(C)$ the functor of twisting by $\widetilde{\eta}$ provides an equivalence $Coh^C_\theta(\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}) \cong Coh^C_{\theta+\eta}(\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)})$ where $\eta = \widetilde{\eta} \bmod pX^*(C)$.

The sheaf of algebras $\widetilde{\mathcal{D}}$ is equivariant with respect to the G action, hence $\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}}$ is equivariant with respect to the action of C on $\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}$. Consider the category $mod^{C,fg}(\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}})$ of C-equivariant coherent sheaves of modules over this C-equivariant sheaf of algebras; here "coherent" refers to coherence as a sheaf of $\mathcal{O}$-modules over the scheme $\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}$. A sheaf $\mathcal{F} \in mod^{C,fg}(\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}})$ carries two commuting actions of $\mathfrak{c}$, α_C and $\alpha_\mathfrak{g}$ (see 1.6.6) whose difference commutes with the action of $\widetilde{\mathcal{D}}$.

It is easy to see (cf. 1.6.6) that every $\mathcal{F} \in mod^{C,fg}(\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}})$ splits as a direct sum $\mathcal{F} = \bigoplus_{\eta \in \mathfrak{c}^*(\mathbb{F}_p)} \mathcal{F}_\eta$ where $\alpha_\mathfrak{g}(x) - \alpha_C(x) - \langle x, \eta \rangle Id$ induces a pro-nilpotent endomorphism of $\mathcal{F}_\eta$. Here by a pro-nilpotent endomorphism we mean one which becomes nilpotent when restricted to any finite nilpotent neighborhood of $\mathcal{B}_e{}^{(1)}$.

Thus we get a decomposition of the category

$$mod^{C,fg}(\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}}) = \bigoplus_{\eta \in \mathfrak{c}^*(\mathbb{F}_p)} mod^{C,fg}_\eta(\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}}). \tag{11}$$

Let $\mathcal{E}$ be a splitting bundle for the Azumaya algebra $\widetilde{\mathcal{D}}$ on $\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}$. We claim that $\mathcal{E}$ admits a C-equivariant structure compatible with the equivariant structure on $\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}} = \mathcal{E}nd(\mathcal{E})$. For $\lambda, \mu \in \mathfrak{h}^*(\mathbb{F}_p)$ the bimodule providing Morita equivalence between the restrictions of $\widetilde{\mathcal{D}}$ to the formal neighborhoods of the preimages of λ and μ under the projections from the spectrum of the center of $\widetilde{\mathcal{D}}$ to $\mathfrak{h}^*$ (see [BMR1, 2.3]) is manifestly G-equivariant, thus it suffices to consider the case $\lambda = -\rho$. Then $\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}}$ is identified with the pull-back of a C-equivariant Azumaya algebra on the formal neighborhood of $e \in \mathfrak{g}^{*(1)}$. We construct compatible C-equivariant splitting bundles on the n-th infinitesimal neighborhood of e for all n by induction in n. The base of induction follows from the fact that every extension

of C by $\mathbb{G}_m$ splits, while the induction step follows from splitting of an extension of C by an additive group. Thus existence of a C-equivariant splitting bundle is established.

We fix such a C-equivariant structure on $\mathcal{E}$; we can and will assume that the restriction of the resulting equivariant $\widetilde{\mathcal{D}}$ module to a finite order neighborhood of $\mathcal{B}_{\mathbb{k},e}$ belongs to $mod^{C,fg}(\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}(1)}})$ (since this restriction is an indecomposable sheaf of $\widetilde{\mathcal{D}}$-modules, this can be achieved by twisting an arbitrarily chosen equivariant lift of $\mathcal{E}$ by a character of C). Then we get a functor $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{E}$ from $\mathcal{C}oh^{C^{(1)}}(\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)})$ to the category $mod^{C,fg}(\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}(1)}})$ of C-equivariant sheaves of modules over a C-equivariant sheaf of algebras. We will compose this functor with the global sections functor $D^b[mod^{C,fg}(\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}(1)}})] \to D^b[mod^{C,fg}(U_{\widehat{e}}^{\widehat{\lambda}})]$. We claim that the composition lands in the full subcategory $D^b[mod^{fg}(U_{\widehat{e}}^{\widehat{\lambda}}, C)]$ and provides the desired equivalence. This follows from the next

Lemma. a) The derived global sections functor provides an equivalence

$$\mathrm{R}\Gamma : D^b[mod^{C,fg}(\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}(1)}})] \to D^b[mod^{C,fg}(U_{\widehat{e}}^{\widehat{\lambda}})].$$

b) The functor $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{E}$ provides an equivalence $\mathcal{C}oh^C(\widehat{\mathcal{B}_{\mathbb{k},e}(1)}) \to mod^{C,fg}(\widetilde{\mathcal{D}}|_{\widehat{\mathcal{B}_{\mathbb{k},e}(1)}})$.

c) Both of these equivalences are compatible with the canonical $\mathfrak{c}^*(\mathbb{F}_p)$-decompositions of categories. In particular for $0 \in \mathfrak{c}^*(\mathbb{F}_p)$, we get equivalences

$$mod^{fg}(U_{\widehat{e}}^{\widehat{\lambda}}, C) \;\cong\; \mathcal{E}coh^{C^{(1)}}(\widehat{\mathcal{B}_{\mathbb{k},e}}^{(1)}).$$

Proof. (a) follows by the argument of Proposition 5.2.1 (the two adjoint functors commute with forgetting the equivariance).

(b) is just the observation that once $\mathcal{E}$ is equivariant, the standard equivalence between coherent sheaves on $\tilde{\mathfrak{g}}$ and modules over the sheaf of algebras $\mathcal{A} = \mathcal{E}nd(\mathcal{E})$ extends to the equivariant setting.

(c) follows from the definition of the decompositions. □

5.3. **Gradings and bases in K-theory.** In this subsection we work over a geometric point $\mathbb{k}$ of R. We reduce the conjectures of [Lu] which motivated this project to a certain property **(⋆)** of exotic sheaves. All substantial proofs of claims in this subsection are

postponed to subsection 5.4. In the next section 6 we will see that property $(\star)$ (thus a proof of Lusztig's conjectures) follows (*for large p*) from the results of [ArkB].

5.3.1. *Construction of gradings.* On the formal neighborhood $\widehat{\mathcal{B}_{\Bbbk,e}}$ of a Springer fiber $\mathcal{B}_{\Bbbk,e}$ in $\tilde{\mathfrak{g}}_{\Bbbk}$, there is a canonical (up to isomorphism) vector bundle $\oplus_{i\in\mathcal{I}_e}\mathcal{E}_i$ which is a minimal projective generator for the heart $\mathcal{E}coh(\widehat{\mathcal{B}_{\Bbbk,e}})$ of the exotic t-structure. We just take $\mathcal{E}_i$'s to be representatives for isomorphism classes of indecomposable summands in the pull back $\mathcal{E}|_{\widehat{\mathcal{B}_{\Bbbk,e}}}$ of any vector bundle $\mathcal{E}$ from Theorem 1.5.1.

By Proposition 5.2.3, vector bundles $\mathcal{E}_i$ admit a $\mathfrak{G}_{\mathfrak{m}}$ equivariant structure. We temporarily fix such a structure in an arbitrary way and let $\widetilde{\mathcal{E}}_i \in \mathcal{C}oh^{\mathfrak{G}_{\mathfrak{m}}}(\widehat{\mathcal{B}_{\Bbbk,e}})$ denote the resulting equivariant bundle. In view of Lemma 5.2.3, any other choice yields an equivariant vector bundle isomorphic to a twist $\widetilde{\mathcal{E}}_i(d)$ by some character $d\in\mathbb{Z}$ of $\mathfrak{G}_{\mathfrak{m}}$.

Consider the restriction of $\widetilde{\mathcal{E}}_i$ to the formal neighborhood of $\mathcal{B}_{\Bbbk,e}$ in $\widetilde{S_{\Bbbk,e}}'$. Since $\mathfrak{G}_{\mathfrak{m}}\subset\widetilde{C}$ acts on $\widetilde{S_{\Bbbk,e}}'$ contracting it to the projective variety $\mathcal{B}_{\Bbbk,e}$, there exists a unique (up to a unique isomorphism), $\widetilde{C}$ equivariant vector bundle on $\widetilde{S_{\Bbbk,e}}'$ whose restriction to the formal neighborhood of $\mathcal{B}_{\Bbbk,e}$ is identified with $\widetilde{\mathcal{E}}_i|_{\widehat{S_{\Bbbk,e}}'}$. We denote this vector bundle by $\widetilde{\mathcal{E}}_i^S \in \mathcal{C}oh^{\mathfrak{G}_{\mathfrak{m}}}(\widetilde{S_{\Bbbk,e}}')$.

To summarize, $\widetilde{\mathcal{E}}_i^S$ is a set of representatives for equivalence classes of indecomposable projective $\mathfrak{G}_{\mathfrak{m}}$-equivariant exotic sheaves on $\widetilde{S_{\Bbbk,e}}'$ modulo $\mathfrak{G}_{\mathfrak{m}}$-shifts.

5.3.2. *Property* $(\star)$. The following statement will be the key to the proof of Lusztig's conjectures.

$$(\star)\quad \begin{cases} \text{There exists a choice of } \mathfrak{G}_{\mathfrak{m}}\text{-equivariant lifts } \widetilde{\mathcal{E}}_i,\ i\in\mathcal{I}_e, \text{ which is} \\ \text{invariant under the action of the centralizer of } e,h \text{ in } G \text{ and such that:} \\ (\mathrm{i}_\star) \qquad \mathrm{Hom}_{\mathcal{C}oh^{\mathfrak{G}_{\mathfrak{m}}}(\widetilde{S_{\Bbbk,e}}')}(\widetilde{\mathcal{E}}_i^S,\widetilde{\mathcal{E}}_j^S(d))=0 \quad \text{for} \quad d>0, \\ (\mathrm{ii}_\star) \qquad \mathrm{Hom}_{\mathcal{C}oh^{\mathfrak{G}_{\mathfrak{m}}}(\widetilde{S_{\Bbbk,e}}')}(\widetilde{\mathcal{E}}_i^S,\widetilde{\mathcal{E}}_j^S) \ = \ \Bbbk^{\delta_{ij}}. \\ \text{Such a choice is unique up to twisting all } \widetilde{\mathcal{E}}_i \text{ by the same character of } \mathfrak{G}_{\mathfrak{m}}. \end{cases}$$

In other words $\mathrm{Hom}_{\mathcal{C}oh(\widetilde{S_{\Bbbk,e}}')}(\widetilde{\mathcal{E}}_i^S,\widetilde{\mathcal{E}}_j^S)$ has no negative $\mathfrak{G}_{\mathfrak{m}}$ weights and the zero weight spaces are spanned by the identity maps.

5.3.3. *Normalizations.* Assuming that $(\star)$ holds for $\Bbbk$ and e, we will reserve notation $\widetilde{\mathcal{E}}_i$ for $\mathfrak{G}_{\mathfrak{m}}$-equivariant vector bundles on $\widehat{\mathcal{B}_{\Bbbk,e}}$, satisfying the above conditions. Recall

that the vector bundle which is a tilting generator for the exotic t-structure has $\mathcal{O}$ as a direct summand, this implies that $\mathcal{E}_{i_0} \cong \mathcal{O}_{\widehat{\mathcal{B}_{\Bbbk,e}}'}$ for some $i_0 \in \mathcal{I}_e$. We will assume that $\widetilde{\mathcal{E}}_{i_0} = \mathcal{O}_{\widehat{\mathcal{B}_{\Bbbk,e}}}(2\dim \mathcal{B}_e)$; since the collection $\widetilde{\mathcal{E}}_i$ is unique up to a simultaneous twist by some character of $\mathfrak{G}_{\mathfrak{m}}$, this fixes the set of isomorphism classes of $\widetilde{\mathcal{E}}_i$ uniquely.

According to Proposition 5.2.2 we can equip the vector bundle $\widetilde{\mathcal{E}}_i$ with some $\widetilde{C}$ equivariant structure compatible with the $\mathfrak{G}_{\mathfrak{m}}$ equivariant structure fixed above. Let $\widetilde{\mathcal{E}}_{i,0}$ denote the resulting $\widetilde{C}$-equivariant vector bundle and set $\widetilde{\mathcal{E}}_{i,\lambda} = \widetilde{\mathcal{E}}_{i,0}(\lambda)$ for $\lambda \in X^*(C)$. As above, from $\widetilde{\mathcal{E}}_{i,\lambda}$ we obtain a $\widetilde{C}$-equivariant vector bundle $\widetilde{\mathcal{E}}^S_{i,\lambda}$ on $\widetilde{S_{\Bbbk,e}}'$. As we vary $i \in \mathcal{I}_e$ and $\lambda \in X^*(C)$, vector bundles $\widetilde{\mathcal{E}}_{i,\lambda}$ on $\widehat{\mathcal{B}_{\Bbbk,e}}$ (resp., $\widetilde{\mathcal{E}}^S_{i,\lambda}$ on $\widetilde{S_{\Bbbk,e}}'$) form a complete list of representatives modulo $\mathfrak{G}_{\mathfrak{m}}$-shifts, of indecomposable exotic projectives in $D^b[\mathcal{C}oh^{\widetilde{C}}(\widehat{\mathcal{B}_{\Bbbk,e}})]$ (resp. $D^b[\mathcal{C}oh^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')]$). Also, we define irreducible exotic objects $\widetilde{\mathcal{L}}_{i,\lambda}$ of $D^b(\mathcal{C}oh^{\widetilde{C}}(\widehat{\mathcal{B}_{\Bbbk,e}}))$ (respectively, $\widetilde{\mathcal{L}}_i$ of $D^b(\mathcal{C}oh^{\mathfrak{G}_{\mathfrak{m}}}(\widehat{\mathcal{B}_{\Bbbk,e}}))$) such that $\mathrm{Ext}^{\bullet}_{D^b\mathcal{C}oh^{\widetilde{C}}(\widehat{\mathcal{B}_{\Bbbk,e}})}(\widetilde{\mathcal{E}}_{i,\lambda}, \widetilde{\mathcal{L}}_{j,\mu}) = \Bbbk^{\delta^i_j\delta^\lambda_\mu}$, $\mathrm{Ext}^{\bullet}_{D^b\mathcal{C}oh^{\mathfrak{G}_{\mathfrak{m}}}(\widehat{\mathcal{B}_{\Bbbk,e}})}(\widetilde{\mathcal{E}}_i, \widetilde{\mathcal{L}}_j) = \Bbbk^{\delta^i_j}$.

5.3.4. *Uniformity of K-groups.* Although not strictly necessary for the the proof of Conjectures, the following result allows a neater formulation of the next Theorem and clarifies the picture

Fix two geometric points of $Spec(R)$: $\Bbbk_0$ of characteristic zero and $\Bbbk$ of characteristic $p > h$. Recall that for a flat Noetherian scheme X over R one has *specialization map* $Sp_X : K(X_{\Bbbk_0}) \to K(X_{\Bbbk})$.

Proposition. The maps $Sp_{\mathcal{B}_e}$, $Sp_{\widetilde{S_e}}$, $Sp_{\widetilde{S_e}'}$ are isomorphisms.

Proof. Tensoring the maps with $\mathbb{Q}$ we get isomorphisms because the modified Chern character map identifies $K(\mathcal{B}_e)\otimes\bar{\mathbb{Q}}_l$ with the dual of l-adic cohomology: in [BMR1] it was shown in Lemmas 7.4.2 and 7.4.1 that the map is injective, however the dimensions are the same by Theorem 7.1.1 and Lemma 7.4.3. The independence of l-adic cohomology on the base field was established by Lusztig ([Lu2] section 24, in particular theorem 24.8 and subsection 24.10).

On the other hand, using the above equivalences of categories we see that over the field $\Bbbk$ the classes of irreducible (respectively, projective) objects, form bases in respective Grothendieck groups. In particular, each of the K-groups is a free abelian group of finite rank and the Ext pairing between $K(\mathcal{B}_{\Bbbk,e})$ and $K(\widetilde{S_{\Bbbk,e}})$ is perfect. The corresponding

statements for $\Bbbk_0$ were proved by Lusztig. It is clear that the specialization map is compatible with this pairing. Thus the pair of maps $Sp_{\mathcal{B}_e}$, $Sp_{\widetilde{S_e}}$ is an example of the following situation. We are given free abelian groups A, A', B, B', all of the same finite rank, maps $F : A \to B$, $F' : A' \to B'$ and perfect pairings $A \times A' \to \mathbb{Z}$, $B \times B' \to \mathbb{Z}$ such that $\langle F(x), F'(y)\rangle = \langle x, y\rangle$ for all $x \in A$, $y \in A'$. It is clear that in this situation F, F' have to be isomorphisms. □

5.3.5. *Reduction of Lusztig's conjectures to* $(\star)$. Lusztig's conjectures from [Lu] will be recalled in detail in sections 5.3.6, 5.3.7 and 5.4.1. In 5.3.6 we will state our precise results and then we will see in 5.3.7 that they imply the following Theorem.

Theorem. (1) If $(\star)$ holds for some geometric point $\Bbbk$ of R', then Conjectures 5.12, 5.16 of [Lu] hold (existence of certain signed bases of K-groups of complex schemes $\mathcal{B}_{\mathbb{C},e}$ and $\widetilde{S_{\mathbb{C},e}}'$). The two bases discussed in Conjecture 5.12 are given by the classes of $\widetilde{\mathcal{L}}^{\Bbbk}_{i,\lambda}(-2\dim \mathcal{B}_e)$, $\widetilde{\mathcal{E}}^{\Bbbk}_{i,\lambda}$, $i \in \mathcal{I}_e$, $\lambda \in X^*(C)$; and the two bases discussed in Conjecture 5.16 are given by the classes of $\widetilde{\mathcal{L}}^{\Bbbk}_i$, $\widetilde{\mathcal{E}}^{\Bbbk}_i$.[14]

(2) If $(\star)$ holds for some $\Bbbk \in FGP$, then Conjecture 17.2 of *loc. cit.* (relation to modular representations over $\Bbbk$; omit the last paragraph in 17.2 on the quantum version[15]) holds for $\Bbbk$.

5.3.6. *A reformulation of Lusztig's conjectures.* Here we formulate a list of properties that naturally appear from the present point of view. Then in 5.3.7 we will recall Lusztig's conjectures and show that they follow from these properties. We will omit [Lu], Conjecture 5.16 since it is similar to *loc. cit.* Conjecture 5.12; the only difference is that 5.12 deals with coherent sheaves equivariant with respect to the torus $\widetilde{C}$, while 5.16 is about sheaves equivariant with respect to a one-parameter subgroup $\mathfrak{G}_{\mathfrak{m}} \subset \widetilde{C}$. So the existence of bases with properties from 5.12 implies the same for 5.16 and Lusztig's uniqueness argument (recalled in footnote (18)) applies equally to both conjectures.

