2009

The spectral curve of a quaternionic holomorphic line bundle over a 2-torus

C Bohle

F Pedit
University of Massachusetts - Amherst, pedit@math.umass.edu

U Pinkall

THE SPECTRAL CURVE OF A QUATERNIONIC HOLOMORPHIC LINE BUNDLE OVER A 2–TORUS

CHRISTOPH BOHLE, FRANZ PEDIT, AND ULRICH PINKALL

1. Introduction

Over the last 20 years algebraically completely integrable systems have been studied in a variety of contexts. On the one hand their theory is interesting from a purely algebraic geometric point of view, on the other hand a number of problems arising in mathematical physics and global differential geometry can be understood in the framework of those integrable systems. This situation has led to a rich cross–fertilization of algebraic geometry, global differential geometry and mathematical physics.

The phase space of an algebro–geometric integrable system consists of moduli of algebraic curves together with their Jacobians, the Lagrangian tori on which the motion of the system linearizes in a direction osculating the Abel image (at some marked point) of the curve. In classical terminology the algebraic curve, usually referred to as *the spectral curve*, encodes the action variables whereas the Jacobian of the curve encodes the angle variables of the system. Since the Jacobian of a curve acts on the Picard variety of fixed degree line bundles, the linear flow in the Jacobian gives rise, via the Kodaira embedding, to a flow of algebraic curves in a projective space. If this projective space is $\mathbb{P}^1$ the correct choice of moduli of curves yields as flows harmonic maps from $\mathbb{R}^2$ to $\mathbb{P}^1 = S^2$. In case those flows are periodic in the Jacobian, one obtains harmonic cylinders or tori in S^2. The action of the Jacobian on harmonic maps is a geometric manifestation of the sinh–Gordon hierarchy in mathematical physics, where one perhaps is more interested in the solutions of the field equation, here the elliptic sinh–Gordon equation, rather than the harmonic maps described by these solutions. It is a remarkable fact [18, 10] that, due to the ellipticity of the harmonic map equation, all harmonic tori arise in this way. It is well known that harmonic maps into S^2 are the unit normal maps of constant mean curvature surfaces in 3–space. Thus, the classical problem of finding all constant mean curvature tori can be rephrased by studying a particular algebro-geometric integrable system [18, 1].

This picture pertains in many other instances, including finite gap solutions of the KdV hierarchy, whose "fields" can be interpreted as the Schwarzian derivatives of curves in $\mathbb{P}^1$, elliptic Toda field equations for the linear groups, which arise in the study of minimal tori in spheres and projective spaces, and Willmore tori in 3 and 4–space. In each of these cases the equations can be rephrased as an algebraically completely integrable system. One of the implications of such a description is the explicit computability of solutions, since linear flows on Jacobians are parametrized by theta functions of the underlying algebraic curve. Moreover, the energy functional — Dirichlet energy, area, Willmore energy etc. — of the

---

Date: April 16, 2009.

All authors supported by DFG SPP 1154 "Global Differential Geometry". Second author additionally supported by Alexander von Humboldt foundation.

corresponding variational problem appears as a residue of a certain meromorphic form on the spectral curve, which makes the functional amenable to algebro–geometric techniques.

In all of the above cases the spectral curves are intimately linked to holonomy representations of holomorphic families of flat connections. For example, the harmonic map equation of a Riemann surface into the 2-sphere can be written as a zero curvature condition on a $\mathbb{C}_*$–family of $\mathrm{SL}(2,\mathbb{C})$–connections [10, 8]. If the underlying surface is a 2–torus, the eigenvalues of the holonomy (based at some point on the torus) give a hyper–elliptic curve. The corresponding eigenlines define a line bundle over this curve which moves linearly in the Picard of the curve as the base point moves over the torus. Perhaps there has been a sentiment that this setup is prototypical for algebraically completely integrable systems which arise in the context of differential geometry. We are learning now [3] that there is a much less confined setting for which the above described techniques are applicable: the geometric classes of surfaces described by zero curvature equations are special invariant subspaces of a more general phase space related to the Davey–Stewartson hierarchy [11, 21, 12, 5, 15].

This more general setting has to do with the notion of a "spectral variety" [6, 16, 13, 14, 7] for a differential operator. For example, if we want to study the spectrum of the Schrödinger operator on a periodic structure (crystal), we need to find a solution (the wave function) for a given energy which is quasi-periodic, that is, gains a phase factor over the crystal, since the physical state only depends on the complex line spanned by the wave function. If the crystal is a 2–D lattice, we obtain a spectral variety given by the energies and the possible phases of the wave functions, which in this case would be an analytic surface. Another example occurs in the computations of tau and correlation functions of massive conformal field theories over a Riemann surface M. In this case one is interested in solutions with monodromy of the Dirac operator with mass

$$D = \begin{pmatrix} \bar\partial & -m \\ m & \partial \end{pmatrix}$$

that is, solutions ψ which satisfy

$$D\psi = 0 \quad \text{and} \quad \gamma^*\psi = \psi h_\gamma$$

for a representation $h\colon \pi_1(M) \to \mathbb{C}_*$ of the fundamental group acting by deck transformations. The spectral variety, parametrizing the possible monodromies, generally is an analytic set.

This last example arises naturally [8] in the study of quaternionic holomorphic line bundles W over a Riemann surface M. Such lines bundles carry a complex structure $J \in \Gamma(\mathrm{End}(W))$ compatible with the quaternionic structure and the holomorphic structure is described by a quaternionic – generally not complex – linear first order operator

$$D = \bar\partial + Q\colon \Gamma(W) \to \Gamma(\bar K W)\,.$$

Here $Q \in \Gamma(\bar K\,\mathrm{End}_-(W))$ is the complex anti–linear part of D and the complex linear part $\bar\partial$ is a complex holomorphic structure on W. The energy of the holomorphic line bundle W is the L^2–norm

$$\mathcal{W}(W,D) = 2\int_M < Q \wedge *Q >$$

of Q which is zero for complex holomorphic structures $D = \bar\partial$. There are two important geometric applications depending on the dimension $h^0(W)$ of the space of holomorphic sections $H^0(W)$. If $h^0(W) = 1$ and the spanning holomorphic section $\psi \in H^0(W)$ has no zeros, the complex structure J can be regarded as a smooth map from the Riemann surface M to S^2 whose Dirichlet energy is $\mathcal{W}$. In the case $h^0(W) = 2$ the ratio of two independent holomorphic sections defines a (branched) conformal immersion from the Riemann surface M to S^4 whose Willmore energy, the average square mean curvature $\int_M H^2$, is given by $\mathcal{W}$. In both cases all the respective maps are described by suitably induced quaternionic holomorphic line bundles [8, 4].

A central geometric feature of both theories is the existence of Darboux transformations [3] which is a first manifestation of complete integrability. These Darboux transforms correspond to holomorphic sections with monodromy, that is, sections $\psi \in H^0(\tilde{W})$ of the pullback bundle $\tilde{W}$ of W to the universal cover of M, which satisfy

$$\gamma^*\psi = \psi h_\gamma$$

for a representation $h\colon \pi_1(M) \to \mathbb{H}_*$.

From here on we only discuss the case when the underlying Riemann surface M is a 2–torus $T^2 = \mathbb{R}^2/\Gamma$. Then the fundamental group is abelian and it suffices to consider holomorphic sections with complex monodromies which are all of the form $h = \exp(\int \omega)$ for a harmonic form $\omega \in \mathrm{Harm}(T^2, \mathbb{C})$. Interpreting $\exp(\int \omega)$ as a (non–periodic) gauge on $\mathbb{R}^2$, a holomorphic section ψ with monodromy $h = \exp(\int \omega)$ gives rise to the section $\psi \exp(-\int \omega) \in \Gamma(W)$ without monodromy on the torus which lies in the kernel of the periodic operator

$$D_\omega = \exp(-\int \omega) \circ D \circ \exp(\int \omega)\colon \Gamma(W) \to \Gamma(\bar{K}W)\,.$$

Therefore, the Darboux transforms are described by the harmonic forms $\omega \in \mathrm{Harm}(T^2, \mathbb{C})$ for which D_ω has a non–trivial kernel, and we call this set the *logarithmic spectrum* $\widetilde{\mathrm{Spec}}(W, D)$ of the quaternionic holomorphic line bundle W. Note that $\widetilde{\mathrm{Spec}}(W, D)$ is invariant under translations by the dual lattice Γ^* of integer period harmonic forms and its quotient under this lattice, the *spectrum*

$$\mathrm{Spec}(W, D) = \widetilde{\mathrm{Spec}}(W, D)/\Gamma^* \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*)\,,$$

is the set of possible monodromies of the holomorphic structure D. The spectrum carries a real structure ρ induced by complex conjugation: if a section ψ has monodromy h the section ψj has monodromy $\bar{h}$.

In the description of surface geometry via solutions to a Dirac operator with potential the spectrum already appeared in the papers of Taimanov [22], and Grinevich and Schmidt [9], and its relevance to the Willmore problem can be seen in [20, 23] and [3, 2].

The present paper analyzes the structure of the spectrum $\mathrm{Spec}(W, D)$ of a quaternionic line bundle W with holomorphic structure D of degree zero over a 2–torus. Due to ellipticity the family D_ω, parametrized over harmonic 1–forms $\mathrm{Harm}(T^2, \mathbb{C})$, is a holomorphic family of Fredholm operators. The minimal kernel dimension of such a family is generic and attained on the complement of an analytic subset. In Sections 2 and 3 we show that for a degree zero bundle over a 2–torus these operators have $\mathrm{index}(D_\omega) = 0$, their generic kernel dimension is zero, and the spectrum $\mathrm{Spec}(W, D) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ is a 1–dimensional analytic set. The spectrum can therefore be normalized $h\colon \Sigma \to \mathrm{Spec}(W, D)$ to a Riemann surface Σ, the *spectral curve*. Moreover, for generic $\omega \in \widetilde{\mathrm{Spec}}(W, D)$ the kernel of D_ω is 1–dimensional and therefore $\ker(D_\omega)$ gives rise to a holomorphic line bundle $\mathcal{L}$, the *kernel bundle*, over the spectral curve Σ. The fiber of $\mathcal{L}$ over a generic point $\sigma \in \Sigma$ is the space of holomorphic sections of W with monodromy $h^\sigma \in \mathrm{Spec}(W, D)$. The real structure ρ on the spectral curve Σ induced by $\rho(h) = \bar{h}$ is covered by multiplication by j on the kernel bundle $\mathcal{L}$ and therefore has no fixed points.

Physical intuition suggests that for large monodromies the spectrum should be asymptotic to the *vacuum* spectrum $\mathrm{Spec}(W, \bar{\partial})$. That this is indeed the case we show in Section 4. Since the vacuum is described by complex holomorphic sections with monodromy it is a translate of the curve

$$\exp(H^0(K)) \cup \exp(\overline{H^0(K)}) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*)$$

with double points along the lattice of real representations. We show that outside a sufficiently large compact subset of $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$ the spectrum is a graph over the vacuum, at least away from the double points. Near a double point the spectrum can have a handle. Depending on whether an infinite or finite number of handles appear, the spectral curve has infinite genus, is connected and has one end, or its genus is finite, it has two ends and at most two components.

For a generic holomorphic line bundle W the spectral curve Σ will have infinite genus and algebro–geometric techniques cannot be applied. This motivates us to study the case of finite spectral genus in Section 5 in more detail. The end behavior of the spectrum then implies that outside a sufficiently large compact set in $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$ none of the double points of the vacuum get resolved into handles. Therefore, we can compactify the spectral curve Σ by adding two points o and ∞ at infinity which are interchanged by the real structure ρ. Because the kernel bundle is asymptotic to the kernel bundle of the vacuum, the generating section $\psi^\sigma \in H^0(\tilde{W})$ with monodromy h^σ of $\mathcal{L}_\sigma$ has no zeros for σ in a neighborhood of o or ∞. We show that the complex structures S^σ defined via $S^\sigma \psi^\sigma = \psi^\sigma i$ are a holomorphic family limiting to $\pm J$ when σ tends to o and ∞, where J is the complex structure of our quaternionic holomorphic line bundle W. Thus, S extends to a T^2-family of algebraic functions $\overline{\Sigma} \to \mathbb{CP}^1$ on the compactification $\overline{\Sigma} = \Sigma \cup \{o, \infty\}$. Pulling back the tautological bundle over $\mathbb{CP}^1$ the family S gives rise to a linear T^2–flow of line bundles in the Picard group of $\overline{\Sigma}$. Finally we give a formula for the Willmore energy $\mathcal{W}$ in terms of the residue of the logarithmic derivative of h and the conformal structure of T^2. This formula allows various interpretations of the Willmore energy including an interpretation as the convergence speed of the spectrum to the vacuum spectrum.

During the preparation of this paper the authors profited from conversations with Martin Schmidt and Iskander Taimanov.

2. The Spectrum of a Quaternionic Holomorphic Line Bundle

In this section we summarize the basic notions of quaternionic holomorphic geometry [8] in as much as they are relevant for the purposes of this paper. We also recall the basic definitions and properties of the spectrum of a holomorphic line bundle which, from a surface geometric point of view, can also be found in [3].

2.1. Preliminaries.

Let W be a quaternionic (right) vector bundle over a Riemann surface M. A complex structure J on W is a section $J \in \Gamma(\mathrm{End}(W))$ with $J^2 = -\mathrm{Id}$, or, equivalently, a decomposition $W = W_+ \oplus W_-$ into real subbundles which are invariant under multiplication by the quaternion i and interchanged by multiplication with the quaternion j: the $\pm i$–eigenbundle of J is $W_\pm$. Note that W_+ and W_- are isomorphic via multiplication by j as vector bundles with complex structures $J_{|W_\pm}$. The degree of the quaternionic bundle W with complex structure J is defined as the degree $\deg W := \deg W_+$ of the underlying complex vector bundle W_+.

A *quaternionic holomorphic* structure on a quaternionic vector bundle W equipped with a complex structure J is a quaternionic linear operator

$$D\colon \Gamma(W) \to \Gamma(\bar{K}W)$$

that satisfies the Leibniz rule $D(\psi\lambda) = (D\psi)\lambda + (\psi d\lambda)''$ for all $\psi \in \Gamma(W)$ and $\lambda\colon M \to \mathbb{H}$ where $(\psi d\lambda)'' = \frac{1}{2}(\psi d\lambda + J * \psi d\lambda)$ denotes the $(0,1)$–part of the W–valued 1–form $\psi d\lambda$ with respect to the complex structures J on W and $*$ on TM^*. Decomposing the operator D into J commuting and anti–commuting parts gives $D = \bar{\partial} + Q$ where $\bar{\partial} = \bar{\partial} \oplus \bar{\partial}$ is the double of a complex holomorphic structure on W_+ and $Q \in \Gamma(\bar{K}\operatorname{End}_-(W))$ is a $(0,1)$–form with values in the complex anti–linear endomorphisms of W (with complex structure post–composition by J), called the *Hopf field*.

The space of holomorphic sections of W is denoted by $H^0(W) = \ker(D)$. Because D is an elliptic operator its kernel $H^0(W)$ is finite dimensional if the underlying surface is compact. The L^2–norm

$$\mathcal{W}(W) = \mathcal{W}(W, D) = 2\int_M < Q \wedge *Q >$$

of the Hopf field Q is called the *Willmore energy* of the holomorphic bundle W where $<,>$ denotes the real trace pairing on $\operatorname{End}(W)$. The special case $Q = 0$, for which $\mathcal{W}(W) = 0$, describes (doubles of) complex holomorphic bundles $W \cong W_+ \oplus W_+$.

2.2. **The quaternionic spectrum of quaternionic holomorphic line bundles.** A holomorphic structure D on W induces a quaternionic holomorphic structure on the pullback bundle $\tilde{W} = \pi^*W$ by the universal covering $\pi\colon \tilde{M} \to M$. The operator D on $\tilde{W}$ is periodic with respect to the group Γ of deck transformations of $\pi\colon \tilde{M} \to M$. The space $H^0(W)$ of holomorphic sections of W are the periodic, that is, Γ–invariant, sections of $\tilde{W}$ solving $D\psi = 0$. In the following we also need to consider solutions with monodromy of $D\psi = 0$. Such solutions are the holomorphic sections of $\tilde{W}$ that satisfy

$$\gamma^*\psi = \psi h_\gamma \qquad \text{for all } \gamma \in \Gamma, \tag{2.1}$$

where $h \in \operatorname{Hom}(\Gamma, \mathbb{H}_*)$ is a representation of Γ. A holomorphic section $\psi \in H^0(\tilde{W})$ satisfying (2.1) for some $h \in \operatorname{Hom}(\Gamma, \mathbb{H}_*)$ is called a *holomorphic section with monodromy* h of W, and we denote the space of all such sections by $H^0_h(\tilde{W})$. By the quaternionic Plücker formula with monodromy (see appendix to [3]) $H^0_h(\tilde{W})$ is a finite dimensional real vector space. Multiplying $\psi \in H^0_h(\tilde{W})$ by some $\lambda \in \mathbb{H}_*$ yields the section $\psi\lambda$ with monodromy $\lambda^{-1}h\lambda$. Unless h is a real representation $H^0_h(\tilde{W})$ is not a quaternionic vector space.

