

University of Massachusetts Amherst  
**ScholarWorks@UMass Amherst**

---

Mathematics and Statistics Department Faculty  
Publication Series

Mathematics and Statistics

---

2001

# Geometric representation theory of restricted lie algebras

I Mirkovic

*University of Massachusetts - Amherst*, [mirkovic@math.umass.edu](mailto:mirkovic@math.umass.edu)

D Rumynin

Follow this and additional works at: [https://scholarworks.umass.edu/math\\_faculty\\_pubs](https://scholarworks.umass.edu/math_faculty_pubs)

 Part of the [Physical Sciences and Mathematics Commons](#)

---

## Recommended Citation

Mirkovic, I and Rumynin, D, "Geometric representation theory of restricted lie algebras" (2001). *TRANSFORMATION GROUPS*. 730.

Retrieved from [https://scholarworks.umass.edu/math\\_faculty\\_pubs/730](https://scholarworks.umass.edu/math_faculty_pubs/730)

This Article is brought to you for free and open access by the Mathematics and Statistics at ScholarWorks@UMass Amherst. It has been accepted for inclusion in Mathematics and Statistics Department Faculty Publication Series by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact [scholarworks@library.umass.edu](mailto:scholarworks@library.umass.edu).

# GEOMETRIC REPRESENTATION THEORY OF RESTRICTED LIE ALGEBRAS OF CLASSICAL TYPE

IVAN MIRKOVIĆ AND DMITRIY RUMYNIN

ABSTRACT. We modify the Hochschild  $\varphi$ -map to construct central extensions of a restricted Lie algebra. Such central extension gives rise to a group scheme which leads to a geometric construction of unrestricted representations. For a classical semisimple Lie algebra, we construct equivariant line bundles whose global sections afford representations with a nilpotent  $p$ -character.

Let  $G$  be a connected simply connected semisimple algebraic group over an algebraically closed field  $\mathbb{K}$  of characteristic  $p$  and  $\mathfrak{g}$  be its Lie algebra. The representation theory of  $\mathfrak{g}$  is connected with the coadjoint orbits through the notion of a  $p$ -character [27, 3, 14, 10]. An irreducible representation  $\rho$  is finite-dimensional and determines a  $p$ -character  $\chi \in \mathfrak{g}^*$  by  $\chi(x)^p \text{Id} = \rho(x)^p - \rho(x^{[p]})$  for each  $x \in \mathfrak{g}$  [27]. There are indications that a geometry stands behind this representation theory, for instance, the Kac-Weisfeiler conjecture proved by Premet [21]. This work has been motivated by an idea of Humphreys that the representations affording  $\chi$  should be related to the Springer fiber  $\mathcal{B}^\chi$ . Some of our intuition comes from algebraic calculations of Jantzen [12, 13]. The most interesting evidence for the relation between Springer fibers and representations of  $\mathfrak{g}$  is now given by Lusztig [17].

The main goal of this paper is to introduce a method for constructing unrestricted representations of  $\mathfrak{g}$  by taking global sections of line bundles on infinitesimal neighborhoods of certain subvarieties of  $\mathcal{B}^\chi$ . A more general approach implementing twisted sheaves of crystalline differential operators will be explained elsewhere.

An attempt to study representations of  $\mathfrak{g}$  with a single  $p$ -character  $\chi$  has led to the notion of a reduced enveloping algebra. We modify this approach by considering a set of  $p$  different  $p$ -characters  $\{0, \chi, 2\chi, \dots, (p-1)\chi\}$  together in Section 1. The category of such representations is closed under tensor products. These are restricted representations of

---

*Date:* July 29, 1998; revised on June 20, 1999.

*1991 Mathematics Subject Classification.* Primary 17B50; Secondary 14M15.

The research was supported by NSF and completed at University of Massachusetts at Amherst.

a central extension  $\mathfrak{g}_\chi$  of  $\mathfrak{g}$  by the multiplicative restricted Lie algebra  $\mathfrak{g}_m$ . One can think of this construction as a multiplicative version of the Hochschild  $\varphi$ -map.

We discuss a geometric machinery necessary for the construction of representations in Sections 2 and 3. In Section 4, we introduce equivariant line bundles and construct representations. This section contains the main result (Theorem 4.3.2) of this paper, which is a geometric construction of unrestricted representations. Section 5 is devoted to various comments on the representations constructed.

Let us briefly explain the construction. The central extension  $\mathfrak{g}_\chi$  defines a central extension  $0 \rightarrow G_m^1 \rightarrow G^\chi \rightarrow G^1 \rightarrow 0$  of the Frobenius kernels of  $G$  and the multiplicative group  $G_m$ . The group scheme  $G^\chi$  acts on the flag variety  $\mathcal{B}$  and preserves the Frobenius neighborhood  $\widehat{Z}$  of any subscheme  $Z$ . For a  $G$ -equivariant line bundle  $\mathcal{F}_\lambda$  on  $\mathcal{B}$ , we construct a  $G^\chi$ -action on  $\mathcal{F}_\lambda|_{\widehat{Z}}$  with a “central charge 1”. Then  $\mathfrak{g}$  will act on the global sections of  $\mathcal{F}_\lambda|_{\widehat{Z}}$  with a  $p$ -character  $\chi$ .

It suffices to construct such an equivariant structure on a subscheme  $\widehat{X}$  that contains  $Z$ . We want to choose  $X$  so that we can put hands on the Frobenius neighborhood  $\widehat{X}$ . We will assume that  $X$  is smooth so that  $G^\chi \times X \rightarrow \widehat{X}$  is the quotient map by the action groupoid  $\mathcal{G}_X^\chi$  arising from the  $G^\chi$ -action on  $\widehat{X}$ .

To construct an equivariant structure, it suffices to split the groupoid  $\mathcal{G}_X^\chi$  as a product of the Frobenius kernel of the multiplicative group  $G_m^1$  and another groupoid  $\mathcal{G}_X^1$ . A necessary condition for this construction is that  $X$  is a subvariety of  $\mathcal{B}^\chi$ .

The groupoid  $\mathcal{G}_X^\chi$  splits canonically over the diagonal. We linearize the requirement that this splitting extends off the diagonal, and study it in terms of Lie algebroids of the above mentioned groupoids.

The authors are greatly indebted to J. Humphreys whose inspiration was crucial for writing this article. The authors would like to thank T. Ekedahl, J. Jantzen, J. Paradowski, G. Seligman, and S. Siegel for various information.

**0.1. Notational conventions.** Let  $\mathbb{F}$  be the prime subfield of an algebraically closed field  $\mathbb{K}$  of characteristic  $p$ .

**0.1.1. Restricted Lie algebras.** The main object of our study is a finite dimensional restricted Lie algebra  $\mathfrak{l}$  over  $\mathbb{K}$ . If  $\mathfrak{l}$  is the Lie algebra of a linear algebraic group, the group is denoted by  $L$ . While discussing a semisimple algebraic group, we denote the group by  $G$  and its Lie algebra by  $\mathfrak{g}$ . Let  $R_{\mathfrak{g}}$  be the set of roots of  $\mathfrak{g}$ ,  $\Delta_{\mathfrak{g}}$  be a set of simple roots,

$W$  be the Weyl group, and  $\Pi$  be the weight lattice. The multiplicative group and its Lie algebra are denoted  $G_m$  and  $\mathfrak{g}_m$ .

0.1.2. *Flag variety.* Let  $\mathcal{B}$  be the flag variety of  $G$ . We think of points of  $\mathcal{B}$  over  $\mathbb{K}$  as Borel subalgebras  $\mathfrak{b}$  in  $\mathfrak{g}$ . Let  $\mathcal{B}_w$  be a Schubert variety for  $w \in W$ . If  $\chi$  is nilpotent then *the Springer fiber*  $\mathcal{B}^\chi$  is a reduced subscheme of  $\mathcal{B}$ , whose points over  $\mathbb{K}$  are those Borel subalgebras on which  $\chi$  vanishes.

0.1.3. *Enveloping algebras.* The universal enveloping algebra of  $\mathfrak{l}$  is  $U(\mathfrak{l})$ . It contains a central Hopf subalgebra  $O$  generated by  $x^p - x^{[p]}$  for all  $x \in \mathfrak{l}$ . For any  $\chi \in \mathfrak{l}^*$ , the reduced enveloping algebra  $U_\chi(\mathfrak{l})$  is a quotient of  $U(\mathfrak{l})$  by the ideal generated by  $x^p - x^{[p]} - \chi(x)^p 1_{U(\mathfrak{l})}$  for all  $x \in \mathfrak{l}$  [25, 3]. The reduced enveloping algebra  $U_0(\mathfrak{l})$  is the restricted enveloping algebra  $u(\mathfrak{l})$ . All  $U_\chi(\mathfrak{l})$  are twisted products of  $u(\mathfrak{l})$  with the field  $\mathbb{K}$  [22].

0.1.4. *p-character.* A representation of  $\mathfrak{l}$  has a  $p$ -character  $\chi \in \mathfrak{l}^*$  if the representation determines a  $U_\chi(\mathfrak{l})$ -module. While working with  $\mathfrak{g}$ , we assume that  $\chi$  is a nilpotent element of  $\mathfrak{g}^*$ . The case of a general  $\chi \in \mathfrak{g}^*$  can be reduced to the nilpotent case.

0.1.5. *Induction.* If  $\mathfrak{m}$  is a restricted Lie subalgebra of  $\mathfrak{l}$  such that  $\chi|_{\mathfrak{m}} = 0$  then the induction functor  $\text{Ind}_{U_\chi(\mathfrak{m})}^{U_\chi(\mathfrak{l})}$  is defined on the category of left  $u(\mathfrak{m})$ -modules.

In particular, for a Borel subalgebra  $\mathfrak{b}$  to  $\mathfrak{g}$ , if  $\chi|_{\mathfrak{b}} = 0$  then all simple modules over  $U_\chi(\mathfrak{b}) = u(\mathfrak{b})$  are one-dimensional and parametrized by the reduced (modulo  $p$ ) weight lattice  $\Lambda$ . The induced module  $Z_{\chi, \mathfrak{b}}(\lambda) = U_\chi(\mathfrak{g}) \otimes_{u(\mathfrak{b})} \mathbb{K}_\lambda$ ,  $\lambda \in \Lambda$ , called a *baby Verma module*, was introduced in [8]. Any irreducible  $U_\chi(\mathfrak{g})$ -module is a quotient of at least one  $Z_{\chi, \mathfrak{b}}(\lambda)$ , though the module  $Z_{\chi, \mathfrak{b}}(\lambda)$  need not have a unique simple quotient, which makes a classification of simple  $\mathfrak{g}$ -modules an interesting problem [3].

## 1. CENTRAL EXTENSIONS

1.1. **Central extensions of Hopf algebras.** Our approach will be explained in this section. The ground field  $\mathbf{k}$  is arbitrary for this section.

