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SCHWARZIAN DERIVATIVES AND FLOWS OF SURFACES

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1. INTRODUCTION

Over the last decades it has been widely recognized that many completely integrable PDE's from mathematical physics arise naturally in geometry. Their integrable character in the geometric context – usually associated with the presence of a Lax pair and a spectral deformation – is nothing but the flatness condition of a naturally occurring connection. Examples of interest to geometers generally come from the integrability equations of special surfaces in various ambient spaces: among the best known examples is the sinh-Gordon equation describing constant mean curvature tori in 3-space. In fact, most of the classically studied surfaces, such as isothermic surfaces, surfaces of constant curvature and Willmore surfaces, give rise to such completely integrable PDE. A common thread in many of these examples is the appearance of a harmonic map into some symmetric space, which is well-known to admit a Lax representation with spectral parameter. Once the geometric problem is formulated this way algebro-geometric integration techniques give explicit parameterizations of the surfaces in question in terms of theta functions.

One contribution of the geometric view of these integrable PDE is a much better understanding of the meaning of the hierarchy of flows associated to these equations. In mathematical physics these hierarchies are obtained by deformations of the Lax operators preserving their eigenvalues. This is rather unsatisfactory from the geometric viewpoint, where one wants to see these flows as geometric deformations on the surfaces. A good example of the geometric derivation of the KdV flows can be found in [17] (see also [21]) and the flows related to the sinh-Gordon equation are explicitly derived from geometric deformations in [18].

Lately two more integrable hierarchies, well known to mathematical physicists, have made their appearance in surface geometry: these are the Novikov–Veselov and Davey–Stewartson equations [13, 23, 11, 12]. They are derived as kernel dimension preserving deformations of the Dirac operator. Since surfaces in 3 and 4 space can be presented by "spinors", these flows in principle give an integrable hierarchy on the space of conformally parameterized tori in 3 and 4 space. The relation to surface geometry leaves a lot to ask for: what is the geometric meaning of these flows? Can one derive them from some geometric principles? What kind of geometry is involved? Even though existing literature [11, 12, 23] uses Euclidean surface geometry, it turns out that these flows are really Möbius invariant. Thus there should be some Möbius invariant setting from which these flows can be derived. However, notwithstanding this, the essential new insight is that there is a completely integrable hierarchy on *all* conformally immersed 2-tori, and not just on special classes given by certain geometric conditions, such as constant mean curvature tori etc. One clearly expects that these new flows reduce in some way to the known ones when restricted to the corresponding special surface classes.

The picture emerging here is that the manifold of conformally parameterized 2-tori in 3-space is stratified by finite dimensional submanifolds, namely those tori stationary with respect to some higher flow in the

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hierarchy. These "finite type" tori can then be parameterized by theta functions on some finite genus Riemann surface. As said, this is just a picture and we feel there still is a lot more explaining to be done to become a useful mathematical theory.

This present note aims to give a self contained, low technology approach to the above mentioned topics. The natural setting of our discussion is Möbius invariant surface geometry. Since we do not claim to have a complete theory, the exposition will at times be sketchy and leave the Reader, hopefully, with some urge to dwell further on the issues.

The most popular integrable hierarchy is the KdV hierarchy. Therefore we begin this paper by deriving this hierarchy as natural flows on holomorphic maps into the Riemann sphere. The basic invariant of a holomorphic map f, its Schwarzian derivative $S_z(f)$ with respect to a coordinate z, turns out to be the function satisfying the equations of the KdV hierarchy. The Schwarzian occurs naturally in a Hill equation

$$\psi_{zz} + \tfrac{1}{2}S_z(f)\psi = 0$$

for some appropriate homogeneous lift $[\psi] = f$. Notice that under a change of coordinates the Schwarzian transforms as

$$S_w(f)dw^2 = (S_z(f) - S_z(w))dz^2\,,$$

which makes it conceptually harder to understand what object the Schwarzian really is (see, however [6]). This approach to KdV is well known [21], but our derivations are closer to the Möbius geometry of conformally immersed Riemann surfaces into the n sphere.

The latter theme gets introduced in the second chapter, where we derive the invariants, structure and integrability equations of a conformal surface $f : M \to S^n$ under the Möbius group. In this case the basic equation defining the invariants is an inhomogeneous Hill equation

$$\psi_{zz} + \tfrac{1}{2}S_z(f)\psi = \kappa\,,$$

for an appropriate lift ψ of f into the forward light cone—we view the conformal n sphere as the projectivized light cone in Lorentz space. The invariant κ is essentially the trace free second fundamental form, or *Hopf differential*, of f, a well known Möbius invariant, and $S_z(f)$ is the *Schwarzian derivative*[1] of the surface f. It transforms under coordinate changes just like the classical Schwarzian above. Of course, if $\kappa = 0$ we reduce to the classical case of holomorphic maps into S^2. The Hopf differential and Schwarzian form a complete set of invariants for surfaces in 3 space with respect to the Möbius group. Their integrability equations, the conformal Gauss and Codazzi equations, are

$$\tfrac{1}{2}S_z(f)_{\bar z} = 3\bar\kappa\kappa_z + \bar\kappa\kappa_z$$
$$\operatorname{Im}(\kappa_{\bar z\bar z} + \tfrac{1}{2}\overline{S_z(f)}\kappa) = 0\,,$$

which gives the fundamental theorem of Möbius invariant surface theory.

To demonstrate the effectiveness of our setup, we discuss a number of natural problems from conformally invariant surface theory: cyclides of Dupin are characterized by holomorphic Schwarzians $S_z(f)$; isothermic surfaces by real Hopf differentials κ; Willmore surfaces are characterized by

$$\kappa_{\bar z\bar z} + \tfrac{1}{2}\overline{S_z(f)}\kappa = 0\,,$$

a stronger condition than the Codazzi equation. From these conditions we easily see the associated families of isothermic and Willmore surfaces, crucial to their completely integrable character. Moreover, unless a

[1]The appearance of Schwarzian derivatives in conformal surface geometry is not new: they were introduced in a closely related context by Calderbank [8] who has also developed, with the first author, an independent and more invariant formulation of the fundamental theorem of Möbius invariant surface theory [6].

surface is isothermic, the Hopf differential alone determines it uniquely up to Möbius transformations. We also discuss results of Thomsen [24] and Richter [19] characterising minimal, respectively constant mean curvature, surfaces in 3-dimensional space forms as isothermic Willmore, respectively constrained Willmore, surfaces and indicate a generalisation to isothermic Willmore surfaces in n-space. We include these sometimes very classical results since they amplify the "retro" charm of our view point on Möbius geometry: all calculations are local and done in a fixed coordinate, just like 100 years ago, but due to our non-redundant set of invariants turn out to be extraordinarily short and efficient.

Having done that much, we finally get to a geometric derivation of the Novikov–Veselov and Davey–Stewartson equations in the last chapter. We follow our initial derivation of the KdV hierarchy in the first chapter to derive the Novikov–Veselov flow for surfaces in n-space. This flow is Möbius invariant by construction and preserves the Willmore energy, at least in 3 space. Moreover, it also preserves isothermic surfaces in any codimension giving a vector version of the modified Novikov–Veselov flow. Again, when the Hopf differential $\kappa = 0$, so that we deal with holomorphic maps into S^2, these flows reduce to the KdV flows. Whereas the Novikov–Veselov flows can be defined in any codimension, the Davey Stewartson hierarchy needs the additional datum of a complex structure in the normal bundle, which is the case for surfaces in 4 space. This flow is defined in term of double derivatives of the Hopf differential and thus simpler then the Novikov–Veselov flow, which is given in triple derivatives of the Hopf differential. Surfaces in 3 space are generally moved into 4 space by the Davey–Stewartson flow, which therefore does not induce flows on surfaces in 3 space. Nevertheless, the flow preserves Willmore surfaces and isothermic surfaces in 4 space.

Even though we mention the word "hierarchies" frequently, we do not really have an explicit scheme to construct all the higher flows for Novikov–Veselov and Davey–Stewartson. This is partially due to the inherently non-explicit character of the higher flows —one needs to solve a $\bar{\partial}$-problem for each flow— and partially due to our insufficient understanding of invariant multilinear differential operators on Riemann surfaces. The latter seems to be an important enough issue to return to. At present there are two attempts in that direction, one [4] using deformation theory on quaternionic holomorphic line bundles for surfaces in $S^4 = \mathbb{HP}^1$, and another [6] using Cartan connections and Möbius structures for surfaces in S^n.

This work began during the conference on "Integrable systems and geometry" in Tokyo in the summer of 2000 and has been completed during January 2001, when all authors met again at the SFB288 at TU-Berlin. We take this opportunity to thank the organizers of the Tokyo conference and the SFB288 for their support.

2. The classical Schwarzian and the KdV hierarchy

2.1. **The classical Schwarzian derivative.** Given a meromorphic function f on a Riemann surface M and a holomorphic coordinate z the *Schwarzian derivative of f with respect to z* is given by the expression

$$S_z(f) = (\frac{f''}{f'})' - \tfrac{1}{2}(\frac{f''}{f'})^2 . \tag{1}$$

One can quickly check that the Schwarzian $S_z(f) = 0$ vanishes if and only if $f = \frac{az+b}{cz+d}$ is fractional linear in z. Moreover, two meromorphic functions f and g are related by a fractional linear transformation

$$g = \frac{af+b}{cf+d}$$

if and only if their Schwarzians $S_z(f) = S_z(g)$ agree. The last property indicates that the Schwarzian is an invariant for maps into $\mathbb{CP}^1$. We will develop this view point further since it motivates most of our constructions on conformally immersed surfaces later on.