Definition 2.1. Let W be a quaternionic line bundle with holomorphic structure D over a Riemann surface M. The *quaternionic spectrum* of W is the subspace

$$\operatorname{Spec}_{\mathbb{H}}(W, D) \subset \operatorname{Hom}(\Gamma, \mathbb{H}_*)/\mathbb{H}_*$$

of conjugacy classes of monodromy representations occurring for holomorphic sections of $\tilde{W}$. In other words, h represents a point in $\operatorname{Spec}_{\mathbb{H}}(W, D)$ if and only if there is a non–trivial holomorphic section $\psi \in H^0(\tilde{W})$ with monodromy h, that is, a solution of

$$D\psi = 0 \quad \text{satisfying} \quad \gamma^*\psi = \psi h_\gamma \quad \text{for all } \gamma \in \Gamma.$$

Our principal motivation for studying the quaternionic spectrum is the observation [3] that Darboux transforms of a conformal immersion $f\colon M \to S^4$ correspond to nowhere vanishing holomorphic sections with monodromy of a certain quaternionic holomorphic line bundle induced by f. From this point of view the quaternionic spectrum arises as a parameter space for the space of Darboux transforms. The quaternionic holomorphic line bundle induced by $f\colon M \to S^4$ is best described when viewing $S^4 = \mathbb{HP}^1$ as the quaternionic projective line. The immersion f is the pull–back L of the tautological line bundle over $\mathbb{HP}^1$ and as such a subbundle $L \subset \underline{\mathbb{H}}^2$ of the trivial quaternionic rank 2 bundle. The induced quaternionic holomorphic line bundle is the quotient bundle $W = V/L$ equipped with the unique holomorphic structure for which constant sections of V project to holomorphic sections of V/L.

The idea of defining spectra of conformal immersions first appears, for tori in $\mathbb{R}^3$, in the work of Taimanov [22], and Grinevich and Schmidt [9]. Their definition leads to the same notion of spectrum although it is based on a different quaternionic holomorphic line bundle associated to a conformal immersion into Euclidean 4–space $\mathbb{R}^4$ via the Weierstrass representation [17].

2.3. **The spectrum of a quaternionic holomorphic line bundle over a 2–torus.** From here on we study the geometry of $\mathrm{Spec}_{\mathbb{H}}(W, D)$ in the case that the underlying surface is a torus $T^2 = \mathbb{R}^2/\Gamma$. Due to the abelian fundamental group $\pi_1(T^2) = \Gamma$ every representation $h \in \mathrm{Hom}(\Gamma, \mathbb{H}_*)$ of the group of deck transformations can be conjugated into a complex representation in $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$. The complex representation $h \in \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ in a conjugacy class in $\mathrm{Hom}(\Gamma, \mathbb{H}_*)$ is uniquely determined up to complex conjugation $h \mapsto \bar{h}$ (which corresponds to conjugation by the quaternion j). In particular, away from the real representations, the map

$$\mathrm{Hom}(\Gamma, \mathbb{C}_*) \to \mathrm{Hom}(\Gamma, \mathbb{H}_*)/\mathbb{H}_*$$

is $2:1$. The lift of the quaternionic spectrum $\mathrm{Spec}_{\mathbb{H}}(W, D)$ of W under this map gives rise to the spectrum of the quaternionic holomorphic line bundle W.

Definition 2.2. Let W be a quaternionic holomorphic line bundle over the torus with holomorphic structure D. Its *spectrum* is the subspace

$$\mathrm{Spec}(W, D) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*)$$

of complex monodromies occurring for non–trivial holomorphic sections of $\tilde{W}$.

By construction, the spectrum is invariant under complex conjugation $\rho(h) = \bar{h}$ and

$$\mathrm{Spec}_{\mathbb{H}}(W, D) = \mathrm{Spec}(W, D)/\rho. \tag{2.2}$$

The study of $\mathrm{Spec}(W, D)$ is greatly simplified by the fact that $G = \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ is an abelian Lie group with Lie algebra $\mathfrak{g} = \mathrm{Hom}(\Gamma, \mathbb{C})$ whose exponential map $\exp\colon \mathfrak{g} \to G$ is induced by the exponential function $\mathbb{C} \to \mathbb{C}_*$. The Lie algebra $\mathrm{Hom}(\Gamma, \mathbb{C})$ is isomorphic to the space $\mathrm{Harm}(T^2, \mathbb{C})$ of harmonic 1–forms: a harmonic 1–form ω gives rise to the period homomorphism $\gamma \in \Gamma \mapsto \int_\gamma \omega$. The image under the exponential map of such a homomorphism is the multiplier $\gamma \mapsto h_\gamma = e^{\int_\gamma \omega}$. The kernel of the group homomorphism exp is the lattice of harmonic forms $\Gamma^* = \mathrm{Harm}(T^2, 2\pi i\mathbb{Z})$ with integer periods. The exponential function thus induces an isomorphism

$$\mathrm{Harm}(T^2, \mathbb{C})/\Gamma^* \cong \mathrm{Hom}(\Gamma, \mathbb{C}_*)\,. \tag{2.3}$$

Rather than $\operatorname{Spec}(W, D)$ we study its logarithmic image which is the Γ^*–invariant subset $\widetilde{\operatorname{Spec}}(W, D) \subset \operatorname{Harm}(T^2, \mathbb{C})$, the *logarithmic spectrum*, with the property that

$$\operatorname{Spec}(W, D) \cong \widetilde{\operatorname{Spec}}(W, D)/\Gamma^*. \tag{2.4}$$

Considering $\widetilde{\operatorname{Spec}}(W, D)$ has the advantage that it is described as the locus of those $\omega \in \operatorname{Harm}(T^2, \mathbb{C})$ for which the operator

$$D_\omega = e^{-\int \omega} \circ D \circ e^{\int \omega}: \quad \Gamma(W) \to \Gamma(\bar{K}W)\,, \tag{2.5}$$

given by $D_\omega \psi = (D(\psi e^{\int \omega}))e^{-\int \omega}$ where $\psi \in \Gamma(W)$, has a non–trivial kernel. Here $e^{\int \omega} \in \operatorname{Hom}(\mathbb{R}^2, \mathbb{C}_*)$ is regarded as a gauge transformation on the universal cover $\mathbb{R}^2$. Nevertheless, the operator D_ω is still well defined on the torus T^2 because the Leibniz rule of a quaternionic holomorphic structure implies

$$D_\omega(\psi) = D\psi + (\psi\omega)''\,. \tag{2.6}$$

The operator D_ω is elliptic but due to the term $(\psi\omega)''$ in (2.6) only a complex linear (rather then quaternionic linear) operator between the complex rank 2 bundles W and $\bar{K}W$, where the complex structure I is given by right multiplication $I(\psi) = \psi i$ by the quaternion i. A section $\psi \in \Gamma(W)$ is in the kernel of D_ω if and only if the section $\psi e^{\int \omega} \in \Gamma(\tilde{W})$ is in the kernel of D, that is, $\psi e^{\int \omega} \in H^0(\tilde{W})$ is holomorphic. Because the section $\psi e^{\int \omega}$ has monodromy $h = e^{\int \omega}$, we obtain the I–complex linear isomorphism

$$\ker D_\omega \to H^0_h(\tilde{W}): \quad \psi \mapsto \psi e^{\int \omega}. \tag{2.7}$$

In particular, a representation $h = e^{\int \omega}$ is in the spectrum $\operatorname{Spec}(W, D)$ if and only if D_ω has non–trivial kernel and therefore

$$\widetilde{\operatorname{Spec}}(W, D) = \{\omega \in \operatorname{Harm}(T^2, \mathbb{C}) \mid \ker D_\omega \neq 0\}. \tag{2.8}$$

Since D_ω is a holomorphic family of elliptic operators the general theory of holomorphic families of Fredholm operators (see Proposition 3.1) implies that the logarithmic spectrum $\widetilde{\operatorname{Spec}}(W, D)$ is a complex analytic subset of $\operatorname{Hom}(\Gamma, \mathbb{C}) \cong \mathbb{C}^2$ and that $\operatorname{Spec}(W, D)$ is a complex analytic subset of $\operatorname{Hom}(\Gamma, \mathbb{C}_*) \cong \mathbb{C}_* \times \mathbb{C}_*$. It turns out that the dimension of $\operatorname{Spec}(W, D) \subset \operatorname{Hom}(\Gamma, \mathbb{C}_*)$ depends on the degree $d = \deg(W)$ of W:

$$\begin{aligned} \dim(\operatorname{Spec}(W, D)) &= 2 && \text{if } d > 0,\\ \dim(\operatorname{Spec}(W, D)) &= 1 && \text{if } d = 0,\\ \dim(\operatorname{Spec}(W, D)) &= 0 && \text{if } d < 0. \end{aligned}$$

The case $d \neq 0$ is dealt with in the following lemma. The case $d = 0$ is the subject of the rest of this paper.

Lemma 2.3. *Let (W, D) be a quaternionic holomorphic line bundle of degree* $\deg(W) \neq 0$ *over a torus T^2. Then*

a) $\operatorname{Spec}(W, D) = \operatorname{Hom}(\Gamma, \mathbb{C}_*)$ *if* $\deg(W) > 0$ *and*
b) $\operatorname{Spec}(W, D)$ *is a finite subset of* $\operatorname{Hom}(\Gamma, \mathbb{C}_*)$ *if* $\deg(W) < 0$.

Proof. The Fredholm index $\operatorname{Index}(D_\omega) = \dim(\ker(D_\omega)) - \dim(\operatorname{coker}(D_\omega))$ (see the proof of Lemma 2.4 in Section 3.1 for more details) of the elliptic operator D_ω depends only on the first order part. Therefore $\operatorname{Index}(D_\omega$ coincides with the Fredholm index of the operator $\bar\partial$ which, by the Riemann Roch theorem, is given by

$$\operatorname{Index}(D_\omega) = \operatorname{Index}(\bar\partial) = 2(d - g + 1) = 2d, \tag{2.9}$$

where d denotes the degree of W and the genus $g = 1$ for the torus T^2. The factor 2 comes from the fact that $\bar\partial$ is the direct sum of complex $\bar\partial$–operators on $W \cong W_+ \oplus W_+$. To prove b) we use the quaternionic Plücker formula with monodromy (see appendix to [3]) which shows that

$$\frac{1}{4\pi}\mathcal{W}(W,D) \geq -d(n+1) + \operatorname{ord}(H))$$

for every $(n+1)$–dimensional linear system $H \subset H^0(\tilde W)$ with monodromy. This implies that for $d < 0$ only a finite number of $h \in \operatorname{Spec}(W,D)$ admit non–trivial holomorphic sections with monodromy. □

In case $W = V/L$ is the quaternionic holomorphic line bundle induced by a conformal immersion $f\colon T^2 \to S^4$ of a torus into the conformal 4–sphere the degree $d = \deg(V/L)$ is half of the normal bundle degree $\deg(\perp_f)$, see [3]. In particular, for immersions into the conformal 3–sphere S^3 the degree of the induced bundle V/L is always zero.

2.4. **Spectral curves of degree zero bundles over 2–tori.** Throughout the rest of the paper we assume that W is a quaternionic holomorphic line bundle of degree zero over a torus T^2. The following lemma shows that in this case the logarithmic spectrum $\widetilde{\operatorname{Spec}}(W,D)$, and hence the spectrum $\operatorname{Spec}(W,D)$, is a 1–dimensional analytic subset. This allows to normalize the spectrum to a Riemann surface, the *spectral curve*.

Lemma 2.4. *Let (W,D) be a quaternionic holomorphic line bundle of degree zero over a torus T^2. Then:*

a) *The logarithmic spectrum $\widetilde{\operatorname{Spec}}(W,D)$ is a 1–dimensional analytic set in $\operatorname{Harm}(T^2,\mathbb{C}) \cong \mathbb{C}^2$ invariant under translations by the lattice Γ^*. Its normalization is the Riemann surface $\tilde\Sigma$ that admits a surjective holomorphic map $\omega\colon \tilde\Sigma \to \widetilde{\operatorname{Spec}}(W,D)$ which, on the complement of a discrete set, is an injective immersion.*
b) *The normalization $\tilde\Sigma$ carries a complex holomorphic line bundle $\tilde{\mathcal{L}}$ which is a holomorphic subbundle (in the topology of C^∞ convergence) of the trivial $\Gamma(W)$–bundle over $\tilde\Sigma$. The fibers of $\tilde{\mathcal{L}}$ satisfy $\tilde{\mathcal{L}}_{\tilde\sigma} \subset \ker D_{\omega(\tilde\sigma)}$ for all $\tilde\sigma \in \tilde\Sigma$ with equality away from a discrete set in $\tilde\Sigma$.*

The fact that $\tilde{\mathcal{L}}$ is a holomorphic line subbundle of the trivial $\Gamma(W)$–bundle with respect to the Frechet topology of C^∞–convergence means that a local holomorphic section $\psi \in H^0(\tilde{\mathcal{L}}_{|U})$ can be viewed as a C^∞–map

$$\psi\colon U \times T^2 \to W \cong T^2 \times \mathbb{C}^2 \quad (\tilde\sigma, p) \mapsto \psi^{\tilde\sigma}(p)$$

which is holomorphic in $\tilde\sigma \in U \subset \tilde\Sigma$. Here W is the trivial $\mathbb{C}^2$–bundle with the complex structure given via quaternionic right multiplication by i.

The lemma is proven in Section 3.1 by asymptotic comparison of the holomorphic families of elliptic operators D_ω and $\bar\partial_\omega$. This analysis shows that the spectrum $\operatorname{Spec}(W,D)$ of D asymptotically looks like the spectrum $\operatorname{Spec}(\bar\partial, W)$ of $\bar\partial$, the *vacuum spectrum.*

Because the spectrum $\operatorname{Spec}(W,D) = \widetilde{\operatorname{Spec}}(W,D)/\Gamma^*$ is the quotient of the logarithmic spectrum by the dual lattice, Lemma 2.4 implies that $\operatorname{Spec}(W,D) \subset \operatorname{Hom}(\Gamma,\mathbb{C}_*)$ also is a 1–dimensional analytic set. The normalization of $\operatorname{Spec}(W,D)$ is the quotient $\Sigma = \tilde\Sigma/\Gamma^*$ with normalization map $h\colon \Sigma \to \operatorname{Spec}(W,D)$ induced from $\omega\colon \tilde\Sigma \to \widetilde{\operatorname{Spec}}(W,D)$ via $h = \exp(\int \omega)$.

Definition 2.5. Let (W, D) be a quaternionic holomorphic line bundle of degree zero over a torus. The *spectral curve* Σ of W is the Riemann surface normalizing the spectrum $h\colon \Sigma \to \operatorname{Spec}(W, D)$.

The spectral curve of a conformal immersion $f\colon T^2 \to S^4$ with trivial normal bundle is [3] the spectral curve of the induced quaternionic holomorphic line bundle $W = V/L$.

As proven in Section 4, the spectral curve Σ of a degree zero quaternionic holomorphic line bundle W is a Riemann surface with either one or two ends, depending on whether its genus is infinite or finite. In the former case it is connected, in the latter case it has at most two components, each containing at least one of the ends.

The bundle $\tilde{\mathcal{L}}$ can be realized as a complex holomorphic subbundle of the trivial $H^0(\tilde{W})$–bundle (with respect to C^∞–convergence on compact subsets on the universal covering $\mathbb{R}^2$) over $\tilde{\Sigma}$ by the embedding

$$\tilde{\mathcal{L}} \to H^0(\tilde{W}) \qquad \psi_{\tilde{\sigma}} \in \tilde{\mathcal{L}}_{\tilde{\sigma}} \mapsto \psi_{\tilde{\sigma}} e^{\int \omega_{\tilde{\sigma}}}.$$

This line subbundle of $H^0(\tilde{W})$ is invariant under the action of Γ^* and hence descends to a complex holomorphic line bundle $\mathcal{L} \to \Sigma$. The fibers of $\mathcal{L}$ satisfy $\mathcal{L}_\sigma \subset H^0_{h^\sigma}(\tilde{W})$ for $\sigma \in \Sigma$ and equality holds away from a discrete set of points in Σ.