The K-group of a torus T is the group algebra of its character lattice $R_T \stackrel{\text{def}}{=} \mathbb{Z}[X^*(T)]$, it contains a *subsemiring* $R_T^+ = \mathbb{Z}_+[X^*(T)]$. Let $\mathfrak{K}_T \stackrel{\text{def}}{=} Frac(R_T)$ be the fraction field

[14]Super index $\Bbbk$ means that we use the sheaves defined over $\Bbbk$. A priori these define elements of K-groups for $\Bbbk$-schemes, however by Proposition 5.3.4 these K-groups are canonically identified for all $\Bbbk$.

[15]The quantum version is closely related to Conjecture 1.7.1 above.

of R_T. Denote by $\partial : R_{\widetilde{C}} \to R_{\mathfrak{G}_\mathfrak{m}}$ the constant coefficient map $\sum_{\nu\in X^*(C)}\ p_\nu[\nu]\mapsto p_0$ ([Lu, 5.9]).

Recall from [Lu, Theorem 1.14.c] that the direct image map gives an embedding $K^{\widetilde{C}}(\mathcal{B}_{\Bbbk,e})\hookrightarrow K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')$. Lusztig's conjectures involve certain involutions β_e on $K^{\widetilde{C}}(\mathcal{B}_{\Bbbk,e})$ and β_S on $K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')$ (denoted $\widetilde{\beta}$, β in [Lu]), a certain pairing $(\ \|\ )$ on $K(\widetilde{S_{\Bbbk,e}}')$ with values in the fraction field $\mathfrak{K}_{\widetilde{C}}$, and a certain element ∇_e of $\mathfrak{K}_{\widetilde{C}}$. We denote by $P\mapsto P^\nu$ the involution of $\mathfrak{K}_{\widetilde{C}}$ corresponding to inversion on $\widetilde{C}$.

The following proposition will be verified in 5.4.4.

Proposition. Let $\Bbbk$ be a geometric point of R such that **(⋆)** holds for $\Bbbk$. Define $\widetilde{\mathcal{E}}_i^S,\ \widetilde{\mathcal{E}}_{i,\lambda}^S$, $\widetilde{\mathcal{L}}_i,\ \widetilde{\mathcal{L}}_{i,\lambda}$ as in 5.3.3.

(A) The following subsets are bases over the ring $R_{\mathfrak{G}_\mathfrak{m}}=\mathbb{Z}[v^{\pm1}]$ (we often omit the super index $\Bbbk$):

$$\mathbf{B}_S^{\Bbbk} \overset{\text{def}}{=} \{[\widetilde{\mathcal{E}}_{i,\lambda}^S];\ i\in\mathcal{I}_e,\ \lambda\in X^*(C)\}\subseteq K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}'),$$
$$\mathbf{B}_e^{\Bbbk} \overset{\text{def}}{=} \{v^{-2\dim\mathcal{B}_e}[\widetilde{\mathcal{L}}_{i,\lambda}];\ i\in\mathcal{I}_e,\ \lambda\in X^*(C)\}\}\subseteq K^{\widetilde{C}}(\mathcal{B}_{\Bbbk,e}).$$

Elements of $\mathbf{B}_S$ are fixed by β_S and elements of $\mathbf{B}_e$ by β_e:

$$\beta_e(v^{-2\dim\mathcal{B}_e}[\widetilde{\mathcal{L}}_{i,\lambda}]) = v^{-2\dim\mathcal{B}_e}[\widetilde{\mathcal{L}}_{i,\lambda}], \tag{12}$$

$$\beta_S([\widetilde{\mathcal{E}}_{i,\lambda}^S]) = [\widetilde{\mathcal{E}}_{i,\lambda}^S]. \tag{13}$$

Both bases satisfy the condition of *asymptotic orthonormality*: [16]

$$(b_1\|b_2)\in\begin{cases} v^{-1}R_C[[v^{-1}]] & \text{if} \quad b_2\notin X^*(C)b_1,\\ 1+v^{-1}R_C[[v^{-1}]] & \text{if} \quad b_2=b_1.\end{cases} \tag{14}$$

(B) The two bases are dual for the pairing $(\ \|\ )$.

(C) $(\mathbf{B}_S\|\mathbf{B}_S)\subseteq \frac{1}{\nabla_e}R_C^+[v^{-1}]\cap R_C^+[[v^{-1}]]$ and $(\mathbf{B}_e\|\mathbf{B}_e)\subseteq R_C[v^{-1}]$. [17]

(D) For $b_1,b_2\in\mathbf{B}_S$, each of the coefficient polynomials $c_{b_1,b_2}^\nu\in\mathbb{Z}_+[v^{-1}],\ \nu\in X^*(C)$, in the expansion $\nabla_e(b_1\|b_2)=\ \sum_{\nu\in X^*(C)}\ c_{b_1,b_2}^\nu[\nu]\in R_C^+[v^{-1}]$, is either even or odd.

[16] To make sense of it we use the embedding $R_{\widetilde{C}}\subset R_C((v^{-1}))$ and the induced embedding of fraction fields.

[17] For basis $\mathbf{B}_e$ the corresponding positivity statement $(\mathbf{B}_e\|\mathbf{B}_e)\subseteq R_C^+[v^{-1}]$ follows from the result of section 5.5 below.

(E) Basis $\mathbf{B}_S$ is the unique $R_{\mathfrak{G}_\mathfrak{m}}$ basis of $K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')$ which is pointwise fixed by β_S, satisfies asymptotic orthonormality, a normalization property $\mathbf{B}_S \ni v^{2\dim(\mathcal{B}_e)}[\mathcal{O}_{\widetilde{S_{\Bbbk,e}}'}]$ and either of positivity properties: $(\mathbf{B}_S\|\mathbf{B}_S) \in R_C^+[[v^{-1}]]$ or $(\mathbf{B}_S\|\mathbf{B}_S) \in \frac{1}{\nabla_e}R_C^+[v^{-1}]$.

(F) If $p = char(\Bbbk) > 0$ then there is a canonical isomorphisms $K^C(\mathcal{B}_{\Bbbk,e}) \underset{\cong}{\xrightarrow{\iota_e}} K^0[mod^{fg}(U^{\widehat{0}}_{\Bbbk,e}, C)]$ which sends $\mathbf{B}_e$ to classes of irreducible modules and $\nabla_e^\vee \mathbf{B}_S$ to classes of indecomposable projective modules.

(G) If $p = char(\Bbbk) > 0$ then for $b_i \in \mathbf{B}_S$, the evaluation of the polynomial $\partial[\nabla_e(b_1\|b_2)]$ at $1 \in \mathfrak{G}_\mathfrak{m}$ is equal to the corresponding entry of the Cartan matrix of $mod^{fg}(U^{\widehat{0}}_{\Bbbk,e}, C)$, i.e., the dimension of the Hom space between the corresponding indecomposable projective objects. The dimension of the Hom space in category $mod^{fg}(U^{\widehat{0}}_{\Bbbk,e})$ is given by the evaluation of the polynomial $\nabla_e(b_1\|b_2)$ at the point $(1,1) \in C\times\mathfrak{G}_\mathfrak{m}$. □

5.3.7. *Proof of Theorem 5.3.5 modulo Proposition 5.3.6.* Regarding K-groups, Lusztig considers the case $\Bbbk = \mathbb{C}$ and defines $\mathbf{B}^\pm_{\mathcal{B}_e} \subseteq K^{\widetilde{C}}(\mathcal{B}_{\Bbbk,e})^{\beta_e}$ and $\mathbf{B}^\pm_{\Lambda_e} \subseteq K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')^{\beta_S}$ by the condition of *asymptotic norm one*: $(b\|b) \in 1+v^{-1}\mathbb{Z}[[v^{-1}]]$. Conjectures [Lu, 5.12] (a,b) assert that these are *signed* $R_{\mathfrak{G}_\mathfrak{m}}$-bases. Parts (c,d) of the Conjecture 5.12 say that signed bases $\mathbf{B}^\pm_{\mathcal{B}_e}, \mathbf{B}^\pm_{\Lambda_e}$ are asymptotically orthonormal and parts (e,f) of the conjecture say that the two bases are dual.

Since the identifications of K-groups in Proposition 5.3.4 are easily shown to be compatible with involutions β_e, β_S and pairing $(\ \|\ )$, we see that if we define $\mathbf{B}^\pm_{\mathcal{B}_e}, \mathbf{B}^\pm_{\Lambda_e}$ in the same way for all $\Bbbk$, what we get will be independent of $\Bbbk$ and the same will hold for validity of conjectures (a-f). However, if **(⋆)** is known for some $\Bbbk$ then conjectures (a-f) follow from **(A)** and **(B)**. The point is that **(A)** implies that for this $\Bbbk$ one has $\mathbf{B}^\pm_{\mathcal{B}_e} = \mathbf{B}^\Bbbk_e \sqcup -\mathbf{B}^\Bbbk_e$ and $\mathbf{B}^\pm_{\Lambda_e} = \mathbf{B}^\Bbbk_S \sqcup -\mathbf{B}^\Bbbk_S$, so these are indeed signed bases.[18]

Part (g) of the Conjecture says that $\partial(\nabla_e b_1\|b_2) \in \pm\mathbb{Z}_{\geq 0}[(-v)^{-1}]$ for $b_1, b_2 \in \mathbf{B}^\pm_{\Lambda_e}$, i.e., for $b_1, b_2 \in \mathbf{B}^\Bbbk_S$. Using the $X^*(C)$-action on $\mathbf{B}^\Bbbk_S$ we see that this is equivalent to the claim that for $b_i \in \mathbf{B}^\Bbbk_S$ all coefficient polynomials $c^\nu_{b_1,b_2}$, $\nu \in X^*(C)$ of $\nabla_e(b_1\|b_2)$ (see **(D)** for

[18] This is a standard argument. Let $\xi \in K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')$ with $(\xi\|\xi) \in 1 + v^{-1}\mathbb{Z}[[v^{-1}]]$. Write ξ as $\sum_{b\in\mathbf{B}_S,\ n\in\mathbb{Z}} c^n_b v^n\ b$. If N is the highest power of v that appears then $\sum_b (c^N_b)^2 \geq 1$ and asymptotic orthonormality of the basis implies that $\sum_b (c^N_b)^2 = 1$ and $N = 0$. If also $\beta_S\xi = \xi$ then $c^{-n}_b = c^n_b$ since $\beta_S(vx) = v^{-1}\beta_S x$. Therefore, $c^n_b \neq 0$ implies $n = 0$ and such b is unique.

notation), are in $\pm\mathbb{Z}_{\geq 0}[(-v)^{-1}]$. This follows from the first claim in **(C)** and the "parity vanishing" statement **(D)**.

Part (h) of the conjecture says that $\mathbf{B}_S^{\Bbbk}$ satisfies the normalization property from **(E)**.

Thus part (1) of the Theorem is established. For part (2) recall that Conjecture 17.2 claims that if a subcategory $\mathfrak{M}\subseteq mod^{C,fg}(U_{\Bbbk,e})$ is a generic block then there exists a bijection $Irr(\mathfrak{M})\underset{\cong}{\to}\mathbf{B}^{\pm}_{\Lambda_e}/\{\pm 1\}$ which is compatible with the action of $X^*(C)$ and identifies the Cartan matrix of $mod^{C,fg}(U_{\Bbbk,e})$ with the matrix $|\partial[\nabla_e(b_1\|b_2)_{(1,-1)}]|,\ b_1,b_2\in\mathbf{B}^{\pm}_{\Lambda_e}$, of absolute values of evaluations at $(1,-1)\in C\times\mathfrak{G}_{\mathfrak{m}}=\widetilde{C}$.

We know that the subcategory $\mathfrak{M}=mod^{C,fg}(U^{\widehat{0}}_{\Bbbk,e})$ is a generic block (see [BG]). Also, Proposition 5.3.6 together with the established part (1) of the Theorem yields bijections $\mathbf{B}^{\pm}_{\Lambda_e}/\{\pm 1\}\cong\mathbf{B}_S\cong Irr(\mathfrak{M})$. Now the difference between Lusztig's formulation and the second sentence in **(G)** is that the former uses evaluation at $-1\in\mathfrak{G}_{\mathfrak{m}}$ and absolute value, while the latter uses evaluation at $1\in\mathfrak{G}_{\mathfrak{m}}$. This is accounted for by the parity vanishing property **(D)**. □

5.3.8. *Pairing* $(-\|-)$ *and Poincare series of sheaves on* $\widetilde{S'_{\Bbbk,e}}$. The next Lemma explains the categorical meaning of the pairing $(-\|-)$. To present it we need another notation.

Let $Rep^+(\widetilde{C})$ be the category of representations U of $\widetilde{C}$ with finite multiplicities and with $\mathfrak{G}_{\mathfrak{m}}$-isotypic components $U_d,\ d\in\mathbb{Z}$, vanishing for $d<<0$. We denote by $[U]$ its image in the K-group $K^0[Rep^+(\widetilde{C})]\cong\ R_C((v))$ where v is the image of the standard representation of $\mathfrak{G}_{\mathfrak{m}}$ in the K-group. This extends to a map $U\mapsto[U]$ from $D^b[Rep^+(\widetilde{C})]$ to $K^0[Rep^+(\widetilde{C})]$.

Now, for $\mathcal{F}\in D^b\mathcal{C}oh^{\widetilde{C}}(\widetilde{S'_{\Bbbk,e}})$ we have $\mathrm{R}\Gamma(\mathcal{F})\in D^b[Rep^+(\widetilde{C})]$ and it is easy to show (see also [Lu]), that $[\mathrm{R}\Gamma(\mathcal{F})]\in R_C((v))$ is Laurent series of a rational function, i.e., it lies in $\mathfrak{K}_{\widetilde{C}}\subset R_C((v))$. Of course, if $\mathcal{G}\in D^b\mathcal{C}oh^{\mathfrak{G}_{\mathfrak{m}}}(\widetilde{S'_{\Bbbk,e}})$ then the same applies to $\mathrm{RHom}(\mathcal{F},\mathcal{G})=\ \mathrm{R}\Gamma[\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{G})]$.

Recall that $P\mapsto P^{\nu}$ denotes the involution of $\mathfrak{K}_{\widetilde{C}}$ corresponding to inversion on $\widetilde{C}$.

Lemma. Let $\mathcal{F},\mathcal{G}\in D^b(Coh^{\widetilde{C}}(\widetilde{S'_{\Bbbk,e}}))$.

a) If $\mathcal{G}$ is set theoretically supported on $\mathcal{B}'_{\Bbbk,e}$ and the class $[\mathcal{G}]$ is invariant under β_e, then $(\mathcal{F}\|\mathcal{G})=\ [\mathrm{RHom}(\mathcal{G},\mathcal{F})]$.

b) If $[\mathcal{G}]$ is invariant under β_S then $(\mathcal{F}\|\mathcal{G})=\ [\mathrm{RHom}(\mathcal{F},\mathcal{G})]^{\nu}$.

c) Let $\mathsf{F}, \mathsf{G} \in D^b\mathcal{C}oh^{\widetilde{C}}(\widehat{\mathcal{B}_{\Bbbk,e}})$ be such that the restrictions of F (respectively, G) to the formal neighborhood of $\mathcal{B}'_{\Bbbk,e}$ in $\widetilde{S_{\Bbbk,e}}'$ is isomorphic to the restriction of $\mathcal{F}$ (respectively, $\mathcal{G}$).

If $[\mathcal{G}]$ is invariant under β_S then

$$\nabla_e(\mathcal{F}\|\mathcal{G}) \;=\; [\mathrm{RHom}(\mathsf{F},\mathsf{G}) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}(\widehat{e})} \Bbbk_e]^{\mathfrak{v}}.$$

The Lemma will be proven in section 5.4.2.

5.4. **Proofs for subsection 5.3.** In 5.4.1 we recall $\beta_e,\ \beta_S,\ (\ \|\ )$ and in 5.4.2 we check formulas for $(b\|c)$ when c is fixed by β_e or β_S. Then we prove in 5.4.3 that β_e fixes the K-class of (a shift of) $\widetilde{\mathcal{L}}_{i,\lambda}$ and β_S fixes the class of $\widetilde{\mathcal{E}}_{i,\lambda}$. This is all the preparation we need for the proof of Proposition 5.3.6 in 5.4.4.

5.4.1. *Involutions β_e, β_S and Υ.* Involutions β_S on $K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')$ and β_e on $K^{\widetilde{C}}(\mathcal{B}_{\Bbbk,e}) = K^{\widetilde{C}}(\mathcal{B}'_{\Bbbk,e})$ are defined in [Lu, section 5.11, page 304] by

$$\beta_S \overset{\mathrm{def}}{=} (-v)^{-\dim\mathcal{B}+2\dim\mathcal{B}_e}\Upsilon T_{w_0}^{-1}\mathbb{D} \quad \text{and} \quad \beta_e \overset{\mathrm{def}}{=} (-v)^{-\dim\mathcal{B}}\Upsilon\circ T_{w_0}^{-1}\circ\mathbb{D}.$$

Here Υ is a certain involution, T_{w_0} is an element of a standard basis for the affine Hecke algebra corresponding to the long element $w_0 \in W$ and $\mathbb{D}$ is the Grothendieck duality functor. Since the direct image for the closed embedding $i : \mathcal{B}'_{\Bbbk,e} \to \widetilde{S_{\Bbbk,e}}'$ intertwines Grothendieck duality functors, we have $i_*\beta_e = v^{-2\dim\mathcal{B}_e}\beta_S i_*$, since i_* is an embedding we write this as $\beta_S = \ v^{2\dim\mathcal{B}_e}\beta_e$.

Actually, $(-v)^{-\dim\mathcal{B}}T_{w_0}^{-1}$ is the effect on the K-group of the action of $\widetilde{w_0} \in \mathbb{B}$ on $D^b(\mathcal{C}oh^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}'))$ (see Theorem 1.3.2.b). Therefore, $\beta_e = \ \Upsilon\widetilde{w_0}\mathbb{D}$.

The only information about Υ (defined in [Lu, 5.7]) that we will use is as follows:

$$\Upsilon = \sum_{s=1}^{l} a_s g_s^*, \text{ with } g_s \in A(\widetilde{C}, \Bbbk\cdot e), \mathrm{ord}(g_s) < \infty \text{ and } a_s \in \mathbb{Q}, \sum_s a_s = 1; \tag{15}$$

$$\overline{\Upsilon} = \overline{T_{w_0}\circ\mathbb{D}}. \tag{16}$$

Here, $A(\widetilde{C}, \Bbbk\cdot e)$ is the group of automorphisms of G normalizing the line $\Bbbk\cdot e$ and $\widetilde{C}$, bar denotes the induced action on the Grothendieck group $K^0(Coh^C(\widetilde{S_{\Bbbk,e}}'))$ and T_{w_0} is the action of $w_0 \in W$ on $K(\widetilde{S_{\Bbbk,e}}')$ from [Lu].

Claim (15) is immediate from the definition of Υ in [Lu] and (16) will be shown in the Appendix A.