Let $f: M \to \mathbb{CP}^1$ be a holomorphic immersion, i.e., $df_p \neq 0$ for all $p \in M$, and fix a holomorphic coordinate z on M. A *lift* of f is a holomorphic map $\psi : M \to \mathbb{C}^2 \setminus \{0\}$ so that $f(p)$ is the line spanned by $\psi(p)$. In other words, ψ is a holomorphic section of the pull-back via f of the tautological line bundle over $\mathbb{CP}^1$. Our assumption that f is an immersion is equivalent to ψ and ψ_z being linearly independent at each point $p \in M$. Using a fixed complex volume form det on $\mathbb{C}^2$ we can find, up to sign, a unique *normalized* lift ψ of f by demanding that

$$\det(\psi, \psi_z) = 1\,. \tag{2}$$

Indeed, two lifts $\tilde{\psi} = \psi\lambda$ are related by a nowhere vanishing scale $\lambda : M \to \mathbb{C}$ and

$$\det(\tilde{\psi}, \tilde{\psi}_z) = \lambda^2 \det(\psi, \psi_z)\,,$$

which proves our assertion. Taking the derivative of (2) we obtain

$$\det(\psi, \psi_{zz}) = 0\,,$$

so that ψ satisfies Hill's equation

$$\psi_{zz} + \tfrac{c}{2}\psi = 0 \tag{3}$$

for some holomorphic function $c : M \to \mathbb{C}$. From our construction it is clear that c depends only on f and the coordinate z. If the holomorphic map f into $\mathbb{CP}^1$ is written in homogeneous coordinates $[f : 1]$ then one easily checks that

$$c = S_z(f) \tag{4}$$

is the Schwarzian derivative. The above mentioned properties of the Schwarzian now follow immediately from (3). Moreover, given c we solve Hill's equation for

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\,,$$

and then $f = [\psi_1 : \psi_2]$ gives a map into $\mathbb{CP}^1$, unique up to the action of $\mathbf{Sl}(2, \mathbb{C})$, with Schwarzian $S_z(f) = c$. Thus the Schwarzian is an infinitesimal invariant, with respect to the Möbius group $\mathbf{PSl}(2, \mathbb{C})$, for maps into $\mathbb{CP}^1$.

To obtain a better understanding what type of object the Schwarzian is, we differentiate with respect to another holomorphic coordinate w on M. An elementary calculation using (3) reveals the transformation formula

$$S_w(f) dw^2 = (S_z(f) - S_z(w)) dz^2\,. \tag{5}$$

Recall that a *projective structure* on M is given by a holomorphic atlas $\mathcal{A}$ whose coordinate transition functions are fractional linear. Given two projective structures $\mathcal{A}$ and $\tilde{\mathcal{A}}$ the holomorphic quadratic differential

$$Q := S_z(\tilde{z}) dz^2\,,$$

where z and $\tilde{z}$ are charts in $\mathcal{A}$ and $\tilde{\mathcal{A}}$, is well-defined by (5). Therefore the space of all projective structures on a Riemann surface is an affine space modelled on the vector space $H^0(K^2)$ of holomorphic quadratic differentials. If we have fixed a projective structure on M, the transformation rule (5) implies that the Schwarzian of a function may be regarded as a holomorphic quadratic differential.

2.2. **The KdV hierarchy.** Consider a deformation $f(t) : M \to \mathbb{CP}^1$ of $f = f(0)$ with corresponding lift $\psi(t)$, which we may assume to be normalized (2). Then ψ and ψ_z are a unimodular basis for $\mathbb{C}^2$ for any z and t, and the infinitesimal variation of ψ can be expressed as

$$\psi_t = a\psi + b\psi_z \,. \tag{6}$$

Here a and b are functions of t and z which have to satisfy

$$b_z = -2a \tag{7}$$

due to (2). For the infinitesimal variation of the Schwarzian, we use Hill's equation (3) and obtain

$$\psi_{tzz} + \tfrac{c_t}{2}\psi + \tfrac{c}{2}\psi_t = 0 \,.$$

Calculating ψ_{tzz} from (6) and (7) the ψ-part of this relation unravels to

$$c_t = b_{zzz} + 2b_z c + bc_z \,. \tag{8}$$

So far we have been describing general deformations of f. To obtain geometrically meaningful deformations the "radial variation" a, and therefore the "tangential variation" b, in (6) have to be chosen geometrically. The most naive choice of variation is *no* radial variation at all, i.e., $a_1 = 0$. Then, up to a constant, $b_1 = 1$ by (7) and from (8) we get

$$c_{t_1} = c_z \,,$$

which is translational flow along z. To obtain a more interesting flow we use the *old* variation of c as the *new* radial variation, i.e., we put

$$a_3 = -\tfrac{1}{2}c_{t_1} \,.$$

Then $b_3 = c$ by (7) and from (8) we obtain

$$c_{t_3} = c_{zzz} + 3cc_z \,,$$

which we recognize as the KdV equation. Continuing our recursion scheme

$$a_{2k+1} = -\tfrac{1}{2}c_{t_{2k-1}} \,,$$

we arrive at an infinite hierarchy of odd order (in z) flows, the KdV hierarchy. For example, the 5^{th}-order flow computes to

$$c_{t_5} = c_{zzzzz} + 5c_{zzz}c + 10c_{zz}c_z + \tfrac{15}{2}c_z c^2 \,.$$

Notice that at each step we have constants of integration, which amounts to considering a given order flow modulo the lower order flows.

A map $f : M \to \mathbb{CP}^1$ has *finite type* n if the $(2n-1)^{st}$–order flow is stationary modulo the lower order flows:

$$c_{t_{2n-1}} + \sum_{k=1}^{n-1} \lambda_k c_{t_{2k-1}} = 0 \,. \tag{9}$$

For example, type 1 maps have $c_z = 0$ or $c = c_0$ constant, and thus are the loxodromes $f = \exp(i\sqrt{2c_0}z)$. In general, finite type n maps are given by θ–functions on a genus $n-1$ Riemann surface, and the flows in the hierarchy become linear flows on the Jacobian of this Riemann surface. The above geometric construction of an integrable hierarchy and its algebraic-geometric integration is folklore by now. Our exposition follows the spirit of [18], where a similar construction was done for the sinh–Gordon hierarchy.

We conclude this paragraph with a remark on the closely related mKdV hierarchy. Whereas the KdV flows commute with the action of Möbius group $\mathbf{PSl}(2,\mathbb{C})$, the mKdV flows arise by "symmetry breaking": fixing a point at infinity on $\mathbb{CP}^1$ we consider $f : M \to \mathbb{C} \subset \mathbb{CP}^1$ where $\mathbb{C}$ now carries the flat Euclidean structure.

In other words, we have reduced the symmetry group of our target space from the Möbius group to the Euclidean group. The fundamental Euclidean invariant of f is now given by the induced metric

$$|df|^2 = f_z\overline{f_z}|dz|^2\,, \qquad f_z\overline{f_z} = e^{2v}$$

where $v : M \to \mathbb{R}$ is a real valued function. Using the formula (1) for $S_z(f)$, it is easy to check that

$$S_z(f) = 2v_{zz} - 2(v_z)^2\,. \tag{10}$$

In particular, this combination of derivatives of the real valued function v is holomorphic. Putting $p := 2v_z$ the above relation reads

$$S_z(f) = p_z - \tfrac{1}{2}p^2\,,$$

which we recognize as the *Miura transform* [15]. Therefore the Miura transform is just the expression of the Möbius invariant Schwarzian of f in terms of the Euclidean invariant induced metric of f. It is a well known fact [15] that the Miura transform interpolates between the mKdV and KdV flows so that p satisfies the mKdV hierarchy.

3. Conformally immersed surfaces and Schwarzians

The constructions of the previous section carry over almost verbatim to conformally immersed surfaces in n-space. As a result we obtain an elegant description of Möbius invariant surface geometry. Later we will indicate how to construct Möbius invariant flows on conformally immersed surfaces.

3.1. The light-cone model.

Following Darboux [9], we linearize the action of the Möbius group on the sphere $S^n \subset \mathbb{R}^{n+1,1}$ via the diffeomorphism

$$S^n \cong \mathbb{P}(\mathcal{L}) : x \leftrightarrow [1 : x] \tag{11}$$

between the sphere and the projectivized light cone where $\mathcal{L} \subset \mathbb{R}^{n+1,1}$ is the null cone in $(n+2)$–dimensional Minkowski space for the quadratic form $\langle v, v\rangle = -v_0^2 + \sum_{k=1}^{n+1} v_k^2$. The projective action of the Lorentz group on $\mathbb{P}(\mathcal{L})$ is by conformal diffeomorphisms giving rise to a double covering of the Möbius group by the Lorentz group.

The isomorphism (11) is a special case of a more general construction which realizes all space forms as conic sections of $\mathcal{L}$: for non-zero $v_0 \in \mathbb{R}^{n+1,1}$, set

$$S_{v_0} = \{v \in \mathcal{L} : \langle v, v_0\rangle = -1\}.$$

S_{v_0} inherits a positive definite metric of constant curvature $-\langle v_0, v_0\rangle$ from $\mathbb{R}^{n+1,1}$ and is a copy of a sphere, Euclidean space or ball according to the sign of $-\langle v_0, v_0\rangle$. For (11), $v_0 = e_0$.

