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SOME RESULTS ABOUT GEOMETRIC WHITTAKER MODEL

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Abstract. Let G be an algebraic reductive group over a field of positive characteristic. Choose a parabolic subgroup P in G and denote by U its unipotent radical. Let X be a G-variety. The purpose of this paper is to give two examples of a situation in which the functor of averaging of ℓ-adic sheaves on X with respect to a generic character $\chi : U \to \mathbb{G}_a$ commutes with Verdier duality. Namely, in the first example we take X to be an arbitrary G-variety and we prove the above property for all $\overline{U}$-equivariant sheaves on X where $\overline{U}$ is the unipotent radical of an opposite parabolic subgroup; in the second example we take $X = G$ and we prove the corresponding result for sheaves which are equivariant under the adjoint action (the latter result was conjectured by B. C. Ngo who proved it for $G = GL(n)$). As an application of the proof of the first statement we reprove a theorem of N. Katz and G. Laumon about local acyclicity of the kernel of the Fourier-Deligne transform.

1. Introduction

1.1. In this paper k will be an algebraically closed field of characteristic $p > 0$. We choose a prime number ℓ which is different from p. By a sheaf on a k-scheme S we mean an ℓ-adic etale sheaf. We denote by $\mathcal{D}^b(S)$ the bounded derived category of such sheaves. For a complex $\mathcal{F} \in \mathcal{D}^b(S)$ we denote by ${}^pH^i(\mathcal{F})$ its i-th perverse cohomology. Recall that for any finite subfield $k' \subset k$ and any non-trivial character $\psi : k' \to \overline{\mathbb{Q}}_l$ we can construct the Artin-Schreier sheaf $\mathcal{L}_\psi$ on $\mathbb{G}_{a,k}$.

Let G denote a connected split reductive group over k. We shall assume that p is sufficiently large (with respect to G) so that for every unipotent subgroup $U \subset G$ with Lie algebra $\mathfrak{u}$ the exponential map $\mathfrak{u} \to U$ is well-defined and is an isomorphism.

Let $m : G \times G \to G$ be the multiplication map. For every $\mathcal{F}, \mathcal{G} \in \mathcal{D}^b(G)$ we shall denote by $\mathcal{F} \star \mathcal{G}$ their "!"-convolution; in other words

$$\mathcal{F} \star \mathcal{G} = m_!(\mathcal{F} \boxtimes \mathcal{G}). \tag{1.1}$$

Similarly, we shall denote by $\mathcal{F} * \mathcal{G}$ the "*"-convolution of $\mathcal{F}$ and $\mathcal{G}$, i.e.

$$\mathcal{F} * \mathcal{G} = m_*(\mathcal{F} \boxtimes \mathcal{G}). \tag{1.2}$$

1.2. **Generic characters.** Let $P \subset G$ be a parabolic subgroup of G with a Levi decomposition $U_P \cdot L$. For a Cartan subgroup T of G contained in P let $\Delta_T(U_P)$ denote the set of roots of T in $\mathfrak{u}_P = \mathrm{Lie}(U_P)$. For every $\alpha, \beta \in \Delta_T(U_P)$ we say that

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$\alpha > \beta$ if $\alpha - \beta \in \Delta_T(U_P)$. Let $\Delta_T^{\min}(U_P)$ be the set of minimal elements with respect to this ordering.

Lemma 1.3. *(a) The natural map* $\oplus_{\alpha\in\Delta_T^{min}(U_P)}\mathfrak{u}_\alpha \to \mathfrak{u}_P^{ab} = \mathfrak{u}_P/[\mathfrak{u}_P,\mathfrak{u}_P]$ *is an isomorphism. In particular,* $\oplus_{\alpha\in\Delta_T^{min}(U_P)}\mathfrak{u}_\alpha$ *generates* $\mathfrak{u}_P$ *as a Lie algebra.*

(b) For a linear functional χ *on* $\mathfrak{u}_P^{ab}$*, the following conditions are equivalent:*
(i) χ *is an* L*-cyclic vector in* $(\mathfrak{u}_P^{ab})^*$*,*
(ii) there is a Cartan subgroup T *in* P *such that* χ *does not vanish on any root space of* T *in* $\mathfrak{u}_P^{ab}$*.*

We say that a homomorphism $\chi : U_P \to \mathbb{G}_a$ is non-degenerate if its differential satisfies (i-ii).