Recall that the spectrum $\operatorname{Spec}(W, D) \subset \operatorname{Hom}(\Gamma, \mathbb{C}_*)$ is invariant under the conjugation $\rho(h) = \bar{h}$. The map ρ lifts to an anti–holomorphic involution $\rho\colon \Sigma \to \Sigma$ of the spectral curve Σ that satisfies $h \circ \rho = \bar{h}$. For every $\sigma \in \Sigma$ and $\psi \in \mathcal{L}_\sigma$, the section ψj is a holomorphic section with monodromy $h^{\rho(\sigma)} = \bar{h^\sigma}$ of $\tilde{W}$. For generic $\sigma \in \Sigma$, we have $\mathcal{L}_{\rho(\sigma)} = H^0_{h^{\rho(\sigma)}}(\tilde{W})$ such that $\psi j \in \mathcal{L}_{\rho(\sigma)}$ and

$$\mathcal{L}_{\rho(\sigma)} = \mathcal{L}_\sigma j\,.$$

By continuity the latter holds for all $\sigma \in \Sigma$. This shows that the anti–holomorphic involution $\rho\colon \Sigma \to \Sigma$ is covered by right multiplication with j acting on $\mathcal{L}$ and thus has no fixed points. The following theorem summarizes the discussion so far:

Theorem 2.6. *The spectrum* $\operatorname{Spec}(W, D)$ *of a quaternionic holomorphic line bundle* (W, D) *of degree zero over a torus is a 1–dimensional complex analytic set. Its spectral curve, the Riemann surface* Σ *normalizing the spectrum* $h\colon \Sigma \to \operatorname{Spec}(W, D)$*, is equipped with a fixed point free anti–holomorphic involution* $\rho\colon \Sigma \to \Sigma$ *satisfying* $h \circ \rho = \bar{h}$*.*

There is a complex holomorphic line bundle $\mathcal{L} \to \Sigma$ *over* Σ*, the* kernel bundle*, which is a subbundle of the trivial bundle* $\Sigma \times H^0(\tilde{W})$ *with the property that* $\mathcal{L}_\sigma \subset H^0_{h^\sigma}(\tilde{W})$ *for all* $\sigma \in \Sigma$ *with equality away from a discrete set in* Σ*. The bundle* $\mathcal{L}$ *is compatible with the real structure* $\rho\colon \Sigma \to \Sigma$ *in the sense that* $\rho^*\mathcal{L} = \mathcal{L} j$*.*

The quotient Σ/ρ with respect to the fixed point free involution ρ is the normalization of the quaternionic spectrum $\operatorname{Spec}_{\mathbb{H}}(W, D)$. If Σ is connected, Σ/ρ is a “non–orientable Riemann surface”.

3. Asymptotic Analysis

This section is concerned with a proof of Lemma 2.4 and some preparatory results needed in Section 4 to analyze the asymptotic geometry of the spectrum $\operatorname{Spec}(W, D)$.

The results of Sections 3 and 4 are obtained by asymptotically comparing the kernels of the holomorphic family of elliptic operators D_ω to those of the holomorphic family of elliptic operators $\bar\partial_\omega$ arising from the "vacuum" $(W,\bar\partial)$. For large multipliers, that is, for multipliers in the complement of a compact subset of $\mathrm{Hom}(\Gamma,\mathbb{C}_*)$, the spectrum $\mathrm{Spec}(W,D)$ is a small deformation of the vacuum spectrum $\mathrm{Spec}(\bar\partial,W)$ away from double points of $\mathrm{Spec}(\bar\partial,W)$ where handles may form. The detailed study of the asymptotic geometry of $\mathrm{Spec}(W,D)$ will then be carried out Section 4.

3.1. **Holomorphic families of Fredholm operators and the proof of Lemma 2.4.**

The following proposition, combined with Corollary 3.9 proven at the end of this section, will yield a proof of Lemma 2.4.

Proposition 3.1. *Let $F(\lambda)\colon E_1 \to E_2$ be a holomorphic family of Fredholm operators between Banach spaces E_1 and E_2 parameterized over a connected complex manifold M. Then the minimal kernel dimension of $F(\lambda)$ is attained on the complement of an analytic subset $N \subset M$. If M is 1–dimensional, the holomorphic vector bundle $\mathcal{K}_\lambda = \ker(F(\lambda))$ over $M\backslash N$ extends through the set N of isolated points to a holomorphic vector subbundle of the trivial E_1–bundle over M.*

Proof. For $\lambda_0 \in M$ there are direct sum decompositions $E_1 = E_1' \oplus K_1$ and $E_2 = E_2' \oplus K_2$ into closed subspaces such that K_1 is the finite dimensional kernel of $F(\lambda_0)$ and E_2' its image. The Hahn–Banach theorem ensures that the finite dimensional kernel of a Fredholm operator can be complemented by a closed subspace. The fact that the image of an operator of finite codimension m is closed follows from the open mapping theorem: for every complement, the restriction of the operator to E_1' can be linearly extended to a bijective operator from $E_1' \oplus \mathbb{C}^m$ to $E_2 = E_2' \oplus K_2$ that respects the direct sum decomposition.

With respect to such a direct sum decomposition, the operators $F(\lambda)$ can be written as

$$F(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},$$

with $A(\lambda)$ invertible for all $\lambda \in U$ contained in an open neighborhood $U \subset M$ of λ_0. For all $\lambda \in U$ we have

$$F(\lambda)\underbrace{\begin{pmatrix} A(\lambda)^{-1} & -A(\lambda)^{-1}B(\lambda) \\ 0 & \mathrm{Id}_C \end{pmatrix}}_{=:G(\lambda)} = \begin{pmatrix} \mathrm{Id}_{E_2'} & 0 \\ C(\lambda)A(\lambda)^{-1} & \underbrace{D(\lambda) - C(\lambda)A(\lambda)^{-1}B(\lambda)}_{=:\tilde F(\lambda)} \end{pmatrix}.$$

Because all $G(\lambda)$ are invertible, the kernel of $F(\lambda)$ is $G(\lambda)$ applied to $\ker(\tilde F(\lambda))$. Because $\tilde F(\lambda)\colon K_1 \to C$ is an operator between finite dimensional spaces whose index coincides with that of $F(\lambda)$, the proposition immediately follows from its finite dimensional version: the analytic set N is given by the vanishing of all $r\times r$–subdeterminants of $\tilde F(\lambda)$, where r is the maximal rank of $\tilde F(\lambda)$ for $\lambda \in U$. For a proof that, in the case of a 1–dimensional parameter domain, the bundle of kernels extends through the points in N, see e.g. Lemma 23 of [4]. □

Corollary 3.2. *If $\mathrm{Index}(F(\lambda)) = 0$, the analytic set $N_0 = \{\lambda \in M \mid \ker(F(\lambda)) \neq 0\}$ can be locally described as the vanishing locus of a single holomorphic function.*

Proof. We may assume $N_0 \neq M$. Then $N_0 = N$, with N as in Proposition 3.1, and therefore locally described as the set of λ for which $\tilde F(\lambda)$ has a non–trivial kernel. But

$\mathrm{Index}(F(\lambda)) = 0$ implies that $\tilde{F}(\lambda)$ is an $r_0 \times r_0$–matrix with $r_0 = \dim(K) = \dim(C)$ such that N_0 is locally given as the vanishing locus of the determinant of $\tilde{F}(\lambda)$. $\square$

In order to show that $\widetilde{\mathrm{Spec}}(W, D) \subset \mathrm{Hom}(\Gamma, \mathbb{C})$ is a complex analytic set, it is sufficient – by the preceding corollary – to extend the family of elliptic operators D_ω to a holomorphic family of Fredholm operators defined on a Banach space containing $\Gamma(W)$ such that the kernels of the extensions coincide with those of D_ω. Corollary 3.9 below implies that $\widetilde{\mathrm{Spec}}(W, D)$ is neither empty nor the whole space and thus has to be 1–dimensional. Moreover, by applying Proposition 3.1 to D_ω pulled back by the normalization $\omega\colon \tilde{\Sigma} \to \widetilde{\mathrm{Spec}}(W, D) \subset \mathrm{Harm}(T^2, \mathbb{C})$, Corollary 3.9 shows that the kernel of D_ω is 1–dimensional for generic $\omega \in \widetilde{\mathrm{Spec}}(W, D)$ and therefore the kernel bundle is a line bundle over $\tilde{\Sigma}$.

Proof of Lemma 2.4. For every integer $k \geq 0$ the operators D_ω extend to bounded operators from the $(k+1)^{th}$–Sobolev space $H^{k+1}(W)$ of sections of W to the k^{th}–Sobolev space $H^k(\bar{K}W)$ of sections of $\bar{K}W$. By elliptic regularity (see e.g. Theorem 6.30 and Lemma 6.22a of [24]), for every k the kernel of the extension to $H^{k+1}(W)$ is contained in the space $\Gamma(W)$ of C^∞–sections and therefore coincides with the kernel of the original elliptic operator $D_\omega\colon \Gamma(W) \to \Gamma(\bar{K}W)$.

The extension of an elliptic operator on a compact manifold to suitable Sobolev spaces is always Fredholm: its kernel is finite dimensional (see e.g. [24], p.258) and so is its cokernel, the space dual to the kernel of the adjoint elliptic operator. The Fredholm index

$$\mathrm{Index}(D_\omega) = \dim(\ker(D_\omega)) - \dim(\mathrm{coker}(D_\omega))$$

of D_ω depends only on the symbol so that by (2.9)

$$\mathrm{Index}(D_\omega) = \mathrm{Index}(\bar{\partial}) = 2d$$

where d is the degree of W.

By assumption $d = 0$ and therefore Corollary 3.2 implies that $\widetilde{\mathrm{Spec}}(W, D)$ is a complex analytic set locally given as the vanishing locus of one holomorphic function in two complex variables. To see that $\widetilde{\mathrm{Spec}}(W, D)$ is 1–dimensional it suffices to check that $\widetilde{\mathrm{Spec}}(W, D)$ is neither empty nor all of $\mathrm{Harm}(T^2, \mathbb{C})$ which will be proven in Corollary 3.9. This corollary also shows that there is $\omega \in \widetilde{\mathrm{Spec}}(W, D)$ for which $\ker(D_\omega)$ is 1–dimensional. Applying Proposition 3.1 to D_ω pulled back to the Riemann surface $\tilde{\Sigma}$ normalizing $\widetilde{\mathrm{Spec}}(W, D)$ shows that, for generic $\omega \in \tilde{\Sigma}$, the kernel of D_ω 1–dimensional. The unique extension of $\ker(D_\omega)$ through the isolated points with higher dimensional kernel thus defines a holomorphic line bundle $\tilde{\mathcal{L}} \to \tilde{\Sigma}$.

Due to elliptic regularity the kernel of the extension of D_ω to the Sobolev space $H^{k+1}(W)$ for every integer $k \geq 0$ is already contained in $\Gamma(W)$. In particular, the kernels of the D_ω do not depend on k and define a line subbundle $\tilde{\mathcal{L}}$ of the trivial $\Gamma(W)$–bundle over $\tilde{\Sigma}$. For every $k \geq 0$ the line bundle $\tilde{\mathcal{L}}$ is a holomorphic line subbundle of the trivial $H^{k+1}(W)$–bundle and therefore a holomorphic line subbundle of the trivial $\Gamma(W)$–bundle where $\Gamma(W)$ is equipped with the Frechet topology of C^∞–convergence. $\square$

3.2. **Functional analytic setting.** In the following we develop the asymptotic analysis needed to prove Corollary 3.9. For further applications in Sections 4 and 5 it will be convenient to view D_ω as a family of unbounded operators on the Wiener space of continuous sections of W with absolutely convergent Fourier series (instead of viewing it as unbounded operators $D_\omega\colon \Gamma(W) \subset L^2(W) \to L^2(W)$ which would be sufficient for proving Corollary 3.9).

In order to work in the familiar framework of function spaces and to simplify the application of Fourier analysis, we fix a uniformizing coordinate z on the torus $T^2 = \mathbb{C}/\Gamma$ and trivialize the bundle W. The latter can be done since the degree of W or, equivalently, of the complex line subbundle $\hat{W} = \{v \in W \mid Jv = vi\}$, is zero. To trivialize W we choose a parallel section $\psi \in \Gamma(\hat{W})$ of a suitable flat connection: let $\hat{\nabla}$ be the unique flat quaternionic connection on W which is complex holomorphic, that is, $\hat{\nabla} J = 0$ and $\hat{\nabla}'' = \bar\partial$, and which has unitary monodromy $\hat{h}_\gamma = e^{-\bar{b}_0\gamma + b_0\bar\gamma}$ when restricted to $\hat{W}$, where $h \in \operatorname{Hom}(\Gamma, \mathbb{C}_*)$. Then the connection $\hat{\nabla} - \bar{b}_0 dz + b_0 d\bar{z}$ on $\hat{W}$ is trivial and we can choose a parallel section $\psi \in \Gamma(\hat{W})$ which in particular satisfies $\bar\partial\,\psi = -\psi b_0 d\bar{z}$.

The choices of uniformizing coordinate z and trivializing section ψ induce isomorphisms

$$C^\infty(T^2, \mathbb{C}^2) \to \Gamma(W), \qquad (u_1, u_2) \mapsto \psi(u_1 + ju_2)$$

and

$$C^\infty(T^2, \mathbb{C}^2) \to \Gamma(\bar{K}W), \qquad (u_1, u_2) \mapsto \psi d\bar{z}(u_1 + ju_2).$$

Moreover, via

$$\omega = (a + \bar{b}_0)dz + (b + b_0)d\bar{z}, \tag{3.1}$$

the space $\operatorname{Harm}(T^2, \mathbb{C})$ is coordinatized by $(a, b) \in \mathbb{C}^2$. Under these isomorphisms the family of operators D_ω defined in (2.5) takes the form

$$D_{a,b} = \bar\partial_{a,b} + M \tag{3.2}$$

with

$$\bar\partial_{a,b} = \begin{pmatrix} \frac{\partial}{\partial\bar{z}} + b & 0 \\ 0 & \frac{\partial}{\partial z} + a \end{pmatrix} \qquad \text{and} \qquad M = \begin{pmatrix} 0 & -\bar{q} \\ q & 0 \end{pmatrix},$$

where $q \in C^\infty(T^2, \mathbb{C})$ is the smooth complex function defined by $Q\psi = \psi j dz q$. We denote by

$$\widetilde{\mathcal{S}} = \{(a, b) \in \mathbb{C}^2 \mid \ker(D_{a,b}) \neq 0\}$$

and

$$\widetilde{\mathcal{S}}_0 = \{(a, b) \in \mathbb{C}^2 \mid \ker(\bar\partial_{a,b}) \neq 0\}$$

the coordinate versions of the logarithmic spectrum $\widetilde{\operatorname{Spec}}(W, D)$ and logarithmic vacuum spectrum $\widetilde{\operatorname{Spec}}(\bar\partial, W)$.

In the following we equip the space $l^1 = l^1(T^2, \mathbb{C}^2)$ of C^0–functions with absolutely convergent Fourier series with the Wiener norm, the l^1–norm

$$\|u\| = \sum_{c \in \Gamma'} |u_{1,c}| + |u_{2,c}| \tag{3.3}$$

of the Fourier coefficients of

$$u(z) = \left(\sum_{c\in\Gamma'} u_{1,c}\, e^{-\bar{c}z + c\bar{z}}, \sum_{c\in\Gamma'} u_{2,c}\, e^{-\bar{c}z + c\bar{z}} \right) \in l^1(T^2, \mathbb{C}^2),$$

where $\Gamma' \subset \mathbb{C}$ denotes the lattice

$$\Gamma' = \{c \in \mathbb{C} \mid -\bar{c}\gamma + c\bar\gamma \in 2\pi i\mathbb{Z} \text{ for all } \gamma \in \Gamma\}.$$

The Banach space l^1 contains $C^\infty = C^\infty(T^2, \mathbb{C}^2)$ as a dense subspace (in fact, it contains all H^2–functions, cf. the proof of the Sobolev lemma 6.22 in [24]). Since the inclusion

$l^1(T^2,\mathbb{C}^2) \to C^0(T^2,\mathbb{C}^2)$ is bounded l^1–convergence implies C^0–convergence and in particular pointwise convergence. We will use those facts in the applications in Sections 4 and 5.

3.3. **$D_{a,b}$ and $\bar\partial_{a,b}$ as closable operators $C^\infty \subset l^1 \to l^1$.** We will show that all operators in the holomorphic families $D_{a,b}$ and $\bar\partial_{a,b}$ are closable as unbounded operators $C^\infty \subset l^1 \to l^1$. The following proposition applied to $D_{a,b}$ and $\bar\partial_{a,b}$ then allows to compare their kernels by comparing their resolvents.