Lusztig defines pairing $(\ :\ )$ on $K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')$ by $([\mathcal{F}]\ :\ [\mathcal{G}]) \stackrel{\text{def}}{=} [\mathrm{R}\Gamma(\mathcal{F} \overset{\mathrm{L}}{\otimes} \mathcal{G})]$ (see [Lu, 2.6]), and uses it to define the pairing $(\ \|\ )$ by

$$([\mathcal{F}]\ \|\ [\mathcal{G}]) \stackrel{\text{def}}{=} (-v)^{\dim \mathcal{B}-2\dim \mathcal{B}_e}([\mathcal{F}] : T_{w_0}\Upsilon[\mathcal{G}]) = v^{-2\dim \mathcal{B}_e}([\mathcal{F}] : \widetilde{w_0}^{-1}\Upsilon[\mathcal{G}])$$

(see [Lu, 5.8]). Since $\beta_e = {\beta_e}^{-1} = \mathbb{D}\widetilde{w_0}^{-1}\Upsilon$ gives $\mathbb{D}\beta_e = \widetilde{w_0}^{-1}\Upsilon$, we have

$$([\mathcal{F}]\|[\mathcal{G}]) = v^{-2\dim \mathcal{B}_e}([\mathcal{F}] : \mathbb{D}\beta_e[\mathcal{G}]) = ([\mathcal{F}] : \mathbb{D}\beta_S[\mathcal{G}]).$$

These pairings on $K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')$ descend to pairings on $K^{C}(\widetilde{S_{\Bbbk,e}}')$ which we denote the same way. We will denote $([\mathcal{F}]\ \|\ [\mathcal{G}])$ simply by $(\mathcal{F}\|\mathcal{G})$.

Remark. The involutions β_e, β_S are K-group avatars of dualities that would fix irreducibles (resp. projectives) corresponding to the fundamental alcove $\mathfrak{A}_0$ The point is that (if one neglects $\mathfrak{G}_{\mathfrak{m}}$ equivariance), the duality $\mathrm{R}\mathcal{H}\mathrm{om}(-,\mathcal{O})$ takes projectives for $\mathfrak{A}_0$ to projectives for $-\mathfrak{A}_0$, and then $\widetilde{w_0}$ returns them to projectives for $\mathfrak{A}_0$.[19] This composition creates a permutation of indecomposable projectives or irreducibles for $\mathfrak{A}_0$, in order to undo this permutation Lusztig describes in terms of the centralizer action a $\mathbb{Z}[v,v^{-1}]$ *linear* involution Υ on the K-group which induces the same permutation. This is a generalization of the relation of the Chevalley involution to duality of irreducible representations of reductive groups.

5.4.2. *Proof of Lemma 5.3.8.* a) β_e-invariance of $[\mathcal{G}]$ gives

$$(\mathcal{F}\|\mathcal{G}) \ = \ ([\mathcal{F}] : (v)^{-2\dim \mathcal{B}_e}\mathbb{D}\beta_e[\mathcal{G}]) \ = \ ([\mathcal{F}] : (v)^{-2\dim \mathcal{B}_e}\mathbb{D}[\mathcal{G}]).$$

However,

$$\mathbb{D}[\mathcal{G}] \ = \ [\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{G}, \Omega_{\widetilde{S_{\Bbbk,e}}'})[\dim \widetilde{S_{\Bbbk,e}}'] \ ] \ = \ v^{2\dim \mathcal{B}_e}\ [\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{G}, \mathcal{O}_{\widetilde{S_{\Bbbk,e}}'})].$$

For the second equality recall that for the standard symplectic form ω on $\tilde{\mathcal{N}}$, restriction $\omega|_{\widetilde{S_{\Bbbk,e}}'}$ is again symplectic, so its top wedge power $\omega^{\dim(\mathcal{B}_e)}|_{\widetilde{S_{\Bbbk,e}}'}$ is a non-vanishing section of the canonical line bundle $\Omega_{\widetilde{S_{\Bbbk,e}}'}$. Now the claim follows since ω is invariant under the action of G and transforms by the tautological character under the action of $\mathbb{G}_m$ by dilations, while $\mathfrak{G}_{\mathfrak{m}}$ acts by a combination of G and square of dilations.

[19] Notice that because of the difference between dimensions of supports, the analogous procedure for irreducibles would use $\mathrm{R}\mathcal{H}\mathrm{om}(-,\mathcal{O})[2\dim \mathcal{B}_e]$ instead of $\mathrm{R}\mathcal{H}\mathrm{om}(-,\mathcal{O})$.

Thus we see that

$$(\mathcal{F}\|\mathcal{G}) \;=\; (\mathcal{F} : \mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{G},\mathcal{O})) \;=\; [\mathrm{R}\Gamma(\mathcal{F} \overset{\mathrm{L}}{\otimes} \mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{G},\mathcal{O}))] \;=\; [\mathrm{RHom}(\mathcal{G},\mathcal{F})].$$

b) β_S-invariance of $[\mathcal{G}]$ gives

$$(\mathcal{F}\|\mathcal{G}) \;=\; ([\mathcal{F}] : \mathbb{D}\beta_S[\mathcal{G}]) \;=\; ([\mathcal{F}] : \mathbb{D}[\mathcal{G}]) \;=\; [\mathrm{R}\Gamma(\mathcal{F} \overset{\mathrm{L}}{\otimes} \mathbb{D}\mathcal{G})] \;=\; [\mathrm{R}\Gamma(\mathbb{D}\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{G}))].$$

It is easy to show that if $\widetilde{C}$ acts linearly on a vector space V and $\mathfrak{G}_{\mathfrak{m}}$ contracts it to the origin for $\mathfrak{G}_{\mathfrak{m}} \ni t \to \infty$, then for any $\mathcal{K} \in D^b[\mathcal{C}oh^{\widetilde{C}}(V)]$ one has $[\mathrm{R}\Gamma(\mathbb{D}\mathcal{K})] = \; [\mathrm{R}\Gamma(\mathcal{K})]^{\mathsf{v}}$. Applying this to the sheaf $\mathcal{K} = (\widetilde{S_{\Bbbk,e}}' \to S_{\Bbbk,e})_*\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{G},\mathcal{F})$ on the space $S_{\Bbbk,e} \cong Z_{\mathfrak{g}}(f)$ we get the result.

c) For a finite dimensional $\widetilde{C}$-module V Lusztig denotes by $\boxed{V} \overset{\mathrm{def}}{=} [\wedge^\bullet V] \in R_{\widetilde{C}}$ the image of the super-module $\wedge^\bullet V$ in $K^{\widetilde{C}}$. So, for $\mathcal{K} \in D^b[Coh^{\widetilde{C}}(V)]$ and $i : 0 \hookrightarrow V$ use of Koszul complex gives

$$[i_*i^*\mathcal{K}] = \; [\mathcal{K} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}(V)} \; \mathcal{O}(V)\otimes_{\Bbbk} \wedge^\bullet V^*] = \; \boxed{V^*}\,[\mathcal{K}] = \; \boxed{V}^{\mathsf{v}}\,[\mathcal{K}].$$

By definition $\nabla_e \overset{\mathrm{def}}{=} \boxed{Z_{\mathfrak{g}}(f)}\,\boxed{\mathfrak{h}}^{-1}$ ([Lu, 3.1]), where $\widetilde{C}$ acts on $\mathfrak{h}$ by $(c,t)h = c^{-2}h$.

As in 5.3.1, there is a unique $\overset{o}{\mathsf{G}} \in D^b[\mathcal{C}oh^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}})]$ which agrees with G on $\widetilde{S_{\Bbbk,e}} \cap \widehat{\mathcal{B}_{\Bbbk,e}}$. Notice that because the restrictions to $\widetilde{S_{\Bbbk,e}}' \cap \widehat{\mathcal{B}_{\Bbbk,e}}$ agree for $\overset{o}{\mathsf{G}}$, G and $\mathcal{G}$, we also have $\overset{o}{\mathsf{G}}|_{\widetilde{S_{\Bbbk,e}}'} \cong \mathcal{G}$. Now, in order to calculate the K-class of $\mathrm{RHom}(\mathsf{F},\mathsf{G}) \otimes_{\mathcal{O}(\widehat{e})} \Bbbk_e \cong \; \mathrm{RHom}(\mathsf{F}, \mathsf{G} \otimes_{\mathcal{O}(\widehat{e})} \Bbbk_e)$, observe that by the definition of $\overset{o}{\mathsf{G}}$ we have $\mathsf{G} \overset{L}{\otimes}_{\mathcal{O}(\widehat{e})} \mathcal{O}(S_{\Bbbk,e} \cap \widehat{e}) \cong \; \overset{o}{\mathsf{G}} \overset{L}{\otimes}_{\mathcal{O}(S_{\Bbbk,e})} \mathcal{O}(S_{\Bbbk,e} \cap \widehat{e})$, and this gives $\mathsf{G} \overset{L}{\otimes}_{\mathcal{O}(\widehat{e})} \Bbbk_e \cong \; \overset{o}{\mathsf{G}} \overset{L}{\otimes}_{\mathcal{O}(S_{\Bbbk,e})} \Bbbk_e$. Similarly, F gives $\overset{o}{\mathsf{F}}$ with analogous properties. Therefore in $K^{\widetilde{C}} = R_{\widetilde{C}}$ one has

$$[\mathrm{RHom}_{\widehat{\mathcal{B}_{\Bbbk,e}}}(\mathsf{F},\mathsf{G}) \overset{L}{\otimes}_{\mathcal{O}(\widehat{e})} \Bbbk_e] = \; [\mathrm{RHom}_{\widehat{\mathcal{B}_{\Bbbk,e}}}(\mathsf{F}, \overset{o}{\mathsf{G}} \overset{L}{\otimes}_{\mathcal{O}(S_{\Bbbk,e})} \Bbbk_e)] = \; [\mathrm{RHom}_{\widehat{\mathcal{B}_{\Bbbk,e}} \cap \widetilde{S_{\Bbbk,e}}}(\mathsf{F}|_{\widetilde{S_{\Bbbk,e}}}, \overset{o}{\mathsf{G}} \overset{L}{\otimes}_{\mathcal{O}(S_{\Bbbk,e})} \Bbbk_e)].$$

When we replace $\mathsf{F}|_{\widetilde{S_{\Bbbk,e}}}$ with $\overset{o}{\mathsf{F}}|_{\widehat{\mathcal{B}_{\Bbbk,e}}}$ we can view this as

$$[\mathrm{RHom}_{\widetilde{S_{\Bbbk,e}}}(\overset{o}{\mathsf{F}}, \overset{o}{\mathsf{G}} \overset{L}{\otimes}_{\mathcal{O}(S_{\Bbbk,e})} \Bbbk_e)] = \boxed{Z_{\mathfrak{g}}(f)^*}\,[\mathrm{RHom}_{\widetilde{S_{\Bbbk,e}}}(\overset{o}{\mathsf{F}}, \overset{o}{\mathsf{G}})] = \frac{\boxed{Z_{\mathfrak{g}}(f)^*}}{\boxed{\mathfrak{h}^*}}\,[\mathrm{RHom}_{\widetilde{S_{\Bbbk,e}}}(\overset{o}{\mathsf{F}}, \overset{o}{\mathsf{G}} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{h})} \Bbbk_0).$$

Now, $\overset{o}{\mathsf{G}} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{h})} \Bbbk_0 \cong \; \overset{o}{\mathsf{G}}|_{\widetilde{S_{\Bbbk,e}}'} \cong \mathcal{G}$. The same observation for $\overset{o}{\mathsf{F}}$ and adjunction give

$$= \; \nabla_e^{\mathsf{v}}\,[\mathrm{RHom}_{\widetilde{S_{\Bbbk,e}}}(\overset{o}{\mathsf{F}}, \mathcal{G})] = \; \nabla_e^{\mathsf{v}}\,[\mathrm{RHom}_{\widetilde{S_{\Bbbk,e}}'}(\mathcal{F}, \mathcal{G})].$$

So, the claim follows from b).

5.4.3. *Proof of invariance of the bases under the involutions.* **(ℵ) Proof of** (13)**.**

(ℵ.i) Reduction to: β_S **preserves** $\oplus_{i,\lambda}\ \mathbb{Q}[\widetilde{\mathcal{E}}^S_{i,\lambda}]$**.** According to [Lu], the restriction of equivariance map induces an isomorphism $K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')/(v-1)\underset{\cong}{\to}K^C(\widetilde{S_{\Bbbk,e}}')$. The $\mathbb{Q}$-vector subspace $\oplus_{i,\lambda}\ \mathbb{Q}[\widetilde{\mathcal{E}}^S_{i,\lambda}]$ in $K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')_{\mathbb{Q}}$ maps isomorphically to $K^C(\widetilde{S_{\Bbbk,e}}')_{\mathbb{Q}}$. In view of (16), the action of β_S on $K^C(\widetilde{S_{\Bbbk,e}}')_{\mathbb{Q}}$ is trivial, so it suffices to see that the vector subspace $\oplus_{i,\lambda}\ \mathbb{Q}[\widetilde{\mathcal{E}}^S_{i,\lambda}]$ is invariant under β_S.

We will factor β_S into $\Upsilon\mathcal{D}$, a functor $\mathcal{D}\mathcal{F}\overset{\text{def}}{=}\widetilde{w_0}\big(\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{O})\big)(4\dim\mathcal{B}_e)$ and Υ which is only defined on the K-group. Indeed, $\beta_S = v^{2\dim\mathcal{B}_e}\beta_e = v^{2\dim\mathcal{B}_e}\Upsilon\widetilde{w_0}\mathbb{D}$ and $\Omega_{\widetilde{S_{\Bbbk,e}}'} = \mathcal{O}_{\widetilde{S_{\Bbbk,e}}'}(2\dim\mathcal{B}_e)$ (see the beginning of 5.4.2), so that $\mathbb{D}\mathcal{F} = \mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{O}_{\widetilde{S_{\Bbbk,e}}'})(2\dim\mathcal{B}_e)[2\dim\mathcal{B}_e]$. We will actually show that $\oplus_{i,\lambda}\ \mathbb{Q}[\widetilde{\mathcal{E}}^S_{i,\lambda}]$ is invariant under both Υ and $\mathcal{D}$.

(ℵ.ii) Invariance under Υ**.** By (15), it suffices to show that for any finite order element g in $A(\widetilde{C},\Bbbk{\cdot}e)$, the pull back g^* permutes $[\widetilde{\mathcal{E}}^S_{i,\lambda}]$'s. Since g is an automorphism commuting with the multiplicative group $\mathfrak{G}_{\mathfrak{m}}$ and fixing the line of e, we see that $\{g^*(\widetilde{\mathcal{E}}_i)\}$ is a set of $\mathfrak{G}_{\mathfrak{m}}$-equivariant vector bundles on $\widehat{\mathcal{B}_{\Bbbk,e}}$ satisfying the properties of $\widetilde{\mathcal{E}}_i$ from **(⋆)**, thus uniqueness part of **(⋆)** implies that for each i there exists some i^g such that $g^*(\widetilde{\mathcal{E}}_i)\cong\widetilde{\mathcal{E}}_{i^g}(d_g)$, where integer d_g does not depend on i. Obviously $d_{g^n}=nd_g$, and therefore $d_g=0$ since g is assumed to have finite order. The isomorphism $g^*(\widetilde{\mathcal{E}}_i)\cong\widetilde{\mathcal{E}}_{i^g}$ implies $g^*(\widetilde{\mathcal{E}}^S_i)\cong\widetilde{\mathcal{E}}^S_{i^g}$ for the corresponding $\mathfrak{G}_{\mathfrak{m}}$-equivariant bundles on $\widetilde{S_{\Bbbk,e}}'$. Since g fixes $S_{\Bbbk,e}$, g^* fixes $\mathfrak{G}_{\mathfrak{m}}$-equivariant vector bundle $\widetilde{\mathcal{E}}_{i_0}=\mathcal{O}_{\widetilde{S_{\Bbbk,e}}'}(2\dim\mathcal{B}_e)$ and by uniqueness in 5.3.3 this implies that g^* permutes the collection of $\widetilde{\mathcal{E}}^S_i$'s and then also the collection of all $\widetilde{\mathcal{E}}^S_{i,\lambda}$'s.

Also note that Υ fixes the K-class of $\widetilde{\mathcal{E}}_{i_0,0}=\mathcal{O}_{\widetilde{S_{\Bbbk,e}}'}(2\dim\mathcal{B}_e)$ since this is true for all relevant g^* and in (15) we have $\sum_s\ a_s=1$.

(ℵ.iii) $\mathcal{D}$ **permutes** $[\widetilde{\mathcal{E}}^S_{i,\lambda}]$**'s.** $\mathcal{D}$ factors to $D^b[\mathcal{C}oh(\widehat{\mathcal{B}_{\Bbbk,e}}')]$ as $\mathcal{D}=\widetilde{w_0}\mathrm{R}\mathcal{H}\mathrm{om}(-,\mathcal{O})$. Part (e) of the Theorem 1.8.2 shows that the dual vector bundles $\mathcal{E}_i^*$ are exactly all indecomposable projectives in the heart of the t-structure $\mathcal{T}^{\widehat{\mathcal{B}_{\Bbbk,e}}',\widetilde{\mathcal{N}}_{\Bbbk}}_{-\mathfrak{A}_0}$ on $D^b(\mathcal{C}oh(\widehat{\mathcal{B}_{\Bbbk,e}}'))$, Since, $b_{-\mathfrak{A}_0,\mathfrak{A}_0}=\widetilde{w_0}$ by example 1.8.1, part (a.2) of the Theorem 1.8.2 now shows that the sheaves $\mathcal{D}\widetilde{\mathcal{E}}_i=\ \widetilde{w_0}(\mathcal{E}_i^*)$ are all indecomposable projectives in the heart of $\mathcal{T}^{\widehat{\mathcal{B}_{\Bbbk,e}}',\widetilde{\mathcal{N}}_{\Bbbk}}_{\mathfrak{A}_0}$. Thus we have $\mathcal{D}\widetilde{\mathcal{E}}_i=\ \widetilde{w_0}(\mathcal{E}_i^*)\cong\ \mathcal{E}_{\check{i}}$ for some permutation $i\mapsto\check{i}$ of the indexing set.[20]

[20]This could also be deduced from [BMR2, Corollary 3.0.11].

Let us now add $\mathfrak{G}_m$-equivariance. Since a $\mathfrak{G}_{\mathfrak{m}}$-equivariant structure on $\widetilde{\mathcal{E}}_i$ is unique up to a twist (Lemma 5.2.3.b), we have $\mathcal{D}\widetilde{\mathcal{E}}_i \cong \widetilde{\mathcal{E}}_{\check{i}}(d_i)$ for some integers d_i. The uniqueness statement in $(\star)$ implies that $d_i = d_j$ for all i, j. On the other hand, it follows from [Lu, 5.14] that β_S sends the class of $\widetilde{\mathcal{E}}^S_{i_0} \cong \mathcal{O}_{\widetilde{S_{\mathbb{k},e}}'}(2\dim\mathcal{B}_e)$ to itself. Since we have already checked that Υ fixes $\mathcal{O}_{\widetilde{S_{\mathbb{k},e}}'}(2\dim\mathcal{B}_e)$ (the last remark in ($\aleph$.ii)), we find the same is true for $\mathcal{D}$, therefore $d_i = 0$ for $i = i_0$ and then the same holds for all i's.