Proof. (a) is clear. For (b), recall that the T-module $\mathfrak{u}_P$ is multiplicity free. So, (ii) implies (i) since it implies that χ is a T-cyclic vector in $(\mathfrak{u}_P^{ab})^*$, and therefore an L-cyclic vector. For the opposite direction we restate (ii) as: for a given Cartan subgroup T some L-conjugate ϕ of χ is not orthogonal to any of the T-root spaces in $\mathfrak{u}_P^{ab}$. This follows from (i) since it is equivalent to: each root space $(\mathfrak{u}_P^{ab})_\alpha$ is not orhogonal to some conjugate of χ. $\square$

Remark. The L-module $\mathfrak{u}_P$ is multiplicity free. So in the case when it is semisimple (which is clearly the case for $p >> 0$), the non-degeneracy is equivalent to the following condition: the restriction of χ to every irreducible L-submodule of $\mathfrak{u}_P^{ab}$ is non-zero.

1.4. Let X be a G-variety. Assume that U is a subgroup of G and $\chi : U \to \mathbb{G}_a$ is a homomorphism. Let $a : U \times X \to X$ denote the action map and let $p : U \times X \to X$ be the projection to the second multiple. Let $\mathrm{Av}_{U,\chi,*} : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ be the functor sending every $\mathcal{F} \in \mathcal{D}^b(X)$ to $a_*(\chi^*\mathcal{L}_\psi \boxtimes \mathcal{F})\left(\overline{\mathbb{Q}}_l[1](\frac{1}{2})\right)^{\otimes \dim U}$. Similarly we define the functor $\mathrm{Av}_{U,\chi,!}$ by replacing a_* by $a_!$. We have the natural morphism $\mathrm{Av}_{U,\chi,!} \to \mathrm{Av}_{U,\chi,*}$. The main result of this paper is the following:

Theorem 1.5. *Let* $U \subset G$ *be the unipotent radical of a parabolic subgroup* $P \subset G$ *and let* $\chi : U \to \mathbb{G}_a$ *be non-degenerate.*

1. *Let* $\overline{P}$ *denote a parabolic subgroup of* G *opposite to* P *and let* $\overline{U}$ *denote its unipotent radical. Let* $\mathcal{F} \in \mathcal{D}^b(X)$ *be* $\overline{U}$*-equivariant. Then the natural morphism*
$$Av_{U,\chi,!}\mathcal{F} \to Av_{U,\chi,*}\mathcal{F}$$
is an isomorphism.
2. *Let* $\mathcal{F} \in \mathcal{D}^b(G)$ *be equivariant with respect to the adjoint action. Then the natural morphism*
$$Av_{U,\chi,!}\mathcal{F} \to Av_{U,\chi,*}\mathcal{F}$$
is an isomorphism.

Remarks.
0. In the above cases the averaging functors preserve perversity: if $\mathcal{F}$ is perverse then $\mathrm{Av}_{U,\chi,!}\mathcal{F}$ is in perverse degrees ≥ 0 and $\mathrm{Av}_{U,\chi,*}\mathcal{F}$ in perverse degrees ≤ 0.
1. The second statement of Theorem 1.5 was communicated to the first author as a conjecture by B. C. Ngo who also proved it for $G = GL(n)$.
2. In the case $G = GL(n)$ a (much more involved) analogue of Theorem 1.5(2) is used in [3] (Theorem 5.1) in order to complete the proof of the geometric Langlands conjecture for $GL(n)$. We believe that both statements of Theorem 1.5 might have something to do with a possible generalization of Theorem 5.1 of [3] to the case of arbitrary reductive group.
3. In the next section we also explain how the main step in the proof of Theorem 1.5(1) allows to reprove one of the main results of [4].
4. Theorem 1.5 also holds when k is an algebraically closed field of characteristic 0 and ℓ-adic sheaves are replaced by holonomic $\mathcal{D}$-modules (in this case one has to replace $\mathcal{L}_\psi$ by the $\mathcal{D}$-module corresponding to the function e^x).

We conclude the introduction with the following conjecture.