1.1.1. Let us consider a Hopf algebra  $U$  and its central Hopf subalgebra  $O$ . Given  $\chi \in \text{Spec } O(\mathbf{k})$ , representations in which  $O$  acts by  $\chi$  are those that can be reduced to the algebra  $U_\chi = U \otimes_O \mathbf{k}(\chi)$ . The algebra  $U_\chi$  is not necessarily a Hopf algebra. The basic idea of the present paper is to replace the study of  $U_\chi$ -modules for a single  $\chi$  with the study of  $U_\chi$ -modules as  $\chi$  runs over a closed subgroup of  $\text{Spec } O$ . One has

more modules but we benefit from having a Hopf algebra rather than a Hopf-Galois extension.

1.1.2. *Proposition.* Let  $O \rightarrow R$  be the natural map where  $R$  is the algebra of functions on the closed subgroup scheme of  $\text{Spec } O$  generated by  $\chi$ . Then  $U \otimes_O R$  is a Hopf algebra.

1.1.3. *Proof.* A subgroup scheme  $X$  gives rise to a surjective Hopf algebra map  $\pi : O \rightarrow \mathcal{O}(X)$ . We need to check that  $A \otimes_O \mathcal{O}(X)$  is a Hopf algebra.

The tensor product  $C = A \otimes_{\mathbf{k}} \mathcal{O}(X)$  is obviously a Hopf algebra. It suffices to check that the ideal  $I$ , generated by all  $x \otimes 1 - 1 \otimes \pi(x)$  with  $x \in O$ , is a Hopf ideal. The latter means that the quotient  $C/I = A \otimes_O \mathcal{O}(X)$  admits a Hopf algebra structure such that the quotient map is a Hopf algebra homomorphism. Being a Hopf ideal includes three axioms that we are checking now.

*Axiom 1:*  $\varepsilon(I) = 0$ . A typical element of  $I$  has a form  $\sum_i a_i(x_i \otimes 1 - 1 \otimes \pi(x_i))b_i$  where  $a_i, b_i \in C$ ,  $x_i \in O$ . Now we compute  $\varepsilon(\sum_i a_i(x_i \otimes 1 - 1 \otimes \pi(x_i))b_i) = \sum_i \varepsilon(a_i)(\varepsilon(x_i) \otimes 1 - 1 \otimes \varepsilon(\pi(x_i)))\varepsilon(b_i) = \sum_i \varepsilon(a_i)(\varepsilon(x_i)1_C - \varepsilon(x_i)1_C)\varepsilon(b_i) = 0$ .

*Axiom 2:*  $S(I) \subseteq I$ . Let us compute  $S(\sum_i a_i(x_i \otimes 1 - 1 \otimes \pi(x_i))b_i) = \sum_i S(b_i)(S(x_i) \otimes 1 - 1 \otimes S(\pi(x_i)))S(a_i) = \sum_i S(b_i)(S(x_i) \otimes 1 - 1 \otimes \pi(S(x_i)))S(a_i) \in I$ .

*Axiom 3:*  $\Delta(I) \subseteq C \otimes I + I \otimes C$ . Let us compute  $\Delta(\sum_i a_i(x_i \otimes 1 - 1 \otimes \pi(x_i))b_i) = \sum_i \Delta(a_i)(x_{i1} \otimes 1 \otimes x_{i2} \otimes 1 - 1 \otimes \pi(x_{i1}) \otimes 1 \otimes \pi(x_{i2}))\Delta(b_i) = \sum_i \Delta(a_i)[\{x_{i1} \otimes 1 - 1 \otimes \pi(x_{i1})\} \otimes x_{i2} \otimes 1 + 1 \otimes \pi(x_{i1}) \otimes \{x_{i2} \otimes 1 - 1 \otimes \pi(x_{i2})\}]\Delta(b_i) \in I \otimes C + C \otimes I$ .  $\square$

1.1.4. In the present paper, we focus on the case of the universal enveloping Hopf algebra of a restricted Lie algebra. The subgroup generated by  $\chi$  is  $\mathbb{F}\chi$ . A quantum linear group  $O_q(G)$  and the unrestricted form of a quantum enveloping algebra  $U_q(\mathfrak{g})$  at a root of unity are other interesting options [22]. However, a closed subgroup of  $G$  or  $\mathbb{C}^{n-r} \times \mathbb{C}^{*r}$  generated by an element is more complicated.

## 1.2. Extensions of restricted Lie algebras.

1.2.1. An exact sequence of restricted Lie algebras  $0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{m} \rightarrow \mathfrak{l} \rightarrow 0$  is called a *central extension* of  $\mathfrak{l}$  if  $\mathfrak{a}$  is a central ideal of  $\mathfrak{m}$ . This terminology is not standard. An additional condition  $\mathfrak{a}^{[p]} = 0$  is required in [6] for a central extension. The reason for this constraint is that such central extensions can be parametrized by the second restricted cohomology group. The important choice of  $\mathfrak{a}$  for us is  $\mathfrak{g}_m$ , which means that we usually have  $\mathfrak{a}^{[p]} = \mathfrak{a}$ .

1.2.2. *Multiplicative Hochschild  $\varphi$ -map.* The Hochschild  $\varphi$ -map [5, 26] provides a central extension of  $\mathfrak{l}$  by the additive Lie algebra for each  $\chi \in \mathfrak{l}^*$ . We modify this construction to obtain a central extension by  $\mathfrak{g}_m$  instead. Given  $\chi \in \mathfrak{l}^*$ , we construct a central extension  $\mathfrak{l}_\chi$ . This extension is trivial as an extension of Lie algebras, i.e.  $\mathfrak{l}_\chi = \mathfrak{l} \oplus \mathbb{K}c$ ; but the  $p$ -structure is twisted by  $\chi$ :

$$(a + \alpha c)^{[p]} = a^{[p]} + (\chi(a)^p + \alpha^p)c \quad (1)$$

The original construction by Hochschild [5, 26] uses a  $p$ -structure  $(a + \alpha c)^{[p]} = a^{[p]} + \chi(a)^p c$ .

1.2.3. *Proposition.* Formula (1) defines a restricted Lie algebra structure.

1.2.4. *Proof.* The operation that we define is obviously  $p$ -linear, i.e.  $(\beta a + \beta \alpha c)^{[p]} = \beta^p(a + \alpha c)^{[p]}$ . Let us denote  $\text{ad}_{\mathfrak{l}}$  by  $\text{ad}$  and  $\text{ad}_{\mathfrak{l}_\chi}$  by  $\text{Ad}$ . Since  $c$  is central,  $\text{Ad}(a + \alpha c) = \text{Ad} a$ . Thus,

$$\begin{aligned} \text{Ad}(a + \alpha c)^{[p]} &= \text{Ad} a^{[p]} = \begin{pmatrix} \text{ad} a^{[p]} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (\text{ad} a)^p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \text{ad} a & 0 \\ 0 & 0 \end{pmatrix}^p = \\ &= (\text{Ad} a)^p = (\text{Ad}(a + \alpha c))^p. \end{aligned}$$

Introducing an independent variable  $T$ , we set  $ns_n(a, b)$  to be a coefficient at  $T^{n-1}$  of  $(\text{ad}(aT + b))^{p-1}(a)$ . By  $S_n$  we denote the result of the similar procedure performed in  $\mathfrak{l}_\chi$ . It is clear that  $S_n(a + \alpha c, b + \beta c) = s_n(a, b)$ . Finally,  $((a + \alpha c) + (b + \beta c))^{[p]} = (a + b)^{[p]} + (\chi(a)^p + \alpha^p + \chi(b)^p + \beta^p)c = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b) + (\chi(a)^p + \alpha^p + \chi(b)^p + \beta^p)c = (a + \alpha c)^{[p]} + (b + \beta c)^{[p]} + \sum_{i=1}^{p-1} S_i(a + \alpha c, b + \beta c)$ .  $\square$

1.2.5. *When is  $\mathfrak{l}_\chi$  split?* The extension  $\mathfrak{l}_\chi$  is split as an extension of Lie algebras but not necessarily as an extension of restricted Lie algebras.

1.2.6. *Lemma.* The splittings of the extension  $\mathfrak{l}_\chi$  are in one-to-one correspondence with  $\beta \in \mathfrak{l}^*$  satisfying the equations

$$\beta([x, y]) = 0, \quad (2)$$

$$\beta(y^{[p]}) = \chi(y)^p + \beta(y)^p \quad (3)$$

for each  $x, y \in \mathfrak{l}$ .

1.2.7. *Proof.* Any splitting  $\mathfrak{l} \rightarrow \mathfrak{l}_\chi$  must be of the form  $y \mapsto y + \beta(y)c$  for some  $\beta \in \mathfrak{l}^*$ . It is a map of Lie algebras if and only if  $\beta([x, y]) = 0$ . The splitting preserves the restricted structure if and only if equation (3) holds.  $\square$

1.2.8. *Corollary.* The canonical map  $\mathfrak{l} \hookrightarrow \mathfrak{l}_\chi$  is a restricted Lie algebra splitting if and only if  $\chi = 0$ .

1.2.9. *Corollary.* If  $\mathfrak{l}$  is perfect (i.e.  $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}$ ) then  $\mathfrak{l}_\chi$  is split if and only if  $\chi = 0$ .

1.3. **Connection with universal enveloping algebra.** Our next objective is to give another, more intrinsic, description of  $\mathfrak{l}_\chi$ . Let  $\psi : \mathfrak{l}^* \rightarrow (\text{Spec } \mathcal{O})(\mathbb{K})$  be the natural map defined by  $(x^p - x^{[p]})(\psi(\chi)) = \chi(x)^p$  for all  $x \in \mathfrak{l}$ ,  $\chi \in \mathfrak{l}^*$ .

1.3.1. *The restricted enveloping algebra of  $\mathfrak{g}_m$ .* The algebra  $u(\mathfrak{g}_m)$  is semisimple [5]. Let  $c$  be a basis element of  $\mathfrak{g}_m$  such that  $c^{[p]} = c$ . The elements  $1, c, \dots, c^{p-1}$  form a basis of  $u(\mathfrak{g}_m)$ . Let us define *the Nielsen polynomial*  $\text{Ni}_\eta(c)$  for  $\eta \in \mathbb{F}$ :

$$\text{Ni}_\eta(c) = \begin{cases} -\sum_{n=1}^{p-1} \frac{c^n}{\eta^n} & \text{if } \eta \neq 0 \\ 1 - c^{p-1} & \text{if } \eta = 0 \end{cases}$$

The elements  $\text{Ni}_\eta(c)$  form a complete system of orthogonal idempotents of  $u(\mathfrak{g}_m)$ . The idempotent  $\text{Ni}_\eta(c)$  corresponds to the character  $\rho_\eta$ , defined by  $\rho_\eta(c) = \eta$ , of  $G_m^1$  and  $\rho_\eta(\text{Ni}_\eta(c)) = 1$ .

1.3.2. *Theorem.* Let  $\chi$  be a non-zero element of  $\mathfrak{l}^*$  and  $\nu = \psi(\chi)$ . Then  $U(\mathfrak{l}) \otimes_{\mathcal{O}} \mathcal{O}(\mathbb{F}\nu)$  is isomorphic to  $u(\mathfrak{l}_\chi)$  as a Hopf algebra.