Proposition 3.3. *Let $F\colon D(F) \subset E \to E$ be a closed operator on a Banach space and let $\sigma(F) = \sigma_1 \dot\cup \sigma_2$ be a decomposition of its spectrum into two closed sets such that there exists an embedded closed curve $\gamma\colon S^1 \to \rho(F) = \mathbb{C}\backslash\sigma(F)$ enclosing σ_1. Then the bounded operator*

$$P := \frac{1}{2\pi i}\int_\gamma (\lambda - F)^{-1} d\lambda$$

is a projection, its range E_1 is a subset of $D(F)$, the kernel $E_2 = \ker(P)$ has $E_2 \cap D(F)$ as a dense subspace and both E_1 and E_2 are invariant subspaces with $\sigma(F_{|E_i}) = \sigma_i$ for $i = 1, 2$. Moreover, P commutes with every operator that commutes with F.

For a proof of this proposition see [19], Theorems XII.5 and XII.6.

Recall that an unbounded operator $F\colon D(F) \subset E \to E$ is closed if the graph norm $\|v\|_F := \|v\| + \|Fv\|$ is a complete norm on its domain $D(F)$. For example, if F is a first order elliptic operator F, the extension of the unbounded operator $F\colon C^\infty \subset L^2 \to L^2$ to the Sobolev space H^1 is closed, because the graph norm is equivalent to the first Sobolev norm (by Gardings inequality, see e.g. [24], 6.29). Elliptic regularity (see e.g. Theorem 6.30 and Lemma 6.22a of [24]) shows that this closure $F\colon H^1 \subset L^2 \to L^2$ has the same kernel as the original operator $F\colon C^\infty \to C^\infty$.

In the following we treat $D_{a,b}$ and $\bar\partial_{a,b}$ as unbounded operators $C^\infty \subset l^1 \to l^1$ and show (in Lemma 3.5 below) that they also admit closures with kernels in C^∞. We will define these closures by taking suitable restrictions of the respective L^2–closures $H^1 \subset L^2 \to L^2$. For this we need the following lemma.

Lemma 3.4. *Let $l^1 = l^1(T^2,\mathbb{C}^2)$ be the Banach space of absolutely convergent Fourier series with values in $\mathbb{C}^2$ equipped with the Wiener norm* (3.3)*. Then:*

a) The multiplication by a 2×2–matrix whose entries are functions with absolutely convergent Fourier series (e.g., C^∞–functions) yields a bounded endomorphism of l^1 whose operator norm is the maximal Wiener norm of the matrix entries.
b) A linear operator $\mathcal{P} \subset l^1 \to l^1$ defined on the subspace $\mathcal{P}$ of Fourier polynomials with the property that all Fourier monomials are eigenvectors extends to a bounded operator if and only if its eigenvalues are bounded. Its operator norm is the supremum of the moduli of its eigenvalues.

Proof. A linear operator F defined on the space of Fourier polynomials with values in the Wiener space of absolutely convergent Fourier series uniquely extends to a bounded endomorphism of Wiener space if it is bounded on the basis v_k, $k \in I$, of Fourier monomials of length 1:

$$\|F\sum_k v_k\lambda_k\| \le \sum_k \|Fv_k\||\lambda_k| \le C\sum_k |\lambda_k| = C\|\sum_k v_k\lambda_k\|,$$

where C is such that $\|Fv_k\| \leq C$. This proves the claim, because a bounded operator defined on a dense subspace of a Banach space has a unique extension to the whole space. The operator norm of the extension is then the supremum of the $\|Fv_k\|$. $\square$

Let F be the extension to the Sobolev space H^1 of one of the operators $D_{a,b}$ or $\bar{\partial}_{a,b}$. The space $D(F) = \{v \in l^1 \cap H^1 \mid F(v) \in l^1\} \supset C^\infty$ does not depend on the choice of F by Lemma 3.4 a) because the operators $D_{a,b}$ and $\bar{\partial}_{a,b}$ differ (3.2) by the bounded endomorphism M of l^1. Similarly, when F is replaced by the extension of another of the operators $D_{a,b}$ and $\bar{\partial}_{a,b}$ in the family, the graph norm of $F\colon D(F) \subset l^1 \to l^1$ is replaced by an equivalent norm on $D(F)$. The space $D^1 := D(F) \subset l^1$ is thus equipped with an equivalence class of norms with respect to which all the operators $D_{a,b}$ and $\bar{\partial}_{a,b}$ extend to bounded operators $D^1 \to l^1$. In order to see that theses extensions are closed it suffices to check that one of the l^1–graph norms is complete. We do this by showing that, for $(a,b) \in \mathbb{C}^2 \backslash \widetilde{\mathcal{S}}_0$, the operator $\bar{\partial}_{a,b}$ extends to a bounded operator $\bar{\partial}_{a,b}\colon D^1 \to l^1$ which is injective, surjective, and has a bounded inverse $(\bar{\partial}_{a,b})^{-1}\colon l^1 \to D^1$.

The kernels of the family of operators $\bar{\partial}_{a,b}$—and hence the logarithmic vacuum spectrum $\widetilde{\mathcal{S}}_0$—are easily understood: the Schauder basis of l^1 given by the Fourier monomials $v_c = (e^{\bar{c}z - c\bar{z}}, 0)$ and $w_c = (0, e^{-\bar{c}z + c\bar{z}})$ with $c \in \Gamma'$ is a basis of eigenvectors of all operators $\bar{\partial}_{a,b}$, $(a,b) \in \mathbb{C}^2$, the eigenvalue of v_c being $b - c$ and that of w_c being $a - \bar{c}$. In particular, the space of (a,b) for which $\bar{\partial}_{a,b}$ has a non–trivial kernel is

$$\widetilde{\mathcal{S}}_0 = (\mathbb{C} \times \Gamma') \cup (\bar{\Gamma}' \times \mathbb{C}). \tag{3.4}$$

The kernel of $\bar{\partial}_{a,b}$ is 1–dimensional for generic $(a,b) \in \widetilde{\mathcal{S}}_0$. Exceptions are the double points $(a,b) \in \bar{\Gamma}' \times \Gamma'$ for which the kernel is 2–dimensional and the corresponding multiplier $h_\gamma = e^{(a+\bar{b}_0)\gamma + (b+b_0)\bar{\gamma}}$ is real (see also the discussion at the beginning of Section 4.1).

For $(a,b) \in \mathbb{C}^2 \backslash \widetilde{\mathcal{S}}_0$, the operator $\bar{\partial}_{a,b}\colon D^1 \to l^1$ is injective. Denote by $G_{a,b}$ the unique bounded endomorphism of l^1 extending the operator on the space $\mathcal{P}$ of Fourier polynomials that is defined by $G_{a,b}(v_c) = (b-c)^{-1} v_c$ and $G_{a,b}(w_c) = (a - \bar{c})^{-1} w_c$. This unique extension exists by Part b) of Lemma 3.4 (and is compact because it can be approximated by finite rank operators). The injective endomorphism $G_{a,b}$ maps l^1 to the space D^1 and is surjective as an operator $G_{a,b}\colon l^1 \to D^1$, because D^1 is the subspace of H^1 of elements

$$u(z) = \sum_{c \in \Gamma'} u_{1,c} v_{-c} + u_{2,c} w_c$$

for which not only

$$\sum_{c \in \Gamma'} |u_{1,c}| + |u_{2,c}| < \infty \quad \text{but also} \quad \sum_{c \in \Gamma'} |b + c||u_{1,c}| + |a - \bar{c}||u_{2,c}| < \infty.$$

In particular, $G_{a,b}$ is not only a compact endomorphisms of l^1, but a bounded operator $(l^1, \|.\|_{l^1}) \to (D^1, \|.\|_{\bar{\partial}_{a,b}})$, because the graph norm of $G_{a,b}(u)$ for $u \in l^1$ is

$$\|G_{a,b}(u)\|_{\bar{\partial}_{a,b}} = \|G_{a,b}(u)\| + \|\,\bar{\partial}_{a,b}\, G_{a,b}(u)\| = \|G_{a,b}(u)\| + \|u\|.$$

Because

$$\bar{\partial}_{a,b}\, G_{a,b} = \mathrm{Id}_{l_1}\,,$$

the injective operator $\bar{\partial}_{a,b}\colon D^1 \to l^1$ is also surjective. This proves

Lemma 3.5. *For $(a,b) \in \mathbb{C}^2 \backslash \widetilde{\mathcal{S}}_0$, the operator $\bar{\partial}_{a,b}$ is a bijective operator from D^1 to l^1 with bounded inverse $G_{a,b} = \bar{\partial}_{a,b}^{-1}\colon l^1 \to D^1$. In particular, for all $(a,b) \in \mathbb{C}^2$ the operators*

$D_{a,b}$ and $\bar\partial_{a,b}$ extend to closed operators $D^1 \subset l^1 \to l^1$, because their graph norms on D^1 are complete.

3.4. **Two lemmas about the asymptotics of $D_{a,b}$.** In the asymptotic analysis we make use of an additional symmetry of the logarithmic vacuum spectrum $\widetilde{\mathcal{S}}_0$, the symmetry induced by the action of the lattice Γ' by quaternionic linear, J–commuting gauge transformations

$$T_c = \begin{pmatrix} e^{-\bar c z + c\bar z} & 0 \\ 0 & e^{\bar c z - c \bar z} \end{pmatrix}, \qquad c \in \Gamma'.$$

Under this gauge the operators $D_{a,b} = \bar\partial_{a,b} + M$ and $\bar\partial_{a,b}$, $(a,b) \in \mathbb{C}^2$, transform according to

$$D_{a+\bar c, b+c} = T_c^{-1}(\bar\partial_{a,b} + MT_{-2c})T_c\,, \tag{3.5}$$

$$\bar\partial_{a+\bar c, b+c} = T_c^{-1}(\bar\partial_{a,b})T_c\,. \tag{3.6}$$

The induced Γ'–action on $\mathbb{C}^2$ is the action of $c \in \Gamma'$ by

$$(a,b) \mapsto (a+\bar c, b+c) \tag{3.7}$$

which is a symmetry of the logarithmic vacuum spectrum $\widetilde{\mathcal{S}}_0$, but not of the logarithmic spectrum$\widetilde{\mathcal{S}}$. In contrast to this, under the Γ'–action by the gauge transformations

$$t_c = \begin{pmatrix} e^{-\bar c z + c\bar z} & 0 \\ 0 & e^{-\bar c z + c \bar z} \end{pmatrix}, \qquad c \in \Gamma',$$

both $D_{a,b}$ and $\bar\partial_{a,b}$, $(a,b) \in \mathbb{C}^2$, transform by the same formula

$$D_{a-\bar c, b+c} = t_c^{-1}(D_{a,b})t_c \tag{3.8}$$

$$\bar\partial_{a-\bar c, b+c} = t_c^{-1}(\bar\partial_{a,b})t_c. \tag{3.9}$$

The induced Γ'–action on $\mathbb{C}^2$ with $c \in \Gamma'$ acting by

$$(a,b) \mapsto (a-\bar c, b+c) \tag{3.10}$$

is thus a symmetry of both $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}_0$. This action is a coordinate version of the Γ^*–action in Section 2 and the corresponding symmetry of $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}_0$ is the periodicity obtained by passing from the spectrum to the logarithmic spectrum.

Instead of $D_{a,b} = \bar\partial_{a,b} + M$ we consider more generally the operators $\bar\partial_{a,b} + MT_{-2c}$ with $c \in \Gamma'$ which also extend to closed operators $D^1 \subset l^1 \to l^1$. If $(a,b) \in \mathbb{C}^2 \backslash \widetilde{\mathcal{S}}_0$, then

$$\bar\partial_{a,b} + MT_{-2c} = (\mathrm{Id} + MT_{-2c}G_{a,b})\,\bar\partial_{a,b} \tag{3.11}$$

where, by Lemma 3.4, the operator $\mathrm{Id} + MT_{-2c}G_{a,b}$ is a bounded endomorphisms of l^1. Corollary 3.7 below show that $\mathrm{Id} + MT_{-2c}G_{a,b}$ is invertible if c is sufficiently large.

The following uniform estimate lies at the heart of the asymptotic analysis.

Lemma 3.6. *Let $\Omega \subset \mathbb{C}^2 \backslash \widetilde{\mathcal{S}}_0$ be compact. For every $\delta > 0$ there exists $R > 0$ such that*

$$\|G_{a,b} M T_{-2c} G_{a,b}\| \le \delta$$

for all $(a,b) \in \Omega$ and all $c \in \Gamma'$ with $|c| \ge R$.

Proof. We use that the operators $\bar\partial_{a,b}$ and $G_{a,b}$ are diagonal with respect to the Schauder basis v_c, w_c of l^1. By Part b) of Lemma 3.4, for all $(a,b) \in \Omega$ the norm of the operator $G_{a,b}$ satisfies

$$\|G_{a,b}\| \leq C := \frac{1}{\min\{C_1, C_2\}},$$

where the minima $C_1 := \min\{|a - \bar c|\} > 0$ and $C_2 := \min\{|b - c|\} > 0$ over all $c \in \Gamma'$ and $(a,b) \in \Omega$ are positive, because $\Omega \subset \mathbb{C}^2 \backslash \widetilde{\mathcal{S}}_0$ is compact.

Choosing R' such that $|a - \bar c| \geq 3C\|M\|/\delta$ and $|b - c| \geq 3C\|M\|/\delta$ for all $(a,b) \in \Omega$ and $c \in \Gamma'$ with $|c| > R'$ yields a decomposition $G_{a,b} = G^0_{a,b} + G^\infty_{a,b}$ with $G^0_{a,b} = G_{a,b} \circ P_{R'}$ and $G^\infty_{a,b} = G_{a,b} - G^0_{a,b}$, where $P_{R'}$ denotes the projection on $\operatorname{Span}\{v_c, w_c \mid |c| \leq R'\}$ with respect to the splitting $l^1 = \operatorname{Span}\{v_c, w_c \mid |c| \leq R'\} \oplus \operatorname{Span}\{v_c, w_c \mid |c| > R'\}$. Then $\|G^\infty_{a,b}\| \leq \frac{\delta}{3C\|M\|}$ and the operator $G^0_{a,b}$ takes values in the image of $P_{R'}$ and vanishes on the kernel of $P_{R'}$.

$M = \begin{pmatrix} 0 & -\bar q \\ q & 0 \end{pmatrix}$ with smooth q so that its Fourier series $q(z) = \sum_{c \in \Gamma'} q_c e^{-\bar c z + c \bar z}$ converges uniformly and there is R'' such that $q^\infty = \sum_{c \in \Gamma', |c| > R''} q_c e^{-\bar c z + c \bar z}$ has Wiener norm $\sum_{c \in \Gamma', |c| > R''} |q_c| < \frac{\delta}{3C^2}$. This yields the decomposition $M = M^0 + M^\infty$ where

$$M^\infty = \begin{pmatrix} 0 & -\bar q_\infty \\ q_\infty & 0 \end{pmatrix} \qquad \text{and} \qquad M^0 = \begin{pmatrix} 0 & -\bar q_0 \\ q_0 & 0 \end{pmatrix}$$

with $q_0 = q - q_\infty$. By Part a) of Lemma 3.4, $\|M^\infty\| \leq \frac{\delta}{3C^2}$ and the operator M^0 is the multiplication by a Fourier polynomial.