In this picture, it is easy to describe the m-spheres (totally umbilic submanifolds) in S^n [20]: they are all given by a decomposition $\mathbb{R}^{n+1,1} = V \oplus V^\perp$ with $V^\perp$ space-like of dimension $(n-m)$. The m-dimensional sphere is then $\mathbb{P}(\mathcal{L} \cap V)$. Viewed as a submanifold $S_{v_0} \cap V$ of the conic section S_{v_0}, one readily shows that this m-sphere has mean curvature vector

$$H_v = -v_0^\perp - \langle v_0^\perp, v_0^\perp\rangle v \tag{12}$$

at v, where $v_0 = v_0^T + v_0^\perp$ according to the decomposition $V \oplus V^\perp$.

3.2. **The Schwarzian derivative of a conformally immersed surface.** Let $f : M \to S^n \subset \mathbb{R}^{n+1}$ be a conformal immersion of a Riemann surface M. A *lift* of f is a map $\psi : M \to \mathcal{L}_+$ into the forward light cone such that the null line spanned by $\psi(p)$ is $f(p)$. Perhaps the most naive choice for a lift is the *Euclidean lift*

$$\phi = (1, f)\,, \tag{13}$$

but any positive scale of this lift will do. Note that the induced metric of a scaled lift $\phi\lambda$, with $\lambda : M \to \mathbb{R}_+$, is given by

$$|d(\phi\lambda)|^2 = \lambda^2 |df|^2\,,$$

where the induced metric $|df|^2$ is computed with respect to the standard round metric on S^n which we also denote by $\langle\,,\,\rangle$. Therefore the various lifts of f give rise to the various metrics in the conformal class of $|df|^2$. From this we see that there is a unique *normalized* lift ψ, with respect to a given holomorphic coordinate z on M, for which the induced metric

$$|d\psi|^2 = |dz|^2 \tag{14}$$

is the standard flat metric of the coordinate z. Since (14) is invariant under Lorentz transformations the normalized lift ψ is Möbius invariant.

If we pass to another holomorphic coordinate w the normalized lift with respect to this new coordinate

$$\tilde{\psi} = \psi |w_z| \tag{15}$$

is obtained by scaling ψ by the length of the Jacobian of the coordinate transformation. This follows immediately from the fact that ψ is null and therefore

$$|d\tilde{\psi}|^2 = |\, d\psi |w_z| + \psi d|w_z|\,|^2 = |d\psi|^2 |w_z|^2 = |dz|^2 |w_z|^2 = |dw|^2\,.$$

A fundamental construction of Möbius invariant surface geometry is the *mean curvature* or *central sphere congruence* [24] which assigns to each point $f(p)$ on the surface the unique 2-sphere $S(p)$ tangent to f at that point and with the same mean curvature vector $H_{S(p)} = H_p$ at $f(p)$ as f. While this is phrased in terms of the Euclidean lift $\phi = (1, f) : M \to S_{e_0}$, the construction is conformally invariant: $S(p)$ corresponds to a decomposition $\mathbb{R}^{n+1,1} = V(p) \oplus V(p)^\perp$ with

$$V(p) = \operatorname{span}\{\phi(p), d\phi_p, e_0^T\}$$

since ϕ is tangent to $V(p) \cap S_{e_0}$ at p. In view of (12), this reads

$$\operatorname{span}\{(1, f(p)), (0, df_p), (-1, -H_p)\} = \operatorname{span}\{(1, f(p)), (0, df_p0), (0, H_p - f(p))\}.$$

However, one readily checks that $(0, H - f) \parallel \phi_{z\bar{z}}$ for any holomorphic coordinate z so that V is a rank 4 subbundle of $M \times \mathbb{R}^{n+1,1}$ of signature $(3, 1)$ given by

$$V = \operatorname{span}\{\phi, d\phi, \phi_{z\bar{z}}\}. \tag{16}$$

It is clear that V so defined is independent of all choices (of lift or holomorphic coordinate) and so is a conformal invariant.

To facilitate calculations we choose the unique section $\hat{\psi} \in \Gamma(V)$ with

$$\langle \hat{\psi}, \hat{\psi} \rangle = 0\,, \qquad \langle \psi, \hat{\psi} \rangle = -1\,, \qquad \text{and} \qquad \langle \hat{\psi}, d\psi \rangle = 0\,. \tag{17}$$

Then, given a holomorphic coordinate z, we have the Möbius invariant framing

$$\psi\,,\ \psi_z\,,\ \psi_{\bar{z}}\,,\ \hat{\psi}\ \in\ \Gamma(V) \tag{18}$$

of the bundle V, or rather $V \otimes \mathbb{C}$. The orthogonality relations of this frame are given by (14) and (17), namely

$$\langle \psi, \psi \rangle = \langle \hat{\psi}, \hat{\psi} \rangle = 0\,, \qquad \langle \psi, \hat{\psi} \rangle = -1 \tag{19a}$$

$$\langle \psi, d\psi \rangle = \langle \hat{\psi}, d\psi \rangle = \langle d\hat{\psi}, \hat{\psi} \rangle = 0\,, \tag{19b}$$

$$\langle \psi_z, \psi_z \rangle = \langle \psi_{\bar z}, \psi_{\bar z} \rangle = 0\,, \tag{19c}$$

$$\langle \psi_z, \psi_{\bar z} \rangle = \tfrac{1}{2}\,. \tag{19d}$$

We immediately obtain the fundamental equation of Möbius invariant surface geometry, an inhomogeneous Hill's equation,

$$\psi_{zz} + \tfrac{c}{2}\psi = \kappa \tag{20}$$

defining the complex valued function c and the section κ of $V^\perp \otimes \mathbb{C}$. In light of (3) and (4) we call c the *Schwarzian derivative* of the immersion f in the coordinate z, and denote it by

$$S_z(f) := c\,. \tag{21}$$

We shall see below that the section κ can be identified with the normal bundle valued *Hopf differential* of the immersion f, suitably scaled. Note that by construction both, c and κ, are Möbius invariants of the immersion f by given coordinate z. For surfaces in 3–space they form a complete set of invariants.

If the Hopf differential $\kappa \equiv 0$ is identically zero, the surface f is totally umbilic and we can view f as a conformal map taking values in $S^2 = \mathbb{CP}^1$. In this case we recover the classical Schwarzian discussed in the previous section 2.1.

Just as in the classical case, it is helpful to understand the behaviour of c and κ under coordinate changes. Let w be another holomorphic coordinate on M. Then the new normalized lift is given by (15). Computing $\tilde{\psi}_{ww}$ and inserting into (20) gives

$$2(\tilde{\kappa} - \kappa) z_w^2 |w_z| + (\tilde{c}|w_z| - c z_w^2 |w_z| + 2 z_w^2 |w_z|_{zz} + 2 z_{ww} |w_z|_z)\psi = 0\,. \tag{22}$$

Taking the $V^\perp$ part we see that

$$\tilde{\kappa} \tfrac{dw^2}{|dw|} = \kappa \tfrac{dz^2}{|dz|}\,. \tag{23}$$

Thus $\kappa \frac{dz^2}{|dz|}$ is a globally defined quadratic differential with values in $LV^\perp \otimes \mathbb{C}$ where L is the real line bundle $(\bar{K}K)^{-1/2}$ of densities of conformal weight 1 [8].

The ψ–part of (22) gives

$$\tilde{c} dw^2 = (c - 2v_{zz} + 2v_z^2) dz^2\,,$$

where we have put $|w_z| = e^v$. From (10) we therefore deduce the same transformation rule

$$S_w(f) dw^2 = (S_z(f) - S_z(w)) dz^2 \tag{24}$$

for the Schwarzian of a conformal immersion as for the Schwarzian of a holomorphic function.

We conclude this section by expressing the Möbius invariants $S_z(f)$ and κ in terms of Euclidean quantities of the immersion $f : M \to S^n$. If

$$|df|^2 = e^{2u} |dz|^2$$

is the induced metric of f in the holomorphic coordinate z on M, then

$$\psi = (1, f) e^{-u}$$

is the normalized lift (14) of f. Let $N_f M \subset f^\perp$ be the Euclidean normal bundle of M. We have an isometric isomorphism $N_f S^n \cong V^\perp$:

$$\xi \mapsto \langle H, \xi\rangle (1, f) + (0, \xi) \tag{25}$$

and so a section $\hat\kappa$ of $N_f S^n \otimes \mathbb{C}$ with

$$\kappa = \langle H, \hat\kappa\rangle(1, f) + (0, \hat\kappa)\,.$$

Inserting into (20) we obtain

$$\hat\kappa\, \tfrac{dz^2}{|dz|} = \frac{II^{(2,0)}}{|df|} \tag{26}$$

and

$$\tfrac{c}{2}\, dz^2 = \langle H, II^{(2,0)}\rangle + (u_{zz} - (u_z)^2) dz^2\,. \tag{27}$$

Here $II^{(2,0)}$ denote the $(2,0)$-part of the normal bundle valued second fundamental form of f. This shows that κ, up to the isomorphism (25), is the trace free part of the second fundamental form, i.e., the normal bundle valued Hopf differential, scaled by the square root of the induced metric. In particular, the *Willmore energy* of the conformal immersion f is given by

$$W(f) = \int |\kappa|^2\,. \tag{28}$$

3.3. **The integrability equations.** Unlike the case of holomorphic maps, where any Schwarzian can be integrated, the invariants for a conformal immersion $f : M \to S^n$ will have to satisfy integrability conditions. Recall that f gives rise to the Möbius invariant splitting

$$M \times \mathbb{R}^{n+1,1} = V \oplus V^\perp\,,$$

where $\mathbb{P}(\mathcal{L} \cap V)$ is the mean curvature sphere congruence. A choice of holomorphic coordinate z on M yields a unique normalized lift (14) which, by Hill's equation (20), defines the Schwarzian derivative $S_z(f)$ and the Hopf differential κ.