1.3.3. *Proof.* The map of Lie algebras  $\mathfrak{l} \rightarrow \mathfrak{l}_\chi$  given by  $a \mapsto a + 0c$  can be extended to a map of Hopf algebras  $\zeta : U(\mathfrak{l}) \rightarrow u(\mathfrak{l}_\chi)$ . Since  $\chi \neq 0$  the algebra  $u(\mathfrak{l}_\chi)$  is generated by  $\mathfrak{l}$  and the map  $\zeta$  is onto.

On the other hand, there is a natural surjective linear map

$$\theta : U(\mathfrak{l}) \rightarrow U(\mathfrak{l}) \otimes_{\mathcal{O}} \mathcal{O}(\mathbb{F}\nu)$$

given by  $y \mapsto y \otimes 1$ . It follows from Proposition 1.1.2 that  $\theta$  is a Hopf algebra map. The kernel of  $\theta$  is generated by some elements of  $\mathcal{O}$ . It suffices to show that for each  $x \in \mathcal{O}$  such that  $\theta(x) = 0$  it holds that  $\zeta(x) = 0$ . Indeed, this condition will imply that  $\ker \theta \subseteq \ker \zeta$ . Thus, there exists a Hopf algebra map  $\kappa : U(\mathfrak{l}) \otimes_{\mathcal{O}} \mathcal{O}(\mathbb{F}\nu) \rightarrow u(\mathfrak{l}_\chi)$ . It is surjective since so is  $\zeta$ . But both algebras have the same dimension  $p^{N+1}$  where  $N$  is the dimension of  $\mathfrak{l}$ . Thus,  $\kappa$  is an isomorphism.

Let  $l_i$  be a basis of  $\mathfrak{l}$ . Then any  $x \in \mathcal{O}$  has a unique representation as a polynomial in  $l_i^p - l_i^{[p]}$ . We assume  $x = \sum_{\mathfrak{t}} a_{\mathfrak{t}} (l^p - l^{[p]})^{\mathfrak{t}} \in \ker \theta$  in multi-index notation. This means that  $\sum_{\mathfrak{t}} a_{\mathfrak{t}} \eta^{|\mathfrak{t}|} (\chi(l)^p)^{\mathfrak{t}} = 0$  for each  $\eta \in \mathbb{F}$ . We can notice that  $\zeta(x) = \sum_{\mathfrak{t}} a_{\mathfrak{t}} c^{|\mathfrak{t}|} (\chi(l)^p)^{\mathfrak{t}} \in u(\mathfrak{l}_\chi)$ . Finally,  $u(\mathfrak{g}_m)$  is a central semisimple subalgebra of  $u(\mathfrak{l}_\chi)$ . The Pierce decomposition of  $u(\mathfrak{l}_\chi)$  is  $u(\mathfrak{l}_\chi) = \bigoplus_{\eta \in \mathbb{F}} u(\mathfrak{l}_\chi) \text{Ni}_\eta(c)$ . Thus,  $\zeta(x) \text{Ni}_\eta(c) = \sum_{\mathfrak{t}} a_{\mathfrak{t}} \eta^{|\mathfrak{t}|} (\chi(x)^p)^{\mathfrak{t}} = 0$  for every  $\eta \in \mathbb{F}$  and, therefore,  $\zeta(x) = 0$ .  $\square$

1.3.4. The theorem clearly fails for  $\chi = 0$ . However, if one thinks that  $\mathbb{F} \cdot 0$  is not just a point but some infinitesimal neighborhood then the theorem is adjustable to the case of  $\chi = 0$ . For instance, the next corollary holds for every  $\chi$ .

1.3.5. *Corollary.*  $u(\mathfrak{l}_\chi)$  is isomorphic to  $\bigoplus_{i \in \mathbb{F}} U_{i\chi}(\mathfrak{l})$  as an algebra.

1.3.6. A representation of  $\mathfrak{l}_\chi$  has a *central charge*  $\eta \in \mathbb{F}$  if  $c$  acts by  $\eta$ . Representations of  $\mathfrak{l}$  affording  $\chi$  are in one-to-one correspondence with restricted representations of  $\mathfrak{l}_\chi$  with a central charge 1.

The next corollary provides an intrinsic construction of  $\mathfrak{l}_\chi$ . Recall that the set of primitive elements of a Hopf algebra  $H$  is  $P(H) = \{h \in H \mid \Delta h = 1 \otimes h + h \otimes 1\}$ . The corollary follows from the fact that  $P(u(\mathfrak{l})) = \mathfrak{l}$ .

1.3.7. *Corollary.*  $\mathfrak{l}_\chi \cong P(U(\mathfrak{l}) \otimes_{\mathcal{O}} \mathcal{O}(\mathbb{F}\nu))$ .

1.3.8. One can describe properties of  $\mathfrak{l}_\chi$  starting from the construction of  $\mathfrak{l}_\chi$  as the set of primitive elements of the Hopf algebra  $U(\mathfrak{l}) \otimes_{\mathcal{O}} \mathcal{O}(\mathbb{F}\nu)$ . The natural map  $U(\mathfrak{l}) \otimes_{\mathcal{O}} \mathcal{O}(\mathbb{F}\nu) \rightarrow u(\mathfrak{l})$ , restricted to the set of primitive elements, is the extension map  $\mathfrak{l}_\chi \rightarrow \mathfrak{l}$ . This extension has a canonical Lie algebra splitting that does not preserve the restricted structure:  $\mathfrak{l} \hookrightarrow U(\mathfrak{l}) \rightarrow U(\mathfrak{l}) \otimes_{\mathcal{O}(\mathfrak{l})} \mathcal{O}(\mathbb{F}\nu)$  has an image in  $\mathfrak{l}_\chi$ . The element  $c$  is also canonical: it is easy to see that for each  $x \in \mathfrak{l}$  such that  $\chi(x) \neq 0$ , the element  $c = \frac{x^p - x^{[p]}}{\chi(x)^p} \in U(\mathfrak{l}) \otimes_{\mathcal{O}} \mathcal{O}(\mathbb{F}\nu)$  is central and independent of  $x$ . It belongs to  $\mathfrak{l}_\chi$  but not to  $\mathfrak{l}$ .

#### 1.4. Harish-Chandra pairs.

1.4.1. A natural question is to try to find a central extension of algebraic groups  $G_m \rightarrow \hat{L}_\chi \rightarrow L$  affording  $\mathfrak{l}_\chi$  on the tangent level. There is no such central extension for a non-zero  $\chi$  and a semisimple group  $G$  because all central extensions of  $G$  are finite.

1.4.2. For a nilpotent  $\chi \in \mathfrak{g}^*$ , it is possible to add a piece of an algebraic group obtaining a restricted Harish-Chandra pair. Let us consider an algebraic group  $S = St_G(\chi)$ , the stabilizer of  $\chi$  in  $G$ . The centralizer  $C_{\mathfrak{g}}(\chi)$  of  $\chi$  contains the Lie algebra  $\mathfrak{s}$  of  $S$ . We define an embedding of Lie algebras  $\theta : \mathfrak{s} \hookrightarrow \mathfrak{g}_\chi$  through the chain of embeddings

$$\mathfrak{s} \hookrightarrow C_{\mathfrak{g}}(\chi) \hookrightarrow \mathfrak{g} \hookrightarrow \mathfrak{g}_\chi. \quad (4)$$

Using the left adjoint action of  $G$ , we define an action of  $S$  on  $\mathfrak{g}_\chi$  by

$$g \cdot (x \oplus \alpha c) = (g \cdot x) \oplus \alpha c \quad (5)$$

for any  $g \in S$ ,  $x \in \mathfrak{g}$ ,  $\alpha \in \mathbb{K}$ . We have to check the following three items to prove that it is a restricted Harish-Chandra pair.



1. *The embedding  $\theta$  is of restricted Lie algebras.* We need the assumption that  $\chi$  is nilpotent, which means that  $C_{\mathfrak{g}}(\chi) \subseteq \ker(\chi)$  by the definition of a nilpotent element. Given  $a \in C_{\mathfrak{g}}(\chi)$ , we observe that  $(a \oplus 0c)^{[p]} = a^{[p]} \oplus \chi(a)^p c = a^{[p]}$ .

2.  *$S$  acts on  $\mathfrak{g}_\chi$  by restricted Lie algebra automorphisms.* Given  $a \in C_{\mathfrak{g}}(\chi)$ ,  $g \in S$ , and  $\alpha \in \mathbb{K}$ , we observe that  $(g \cdot (a \oplus \alpha c))^{[p]} = (g \cdot a)^{[p]} \oplus (\chi(g \cdot a)^p + \alpha^p) c = g \cdot (a^{[p]} \oplus ((g^{-1} \cdot \chi)(a) + \alpha^p) c) = g \cdot (a \oplus \alpha c)^{[p]}$ .

3. *The actions of  $\mathfrak{s}$  on  $\mathfrak{g}_\chi$ , induced by the action of  $S$  and the embedding  $\mathfrak{s} \hookrightarrow \mathfrak{g}_\chi$ , are the same.* It is true because the representation of  $\mathfrak{g}$  on  $\mathfrak{g}_\chi$  is the sum of trivial and adjoint representations.

## 2. FROBENIUS MORPHISM

The main object of study in this section is a Noetherian algebraic scheme  $X$  over  $\mathbb{K}$ . We view  $X$  from the two viewpoints. On the one hand,  $X$  is a ringed topological space. On the other hand,  $X$  is a functor mapping a commutative  $\mathbb{K}$ -algebra  $R$  to the set  $X(R)$  of points over  $R$ .

### 2.1. Properties of Frobenius morphisms.

2.1.1. *Definition.* Let  $X^{(n)}$  be  $X$  as a scheme (i.e.  $X^{(n)} = X$  as a topological space and  $\mathcal{O}_{X^{(n)}} = \mathcal{O}_X$  as a sheaf of rings) with the new structure over the field:  $X \longrightarrow \text{Spec } \mathbb{K} \xrightarrow{x^{p^{-n}}} \text{Spec } \mathbb{K}$ . *The Frobenius morphism  $F_X$ ,* defined on the level of functions by  $f \mapsto f^p$ , is a morphism of  $\mathbb{K}$ -schema:  $F_X : X \longrightarrow X^{(1)}$ .

2.1.2. *Frobenius morphism for a smooth scheme.* The Frobenius morphism  $F_X$  is never smooth. It is flat if and only if  $X$  is smooth by Kunz theorem [16]. The following proposition is a technical fact about the Frobenius morphism, crucial for further study. It would be interesting to know whether Proposition 2.1.3 holds true for some singular variety.

Intuitively, the proposition says that the Frobenius map is locally surjective on points over rings. It holds if one replaces the Frobenius map by any faithfully flat finitely presented map.

2.1.3. *Proposition.* Let  $X$  be a smooth algebraic variety and  $R$  be a commutative  $\mathbb{K}$ -algebra. For each  $h \in X^{(1)}(R)$  there exist a faithfully flat finitely presented  $R$ -algebra  $\tilde{R}$  and  $y \in X(\tilde{R})$  such that  $F_X(\tilde{R})(y) = X^{(1)}(\varphi)(h)$  where  $\varphi : R \rightarrow \tilde{R}$  is the natural map.