We can transport $\mathcal{D}[\widetilde{\mathcal{E}}_i] = [\widetilde{\mathcal{E}}_{\check{i}}]$ to $\widetilde{S_{\mathbb{k},e}}'$ to get $\mathcal{D}[\widetilde{\mathcal{E}}^S_i] = [\widetilde{\mathcal{E}}^S_{\check{i}}]$. Similarly, uniqueness of a torus equivariant structure (up to a twist) gives $\mathcal{D}[\widetilde{\mathcal{E}}^S_{i,\lambda}] \cong [\widetilde{\mathcal{E}}^S_{\check{i},\nu(i,\lambda)}]$ for some $\nu(i,\lambda) \in \Lambda$. We will write this as $\mathcal{D}[\widetilde{\mathcal{E}}^S_{i,\lambda}] \cong [\widetilde{\mathcal{E}}^S_{(i,\lambda)^\vee}]$

($\aleph\aleph$) Proof of (12)**.** Recall from ($\aleph$.i) that $\beta_e = v^{-2\dim\mathcal{B}_e}\beta_S = \Upsilon v^{-2\dim\mathcal{B}_e}\mathcal{D}$. In particular, β_e acts on $K^C(\widetilde{S_{\mathbb{k},e}}')$ the same as β_S, i.e., trivially. Therefore, as in the proof of (13) we only need that $\oplus_{i,\lambda}\, v^{-2\dim\mathcal{B}_e}\mathbb{Q}[\widetilde{\mathcal{L}}_{i,\lambda}] \subseteq K^{\widetilde{C}}(\widehat{\mathcal{B}_{\mathbb{k},e}})$ be invariant under β_e, and this will follow from more detailed information: $v^{-2\dim\mathcal{B}_e}[\widetilde{\mathcal{L}}_{i,\lambda}]$ are permuted by (i) finite order elements of $A(\widetilde{C}, \mathbb{k}e)$ and (ii) $v^{-2\dim\mathcal{B}_e}\mathcal{D} = \widetilde{w_0}\mathbb{D}$.

Since we have checked that finite order elements of $A(\widetilde{C}, \mathbb{k}e)$ permute $\widetilde{\mathcal{E}}_{i,\lambda}$'s it follows that they also permute $\widetilde{\mathcal{L}}_{i,\lambda}$'s, hence also $v^{-2\dim\mathcal{B}_e}[\widetilde{\mathcal{L}}_{i,\lambda}]$. On the other hand,

$$\mathrm{RHom}(\mathcal{D}\widetilde{\mathcal{L}}_{i,\lambda}, \widetilde{\mathcal{E}}_{j,\mu}) \cong \mathrm{RHom}(\mathcal{D}\widetilde{\mathcal{E}}_{j,\mu}, \widetilde{\mathcal{L}}_{i,\lambda}) \cong \mathbb{k}^{\delta^{(i,\lambda)}_{(j,\mu)^\vee}}$$

gives

$$\mathbb{k}^{\delta^{(i,\lambda)^\vee}_{(j,\mu)}} \cong \mathrm{RHom}(\mathcal{D}\widetilde{\mathcal{L}}_{i,\lambda}, \widetilde{\mathcal{E}}_{j,\mu})^* \cong \mathrm{RHom}(\widetilde{\mathcal{E}}_{j,\mu}, \mathcal{D}\widetilde{\mathcal{L}}_{i,\lambda} \otimes \Omega_{\widetilde{S_{\mathbb{k},e}}'}[2\dim\mathcal{B}_e]),$$

hence $\mathcal{D}(\widetilde{\mathcal{L}}_{i,\lambda}) = \widetilde{\mathcal{L}}_{(i,\lambda)^\vee} \otimes \Omega^{-1}_{\widetilde{S}'_e}[-2\dim\mathcal{B}_e]$. Thus

$$(v^{-2\dim\mathcal{B}_e}\mathcal{D})\big(v^{-2\dim\mathcal{B}_e}[\widetilde{\mathcal{L}}_{i,\lambda}]\big) = \mathcal{D}[\widetilde{\mathcal{L}}_{i,\lambda}] = v^{-2\dim\mathcal{B}_e}[\widetilde{\mathcal{L}}_{(i,\lambda)^\vee}].$$

5.4.4. *Proof of Proposition 5.3.6.* **(A)** We know that $\mathbf{B}_S$ and $\mathbf{B}_e$ are sets of representatives – modulo $\mathfrak{G}_{\mathfrak{m}}$ shifts – of isomorphism classes of respectively, indecomposable projective objects in $\mathcal{E}coh^{\widetilde{C}}(\widetilde{S_{\mathbb{k},e}}')$ and of irreducible objects in $\mathcal{E}coh^{\widetilde{C}}(\widehat{\mathcal{B}_{\mathbb{k},e}}')$. Since the exotic t-structure is bounded, they form bases in the respective Grothendieck groups $K^{\widetilde{C}}(\widetilde{S_{\mathbb{k},e}}')$ and $K^{\widetilde{C}}(\widehat{\mathcal{B}_{\mathbb{k},e}}')$ over the ring $R_{\mathfrak{G}_{\mathfrak{m}}}$. Pointwise invariance of $\mathbf{B}_S$ and $\mathbf{B}_e$ under the involutions β_S and β_e has been proved in the previous subsection 5.4.3.

Lemma 5.3.8.b) implies that for $b_i \in \mathbf{B}_S$ one has $(b_1\|b_2) = [\mathrm{RHom}(b_1, b_2)]^\vee$ (because β_S fixes b_2). Since b_i are projective objects this is really $[\mathrm{Hom}(b_1, b_2)]^\vee$, so it lies in $R^+_C[v^{\pm 1}]$.

Now, property **(⋆)** says that the algebra $\mathcal{A} = End_{Coh(\widetilde{S_{\Bbbk,e}}')}(\bigoplus_i \widetilde{\mathcal{E}}_i^S)$ equipped with the grading coming from the $\mathfrak{G}_\mathfrak{m}$ equivariant structure on $\widetilde{\mathcal{E}}_i^S$ has no components of negative degree and the component of degree zero is spanned by identity endomorphisms of $\widetilde{\mathcal{E}}_i^S$'s. This is the same as saying that if $X^*(C)b_1 \neq X^*(C)b_2$ then $[\mathrm{RHom}(b_1, b_2)] \in vR_C^+[[v]]$ and if $b_1 = b_2$ then $[\mathrm{RHom}(b_1, b_2)] \in 1 + R_C^+[[v]]$. So, we have established for the basis $\mathbf{B}_S$ the asymptotic orthonormality property and also a positivity property $(\mathbf{B}_S||\mathbf{B}_S) \subseteq R_C^+[[v^{-1}]]$.

Similarly, Lemma 5.3.8.a) implies that for $b_i \in \mathbf{B}_e$ one has $(b_1\|b_2) = [\mathrm{RHom}(b_1, b_2)]$, because β_e fixes b_2. The properties of the $\mathfrak{G}_\mathfrak{m}$-grading of $\mathcal{A}$ imply that the $\mathfrak{G}_\mathfrak{m}$-grading on $\mathrm{Ext}^\bullet_\mathcal{A}[\bigoplus_i \widetilde{\mathcal{L}}_i^S, \bigoplus_i \widetilde{\mathcal{L}}_i^S)]$ has no positive $\mathfrak{G}_\mathfrak{m}$-degrees and the component of degree zero is spanned by identity maps. This is the same as as saying that if $X^*(C)b_1 \neq X^*(C)b_2$ then $[\mathrm{RHom}(b_1, b_2)] \in v^{-1}R_C[[v^{-1}]]$ and if $b_1 = b_2$ then $[\mathrm{RHom}(b_1, b_2)] \in 1 + v^{-1}R_C[[v^{-1}]]$.

(B) Since β_S fixes $\widetilde{\mathcal{E}}_{j,\mu}^S$, Lemma 5.3.8.b) and Calabi-Yau property of $\tilde{\mathfrak{g}}$ give

$$(\widetilde{\mathcal{L}}_{i,\lambda}\|\widetilde{\mathcal{E}}_{j,\mu}^S) = [RHom(\widetilde{\mathcal{L}}_{i,\lambda}, \widetilde{\mathcal{E}}_{j,\mu}^S)] = [\mathrm{RHom}_\Bbbk\left(\mathrm{RHom}(\widetilde{\mathcal{E}}_{j,\mu}^S, \widetilde{\mathcal{L}}_{i,\lambda} \otimes \Omega_{\widetilde{S_{\Bbbk,e}}'}[\dim \widetilde{S_{\Bbbk,e}}']), \Bbbk\right)]$$

$$= \left(v^{2\dim \mathcal{B}_e}[\mathrm{RHom}(\widetilde{\mathcal{E}}_{j,\mu}^S, \widetilde{\mathcal{L}}_{i,\lambda})]\right)^\vee = v^{-2\dim \mathcal{B}_e}[\Bbbk^{\delta_j^i \delta_\mu^\lambda}] = \delta_j^i \delta_\mu^\lambda \, v^{-2\dim \mathcal{B}_e}.$$

(C) In **(A)** we have already checked that $(\mathbf{B}_S\|\mathbf{B}_S) \subseteq R_C^+[[v^{-1}]]$. Recall that $\widetilde{\mathcal{E}}_{i,\lambda}^S$ was constructed so that on $\widehat{\widetilde{S_{\Bbbk,e}}'} \cap \widehat{\mathcal{B}_{\Bbbk,e}}$ it coincides with a certain projective exotic object $\widetilde{\mathcal{E}}_{i,\lambda} \in \mathcal{E}coh^{\widetilde{C}}(\widehat{\mathcal{B}_{\Bbbk,e}})$ (see 5.3.3). So, because β_S fixes $\widetilde{\mathcal{E}}_{j,\mu}^S$, Lemma 5.3.8.c) gives

$$\nabla_e(\widetilde{\mathcal{E}}_{i,\lambda}^S\|\widetilde{\mathcal{E}}_{j,\mu}^S) = [\mathrm{RHom}(\widetilde{\mathcal{E}}_{i,\lambda}, \widetilde{\mathcal{E}}_{j,\mu}) \overset{L}{\otimes}_{\mathcal{O}(\widehat{e})} \Bbbk_e] = [\mathrm{RHom}(\widetilde{\mathcal{E}}_{i,\lambda}, \widetilde{\mathcal{E}}_{j,\mu} \overset{L}{\otimes}_{\mathcal{O}(\widehat{e})} \Bbbk_e)].$$

Here, $\widetilde{\mathcal{E}}_{j,\mu} \overset{L}{\otimes}_{\mathcal{O}(\widehat{e})} \Bbbk_e$ is exotic, i.e., under the equivalence $D^b(Coh(\tilde{\mathfrak{g}})) \cong D^b(mod^{fg}(A))$ (restricted to $\widehat{\mathcal{B}_{\Bbbk,e}}$), the object $\widetilde{\mathcal{E}}_{j,\mu} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}(\widehat{e})} \Bbbk_e$ corresponds to a module rather than a complex of modules. The reason is that the algebra A is flat over $\mathcal{O}(\mathfrak{g})$ (see Lemma 1.5.3), hence the same is true for its projective modules. Therefore, the result is just $[\mathrm{Hom}(\widetilde{\mathcal{E}}_{i,\lambda}, \widetilde{\mathcal{E}}_{j,\mu} \overset{L}{\otimes}_{\mathcal{O}(\widehat{e})} \Bbbk_e)]$ which lies in $R_C^+[v^{\pm 1}]$. However, as $\nabla_e \in 1 + v^{-1}R_C[v^{-1}]$ ([Lu, Lemma 3.2]), from $(\mathbf{B}_S\|\mathbf{B}_S) \subseteq R_C^+[[v^{-1}]]$ we now get $\nabla_e(\mathbf{B}_S\|\mathbf{B}_S) \subseteq R_C^+[v^{-1}]$.

The second claim follows from asymptotic orthonormality from **(A)** and the fact $(K^{\widetilde{C}}(\mathcal{B}'_{\Bbbk,e})\|K^{\widetilde{C}}(\mathcal{B}'_{\Bbbk,e})) \subseteq R_{\widetilde{C}}$ which is checked in [Lu].

(D) Recall from 5.2.2 that for a certain $\boldsymbol{z} \in Z(G)$ the element $\boldsymbol{m} = (\phi(-1)\boldsymbol{z}, -1)$ of $\widetilde{C}$ acts trivially on $\tilde{\mathfrak{g}}$. This implies that it acts on any $\widetilde{\mathcal{E}}_{i,\lambda}$ by a scalar $\varepsilon_{i,\lambda}$.[21] For $\nu \in X^*(C)$ and $d \in \mathbb{Z}$ the coefficient of v^d in $c^{\nu}_{b_1,b_2} \in \mathbb{Z}[v^{\pm 1}]$ is the dimension of $\mathrm{Hom}_{\widetilde{C}}[\widetilde{\mathcal{E}}_{i,\lambda}, \widetilde{\mathcal{E}}_{j,\mu+\nu}(d) \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \Bbbk_e]$. If this is not zero then $(-1)^d = \varepsilon_{j,\mu+\nu}\varepsilon^{-1}_{i,\lambda}$ since $\boldsymbol{m} \in \widetilde{C}$ acts on $\mathrm{Hom}[\mathcal{E}_{i,\lambda}, \widetilde{\mathcal{E}}_{j,\mu+\nu}(d) \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \Bbbk_e]$ by $(-1)^d \varepsilon_{j,\mu+\nu}\varepsilon^{-1}_{i,\lambda}$.

(E) The normalization property is a part of the definition of $\mathbf{B}_S$ in 5.3.3, and we have already checked that $\mathbf{B}_S$ satisfies all other properties. Any $R_{\mathfrak{G}_{\mathfrak{m}}}$-basis $\mathbf{B}$ of $K^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')$ which is pointwise fixed by β_S and satisfies asymptotic orthonormality is of the form $\epsilon_b b,\ b \in \mathbf{B}_S$, for some $\epsilon \in \{\pm 1\}$, this much was established immediately after the statement of Proposition 5.3.6. If $\mathbf{B}$ satisfies normalization property then $\epsilon_{i_0} = 1$. Now either of positivity properties for the pairing $(\ \|\ )$ implies $\epsilon = 1$. The reason is that the equivalence relation $\sim$ on $\mathbf{B}_S$ generated by $b_1 \sim b_2$ if $(b_1 \| b_2) \neq 0$, is transitive since $(\widetilde{\mathcal{E}}^S_{i,\lambda} \| \widetilde{\mathcal{E}}^S_{j,\mu}) \neq 0$ is equivalent to $\mathrm{Hom}_{D^b[\mathcal{C}oh^{\widetilde{C}}(\widetilde{S_{\Bbbk,e}}')]}(\widetilde{\mathcal{E}}^S_{i,\lambda} \| \widetilde{\mathcal{E}}^S_{j,\mu}) \neq 0$ and the category $\mathcal{E}coh^{\mathbb{C}}(\widetilde{S_{\Bbbk,e}}')$ is indecomposable (because $D^b\mathcal{C}oh(X)$ is indecomposable for a connected variety X and $D^b\mathcal{C}oh(\widetilde{S_{\Bbbk,e}}') \cong D^b[mod^{fg}(A_S)] \cong D^b[\mathcal{E}coh^C(\widetilde{S_{\Bbbk,e}}')]$). Notice also that the last claim is equivalent to indecomposability of $\mathcal{E}coh^C(\widehat{\mathcal{B}_{\Bbbk,e}})$ and then the corresponding statement in representation theory is well known (see [BG]).

(F) The equivalence $\mathcal{E}coh^C(\widehat{\mathcal{B}_{\Bbbk,e}}) \xrightarrow{\iota_e} mod^{fg}(U^{\widehat{0}}_{\Bbbk,\widehat{e}}, C)$ from Theorem 1.6.7(c) provides compatible bijections (of isomorphism classes) of irreducibles and indecomposable projectives and an isomorphism $K^C(\widehat{\mathcal{B}_{\Bbbk,e}}) \underset{\cong}{\xrightarrow{\iota_e}} K^0[mod^{fg}(U^{\widehat{0}}_{\widehat{e}}, C)]$ which we can view as $K^C(\mathcal{B}_{\Bbbk,e}) \underset{\cong}{\xrightarrow{\iota_e}} K^0[mod^{fg}(U^{\widehat{0}}_{e}, C)]$. The list of irreducibles and their projective covers in $\mathcal{E}coh^C(\widehat{\mathcal{B}_{\Bbbk,e}})$ is given by images $\mathcal{L}_{i,\lambda}, \mathcal{E}_{i,\lambda}$ of the corresponding objects $\widetilde{\mathcal{L}}_{i,\lambda}, \widetilde{\mathcal{E}}_{i,\lambda}$ of $\mathcal{E}coh^{\widetilde{C}}(\widehat{\mathcal{B}_{\Bbbk,e}})$, and we denote by $L_{i,\lambda}, E_{i,\lambda}$ their images in $mod^{fg}(U^{\widehat{0}}_{\Bbbk,\widehat{e}}, C)$. The projective cover of $L_{i,\lambda}$ in $mod^{fg}(U^{\widehat{0}}_{\Bbbk,e}, C)$ is the restriction $E_{i,\lambda} \overset{L}{\otimes}_{\mathcal{O}(\widehat{e})} \Bbbk_e$. So, it remains to notice that the K-class of the restriction $\widetilde{\mathcal{E}}_{i,\lambda} \otimes_{\mathcal{O}(\widehat{e})} \Bbbk_e$ is $\nabla^{\nu}_e[\widetilde{\mathcal{E}}^S_{i,\lambda}]$. This calculation we repeat from part (c) of 5.4.2. We use an intermediate object $(\widetilde{\mathcal{E}}_{i,\lambda})^o \in \mathcal{E}coh(\widetilde{S_{\Bbbk,e}})$, by its definition $\widetilde{\mathcal{E}}_{i,\lambda}|_{S_{\Bbbk,e}} = (\widetilde{\mathcal{E}}_{i,\lambda})^o|_{\widehat{\mathcal{B}_{\Bbbk,e}}}$,

[21] If G coincides with its adjoint quotient $\overline{G}$ then $\boldsymbol{m}^2 = 1$, so since $\widetilde{\mathcal{E}}_{i,\lambda}$ is indecomposable the claim is true and $\varepsilon_{i,\lambda} = \pm 1$. By definitions in 5.3.1-5.3.3 if G is replaced by its adjoint quotient $\overline{G}$ the collections $\mathcal{E}_i,\ \widetilde{\mathcal{E}}_i,\ i \in \mathcal{I}_e$, do not change, and if $\overline{C}$ is the image of C in $\overline{G}$ then $B^{\overline{G}} \overset{\mathrm{def}}{=} \{\widetilde{\mathcal{E}}_{i,\lambda};\ i \in \mathcal{I}_e,\ \lambda \in X^*(\overline{C})\}$ and the corresponding object for G are related by $B^G \cong B^{\overline{G}} \times_{X^*(\overline{C})} X^*(C)$. This implies the general case. We also see that $\varepsilon_{i,\lambda} = \varepsilon_{i,0}\ \langle \lambda, \overline{m} \rangle$.

so one gets $\widetilde{\mathcal{E}}_{i,\lambda}|_e = (\widetilde{\mathcal{E}}_{i,\lambda})^o|_e$, hence $[\widetilde{\mathcal{E}}_{i,\lambda}|_e] = [(\widetilde{\mathcal{E}}_{i,\lambda})^o|_e] = \boxed{Z_{\mathfrak{g}}(f)^*}[(\widetilde{\mathcal{E}}_{i,\lambda})^o]$. Also, $\widetilde{\mathcal{E}}^S_{i,\lambda} = (\widetilde{\mathcal{E}}_{i,\lambda})^o \otimes_{\mathcal{O}(\mathfrak{h})} \mathbb{k}_0$ gives $[\widetilde{\mathcal{E}}^S_{i,\lambda}] = \boxed{\mathfrak{h}^*}[(\widetilde{\mathcal{E}}_{i,\lambda})^o]$, hence $[\widetilde{\mathcal{E}}_{i,\lambda}|_e] = \boxed{Z_{\mathfrak{g}}(f)^*}\boxed{\mathfrak{h}^*}^{-1}[\widetilde{\mathcal{E}}^S_{i,\lambda}] = \nabla^{\mathsf{v}}_e[\widetilde{\mathcal{E}}^S_{i,\lambda}]$.