For $|c| \geq R := R' + R''$ we have $G^0_{a,b} M^0 T_{-2c} G^0_{a,b} = 0$ and hence, because the shift operator T_{-2c} is an isometry of l^1,

$$\|G_{a,b} M T_{-2c} G_{a,b}\| = \|G^\infty_{a,b} M T_{-2c} G_{a,b} + G^0_{a,b} M T_{-2c} G^\infty_{a,b} + G^0_{a,b} M^\infty T_{-2c} G^0_{a,b}\| \leq \delta.$$

□

Corollary 3.7. *For every $\delta > 0$ and every compact $\Omega \subset \mathbb{C}^2 \backslash \widetilde{\mathcal{S}}_0$ there exists $R > 0$ such that for all $c \in \Gamma'$ with $|c| \geq R$ and $(a,b) \in \Omega$ the operator $\bar\partial_{a,b} + M T_{-2c}$ is invertible and*

$$\|G_{a,b} - (\bar\partial_{a,b} + M T_{-2c})^{-1}\| \leq \delta.$$

Proof. We use that if an endomorphisms F of a Banach space satisfies $\|F^n\| < 1$ for some power n, the operator $\operatorname{Id} + F$ is invertible with inverse given by the Neumann series $\sum_{k=0}^\infty (-1)^k F^k$, and

$$\|(\operatorname{Id} + F)^{-1}\| \leq \frac{1}{1 - \|F^n\|}(1 + \|F\| + ... + \|F^{n-1}\|). \tag{3.12}$$

By Lemma 3.6 we can choose $R > 0$ such that, for all for $c \in \Gamma'$ with $|c| > R$ and $(a,b) \in \Omega$,

$$\|G_{a,b} M T_{-2c} G_{a,b}\| \leq \min\{\tfrac{1}{2\|M\|}, \tfrac{\delta}{2(1 + G\|M\|)}\},$$

where $G = \max_{(a,b) \in \Omega} \|G_{a,b}\|$. Because T_{-2c} is an isometry of l^1, for $c \in \Gamma'$ with $|c| > R$ and $(a,b) \in \Omega$, the operator $M T_{-2c} G_{a,b}$ satisfies $\|(M T_{-2c} G_{a,b})^2\| < 1/2$. Thus $\operatorname{Id} + M T_{-2c} G_{a,b}$

and therefore $\bar\partial_{a,b} + MT_{-2c} = (\mathrm{Id} + MT_{-2c}G_{a,b})\bar\partial_{a,b}$ are invertible and

$$\begin{aligned}\|G_{a,b} - (\bar\partial_{a,b} + MT_{-2c})^{-1}\| &= \|G_{a,b}(\mathrm{Id} - (\mathrm{Id} + MT_{-2c}G_{a,b})^{-1})\| \\ &= \|G_{a,b}MT_{-2c}G_{a,b}(\sum_{k=0}^{\infty}(-1)^k(MT_{-2c}G_{a,b})^k)\| .\end{aligned}$$

Together with (3.12) this implies

$$\|G_{a,b} - (\bar\partial_{a,b} + MT_{-2c})^{-1}\| \le \|G_{a,b}MT_{-2c}G_{a,b}\| \frac{1 + \|G_{a,b}\|\|M\|}{1 - \|(MT_{-2c}G_{a,b})^2\|} < \delta.$$

□

The preceding corollary shows in particular that, for every $(a, b) \notin \widetilde{\mathcal{S}}_0$, the operator $\bar\partial_{a,b} + MT_{-2c} = (\mathrm{Id} + MT_{-2c}G_{a,b})\bar\partial_{a,b}$ is invertible if $c \in \Gamma'$ is large enough and moreover that $(\bar\partial_{a,b} + MT_{-2c})^{-1}$ converges to $G_{a,b} = (\bar\partial_{a,b})^{-1}$ when $|c| \to \infty$. The convergence is uniform for $(a, b) \in \Omega$ in a compact set $\Omega \subset \mathbb{C}^2 \setminus \widetilde{\mathcal{S}}_0$. This is needed in the proof of the following lemma.

Lemma 3.8. *Let $\delta > 0$ and $\Omega \subset \mathbb{C}^2 \setminus \widetilde{\mathcal{S}}_0$ compact. Then there exists $R > 0$ such that*

1.) *$\bar\partial_{a,b} + MT_{-2c}$ is invertible for all $(a, b) \in \Omega$ and all $c \in \Gamma'$ with $|c| > R$.*
2.) *For every "transversal circle" $\gamma = \{(a+\lambda, b+\lambda) \mid |\lambda| = \epsilon\}$ in Ω with radius $\epsilon > 0$, center $(a, b) \in \mathbb{C}^2$, and for all $c \in \Gamma'$ with $|c| > R$, the operators*

$$P^c_\gamma = \frac{1}{2\pi i}\int_{|\lambda|=\epsilon}(\bar\partial_{a+\lambda,b+\lambda} + MT_{-2c})^{-1}d\lambda \qquad \textit{and} \qquad P^\infty_\gamma = \frac{1}{2\pi i}\int_{|\lambda|=\epsilon} G_{a+\lambda,b+\lambda}d\lambda$$

are projections and satisfy $\|P^c_\gamma - P^\infty_\gamma\| < \delta$.

The operator P^∞_γ projects to the finite dimensional sum $\mathrm{im}(P^\infty_\gamma) = \bigoplus_{(\tilde a,\tilde b)\in D}\ker(\bar\partial_{\tilde a,\tilde b})$ with D the "transversal disc" $D = \{(a + \lambda, b + \lambda) \in \mathbb{C}^2 \mid |\lambda| < \epsilon\}$ bounded by the circle γ. If $\delta < 1$, for every $c \in \Gamma'$ with $|c| > R$ the operator P^c_γ projects to a finite dimensional space whose dimension coincides with that of $\mathrm{im}(P^\infty_\gamma)$ and which is spanned by the kernels of all iterates of $\bar\partial_{\tilde a,\tilde b} + MT_{-2c}$ with $(\tilde a, \tilde b) \in D$.

The notion *transversal circle* and *transversal disc* reflects the fact that, away from double points of $\widetilde{\mathcal{S}}_0$, the intersection of a transversal disc D with $\widetilde{\mathcal{S}}_0$ is transversal.

Proof. By Corollary 3.7, there is $R > 0$ such that for all $(a, b) \in \Omega$ and all $c \in \Gamma'$ with $|c| > R$ the operator $\bar\partial_{a,b} + MT_{-2c}$ is invertible and $\|G_{a,b} - (\bar\partial_{a,b} + MT_{-2c})^{-1}\| < 2\delta/\operatorname{diam}(\Omega)$. For every transversal circle $\gamma = \{(a+\lambda, b+\lambda) \mid |\lambda| = \epsilon\} \subset \Omega$ and every $c \in \Gamma'$ with $|c| > R$, the operators P^c_γ and P^∞_γ are then well defined and satisfy $\|P^c_\gamma - P^\infty_\gamma\| < 2\epsilon\delta/\operatorname{diam}(\Omega) < \delta$. Proposition 3.3 shows that they are projection operators.

As one can easily check by evaluation on the Fourier monomials v_c and w_c, the operator P^∞_γ projects to the space spanned by the kernels of $\bar\partial_{\tilde a,\tilde b}$ for all $(\tilde a, \tilde b) \in D \cap \widetilde{\mathcal{S}}_0$. This space is finite dimensional, because $D \cap \widetilde{\mathcal{S}}_0$ is a finite set.

If $\delta < 1$, for every $c \in \Gamma'$ with $|c| > R$ the operator $(\mathrm{Id} - P^c_\gamma + P^\infty_\gamma)$ is invertible and maps $\mathrm{im}(P^c_\gamma) = \ker(\mathrm{Id} - P^c_\gamma)$ to a subspace of $\mathrm{im}(P^\infty_\gamma)$. Similarly, $(\mathrm{Id} - P^\infty_\gamma + P^c_\gamma)$ is invertible and maps $\mathrm{im}(P^\infty_\gamma) = \ker(\mathrm{Id} - P^\infty_\gamma)$ to a subspace of $\mathrm{im}(P^c_\gamma)$. This shows that $\mathrm{im}(P^c_\gamma)$ and

$\mathrm{im}(P_\gamma^\infty)$ have the same dimension. In particular, the space $\mathrm{im}(P_\gamma^c)$ is also finite dimensional. By Proposition 3.3 the finite dimensional image of P_γ^c is an invariant subspace for the operator $\bar\partial_{a,b} + MT_{-2c}$ and the restriction of $\bar\partial_{a,b} + MT_{-2c}$ to this subset has spectrum contained in $\{|\lambda| < \epsilon\}$. The image of P_γ^c is thus the direct sum of the kernels of all iterates of $\bar\partial_{\tilde a,\tilde b} + MT_{-2c}$ with $(\tilde a, \tilde b) \in D$. □

Corollary 3.9. *There is a point $(a,b) \in \mathbb{C}^2$ such that $\ker(D_{a,b})$ is 0–dimensional and there is a point $(\tilde a, \tilde b) \in \mathbb{C}^2$ such that $\ker(D_{\tilde a,\tilde b})$ is 1–dimensional.*

Proof. Let $(a,b) \in \widetilde{\mathcal{S}}_0$ be a point such that the kernel of $\bar\partial_{a,b}$ is 1–dimensional and choose $\epsilon > 0$ such that the closure of the transversal disc $D = \{(a+\lambda, b+\lambda) \in \mathbb{C}^2 \mid |\lambda| < \epsilon\}$ intersects $\widetilde{\mathcal{S}}_0$ in (a,b) only. By Lemma 3.8 there is $R > 0$ such that for all $c \in \Gamma'$ with $|c| > R$ there is a unique point $(\tilde a, \tilde b) \in D$ for which the kernel of $\bar\partial_{\tilde a,\tilde b} + MT_{-2c}$ not trivial, but 1–dimensional. By (3.5) this implies that, for all $c \in \Gamma'$ with $|c| > R$, there is a unique point $(\tilde a, \tilde b) \in \widetilde{\mathcal{S}}$ contained in the disc $\{(a+\bar c+\lambda, b+c+\lambda) \in \mathbb{C}^2 \mid |\lambda| < \epsilon\}$ and the kernel of $D_{\tilde a,\tilde b}$ at that point $(\tilde a, \tilde b) \in \widetilde{\mathcal{S}}$ is 1–dimensional. □

Corollary 3.9 completes the above proof of Lemma 2.4 (and hence the proof of Theorem 2.6).

4. Asymptotic Geometry of Spectral Curves

We investigate the asymptotic geometry of the spectrum $\mathrm{Spec}(W,D)$, the spectral curve Σ, and the kernel bundle $\mathcal{L} \to \Sigma$ of a quaternionic holomorphic line bundle $(W, D = \bar\partial + Q)$ of degree zero over a 2–torus. We show that the spectrum $\mathrm{Spec}(W,D)$ is asymptotic to the vacuum spectrum $\mathrm{Spec}(W,\bar\partial)$. As a consequence asymptotically the spectral curve Σ is bi–holomorphic to a pair of planes joined by at most countably many handles.

4.1. **Statement of the main result.** Recall that the vacuum spectrum $\mathrm{Spec}(W,\bar\partial)$ is a real translate of

$$\exp(H^0(K)) \cup \exp(H^0(\bar K)),$$

see (3.4) or Section 3.2 of [3]. Its double points are the real representations

$$\mathrm{Spec}(W,\bar\partial) \cap \mathrm{Hom}(\Gamma, \mathbb{R}_*).$$

If $\exp(\int\omega) = \exp(\int\bar\eta) \in \mathrm{Hom}(\Gamma,\mathbb{C}_*)$ with ω, $\eta \in H^0(K)$, then $\omega = \eta$ and $\omega - \bar\omega \in \Gamma^*$ since 1-forms in the dual lattice are always imaginary. On the other hand, for every $\omega \in \mathrm{Harm}(T^2,\mathbb{C})$ the representation $\exp(\int\omega) \in \mathrm{Hom}(\Gamma,\mathbb{C}_*)$ has a unique decomposition $\exp(\int\omega) = \exp(\int\frac{\omega+\bar\omega}{2})\exp(\int\frac{\omega-\bar\omega}{2})$ into $\mathbb{R}_+$– and S^1–representations the latter of which is real if and only if $\omega - \bar\omega \in \Gamma^*$, that is, if it is a $\mathbb{Z}_2$–representation. This shows that the subgroup

$$G' = \{h \in \mathrm{Hom}(\Gamma,\mathbb{R}_*) \mid h = \exp(\textstyle\int\omega) \text{ with } \omega \in H^0(K),\ \omega - \bar\omega \in \Gamma^*\}$$

of $G = \mathrm{Hom}(\Gamma,\mathbb{C}_*)$ acts simply transitive on the set of vacuum double points.

We now come to the main theorem describing the structure and asymptotics of the spectrum and the kernel line bundle.

Theorem 4.1. *Let (W,D) be a quaternionic holomorphic line bundle of degree zero over a torus. Then (with respect to a fixed bi–invariant metric on $\mathrm{Hom}(\Gamma,\mathbb{C}_*)$):*

(1) *For every $\epsilon > 0$, there exists a compact set $\Omega \subset \operatorname{Hom}(\Gamma, \mathbb{C}_*)$ and a neighborhood U of a vacuum double point in $\operatorname{Spec}(W, \bar\partial) \cap \operatorname{Hom}(\Gamma, \mathbb{R}_*)$ such that*
   a) *in the complement of Ω, the spectrum $\operatorname{Spec}(W, D)$ is contained in an ϵ–tube around the vacuum spectrum $\operatorname{Spec}(W, \bar\partial)$;*
   b) *in the complement of Ω and away from the neighborhood $\bigcup_{h \in G'} hU$ of the vacuum double points, the spectrum $\operatorname{Spec}(W, D)$ is a "graph" over $\operatorname{Spec}(W, \bar\partial)$. More precisely, $\operatorname{Spec}(W, D)$ is locally a graph over a real translate of $\exp(H^0(K))$ or $\exp(H^0(\bar K))$ with respect to coordinates induced, via the exponential map, from the splitting $\operatorname{Hom}(\Gamma, \mathbb{C}) \cong \operatorname{Harm}(T^2, \mathbb{C}) = H^0(K) \oplus H^0(\bar K)$;*
   c) *in the neighborhood hU, $h \in G'$, of a vacuum double point that is contained in the complement of Ω, the intersection of the spectrum $\operatorname{Spec}(W, D)$ with hU either consists of one handle, that is, is equivalent to an annulus, or it consists of two transversally immersed discs which have a double point at a real multiplier.*

   In particular, in the complement of Ω, the spectrum $\operatorname{Spec}(W, D)$ is non–singular except at real points which are transversally immersed double points. Moreover, for all $h \in \operatorname{Spec}(W, D) \cap (\operatorname{Hom}(\Gamma, \mathbb{C}_*) \backslash \Omega)$ either

$$\dim(H^0_h(\tilde W)) = 1 \qquad \text{or} \qquad \dim(H^0_h(\tilde W)) = 2$$

   depending on whether $h \in (\operatorname{Hom}(\Gamma, \mathbb{C}_*) \backslash \operatorname{Hom}(\Gamma, \mathbb{R}))$ or $h \in \operatorname{Hom}(\Gamma, \mathbb{R})$.

(2) *The spectral curve Σ is the union*

$$\Sigma = \Sigma_{cpt} \cup \Sigma_\infty$$

   of two ρ–invariant Riemann surfaces Σ_{cpt} and Σ_∞ with the following properties:
   a) *both Σ_{cpt} and Σ_∞ have a boundary consisting of two circles along which they are glued, that is, $\partial \Sigma_{cpt} = -\partial \Sigma_\infty$;*
   b) *Σ_{cpt} is compact with at most two components each of which has a non–empty boundary;*
   c) *Σ_∞ consists of two planes, each with a disc removed, which are joined by a countable number of handles.*

   In particular, either the spectral curve Σ has infinite genus, one end and is connected, or it has finite genus, two ends and at most two connected components, each containing an end. In the finite genus case, both ends are interchanged by the involution $\rho \colon \Sigma \to \Sigma$.

(3) *Given $\epsilon > 0$ and $\delta > 0$, the compact set Ω and the open set U in (1) can be chosen with the following additional properties:*
   a) *defined on the preimage V under $\tilde\Sigma \to \operatorname{Spec}(W, D)$ of*

$$\operatorname{Spec}(W, D) \cap \big(\operatorname{Hom}(\Gamma, \mathbb{C}_*) \backslash (\Omega \cup \bigcup_{h \in G'} hU)\big),$$

   there is a holomorphic section ψ of $\tilde{\mathcal{L}} \to \tilde\Sigma$ that is "δ–close to a vacuum solution". By this we mean that

$$\|\psi - \varphi\| < \delta$$

   for φ a nowhere vanishing, locally constant section defined over V of the trivial $\Gamma(W)$–bundle over $\tilde\Sigma$ that for every $\tilde\sigma \in V \subset \tilde\Sigma$ solves $\bar\partial_\omega \varphi^{\tilde\sigma} = 0$ with $\omega \in \widetilde{\operatorname{Spec}}(W, \bar\partial)$ satisfying $\|\omega(\tilde\sigma) - \omega\| < \epsilon$.
   b) *Let $h \in G'$ such that the preimage under $\tilde\Sigma \to \operatorname{Spec}(W, D)$ of*

$$\operatorname{Spec}(W, D) \cap (\operatorname{Hom}(\Gamma, \mathbb{C}_*) \backslash \Omega) \cap hU$$

is the sum of two discs. Then ψ holomorphically extends through these discs and the extension is again δ–close to vacuum solutions.