To compute the integrability conditions on the Schwarzian and the Hopf differential we work with the orthogonal frame

$$F = (\psi, \psi_z, \psi_{\bar z}, \hat\psi, \xi) \tag{29}$$

of $V \oplus V^\perp$ where ξ denotes a section of $V^\perp$. Clearly, the integrability equations for this frame are

$$F_{z\bar z} = F_{\bar z z}$$

or, by using reality conditions,

$$\psi_{zz\bar z} = \psi_{z\bar z z}\,, \tag{30a}$$

$$\operatorname{Im} \hat\psi_{z\bar z} = 0\,, \tag{30b}$$

$$\operatorname{Im} \xi_{z\bar z} = 0\,. \tag{30c}$$

To evaluate these equations, we will make frequent use of the formula

$$\phi = -\langle \phi, \hat\psi\rangle\psi - \langle\phi, \psi\rangle\hat\psi + 2\langle\phi, \psi_{\bar z}\rangle\psi_z + 2\langle\phi,\psi_z\rangle\psi_{\bar z}\,,$$

which expresses a section ϕ of V in the basis (18). Together with (20) we therefore get

$$\psi_{zz} = -\tfrac{c}{2}\psi + \kappa\,, \qquad \tfrac{c}{2} = \langle \psi_{zz}, \hat\psi\rangle = -\langle\psi_z, \hat\psi_z\rangle\,.$$

Moreover,

$$\psi_{z\bar z} = q\psi + \tfrac{1}{2}\hat\psi\,, \qquad q = -\langle \psi_{z\bar z}, \hat\psi\rangle = \langle \psi_z, \hat\psi_{\bar z}\rangle\,,$$

for some real valued function q. Therefore

$$\hat\psi_z = -c\psi_{\bar z} + 2q\psi_z + \chi$$

where χ is a section of $V^\perp \otimes \mathbb{C}$. We now evaluate the first integrability equation (30a)

$$(\kappa - \tfrac{c}{2}\psi)_{\bar z} = (q\psi + \tfrac{1}{2}\hat\psi)_z$$

which, using the last relation, unravels to

$$\kappa_{\bar z} = 2q\psi_z + (\tfrac{1}{2}c_{\bar z} + q_z)\psi + \tfrac{1}{2}\chi\,.$$

To compare V and $V^\perp$-parts of this equation, we note that the z-derivative of a section ξ in $V^\perp \otimes \mathbb{C}$ decomposes as

$$\xi_z = D_z\xi + \langle \xi, \chi\rangle\psi - 2\langle \xi, \kappa\rangle\psi_{\bar z}\,, \tag{31}$$

where D denotes the connection in the bundle $V^\perp$. Therefore,

$$D_{\bar z}\kappa = \tfrac{1}{2}\chi\,, \qquad q = -\langle \bar\kappa, \kappa\rangle\,, \qquad \tfrac{1}{2}c_{\bar z} + q_z = \langle \kappa, \bar\chi\rangle$$

and (30a) is equivalent to the *conformal Gauss equation*

$$\tfrac{1}{2}c_{\bar z} = 3\langle D_z\bar\kappa, \kappa\rangle + \langle \bar\kappa, D_z\kappa\rangle\,.$$

The second integrability condition (30b) becomes

$$0 = \mathrm{Im}\hat\psi_{z\bar z} = \mathrm{Im}(-c_{\bar z}\psi_{\bar z} - c\psi_{\bar z\bar z} + 2q_{\bar z}\psi_z + 2q\psi_{z\bar z} + \chi_{\bar z})\,.$$

By inserting the appropriate terms and keeping track of reality conditions, we therefore conclude that (30b) is equivalent to the *conformal Codazzi equation*

$$\mathrm{Im}(D_{\bar z}D_{\bar z}\kappa + \tfrac{1}{2}\bar c\kappa) = 0\,.$$

It is easy to see that for the remaining integrability equation (30c) only the $V^\perp$-part gives new conditions. Using (31) we immediately obtain the *conformal Ricci equation*

$$D_{\bar z}D_z\xi - D_zD_{\bar z}\xi - 2\langle \xi, \kappa\rangle\bar\kappa + 2\langle \xi, \bar\kappa\rangle\kappa = 0$$

for a section ξ of $V^\perp$. Note that

$$R^D_{\bar z z}\xi = D_{\bar z}D_z\xi - D_zD_{\bar z}\xi$$

is the curvature of the normal bundle $V^\perp$.

For further reference we collect the fundamental equations for a conformal immersion $f : M \to S^n$ in a given holomorphic coordinate z on M: keeping reality conditions in mind the equations of the frame (29) are given by

$$\begin{aligned} \psi_{z\bar z} &= -\langle \kappa, \bar\kappa\rangle\psi + \tfrac{1}{2}\hat\psi\,,\\ \psi_{zz} &= -\tfrac{c}{2}\psi + \kappa\,,\\ \hat\psi_z &= -2\langle \kappa, \bar\kappa\rangle\psi_z - c\psi_{\bar z} + 2D_{\bar z}\kappa\,,\\ \xi_z &= D_z\xi + 2\langle \xi, D_{\bar z}\kappa\rangle\psi - 2\langle \xi, \kappa\rangle\psi_{\bar z}\,. \end{aligned} \tag{32}$$

The resulting integrability conditions, the conformal Gauss, Codazzi and Ricci equations, are

$$\tfrac{1}{2}c_{\bar z} = 3\langle D_z\bar\kappa, \kappa\rangle + \langle \bar\kappa, D_z\kappa\rangle\,, \tag{33a}$$

$$\mathrm{Im}(D_{\bar z}D_{\bar z}\kappa + \tfrac{1}{2}\bar c\kappa) = 0\,, \tag{33b}$$

$$R^D_{\bar z z}\xi = D_{\bar z}D_z\xi - D_zD_{\bar z}\xi = 2\langle \xi, \kappa\rangle\bar\kappa - 2\langle \xi, \bar\kappa\rangle\kappa\,. \tag{33c}$$

These equations can be given an invariant co-ordinate free meaning using the technology of Cartan connections [6]. However this is not our aim here, where we intend mostly to focus on classical surface theory in 3-space. In that case the conformal Ricci equation is vacuous and we get the following fundamental theorem of conformal surface theory:

Theorem 3.1. *Let M be a Riemann surface and let $L = (K\bar K)^{-1/2}$ denote the 1-density bundle, where K is the canonical bundle of M.*

If $f_k : M \to S^3$ are two conformal immersions inducing the same Hopf differentials and Schwarzians, then there is a Möbius transformation $T : S^3 \to S^3$ with $Tf_1 = f_2$.

Conversely, let $\kappa\frac{dz^2}{|dz|} \in \Gamma(LK^2)$ be a quadratic differential with values in L and c be a Schwarzian derivative, i.e., c transforms according to (10)*. If κ and c satisfy the conformal Gauss and Codazzi equations then there exists a conformal immersion $f : M \to S^3$ with Möbius holonomy, Hopf differential κ, and Schwarzian derivative c.*

3.4. **Applications to old and new results in conformal surface theory.** At this stage it is worthwhile to pause and apply our conformally invariant setup to discuss some results of conformal surface theory. We content ourselves by showing the basic ideas and will work locally, leaving the global discussion as an exercise to the interested Reader.

The simplest question one can ask in any geometry is to characterize the homogeneous surfaces, i.e., surfaces which are 2-parameter orbits of the symmetry group under consideration. In Möbius geometry such surfaces are called the *cyclides of Dupin.* Clearly, such a surface must have Hopf differential and Schwarzian constant in the coordinates given by the 2-parameter group. Due to the transformation rule (21) the constancy of the Schwarzian has the invariant meaning that the Schwarzian is holomorphic (this amounts to the flatness of a certain normal Cartan connection on M [6]).

Theorem 3.2. *Let $f : M \to S^3$ be a conformal immersion, not contained in a 2-sphere, whose Schwarzian is holomorphic. Then f is a cyclide of Dupin.*

Proof. It suffices to show that κ and $S_w(f)$ are constant in an appropriate holomorphic coordinate w. Since c is holomorphic the conformal Gauss equation (33a) reads

$$(\kappa\bar\kappa^3)_z = 0\,,$$

expressing the fact that $\bar\kappa\kappa^3$ is holomorphic. By our assumption f is not contained in a 2-sphere so that κ is non-zero. Therefore

$$dw = (\bar\kappa\kappa^3)^{1/4}dz$$

defines a new holomorphic coordinate w on M. Due to the transformation rule (23) for sections of LK^2 we see that

$$\kappa = (w_z)^{3/2}\bar{w_z}^{-1/2}\tilde\kappa\,.$$

Combining these two equations gives $\tilde\kappa = 1$ and the conformal Codazzi equation (33b) forces $\tilde c$ to be imaginary, and thus constant. ∎

A further basic question to ask is how much of the invariants of an immersed surface are actually needed to determine the surface up to symmetry. In the Euclidean setup this is the Bonnet problem: generically the induced metric and the mean curvature determine a surface up to rigid motions. In Möbius geometry one expects that generically the Hopf differential should determine a surface up to Möbius transformations.