2.1.4. *Proof.* The Frobenius morphism  $F_X$  is flat by the Kunz theorem. It is faithfully flat because it is surjective on the level of points over  $\mathbb{K}$ .

We assume that  $X$  is affine without loss of generality since the question is local. Denote  $A = \mathcal{O}(X)$ ; the Frobenius morphism is given by the  $p$ -th power map  $F : A^{(1)} \rightarrow A$ , where  $A^{(1)} = \mathcal{O}(X^{(1)})$  by definition. The point  $h$  is a map of  $\mathbb{K}$ -algebras  $A^{(1)} \rightarrow R$ . The  $R$ -algebra  $\tilde{R} = R \otimes_{A^{(1)}} A$  is faithfully flat by [2, 1.2.2.5] and obviously finitely presented (it has the same generators and relations over  $R$  as  $A$  over  $A^{(1)}$ ). The natural map  $A \rightarrow \tilde{R}$  is the point  $y$  we are looking for.  $\square$

**2.2. Frobenius neighborhoods.** We define Frobenius neighborhoods and consider Frobenius kernels as an example of this phenomenon.

2.2.1. *Definition.* Let  $Y$  be a closed subscheme of  $X$ ; then  $Y^{(1)}$  is naturally a subscheme of  $X^{(1)}$ . Our main concern in this section is the inverse image subscheme  $F_X^{-1}(Y^{(1)})$ , *the Frobenius neighborhood of  $Y$  in  $X$* . We denote it by  $\hat{Y}$ . This notation is ambiguous because it is unclear in which  $X$  it is taken.

Assume  $Y$  is a closed subscheme of an affine scheme  $X$ , determined by equations  $f_1 = 0, \dots, f_m = 0$ . The ideal of  $\hat{Y}$  is generated by  $f_i^p$ . Thus,  $\hat{Y}$  lies in the  $p$ -th infinitesimal neighborhood of  $Y$  and contains the first infinitesimal neighborhood.

2.2.2. *Frobenius kernels.* An interesting choice of  $X$  and  $Y$  is  $X = L$ , an algebraic group, and  $Y = \{e\}$ , the reduced identity element. The functoriality of Frobenius morphism implies that  $L^{(1)}$  is an algebraic group and  $F_L$  is a map of algebraic group schema. The neighborhood  $\hat{Y}$  is the kernel of  $F_L$ , which is an infinitesimal finite group scheme (called *the first Frobenius kernel*). It will be denoted  $L^1$ .

Since  $\mathcal{O}(L^1) \cong u(\mathfrak{l})^*$  [11, 1.9.6], the  $u(\mathfrak{l})$ -modules coincide with  $\mathcal{O}(L^1)$ -comodules, i.e. with  $L^1$ -modules.

2.2.3. *Frobenius neighborhoods in an  $L$ -variety.* The Frobenius kernel  $L^1$  acts on the Frobenius neighborhood  $\hat{Y}$  of any subvariety  $Y$  because of the functoriality of the Frobenius morphism. To prove this, pick  $g \in L^1(R)$  and  $x \in \hat{Y}(R)$ . We have to show that  $gx \in \hat{Y}(R)$ . The latter means that  $F_Z(gx) \in Y^{(1)}(R)$ . But  $F_Z(gx) = F_L(g)F_Z(x)$  because of the functoriality. Now we finish the computation  $F_Z(gx) = F_L(g)F_Z(x) = 1_L F_Z(x) = F_Z(x) \in Y^{(1)}(R)$ .

2.2.4. *Central extensions  $L^\chi$  of Frobenius kernels.* It is interesting that  $U_\chi(\mathfrak{l})$ -modules can be also understood in a similar spirit. They are representations of a certain central extension of  $L^1$ . The central extension

$$0 \rightarrow \mathfrak{g}_m \rightarrow \mathfrak{l}_\chi \rightarrow \mathfrak{l} \rightarrow \mathcal{O}$$

gives rise to an exact sequence in the category of Hopf algebras

$$\mathbb{K} \rightarrow u(\mathfrak{g}_m) \xrightarrow{\alpha} u(\mathfrak{l}_\chi) \rightarrow u(\mathfrak{l}) \rightarrow \mathbb{K}.$$

It is central in a sense that  $u(\mathfrak{g}_m)$  lies in the center of  $u(\mathfrak{l}_\chi)$ . The kernel of  $\alpha$  is an ideal generated by  $\mathfrak{g}_m$  inside  $u(\mathfrak{l}_\chi)$ . We dualize the sequence:

$$\mathbb{K} \leftarrow u(\mathfrak{g}_m)^* \xleftarrow{\beta} u(\mathfrak{l}_\chi)^* \leftarrow u(\mathfrak{l})^* \leftarrow \mathbb{K}. \quad (6)$$

The centrality of  $\mathfrak{g}_m$  in  $\mathfrak{l}_\chi$  amounts to the fact that  $b_1 \otimes \beta(b_2) = \beta(b_1) \otimes b_2$  for each  $b \in u(\mathfrak{l}_\chi)^*$ . The algebra extension  $u(\mathfrak{l}_\chi)^* \supseteq u(\mathfrak{l})^*$  is  $u(\mathfrak{g}_m)^*$ -Galois [19, 22]. Noticing that  $u(\mathfrak{g}_m)^* \cong \mathbb{K}\mathbb{Z}_p$ , the Galois condition means that  $u(\mathfrak{l}_\chi)^*$  is a  $\mathbb{Z}_p$ -graded algebra such that  $u(\mathfrak{l}_\chi)_s^* u(\mathfrak{l}_\chi)_{s-1}^* = u(\mathfrak{l}_\chi)_e^* = u(\mathfrak{l})^*$  for all  $s \in \mathbb{Z}_p$  where  $e \in \mathbb{Z}_p$  is the identity element [19]. Applying the functor  $\text{Spec}$  to sequence (6), we arrive at a central extension of finite infinitesimal group schema:

$$1 \rightarrow G_m^1 \rightarrow L^\chi \xrightarrow{\pi} L^1 \rightarrow 1$$

where  $L^\chi$  is the spectrum of  $u(\mathfrak{l}_\chi)^*$ , by definition.

2.2.5. *Lemma.* For each  $\eta \in \mathbb{F}$ , there exists an invertible element  $f \in u(\mathfrak{l}_\chi)^*$  such that  $f(xa) = a^\eta f(x)$  for all  $a \in G_m^1(R)$ ,  $x \in L^\chi(R)$ , and any commutative  $\mathbb{K}$ -algebra  $R$  (note that  $a^\eta$  is well-defined since  $a \in G_m^1(R) = \{r \in R \mid r^p = 1\}$ ).

2.2.6. *Proof.* The element  $\rho = \rho_\eta \in u(\mathfrak{g}_m)^*$  is group-like (i.e.  $\Delta(\rho) = \rho \otimes \rho$ ) since it is a representation. Rewriting  $f(xa) = a^\eta f(x) = f(x)\rho(a)$ , we realize that we are looking for an invertible element  $f$  such that  $f_1 \otimes \beta(f_2) = f \otimes \rho$ , i.e.  $f$  is homogeneous of degree  $\rho$ . The algebra  $u(\mathfrak{l}_\chi)^*$  is local. As a result,  $f$  is invertible if and only if  $\varepsilon(f) \neq 0$ . The Galois condition [19, Theorem 8.1.7] implies that

$$1 \in u(\mathfrak{l}_\chi)_\rho^* u(\mathfrak{l}_\chi)_{\rho^{-1}}^* \quad (7)$$

where  $u(\mathfrak{l}_\chi)_\rho^*$  denotes the subspace of  $\rho$ -homogeneous elements. If no such  $f$  exists then  $\varepsilon(u(\mathfrak{l}_\chi)_\rho^*) = 0$ , which contradicts (7).  $\square$

2.2.7. *Harish-Chandra pairs.* The Harish-Chandra pair  $(S, \mathfrak{g}_\chi)$ , constructed in 1.4.2, is a central extension of another pair  $(S, \mathfrak{g})$ , which can be interpreted as a Frobenius neighborhood  $\widehat{S}$  of  $S$  in  $G$  since they have the same categories of representations. Similarly, one can interpret the pair  $(S, \mathfrak{g}_\chi)$  as a central extension of  $\widehat{S}$  by  $G_m^1$ .

## 3. GROUPOIDS

**3.1. Basics.** We discuss groupoids and their relevance to Frobenius neighborhoods. We follow [18] for groupoid and Lie algebroid terminology.

**3.1.1. Groupoid scheme.** A groupoid  $J$  over a scheme  $X$  is a scheme  $J$  over  $X \times X$ , equipped with morphisms

$$m : J^{[2]} = J \times_X J \rightarrow J, \quad \iota : X \rightarrow J, \quad {}^{-1} : J \rightarrow J$$

of multiplication, identity that is a closed embedding, and inversion such that for any commutative ring  $R$  the set  $J(R)$  is a groupoid with the base  $X(R)$  under the structure maps  $m(R)$ ,  $\iota(R)$ , and  ${}^{-1}(R)$ . Moreover, for any algebra homomorphism  $\mu : R \rightarrow R'$ , the map  $J(\mu) : J(R) \rightarrow J(R')$  must be a map of groupoids. If the  $X \times X$ -structure on  $J$  is given by  $(\mathfrak{A}, \mathfrak{P}) : J \rightarrow X \times X$  then the fiber product  $J^{[2]} = J \times_X J$  is taken using  $\mathfrak{P}$  in the first position and  $\mathfrak{A}$  in the second position.

**3.1.2. Quotients.** A groupoid  $J$  over  $X$  acts on an  $X$ -scheme  $Y$  if a morphism

$$\star : (J \xrightarrow{\mathfrak{P}} X) \times_X Y \rightarrow Y$$

is given satisfying associativity and unitarity conditions. For any  $\mathbb{K}$ -algebra  $R$ , an equivalence relation  $\sim$  on  $Y(R)$  is

$$x \sim y \iff \exists g \in J(R) \mid g \star x = y.$$

Then  $Y/J$  is a sheaf in the flat topology on the category of  $\mathbb{K}$ -algebras associated to the presheaf  $R \mapsto Y(R)/\sim$ .

If  $Y = X$  and  $\star = \mathfrak{A}$  then  $X/J$  is a quotient by a groupoid as defined in [2].

**3.1.3. Action groupoid.** A group scheme  $L$  action on a scheme  $Y$  gives rise to the action groupoid  $J_X$  for each closed subscheme  $X$  of  $Y$ . Note that  $X$  need not be invariant under the  $L$ -action. If  $a : L \times Y \rightarrow Y$  is the action map then  $J_X$  is the inverse image scheme:  $J_X = (a|_{L \times X})^{-1}(X)$ . In other words,  $J_X(R) = \{(g, x) \in L(R) \times X(R) \mid g \cdot x \in X(R)\}$ . The product  $m((g, x), (h, y)) = (gh, y)$  is defined whenever  $x = h \cdot y$ .