Conjecture 1.6. *Let U and χ be as above. For any irreducible perverse sheaf $\mathcal{F} \in \mathcal{D}^b(G)$ equivariant with respect to the adjoint action, $\mathrm{Av}_{U,\chi,!}\mathcal{F}$ is an irreducible perverse sheaf or zero.*

2. Proof of Theorem 1.5(1)

2.1. **Cleanness.** Let Z be an algebraic variety over k and let $j : Z_0 \to Z$ be an open embedding. We shall say that $\mathcal{G} \in \mathcal{D}^b(Z_0)$ is *clean with respect to j* if the natural map $j_!\mathcal{G} \to j_*\mathcal{G}$ is an isomorphism.

Let X be any $\overline{P}$-variety. Consider the variety $G \underset{\overline{P}}{\times} X$. We have the natural open embedding $j : U \times X \to G \underset{\overline{P}}{\times} X$. We will prove Theorem 1.5(1) by a series of reductions. We claim that Theorem 1.5(1) follows from

Theorem 2.2. *Let X be a $\overline{P}$-variety and let $\mathcal{F} \in \mathcal{D}^b(X)$ be $\overline{U}$-equivariant. Then the sheaf $\chi^*\mathcal{L}_\psi \boxtimes \mathcal{F}$ is clean with respect to j. In other words the natural morphism*

$$j_!(\chi^*\mathcal{L}_\psi \boxtimes \mathcal{F}) \to j_*(\chi^*\mathcal{L}_\psi \boxtimes \mathcal{F}) \tag{2.1}$$

is an isomorphism.

2.3. **Theorem 2.2 implies Theorem 1.5(1).** Indeed if X is a G-variety then we have the natural proper map $b : G \underset{\overline{P}}{\times} X \to X$ sending every $(g, x) \mathrm{mod} \overline{P}$ to $g(x)$. Moreover, we have $b \circ j = a$ (recall that $a : U \times X \to X$ denotes the action map). Hence Theorem 2.2 and the fact that b is proper imply that

$$\mathrm{Av}_{U,\chi,!}\mathcal{F} = b_!(j_!(\chi^*\mathcal{L}_\psi \boxtimes \mathcal{F})) = b_*(j_*(\chi^*\mathcal{L}_\psi \boxtimes \mathcal{F})) = \mathrm{Av}_{U,\chi,*}\mathcal{F}.$$

It remains to prove Theorem 2.2. Note that in the formulation of Theorem 2.2 we do not need X to be a G-variety but only a $\overline{P}$-variety.

2.4. **A reformulation of the Theorem 2.2.** Let $\pi : G\times X \to G\underset{\overline{P}}{\times} X$ be the natural projection. Also let $\widetilde{j} : U\cdot\overline{P}\times X \to G\times X$ be the natural embedding. It follows from the smooth base change theorem that it is enough to show that the natural map

$$\widetilde{j}_!\pi^*(\chi^*\mathcal{L}_\psi \boxtimes \mathcal{F}) \to \widetilde{j}_*\pi^*(\chi^*\mathcal{L}_\psi \boxtimes \mathcal{F})$$

is an isomorphism (note that we have the natural identification $U\cdot\overline{P}\times X$ with $\pi^{-1}(U\times X)$).

The sheaf $\pi^*(\chi^*\mathcal{L}_\psi \boxtimes \mathcal{F})$ is obviously (U,χ)-equivariant with respect to the U-action by multiplication on the left. We claim that it is also $\overline{U}$-equivariant with respect to multiplication on the right, i.e. with respect to the $\overline{U}$-action on $U\cdot\overline{P}\times X$ given by $\overline{u} : (u,\overline{p},x) \mapsto (u,\overline{pu},x)$. Indeed, the map π from $U\cdot\overline{P}\times X$ to $U\times X$ is given by $\pi : (u,\overline{p},x) \mapsto (u,\overline{p}(x))$ (since the action of $\overline{P}$ on $G\times X$ is given by $\overline{p} : (g,x) \mapsto (g\overline{p}^{-1},\overline{p}x)$). Thus

$$\pi(u,\overline{pu},x) = (u,\overline{pu}(x)) = (u,\overline{pup}^{-1}(\overline{p}(x)))$$

and our statement follows from $\overline{U}$-equivariance of $\mathcal{F}$.

Hence we see that Theorem 2.2 follows from the following lemma.