4.2. **Proof of the main result.** Theorem 4.1 is a consequence of the following three lemmas below: Lemma 4.2 shows that, for large multipliers, the spectrum is contained in an ϵ–tube around the vacuum spectrum with respect to a bi–invariant metric on $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$. Lemma 4.3 shows that, for large multipliers and away from double points of the vacuum, the spectrum is a graph over the vacuum spectrum. Lemma 4.6 shows that, for large multipliers, in a neighborhood of a vacuum double point the spectrum either consists of an annulus or of a pair of discs with a double point.

We continue using the (a, b)–coordinates (3.1) on the Lie algebra $\mathrm{Hom}(\Gamma, \mathbb{C}) \cong \mathrm{Harm}(T^2, \mathbb{C})$ of $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$. In these coordinates the logarithmic vacuum spectrum is

$$\widetilde{\mathcal{S}}_0 = (\mathbb{C} \times \Gamma') \cup (\bar{\Gamma}' \times \mathbb{C})$$

with the set of double points $\bar{\Gamma}' \times \Gamma'$, see (3.4). The preimages under the exponential map of points in $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$ are the orbits of the Γ'–action $(a, b) \mapsto (a - \bar{c}, b + c)$ for $c \in \Gamma'$ on $\mathbb{C}^2$, see (3.10). In order to define fundamental domains for this group action, we fix a basis c_1, c_2 of Γ' of vectors of minimal length. Then both

$$\begin{aligned} A &= \{(a, b) \mid a \in \mathbb{C} \text{ and } b = \lambda_1 c_1 + \lambda_2 c_2 \text{ with } \lambda_i \in [-1/2, 1/2]\} \quad \text{and} \\ B &= \{(a, b) \mid b \in \mathbb{C} \text{ and } a = \lambda_1 \bar{c}_1 + \lambda_2 \bar{c}_2 \text{ with } \lambda_i \in [-1/2, 1/2]\} \end{aligned} \tag{4.1}$$

are fundamental domains for the Γ'–action (3.10): orbits of generic points $(a, b) \in \mathbb{C}^2$ intersect A and B in a single point, only the orbits of boundary points of A and B intersect several times. For understanding $\mathrm{Spec}(W, D) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ it is sufficient to study the intersection of $\widetilde{\mathcal{S}}$ with the fundamental domain B. To understand the intersection with A it is sufficient to apply the involution ρ, in our coordinates given by $(a, b) \mapsto (\bar{b}, \bar{a})$, which interchanges A and B and leaves $\widetilde{\mathcal{S}}$ invariant.

To investigate the intersection $\widetilde{\mathcal{S}} \cap B$ we use (3.7) and (3.10): under the Γ'–action by the gauge transformation $t_c T_c$ the operator $D_{a,b}$ transforms according to

$$D_{a,b+2c} = t_c^{-1} T_c^{-1} (\bar{\partial}_{a,b} + M T_{-2c}) t_c T_c \qquad \text{for every} \quad c \in \Gamma' \tag{4.2}$$

while the fundamental domain B is invariant under the action $(a, b) \mapsto (a, b+2c)$ of $c \in \Gamma'$.

The following lemma shows that, for large multipliers, the spectrum $\mathrm{Spec}(W, D)$ lies in an ϵ–tube around the vacuum spectrum $\mathrm{Spec}(W, \bar{\partial})$.

Lemma 4.2. *For every $\epsilon > 0$, there is a compact subset of B in the complement of which the intersection of $\widetilde{\mathcal{S}}$ with B is contained in an ϵ–tube around $\widetilde{\mathcal{S}}_0$.*

Proof. For $\epsilon > 0$ with $2\epsilon < \min\{|c| \mid c \in \Gamma' \backslash \{0\}\}$ we define

$$\widetilde{\mathcal{S}}_0^\epsilon = \{(a, b) \mid |a - \bar{c}| < \epsilon \text{ for } c \in \Gamma'\} \cup \{(a, b) \mid |b - c| < \epsilon \text{ for } c \in \Gamma'\}. \tag{4.3}$$

For $B' = \{(a, b) \in B \mid b = \lambda_1 c_1 + \lambda_2 c_2 \text{ with } \lambda_i \in [-1, 1]\}$ we have $B = \bigcup_{c \in \Gamma'} (B' + (0, 2c))$. The set $\Omega = B' \backslash \widetilde{\mathcal{S}}_0^\epsilon$ is compact and does not intersect $\widetilde{\mathcal{S}}_0$. Hence, by Corollary 3.7, there is $R > 0$ such that for all $(a, b) \in \Omega$ and every $c \in \Gamma'$ with $|c| > R$, the kernel of $\bar{\partial}_{a,b} + M T_{-2c}$ and, by (4.2), that of $D_{a,b+2c}$ is zero. This shows that $\widetilde{\mathcal{S}}$ does not intersect $\bigcup_{c \in \Gamma'; |c| > R} (\Omega + (0, 2c))$ or, equivalently, that the intersection of $\widetilde{\mathcal{S}}$ with the subset $\bigcup_{c \in \Gamma'; |c| > R} (B' + (0, 2c))$ of B is contained in the ϵ–tube $\widetilde{\mathcal{S}}_0^\epsilon$ around $\widetilde{\mathcal{S}}_0$. $\square$

The next lemma and remark show that, away from the double points of the vacuum spectrum $\operatorname{Spec}(\bar\partial, W)$, for large enough multipliers the spectrum $\operatorname{Spec}(W, D)$ is an arbitrarily small deformation of $\operatorname{Spec}(\bar\partial, W)$.

Lemma 4.3. *For $\epsilon > 0$ and $\delta > 0$ with $2\epsilon < \min\{|c| \mid c \in \Gamma'\backslash\{0\}\}$ and $\delta < 1$ there exists $R > 0$ such that:*

a) The intersection of $\widetilde{\mathcal{S}}$ with

$$\{|a| < \epsilon\} \times \big(\bigcup_{c\in\Gamma';|c|>R} (\Delta_\epsilon + 2c)\big) \subset \mathbb{C}^2$$

is the graph of a holomorphic function $b \mapsto a(b)$ defined on $\bigcup_{c\in\Gamma';|c|>R}(\Delta_\epsilon + 2c) \subset \mathbb{C}$ with $\Delta_\epsilon = \{b = \lambda_1 c_1 + \lambda_2 c_2 \mid \lambda_j \in [-1, 1]$ and $|b - c| > \epsilon$ for all $c \in \Gamma'\}$. For all points $(a, b) \in \mathbb{C}^2$ contained in this graph, the kernel of $D_{a,b}$ is 1–dimensional and, in particular, the resulting multiplier is non–real, that is, an element of $\operatorname{Hom}(\Gamma, \mathbb{C}_*)\backslash \operatorname{Hom}(\Gamma, \mathbb{R}_*)$.

b) The bundle $\tilde{\mathcal{L}} \to \tilde\Sigma$ admits a holomorphic section ψ which is defined over the preimage under the normalization map $\tilde\Sigma \to \widetilde{\mathcal{S}}$ of the subset described in a) and has the property that, for every $\tilde\sigma$ in this preimage, the section $\psi^{\tilde\sigma} \in \tilde{\mathcal{L}}_{\tilde\sigma}$ satisfies

$$\|\psi^{\tilde\sigma} - \psi^o\| < \delta,$$

where $\|\,\|$ denotes the Wiener norm and $\psi^o = (0, 1)$ is the Fourier monomial contained in the kernel of $\bar\partial_{0,b}$ for all $b \in \mathbb{C}$.

Remark 4.4. The analogous result for the a–plane is obtained by applying the anti–holomorphic involution ρ: for ϵ, δ, and R as in Lemma 4.3, the intersection of $\widetilde{\mathcal{S}}$ with the set $\bigcup_{c\in\Gamma';|c|>R}(\bar\Delta_\epsilon+2\bar c)\times\{|b| < \epsilon\}$ is the graph of a function $a \mapsto b(a)$ over $\bigcup_{c\in\Gamma';|c|>R}(\bar\Delta_\epsilon+2\bar c)$. Setting $\psi^{\tilde\sigma} = -\psi^{\rho(\tilde\sigma)}j$, the holomorphic section ψ from Lemma 4.3, b) yields a holomorphic section of $\tilde{\mathcal{L}}$ defined on the part of $\tilde\Sigma$ which is graph over $\bigcup_{c\in\Gamma';|c|>R}(\bar\Delta_\epsilon + 2\bar c)$. This section satisfies $\|\psi^{\tilde\sigma} - \psi^\infty\| < \delta$ for $\psi^\infty = -\psi^o j = (1, 0)$.

Proof. Let $\tilde\epsilon = \frac{1}{5}\epsilon$ and $\Omega = B^\epsilon\backslash \widetilde{\mathcal{S}}_0^{\tilde\epsilon}$ with $\widetilde{\mathcal{S}}_0^{\tilde\epsilon}$ as defined in (4.3) and

$$B^\epsilon = \{(a, b) \in B \mid \operatorname{dist}((a, b), B') \le \epsilon\},$$

where as above $B' = \{(a, b) \in B \mid b = \lambda_1 c_1 + \lambda_2 c_2$ with $\lambda_i \in [-1, 1]\}$. By Lemma 3.8 we can chose $R > 0$ such that, for every $c \in \Gamma'$ with $|c| > R$, the operator $\bar\partial_{a,b} + MT_{-2c}$ is invertible for all $(a, b) \in \Omega$ and $\|P^c_\gamma - P^\infty_\gamma\| < \delta$ for all transversal circles $\gamma \subset \Omega$. As in the proof of Lemma 4.2, the intersection of $\widetilde{\mathcal{S}}$ with $\bigcup_{c\in\Gamma';|c|>R}(B^\epsilon + (0, 2c))$ is then contained in the $\tilde\epsilon$–tube $\widetilde{\mathcal{S}}_0^{\tilde\epsilon}$ around $\widetilde{\mathcal{S}}_0$.

For every $b \in \Delta^{4\tilde\epsilon}_{2\tilde\epsilon}$ with

$$\Delta^{\epsilon_2}_{\epsilon_1} = \{b \in \mathbb{C} \mid \operatorname{dist}(b, \Delta) \le \epsilon_2 \text{ and } |b - c| > \epsilon_1 \text{ for all } c \in \Gamma'\}, \tag{4.4}$$

where $\Delta = \{b = \lambda_1 c_1 + \lambda_2 c_2 \mid \lambda_j \in [-1, 1]\}$, the transversal circle $\gamma_b = \{(\lambda, b+\lambda) \mid |\lambda| = \tilde\epsilon\}$ is contained in Ω. The operator $P^\infty_{\gamma_b}$ projects to the one dimensional kernel of $\bar\partial_{0,b}$. Because $\delta < 1$, Lemma 3.8 implies that, if $c \in \Gamma'$ with $|c| > R$, the image of the projection P^c_γ is the 1–dimensional kernel of $\bar\partial_{\tilde a,\tilde b} + MT_{-2c}$. Here $(\tilde a, \tilde b) \in D$ is the unique point in the transversal disc $D = \{(\lambda, b + \lambda) \mid |\lambda| < \tilde\epsilon\}$ for which the kernel of $\bar\partial_{\tilde a,\tilde b} + MT_{-2c}$ is non–trivial.

By (4.2), for every $b \in \bigcup_{c\in\Gamma';|c|>R}(\Delta_{2\tilde\epsilon}^{4\tilde\epsilon}+2c) \subset \mathbb{C}$ the disc $\{(\lambda, b+\lambda) \in \mathbb{C}^2 \mid |\lambda| < \tilde\epsilon\}$ contains a unique point in $\widetilde{\mathcal{S}}$. This defines a holomorphic function λ on $\bigcup_{c\in\Gamma';|c|>R}(\Delta_{2\tilde\epsilon}^{4\tilde\epsilon}+2c)$ with $|\lambda| < \tilde\epsilon$ and such that every point in the intersection $\widetilde{\mathcal{S}} \cap \big(\{|a| < \epsilon\} \times \bigcup_{c\in\Gamma';|c|>R}(\Delta_{3\tilde\epsilon}^{3\tilde\epsilon}+2c)\big)$ is of the form $(\lambda(b), b+\lambda(b))$ for some $b \in \bigcup_{c\in\Gamma';|c|>R}(\Delta_{2\tilde\epsilon}^{4\tilde\epsilon}+2c)$. The Cauchy integral formula for the first derivative of λ (applied to circles of radius $2\tilde\epsilon$) implies that the differential of λ restricted to $\bigcup_{c\in\Gamma';|c|>R}(\Delta_{4\tilde\epsilon}^{2\tilde\epsilon}+2c)$ is bounded by $1/2$. Because any two points b_0 and b_1 in $\bigcup_{c\in\Gamma';|c|>R}(\Delta_{4\tilde\epsilon}^{2\tilde\epsilon}+2c)$ can be joined by a curve of length $l \le \frac{\pi}{2}|b_0-b_1|$, the following proposition shows that the map $b \mapsto b+\lambda(b)$ restricted to $\bigcup_{c\in\Gamma';|c|>R}(\Delta_{4\tilde\epsilon}^{2\tilde\epsilon}+2c)$ is injective:

Proposition 4.5. *Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^n$ with $\|Df_x - \mathrm{Id}\,\| \le \epsilon$ for $\epsilon > 0$ independent of $x \in U$. If any two points x_0, $x_1 \in U$ can be joined by a curve of length $l \le C|x_0 - x_1|$ with $\epsilon < 1/C$, then f is injective and therefore a diffeomorphism from U to the open set $f(U)$.*

Proof. Assume $f(x_0) = f(x_1)$ with $x_0 \ne x_1$. Let $\gamma\colon [0,1] \to U$ be a curve of length $l \le C|x_0 - x_1|$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$ and constant speed $|\gamma'(t)| = l$. Then

$$|x_0 - x_1| = |\int_0^1 (f(\gamma(t)) - \gamma(t))'dt| \le l\int_0^1 \|Df_{\gamma(t)} - \mathrm{Id}\,\|dt \le C|x_0 - x_1|\epsilon < |x_0 - x_1|.$$

This contradicts the assumption that $x_0 \ne x_1$ such that f is injective. Because $\epsilon < 1/C$ and $1/C < 1$, the inverse function theorem implies that f is a local diffeomorphism. $\square$

For every $c \in \Gamma'$ with $|c| > R$, the image of the boundary of $(\Delta_{4\tilde\epsilon}^{2\tilde\epsilon}+2c)$ under the map $b \mapsto b+\lambda(b)$ is contained in $(\Delta_{3\tilde\epsilon}^{3\tilde\epsilon}+2c)\backslash(\Delta_{\epsilon}^{\tilde\epsilon}+2c)$ such that the image of $(\Delta_{4\tilde\epsilon}^{2\tilde\epsilon}+2c)$ is a subset of $(\Delta_{3\tilde\epsilon}^{3\tilde\epsilon}+2c)$ which contains $(\Delta_{\epsilon}^{\tilde\epsilon}+2c)$. Therefore, the injective function $b \mapsto b+\lambda(b)$ maps $\bigcup_{c\in\Gamma';|c|>R}(\Delta_{4\tilde\epsilon}^{2\tilde\epsilon}+2c)$ onto a set containing $\bigcup_{c\in\Gamma';|c|>R}(\Delta_{\epsilon}^{\tilde\epsilon}+2c)$. Taking its inverse function μ restricted to $\bigcup_{c\in\Gamma';|c|>R}(\Delta_{\epsilon}^{\tilde\epsilon}+2c)$ yields a representation of the intersection of $\widetilde{\mathcal{S}}$ with $\{|a| < \tilde\epsilon\} \times \bigcup_{c\in\Gamma';|c|>R}(\Delta_{\epsilon}^{\tilde\epsilon}+2c)$ as the graph of the function $a(b) = b - \mu(b)$ over $\bigcup_{c\in\Gamma';|c|>R}(\Delta_{\epsilon}^{\tilde\epsilon}+2c)$.