(G) In order to avoid the dg-setting,[22] we will pass here from exotic sheaves to A-modules by means of the equivalence $D^b[\mathcal{C}oh^T(\widehat{\mathcal{B}_{\mathbb{k},e}})] \underset{\cong}{\overset{\boldsymbol{\mathcal{A}}}{\to}} D^b[mod^{T,fg}(A_{\mathbb{k},\widehat{e}})]$, $\boldsymbol{\mathcal{A}} \overset{\text{def}}{=} \mathrm{RHom}(\mathcal{E}|_{\widehat{\mathcal{B}_{\mathbb{k},e}}}, -)$, where T could be $\{1\}$, C or $\widetilde{C}$.

Let us start with the non-equivariant statement, i.e., Hom in $U^{\widehat{0}}_{\mathbb{k},\widehat{e}}$-modules. We are really interested in the composition of equivalences $mod^{fg}(U^{\widehat{0}}_{\mathbb{k},\widehat{e}}) \cong \mathcal{E}coh(\widehat{\mathcal{B}_{\mathbb{k},e}}) \underset{\cong}{\overset{\mathbf{F}}{\to}} mod^{fg}(A_{\mathbb{k},\widehat{e}})$, due to compatibility with the action of $\mathcal{O}(\mathfrak{g}\times_{\mathfrak{h}/W}\mathfrak{h})$ it restricts to an equivalence $mod^{fg}(U^{\widehat{0}}_{\mathbb{k},e}) \cong mod^{fg}(A_{\mathbb{k},e})$.

We will start as in **(C)**, so $\widetilde{\mathcal{E}}_{j,\mu} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e$ is an exotic sheaf and

$$\nabla_e(\widetilde{\mathcal{E}}^S_{i,\lambda} \| \widetilde{\mathcal{E}}^S_{j,\mu}) \;=\; [\mathrm{RHom}_{\widehat{\mathcal{B}_{\mathbb{k},e}}}(\widetilde{\mathcal{E}}_{i,\lambda}, \widetilde{\mathcal{E}}_{j,\mu} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e)] \;=\; [\mathrm{RHom}_{A_{\mathbb{k},\widehat{e}}}(\mathbf{F}\widetilde{\mathcal{E}}_{i,\lambda}, \mathbf{F}\widetilde{\mathcal{E}}_{j,\mu} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e)].$$

By adjunction in sheaves of A-modules,

$$\nabla_e(\widetilde{\mathcal{E}}^S_{i,\lambda} \| \widetilde{\mathcal{E}}^S_{j,\mu}) \;=\; \mathrm{RHom}_{A_{\mathbb{k},e}}(\mathbf{F}\widetilde{\mathcal{E}}_{i,\lambda} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e, \mathbf{F}\widetilde{\mathcal{E}}_{j,\mu} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e)].$$

So, the evaluation $\nabla_e(\widetilde{\mathcal{E}}^S_{i,\lambda} \| \widetilde{\mathcal{E}}^S_{j,\mu})(1_C, 1_{\mathfrak{G}_{\mathfrak{m}}})$ is the image of $\mathrm{Hom}_{A_{\mathbb{k},e}}(\mathbf{F}\widetilde{\mathcal{E}}_{i,\lambda} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e, \mathbf{F}\widetilde{\mathcal{E}}_{j,\mu} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e)$ in $K^0(mod^{fg}(\mathbb{k}))$, i.e., the dimension of this vector space.

It remains to notice that $\mathbf{F}\widetilde{\mathcal{E}}_{i,\lambda} \otimes_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e$ is a projective cover of $\mathbf{F}\widetilde{\mathcal{L}}_{i,\lambda}$ in $mod^{fg}(A_{\mathbb{k},e})$. Since $\mathbf{F}\widetilde{\mathcal{E}}_{i,\lambda}$ is a projective cover of $\mathbf{F}\widetilde{\mathcal{L}}_{i,\lambda}$ in $mod^{fg}A_{\mathbb{k},\widehat{e}}$, it is projective over $\mathbb{k}$, hence

$$\mathbf{F}(\widetilde{\mathcal{E}}_{i,\lambda} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e) \cong \mathbf{F}\widetilde{\mathcal{E}}_{i,\lambda} \overset{L}{\otimes}_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e \cong \mathbf{F}\widetilde{\mathcal{E}}_{i,\lambda} \otimes_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e.$$

Since $\mathcal{A}\widetilde{\mathcal{L}}_{i,\lambda}$ is irreducible in $mod(A|_{\widehat{e}})$, it is supported scheme theoretically on e, therefore we find by adjunction that $\mathbf{F}\widetilde{\mathcal{E}}_{i,\lambda} \otimes_{\mathcal{O}(\mathfrak{g})} \mathbb{k}_e$ is a projective cover of $\mathbf{F}\widetilde{\mathcal{L}}_{i,\lambda}$ in $mod^{fg}A_{\mathbb{k},e}$.

If one is interested in maps in $mod^{fg}(U^{\widehat{0}}_e, C)$ only, one uses equivariant equivalences $mod^{fg}(U^{\widehat{0}}_{\mathbb{k},\widehat{e}}, C) \cong \mathcal{E}coh^C(\widehat{\mathcal{B}_{\mathbb{k},e}}) \underset{\cong}{\overset{\mathbf{F}}{\to}} mod^{C,fg}(A_{\mathbb{k},\widehat{e}})$ and one also needs to take C-invariants in the above calculation. This has the effect of applying ∂ to $\nabla_e(\widetilde{\mathcal{E}}^S_{i,\lambda} \| \widetilde{\mathcal{E}}^S_{j,\mu})$. $\square$

[22] The above localization of $A^0_{\mathbb{k},\widehat{e}}$-modules to coherent sheaves on $\widehat{\mathcal{B}_{\mathbb{k},e}}'$ specializes to a localization of the category of $A^0_{\mathbb{k},e}$-modules on the Springer fiber, however one is forced to use the dg-version of the Springer fiber $\mathcal{B}'_{\mathbb{k},e}$ [Ri2].

5.5. **Koszul property.** This subsection is not used in the rest of the text. Set $\mathcal{A}_e = \mathrm{End}(\oplus \mathcal{E}_i^S)$ where the vector bundles $\mathcal{E}_i^S$ on $\widetilde{S_{\Bbbk,e}}'$ are as above. The $\mathfrak{G}_{\mathtt{m}}$-equivariant structure $\widetilde{\mathcal{E}}_i^S$ on $\mathcal{E}_i^S$ introduced in 5.3.3 equips $\mathcal{A}_e$ with a grading.

Proposition. Properties $\mathrm{i}_\star$, $\mathrm{ii}_\star$ of 5.3.2 imply that the graded algebra $\mathcal{A}_e$ is a Koszul quadratic algebra.

Proof. [(23)] For two graded modules M, N over $\mathcal{A}_e$ let $Ext_j^i(M,N)$ denote the component of inner degree j in $Ext^i_{\mathcal{A}_e}(M,N)$. Then $\mathrm{i}_\star$, $\mathrm{ii}_\star$ imply that $Ext_j^i(\widetilde{\mathcal{L}}_1, \widetilde{\mathcal{L}}_2) = 0$ for $j < i$ where $\widetilde{\mathcal{L}}_1$, $\widetilde{\mathcal{L}}_2$ are irreducible graded $\mathcal{A}_e$-modules concentrated in graded degree zero.

The canonical line bundle of $\widetilde{S}_e'$ admits a trivialization which transforms under the action of $\mathfrak{G}_{\mathtt{m}}$ by the $2d_e$-th power of the tautological character. So, Serre duality shows that for finite dimensional graded $\mathcal{A}_e$ modules we have

$$Ext_j^i(M,N) = Ext^{2d_e-i}_{2d_e-j}(N,M)^*.$$

Thus we see that $Ext_j^i(\widetilde{\mathcal{L}}_1, \widetilde{\mathcal{L}}_2) = 0$ for $j \neq i$, which is one of characterizations of Koszul algebras. □

5.5.1. *Remark.* For $e = 0$ the work of S. Riche [Ri2] provides a representation theoretic interpretation of the algebra $\kappa(\mathcal{A}_e)$ which is Koszul dual to $\mathcal{A}_e$. It would be interesting to generalize this to nonzero nilpotents.

When e is of principal Levi type, the relation between the parabolic semi-infinite module over the affine Hecke algebra and $K(Coh^{\widetilde{C}}(\widetilde{S}_e'))$ (see [Lu], sections 9, 10) suggests that the category of $\kappa(\mathcal{A}_e)$-modules can be identified with the category of perverse sheaves on the parabolic semi-infinite flag variety of the Langlands dual group. For $e = 0$ this follows from the result of [ABBGM] compared with [Ri2].

6. Grading that satisfies property (⋆)

In subsection 6.1 we reduce verification of property **(⋆)** (see 5.3.2), to the case of a characteristic zero base field. From then on until the end of the section we work over the field $\Bbbk = \bar{\mathbb{Q}}_l$ of characteristic zero.

[23] The proof is due to Dmitry Kaledin.

Our goal is to construct a $\mathfrak{G}_{\mathfrak{m}}$-equivariant structure on projective exotic sheaves that satisfies property $(\star)$. For this we use a derived equivalence between the category of G-equivariant coherent sheaves on $\widetilde{\mathcal{N}}$ and of certain perverse constructible sheaves on the affine flag variety $\mathcal{F}l$. In the new setting the $\mathfrak{G}_{\mathfrak{m}}$-structure is related to Frobenius (Weil) structure on l-adic sheaves, which we choose to be pure of weight zero. In 6.2 we compare the exotic t-structure on coherent sheaves to the standard t-structure on perverse constructible sheaves on $\mathcal{F}l$, this involves a new notion of *perversely exotic* G-equivariant coherent sheaves. In 6.3 we reduce $(\star)$ to a property $(\star\star\star)$ which is stated in terms of G-equivariant sheaves. Finally, in 6.4 we verify $(\star\star\star)$.

6.1. **Lusztig's conjectures for $p \gg 0$.**

6.1.1. *Proposition.* If $(\star)$ holds in characteristic zero, it holds for almost all positive characteristics.

Proof. By Proposition 5.2.3(c) the choice of a graded lift of indecomposable projectives and irreducibles in characteristic zero defines such a choice in almost all prime characteristics. We claim that the required properties are inherited from characteristic zero to almost all prime characteristics. Indeed, the fact that the given choice of graded lifts satisfies the positivity requirement amounts to vanishing of the components of negative degree in the Hom space between indecomposable projective modules. Since the sum of these components is easily seen to be a finite R' module (here we use the fact that this Hom space is a finite module over the center $\mathcal{O}(\widetilde{S}'_{eR'})$), it vanishes after a finite localization provided that its base change to a characteristic zero field vanishes. Invariance of the graded lifts under the action of the centralizer clearly holds in large positive characteristic if it holds in characteristic zero (notice that the centralizer acts on the set of isomorphism classes of (graded) modules through its group of components, which is the same in almost all characteristics).

Uniqueness of the graded lifts with required properties amounts to non-vanishing of components of degree minus one in Ext^1 between certain pairs of irreducibles (see the proof of 6.2.1 below). After possibly replacing R' with its localization we can assume that Ext^1 between the "extended irreducible" modules over R' are flat over R', thus dimensions of each graded component in Ext^1 between the corresponding irreducibles over every geometric point of R' is the same. $\square$

6.1.2. *The final form of the results.* Since **(⋆)** for $\Bbbk = \mathbb{C}$ will be established in the remainder of this section, the proposition implies that there exists a quasifinite R-domain R' such that for all geometric points $\Bbbk$ of R' property **(⋆)** holds and therefore so do all claims **(A)**-**(G)** from Proposition 5.3.6. In particular this establishes the following version of Lusztig conjectures.

Theorem. (1) Conjectures 5.12, 5.16 of [Lu] (existence of certain signed bases of K-groups of complex schemes $\mathcal{B}_{\mathbb{C},e}$ and $\widetilde{S_{\mathbb{C},e}}'$) hold.

(2) The part of Conjecture 17.2 of *loc. cit.* concerning modular representations holds for all finite characteristic geometric points $\Bbbk$ of R'.

6.2. **Perverse t-structures on A^0-modules.** Recall that the triangulated category $D^b[Coh^G(\mathcal{N})]$ carries a certain t-structure called *perverse coherent t-structure of middle perversity* [B6] (see also [ArinB] for the general theory of such t-structures). As above let A^0 be $\mathrm{End}(\mathcal{E}|_{\widetilde{\mathcal{N}}})$ for the vector bundle $\mathcal{E}$ from Theorem 1.5.1. This is a $\mathcal{O}(\mathcal{N})$-algebra equipped with a $G \times \mathfrak{G}_{\mathfrak{m}}$-action. This allows us to define a perverse coherent t-structure $\mathcal{T}^{\mathbf{G}}_{pc}(A^0)$ of middle perversity on $D^b[mod^{\mathbf{G},fg}(A^0)]$ where $\mathbf{G}$ is one of the groups $G, G\times\mathbb{G}_m$ or $\mathfrak{G}$. These are characterized by the requirement that the forgetful functor to $D^b[Coh^G(\mathcal{N})]$ is t-exact when the target category is equipped with the perverse coherent t-structure of middle perversity.

Recall the equivalence of derived categories of coherent and constructible sheaves

$$\mathbf{\Phi}: \ D^b(Coh^G(\widetilde{\mathcal{N}})) \underset{\cong}{\to} D^b(Perv_{\mathcal{F}l}), \tag{17}$$

constructed in [ArkB], where $D^b(Perv_{\mathcal{F}l})$ is the derived category of *anti-spherical* perverse sheaves on the affine flag variety of the dual group.

6.2.1. *Theorem.* The composed equivalence $\mathbf{\Phi}_{A^0}$

$$\mathbf{\Phi}_{A^0} \ \stackrel{\mathrm{def}}{=} \ [\ D^b(mod^{G,fg}(A^0)) \underset{\cong}{\to} D^b(Coh^G(\widetilde{\mathcal{N}})) \ \underset{\cong}{\stackrel{\mathbf{\Phi}}{\to}} \ D^b(Perv_{\mathcal{F}l})\ ].$$

sends the perverse coherent t-structure of middle perversity $\mathcal{T}^G_{pc}(A^0)$ to the tautological t-structure on $D^b(Perv_{\mathcal{F}l})$.

Proof. In Lemma 6.2.4 below, we show that the t-structure on $D^b(Coh^G(\widetilde{\mathcal{N}}))$ that comes from $D^b(Perv_{\mathcal{F}l})$ satisfies a certain property and in Lemma 6.2.3 we show that the only

t-structure on $D^b(Coh^G(\widetilde{\mathcal{N}}))$ that could satisfy this property is the one coming from the t-structure $\mathcal{T}^G_{pc}(A^0)$ on $D^b[mod^{G,\mathfrak{g}}(A^0)]$.

6.2.2. *Perversely exotic t-structures.* We will say that a t-structure on a triangulated category $\mathcal{C}$ is *compatible with a thick triangulated subcategory* $\mathcal{C}'$ if there exist t-structures on $\mathcal{C}'$, $\mathcal{C}/\mathcal{C}'$ such that the embedding and projection functors are t-exact (cf. [BBD]). Inductively one extends this definition to the definition of a t-structure compatible with a filtration by thick triangulated subcategories.

By a *support filtration* on $D^b(Coh^G(\widetilde{\mathcal{N}}))$ we will mean the filtration by full subcategories of complexes supported (set theoretically) on the preimage of the closure of a given G orbit in $\mathcal{N}$ (we fix a complete order on the set of orbits compatible with the adjunction partial order).

Finally, we say that a t-structure on $D^b(Coh^G(\widetilde{\mathcal{N}}))$, is *perversely exotic* if it is

(1) compatible with the support filtration;
(2) braid positive (see 1.4.1);
(3) such that the functor π_* is t-exact when the target category $D^b(Coh^G(\mathcal{N}))$ is equipped with perverse coherent t-structure of middle perversity.

Uniqueness of such t-structure follows from:

6.2.3. *Lemma.* A perversely exotic t-structure $\mathcal{T}$ on $D^b(Coh^G(\widetilde{\mathcal{N}}))$ corresponds under the equivalence $D^b(Coh^G(\widetilde{\mathcal{N}})) \cong D^b[mod^{G,fg}(A^0)]$ to $\mathcal{T}^G_{pc}(A^0)$, the perverse coherent t-structure of middle perversity.

Proof. It is a standard fact that for a triangulated category $\mathcal{C}$, a thick subcategory $\mathcal{C}'$ and t-structures $\mathcal{T}'$ on $\mathcal{C}'$, $\mathcal{T}''$ on $\mathcal{C}/\mathcal{C}'$ a t-structure $\mathcal{T}$ on $\mathcal{C}$ compatible with $\mathcal{T}'$, $\mathcal{T}''$ is unique if it exists. Thus uniqueness of an exotic t-structure implies uniqueness of perversely exotic t-structure.

On the other hand, the t-structure corresponding to $\mathcal{T}^G_{pc}(A^0)$ is perversely exotic as is clear from the fact that the t-structure corresponding to the tautological one on $D^b(mod^{G,fg}(A))$ is exotic (the last fact is the definition of A as endomorphism of the exotic tilting generator $\mathcal{E}$). $\square$

6.2.4. *Lemma.* The t-structure on $D^b(Coh^G(\widetilde{\mathcal{N}}))$ which under the equivalence [ArkB] corresponds to the perverse t-structure on $D^b(Perv_{\mathcal{F}l})$, is perversely exotic.

Proof. Properties (1) and (3) are satisfied by [ArkB], Theorem 4(a) and Theorem 2 respectively. We now deduce property (2) from the results of [B4]. In *loc. cit.* it is shown that the t-structure corresponding to the one of $Perv_{\mathcal{F}l}$ can be characterized as follows:

$$D^{\geq 0} = \langle \Delta^\lambda[-d] \rangle_{d\geq 0,\ \lambda\in\Lambda} \quad \text{and} \quad D^{\leq 0} = \ \langle \nabla_\lambda[d] \rangle_{d\geq 0,\ \lambda\in\Lambda},$$

where $\langle\ ,\ \rangle$ denotes the full subcategory generated under extensions and Δ^λ, ∇_λ, $\lambda \in \Lambda$, are certain explicitly defined objects in $D^b(Coh^G(\widetilde{\mathcal{N}}))$.