**3.1.4. Product groupoid.** Given a groupoid  $J$  over a scheme  $Y$  and a group scheme  $L$ , one can form a product groupoid  $J \times L$  over  $Y$ . It is the product scheme with the structure maps

$$\begin{aligned} \mathfrak{A}'(g, l) &= \mathfrak{A}(g), \quad \mathfrak{P}'(g, l) = \mathfrak{P}(g), \quad m'((g, l), (g', l')) = (m(g, g'), ll'), \\ l'(x) &= (\iota(x), 1_L), \quad (x, l)^{-1} = (x^{-1}, l^{-1}) \end{aligned}$$

for all  $g, g' \in J(R)$ ,  $l, l' \in L(R)$ , and  $x \in Y(R)$ .

3.1.5. *Central extension of a groupoid.* A central extension by an Abelian group scheme  $A$  of groupoid  $J$  over  $X$  is a quotient map  $\pi : J' \rightarrow J$  that is a morphism of groupoids over  $\text{Id}_X$ . Moreover, an isomorphism must be given between the kernel  $\pi^{-1}(\iota(X))$  and the group scheme  $A \times X$ , and the following centrality condition holds. The equality

$$m(g, (a, \mathfrak{P}(g))) = m((a, \mathfrak{A}(g)), g) \quad (8)$$

must hold for each  $g \in \mathcal{L}^X(R)$ ,  $a \in A(R)$ .

3.1.6. *Example.* Let an algebraic group  $L$  act on an algebraic variety  $Y$ . The central extension  $L^\chi$  acts on  $Y$  through  $L^\chi \rightarrow L^1 \hookrightarrow L$ . For each  $X$ , a subscheme of  $Y$ , the action groupoid  $\mathcal{L}_X^\chi$  of  $L^\chi$  is a central extension of the action groupoid  $\mathcal{L}_X^1$  of  $L^1$ :

$$G_m^1 \times X \rightarrow \mathcal{L}_X^\chi \rightarrow \mathcal{L}_X^1. \quad (9)$$

3.1.7. *Proposition.* If  $Y$  is a homogeneous  $L$ -variety and  $X$  is a smooth subvariety then  $\widehat{X}$  is isomorphic to both the quotient of  $L^1 \times X$  by the groupoid  $\mathcal{L}_X$  and the quotient of  $L^\chi \times X$  by the groupoid  $\mathcal{L}_X^\chi$  for each  $\chi \in \mathfrak{l}^*$ .

3.1.8. *Proof.* There is at least one point in  $Y(\mathbb{K})$  since  $\mathbb{K}$  is algebraically closed. Thus, we can assume that  $Y = L/H$  for some closed subgroup  $H$ . To treat  $L^\chi$  and  $L^1$  together, we speak about an infinitesimal group scheme  $L^\zeta$  and a groupoid  $\mathcal{L}_X^\zeta$ . First we show that the action is a quotient map and then we write down an action of the groupoid  $\mathcal{L}_X^\zeta$ .

Thinking of schemes as functors from the category of  $\mathbb{K}$ -algebras to the category of sets, we notice that the image of the action  $L^\zeta \cdot X$  is a subfunctor of  $\widehat{X}$ . We need to show that it is a ‘‘plump’’ subfunctor [2, 3.1.1.4], which means that  $\widehat{X}$  is a sheaf associated to  $L^\zeta X$ . Reiterating the argument before Proposition 3.1.7, we notice that  $F_Y(gx) = F_L(\bar{g})F_Y(x)$  for each  $g \in L^\zeta(R)$  (where  $\bar{g}$  is the image of  $g$  in  $L^1(R)$ ),  $x \in X(R)$  and every  $\mathbb{K}$ -algebra  $R$ . Thus,  $\widehat{X}(R) \supseteq L^\zeta(R) \cdot X(R)$ . If  $\widetilde{L^\zeta X}$  is a sheaf associated to  $L^\zeta X$  then  $\widehat{X}(R) \supseteq \widetilde{L^\zeta X}(R) \supseteq L^\zeta(R)X(R)$  for any ring  $R$  since  $L^\zeta \cdot X$  is a subfunctor of a sheaf.

Let us pick  $y \in \widehat{X}(R)$ . We will construct a chain of faithfully flat finitely presented algebras  $R \rightarrow R_1 \rightarrow R_2 \rightarrow R_3 \rightarrow R_4$  and elements  $g_4 \in L^\zeta(R_4)$  and  $x_4 \in X(R_4)$  such that  $y_4 = g_4 x_4$  where  $y_i = \widehat{X}(\pi_i)(y)$  for  $\pi_i : R \rightarrow R_i$ . This proves that the action  $L^\zeta \times X \rightarrow \widehat{X}$  is a quotient map.

By Proposition 2.1.3, there exist  $R_1$  and  $x_1 \in X(R_1)$  such that  $F(x_1) = F(y_1)$ . We should notice that this is the place that we use the assumption of  $X$  being smooth. By the definition of  $L/H$ , there exist

$R_2$  and  $a_2, b_2 \in L(R_2)$  such that  $y_2 = a_2 H(R_2)$  and  $x_2 = b_2 H(R_2)$ . The elements  $F_L(a_2)$  and  $F_L(b_2)$  lie in the same coset. Thus, there exists  $z_2 \in H^{(1)}(R_2)$  such that  $F_L(a_2) = F_L(b_2)z_2$ . By Proposition 2.1.3 used for  $H$  (any algebraic group is smooth!), there exist  $R_3$  and  $h_3 \in H(R_3)$  such that  $z_3 = F_H(h_3) = F_L(h_3)$ . Thus,  $F(a_3) = F(b_3)F(h_3) = F(b_3 h_3)$ . Let  $f_3 = a_3 h_3^{-1} b_3^{-1}$ . It is clear that  $f_3 \in L^1(R_3)$  and  $f_3 x_3 = y_3$ .

If  $? = 1$  then we set  $R_4 = R_3$  and  $g_4 = f_3$ . If, on the other hand,  $? = \chi$  then there exists  $g_4 \in G^\chi(R_4)$  such that  $\widetilde{g_4} = f_4$  since  $G^\chi \rightarrow G^1$  is a quotient map. This proves that  $\widehat{X} = L^? \cdot X$ .

We define an  $\mathcal{L}_X^?$ -action on  $L^? \times X$  by

$$(g, x) \star (h, x) = (hg^{-1}, gx).$$

$L^? X$  is a quotient functor of  $L^? \times X$  by the relation

$$(g, x) \sim (h, y) \iff gx = hy.$$

But it is equivalent to the condition  $(h, y) = t \star (g, x)$  where  $t = (h^{-1}g, x) \in \mathcal{L}_X^?(R)$ .

Thus,  $L^? \cdot X$  is a quotient functor of  $L^? \times X$  by the groupoid  $\mathcal{L}_X$  or  $\mathcal{L}_X^\chi$  correspondently. This implies that  $\widehat{X} = \widetilde{L^? \cdot X}$  is a quotient sheaf  $L^? \times X / \mathcal{L}_X^?$  on the category of  $\mathbb{K}$ -algebras.  $\square$

### 3.2. Lie Algebroids.

We discuss Lie theory of groupoids.

**3.2.1. Definition.** Intuitively, a Lie algebroid is a tangent structure to a groupoid [18]. In positive characteristic, such structure is equipped with a  $p$ -th power map that was axiomatized by Hochschild [7].

A *restricted Lie algebroid*  $\mathbb{L}$  on a scheme  $X$  is a quasicoherent  $\mathcal{O}_X$ -module that carries a structure of a sheaf of restricted Lie algebras over  $\mathbb{K}$ . It must be equipped with an anchor map  $\mathcal{A} : \mathbb{L} \rightarrow TX$  that is a morphism of both  $\mathcal{O}_X$ -modules and sheaves of restricted Lie algebras. Furthermore, it must satisfy the following identities for sections  $u \in \mathcal{O}_X(V)$ ,  $x, y \in \mathbb{L}(V)$  on an open subset  $V$  of  $X$ :

$$\begin{aligned} [x, uy] &= u[x, y] + \mathcal{A}(x)(u)y, \\ (ux)^{[p]} &= u^p x^{[p]} + \mathcal{A}(ux)^{p-1}(u)x. \end{aligned} \quad (10)$$

For instance, a restricted Lie algebra is a Lie algebroid over a point. Another example of a Lie algebroid is the tangent bundle  $TX$ . The first relation of (10) is obvious in this case. The second one follows from Hochschild's lemma [7, Lemma 1].

3.2.2. *Lie algebroid of a groupoid.* The Lie algebroid of a groupoid scheme  $J$  over  $X$  is the normal sheaf  $N_{J|X}$  to the identity morphism  $\iota : X \rightarrow J$ . Quoting [1], “one defines the Lie bracket and projection by usual formulas”, which one can find in [23].

3.2.3. *Lie algebroid of an action groupoid.* We consider a group scheme  $L$  acting on a scheme  $Y$ . We would like to understand the Lie algebroid  $\mathbb{L}_X$  of the action groupoid of  $L$  on  $X$  for a closed subscheme  $X \subseteq Y$  (see 3.1.3). It is easy to see that  $\mathbb{L}_Y = \mathcal{O}_Y \otimes \mathfrak{l}$  as a sheaf with operations easily computable by formulas (10).

In general, for on an open affine  $V \subseteq X$ , pick an affine open subset  $V' \subseteq Y$  such that  $V = X \cap V'$ , then

$$\mathbb{L}_X(V) = \{v|_V \mid v \in (\mathcal{O}_Y \otimes \mathfrak{l})(V') \text{ \& } \mathcal{A}(v) \text{ is tangent to } X\}. \quad (11)$$

3.2.4. *Central extensions of Lie algebroids.* Let  $L$  be an algebraic group acting on a variety  $Y$ . Assume  $X$  is a subscheme of  $Y$ . We have a central extension (9) of action groupoids  $\pi : \mathcal{L}_X^\chi \rightarrow \mathcal{L}_X$ . Their tangent Lie algebroids  $\mathbb{L}_{\chi,X}$  and  $\mathbb{L}_X$  form a central extension of restricted Lie algebroids on  $X$ :

$$0 \rightarrow \mathfrak{g}_m \otimes \mathcal{O}_X \rightarrow \mathbb{L}_{\chi,X} \xrightarrow{d\pi} \mathbb{L}_X \rightarrow 0. \quad (12)$$

3.2.5. *Proposition.* Let  $L$  be a linear algebraic group and  $Y$  be a homogeneous  $L$ -variety. For any smooth subvariety  $X$ , the Lie algebroid  $\mathbb{L}_X$  is a vector subbundle of  $\mathcal{O}_X \otimes \mathfrak{l}$ . Similarly,  $\mathbb{L}_{\chi,X}$  is a vector subbundle of  $\mathcal{O}_X \otimes \mathfrak{l}_\chi$  for each  $\chi \in \mathfrak{l}^*$ .

3.2.6. *Proof.* To prove the first statement, we show that the quotient sheaf  $\mathcal{O}_X \otimes \mathfrak{l} / \mathbb{L}_X$  is locally free. Then  $\mathcal{O}_X \otimes \mathfrak{l}$  is locally a direct sum of  $\mathbb{L}_X$  and the quotient sheaf since vector bundles are projective objects in the category of  $\mathcal{O}$ -modules on an affine variety by the Serre theorem.