For $b \in \bigcup_{c\in\Gamma';|c|>R}(\Delta_{\epsilon}^{\tilde\epsilon}+2c)$ take $b' \in \Delta_{4\tilde\epsilon}^{2\tilde\epsilon}$ and $c \in \Gamma'$ with $|c| > R$ such that $\mu(b) = b'+2c$. Then, by definition of $P^c_{\gamma_{b'}}$ and (4.2),

$$P_b := t_c^{-1}T_c^{-1}(P^c_{\gamma_{b'}})t_cT_c = \frac{1}{2\pi i}\int_{|\lambda|=\tilde\epsilon} D^{-1}_{\lambda,\mu(b)+\lambda}d\lambda. \tag{4.5}$$

In particular, the definition of P_b does not depend on the choice of the representation $\mu(b) = b'+2c$. Analogously, we define

$$P_b^\infty = t_c^{-1}T_c^{-1}(P^\infty_{\gamma_{b'}})t_cT_c = \frac{1}{2\pi i}\int_{|\lambda|=\tilde\epsilon} \bar\partial^{-1}_{\lambda,\mu(b)+\lambda}d\lambda\,.$$

This projection operator is independent of b: its kernel contains all Fourier monomials except the constant section $\psi^o = (0,1) \in C^\infty(T^2,\mathbb{C}^2)$ which spans its image, that is, $\psi^o = P_b^\infty(\psi^o)$. Since we have chosen R such that $\|P_b - P_b^\infty\| < \delta$, the section $\psi^{\tilde\sigma} := P_b(\psi^o)$ with $\tilde\sigma \in \tilde\Sigma$ corresponding to $(a(b), b) \in \widetilde{\mathcal{S}}$ satisfies $\|\psi^{\tilde\sigma} - \psi^o\| < \delta$. $\square$

The following lemma shows that, for large enough multipliers, in a neighborhood of a vacuum double point the spectrum $\mathrm{Spec}(W,D)$ either has a double point or the vacuum double point resolves into a handle.

Lemma 4.6. *Let $\epsilon > 0$ and $\delta > 0$ with $2\epsilon < \min\{|c| \mid c \in \Gamma'\backslash\{0\}\}$ and $\delta < 1$. Then there exists $R > 0$ such that:*

a) For every $c_0 \in \Gamma'$ with $c_0 > 2R$, the intersection of $\widetilde{\mathcal{S}}$ with the polydisc

$$\{(a,b) \in \mathbb{C}^2 \mid |a| < \epsilon \text{ and } |b - c_0| < \epsilon\}$$

is either bi–holomorphic to an annulus or to a pair of transversally intersecting immersed discs with one intersection point. Each disk is a graph over one of the coordinate planes. For a double point (a,b) of the spectrum $\widetilde{\mathcal{S}}$ contained in the polydisc, the kernel of the operator $D_{a,b}$ is 2–dimensional and the corresponding multiplier is real; for all other $(a,b) \in \widetilde{\mathcal{S}}$ contained in the polydisc, the kernel of $D_{a,b}$ is 1–dimensional and the corresponding multiplier is non–real, that is, an element of $\operatorname{Hom}(\Gamma, \mathbb{C}_*)\backslash \operatorname{Hom}(\Gamma, \mathbb{R}_*)$.

b) The intersection of $\widetilde{\mathcal{S}}$ with $\{|a| < \epsilon\} \times \left(\bigcup_{c\in\Gamma';|c|>R}(\Delta_{\epsilon/2} + 2c)\right) \subset \mathbb{C}^2$ is the graph of a function $b \mapsto a(b)$ and the holomorphic section ψ of $\tilde{\mathcal{L}}$ defined (as in Lemma 4.3) over this set extends holomorphically through the discs around double points which are graphs over the b–plane. The extension satisfies $\|\psi^{\tilde{\sigma}} - \psi^o\| < \delta$.

Remark 4.7. As in Remark 4.4, the analogous result for the part of the spectral curve that is a graph over the a–plane can be obtained by applying the involution ρ.

Proof. Let $\tilde{\epsilon} = \frac{\epsilon}{10}$ and $\Omega = B^{14\tilde{\epsilon}}\backslash \widetilde{\mathcal{S}}_0^{\tilde{\epsilon}}$ with B^ϵ and $\widetilde{\mathcal{S}}_0^{\tilde{\epsilon}}$ as in the proof of Lemma 4.3. By Lemma 3.8, we can chose $R > 0$ such that for every $c \in \Gamma'$ with $|c| > R$ the operator $\bar{\partial}_{a,b} + MT_{-2c}$ is invertible for all $(a,b) \in \Omega$ and $\|P^c_\gamma - P^\infty_\gamma\| < \delta$ for all transversal circles $\gamma \subset \Omega$. As in the proof of Lemma 4.3, the intersection of the logarithmic spectrum $\widetilde{\mathcal{S}}$ with $\bigcup_{c\in\Gamma';|c|>R}(B^{14\tilde{\epsilon}} + (0,2c))$ is then contained in the $\tilde{\epsilon}$–tube $\widetilde{\mathcal{S}}_0^{\tilde{\epsilon}}$ around the vacuum $\widetilde{\mathcal{S}}_0$ and the intersection of $\widetilde{\mathcal{S}}$ with $\{|a| < \epsilon\} \times (\bigcup_{c\in\Gamma';|c|>R}(\Delta^\epsilon_{\epsilon/2} + 2c))$ is a graph over $\bigcup_{c\in\Gamma';|c|>R}(\Delta^\epsilon_{\epsilon/2} + 2c)$, with $\Delta^{\epsilon_2}_{\epsilon_1}$ as in (4.4).

For $c_0 \in \Gamma'$, not in the same connected component of $\mathbb{C}\backslash \bigcup_{c\in\Gamma';|c|>R}(\Delta^\epsilon_{\epsilon/2} + 2c)$ than the origin, we examine the intersection $\widetilde{\mathcal{S}} \cap \{(a,b) \in \mathbb{C}^2 \mid |a| < \epsilon \text{ and } |b - c_0| < \epsilon\}$. The part of $\widetilde{\mathcal{S}}$ contained in $\{(a,b) \in \mathbb{C}^2 \mid |a| < \epsilon \text{ and } \epsilon/2 < |b - c_0| < \epsilon\}$ is then the graph of a function $b \mapsto a(b)$ over $\{b \mid \epsilon/2 < |b - c_0| < \epsilon\}$ and, by Remark 4.4 and (3.10), the intersection of $\widetilde{\mathcal{S}}$ with $\{(a,b) \in \mathbb{C}^2 \mid \epsilon/2 < |a| < \epsilon \text{ and } |b - c_0| < \epsilon\}$ is the graph of a function $a \mapsto b(a)$ over $\{a \mid \epsilon/2 < |a| < \epsilon\}$.

We decompose $c_0 = c' + 2c''$ into $c'' \in \Gamma'$ with $|c''| > R$ and $c' = l_1c_1 + l_2c_2$ for $l_1, l_2 \in \{0, \pm 1\}$. For $|x| < \frac{\epsilon}{2}$ define the transversal circle

$$\tilde{\gamma}_x = \left\{(0,c') + (x,-x) + (\lambda,\lambda) \mid |\lambda| = \frac{\epsilon}{2} + \tilde{\epsilon}\right\}$$

in Ω and the corresponding projection operator

$$\tilde{P}_x = t^{-1}_{c''}T^{-1}_{c''}(P^{c''}_{\tilde{\gamma}_x})t_{c''}T_{c''} = \frac{1}{2\pi i}\int_{|\lambda|=\frac{\epsilon}{2}+\tilde{\epsilon}} D^{-1}_{x+\lambda,c_0-x+\lambda}d\lambda.$$

Moreover, for x with $\tilde{\epsilon} < |x| < \frac{\epsilon}{2}$ define the transversal circles

$$\gamma^1_x = \left\{(0,c') + (x,-x) + (-x+\mu_1, -x+\mu_1) \mid |\mu_1| = \tilde{\epsilon}\right\}$$
$$\gamma^2_x = \left\{(0,c') + (x,-x) + (x+\mu_2, x+\mu_2) \mid |\mu_2| = \tilde{\epsilon}\right\}$$

in Ω and the corresponding projection operators

$$P^1_x = t^{-1}_{c''}T^{-1}_{c''}(P^{c''}_{\gamma^1_x})t_{c''}T_{c''} = \frac{1}{2\pi i}\int_{|\mu_1|=\tilde\epsilon} D^{-1}_{\mu_1,c_0-2x+\mu_1}d\mu_1 \quad \text{and}$$

$$P^2_x = t^{-1}_{c''}T^{-1}_{c''}(P^{c''}_{\gamma^2_x})t_{c''}T_{c''} = \frac{1}{2\pi i}\int_{|\mu_2|=\tilde\epsilon} D^{-1}_{2x+\mu_2,c_0+\mu_2}d\mu_2.$$

Using the holomorphicity of the resolvent in the definition of P^c_γ, by Stokes theorem we obtain

$$\tilde P_x = P^1_x + P^2_x \tag{4.6}$$

for all x with $\tilde\epsilon < |x| < \frac{\epsilon}{2}$.

By Lemma 3.8, for all $|x| < \frac{\epsilon}{2}$ the operator $\tilde P_x$ projects to a 2–dimensional space which contains the span of the kernels of $D_{a,b}$ for all $(a,b) \in \big\{(0,c_0)+(x,-x)+(\lambda,\lambda) \mid |\lambda| < \frac{\epsilon}{2}+\tilde\epsilon\big\}$. For $\tilde\epsilon < |x| < \frac{\epsilon}{2}$, the operator P^1_x projects to the 1–dimensional kernel of $D_{a,b}$ with $(a,b) \in \big\{(0,c_0)+(x,-x)+(-x+\mu_1,-x+\mu_1) \mid |\mu_1| < \tilde\epsilon\big\}$ the unique point for which $D_{a,b}$ has a non–trivial kernel. Analogously, P^2_x projects to the 1–dimensional kernel of $D_{a,b}$ for a unique $(a,b) \in \big\{(0,c_0)+(x,-x)+(x+\mu_2,x+\mu_2) \mid |\mu_2| < \tilde\epsilon\big\}$.

This gives rise to a holomorphic family of polynomials $p_x(\lambda) = \lambda^2 + p_1(x)\lambda + p_2(x)$ (the determinants of the operators $D_{x+\lambda,c_0-x+\lambda}$ restricted to the 2–dimensional images of $\tilde P_x$) defined on $\{x \mid |x| < \frac{\epsilon}{2}\}$ whose zeros describe those λ with $|\lambda| < \frac{\epsilon}{2}+\tilde\epsilon$ for which $\ker(D_{x+\lambda,c_0-x+\lambda}) \neq \{0\}$. If $\tilde\epsilon < |x| < \frac{\epsilon}{2}$, then (4.6) implies that, corresponding to the images of P^1_x and P^2_x, the polynomial p_x has two different zeroes

$$\lambda_1(x) = -x+\mu_1(x) \quad \text{and} \quad \lambda_2(x) = x+\mu_2(x) \tag{4.7}$$

with $|\mu_k(x)| < \tilde\epsilon$. The discriminant $q(x) = p_1(x)^2 - 4p_2(x)$ of p_x vanishes exactly at those x for which both zeroes coincide. Its total vanishing order on the set $\{x \mid |x| < \frac{\epsilon}{2}\}$ is given by the winding number $\frac{1}{2\pi i}\int_{|x|=2\tilde\epsilon} d(\log(q))$ of q restricted to $|x| = 2\tilde\epsilon$. By (4.7), the discriminant $q(x) = (\lambda_1(x)+\lambda_2(x))^2 - 4\lambda_1(x)\lambda_2(x)$ restricted to $|x| = 2\tilde\epsilon$ is homotopy equivalent to $4x^2$ (the discriminant for the vacuum spectrum). Thus the total vanishing order of the discriminant q on the disc $|x| < \epsilon/2$ is two with zeros located in the disc $|x| < 2\tilde\epsilon$.

If the discriminant q has two zeros of order one, the intersection of $\widetilde{\mathcal S}$ with the open set $U = \{(0,c_0)+(x,-x)+(\lambda,\lambda) \mid |x| < 4\tilde\epsilon, |\lambda| < \frac{\epsilon}{2}+\tilde\epsilon\}$ is non–singular and its projection to the disc $\{|x| < 4\tilde\epsilon\}$ is a branched 2–fold covering with two branch points of order one. Thus $\widetilde{\mathcal S}\cap U$ is an annulus whose subsets over $3\tilde\epsilon < |x| < 4\tilde\epsilon$, by (4.7), are contained in the sets $\{(a,b) \mid |a| < \epsilon \text{ and } \epsilon/2 < |b-c_0| < \epsilon\}$ and $\{(a,b) \mid \epsilon/2 < |a| < \epsilon \text{ and } |b-c_0| < \epsilon\}$ and therefore graphs over the a– and b–planes. Because U contains the intersection of the polydisc $\{(a,b) \in \mathbb C^2 \mid |a| \le \epsilon/2 \text{ and } |b-c_0| \le \epsilon/2\}$ with the $\tilde\epsilon$–tube around $\widetilde{\mathcal S}_0$, we obtain that the intersection of $\widetilde{\mathcal S}$ with $\{(a,b) \in \mathbb C^2 \mid |a| < \epsilon \text{ and } |b-c_0| < \epsilon\}$ is an annulus.

In case the discriminant q has one zero of order two, the intersection of $\widetilde{\mathcal S}$ with the open set $U = \{(0,c_0)+(x,-x)+(\lambda,\lambda) \mid |x| < 4\tilde\epsilon, |\lambda| < \frac{\epsilon}{2}+\tilde\epsilon\}$ is a 2–fold covering of $\{|x| < 4\tilde\epsilon\}$ with one double point over the zero of q. Thus, the intersection $\widetilde{\mathcal S}\cap U$ (and therefore also $\widetilde{\mathcal S}\cap\{(a,b) \in \mathbb C^2 \mid |a| < \epsilon \text{ and } |b-c_0| < \epsilon\}$) is normalized by two immersed discs which, near their boundaries and hence everywhere, are graphs over the a– and b–plane, respectively. The two discs intersect transversally, because, by the Cauchy integral formula for the first derivative, they are graphs of functions $a \mapsto b(a)$ and $b \mapsto a(b)$ with small derivatives.

I order to see that a double point of $\widetilde{\mathcal{S}}$ in $\{(a,b) \in \mathbb{C}^2 \mid |a| < \epsilon \text{ and } |b - c_0| < \epsilon\}$ gives rise to a real multiplier, we note that the involution $(a,b) \mapsto (\bar{b}, \bar{a}) + (-\bar{c}_0, c_0)$ leaves both $\widetilde{\mathcal{S}}$ and $\{(a,b) \in \mathbb{C}^2 \mid |a| < \epsilon \text{ and } |b - c_0| < \epsilon\}$ invariant. Because there is at most one double point of $\widetilde{\mathcal{S}}$ in the polydisc, the double point is a fixed point of this involution and hence gives rise to a real multiplier. The kernel of $D_{a,b}$ at the double point is thus 2–dimensional and coincides with the image of $\tilde{P}_x$ for x a zero of the discriminant q.

For all non–singular points $(a,b) \in \widetilde{\mathcal{S}}$ contained in the polydisc $\{(a,b) \in \mathbb{C}^2 \mid |a| < \epsilon \text{ and } |b - c_0| < \epsilon\}$, the kernel of $D_{a,b}$ is 1–dimensional, because the vanishing order of $p_x(\lambda)$ seen as a function of two variables is greater or equal to the kernel dimension. This completes the proof of part a) of the lemma.

To prove part b) of the lemma, we assume that the intersection of $\widetilde{\mathcal{S}}$ with the polydisc $\{(a,b) \in \mathbb{C}^2 \mid |a| < \epsilon \text{ and } |b - c_0| < \epsilon\}$ consists of two discs with one double point. The functions $\lambda_1(x)$ and $\lambda_2(x)$ describing the roots of p_x for $x \in \{x \mid \tilde{\epsilon} < |x| < \frac{\epsilon}{2}\}$ then extend to $\{x \mid |x| < \frac{\epsilon}{2}\}$ and define parametrizations of the normalization $\tilde{\Sigma}$ of $\widetilde{\mathcal{S}}$ in a neighborhood of $(0, c_0)$.

For every $x \in \{x \in \mathbb{C} \mid |x| < \frac{\epsilon}{2}\}$, the operator $\tilde{P}_x$ projects to the sum $\tilde{\mathcal{L}}_{\sigma_1(x)} \oplus \tilde{\mathcal{L}}_{\sigma_2(x)}$, where $\sigma_j(x) \in \tilde{\Sigma}$, $j = 1,2$ are lifts of $(0,c_0)+(x,-x)+(\lambda_j(x),\lambda_j(x)) \in \widetilde{\mathcal{S}}$. Denote by $\psi_j^{\sigma_j}$, $j = 1,2$, nowhere vanishing local holomorphic sections of $\tilde{\mathcal{L}}$ defined on $\{\sigma_j(x) \in \tilde{\Sigma} \mid |x| < \frac{\epsilon}{2}\}$. Then $\tilde{P}_x\psi^o = \psi_1^{\sigma_1(x)} f_1(x) + \psi_2^{\sigma_2(x)} f_2(x)$ for holomorphic functions f_1, f_2, where as in Lemma 4.3, we set $\psi^o = (0,1)$. By (4.6) we have

$$P_x^1 \psi^o = \psi_1^{\sigma_1(x)} f_1(x) \tag{4.8}$$

for all $x \in \{x \mid \tilde{\epsilon} < |x| < \frac{\epsilon}{2}\}$.