Lemma 2.5. *Consider the action of $U\times\overline{U}$ on $U\cdot\overline{P}\subset G$ given by left and right multiplications. For any variety X, if $\mathcal{G}\in\mathcal{D}^b(U\cdot\overline{P}\times X)$ is (U,χ)-equivariant on the left and $\overline{U}$-equivariant on the right, then the natural map given by the inclusion $\widetilde{j} : U\cdot\overline{P}\times X \to G\times X$,*

$$\widetilde{j}_!\mathcal{G} \to \widetilde{j}_*\mathcal{G},$$

is an isomorphism.

Proof. Let Z denote the complement of $U\cdot\overline{P}$ in G and let i be the natural embedding of $Z\times X$ to $G\times X$. Since $\widetilde{j}_*\mathcal{G}$ is also (U,χ)-equivariant on the left and $\overline{U}$-equivariant on the right it is enough to show that for every complex $\mathcal{H}$ on $G\times X$ with the above equivariance properties we have $i^*\mathcal{H}=0$. However, it is clear that this follows from:

Lemma 2.6. *Let $g\in Z$. Let $S_g\in U\times\overline{U}$ denote the set of all pairs $(u,\overline{u})$ such that $ug\overline{u}=g$. Let also U_g be the projection of S_g to U. Then the restriction of χ to U_g is non-trivial.*

Proof. Indeed, assume that for some $g\in G$ the restriction $\chi|_{U_g}$ is trivial. Choose a pair of opposite Borel subgroups $(B,\overline{B})$ of G such that $U\subset B$, $\overline{U}\subset\overline{B}$. Let $T=B\cap\overline{B}$ and let $w\in W$ be such that $g\in B\widetilde{w}\overline{B}$ where $\widetilde{w}$ is any representative of w in the normalizer of T. We must show that $w\in W_M$ where $W_M\subset W$ is the Weyl group of $M=P\cap\overline{P}$. We have $U_g=U\cap\widetilde{w}\overline{U}\widetilde{w}^{-1}$ (note that this intersection does not change when we multiply $\widetilde{w}$ on the right by any element of M; hence it depends in fact only on the class of w modulo W_M). Let $\mathfrak{u}_g = \mathrm{Lie}(U_g)$. Since $\chi|_{U_g} = 0$ it follows that for every $\alpha \in \Delta_T^{\min}(U_P)$ we have $\mathfrak{u}_\alpha \not\subset \mathfrak{u}_g$. Hence for every $\alpha \in \Delta_T^{\min}(U_P)$ we have $\mathfrak{u}_\alpha \in \mathrm{Lie}(U \cap \widetilde{w}P\widetilde{w}^{-1})$. Since $\mathfrak{u}_\alpha$ generate $\mathfrak{u}$ when α runs over $\Delta_T^{\min}(U_P)$ it follows that $\mathfrak{u} \subset \mathrm{Lie}(\widetilde{w}P\widetilde{w}^{-1})$ which implies that $w \in W_M$. $\square$

$\square$

Corollary 2.7. *Let j denote the open embedding of P into $G/\overline{U}$. Let $\mathcal{F}$ be any (U,χ)-equivariant sheaf on P (with respect to the left multiplication action). Then the natural morphism*

$$j_!\mathcal{F} \to j_*\mathcal{F}$$

is an isomorphism. In other words, every (U,χ)-equivariant sheaf on P is clean with respect to j.

Proof. Let L be the Levi factor of P. The isomorphism $L \simeq \overline{P}/\overline{U}$ gives rise to a natural action of P on L. Since the action of U on L is trivial it follows that every $\mathcal{F} \in \mathcal{D}^b(L)$ is automatically $\overline{U}$-equivariant.

We have the natural identifications $U \times L \simeq P$ (by multiplication map) and $G \underset{\overline{P}}{\times} L \simeq G/\overline{U}$ (sending every $(g,l) \mod \overline{P}$ to $gl \mod \overline{U}$). Under these identification the embedding $j : P \to G/\overline{U}$ becomes equal to the natural embedding $U \times L \to G \underset{\overline{P}}{\times} L$ considered in Theorem 2.2 (for $X = L$). Also the fact that $\mathcal{F}$ is (U,χ)-equivariant implies that as a sheaf on $U \times L$ it can be decomposed as $\mathcal{F} = \chi^*\mathcal{L}_\psi \boxtimes \mathcal{F}'$ for some $\mathcal{F}' \in \mathcal{D}^b(L)$. Hence Corollary 2.7 is a particular case of Theorem 2.2. $\square$