Let  $J$  be the action groupoid of  $L$  on  $X$ . The groupoid  $\mathcal{L}_X$  is the Frobenius neighborhood of  $\iota(X)$  in the groupoid  $J$ . This can be easily seen because of the functoriality of Frobenius morphism: points of both  $\mathcal{L}_X$  and  $\widehat{\iota(X)}$  over  $R$  are such  $(g, x) \in L(R) \times X(R)$  that  $F_L(g) = 1$ . Thus, the Lie algebroids of  $J$  and  $\mathcal{L}_X$  coincide, since a normal bundle is determined by the first order neighborhood that is contained in the Frobenius neighborhood. The quotient sheaf  $\mathcal{O}_X \otimes \mathfrak{l} / \mathbb{L}_X$  is the normal bundle  $N_{L \times X | J}$  restricted to  $X$ , which is a subvariety of  $J$  under  $\iota$ . It suffices to show that  $J$  is smooth, since a restriction of a locally free sheaf is locally free and a normal sheaf of an embedding of smooth varieties is locally free.

Since  $Y$  is an  $L$ -homogeneous variety, the action morphism  $a : L \times X \rightarrow Y$  is a submersion and, therefore, smooth by [4, Proposition 3.10.4]. The morphism  $\mathfrak{A} : J = a^{-1}(X) \rightarrow X$  is smooth, being a base change of  $a$  [4, Proposition 3.10.1]. Since  $X$  is smooth, then so is  $J$ .

Now we prove the second statement. The sheaf  $\mathbb{L}_{\chi, X}$  is a direct sum of  $\mathbb{L}_X$  and the trivial sheaf  $\mathcal{O}_X \otimes \mathfrak{g}_m$ . Thus, the quotient sheaves  $(\mathcal{O}_X \otimes \mathfrak{l})/\mathbb{L}_X$  and  $(\mathcal{O}_X \otimes \mathfrak{l}_{\chi})/\mathbb{L}_{\chi, X}$  are the same.  $\square$

### 3.3. Split extensions.

3.3.1. *Definition.* A central extension of groupoids  $\mathcal{G}' \rightarrow \mathcal{G}$  by an Abelian group scheme  $A$  is called *split* if it is isomorphic to the extension  $\mathcal{G} \times A \rightarrow \mathcal{G}$ .

3.3.2. *Lemma* [23]. The following statements about a central extension of groupoids  $\mathcal{G}' \xrightarrow{\pi} \mathcal{G}$  by  $A$  over a scheme  $Y$  are equivalent.

1. The extension is split.
2. There exists a groupoid map  $\mu : \mathcal{G} \rightarrow \mathcal{G}'$  such that  $\pi \circ \mu = \text{Id}_{\mathcal{G}}$ .
3. There exists a groupoid map  $\nu : \mathcal{G}' \rightarrow A \times X \times X$  such that  $\nu(g, x) = (g, x, x)$  for each  $(g, x) \in \ker \pi(R)$ .
4. There exists a groupoid map  $\xi : \mathcal{G}' \rightarrow A$ , lying over the morphism  $Y \rightarrow \text{Spec } \mathbb{K}$ , such that  $\xi(g, x) = g$  for each  $(g, x) \in \ker \pi(R)$ .

3.3.3. *Theorem.* Let a linear algebraic group  $L$  act on a smooth algebraic variety  $Y$  over  $\mathbb{K}$ . Let  $X$  be a smooth subvariety of  $Y$  and  $\chi \in \mathfrak{l}^*$  such that the canonical splitting of morphism  $d\pi_X$  in (12) is a map of restricted Lie algebras. Then the central extension (9) of action groupoids  $\pi_X : \mathfrak{L}_X^{\chi} \rightarrow \mathfrak{L}_X$  is split.

3.3.4. *Proof.* A Hopf algebroid  $H(J)$  of a groupoid  $J$  over  $X$  is the push-forward sheaf  $(\mathfrak{A}, \mathfrak{P})_{\circ}(\mathcal{O}_J)$ . It is a sheaf of commutative algebras on  $X \times X$ , whose local structure is described in [24]. If the morphism  $(\mathfrak{A}, \mathfrak{P})$  is affine, which is the case with action groupoids of affine group schema, then the groupoid can be recovered from its Hopf algebroid as a relative spectrum. The morphism  $\pi_X$  determines a morphism of Hopf algebroids  $\pi_X^{\#} : H(\mathfrak{L}_X) \rightarrow H(\mathfrak{L}_X^{\chi})$ . Thus, to split  $\pi_X$ , it suffices to construct a morphism of Hopf algebroids splitting  $\pi_X^{\#}$ .

The splitting of restricted Lie algebroids determines a morphism of restricted enveloping  $\mathcal{O}_X$ -algebras  $\zeta : u(\mathbb{L}_X) \rightarrow u(\mathbb{L}_{\chi, X})$ . The left dual morphism  ${}^*\zeta$  is the splitting of  $\pi_X^{\#}$  [24, Corollary 12] because of canonical isomorphisms  $H(\mathfrak{L}_X) \cong {}^*u(\mathbb{L}_X)$  and  $H(\mathfrak{L}_X^{\chi}) \cong {}^*u(\mathbb{L}_{\chi, X})$ .

The argument in [24] is local but the canonical isomorphisms are defined globally since the construction behaves well under localizations. The “ $O$ -good” condition, used in [24], is that the quotient sheaves



$(\mathcal{O}_X \otimes \mathfrak{l})/\mathbb{L}_X$  and  $(\mathcal{O}_X \otimes \mathfrak{l}_\chi)/\mathbb{L}_{\chi,X}$  are locally free. It is shown in the proof of Proposition 3.2.5.  $\square$

3.3.5. Now we choose a connected simply-connected semisimple algebraic group  $G$  as the algebraic group  $L$ . The functional  $\chi$  is nilpotent. The  $G$ -homogeneous variety is the flag variety  $\mathcal{B}$ .  $X$  is a subscheme of  $\mathcal{B}$ . We use notation  $\pi_X : \mathcal{G}_X^\chi \rightarrow \mathcal{G}_X$  for the central extension (9) of action groupoids and  $d\pi_X : \mathbb{G}_{\chi,X} \rightarrow \mathbb{G}_X$  for the central extension (12) of Lie algebroids.

3.3.6. *Infinitesimal splitting condition.* The Lie algebroid  $\mathbb{G}_{\chi,X}$  is equal to  $\mathbb{G}_X \oplus (\mathfrak{g}_m \otimes \mathcal{O}_X)$ . The inclusion  $\gamma_X$  of  $\mathbb{G}_X$  into  $\mathbb{G}_{\chi,X}$  is a splitting on the level of Lie algebroids. We say that *the infinitesimal splitting condition* holds for a subvariety  $X$  if  $\gamma_X$  is a morphism of restricted Lie algebroids. The infinitesimal splitting condition implies that  $X$  is a subscheme of  $\mathcal{B}^\times$ , which is equivalent to  $\gamma_X$  being a splitting on the diagonal by Corollary 1.2.8. We are going to use the action map  $\mu : \mathfrak{g} \rightarrow T\mathcal{B}$  in the next proposition.

3.3.7. *Proposition.* Let  $X$  be a subscheme of  $\mathcal{B}^\times$  such that the following condition holds for each Borel subalgebra  $\mathfrak{b} \in X(\mathbb{K})$ : if  $y$  is an element of  $\mathfrak{g}$  such that the tangent vector  $\mu(y)_\mathfrak{b}$  defined by  $y$  at the point  $\mathfrak{b}$  is tangent to  $X$  then  $\chi(y) = 0$ . Under this condition the map  $\gamma_X : \mathbb{G}_X \rightarrow \mathbb{G}_{\chi,X}$  is a morphism of restricted Lie algebroids.

3.3.8. *Proof.* Let  $V$  be an open subset of  $X$ . Pick  $\sum_i F_i \otimes x_i$  with  $F_i \in \mathcal{O}_X(V)$ ,  $x_i \in \mathfrak{g}$  such that  $\mathcal{A}(\sum_i F_i \otimes x_i)$  is tangent to  $X$ . Denoting the  $p$ -th power in  $\mathbb{G}_{\chi,X}$  by  $^{(p)}$ , we compute by formulas (10).

$$\begin{aligned} \left(\sum_i F_i \otimes x_i\right)^{(p)} &= \sum_i (F_i \otimes x_i)^{(p)} + \dots = \\ \sum_i (F_i^p \otimes x_i^{(p)} + \mathcal{A}(F_i x_i)^{p-1}(F_i x_i)) &+ \dots = \\ \sum_i (F_i^p \otimes \chi(x_i)^p c + F_i^p \otimes x_i^{[p]} + \mathcal{A}(F_i x_i)^{p-1}(F_i x_i)) &+ \dots = \\ \left(\sum_i F_i \otimes x_i\right)^{[p]} + \left(\sum_i F_i \chi(x_i)\right)^p \otimes c & \end{aligned}$$

where  $\dots$  denote the terms coming from the formula for  $p$ -th degree of a sum in an associative algebra. These terms depend on the adjoint representation only and, therefore, are the same for  $^{(p)}$  and  $^{[p]}$ .

This argument shows that we have to check that  $\sum_i F_i \chi(x_i) = 0$ . We check this condition pointwise. Pick  $\mathfrak{b} \in X(\mathbb{K})$ . Let  $y = \sum_i F_i(\mathfrak{b})x_i \in \mathfrak{g}$ . It suffices to deduce  $\chi(y) = 0$  from  $\mathcal{A}(y)$  being tangent to  $X$ , which is the assumption of the proposition.  $\square$

3.3.9. *Lemma.* Any partial flag variety lying in  $\mathcal{B}^\times$  satisfies the infinitesimal splitting condition.

3.3.10. *Proof.* Pick a parabolic subalgebra and a Borel subalgebra  $\mathfrak{p} = \text{Lie } P \supseteq \mathfrak{b}$ . Assume that the partial flag variety  $X = P \cdot \mathfrak{b}$  lies in  $\mathcal{B}^\times$ . This implies that  $\chi$  vanishes on  $\mathfrak{p}$ . But the vector field  $\mu(y)$  is tangent to  $X$  if and only if  $y \in \mathfrak{p}$ . We are done by Proposition 3.3.7.  $\square$

3.3.11. If  $X$  is not a partial flag variety then the tangency to  $X$  condition is difficult to put hands on. But if  $\mu(y)$  is tangent to  $X \subseteq \mathcal{B}^\times$  then it is also tangent to  $\mathcal{B}^\times$ , which implies that  $\chi([y, \mathfrak{b}]) = 0$  for each  $\mathfrak{b} \in X(\mathbb{K})$ . We investigate when the latter condition implies  $\chi(y) = 0$ .

3.3.12. *Lemma.* If every  $\mathfrak{b} \in X(\mathbb{K})$  contains an element  $h$  such that  $\text{ad}^*(h)\chi = \chi$  then  $X$  satisfies the infinitesimal splitting condition.