Moreover, one can deduce from [B4, Proposition 7] that $\nabla_\lambda = \widetilde{w}_\lambda(\mathcal{O})$, $\Delta_\lambda = \widetilde{w}_\lambda^{-1}\mathcal{O}$ where w_λ is any representative of the coset $\lambda W \subset W_{aff}$ and $\widetilde{w}$ denotes the canonical representative in $\mathbb{B}_{\text{aff}}$ of $w \in W_{aff}$.

Now Proposition 2.1.2(a) yields exact triangles available for any $\mathcal{F} \in D^b(Coh^G(\widetilde{\mathcal{N}}))$:

$$\widetilde{s_\alpha}^{-1}\mathcal{F} \to \widetilde{s_\alpha}\mathcal{F} \to \mathcal{F} \oplus \mathcal{F}[1].$$

Thus if $\ell(s_\alpha w_\lambda) < \ell(w_\lambda)$ (where ℓ is the length function on W_{aff}), then $\widetilde{s}_\alpha{}^{-1}\nabla_\lambda = \nabla_{s_\alpha\lambda}$. so we have an exact triangle:

$$\nabla_{s_\alpha\lambda} \to \widetilde{s_\alpha}(\nabla_\lambda) \to \nabla_\lambda \oplus \nabla_\lambda[1],$$

which shows that $\widetilde{s_\alpha}\nabla_\lambda \in D^{\leq 0}$. Also, if $\ell(s_\alpha w_\lambda) > \ell(w_\lambda)$, then $\widetilde{s_\alpha}\nabla_\lambda \cong \nabla_{s_\alpha(\lambda)}$. Thus $\widetilde{s}_\alpha : D^{\leq 0} \to D^{\leq 0}$ which implies braid positivity property (2). □

6.2.5. *Remark.* A more conceptual proof of braid positivity property (2) in the last lemma follows from the paper [B5] (in preparation; see announcement in [B3]). It permits to relate the $\mathbb{B}_{\text{aff}}$ action described above to a standard action on the category of constructible sheaves on the affine flag space $\mathcal{F}l$. In the latter case the generator $\widetilde{s}_\alpha$ acts by convolution with a constructible sheaf $j_{s_\alpha *}$ (in the notations of, say, [ArkB]), i.e. the $*$ extension of the constant sheaf shifted by 1 on the Iwahori orbit corresponding to s_α. It is well known that convolution with such a sheaf is right exact with respect to the perverse t-structure (see e.g. [BeBe2]).

In fact, these considerations have led us to the notion of a braid positive t-structure, which was introduced as a way to relate modular representations to perverse sheaves on

the affine flag space. We have chosen to present the above ad hoc argument to minimize references to yet unpublished results.

6.3. **Reduction to a G_e-equivariant setting.** We consider the algebra $A^0_e \stackrel{\text{def}}{=} A^0 \otimes_{\mathcal{O}_\mathfrak{g}} \Bbbk_e$. It is graded by means of the action of $\mathfrak{G}_\mathfrak{m} \subseteq \widetilde{C}$ as above.

6.3.1. *Reduction to a property of A^0_e-modules.*

Lemma. Property $(\star)$ follows from:

$(\star\star)$ there exists a G_e-invariant choice of a graded lifting $\widetilde{L}$ for every irreducible representation L of A^0_e, such that:

(1) Components of nonnegative weight in $\mathrm{Ext}^1_{mod(A^0_e)}(\widetilde{L}_1, \widetilde{L}_2)$ vanish for $L_i \in Irr(A^0_e)$.
(2) Consider the preorder on the quotient of the set of irreducible representations of A^0_e by the action of G_e, generated by: $\alpha_1 \leq \alpha_2$ if for some representatives L_i of α_i the component of degree –1 in $\mathrm{Ext}^1_{mod(A^0_e)}(\widetilde{L}_1, \widetilde{L}_2)$ does not vanish. This preorder is actually a transitive equivalence relation, i.e. $\alpha_1 \leq \alpha_2$ for all $\alpha_i \in Irr(A^0_e)/G_e$.

Proof. **(i) Existence.** Set $A^0_S = A^0 \otimes_{\mathcal{O}_\mathfrak{g}} \mathcal{O}(S_{\Bbbk,e})$. Then we have $D^b[mod^{fg}(A^0_S)] \cong D^b(Coh(\widetilde{S_{\Bbbk,e}}))$ and the same holds with equivariance under $\mathfrak{G}_\mathfrak{m}$ or $\widetilde{C}$. Recall that $(\star)$ (see 5.3.2, involves a $G_{e,h}$-invariant choice of graded liftings $\widetilde{\mathcal{E}}^S_i$ of exotic sheaves $\mathcal{E}^S_i$, i.e., a $G_{e,h}$-invariant choice of graded liftings of indecomposable projective A^0_S-modules. This is equivalent to a $G_{e,h}$-invariant choice of graded liftings $\widetilde{L}_i$ of irreducible modules L_i supported at e.

Since $(\star\star)$ provides a choice with stronger equivariance, it remains to check that the choice of $\widetilde{L}_i$ satisfying the vanishing property (1) of $(\star\star)$ yields a choice of $\widetilde{\mathcal{E}}^S_i$ satisfying the vanishing requirements of $(\star)$, i.e., $\oplus_{i,j} \mathrm{Hom}_{\mathcal{C}oh(\widetilde{S_{\Bbbk,e}}')}(\widetilde{\mathcal{E}}^S_i, \widetilde{\mathcal{E}}^S_j)$ has no negative $\mathfrak{G}_\mathfrak{m}$ weights and zero weights are spanned by identity maps. By a standard argument this property from $(\star)$ is equivalent to saying that

$$\mathrm{Ext}^1_{A^0_{S_e}}(\widetilde{L}_i, \widetilde{L}_j(d)) = 0 \quad \text{for} \quad d \geq 0.$$

If $L_i \ncong L_j$ then any A^0_S-module which is an extension of L_i by L_j is actually an A^0_e module, because the action of a regular function on $S_{\Bbbk,e}$ vanishing at e on such an extension factors through a map $L_i \to L_j$, such map is necessarily zero. On the other hand, if $L_i \cong L_j$ and

an extension $0 \to L_j \to M \to L_i \to 0$ is such that M does not factor through A^0_e, then some function as above induces a nonzero map $L_i \to L_j$. Since $\mathfrak{G}_{\mathfrak{m}}$ acts on the ideal in $\mathcal{O}(S_{\Bbbk,e})$ by positive weights, we see that the class of the extension has negative weight.

(ii) Uniqueness. Finally, we will see that the uniqueness statement in **(⋆)** follows from property (2). Any graded lifting of L_i's is of the form $\widetilde{L}_i(d_i)$ for some integers d_i. If it satisfies the requirements, then d_i is clearly monotone with respect to our preorder. Thus property (2) implies that $d_i = d_j$ for all i, j. □

6.3.2. *Category $mod^{G_e,fg}(A^0_e)$ of G_e-equivariant A^0_e-modules.* Notice that $mod^{G_e,fg}(A^0_e) \cong mod^{G,fg}(A^0|_{O_e})$, where O_e is the G-orbit of e and the category in the right hand side is the category of equivariant quasicoherent sheaves of modules over the sheaf of algebras $A^0|_{O_e}$. This category has a graded version $mod^{\mathfrak{G},fg}(A^0|_{O_e})$, compatible with the graded version $mod^{\mathfrak{G},fg}(A^0) \cong D^b[\mathcal{C}oh^{\mathfrak{G}}(\widetilde{\mathcal{N}})]$ considered above. In terms of the stabilizer $\mathfrak{G}_e$ of e in $\mathfrak{G}$ this is $mod^{\mathfrak{G}_e,fg}(A^0_e)$.

Tensor category $Rep(G_e)$ clearly acts on the category $mod^{G_e,fg}(A^0_e)$ where for $V \in Rep(G_e)$ and $M \in mod^{G_e,fg}(A^0_e)$ one equips the tensor product $V \otimes M$ with the diagonal action of G_e. We will now see that a tensor subcategory $Rep^{ss}(G_e)$ of semisimple representations of G_e acts on $mod^{\mathfrak{G}_e,fg}(A^0_e)$.

We use morphisms $SL_2 \xrightarrow{\varphi} G$ and $\mathfrak{G}_{\mathfrak{m}} \xrightarrow{\mathbf{i}} \mathfrak{G}$, $\mathbf{i}(t) = (\phi(t), t^{-1})$, chosen in 5.2.2. Notice that $(g,t) \in \mathfrak{G} = G \times \mathbb{G}_m$ lies in $\mathfrak{G}_e$ iff $e = t^{2\cdot} g e = {}^{g\phi(t)}e$, i.e., $g\phi(t) \in G_e$. So, $\mathfrak{G}_e$ contains $G_e \cdot \mathbf{i}(\mathfrak{G}_m)$ and this is equality since $\widetilde{g} = g\phi(t) \in G_e$ implies that $(g,t) = (\widetilde{g}, 1) \cdot \mathbf{i}(t^{-1})$. We have exact sequence $0 \to \mathfrak{G}_m \subseteq \mathfrak{G}_e \xrightarrow{p} G_e \to 0$ for $p(g,t) = g\phi(t)$ and maximal reductive subgroups of G_e and $\mathfrak{G}_e$ can be chosen as the stabilizer $G_\varphi = Z_G(Im(\varphi))$ of φ in G and $G_\varphi \cdot \mathbf{i}(\mathfrak{G}_{\mathfrak{m}})$. Now $G_\varphi \cdot \mathbf{i}(\mathfrak{G}_{\mathfrak{m}}) \xrightarrow{p} G_\varphi$ gives a tensor functor

$$Rep^{ss}(G_e) \cong Rep(G_\varphi) \xrightarrow{p^*} Rep[G_\varphi \cdot \mathbf{i}(\mathfrak{G}_{\mathfrak{m}})] \xleftarrow{\cong} Rep^{ss}(\mathfrak{G}_e).$$

Lemma. (a) For any $M \in Irr^{G_e}(A^0_e)$, restriction to A^0_e is a multiple of a sum over some G_e-orbit $\mathcal{O}_M$ in $Irr(A^0_e)$.

(b) For $M_1, M_2 \in Irr^{G_e}(A^0_e)$, the space $\mathrm{Hom}_{A^0_e}(M_1, M_2)$ is a semi-simple G_e-module.

Proof. (a) For an irreducible A^0_e-module L denote by $G_{e,L} \subset G_e$ the stabilizer of the isomorphism class of L. Then $G_{e,L}$ is a finite index subgroup in G_e, and L can be

equipped with a compatible projective action of $G_{e,L}$, i.e., an action of a central extension $0 \to \mathcal{A}_L \to \mathcal{G}_{e,L} \to G_{e,L} \to 0$ by a finite abelian group $\mathcal{A}_L$.[24]

The subgroup $\mathcal{A}_L$ acts on L by a character χ_L. For any irreducible representation ρ of $\mathcal{G}_{e,L}$, with $\mathcal{A}_L$ acting by χ_L^{-1}, we get an irreducible object $\rho \otimes L$ in $mod^{G_{e,L}}(A_e^0)$.

Then $Ind_{G_{e,L}}^{G_e}(\rho \otimes L)$ is an irreducible object of $mod^{G_e}(A_e^0)$, and all irreducible objects M arise this way.

Now for $M = Ind_{G_{e,L}}^{G_e}(\rho \otimes L)$, we have

$$M|_{A_e^0} = \bigoplus_{g \in G_e/G_{e,L}} {}^g\rho \otimes {}^gL|_{A_e^0} \cong \Bbbk^{\oplus \dim(\rho)} \otimes \bigoplus_{K \in G_e \cdot L} K$$

To check (b) observe that $\mathrm{Hom}_{A_e^0}(M_1, M_2) = 0$ if $\mathcal{O}_{M_1} \neq \mathcal{O}_{M_2}$. Thus assume that $\mathcal{O}_{M_1} = \mathcal{O}_{M_2}$ is the orbit of $L \in Irr(A_e^0))$, then we get:

$$\bigoplus_{g,h \in G_e/G_{e,L}} \mathrm{Hom}_\Bbbk[{}^g\rho, {}^h\rho] \otimes \mathrm{Hom}_{A_e^0}({}^gL, {}^hL) = \bigoplus_{g \in G_e/G_{e,L}} \mathrm{Hom}_\Bbbk[\rho, \rho] = Ind_{G_{e,L}}^{G_e} \mathrm{End}_\Bbbk[\rho].$$

Since ρ is a semisimple $\mathcal{G}_{e,L}$-module, $\mathrm{End}_\Bbbk(\rho)$ is a semisimple $G_{e,L}$ module. As $G_{e,L}$ has finite index in G_e, it follows that $Ind_{G_{e,L}}^{G_e}\ \mathrm{End}_\Bbbk[\rho]$ is also semisimple.

6.3.3. *Reduction to a property of G_e equivariant A_e^0-modules.*

Lemma. **($\star\star$)** follows from the following.

($\star\star\star$) there exists a choice of a graded lifting $\widetilde{L} \in mod^{\mathfrak{G}_e}(A_e^0)$ for every irreducible object L of $mod^{G_e}(A_e^0)$, such that:

- ($0_{\star\star\star}$) For $L \in Irr^{G_e}(A_e^0)$ and any irreducible representation V of G_e we have $V \otimes \widetilde{L} \cong \sum \widetilde{L}_i$ for some $L_i \in Irr^{G_e}(A_e^0)$.
- ($1_{\star\star\star}$) components of nonnegative weight in $\mathrm{Ext}^1_{A_e^0}(\widetilde{L}_1, \widetilde{L}_2)$ vanish for $L_1, L_2 \in Irr^{G_e}(A_e^0)$.
- ($2_{\star\star\star}$) Consider the preorder $\leq$ on the set of irreducible objects in $mod^{G_e}(A_e^0)$ generated by: $L_1 \leq L_2$ if component of degree (-1) in $\mathrm{Ext}^1_{A_e^0}(\widetilde{L}_1, \widetilde{L}_2)$ does not vanish. This partial preorder is actually a transitive equivalence relation, i.e. $L_1 \leq L_2$ for all (L_1, L_2).

[24] V. Ostrik has informed us that he can prove that in fact this extension can be assumed to be trivial provided G is simply connected. (An equivalent statement is that the set of "centrally extended points" appearing in [BO] is actually a plain finite set with a G_e action.) We neither prove nor use this fact here.

Proof. Property $(0_{\star\star\star})$ is equivalent to saying that for $L_i \in Irr^{G_e}(A_e^0)$, the multiplicative group $\mathfrak{G}_{\mathfrak{m}}$ acts trivially on $\mathrm{Hom}_{A_e^0}(\widetilde{L}_1, \widetilde{L}_2)$. Indeed, if $(0_{\star\star\star})$ holds then $\mathfrak{G}_{\mathfrak{m}}$ acts trivially on $\mathrm{Hom}_{A_e^0}(\widetilde{L}_1, \widetilde{L}_2)$, because for any $\sigma \in Irr(G_e)$, $\mathrm{Hom}_{G_e}[\sigma, \mathrm{Hom}_{A_e^0}(L_1, L_2)] = \mathrm{Hom}_{mod^{G_e}(A_e^0)}(\sigma \otimes L_1, L_2)$ and $\sigma \otimes \widetilde{L}_1$ is a sum of $\widetilde{L}$'s.

Consequently, a choice of graded lifts of G_e equivariant irreducibles defines uniquely a choice of graded lifts of irreducible A_e^0 modules, such that the forgetful functor sends the graded lift of an equivariant irreducible $\widetilde{M}$ to a sum of graded lifts of non-equivariant irreducibles $\widetilde{L}_i$.

It is clear that G_e permutes the isomorphism classes of those $\widetilde{L}_i$.

Suppose that $\mathrm{Ext}^1(\widetilde{L}_1, \widetilde{L}_2(d)) \neq 0$, for some L_1, L_2, $d > 0$. Fix M_1, M_2 such that L_1, L_2 are direct summands in M_1, M_2 considered as A_e^0 modules. Then $\mathrm{Ext}^1_{A_e^0}(M_1, M_2(d)) \neq 0$. The space $\mathrm{Ext}^1_{A_e^0}(M_1, M_2)$ carries a (not necessarily semi-simple) G_e action, and for an irreducible representation ρ of G_e we have an embedding of graded spaces $\mathrm{Hom}(\rho, \mathrm{Ext}^1_{A_e^0}(M_1, M_2)) \to \mathrm{Ext}^1_{mod^{G_e}(A_e^0)}(\rho \otimes M_1, M_2)$. The latter embedding can be obtained as follows: given an element in the source space we get an extension of A_e^0 modules $0 \to M_2 \to M \to \rho \otimes M_1 \to 0$. Twisting this module by $z \in G_e$ we obtain an isomorphic extension; since $\mathrm{Hom}_{A_e^0}(M_1, M_2) = 0$ we actually get a unique isomorphism $M^z \cong M$ compatible with the given equivariant structures on $\rho \otimes M_1$, M_2. Thus we get a G_e equivariant structure on M.

Since $\mathfrak{G}_{\mathfrak{m}}$ acts on the Lie algebra of the unipotent radical of G_e by positive weights, the G_e submodules generated by the degree d components in $\mathrm{Ext}^1_{A_e^0}(M_1, M_2)$ is concentrated in positive degrees. This subspace has an irreducible subrepresentation ρ, which produces a nonzero $\mathrm{Ext}^1(\rho \otimes M_1, M_2)$ of positive degree contradicting properties $(0_{\star\star\star})$,$(1_{\star\star\star})$. To prove property $(2_{\star\star\star})$ it is enough to show that if M_1, M_2 are irreducible objects in $mod^{G_e}(A_e^0)$ such that $\mathrm{Hom}_{A_e^0}(M_1, M_2) = 0$, and $\mathrm{Ext}^1_{mod^{\mathfrak{G}}(A_e^0)}(\widetilde{M}_1, \widetilde{M}_2(1)) \neq 0$, then $\mathrm{Ext}^1_{mod^{\mathfrak{G}_{\mathfrak{m}}}(A_e^0)}(\widetilde{M}_1, \widetilde{M}_2(1)) \neq 0$. It suffices to check that applying the forgetful functor $mod^{G_e}(A_e^0) \to mod(A_e^0)$ to a nontrivial extension $0 \to M_2 \to M \to M_1 \to 0$ we get a nontrivial extension. However, if there exists an A_e^0 invariant splitting $M_1 \to M$, then its image has to be invariant under G_e, since $\mathrm{Hom}_{A_e^0}(M_1, M_2) = 0$ and the isomorphism class of the A_e^0-module M_1 is G_e invariant. Thus existence of a non-equivariant splitting implies the existence of an equivariant splitting. $\square$

6.4. **End of the proof.** Here we prove **(⋆⋆⋆)**, the proof is based on the equivalence $\mathbf{\Phi} : D^b[\mathcal{C}oh^G(\widetilde{\mathcal{N}})] \underset{\cong}{\to} D^b(Perv_{\mathcal{F}l}))$ from [ArkB], which we use in the form $\mathbf{\Phi}_{A^0} : D^b[mod^{G,fg}(A^0)] \underset{\cong}{\to} D^b(Perv_{\mathcal{F}l})$ (see Theorem 6.2.1).