However, let us recall the notion of *isothermic* surfaces [1, 7, 10]. Such surfaces are conformally parameterized by their curvature lines away from umbilic points. Put differently, away from umbilics there are holomorphic coordinates in which the Hopf differential κ is *real valued*. In this case the conformal Gauss and Codazzi equations simplify to

$$c_{\bar z} = 4(\kappa^2)_z \,,$$
$$\operatorname{Im}(\kappa_{\bar z\bar z} + \tfrac{1}{2}\bar c\kappa) = 0\,.$$

Assuming κ non-zero the two equations combine to Calapso's equation [7]

$$\triangle \tfrac{\kappa_{xy}}{\kappa} + 8(\kappa^2)_{xy} = 0$$

where $z = x + iy$. The conformal Gauss and Codazzi equations are invariant under deformations of the Schwarzian by

$$c_r = c + r$$

where $r \in \mathbb{R}$ is a real parameter, the spectral parameter of algebro-geometric integrable system theory. By theorem 3.1 we thus see that an isothermic surface comes in an associated family parameterized by $\mathbb{R}$: these are the T-transforms of Calapso and Bianchi [2, 7]. Note that since all c_r are distinct the surfaces in the associated family are non-congruent with respect to the Möbius group. Therefore the Hopf differential alone does not determine isothermic surfaces. But these surfaces are the only exceptions:

Theorem 3.3. *Let $f_k : M \to S^3$ be two non-congruent conformal immersions inducing the same Hopf differentials. Then f_1 and f_2 are isothermic surfaces in the same associated family.*

Proof. Since $\kappa = \kappa_k$ the conformal Codazzi equation implies that the difference of the Schwarzians

$$c dz^2 := c_1 dz^2 - c_2 dz^2 \in H^0(K^2)$$

is a holomorphic quadratic differential. Since the f_k are non-congruent, c is not identically zero, and so we can choose holomorphic coordinates such that

$$c = 1.$$

The conformal Codazzi equation now implies

$$\operatorname{Im}(\kappa) = 0\,,$$

so that both surfaces are isothermic and in the same associated family. □

We conclude this section with a brief discussion of (constrained) Willmore surfaces: it can be checked that a conformal immersion $f : M \to S^n$ is Willmore if and only if

$$\operatorname{Re}(D_{\bar z}D_{\bar z}\kappa + \tfrac{1}{2}\bar c\kappa) = 0\,.$$

Therefore the integrability conditions for a Willmore surface are

$$D_{\bar z}D_{\bar z}\kappa + \tfrac{1}{2}\bar c\kappa = 0\,,$$
$$\tfrac{1}{2}c_{\bar z} = 3\langle D_z\bar\kappa, \kappa\rangle + \langle \bar\kappa, D_z\kappa\rangle\,,$$
$$R^D_{\bar z z}\xi = 2\langle \xi, \kappa\rangle\bar\kappa - 2\langle \xi, \bar\kappa\rangle\kappa\,.$$

More generally, as we shall see below, a surface is *constrained Willmore*—that is, it extremizes the Willmore functional with respect to variations through conformal immersions—if

$$D_{\bar z}D_{\bar z}\kappa + \tfrac{1}{2}\bar c\kappa = \mathrm{Re}(\bar q\kappa) \tag{34}$$

for some holomorphic quadratic differential $qdz^2 \in H^0(K^2)$. These systems have the symmetry

$$\kappa_\lambda = \lambda\kappa, \qquad c_\lambda = c + (\lambda^2 - 1)q \qquad q_\lambda = \lambda^2 q \tag{35}$$

for unitary $\lambda \in S^1$, which describes the associated family of (constrained) Willmore surfaces in our setup. This spectral deformation is particularly simple for Willmore surfaces (q=0): here the Schwarzian is fixed and the scaled Hopf differential $\kappa\frac{dz^2}{|dz|} \in \Gamma(LK^2)$ is rotated by a phase. All this is reminiscent of the theory of constant mean curvature surfaces in 3-space where the associated family is obtained by rotating the Euclidean Hopf differential by a phase and fixing the induced metric. As we shall see, this similarity is not a coincidence.

A classical result of Thomsen [24] characterizes isothermic Willmore surfaces in 3-space as minimal surfaces in some space-form (conic section). More generally, Richter [19] proves that constant mean curvature surfaces in space forms are isothermic and constrained Willmore. These results are easy to see in our setup: from (12) we see that a surface has mean curvature H in the conic section S_{v_0} exactly when $H = -\langle v_0, N\rangle$ for N a unit length section of $V^\perp$ (in particular, a surface is minimal if V contains a constant section v_0). Moreover, the metric induced from S_{v_0} is given by

$$e^{2u}|dz|^2 = \frac{|dz|^2}{\langle\psi, v_0\rangle^2}. \tag{36}$$

Suppose then that f has constant mean curvature H in S_{v_0} with induced metric $e^{2u}|dz|^2$. We write

$$v_0 = a\psi + b\psi_z + \bar b\psi_{\bar z} + \hat a\hat\psi - HN \tag{37}$$

for real valued functions a, $\hat a$ and a complex valued function b. Writing out $v_{0,z}$ in terms of our frame and using (32), we see that the constancy of v_0 amounts to

$$b\kappa + 2\hat a\kappa_{\bar z} = 0 \tag{38a}$$

$$\hat a_z + \tfrac{1}{2}\bar b = 0 \tag{38b}$$

$$\bar b_z - c\hat a + 2H\kappa = 0 \tag{38c}$$

$$a + b_z - 2\hat a|\kappa|^2 = 0 \tag{38d}$$

$$a_z - \tfrac{c}{2}b - \bar b|\kappa|^2 - 2H\kappa_{\bar z} = 0 \tag{38e}$$

From (36) we have $\hat a = e^{-u}$ and then (38a) and (38b) yield

$$e^{-2u}(e^u\kappa)_{\bar z} = 0$$

so that the Hopf differential $e^u\kappa dz^2$ is holomorphic. Thus, away from umbilics, we can choose the holomorphic coordinate so that $e^u\kappa = 1$ or, equivalently, that $\hat a = \kappa$. In particular, κ is real and the surface is isothermic. Now (38b) gives $b = -2\kappa_{\bar z}$ while (38d) gives $a = 2(\kappa^3 + k_{z\bar z})$. This leaves (38c) which reads

$$\kappa_{\bar z\bar z} + \tfrac{\bar c}{2}\kappa = H\kappa$$

so that ψ is Willmore if $H = 0$ and constrained Willmore otherwise, while, in view of the Gauss equation, (38e) is an identity.

Conversely, given a surface with κ real and

$$\kappa_{\bar z\bar z} + \tfrac{\bar c}{2}\kappa = H\kappa \tag{39}$$

for some constant H we define v_0 by (37) with the same choice of coefficients as above and conclude from (38) that v_0 is constant whence ψ gives a surface of constant mean curvature H in the space form S_{v_0} of constant curvature K given by

$$K = -\langle v_0, v_0\rangle = -H^2 + 4\kappa^4 + 4\kappa\kappa_{z\bar{z}} - 4|\kappa_z|^2 \tag{40}$$

(see [16] for a recent analysis of the soliton theory of this equation).

The isothermic spectral deformation $c_r = c + r$, $r \in \mathbb{R}$, preserves the class of constant mean curvature surfaces: in view of (39), we have

$$\kappa_{\bar{z}\bar{z}} + \tfrac{\bar{c_r}}{2}\kappa = (H + \tfrac{r}{2})\kappa$$

so that the corresponding surface ψ_r has constant mean curvature $H_r = H + \frac{r}{2}$ is a space form of curvature K_r where, in view of (40), $K_r + H_r^2$ is independent of r. Moreover, all these surfaces are isometric with metric $\kappa^{-2}|dz|^2$. Thus our spectral deformation recovers the family of constant mean curvature surfaces related by the Lawson correspondence [14].

Moreover, the spectral symmetry (35) of constrained Willmore surfaces preserves the isothermic condition: if κ is real for a coordinate z then, by (23), κ_λ is real in the rotated coordinate $w = \sqrt{\lambda}z$.

For constant mean curvature surfaces, these two deformations can be combined to give an action of $\mathbb{C}\setminus\{0\}$ on such surfaces. Indeed, for a surface of constant mean curvature H in a space form of curvature K , we have seen that there is a coordinate z for which κ is real and $q = H$. For $\lambda \in \mathbb{C}\setminus\{0\}$, set

$$\kappa_\lambda = \tfrac{\lambda}{|\lambda|}\kappa \qquad c_\lambda = c + 2(\lambda - 1)q \qquad q_\lambda = \lambda q.$$

The fact that $\bar{q}\kappa$ is real ensures that each $\kappa_\lambda, c_\lambda, q_\lambda$ solves (33) and (34). Further, in the rotated coordinate $w = \sqrt{\frac{\lambda}{|\lambda|}}z$, κ_λ is real (in fact $\kappa_\lambda dz^2 = \kappa dw^2$) while $q_\lambda dz^2 = |\lambda| H dw^2$ so that, with respect to w, we have a solution of (39) with $H_\lambda = |\lambda|H$. From (40), we conclude that the surface ψ_λ has constant mean curvature $|\lambda|H$ and metric $\kappa^{-2}|dz|^2$ in a space form of curvature $K_\lambda = K + (1 - |\lambda|^2)H$. In particular, for $\lambda \in S^1$, we have a family of isometric surfaces in a fixed space form with fixed H and rotated Hopf differential—this is the classical associated family. For minimal surfaces, the action of $\mathbb{R}\setminus\{0\}$ is trivial but when $H \neq 0$ we get (part of) the Lawson deformation so that these two well-known symmetries are unified in a single $\mathbb{C}\setminus\{0\}$ action.