3.3.13. *Proof.* We just need to note that the pairing  $\mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{K}$  is  $\mathfrak{g}$ -invariant. Since  $\chi([y, \mathfrak{b}]) = 0$ , for the choice of  $h$  as explained,  $0 = \chi([y, h]) = \text{ad}^*h(\chi)(y) = \chi(y)$ .  $\square$

#### 4. EQUIVARIANT SHEAVES AND REPRESENTATIONS

We introduce the geometric construction of  $U_\chi(\mathfrak{g})$ -modules in this section.

4.1. **Equivariant sheaves.** Sheaves equivariant for groupoids provide a proper framework for constructing  $U_\chi(\mathfrak{g})$ -modules.

4.1.1. *Definition.* We consider a groupoid  $J$  over an algebraic scheme  $Y$ . We notice that a groupoid structure gives rise to three maps  $t_1, t_2, m : J^{[2]} \rightarrow J$ . The maps  $t_1$  and  $t_2$  are the projections to the first and second component. A  $J$ -equivariant sheaf is an  $\mathcal{O}$ -module  $\mathcal{F}$  on  $Y$  with an additional structure, namely, an isomorphism  $I : \mathfrak{P}^\circ(\mathcal{F}) \rightarrow \mathfrak{A}^\circ(\mathcal{F})$  of  $\mathcal{O}$ -modules on  $J$  such that

$$I|_{i(Y)} : \mathcal{F} = \mathfrak{P}^\circ(\mathcal{F})|_{i(Y)} \rightarrow \mathfrak{A}^\circ(\mathcal{F})|_{i(Y)} = \mathcal{F}$$

is the identity map and  $t_1^\circ I \circ t_2^\circ I = m^\circ I$ . The inverse images are taken in the category of  $\mathcal{O}$ -modules.

4.1.2. *Action on fibers.* A  $J$ -equivariant structure gives rise to the action of  $J$  on the fibers. Indeed, for each  $g \in J$  one obtains an isomorphism  $I_g : \mathcal{F}_{\mathfrak{P}(g)} = (\mathfrak{P}^\circ \mathcal{F})_g \rightarrow \mathcal{F}_{\mathfrak{A}(g)} = (\mathfrak{A}^\circ \mathcal{F})_g$ .

4.1.3. An  $L$ -equivariant bundle  $\mathcal{F}$  may be utilized to construct a large family of  $L^1$ -modules. Let  $X$  be a subscheme of  $Y$ . Then  $\Gamma(\widehat{X}, \mathcal{F}|_{\widehat{X}})$  carries a structure of an  $L^1$ -module (and, therefore, a  $u(\mathfrak{l})$ -module).

## 4.2. Central charge of an equivariant vector bundle.

4.2.1. *Definition.* Let  $J' \xrightarrow{\pi} J$  be a central extension by  $G_m^1$  of groupoids over a scheme  $X$ . We say that a  $J'$ -equivariant vector bundle  $\mathcal{F}$  has a *central charge*  $\eta \in \mathbb{F}$  if  $G_m^1$  acts on  $\mathcal{F}$  by the character  $\rho_\eta$ .

4.2.2. If the extension of groupoids  $\pi : J' \rightarrow J$  is split, we can modify a  $J$ -equivariant structure on a bundle  $\mathcal{F}$  into a  $J'$ -equivariant structure with any central charge  $\eta \in \mathbb{F}$ . Let  $\mathcal{O}_X^\eta$  be the trivial line bundle with a  $G_m^1$ -equivariant structure given by  $\rho_\eta$ . Thus, the tensor product  $\mathcal{O}_X^\eta \otimes \mathcal{F}$  carries a canonical  $J'$ -equivariant structure with central charge  $\eta$ .

4.2.3. *Theorem.* Let  $Y$  be a homogeneous  $L$ -variety and  $\mathcal{F}$  be an  $L^1$ -equivariant vector bundle on  $Y$ . We consider  $\chi \in \mathfrak{l}^*$  and a smooth subvariety  $X$  of  $Y$  such that the central extension (9) of action groupoids  $\pi_X : \mathcal{L}_X^\chi \rightarrow \mathcal{L}_X$  is split. Then  $\mathcal{F}|_{\widehat{X}}$  admits an  $L^\chi$ -equivariant structure with any central charge  $\mu \in \mathbb{F}$ .

4.2.4. *Proof.* It suffices to exhibit an  $L^\chi$ -equivariant structure with a central charge  $\mu$  on  $\mathcal{O}_{\widehat{X}}$  since a tensor product of two equivariant vector bundles has a natural equivariant structure so that central charges add. Thus,  $\mathcal{F}|_{\widehat{X}} \otimes \mathcal{O}_{\widehat{X}} \cong \mathcal{F}|_{\widehat{X}}$  admits an  $L^\chi$ -equivariant structure with a central charge  $\mu + 0$ .

By Proposition 3.1.7,  $\widehat{X}$  is isomorphic to the quotient  $(L^\chi \times X)/\mathcal{L}_X^\chi$ . The bundle  $\mathcal{O}_{L^\chi \times X}$  admits an  $\mathcal{L}_X^\chi$ -equivariant structure, called  $I$ , with a central charge  $\mu \in \mathbb{F}$  by the argument in 4.2.2 because the extension  $\pi_X$  is split.

The non-trivial part of the proof is to comprehend the quotient  $(L^\chi \times X \times \mathbb{K})/\mathcal{L}_X^\chi$ . The quotient  $(L^\chi \times X \times \mathbb{K})/G_m^1$  is the trivial line bundle on  $L^1 \times X$  because there exists a  $G_m^1$ -equivariant global section  $s : L^\chi \times X \rightarrow \mathbb{K}$ , defined by  $s(g, x) = f(g)$  where a function  $f$  is given by Lemma 2.2.5 with  $\eta = -\mu$ . Finally, we observe that  $(L^\chi \times X \times \mathbb{K})/\mathcal{L}_X^\chi = ((L^\chi \times X \times \mathbb{K})/G_m^1)/\mathcal{L}_X \cong (L^1 \times X \times \mathbb{K})/\mathcal{L}_X \cong \widehat{X} \times \mathbb{K}$  is the trivial line bundle on  $\widehat{X}$ , which inherits a  $L^\chi$ -equivariant structure with a central charge  $\mu$  from a  $\mathcal{O}_{L^\chi \times X}$ .  $\square$

## 4.3. Construction of representations.

4.3.1. We consider a nilpotent functional  $\chi \in \mathfrak{g}^*$ . We say that a subscheme  $X$  of  $\mathcal{B}$  is  $\chi$ -nice if it is a smooth subvariety and satisfies the infinitesimal splitting condition for  $\chi$ . Every  $\chi$ -nice subvariety is a subvariety of the Springer fiber  $\mathcal{B}^\chi$  by Corollary 1.2.8.

4.3.2. *Theorem.* Let us consider subschema  $Z \subseteq X \subseteq \mathcal{B}$  such that  $X$  is  $\chi$ -nice. If  $\mathcal{F}_\lambda$  is a  $G$ -equivariant line bundle on  $\mathcal{B}$  then the space of sections  $\Gamma(\widehat{Z}, \mathcal{F}_\lambda)$  has a canonical structure of a  $U_\chi(\mathfrak{g})$ -module (the Frobenius neighborhood of  $Z$  is taken inside  $\mathcal{B}$ ).

4.3.3. *Proof.* Theorem 3.3.3 and Theorem 4.2.3 imply that the line bundle  $\mathcal{F}_\lambda|_{\widehat{Z}}$  has a  $G^X$ -equivariant structure with central charge 1. Therefore,  $\Gamma(\widehat{Z}, \mathcal{F}_\lambda)$  is a  $u(\mathfrak{g}_\chi)$ -module with central charge 1, which is canonically a  $U_\chi(\mathfrak{g})$ -module.  $\square$

4.3.4. *Examples.* We would like to compile a list of known  $\chi$ -nice subschema. Any partial flag subvariety in  $\mathcal{B}^x$  is  $\chi$ -nice by Lemma 3.3.9. One can check by a straightforward calculation that any nilpotent element of  $\mathfrak{sl}_5$  satisfies the condition of Lemma 3.3.12. Thus, this lemma guarantees that all smooth subvarieties of  $\mathcal{B}^x$  are  $\chi$ -nice for each nilpotent  $\chi$  if  $\mathfrak{g}$  is of type  $A_1, A_2, A_3, A_4$ , or  $B_2$ .

4.3.5. *Stabilizer action.* If  $S_1$  is a subgroup of the stabilizer of  $\chi$  in  $G$  such that  $Y$  is  $S_1$ -invariant then  $S_1$  also acts on the vector space  $\Gamma(\widehat{Y}, \mathcal{F}_\lambda)$ . It is plausible that one can combine the actions of  $S_1$  and  $\mathfrak{g}_\chi$  to obtain a representation of the Harish-Chandra pair  $(S_1, \mathfrak{g}_\chi)$ , which is a subpair of  $(S, \mathfrak{g}_\chi)$  constructed in 1.4.2.

## 5. CONCLUDING REMARKS

### 5.1. Geometric modules.

5.1.1. *The category of geometric modules.* Though the components of  $\mathcal{B}^x$  need not be  $\chi$ -nice, we introduce a standard category of modules. Consider a category  $\mathfrak{C}$  whose objects are pairs  $(Z, \lambda)$  where  $Z$  is a subscheme of  $\mathcal{B}$  contained in a  $\chi$ -nice subscheme and  $\lambda$  is a weight. The morphism set  $\text{Hom}_{\mathfrak{C}}((Z, \lambda), (Z', \lambda'))$  consists of one element if  $Z \supseteq Z'$  and  $\lambda = \lambda'$  and is empty otherwise. There is a functor  $(Z, \lambda) \mapsto \Gamma(\widehat{Z}, \mathcal{F}_\lambda)$  from  $\mathfrak{C}$  to  $U_\chi(\mathfrak{g})\text{-Mod}$ . A morphism in  $\mathfrak{C}$  goes to the restriction morphism of the global sections. The Abelian subcategory of  $U_\chi(\mathfrak{g})\text{-Mod}$ , generated by the image of  $\mathfrak{C}$ , will be called *the category of geometric modules* and denoted  $U_\chi(\mathfrak{g})\text{-Geom}$ . A module  $M$  is called *geometric* if it is isomorphic to an object in  $U_\chi(\mathfrak{g})\text{-Geom}$ . A filtration (submodule, subquotient) of a geometric module  $M$  is called *geometric* if it exists on an object of  $U_\chi(\mathfrak{g})\text{-Geom}$  isomorphic to  $M$ .

5.1.2. *Question.* Are simple  $U_\chi(\mathfrak{g})$ -modules geometric?

5.1.3. *Parabolic induction.* We want to identify some of the geometric modules with modules constructed by algebraic methods. Let  $P$  be a parabolic subgroup containing a Borel subgroup  $B$ . Let  $U$  be the unipotent radical of the opposite parabolic.  $PU$  is a dense open subset of  $G$ , isomorphic to  $P \times U$ . It follows that  $\widehat{P/B} \cong P \times U^1$ . The condition  $P/B \subseteq \mathcal{B}^\chi$  is equivalent to  $\chi|_{\mathfrak{p}} = 0$  where  $\mathfrak{p}$  is the Lie algebra of  $P$ . The following proposition makes sense since  $u(\mathfrak{p}) = U_\chi(\mathfrak{p}) \subseteq U_\chi(\mathfrak{g})$ .