6.4.1. *The choice of grading.* Equivalence $\mathbf{\Phi}_{A^0}$ makes category $mod^{G_e,fg}(A^0_e) = Coh^G(A^0|_{O_e})$ a full subcategory in a Serre quotient category of $Perv_{\mathcal{F}l}$. We will now show that property **(⋆⋆⋆)** holds when the graded lifting $\widetilde{L}$ of irreducibles L in $mod^{G_e}(A^0_e)$ is chosen so that it corresponds to pure Weil structure of weight zero. What is meant by this is the following.

First, it is shown in [ArkB] that the Frobenius functor corresponding to a finite field $\mathcal{F}_q$ on the perverse sheaves category, corresponds to the functor $\mathcal{G} \mapsto \mathbf{q}^*(\mathcal{G})$ on coherent sheaves, where $\mathbf{q} : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$ by $\mathbf{q}(\mathfrak{b}, x) = (\mathfrak{b}, qx)$. The same then applies to $\mathcal{F} \in D^b[mod^{G,fg}(A^0)]$ with $\mathbf{q} : \mathcal{N} \to \mathcal{N}$ by $\mathbf{q}(x) = q{\cdot}x$.

Thus, for a perverse coherent sheaf $\mathcal{F}$ of A^0-modules, a Weil structure on the perverse sheaf $\mathbf{\Phi}_{A^0}\mathcal{F}$ is the same as an isomorphism $\mathcal{F} \underset{\cong}{\to} \mathbf{q}^*(\mathcal{F})$. In particular this shows that any $\mathfrak{G}_{\mathfrak{m}}$-equivariant structure on $\mathcal{F}$ defines a Weil structure on $\mathbf{\Phi}_{A^0}\mathcal{F}$. Notice that the resulting functor from $D^b(Coh^{G\times\mathbb{G}_m}(\widetilde{\mathcal{N}}))$ to Weil complexes on $\mathcal{F}l$ sends the twist by the tautological $\mathbb{G}_m$ character $M \mapsto M(1)$ to the square root of Weil twist $\mathcal{F} \mapsto F(\frac{1}{2})$ acting on Weil sheaves.

$M(1)$ here stands for the graded module $M(1)^i = M^{i+1}$. This functor is compatible with the functor $\mathcal{F} \mapsto \mathcal{F}(\frac{1}{2})$ on Weil perverse sheaves under the equivalence (17).

It is shown in [B4] that when $\mathbf{\Phi}_{A^0}\mathcal{F}$ is an irreducible perverse sheaf, any $\mathfrak{G}_{\mathfrak{m}}$-equivariant structure on $\mathcal{F}$ induces a pure Weil structure on $\mathbf{\Phi}_{A^0}\mathcal{F}$ and there is a unique $\mathfrak{G}_{\mathfrak{m}}$-equivariant structure on $\mathcal{F}$ such that the corresponding Weil structure on $\mathbf{\Phi}_{A^0}\mathcal{F}$ is pure of weight zero.[25] It is also proven in *loc. cit.* that for $\mathcal{F}, \mathcal{G} \in D^b(Coh^{G\times\mathfrak{G}_{\mathfrak{m}}}(\widetilde{\mathcal{N}}))$ the isomorphism

$$\mathrm{Hom}_{D^b(Coh^G(\widetilde{\mathcal{N}}))}(\mathcal{F}, \mathcal{G}) \;\cong\; \mathrm{Hom}_{D^b(Perv_{\mathcal{F}l})}(\mathbf{\Phi}(\mathcal{F}), \mathbf{\Phi}(\mathcal{G}))$$

takes the grading induced by the $\mathfrak{G}_{\mathfrak{m}}$-equivariant structure into the grading by Frobenius weights.

[25] [B4] provides also a more direct way to describe the resulting $G \times \mathfrak{G}_{\mathfrak{m}}$ equivariant sheaves.

6.4.2. *Property* $(1_{\star\star\star})$ *and Purity Theorem.* Purity Theorem of [BBD] implies that Ext^1 between two pure weight zero Weil sheaves in $Perv_{\mathcal{F}l}$ has weights < 0. Thus for the above graded lifts $\widetilde{L}$ of irreducible equivariant perverse coherent sheaves of A^0-modules, Ext^1 between two such objects has weights < 0. It is not hard to check that this property is inherited by a quotient category, thus property $(1_{\star\star\star})$ follows.

6.4.3. *Property* $(2_{\star\star\star})$ *and definition of cells.* Property $(2_{\star\star\star})$ says that for any L, L' in $Irr^{G_e}(A_e^0)$, there exists a sequence of irreducible objects $L_0 = L, L_1, \ldots, L_n = L'$ such that the component of degree -1 in $\mathrm{Ext}^1_{mod^{G_e}(A_e^0)}(L_{i-1}, L_i)$ is nontrivial.

Recall that by [ArkB, Theorem 4(a)] the support filtration on $D^b(Coh^G(\widetilde{\mathcal{N}}))$ is identified with the (left) cell filtration on $D^b(Perv_{\mathcal{F}l})$. In particular the irreducible objects in the subquotient piece of the filtration corresponding to a given nilpotent orbit O_e are in bijection with elements in a *canonical left cell* in the two-sided cell in W_{aff} attached to O_e. Furthermore, the definition of a left cell implies the following. For any two irreducible objects L, L' in the same left cell, there exists a sequence of irreducible objects $L_0 = L, L_1, \ldots, L_n = L'$ such that for any step $M = L_{i-1}$ and $N = L_i$ in the chain, there is a simple affine root α such that N is a direct summand in the perverse sheaf $\pi^*\pi_* M[1]$, where π stands for the projection $\mathcal{F}l \to \mathcal{F}l_\alpha$ to the partial flag variety of the corresponding type.

This implies that $\pi_* M$ is a semisimple perverse sheaf on $\mathcal{F}l_\alpha$ and that the relation of α to M is such that we have a canonical extension of Weil perverse sheaves

$$0 \to \pi^*\pi_* M[1](\frac{1}{2}) \ \to \ \mathcal{F} \ \to \ M \to 0. \tag{18}$$

Here $\mathcal{F} = \ J^*_{s_\alpha} \star M$ where $\star$ denotes the convolution of constructible sheaves on $\mathcal{F}l$ and $J^*_{s_\alpha}$ is the $*$ extension of the (pure weight zero perverse) constant sheaf on the Schubert cell corresponding to s_α.

So, it suffices to see that in each of the above steps the component of degree -1 in $\mathrm{Ext}^1_{Perv^e_{\mathcal{F}l}}(M, N)$ is nontrivial, where $Perv^e_{\mathcal{F}l} \stackrel{\mathrm{def}}{=} Perv_{\mathcal{F}l}/Perv^{<e}_{\mathcal{F}l}$ for the Serre subcategory $Perv^{<e}_{\mathcal{F}l}$ generated by irreducible objects belonging to smaller cells. Exact sequence (18) gives

$$\mathrm{Hom}_{Perv^e_{\mathcal{F}l}}[\mathcal{F}, N] \to \mathrm{Hom}_{Perv^e_{\mathcal{F}l}}[\pi^*\pi_* M[1](\frac{1}{2}), N] \to \mathrm{Ext}^1_{Perv^e_{\mathcal{F}l}}[M, N].$$

The middle term is nonzero since N is a summand of $\pi^*\pi_* M[1]$. It has weight -1 because M, N are pure of weight zero, hence $\pi^*\pi_* M[1]$ and $\mathrm{Hom}_{Perv^e_{\mathcal{F}l}}[\pi^*\pi_* M[1], N]$ are also pure

of weight zero. So it suffices to see that

$$\mathrm{Hom}_{Perv^e_{\mathcal{F}l}}(\mathcal{F}, N) = 0. \tag{19}$$

To check (19) notice that a nonzero element of the Hom space corresponds to a quotient $\mathcal{F}'$ of $\mathcal{F}$ in $Perv_{\mathcal{F}l}$, such that the only irreducible constituent of $\mathcal{F}'$ which does not belong to $Perv_{<e}(\mathcal{F}l)$ is N. Since M is not in $Perv_{<e}(\mathcal{F}l)$, the exact sequence (18) shows that such quotient $\mathcal{F}'$ is necessarily of the form $\pi^*\mathcal{F}''[1]$ for some semi-simple perverse sheaf $\mathcal{F}''$ on the partial affine flag variety $\mathcal{F}l_\alpha$. We have:

$$\mathrm{Hom}(\mathcal{F}, \pi^*\mathcal{F}''[1]) = \mathrm{Hom}(\mathcal{F}, \pi^!\mathcal{F}''[-1]) = \mathrm{Hom}(\pi_*\mathcal{F}, \mathcal{F}''[-1])$$

$$= \mathrm{Hom}(\pi_* M[1], \mathcal{F}''[-1]),$$

where we used the identity $\pi_*(J^*_{s_\alpha} \star \mathcal{G}) = \pi_*\mathcal{G}[1]$ for $\mathcal{G} = M$. Finally, since π_*M and $\mathcal{F}'$ are perverse sheaves $\mathrm{Hom}(\pi_*M[1], \mathcal{F}''[-1]) = \mathrm{Ext}^{-2}(\pi_*M, \mathcal{F}'') = 0$.

6.4.4. *Property* $(0_{\star\star\star})$ *and Gabber's theorem.* Property $(0_{\star\star\star})$ claims that the class of semisimple objects of $mod^{\mathfrak{G}_e, fg}(A^0_e)$ whose irreducible constituents are the particular lifts $\widetilde{L}$ (chosen in 6.4.1) of irreducibles L in $mod^{G_e}(A^0_e)$, is invariant under the action of $Rep^{ss}(G_e)$.

It is explained in [ArkB] that under the equivalence $\mathbf{\Phi}: \ D^b(Coh^G(\widetilde{\mathcal{N}})) \underset{\cong}{\to} D^b(Perv_{\mathcal{F}l})$, the action $\mathcal{G} \mapsto V \otimes \mathcal{G}$ of $V \in Rep(G)$ on the source, corresponds on the target to the action of a *central functor* $\mathcal{Z}_V$ described in [Ga]. This is then also true for the equivalence $\mathbf{\Phi}_{A^0}: D^b(mod^{G, fg}(A^0)) \underset{\cong}{\to} D^b(Perv_{\mathcal{F}l})$. Since the central functors are defined by means of a nearby cycles functor, thus they carry the canonical *monodromy* automorphism $\mathfrak{M}$.

To any $M \in mod^{G_e, fg}(A^0_e)$ one associates a G-equivariant vector bundle $\mathcal{M}$ on the nilpotent orbit O_e and its intersection cohomology extension $\mathrm{IC}(\mathcal{M})$ which lies in the heart of the perverse t-structure of middle perversity on $D^b(mod^{G, fg}(A^0))$ and has support $\overline{O_e}$ (see [ArinB]). We will denote $\mathrm{IC}(\mathcal{M})$ just by $\mathrm{IC}(M)$, then $M \mapsto \mathrm{IC}(M)$ is a bijection of irreducibles in $mod^{G_e, fg}(A^0_e)$ and those irreducibles in the heart of the perverse t-structure that have support $\overline{O_e}$ (ibid).

For $V \in Rep(G)$ we have $V \otimes \mathrm{IC}(M) = \mathrm{IC}(V|_{G_e} \otimes M)$. Moreover, for any semisimple subquotient ρ of $V|_{G_e}$, the tensor product $\rho \otimes M$ is semisimple, so $\mathrm{IC}(\rho \otimes M)$ is semisimple. It is also a subquotient of $V \otimes \mathrm{IC}(M)$ (ibid).

The G_e-module $V|_{G_e}$ carries a nilpotent endomorphism given by the action of e and we denote by $F^i(V)$ the corresponding Jacobson-Morozov-Deligne filtration, and $gr_i(V) = F^i(V)/F^{i+1}(V)$. By definition of this filtration the graded pieces $gr_i(V)$ are semisimple G_e-modules, thus $\mathrm{IC}(gr_i(V) \otimes M)$ is a semisimple subquotient of $V \otimes \mathrm{IC}(M)$.

Now we pass to $\mathfrak{G}_m$-equivariant objects. By the same formalism, if we start with $\widetilde{M} \in mod^{\mathfrak{G}_e, fg}(A_e^0)$ with the underlying object M in $mod^{G_e, fg}(A_e^0)$, we get a graded lift $\mathrm{IC}(\widetilde{M})$ of $\mathrm{IC}(M)$ that lies in the perverse heart of $D^b(mod^{G\times\mathfrak{G}_\mathfrak{m}, fg}(A^0))$. As was explained in 6.4.1, the $\mathfrak{G}_\mathfrak{m}$-equivariant structure $\widetilde{M}$ induces a Weil structure on the perverse sheaf $\mathbf{\Phi}_{A^0}\big(\mathrm{IC}(M)\big)$; we will denote the corresponding Weil sheaf by $\mathbf{\Phi}_{A^0}\big(\mathrm{IC}(\widetilde{M})\big)$. We will combine this with the action of semisimple representations ρ of G_e on $mod^{\mathfrak{G}_e, fg}(A_e^0)$ in order to produce Weil sheaves $\mathbf{\Phi}_{A^0}\big(\mathrm{IC}(\rho \otimes \widetilde{M})\big)$. Now property $(0_{\star\star\star})$ is the part b) of the following Lemma.

Lemma. Let $\widetilde{M} \in mod^{\mathfrak{G}_e, fg}(A_e^0)$ be such that the Weil structure on $\mathbf{\Phi}_{A^0}\big(\mathrm{IC}(\widetilde{M})\big)$ is pure of weight zero.

a) For any $V \in Rep(G)$, the Weil structure on $\mathbf{\Phi}_{A^0}\big(\mathrm{IC}(gr_i(V) \otimes \widetilde{M})\big)$ is pure of weight i.

b) For any semisimple representation ρ of G_e the Weil structure on $\mathbf{\Phi}_{A^0}\big(\mathrm{IC}(\rho \otimes \widetilde{M})\big)$ is pure of weight zero.

Proof. a) We consider the nilpotent endomorphism $\underline{e}$ of the G_e-module $V|_{G_e} \otimes M$ given by the action of e on $V|_{G_e}$. It induces a nilpotent endomorphism of $V \otimes \mathrm{IC}(M) = \mathrm{IC}(V \otimes M)$ which can be used to define a Deligne-Jacobson-Morozov filtration on $V \otimes \mathrm{IC}(M)$. The induced filtration on the fiber $V \otimes \mathrm{IC}(M)|_e \cong V|_{G_e} \otimes M$ is just the Deligne-Jacobson-Morozov filtration for $\underline{e}$ because formation of Deligne-Jacobson-Morozov filtration commutes with exact functors and restriction to $e \in \overline{O_e}$ is exact on perverse sheaves supported in $\overline{O_e}$, Thus the semi-simple subquotient $\mathrm{IC}(gr_i(V) \otimes \widetilde{M})$ of $V \otimes \mathrm{IC}(\widetilde{M})$ is actually a subquotient of $gr_i(V \otimes \mathrm{IC}(\widetilde{M}))$.

According to [ArkB], $\underline{e}$ induces on

$$\mathbf{\Phi}_{A^0}\big(\mathrm{IC}(V|_{G_e} \otimes \mathcal{M})\big) = \mathbf{\Phi}_{A^0}\big(V \otimes \mathrm{IC}(M)\big) = \mathcal{Z}_V\big(\mathbf{\Phi}_{A^0}\ \mathrm{IC}(M)\big)$$

the endomorphism given by the action of the logarithm of monodromy $\log \mathfrak{M}$ on the functor $\mathcal{Z}_V$. Now the Lemma follows from Gabber's Theorem asserting that the monodromy filtration (i.e., the Deligne-Jacobson-Morozov filtration for the logarithm of monodromy)

coincides with the weight filtration on the nearby cycles of a pure weight zero sheaf, cf. [BeBe1].

b) Any irreducible representation ρ of G_e is a subquotient of $V|_{G_e}$ for some $V \in Rep(G)$, hence a subquotient of some $gr_i(V|_{\mathfrak{G}_e})$. The definition of Deligne-Jacobson-Morozov filtration implies that the natural $\mathbb{G}_m$ action on $gr_i(V|_{\mathfrak{G}_e})$ is by the character $t \mapsto t^i$, so part a) implies that the Weil sheaf $\mathrm{IC}(\rho \otimes \widetilde{M})(i)$ is a subquotient in $gr_i(V \otimes \mathrm{IC}(\widetilde{M}))$. Thus $\mathbf{\Phi}_{A^0}\big(\mathrm{IC}(\rho \otimes \widetilde{M})(i)\big)$ has weight i and then $\mathbf{\Phi}_{A^0}\big(\mathrm{IC}(\rho \otimes \widetilde{M})\big)$ has weight zero. $\square$

Appendix A. **Involutions on homology of Springer fibers**

Our goal here is to prove equality (16) from 5.4.1. The result can be viewed as a generalization of the fact that a Chevalley involution (i.e. an involution which sends every element of some Cartan subgroup to its inverse) sends every irreducible representation of an algebraic group to its dual.

A.1. **Cohomology of a Springer fiber as a module for the extended centralizer.** All cohomology spaces in this subsection are taken with coefficients in $\mathbb{C}$ in the classical topology or coefficients in $\bar{\mathbb{Q}}_l$ in the l-adic setting.

Let ι be an involution of G which induces conjugation with w_0 on the abstract Weyl group (e.g. a Chevalley involution). Let $\overset{\bullet}{G}$ denote the semi-direct product $\{1, \iota\} \ltimes G$. It is well known that ι as above is unique up to composition with an inner automorphism, thus the group $\overset{\bullet}{G}$ is defined uniquely (up to an isomorphism).

Fix a nilpotent $e \in \mathfrak{g}$ and set $d_e = \dim(\mathcal{B}_e)$. Let G_e be the centralizer of e in G and $\overset{\bullet}{G}_e$ be the stabilizer of e in $\overset{\bullet}{G}$. Set $\Gamma = \pi_0(G_e)$ and $\overset{\bullet}{\Gamma} = \pi_0(\overset{\bullet}{G}_e)$. It is easy to see that $\overset{\bullet}{G}_e$ intersects the non-identity component of $\overset{\bullet}{G}$, thus $\overset{\bullet}{\Gamma}/\Gamma \cong \mathbb{Z}_2$. Let ϵ be the nontrivial character of $\overset{\bullet}{\Gamma}/\Gamma$.

The group $\overset{\bullet}{G}_e$ acts on the Springer fiber $\mathcal{B}_{\mathbb{k},e}$ thus $\overset{\bullet}{\Gamma}$ acts on its cohomology. We denote this action by η. We consider also another action of $\overset{\bullet}{\Gamma}$ on $H^*(\mathcal{B}_{\mathbb{k},e})$: the two actions coincide on the subgroup $\Gamma \subseteq \overset{\bullet}{\Gamma}$, while on elements of $\overset{\bullet}{\Gamma} \setminus \Gamma$ they differ by the action of $w_0 \in W$ (where W acts via the Springer representation). We denote this new action of $\overset{\bullet}{\Gamma}$ on $H^\bullet(\mathcal{B}_{\mathbb{k},e})$ by ψ.