For M a torus, any holomorphic quadratic differential is of the form qdz^2 with q constant so that we conclude with Richter [19] that *all* constrained Willmore isothermic tori have constant mean curvature in some space form. However, without this global hypothesis, there remains the possibility of isothermic constrained Willmore surfaces that are not spectral deformations of isothermic Willmore surfaces.

One might ask if a version of Thomsen's result holds in n-space. We restrict attention to isothermic Willmore surfaces of which obvious examples, in addition to the above mentioned minimal surfaces, are cylinders in $\mathbb{R}^4$ over elastic curves in $\mathbb{R}^3$, cones over elastic curves in S^3 and rotational surfaces about the plane at infinity of elastic curves in hyperbolic 3-space H^3. These surfaces are clearly isothermic and, by the principle of symmetric criticality, Willmore.

Theorem 3.4. *Let $f : M \to S^n$ be an isothermic Willmore surface with $n \geq 3$. Then either f is minimal in 3-space, or an isothermic Willmore surfaces in 4-space described by an ODE: the equation describing elastica or a Painleve-type equation for the Hopf differential with radial symmetry.*

We begin by collecting the integrability equations for isothermic Willmore surfaces: since $\kappa = \bar{\kappa}$ is real, Ricci's equation immediately implies that the normal bundle is flat. We can therefore assume $V^\perp = M \times \mathbb{R}^{n-2}$

and D is just directional derivative. The two remaining integrability conditions, the Gauss and Codazzi equations, become

$$\kappa_{zz} + \tfrac{c}{2}\kappa = 0\,,$$
$$c_{\bar z} = 4\langle \kappa, \kappa\rangle_z\,,$$

for the $\mathbb{C}$-valued Schwarzian c and the $\mathbb{R}^{n-2}$-valued Hopf differential κ. Differentiating the Codazzi equation twice with respect to $\bar z$ and keeping reality conditions and the Gauss equation in mind, we obtain

$$\operatorname{Im}\langle \kappa, \kappa\rangle_z \kappa_{\bar z} = 0\,. \tag{41}$$

First note that this is automatically satisfied if $n = 3$, in which case we already know that f is minimal. From now on we assume that $n \geq 4$. If κ is a nonzero constant, then Gauss' equation implies that $c = 0$ and by using the frame equations (32), one can check that f is the Clifford torus. The case $\kappa = 0$ results, as we already have discussed, in a holomorphic map f into the 2-sphere. Therefore we may assume that κ is non-constant. If κ has constant length $\langle \kappa, \kappa\rangle \neq 0$ then $\langle \kappa_z, \kappa\rangle = 0$, and differentiating yields

$$0 = \langle \kappa_{zz}, \kappa\rangle + \langle \kappa_z, \kappa_z\rangle = -\tfrac{c}{2}\langle \kappa, \kappa\rangle + \langle \kappa_z, \kappa_z\rangle\,.$$

From the Gauss and Codazzi equations we see that c is holomorphic and that $\langle \kappa_z, \kappa_z\rangle$ is anti-holomorphic. Therefor both are constant. After a rotation of the coordinate we may assume that c is a real constant. Then κ is contained in a rotated $\mathbb{R}^{n-2}$ in $\mathbb{C}^{n-2}$ and has constant length. Decomposing the complex Codazzi equation according to the splitting $\mathbb{C}^{n-2} = \mathbb{R}^{n-2} \oplus i\mathbb{R}^{n-2}$, we get

$$\kappa_{xx} - \kappa_{yy} + 2c\kappa = 0\,, \qquad \text{and} \qquad \kappa_{xy} = 0\,.$$

The latter implies $\kappa(x, y) = k_1(x) + k_2(y)$ which, inserted into the former, gives

$$f''(x) + 2cf(x) = g''(y) - 2cg(y) = \text{constant}\,.$$

So k_1 and k_2 solve linear ordinary differential equations which can be explicitly integrated. If $c = 0$ only constant κ's are solutions of constant length. If $c > 0$ then any constant length solution is of the form

$$\kappa(x, y) = \cos(\sqrt{2c}\,x)\kappa_1 + \sin(\sqrt{2c}\,x)\kappa_2$$

for $\kappa_1,\ \kappa_2 \in \mathbb{R}^{n-2}$ two orthogonal vectors of same length. The corresponding surfaces are the rotational surfaces, cylinders and cones over free elastic helixes in the respective 4-dimensional space forms.

We now may assume that the length of κ is non-constant. In this case (41) is equivalent to

$$\kappa = k(g) \qquad \text{with} \qquad g = \langle \kappa, \kappa\rangle\,,$$

for some $\mathbb{R}^{n-2}$-valued function k of one variable. Inserting into the Codazzi equation yields the second order differential equation

$$g_z^2 k'' + g_{zz} k' + \tfrac{c}{2} k = 0$$

for the function k. This shows that the image of κ is contained in an at most 2-dimensional parallel subbundle of $E \subset V^\perp$. Using the fourth frame equation in (32), we deduce that the complementary subbundle $E^\perp \subset V^\perp$ is a constant subspace in the ambient space. We therefore conclude that f takes values in S^4.

Assuming from now on that our isothermic Willmore surface f is fully contained in 4-space, its trivial normal bundle has a well-defined 90-degree rotation J. Since $\langle \kappa, J\kappa\rangle = 0$, the Codazzi equation gives

$$\langle \kappa_z, J\kappa\rangle_z = 0\,.$$

From this we deduce that

$$\langle \kappa_z, J\kappa\rangle = \bar h^{-1}\,, \qquad h_{\bar z} = 0$$

for a holomorphic function h. Note that h is nonzero, otherwise κ would have values in a 1-dimensional parallel subbundle of the normal space and thus f would not be full in 4-space. Since $\kappa = k(g)$ we get

$$\tfrac{h}{|h|^2} = \langle \kappa_z, J\kappa\rangle = \langle k'(g), Jk(g)\rangle g_z$$

implying that h is a real multiple of $g_z = \langle \kappa, \kappa\rangle_z$. Multiplying (41) by this real multiple gives

$$\operatorname{Im} h\kappa_{\bar z} = 0\,.$$

But this is equivalent to κ depending only on $\operatorname{Re} w$, where

$$w_z = h$$

is a new holomorphic coordinate. We may thus assume that

$$\kappa = k(w + \bar w)$$

for some $\mathbb{R}^2$-valued function of one variable, which we again call k. Inserting back into the Codazzi equation yields

$$h^2 k'' + h_z k' + \tfrac{c}{2} k = 0 \qquad \text{or} \qquad k'' + \tfrac{h_z}{h^2} k' + \tfrac{c}{2h^2} k = 0\,.$$

Now f is assumed to be full in 4-space so that k and k' are linearly independent. Since k is real, we conclude that

$$h_z = \alpha h^2\,, \qquad \alpha \in \mathbb{R}$$

and then that

$$\frac{c}{2h^2} = \rho(w + \bar w)\,. \tag{42}$$

The first relation tells us that $h = \frac{1}{az+b}$ is a special fractional linear coordinate change. Taking $\bar z$ derivative of the second relation and applying the Gauss equation unravels to

$$\rho' = \frac{2\langle k, k\rangle'}{|h|^2}\,.$$

We have two cases to discuss: first, if h is constant the last relation can be integrated to

$$\rho = 2\frac{\langle k, k\rangle + \gamma}{|h|^2}\,, \qquad \gamma \in \mathbb{R}\,,\ \ h \in \mathbb{C}\,.$$

This yields the formula

$$c = 2\frac{h}{\bar h}(\langle k, k\rangle + \gamma) \tag{43}$$

for the Schwarzian of our surface f. Inserting back into the Codazzi equation gives the second order ODE for spatial elastica

$$k'' + \frac{h}{\bar h}(\langle k, k\rangle + \gamma)k = 0 \tag{44}$$

the Hopf differential $\kappa = k(hz + \bar h\bar z)$ has to satisfy.

In the second case the fractional linear coordinate change is, after a translation and real scale, of the form

$$h(z) = \frac{1}{z}\,.$$

Then $w + \bar w = 2\ln|z|$ and

$$\rho' = 2\langle k, k\rangle |z|^2\,, \qquad k'' - k + \rho k = 0\,,$$

is and ODE for the Hopf differential k depending on the radius only. This case is similar to the intrinsic rotational constant mean curvature surfaces [3, 22, 25].

Finally, solving the respective ODE's determines the Hopf differential and the Schwarzian is computed from (42). Inserting the invariants into the frame equations (32) allows to determine the surface f. We leave the detailed calculations to the interested Reader.

4. The Novikov–Veselov and Davey–Stewartson flows

Over the last 15 years a number of papers have been written on *the soliton theory of surfaces.* In all cases only special surface classes, like constant (mean) curvature, isothermic or Willmore surfaces, were considered. In case the underlying surface is a 2-torus it has been shown that those surfaces allow an infinite hierarchy of commuting flows. Surfaces stationary under a finite number of flows could be explicitly constructed by noticing that those flows are given by the linear flows on the Jacobian of some auxiliary Riemann surface, the so-called *spectral curve.*

More recently, Konopelchenko and his collaborators [11, 12, 13, 23] have shown how to construct hierarchies of commuting flows on the space of *all* conformally immersed tori in 3 and 4-dimensional spheres based on the modified Novikov–Veselov and Davey–Stewartson hierarchies.