2.8. **Application to Katz-Laumon theorem.** Consider the variety $\mathbb{A}^1 \times \mathbb{G}_m$ with coordinates (x,y). Let $f : \mathbb{A}^1 \times \mathbb{G}_m \to \mathbb{A}^1$ be given by $f(x,y) = \frac{x}{y}$. Let also $i : \mathbb{A}^1 \times \mathbb{G}_m \to \mathbb{A}^2$ denote the natural embedding and let $\pi : \mathbb{A}^1 \times \mathbb{G}_m \to \mathbb{G}_m$ be the projection to the second variable. The following theorem is proved in [4].

Theorem 2.9. *For every $\mathcal{F} \in \mathcal{D}(\mathbb{G}_m)$ the natural map*

$$i_!(f^*\mathcal{L}_\psi \otimes \pi^*\mathcal{F}) \to i_*(f^*\mathcal{L}_\psi \otimes \pi^*\mathcal{F}) \tag{2.2}$$

is an isomorphism.

Below we explain that Theorem 2.9 may be viewed as a particular case of Corollary 2.7.

Proof. Take now $G = SL(2)$ and let P and $\overline{P}$ be respectively the subgroups of lower-triangular and upper-triangular matrices, with unipotent radicals U and $\overline{U}$. We denote the natural isomorphism between U and $\mathbb{G}_a$ by χ.

Let us identify $G/\overline{U}$ with $\mathbb{A}^2\backslash\{0\}$ by $g\overline{U} \mapsto g(e_1)$ for the first standard basis vector e_1 of $\mathbb{A}^2$. Then $P \subset G/\overline{U}$ is identified with $\mathbb{A}^1 \times \mathbb{G}_m \subset \mathbb{A}^2\backslash\{0\}$ by $\begin{pmatrix} \lambda & 0 \\ t & \lambda^{-1} \end{pmatrix} \leftrightarrow (t, \lambda^{-1})$. The sheaf $f^*\mathcal{L}_\psi \otimes \pi^*\mathcal{F}$ is (U,χ)-equivariant, so by Corollary 2.7 this sheaf is clean for the embedding $\mathbb{A}^1 \times \mathbb{G}_m \subset \mathbb{A}^2 \backslash \{0\}$. It remains to observe that the resulting sheaf on $\mathbb{A}^2 \backslash \{0\}$ is clean for the embedding into $\mathbb{A}^2$ since the cone of the canonical map between the shriek and star direct images is zero – it is a (U, χ)-equivariant sheaf supported at a point $\{0\}$. $\square$

3. Proof of Theorem 1.5(2)

3.1. **Horocycle transform.** Let P be a parabolic subgroup in G and let Y_P denote the variety of all parabolic subgroups of G which are conjugate to P. We also denote by W_P the variety of P-horocycles, i.e., the pairs $(Q \in Y_P, x \in G/U_Q)$ where U_Q denotes the unipotent radical of Q (see section Section 3.3 below for a more direct definition of W_P). We have the natural map $p : W_P \to Y_P$.

We also have the natural morphisms $\alpha : G \times Y_P \to G$ and $\beta : G \times Y_P \to W_P$ where α is just the projection to the first multiple and β sends (g, Q) to $(Q, g \mod U_Q)$. We define two functors $\mathcal{R}_P : \mathcal{D}^b(G) \to \mathcal{D}^b(W_P)$ and $\mathcal{S}_P : \mathcal{D}^b(W_P) \to \mathcal{D}^b(G)$ by setting

$$\mathcal{R}_P(\mathcal{F}) = \beta_! \alpha^*(\mathcal{F}) \otimes \left(\overline{\mathbb{Q}}_l[1](\frac{1}{2}) \right)^{\otimes \dim U_P}$$

and

$$\mathcal{S}_P(\mathcal{G}) = \alpha_! \beta^*(\mathcal{G}) \otimes \left(\overline{\mathbb{Q}}_l[1](\frac{1}{2}) \right)^{\otimes \dim U_P}.$$

The following lemma is proved in [5] when Q is a Borel subgroup in G.