5.1.4. *Proposition.* The  $U_\chi(\mathfrak{g})$ -module  $\Gamma(\widehat{P/B}, \mathcal{F}_\lambda)$  is isomorphic to  $\text{Ind}_{U_\chi(\mathfrak{p})}^{U_\chi(\mathfrak{g})}(\text{Ind}_B^P(\mathbb{K}_{-w_0 \bullet \lambda}))^*$  where  $w_0$  is the longest element of the Weyl group of the Levi factor of  $P$  and  $\bullet$  is the dot action.

5.1.5. *Proof.* The Frobenius neighborhood  $\Sigma$  of the point  $P$  in  $G/P$  is isomorphic to  $G^\chi/P^\chi$ . The  $P/B$ -bundle  $\widehat{P/B} \xrightarrow{\pi} \Sigma$  is the restriction of the natural one to  $\Sigma$ . Thus,

$$\begin{aligned} \Gamma(\widehat{P/B}, \mathcal{F}_\lambda) &= \Gamma(\Sigma, \pi_* \mathcal{F}_\lambda) = \Gamma(\Sigma, G^\chi \times_{P^\chi} \text{Ind}_B^P \mathbb{K}_\lambda) = \\ &= \text{Ind}_{U_\chi(\mathfrak{p})}^{U_\chi(\mathfrak{g})}((\text{Ind}_B^P \mathbb{K}_\lambda)^*)^* \cong \text{Ind}_{U_\chi(\mathfrak{p})}^{U_\chi(\mathfrak{g})}(\text{Ind}_B^P(\mathbb{K}_{-w_0 \bullet \lambda}))^*. \quad \square \end{aligned}$$

5.1.6. *The subregular orbit of  $\mathfrak{sl}_3$ .* We explicate a geometric reason for a baby Verma module to have more than one simple quotient.

Let us look at the subregular nilpotent orbit of  $\mathfrak{sl}_3$ . Let us assume that  $p \neq 3$  to identify  $\mathfrak{g}$  and  $\mathfrak{g}^*$ . Choosing a matrix  $A$  with  $A_{ij} = 0$  except  $A_{13} = 1$  as  $\chi$ , we take the standard Borel subalgebra  $\mathfrak{b}$  to be the intersection of the two components  $Y_1, Y_2$  of  $\mathcal{B}^\chi$ , which is a Dynkin curve in this case [9]. Now there are non-zero restriction morphisms

$$\Gamma(\widehat{\mathcal{B}^\chi}, \mathcal{F}_\lambda) \xrightarrow{i} \Gamma(\widehat{Y}_1, \mathcal{F}_\lambda) \oplus \Gamma(\widehat{Y}_2, \mathcal{F}_\lambda) \rightarrow \Gamma(\widehat{\mathfrak{b}}, \mathcal{F}_\lambda)$$

for a weight  $\lambda$  inside the lowest dominant alcove. The direct summands in the middle are distinct irreducible  $U_\chi(\mathfrak{g})$ -modules by Proposition 5.1.4 and [15, 12]. Therefore, the socle of  $\Gamma(\widehat{\mathfrak{b}}, \mathcal{F}_\lambda)$  is not simple. Thus, baby Verma  $U_\chi(\mathfrak{g})$ -modules  $Z_{\chi, \mathfrak{b}}(\lambda)$  with this  $\mathfrak{b}$ , which are isomorphic to  $\Gamma(\widehat{\mathfrak{b}}, \mathcal{F}_{-w_0 \lambda})$ , do not have a unique simple quotient.

Another interesting observation is that  $\Gamma(\widehat{\mathcal{B}^\chi}, \mathcal{F}_\lambda)$  has no natural  $U_\chi(\mathfrak{g})$ -module structure since the embedding  $i$  is not an isomorphism.

## 5.2. Deformations of modules.

5.2.1. If  $\mathcal{B}^\chi \subseteq \mathcal{B}^\eta$  then a geometric  $U_\chi(\mathfrak{g})$ -module can have a structure of  $U_\eta(\mathfrak{g})$ -module. By Theorem 4.3.2, it suffices to ensure that a  $\chi$ -nice subscheme  $Z$  is  $\eta$ -nice. Similarly, a geometric filtration of a  $U_\chi(\mathfrak{g})$ -module turns out to be a filtration by  $U_\eta(\mathfrak{g})$ -modules of the corresponding  $U_\eta(\mathfrak{g})$ -module.

In the particular case of  $\eta = 0$ , every geometric  $U_\chi(\mathfrak{g})$ -module admits a structure of a restricted  $\mathfrak{g}$ -module since any smooth subscheme is 0-nice. If Question 5.1.2 has an affirmative answer then any simple  $U_\chi(\mathfrak{g})$ -module has a structure of  $u(\mathfrak{g})$ -module and the dimension of a simple  $U_\chi(\mathfrak{g})$ -module is a sum of dimensions of some simple  $u(\mathfrak{g})$ -modules. The case of  $\mathfrak{so}_5$  has been worked out in [23].

5.2.2. Let us consider a family of nilpotent elements  $\chi(t)$  and a smooth subvariety  $Z \subseteq \mathcal{B}$  such that  $\mathcal{B}^{\chi(t)}$  contains  $Z$  for each value of the parameter  $t$ . If one can further ensure that  $Z$  is  $\chi(t)$ -nice for each  $t$ , then we obtain a family of  $\mathfrak{g}$ -module structures on the vector space  $\Gamma(\widehat{Z}, \mathcal{F}_\lambda)$  for each  $\lambda \in \Pi$ . The  $p$ -character of the action at  $t$  is  $\chi(t)$ .

### 5.3. Kac-Weisfeiler Conjecture.

5.3.1. **Question.** Let  $X$  be a closed subvariety of a projective algebraic variety  $Y$  and  $\mathcal{F}$  be a line bundle on  $Y$ . When is it true that the dimension of  $\Gamma(\widehat{X}, \mathcal{F})$  is divisible by  $p^{\text{codim } X}$ ?

5.3.2. Affirmative answers to Questions 5.3.1 and 5.1.2 would prove the Kac-Weisfeiler Conjecture because the dimension formula [9, 6.7] implies that the codimension of  $\mathcal{B}^\chi$  in  $\mathcal{B}$  is equal to  $\frac{1}{2}\dim G \cdot \chi$ .

Conversely, if a component of  $\mathcal{B}^\chi$  is  $\chi$ -nice then the Kac-Weisfeiler conjecture, being the Premet theorem now [21], implies an affirmative answer to Question 5.3.1 for a component of  $\mathcal{B}^\chi$  as  $X$  and  $Y = \mathcal{B}$ . Thus, Question 5.3.1 may be regarded as a geometric version of the Kac-Weisfeiler conjecture.

## REFERENCES

- [1] A. Beilinson, A. Bernstein, *A proof of Jantzen conjectures*, Advances in Soviet Mathematics **16** (part 1) (1993), 1-49.
- [2] M. Demazure, P. Gabriel, *Groupes Algébriques*, North-Holland Publishing Company, Amsterdam, 1970.
- [3] E. M. Friedlander, B. Parshall, *Modular representation theory of Lie algebras*, Amer. J. Math. **110** (1988), 1055-1094.
- [4] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer Verlag, Berlin, 1977.
- [5] G. Hochschild, *Representations of restricted Lie algebras of characteristic  $p$* , Proc. Amer. Math. Soc. **5** (1954), 603-605.
- [6] G. Hochschild, *Cohomology of restricted Lie algebras*, Amer. J. Math. **76** (1954), 555-580.
- [7] G. Hochschild, *Simple Lie algebras with purely inseparable splitting fields of exponent 1*, Trans. Amer. Math. Soc. **79** (1955), 477-489.
- [8] J. E. Humphreys, *Modular representations of classical Lie algebras and semisimple groups*, J. Algebra **19** (1971), 51-79.

- [9] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Amer. Math. Soc., Providence, 1995.
- [10] J. E. Humphreys, *Modular representations of simple Lie algebras*, Bull. Amer. Math. Soc. (N.S.) **35** (1998), 105-122.
- [11] J. C. Jantzen, *Representations of Algebraic Groups*, Academic Press, Orlando, 1987.
- [12] J. C. Jantzen, *Subregular nilpotent representations of  $\mathfrak{sl}_n$  and  $\mathfrak{so}_{2n+1}$* , Math. Proc. Cambridge Philos. Soc. **126** (1999), 223-257.
- [13] J. C. Jantzen, *Representations of  $\mathfrak{so}_5$  in prime characteristic*, University of Aarhus preprint series **13**, July 1997.
- [14] J. C. Jantzen, *Representations of Lie algebras in prime characteristic*, University of Aarhus preprint series **1**, January 1998.
- [15] V. Kac, *On irreducible representations of Lie algebras of classical type* [Russian], Uspekhi Mat. Nauk **27** (1972), 237-238.
- [16] E. Kunz, *Characterizations of regular local rings in characteristic  $p$* , Amer. J. Math. **41** (1969), 772-784.
- [17] G. Lusztig, *Bases in equivariant  $K$ -theory*, Represent. Theory **2** (1998), 298-369.
- [18] K. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, Cambridge University Press, Cambridge, 1987.
- [19] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conf. Ser. in Math. **82**, Amer. Math. Soc., Providence, 1993.
- [20] G. Nielsen, *A determination of the minimal right ideals in the enveloping algebra of a Lie algebra of classical type*, Ph.D. dissertation, Univ. of Wisconsin, 1963.
- [21] A. Premet, *Irreducible representations of Lie algebras of reductive groups and the Kac-Weisfeiler conjecture*, Invent. Math. **121** (1995), 79-117.
- [22] D. Rumynin, *Hopf-Galois extensions with central invariants and their geometric properties*, Algebr. Represent. Theory, **1** (1998), 353-381.
- [23] D. Rumynin, *Modular Lie algebras and their representations*, Ph.D. dissertation, Univ. of Massachusetts at Amherst, 1998.
- [24] D. Rumynin, *Duality for Hopf algebroids*, Journal of Algebra, to appear.
- [25] H. Strade, R. Farnsteiner, *Modular Lie algebras and their representations*, Marcel Dekker, New York, 1988.
- [26] J. B. Sullivan, *Lie algebra cohomology at irreducible modules*, Illinois J. Math. **23** (1979), 363-373.
- [27] B. Yu. Weisfeiler and V. G. Kac, *On irreducible representations of Lie  $p$ -algebras*, Funct. Anal. Appl. **5** (1971), 111-117.

DEPT. OF MATH. AND STAT., LGRT, UMASS, AMHERST, MA, 01003, USA  
*E-mail address:* mirkovic@math.umass.edu

MATHEMATICS DEPT., UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL,  
 U.K.  
*E-mail address:* rumynin@maths.warwick.ac.uk