Notice that unlike the original action, ψ commutes with the action of W in all cases.

Proposition. Let ρ be an irreducible constituent of the $\dot{\Gamma}$-module $(H^{2i}(\mathcal{B}_{\Bbbk,e}), \psi)$. Then $\rho \otimes \epsilon^{d_e - i}$ is a constituent of $(H^{2d_e}(\mathcal{B}_{\Bbbk,e}), \psi)$.

Remark. Validity of the proposition for the groups such that w_0 is central in W is equivalent to the result of [Sp]. The method of [Sp] is based on Shoji's orthogonality formula for Green functions and is quite different from the present one.

Proof. It is well-known that any irreducible representation of Γ which occurs in $H^i(\mathcal{B}_{\Bbbk,e})$ for some i occurs also in $H^{d_e}(\mathcal{B}_{\Bbbk,e})$. Thus the Proposition follows from the following

Lemma. The extension $\dot{\Gamma}$ acts on $(H^{2i} \otimes H^{2j})^{\Gamma}$ by the character $\epsilon^{\otimes\, i+j}$.

Proof. We will deduce the lemma from some known properties of equivariant Borel-Moore homology of the Steinberg variety of triples $St \stackrel{\text{def}}{=} \widetilde{\mathcal{N}} \times_{\mathfrak{g}} \widetilde{\mathcal{N}}$. Let $H^{\bullet}_{BM}$ denote Borel-Moore homology, i.e. derived global sections of the Verdier dualizing sheaf (for convenience we use cohomological grading despite the term "homology").[26]

It is well known (see e.g. [Lu1] Corollary 6.4 for a much stronger result) that

$$H^{2i,G}_{BM}(St) \cong \mathbb{C}[W] \otimes Sym^{i+d}(\mathfrak{h}^*),$$

where $d = 2 \dim \mathcal{B}$ and odd degree homology vanishes.

On the right hand side of the last isomorphism we have a natural action of W (by conjugation on the first factor and by the reflection representation on $\mathfrak{h}^*$) and of the group of outer automorphisms of G. Standard considerations show that the automorphism $\iota \circ w_0$ acts trivially on $\mathbb{C}[W]$ and by $(-1)^i$ on $Sym^i(\mathfrak{h}^*)$.

Let $\varpi : St \to \mathcal{N}$ be the projection. Let O be the G-orbit of e. We reduce the equivariance of Borel-Moore homology from G to G_e

$$H^{G,i}_{BM}(\varpi^{-1}O) = H^{G_e, i+2(d-2d_e)}_{BM}(\mathcal{B}_{\Bbbk,e^2}),$$

and then to the maximal torus C in the identity component G^0_e of G_e,

$$H^{G_e,j}_{BM}(\mathcal{B}_{\Bbbk,e^2}) = H^{G^0_e,j}_{BM}(\mathcal{B}_{\Bbbk,e^2})^{\Gamma} \quad \text{and} \quad H^{G^0_e,j}_{BM}(\mathcal{B}_{\Bbbk,e^2}) = H^{C,j}_{BM}(\mathcal{B}_{\Bbbk,e^2})^{W(G^0_e)}.$$

[26]Since the Verdier dualizing sheaf admits a canonical lifting to the equivariant derived category, equivariant Borel-Moore homology is also defined (cf. [Lu1, 1.1] for a slightly more elementary definition).

Here $W(G_e^0)$ is the Weyl group of G_e^0. Also, $H_{BM}^{2k+1}({\mathcal{B}_{\Bbbk,e}}^2) = 0$ and

$$H^*({\mathcal{B}_{\Bbbk,e}}^2) \;=\; H_C^*({\mathcal{B}_{\Bbbk,e}}^2) \otimes_{H_C^\bullet(pt)} H^0(pt).$$

The last two isomorphisms follow from the existence of a C-invariant stratification of $\mathcal{B}_{\Bbbk,e}$ where each stratum X_i is a C equivariant vector bundle over a space Z_i such that C acts trivially on Z_i and $H^{2k+1}(Z_i) = 0$ [DLP].

In particular, odd degree components in $H_{BM}^G(\varpi^{-1}O)$ vanish. This argument applies to other orbits, thus we see that the Cousin spectral sequence for $H_{BM}^G(St)$ corresponding to the stratification by the preimages of G-orbits under ϖ degenerates, thus we get a canonical filtration on $H_{BM}^G(St)$ whose associated graded pieces are equivariant Borel-Moore homology spaces of the preimages of G-orbits under ϖ.

In particular, one of the pieces is $H_{BM}^G(\varpi^{-1}O)$. The above isomorphisms show that $H_{BM}({\mathcal{B}_{\Bbbk,e}}^2)^\Gamma \;=\; [H_{BM}(\mathcal{B}_{\Bbbk,e})^{\otimes 2}]^\Gamma$ is naturally a quotient of $H_{BM}^G(\varpi^{-1}O)$. Thus $[H_{BM}(\mathcal{B}_{\Bbbk,e})^{\otimes 2}]^\Gamma$ is a subquotient of $H_{BM}^G(St)$.

For $s \in \overset{\bullet}{G}_e \setminus G_e$ the action of $\psi(s)$ on $H_{BM}({\mathcal{B}_{\Bbbk,e}}^2)^\Gamma$ is clearly compatible with the action of $w_0 \circ \iota$ on $H_{BM}^G(St)$. Thus the restriction of this action to $[H_{BM}^{2i}(\mathcal{B}_{\Bbbk,e}) \otimes H_{BM}^{2j}(\mathcal{B}_{\Bbbk,e})]^\Gamma$ equals $(-1)^{i+j+d-2d_e+d} = (-1)^{i+j}$.

However, since $\mathcal{B}_{\Bbbk,e}$ is compact, Borel-Moore homology coincides with homology

$$H_{BM}^{-k}(\mathcal{B}_{\Bbbk,e}) = H_k(\mathcal{B}_{\Bbbk,e}) = H^k(\mathcal{B}_{\Bbbk,e})^*,$$

which yields

$$\psi(\gamma)|_{[H^{2i}(\mathcal{B}_{\Bbbk,e})\otimes H^{2j}(\mathcal{B}_{\Bbbk,e})]^\Gamma} \;=\; \epsilon^{\otimes i+j}(\gamma) \cdot Id \quad \text{for } \gamma \in \overset{\bullet}{\Gamma}. \tag{20}$$

and thereby finishes the proof. □

A.2. **The proof of** (16) **for distinguished nilpotents.** In this subsection we assume that e is distinguished. In this case the torus C is trivial; thus we are dealing with the group $K^0(Coh(\mathcal{B}_{\Bbbk,e}))$. The result of [DLP] implies that it is a free abelian group and the Chern character map induces an isomorphism $ch : K^0(Coh(\mathcal{B}_{\Bbbk,e})) \otimes \mathbb{C} \underset{\cong}{\to} H_{BM}^\bullet(\mathcal{B}_{\Bbbk,e})$.

A.2.1. *Lemma.* a) The Chern character map intertwines Grothendieck-Serre duality $\mathbb{D}$ on $K^0(Coh(\mathcal{B}_{\Bbbk,e}))$ and the involution σ on $H_{BM}^\bullet(\mathcal{B}_{\Bbbk,e})$ such that $\sigma = (-1)^i$ on H_{BM}^{2i},

$$ch \circ \mathbb{D} \;=\; \sigma \circ ch.$$

b) The action of $\mathbb{B} \subset \mathbb{B}_{\mathrm{aff}}$ on $K^0(Coh(\mathcal{B}_{\Bbbk,e})) = K^0(Coh_{\mathcal{B}_{\Bbbk,e}}(\widetilde{\mathcal{N}}))$ induced by the action of $\mathbb{B}_{\mathrm{aff}}$ on the category $D^b[Coh_{\mathcal{B}_{\Bbbk,e}}(\widetilde{\mathcal{N}})]$ factors through W and corresponds under ch to the Springer action.

Proof. (a) follows from triviality of the canonical class. (b) is clear from Theorem 1.3.2(b) above. □

A.2.2. Recall that $\overline{\Upsilon} = \sum_{s=1}^{l} a_s g_s^*$ with $g_s \in A(\widetilde{C}, \Bbbk \cdot e)$, $\mathrm{ord}(g_s) < \infty$ and $a_s \in \mathbb{Q}$ (15). It is immediate from the definition of Υ in [Lu] that the automorphisms g_s of G lie in the outer class of the Chevalley involution. Thus g_s can be considered as an element in $\overset{\bullet}{G}_e \setminus G_e$.

Let $\overline{g_s} \in \overset{\bullet}{\Gamma} \setminus \Gamma$ denote the image of g_s in $\overset{\bullet}{\Gamma}$, and set $\upsilon = \sum a_s \overline{g_s} \in \mathbb{Q}[\overset{\bullet}{\Gamma}]$. It is clear from the definitions that the Chern character map ch intertwines $\overline{\Upsilon}$ with $\eta(\upsilon)$, the natural action of υ on $H^*(\mathcal{B}_{\Bbbk,e})$.

The definition of the modified action ψ and the fact that $\overline{g_s} \in \overset{\bullet}{\Gamma} \setminus \Gamma$ show that $\psi(\upsilon) = w_0 \cdot \eta(\upsilon)$, where w_0 acts via the Springer action. By Lemma A.2.1(b) the endomorphism $\psi(\upsilon)$ is compatible with $T_{w_0}^{-1} \cdot \overline{\Upsilon}$ under the Chern character map. Thus, in view of Lemma A.2.1(a), we will be done if we check that

$$\psi(\upsilon) \;=\; (-1)^i \qquad \text{on } H^{2i}(\mathcal{B}_{\Bbbk,e}). \tag{21}$$

Notice that Proposition A.1 shows that (21) holds for all i provided that it holds for $i = d_e$.

This latter fact has almost been checked by Lusztig. More precisely, [Lu] implies that

$$\psi(\upsilon)|_{H^{2d_e}(\mathcal{B}_{\Bbbk,e})} = \pm 1.$$

When e is not of type $E_8(b_6)$, then this is clear from Proposition 5.2 and definition of Υ in 5.7. If e is of type $E_8(b_6)$, then this follows from part IV of the proof of Proposition 5.2 and definition in 5.7 (all references are to [Lu]). Thus it remains to show that the sign in the last displayed equality equals $(-1)^{d_e}$.

To see this observe that the homomorphism $(\mathcal{B}_{\Bbbk,e} \overset{i}{\to} \mathcal{B})^* : H^{2d_e}(\mathcal{B}) \to H^{2d_e}(\mathcal{B}_{\Bbbk,e})$ is nonzero because the cohomology class of an algebraic cycle is nonzero. The map i^* is

obviously equivariant with respect to the action of automorphisms preserving e, and it is well known that this map is W-equivariant.

Thus it intertwines $\psi(v)$ with $w_0 \cdot \sum a_s g_s$, where w_0 acts via the canonical (Springer) action of W on $H^*(\mathcal{B})$ and the action of g_s comes from its action of $\mathcal{B}$. So we will be done if we show that this endomorphism coincides with $(-1)^i$ on $H^{2i}(\mathcal{B})$.

Since G is connected, each g_s acts in fact by the identity map. Also it is well known that w_0 acts by $(-1)^i$ on H^{2i}. So, we are done because we find in (15) that $\sum\limits_s a_s = 1$.

A.3. **The general case.** Let now $e \in \mathfrak{g}$ be an arbitrary nilpotent. We fix an $sl(2)$ triple (e, h, f) containing e and let $\varphi : SL(2) \to G$ be a homomorphism such that the image of $d\varphi$ is spanned by (e, h, f). We can and will assume that $Im(\varphi)$ commutes with C. There exists an element σ in the image of φ such that $Ad(\sigma) : e \mapsto -e$.

Recall that $K^C(\mathcal{B}_{\Bbbk,e})$ is a free module over $K^0(Rep(C)) = \mathbb{Z}[X^*(C)]$ and $K(\mathcal{B}_{\Bbbk,e}) \cong K^C(\mathcal{B}_{\Bbbk,e}) \otimes_{K^0(Rep(C))} K^0(Vect)$. So, an involution of a free $\mathbb{Z}[X^*(C)]$-module M which induces identity on the quotient $M \otimes_{\mathbb{Z}[X^*(C)]} \mathbb{Z}$ is itself equal to identity. Thus it is enough to check that an analogue of (16) holds in the non-equivariant K-group.

Furthermore, it suffices to check that this identity holds when the base field $\Bbbk$ is of positive characteristic $p > h$. In this case the equivalence of [BMR1] provides an isomorphism $K(\mathcal{B}_{\Bbbk,e}) \cong K^0(mod_e^{0,fg}(U))$.

We will identify the two groups by means of this isomorphism. By the result of [BMR2, $\mathcal{S}3$], the involution $T_{w_0} \circ \mathbb{D}$ on the left hand side corresponds to the map $[M] \mapsto \sigma^*[M^*]$ on the right hand side, where for $M \in mod_e^{0,fg}(U)$ we let M^* denote the dual $\mathfrak{g}$ module (which happens to lie in $mod_{-e}^{0,fg}(U)$). Thus we are reduced to showing the equality in $K^0(mod_e^{0,fg}(U))$:

$$[\sigma^*(M^*)] = \Upsilon[M], \tag{22}$$

where we set $\Upsilon[M] = \sum a_s[g_s^*(M)]$, with a_s, g_s being as in (15).

We will actually show an equality stronger than (22). Namely, consider the category $mod_e^{C,0,fg}(U)$ of modules equipped with a compatible grading by the weights of C. We will show that for M in this category equality (22) holds in $K^0(mod_e^{C,0,fg}(U))$.

We have the Levi subalgebra $\mathfrak{l} = \mathfrak{z}(C) \subset \mathfrak{g}$ such that $e \in \mathfrak{l}$ is distinguished. By the previous subsection we can assume that the equality is known for $(e, \mathfrak{l})$. We claim that the restriction functor from $mod_e^{C,0,fg}(U)$ to $mod_e^{C.0,fg}(U(\mathfrak{l}))$ induces an injective map on K-groups. This follows from the well-known fact that an irreducible module in $mod_e^{C,0,fg}(U(\mathfrak{g}))$ is uniquely determined by its highest weight component which is an irreducible object in $mod_e^{C,0}(U(\mathfrak{l}))$. [We use an ordering on weights corresponding to a choice of a parabolic with Levi L].

It is clear that this restriction functor is compatible with the duality functor. It is also immediate from the definition in [Lu, 5.7] that it is compatible with the involution Υ. Thus (16) for $e \in \mathfrak{g}$ follows from (16) for $e \in \mathfrak{l}$. □

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Appendix B. **A result on component groups, by Eric Sommers**

Here, G is a reductive algebraic group over the algebraically closed field $\Bbbk$ and $\mathfrak{g}$ its Lie algebra. As in Section 5.2.2, we are given a homomorphism

$$\varphi : SL_2(\Bbbk) \to G$$

and the characteristic of $\Bbbk$ is at least $3h-3$.

Let

$$s = \varphi \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$$

and

$$e = d\varphi \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right).$$

It is clear that $s \in G_e$ and $e \in \mathfrak{g}_s$.

Recall that $\phi : \mathbb{G}_m \to G$ is defined as $\phi(t) = \varphi \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix} \right)$.

B.0.1. *Proposition.* If G is semisimple and adjoint, then s belongs to the identity component of G_e.

B.0.2. *Remark.* After this appendix was written and made available in a preprint form, we learned that the result was also proved by A. Premet [Pr1, Lemma 2.1]

Proof. Let $x \in G$ be an arbitrary semisimple element commuting with e. The conjugacy class of the image of x in $G_e/(G_e)^0$ is determined by the G-orbit of the pair $(e, \mathfrak{l}'')$ where $\mathfrak{l}''$ is any Levi subalgebra of $\mathfrak{g}_x$ such that $e \in \mathfrak{l}''$ is distinguished. More precisely, two semisimple elements commuting with e have conjugate image in $G_e/(G_e)^0$ if and only if the corresponding pairs as above are G-conjugate. This result is true in any good characteristic by [MS], [Pr]. In the case where $x = 1$, the G-orbit of such pairs includes

$(e, \mathfrak{l})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ such that $e \in \mathfrak{l}$ is distinguished. Hence, an arbitrary x as above lies in the identity component of G_e if and only if e is distinguished in $\mathfrak{l}'' \subset \mathfrak{g}_x$ where $\mathfrak{l}''$ is a Levi subalgebra of $\mathfrak{g}$ (and not only of $\mathfrak{g}_x$).

Now as in Section 5.2.2, let C be a maximal torus in the centralizer of the image of φ in G. Then with the assumption on the characteristic of $\Bbbk$, C is a maximal torus of G_e and thus e is distinguished in the Levi subalgebra $\mathfrak{l} = Z_{\mathfrak{g}}(C)$ of $\mathfrak{g}$ (see [Ca]). We then also have that the orbit of e in $\mathfrak{l}$ is an even nilpotent orbit. In other words, if we pick a maximal torus of $L = Z_G(C)$ containing the image of ϕ, then each root of L paired with the co-character ϕ is an even integer. Thus $s = \phi(-1)$ acts trivially on $\mathfrak{l}$, and hence $\mathfrak{l} \subset \mathfrak{l}' := \mathfrak{g}_s$.

On the other hand, since $C \subset G_s$ we have that $Z_{\mathfrak{l}'}(C)$ is a Levi subalgebra of $\mathfrak{l}'$. But by the previous paragraph, $\mathfrak{l} = Z_{\mathfrak{g}}(C) \subset \mathfrak{l}'$, so $\mathfrak{l} = Z_{\mathfrak{l}'}(C)$. Therefore $\mathfrak{l}$ is a Levi subalgebra of both $\mathfrak{g}_s$ and $\mathfrak{g}$, and we can conclude by the first paragraph that s lies in the identity component of G_e.

□

B.0.3. *Remark.* A similar result holds in all good characteristics for $s = \phi(-1)$, where ϕ is an associated co-character of a nilpotent element e. In this case, C is defined to be the maximal torus in the simultaneous centralizer in G of e and the image of ϕ. Then e is distinguished in $Z_{\mathfrak{g}}(C)$ as before and by [Pr] or [Ja] ϕ corresponds to a weighted Dynkin diagram arising in characteristic zero for a distinguished element for the corresponding Levi subalgebra. Therefore, it remains true that s acts trivially on $Z_{\mathfrak{g}}(C)$ and the proof goes through.

B.0.4. *Corollary.* For reductive G, $sz \in C$ for some $z \in Z(G)$, where C is as above.

Proof. As $G/Z(G)$ is semisimple and adjoint, it amounts to showing that $s \in C$ when G is semisimple and adjoint. Assume the latter. We know that s centralizes C by definition. Then since s is in the identity component of G_e by the proposition, we know that s belongs to the centralizer of C in the identity component of G_e. That centralizer is equal to C itself, being the centralizer of a maximal torus in a connected group. Hence $s \in C$.

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