Lemma 3.2. *The identity functor is a direct summand of $\mathcal{S}_P \circ \mathcal{R}_P$.*

Proof. Let $\mathcal{T}_Q = \{(Q \in Y_P, u \in U_Q)\}$. We have the natural map $p_P : \mathcal{T}_Q \to G$ sending every (Q, u) to u (clearly the image of p_P lies in the set of unipotent elements in G). Let $\mathbf{Spr}_P = (p_P)_! \overline{\mathbb{Q}}_l[2 \dim Y_P](\dim Y_P)$. It is known (cf. [2]) that $\mathbf{Spr}_P$ is perverse and that it contains the skyscraper sheaf δ_e at the unit element $e \in G$ as a direct summand. We set $\mathbf{Spr}_P = \delta_e \oplus \mathbf{Spr}'_P$.

On the other hand, arguing as in [5] we can show that for every $\mathcal{F} \in \mathcal{D}^b(G)$ we have a canonical isomorphism

$$\mathcal{S}_P \circ \mathcal{R}_P(\mathcal{F}) = \mathcal{F} \star \mathbf{Spr}_P. \tag{3.1}$$

Hence

$$\mathcal{S}_P \circ \mathcal{R}_P(\mathcal{F}) = \mathcal{F} \oplus (\mathcal{F} \star \mathbf{Spr}'_P) \tag{3.2}$$

which finishes the proof. $\square$

3.3. **Another definition of W_P.** One can identify W_P with $(G/U_P \times G/U_P)/M$ where $M = P/U$ acts on $G/U_P \times G/U_P$ diagonally. The identification is given by the map

$$(x_1 \mod U_P, x_2 \mod U_P) \mapsto (x_2 P x_2^{-1}, x_1 x_2^{-1} \mod U_P).$$

Under this identification the natural left and right G-actions on $(G/U_P \times G/U_P)/M$ give two actions of G on W_P, which we still call the "left" and "right" action. The left action is just the natural G-action in the fibers of p. The right action is given by

$$g : (Q, x) \mapsto (gQg^{-1}, xg^{-1} \mod gU_Q g^{-1}).$$

The corresponding adjoint action is given by

$$g : (Q, x) \mapsto (gQg^{-1}, gxg^{-1} \mod gU_Q g^{-1}).$$

We now claim the following

Theorem 3.4. *Let P be a parabolic subgroup in G and let U be its unipotent radical. Let $\mathcal{G} \in \mathcal{D}^b(W_{\overline{P}})$ be equivariant with respect to the adjoint action. Then for every non-degenerate character $\chi : U \to \mathbb{G}_a$ the natural map*

$$\mathrm{Av}_{U,\chi,!}\mathcal{G} \to \mathrm{Av}_{U,\chi,*}\mathcal{G}$$

is an isomorphism (here averaging is performed with respect to the left action).

Let us explain why Theorem 3.4 implies Theorem 1.5(2). Let $\mathcal{F} \in \mathcal{D}^b(G)$ be equivariant with respect to the adjoint action. We need to prove that the map

$$\mathrm{Av}_{U,\chi,!}\mathcal{F} \to \mathrm{Av}_{U,\chi,*}\mathcal{F} \tag{3.3}$$

is an isomorphism. Since by Lemma 3.2 $\mathcal{F}$ is a direct summand of $\mathcal{S}_{\overline{P}} \circ \mathcal{R}_{\overline{P}}(\mathcal{F})$ it is enough to show that (3.3) holds for the latter. It follows from the fact that α is a proper morphism that we have the natural isomorphisms of functors

$$\mathrm{Av}_{U,\chi,!} \circ \mathcal{S}_{\overline{P}} \simeq \mathcal{S}_{\overline{P}} \circ \mathrm{Av}_{U,\chi,!} \text{ and } \mathrm{Av}_{U,\chi,*} \circ \mathcal{S}_{\overline{P}} \simeq \mathcal{S}_{\overline{P}} \circ \mathrm{Av}_{U,\chi,*}.$$

Hence it is enough to show that (3.3) holds for $\mathcal{R}_{\overline{P}}(\mathcal{F})$. However, it is clear that $\mathcal{R}_{\overline{P}}$ maps ad-equivariant complexes to ad-equivariant ones which finishes the proof by Theorem 3.4.