Following our construction of the KdV-hierarchy, we are now going to give a manifestly conformally invariant construction of such flows and show how the special surface classes mentioned above are invariant under them.

4.1. **Deformations of conformal immersions.** Let $M = T^2$ be a 2-torus and $X \in H^0(TM)$ a holomorphic vector field so that $\frac{\partial}{\partial z} = \frac{1}{2}(X - iJX)$ for a holomorphic coordinate z on the universal cover $\mathbb{R}^2$. Given a deformation $f(t): M \to S^n$ of conformal immersions with $f = f(0)$, we get the unique normalized lifts

$$\psi(t): M \to \mathcal{L}_+$$

from (14) satisfying

$$|d\psi(t)|^2 = |dz|^2 \,.$$

Therefore we have the canonical frame (18) of $V(t)$ for all times t. The deformation of this frame is determined by the variation

$$\psi_t = a\psi + b\psi_z + \bar{b}\psi_{\bar{z}} + \sigma \tag{45}$$

of the normalized lift ψ. Here $\sigma(t) \in \Gamma(V(t)^\perp)$ is the normal variation, the complex valued function $b(t): M \to \mathbb{C}$ determines the tangential and $a(t): M \to \mathbb{R}$ the radial variation. Since ψ is null, there is no $\hat{\psi}$ component. For the deformation $f(t)$ to remain conformal, i.e.

$$|d\psi|^2_t = 0\,, \tag{46}$$

the radial and tangential deformations a and b have to be determined by the normal deformation σ. To compute those relations, we need

$$\begin{aligned}(\psi_z)_t &= (\psi_t)_z \\ &= (a_z - \tfrac{c}{2}b - \bar{b}|\kappa|^2 + 2\langle\sigma, D_{\bar{z}}\kappa\rangle)\psi + (a + b_z)\psi_z + (\bar{b}_z - 2\langle\sigma,\kappa\rangle)\psi_{\bar{z}} + \tfrac{\bar{b}}{2}\hat{\psi} + (b\kappa + D_z\sigma)\end{aligned} \tag{47}$$

where we have used the frame equations (32). Inserting into (46) gives

$$\bar{b}_z = 2\langle\sigma,\kappa\rangle \qquad \text{and} \qquad a = -\mathrm{Re}\, b_z\,, \tag{48}$$

of which the first is a $\bar{\partial}$-problem solvable over a torus if and only if the right hand side has vanishing integral $\int_M \langle\sigma,\kappa\rangle = 0$.

Using (48) we can eliminate a in (47) and rewrite

$$(\psi_z)_t = (-\tfrac{1}{2}b_{zz} - \tfrac{c}{2}b - \bar{b}|\kappa|^2 + \langle\sigma, D_{\bar z}\kappa\rangle - \langle D_{\bar z}\sigma, \kappa\rangle)\psi + (\operatorname{Im} b_z)\psi_z + \tfrac{\bar b}{2}\hat\psi + (b\kappa + D_z\sigma)\,. \tag{49}$$

(Here $\mathrm{Im} b_z = \frac{1}{2}(b_z - \bar{b_z})$ and so is pure imaginary.)

We now can calculate the remaining deformations. First note that

$$\hat\psi_t = (\operatorname{Re} b_z)\hat\psi - 4\operatorname{Re}(\langle\hat\psi, \psi_{\bar z t}\rangle\psi_z) + \tau \tag{50}$$

where, using (49), we have

$$2\langle\hat\psi, \psi_{zt}\rangle = b_{zz} + cb + 2\bar b|\kappa|^2 - 2\langle\sigma, D_{\bar z}\kappa\rangle + 2\langle D_{\bar z}\sigma, \kappa\rangle\,.$$

Moreover, the normal variation τ is obtained from (32),

$$\tau = (\hat\psi_t)^\perp = 2((\psi_{zt\bar z})^\perp + |\kappa|^2\sigma)\,,$$

which unravels to

$$\tfrac{\tau}{2} = D_{\bar z}D_z\sigma + 2\langle\sigma, \bar\kappa\rangle\kappa + |\kappa|^2\sigma + 2\operatorname{Re}(bD_{\bar z}\kappa)\,.$$

Note that the latter is indeed real due to the Ricci equation (33c). Finally, if ξ is normal then its tangential deformation calculates to

$$(\xi_t)^T = \langle\xi, \tau\rangle\psi + \langle\xi, \sigma\rangle\hat\psi - 4\operatorname{Re}(\langle\xi, b\kappa + D_z\sigma\rangle\psi_{\bar z})\,. \tag{51}$$

To calculate the deformations of the Hopf differential κ and the Schwarzian c, we use Hill's equation (20) and obtain

$$\kappa_t = \psi_{tzz} + \tfrac{c_t}{2}\psi + \tfrac{c}{2}\psi_t\,.$$

Taking the $V^\perp$ part, together with (49) and (32), gives

$$D_t\kappa = D_zD_z\sigma + \tfrac{c}{2}\sigma + (\mathrm{Im} b_z + b_z)\kappa + bD_z\kappa + \bar bD_{\bar z}\kappa\,. \tag{52}$$

Similarly, using (50) and taking an inner product with $\hat\psi$ we get

$$\begin{aligned}\tfrac{1}{2}c_t =& \tfrac{1}{2}b_{zzz} + cb_z + \tfrac{1}{2}(bc_z + \bar b c_{\bar z}) + 8\langle\sigma, \bar\kappa\rangle\langle\kappa, \kappa\rangle + \\ & 3\langle D_{\bar z}D_z\sigma, \kappa\rangle - \langle\sigma, D_{\bar z}D_z\kappa\rangle - 3\langle D_z\sigma, D_{\bar z}\kappa\rangle + \langle D_{\bar z}\sigma, D_z\kappa\rangle\,.\end{aligned} \tag{53}$$

For later use, we need to record the deformation of the Willmore energy (28). Up to exact forms

$$\tfrac{1}{2}|\kappa|^2_t = \operatorname{Re}\langle D_t\kappa, \bar\kappa\rangle \equiv \operatorname{Re}\langle D_zD_z\sigma + \tfrac{c}{2}\sigma, \bar\kappa\rangle \equiv \langle\sigma, D_zD_z\bar\kappa + \tfrac{c}{2}\bar\kappa\rangle\,,$$

where we used (52) together with the product rule. Note that due to Codazzi's equation (33b) the last expression is real. Therefore the Willmore energy deforms by

$$W_t = 2\int\langle\sigma, D_zD_z\bar\kappa + \tfrac{c}{2}\bar\kappa\rangle\,. \tag{54}$$

Note that $W_t = 0$ for compactly supported variations if and only if

$$D_{\bar z}D_{\bar z}\kappa + \tfrac{1}{2}\bar c\kappa = \operatorname{Re}(\bar q\kappa)$$

for some holomorphic quadratic differential qdz^2. This follows from the constraint $\int\langle\sigma, \kappa\rangle = 0$ on the normal variations (48), expressing the fact that we deform via *conformal* immersions $f(t) : M \to S^n$ rather then *all* immersions. The resulting critical points are the *constrained Willmore* immersions. Every Willmore immersion, $q = 0$, clearly is constrained Willmore.

4.2. **The KdV flows revisited.** As a warm up, we make contact with our discussion of the KdV flows earlier on. We already have indicated that holomorphic maps into $\mathbb{CP}^1$ are characterized by vanishing Hopf differential $\kappa \equiv 0$. In this case the $\bar\partial$-problem (48) has the holomorphic solution $b = c$. Inserting into the variation of the Schwarzian (53), we immediately obtain the KdV equation

$$c_t = c_{zzz} + 3c_z c\,.$$

4.3. **The Novikov–Veselov flows.** Adapting our recursive scheme in the construction of the KdV hierarchy in Section 2.2 to deformations of conformal immersion, we outline the first steps in the construction of a commuting hierarchy of odd-order flows on conformally immersed tori. For these flows to restrict to the known flows on say Willmore surfaces, they should at least preserve the Willmore energy (28). Contrary to the existing literature [23], where a hierarchy of flows for surfaces in 3-space is described in terms of Euclidean data, our flows are Möbius invariant from the onset.

Any conformal deformation $f(t): M \to S^n$ is determined, up to the solution of a $\bar\partial$-problem (48), by the normal variation (45). The most naive flow, which we will label as the first flow, has no normal variation, i.e., $\sigma = 0$, and thus $b = 1$. Then (52), (53) give

$$D_{t_1}\kappa = (D_z + D_{\bar z})\kappa \qquad \text{and} \qquad c_{t_1} = \tfrac{1}{2}(c_z + c_{\bar z})\,,$$

which is just translational flow in X direction.