3.5. The rest of this section is occupied by the proof of Theorem 3.4.

Let $Y^0_{\overline{P}}$ denote the open U-orbit on $Y_{\overline{P}}$ and let $W^0_{\overline{P}}$ denote its preimage in $W_{\overline{P}}$.

First of all we claim that both $\mathrm{Av}_{U,\chi,*}\mathcal{G}$ and $\mathrm{Av}_{U,\chi,!}\mathcal{G}$ are equal to the extension by zero of their restriction to $W^0_{\overline{P}}$. Indeed we must show that the $*$-restriction of either of these sheaves to the fiber of $p : W_{\overline{P}} \to Y_{\overline{P}}$ over any parabolic Q which is not opposite to P is equal to zero. Let us denote this restriction by $\mathcal{H}$. This is a complex of sheaves on $p^{-1}(Q) = G/U_Q$. The fact that $\mathcal{G}$ is equivariant with respect to the adjoint action implies that both $\mathrm{Av}_{U,\chi,*}\mathcal{G}$ and $\mathrm{Av}_{U,\chi,!}\mathcal{G}$ are equivariant with respect to the adjoint action of U. Hence $\mathcal{H}$ is equivariant with respect to the left action of $U \cap U_Q$. On the other hand, it is clear that $\mathcal{H}$ is (U, χ)-equivariant with respect to the left action of U. Thus our statement follows from the following result which is equivalent to Lemma 2.6: let Q be as above (i.e. Q is conjugate to $\overline{P}$ but it is not in the generic position with respect to P); then the restriction of χ to $U \cap U_Q$ is non-trivial.

It remains to show that the map $\mathrm{Av}_{U,\chi,!}\mathcal{G} \to \mathrm{Av}_{U,\chi,*}\mathcal{G}$ is an isomorphism when restricted to $W^0_{\overline{P}}$.

The map $u \mapsto u\overline{P}u^{-1}$ is an isomorphism between U and $Y^0_{\overline{P}}$. Let $\kappa : W^0_{\overline{P}} \to U$ be the composition of the natural projection $W^0_{\overline{P}} \to Y^0_{\overline{P}}$ with this isomorphism. Define now a new G-action on $W^0_{\overline{P}}$ (denoted by $(g,w) \mapsto g \times w$) by

$$g \times w = \kappa(w) g \kappa(w)^{-1}(w)$$

(in the right hand side we use the standard left action of G on $W_{\overline{P}}$).

To finish the argument we need the following general (and basically tautological) result:

Lemma 3.6. *a) Let H be an algebraic group, and X be an algebraic variety equipped with two actions ϕ_1, ϕ_2 of H. Suppose that the two actions differ by a conjugation, i.e. there exists a morphism of algebraic varieties $c : X \to H$, such that*

$$\phi_1(g)(x) = \phi_2(c(x) \cdot g \cdot c(x)^{-1})(x)$$

for all $g \in H$, $x \in X$. Then for any character $\chi : H \to \mathbb{G}_a$ we have canonical isomorphisms of the averaging functors corresponding to the two actions:

$$Av^{\phi_1}_{H,\chi,!} = Av^{\phi_2}_{H,\chi,!},$$

$$Av^{\phi_1}_{H,\chi,*} = Av^{\phi_2}_{H,\chi,*}.$$

b) Let H_1, H_2 be two algebraic groups, ϕ_i be an action of H_i on an algebraic variety X_i (where $i = 1,2$). Let $f : X_1 \to X_2$ be a morphism, and assume that there exists a morphism $s : H_1 \times X_1 \to H_2$, such that

$$f(\phi_1(h_1)(x_1)) = \phi_2(s(h_1,x_1))(f(x))$$

for $x_1 \in X_1$, $h_1 \in H_1$. Then for any H_2-equivariant complex of constructible sheaves on X_2 the complex $f^*(X)$ is also H_1 equivariant. □

Part (a) of Lemma 3.6 shows that both averaging functors $\mathrm{Av}_{U,\chi,!}$ and $\mathrm{Av}_{U,\chi,*}$ do not change when we replace the old action by the new one. Also, since our $\mathcal{G}$ is equivariant with respect to the adjoint action it follows that $\mathcal{G}|_{W^0_{\overline{P}}}$ is also equivariant with respect to the new action of $\overline{U}$ by part (b) of Lemma 3.6. The statement now follows from Theorem 1.5(1).

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