Our previous scheme—the new normal deformation is given by the old variation of the invariants—suggests $\sigma = D_{t_1}\kappa$ for the construction of the next flow up. This is certainly reasonable, since both, κ and σ, are normal bundle valued. Since σ has to be *real* and the resulting flow should preserve the Willmore energy (28),

$$\sigma = \operatorname{Re} D_z\kappa \tag{55}$$

turns out to be the right choice for the third flow. Let us first check that we can solve the $\bar\partial$-problem $\bar b_z = 2\langle\sigma,\kappa\rangle$: modulo exact forms, we obtain

$$2\langle \operatorname{Re} D_z\kappa, \kappa\rangle \equiv \langle D_{\bar z}\bar\kappa, \kappa\rangle \equiv -\langle \bar\kappa, D_{\bar z}\kappa\rangle \equiv 0\,,$$

where the last identity uses the Gauss equation (33a). Thus we can solve for b, and hence for a, in (48) to get the variation (45) of ψ respectively f. The solution b of the $\bar\partial$-problem on the torus M is determined only up to a constant for each time t. To obtain a well defined flow in t, we normalize b so that $\int b = 0$.

To see whether the Willmore energy is preserved, we calculate (54) the expression

$$\langle \operatorname{Re} D_z\kappa, D_{\bar z}D_{\bar z}\kappa + \tfrac{1}{2}\bar c\kappa\rangle = \operatorname{Re}\langle D_z\kappa, D_{\bar z}D_{\bar z}\kappa + \tfrac{1}{2}\bar c\kappa\rangle\,, \tag{56}$$

where we note that $D_{\bar z}D_{\bar z}\kappa + \frac{1}{2}\bar c\kappa$ is real by Codazzi's equation (33b). Up to exact forms

$$\begin{aligned}
-\langle D_z\kappa, D_{\bar z}D_{\bar z}\kappa + \tfrac{1}{2}\bar c\kappa\rangle &\equiv \langle D_{\bar z}D_z\kappa, D_{\bar z}\kappa\rangle + \tfrac{1}{4}\bar c_z\langle\kappa,\kappa\rangle \\
&\equiv \langle R^D_{\bar z z}\kappa, D_{\bar z}\kappa\rangle + (\tfrac{3}{2}\langle D_{\bar z}\kappa,\bar\kappa\rangle + \tfrac{1}{2}\langle\kappa, D_{\bar z}\bar\kappa\rangle)\langle\kappa,\kappa\rangle \\
&\equiv \tfrac{3}{2}\langle R^D_{\bar z z}\kappa, D_{\bar z}\kappa\rangle\,.
\end{aligned}$$

Here we used Gauss' equation (33a), the Ricci equation (33c) and the formula

$$\langle\kappa,\kappa\rangle\langle\kappa,\bar\kappa\rangle_{\bar z} = -\langle R^D_{\bar z z}\kappa, D_{\bar z}\kappa\rangle + (3\langle D_{\bar z}\kappa,\bar\kappa\rangle + \langle\kappa, D_{\bar z}\bar\kappa\rangle)\langle\kappa,\kappa\rangle\,.$$

Therefore we conclude that our third order flow, which we will call the *Novikov–Veselov flow*, preserves the Willmore energy in case flatness of the normal bundle is preserved. This is certainly the case for surfaces in 3-space and also, as we will see later on, for isothermic surfaces in any codimension.

At present we are unable to say more about the general codimension case for the Novikov–Veselov flow. This is mainly due to our low technology setup, which makes any discussion of the normal bundle flow a notational debauch.

There is still much to do here: first we must see that our flow coincides with that defined in Euclidean terms in the existing literature [13, 23]. Further a recursive scheme for constructing higher flows is required. For $n = 3$, the first question is answered in the affirmative by Richter [19] who proposes also $\sigma_{2n+1} = \mathrm{Re}(D_z)^n\kappa$ for the higher flows. We shall return to these matters elsewhere.

4.4. **The Davey–Stewartson flows.** In the previous section we indicated how to construct a hierarchy of Möbius invariant odd order flows on immersed tori in n-space. To obtain even order flows one needs more structure in the normal bundle, which is available for surfaces in 4-space. There, the normal bundle has a complex structure J which we can use to define a zero order flow, namely rotation in the normal bundle by J. The corresponding deformation of the Hopf differential is given by

$$D_{t_0}\kappa = J\kappa$$

and we take

$$\sigma = \mathrm{Re}\, J\kappa$$

as our normal deformation for the construction of the second order flow. To obtain the conformal deformation $f(t) : M \to S^4$, we have to solve for the tangential variation b in (45). As before, this amounts to solving the $\bar\partial$-problem $\bar b_z = 2\langle\sigma,\kappa\rangle$. Now

$$2\langle\sigma,\kappa\rangle = \langle J\kappa,\kappa\rangle + \langle J\bar\kappa,\kappa\rangle = \langle J\bar\kappa,\kappa\rangle$$

since J is skew. But Ricci's equation (33c) implies

$$\int\langle J\bar\kappa,\kappa\rangle = \pi\deg N_f M\,,$$

so that the $\bar\partial$-problem (48) is solvable if and only if the normal bundle of f has degree zero. This condition is preserved under any continuous deformation, and we obtain a unique second order flow by normalizing $\int b = 0$.

This flow also preserves the Willmore energy (54): up to exact forms

$$\langle\mathrm{Re}\, J\kappa, D_{\bar z}D_{\bar z}\kappa + \tfrac{1}{2}\bar c\kappa\rangle = \mathrm{Re}\langle J\kappa, D_{\bar z}D_{\bar z}\kappa\rangle \equiv \mathrm{Re}\langle D_{\bar z}\kappa, JD_{\bar z}\kappa\rangle = 0\,,$$

where we used that $DJ = 0$ and J is skew.

Again there is more to be done, both in making contact with the existing literature [11, 12] and constructing higher flows.

4.5. **Flows on isothermic surfaces.** As an explicit example of the above discussions, we apply the Novikov–Veselov and Davey–Stewartson flow to isothermic surfaces in S^n [5]. Recall that $f : M \to S^n$ is isothermic if the Hopf differential is real, i.e., if $\kappa = \bar\kappa$. Note that in this case Ricci's equation (33c) immediately implies the flatness of the normal bundle. Thus we can define a "vector" version of the Novikov–Veselov flow and the Davey–Stewartson flow in 4-space on isothermic surfaces, both preserving the Willmore energy, once we have shown that the isothermic condition is preserved under the flows.

First we note that in the isothermic case the $\bar{\partial}$-problem can be explicitly solved for both flows. Since κ is real, the integrability equations for an isothermic surface become

$$\begin{aligned} c_{\bar z} &= 4\langle \kappa, \kappa\rangle_z\,, \\ \operatorname{Im}\,(D_{\bar z} D_{\bar z}\kappa + \tfrac{1}{2}\bar{c}\kappa) &= 0\,, \\ R^D &= 0\,. \end{aligned}$$

As discussed in section 4.3 the Novikov–Veselov flow is (up to the factor of 2) given by the normal variation $\sigma = 2\mathrm{Re}\, D_z\kappa$. Therefore

$$\bar{b}_z = 2\langle \sigma, \kappa\rangle = \langle \kappa, \kappa\rangle_z + \langle \kappa, \kappa\rangle_{\bar z} = (\langle \kappa, \kappa\rangle + \tfrac{\bar c}{4})_z$$

and we obtain the explicit solution

$$b = \langle \kappa, \kappa\rangle + \tfrac{c}{4}$$

of the $\bar{\partial}$-problem (48). To see that the Novikov–Veselov flow preserves isothermic surfaces, we need to check whether $\kappa = \bar\kappa$ is preserved under the flow. Recalling (52) and the fact that κ is real, we immediately calculate

$$\begin{aligned} \operatorname{Im}(D_{t_3}\kappa) &= \operatorname{Im}(D_z D_z D_z\kappa + D_z D_z D_{\bar z}\kappa + \tfrac{c}{2}(D_z + D_{\bar z})\kappa + 2b_z\kappa) \\ &= (D_z + D_{\bar z})\operatorname{Im}(D_z D_z\kappa + \tfrac{c}{2}\kappa) - \tfrac{1}{2}\operatorname{Im}(c_{\bar z} - 4\langle\kappa,\kappa\rangle_z)\kappa \\ &= 0\,, \end{aligned}$$

where we used the integrability equations. Therefore isothermic surfaces are preserved under the Novikov–Veselov flow and

$$D_{t_3}\kappa = \operatorname{Re}(2D_z D_z D_z\kappa + \tfrac{3}{2}cD_z\kappa + \tfrac{3}{4}c_z\kappa + 2(\langle\kappa,\kappa\rangle D_z\kappa - \langle\kappa, D_z\kappa\rangle\kappa))\,,$$

which is a modified vector version of the usual Novikov–Veselov equation found in the literature [23]. Notice that the last term

$$\langle\kappa,\kappa\rangle D_z\kappa - \langle\kappa, D_z\kappa\rangle\kappa = 0$$

for surfaces in 3-space.

A similar, but simpler, computation shows that the Davey–Stewartson flow preserves isothermic surfaces in 4-space. In this case the normal deformation

$$\sigma = \operatorname{Re} J\kappa = J\kappa$$

since κ is real. The $\bar{\partial}$-problem

$$\bar{b}_z = 2\langle\sigma,\kappa\rangle = 2\langle J\kappa,\kappa\rangle = 0$$

is solved by $b = 0$. Using again (52) we calculate

$$\operatorname{Im} D_{t_2}\kappa = \operatorname{Im}(D_z D_z J\kappa + \tfrac{c}{2}J\kappa) = 0\,.$$

This gives

$$D_{t_2}\kappa = J(D_z D_z\kappa + \tfrac{c}{2}\kappa)$$

for the Davey–Stewartson flow on κ for isothermic surfaces in 4-space.

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