Because $x \mapsto c_0 - x + \lambda_1(x)$ is bijective near the boundary of $\{x \mid |x| < \frac{\epsilon}{2}\}$ (which parametrizes a piece of $\widetilde{\mathcal{S}}$ which is a graph over the b–plane) it is bijective everywhere. It maps $\{x \mid |x| < \frac{\epsilon}{2}\}$ onto an open subset of $\{b \mid |b - c_0| < \epsilon + \tilde{\epsilon}\}$ which contains $\{b \mid |b - c_0| < 9\tilde{\epsilon}\}$. Denote by $b \mapsto x(b)$ the inverse of $x \mapsto c_0 - x + \lambda_1(x)$ restricted to $\{b \mid |b - c_0| < 9\tilde{\epsilon}\}$. Then $P_b = P^1_{x(b)}$ for all $b \in \{b \mid 5\tilde{\epsilon} < |b - c_0| < 9\tilde{\epsilon}\}$, where P_b is the operator defined in (4.5). This allows to extend the holomorphic section $\psi^{\tilde{\sigma}} = P_b\psi^o$ defined in Lemma 4.3 to the disc $\{\tilde{\sigma}_1(x(b)) \in \tilde{\Sigma} \mid |b - c_0| < 9\tilde{\epsilon}\}$ whose image in $\widetilde{\mathcal{S}}$ is a graph over $\{b \mid |b - c_0| < 9\tilde{\epsilon}\}$: for the points $\tilde{\sigma} \in \tilde{\Sigma}$ over b in the annulus $5\tilde{\epsilon} < |b - c_0| < 9\tilde{\epsilon}$ we have $\psi^{\sigma_1(x(b))} = P_b\psi^o = P^1_{x(b)}\psi^o$ such that, by (4.8), the section $\psi_1^{\sigma_1(x(b))} f_1(x(b))$ defines an extension to the disc over $\{b \mid |b - c_0| < 9\tilde{\epsilon}\}$. The maximum principle implies that this extension still satisfies $\|\psi^{\tilde{\sigma}} - \psi^0\| < \delta$. □

Proof of Theorem 4.1. Parts 1) and 3) of the theorem are mere reformulations of Lemmas 4.2, 4.3 and 4.6 and Remarks 4.4 and 4.7. The decomposition $\Sigma = \Sigma_{cpt} \cup \Sigma_\infty$ in Part 2) is also an immediate consequence of Lemmas 4.3 and 4.6 and Remarks 4.4 and 4.7, because the spectral curve Σ cannot have compact components: on such a compact component the harmonic function $\log|h_\gamma|$ had to be constant for all $\gamma \in \Gamma$. But this would imply that the normalization map $h\colon \Sigma \to \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ is constant on this component which is impossible for an analytic set of dimension one. Therefore, each component of Σ contains at least one end for which h goes to infinity and the number of components of Σ is bounded by the number of ends.

Asymptotically, away form the vacuum double points, the spectrum $\text{Spec}(W, D)$ is a small deformation of the vacuum $\text{Spec}(W, \bar\partial)$ and hence bi–holomorphic to two planes with neighborhoods around the vertices of of $\mathbb{Z}^2$–lattices removed. The number of ends of Σ depends on the number of handles in $\text{Spec}(W, D)$ near large vacuum double points: if there are infinitely many handles the spectral curve Σ has one end, infinite genus and is connected. If the number of handles is finite, then Σ has two ends, at most two components each of which contains an end, and has finite genus. □

4.3. **The connection approach to the spectral curve.** The spectral curve Σ of a quaternionic holomorphic line bundle (W, D) of degree zero contains a subset $\Sigma_\nabla \subset \Sigma$ that can be characterized in terms of flat connections adapted to the quaternionic holomorphic structure D. This point of view is advantageous when studying spectral curves of finite genus.

Definition 4.8. For a quaternionic holomorphic line bundle W of degree zero over a torus, we define $\Sigma_\nabla \subset \Sigma$ to be the subset of all points $\sigma \in \Sigma$ for which non-zero elements $\psi^\sigma \in \mathcal{L}_\sigma$ in the fiber over σ of the kernel line bundle $\mathcal{L} \to \Sigma$ are nowhere vanishing holomorphic sections with monodromy of W.

Lemma 4.9. *The subset Σ_∇ is a non–empty open neighborhood of the ends of Σ, that is, the complement $\Sigma \setminus \Sigma_\nabla$ is compact.*

Proof. The fact that $\tilde{\mathcal{L}}$ is a subbundle in the Frechet topology of C^∞–convergence implies that every point $\sigma \in \Sigma_\nabla$ has a neighborhood in Σ on which the non–trivial elements of $\mathcal{L} \to \Sigma$ are nowhere vanishing sections with monodromy of W. This shows that Σ_∇ is open.

To see that Σ_∇ is a neighborhood of the ends, note that the holomorphic section ψ, which has been constructed in Lemmas 4.3 and 4.6, is nowhere vanishing. Therefore it is sufficient to check that, for a point σ on a handle joining the two planes in Σ_∞ and corresponding to a large enough multiplier, a non–trivial section $\psi \in \mathcal{L}_\sigma$ is nowhere vanishing. Using the notation in the proof of Lemma 4.6 a non–trivial section $\psi \in \tilde{\mathcal{L}}_{\tilde\sigma}$ with $\tilde\sigma \in \tilde\Sigma$ close to a large vacuum double point $(0, c)$ can be written as $\psi = \tilde P_x(\psi^\infty u_a + \psi^o u_b)$ for some point x and u_a, $u_b \in \mathbb{C}$. Without loss of generality, we can assume $|u_a| + |u_b| = 1$. Applying Lemma 4.6 with $\delta = 1/2$ now shows that, in a neighborhood of a large enough vacuum double point,

$$\|\psi - \psi^\infty u_a - \psi^o u_b\| < \delta(|u_a| + |u_b|) < \tfrac{1}{2}$$

which implies that the section ψ has no zeroes. □

Remark 4.10. There are two important special cases of quaternionic holomorphic line bundles (W, D) of degree zero for which $\Sigma_\nabla = \Sigma$. The first is the case when the bundle (W, D) carries a flat connection ∇ which is a Willmore connection [8] and adapted to D, that is, which satisfies the Willmore condition $d^\nabla * Q = 0$ and $D = \nabla''$. The spectral curve can then be interpreted as the holonomy eigenline curve of the associated family ∇^μ of flat connections [8] which means that there is a holomorphic function $\mu\colon \Sigma \to \mathbb{C}_*$, a 2–fold branched covering, such that every non–trivial element $\psi^\sigma \in \mathcal{L}_\sigma$ in the fiber of $\mathcal{L}$ over $\sigma \in \Sigma$ is a ∇^{μ_σ}–parallel section and hence a nowhere vanishing holomorphic section with monodromy of W, see [8] or Section 6 of [2]. Note that trivial Willmore connections on a rank 1 bundle correspond to harmonic maps from T^2 to S^2.

The second is the case when the quaternionic holomorphic line bundle W is the bundle $W = V/L$ induced by a conformal immersion $f\colon T^2 \to S^4$ with Willmore functional

$\mathcal{W} < 8\pi$. In this situation $\Sigma_\nabla = \Sigma$ is essentially a consequence of Lemma 2.8 of [3]. What remains to be verified is that a non–trivial section $\psi^{\sigma_0} \in \mathcal{L}_{\sigma_0}$ over a point $\sigma_0 \in \Sigma$ belonging to the trivial multiplier $h^{\sigma_0} = 1$ is nowhere vanishing. If such a section ψ^{σ_0} had a zero p one could construct a 2–dimensional linear system with Jordan monodromy all of whose sections vanish at the point p by taking the span of ψ^{σ_0} and $\frac{\partial \psi^\sigma}{\partial \sigma}_{|\sigma=\sigma_0} + \pi\varphi$. Here ψ^σ is a local holomorphic section of $\mathcal{L}$ and $\pi\varphi$ the projection to $W = V/L$ of a parallel section of V such that $\frac{\partial \psi^\sigma}{\partial \sigma}_{|\sigma=\sigma_0} + \pi\varphi$ vanishes at p. The quaternionic Plücker formula with monodromy [3] would then contradict $\mathcal{W} < 8\pi$.

Definition 4.11. For $\sigma \in \Sigma_\nabla$ define the quaternionic connection ∇^σ and the complex structure $S^\sigma \in \Gamma(\mathrm{End}(W))$ on W by setting

$$\nabla^\sigma \psi^\sigma = 0 \quad \text{and} \quad S^\sigma \psi^\sigma = \psi^\sigma i\,, \tag{4.9}$$

where $\psi^\sigma \in \mathcal{L}_\sigma$ is a non–trivial element of the fiber $\mathcal{L}_\sigma$ and therefore a nowhere vanishing holomorphic section with monodromy of W.

By definition, the connection ∇^σ is flat and compatible with S^σ and D, i.e., for $\sigma \in \Sigma_\nabla$

$$\nabla^\sigma S^\sigma = 0 \qquad \text{and} \qquad D = (\nabla^\sigma)''. \tag{4.10}$$

The real structure $\rho\colon \Sigma \to \Sigma$ leaves ∇^σ invariant and changes the sign of S^σ, that is,

$$\nabla^{\rho(\sigma)} = \nabla^\sigma \qquad \text{and} \qquad S^{\rho(\sigma)} = -S^\sigma. \tag{4.11}$$

By choosing a local holomorphic section ψ^σ of the holomorphic line bundle $\mathcal{L}$ we obtain:

Lemma 4.12. *The connection ∇^σ and the complex structure S^σ depend holomorphically on $\sigma \in \Sigma_\nabla$ in the sense that*

$$(S^\sigma)' = (S^\sigma)^{\cdot} S^\sigma \qquad \textit{and} \qquad (\nabla^\sigma)' = (\nabla^\sigma)^{\cdot} S^\sigma, \tag{4.12}$$

where $^{\cdot}$ and $'$ denote the derivatives with respect to the t– and s–coordinates for $x = t + is$ a local holomorphic chart on $U \subset \Sigma_\Sigma$.

The holomorphic family S^σ of complex structures on W defined for $\sigma \in \Sigma_\nabla$ can be interpreted as a family of holomorphic maps $S_p\colon \Sigma_\nabla \to \mathbb{P}_\mathbb{C}(W_p) \cong \mathbb{CP}^1$ parametrized over the torus T^2 . For this we identify

$$\mathbb{P}_\mathbb{C}(W_p) \cong \{S_p \in \mathrm{End}(W_p) \mid S_p^2 = -\mathrm{Id}\}$$

by identifying the complex line $v\mathbb{C}$ in W_p with the quaternionic endomorphism S_p whose i–eigenspace is $v\mathbb{C}$.

Theorem 4.13. *Let (W, D) be a quaternionic holomorphic line bundle of degree zero over a torus with spectral curve Σ. For every $p \in T^2$, the evaluation S_p^σ at p of the complex structure defined in* (4.9) *for all $\sigma \in \Sigma_\nabla$ uniquely extends to a holomorphic map*

$$S_p\colon \Sigma \to \mathbb{P}_\mathbb{C}(W_p) \cong \mathbb{CP}^1. \tag{4.13}$$

If $\Sigma_\nabla = \Sigma$, this T^2–family of holomorphic maps glues to a C^∞–map

$$S\colon \Sigma \times T^2 \to \mathbb{CP}^1.$$

Proof. Since Σ_∇ is non–empty the evaluation at $p \in T^2$ of a non–trivial local holomorphic section ψ^σ of $\mathcal{L}$ does not vanish identically. Thus we can holomorphically extend $\sigma \mapsto \psi^\sigma_p \mathbb{C}$ across the isolated zeros of $\sigma \mapsto \psi^\sigma_p$. This shows that for $p \in T^2$ fixed, S_p can be holomorphically extended from Σ_∇ to Σ.

Recall that $\tilde{\mathcal{L}}$ is a holomorphic subbundle of $\Gamma(W)$ in the C^∞–topology by Lemma 2.4. We chose a trivial connection on W to identify $\mathbb{P}_\mathbb{C}(W_p) \cong \mathbb{CP}^1$ for all $p \in T^2$. Then $\Sigma_\nabla = \Sigma$ implies that $S\colon \Sigma \times T^2 \to \mathbb{CP}^1$ is smooth. □

If $\Sigma_\nabla \neq \Sigma$, the map S is not necessarily continuous as a map depending on two variables: bubbling phenomena might occur at the points $(\sigma, p) \in \Sigma \times T^2$ for which a non–trivial $\psi^\sigma \in \mathcal{L}_\sigma$ is a holomorphic section with monodromy of W with a zero at $p \in T^2$.

5. Spectral Curves of Finite Genus and the Willmore Functional

We now come to the case where the spectral curve Σ of a quaternionic holomorphic line bundle of degree zero over a torus has finite genus and thus can be compactified by adding two points $\{o, \infty\}$. Theorem 5.4 then will show that the T^2–family (4.13) of holomorphic maps $S_p\colon \Sigma \to \mathbb{CP}^1$ extends to a family of algebraic functions

$$S_p\colon \Sigma \cup \{o, \infty\} \to \mathbb{CP}^1$$

on the compactification of Σ. Moreover, the T^2–family of complex holomorphic line bundles corresponding to S_p for $p \in T^2$ move linearly in the Jacobian of the compactified spectral curve.

Important examples of conformal immersions $f\colon T^2 \to S^4$ with degree zero normal bundle for which the induced quaternionic holomorphic line bundle $W = V/L$ has finite spectral genus are constrained Willmore tori with trivial normal bundle, see [2].

5.1. **Asymptotics of finite genus spectral curves.** We say that a quaternionic holomorphic line bundle W of degree zero over a torus has *finite spectral genus* if its spectral curve is of finite genus.

In general, the two planes in the Σ_∞–part of the decomposition $\Sigma = \Sigma_{cpt} \cup \Sigma_\infty$ of Theorem 4.1 are joined by an infinite number of handles accumulating at the end. In the finite genus case there is a compact set outside of which there are no handles. A spectral curve Σ of finite genus can thus be compactified by adding two points $\{o, \infty\}$ at infinity. The real structure $\rho\colon \Sigma \to \Sigma$ extends to the compactification $\Sigma \cup \{o, \infty\}$ and interchanges o and ∞.

The compact component Σ_{cpt} in the decomposition $\Sigma = \Sigma_{cpt} \cup \Sigma_\infty$ of a finite genus spectral curve can be chosen large enough such that there are no handles joining the two planes in Σ_∞. Depending on whether Σ_{cpt} has one or two connected components, the compactification $\Sigma \cup \{o, \infty\}$ is connected or consists of two connected Riemann surfaces of genus at least one by Corollary 5.2 below. Since the Σ_∞–component contains no handles, it is the disconnected sum of two punctured discs, the neighborhoods of the added points ∞ and o, which in the logarithmic picture are graphs over the a– or b–planes, respectively. More precisely:

Lemma 5.1. *Let W be a quaternionic holomorphic line bundle of degree zero over a torus $T^2 \cong \mathbb{C}/\Gamma$ with spectral curve Σ of finite genus. Then there is a punctured neighborhood U_o of one of the points at infinity, in the following called o, parameterized by the punctured disc $\{x \in \mathbb{C}_* \mid |x| < r\}$ for some $r > 0$ such that the restriction of the normalization map $h\colon \Sigma \to \mathrm{Spec}(W, D) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ to U_o is of the form*

$$h^x_\gamma = \exp((\bar{b}_0 + a(x))\gamma + (b_0 + 1/x)\bar{\gamma}), \qquad \gamma \in \Gamma,$$

where $b_0 \in \mathbb{C}$ and $x \mapsto a(x)$ is a holomorphic function with $a(0) = 0$. Similarly, the other point at infinity, in the following called ∞, has a punctured neighborhood U_∞ parameterized by $\{x \in \mathbb{C}_* \mid |x| < r\}$ such that the restriction of h to U_∞ is

$$h^x_\gamma = \exp((\bar{b}_0 + 1/x)\gamma + (b_0 + b(x))\bar{\gamma}), \qquad \gamma \in \Gamma,$$

where $b_0 \in \mathbb{C}$ and $x \mapsto b(x)$ is a holomorphic function with $b(0) = 0$.

The open sets U_o and U_∞ can be chosen small enough such that the sections $\tilde{\sigma} \mapsto \psi^{\tilde{\sigma}}$ of $\tilde{\mathcal{L}}$ constructed in Lemmas 4.3 and 4.6 and Remarks 4.4 and 4.7 are defined on the respective preimages $\tilde{U}_o$, $\tilde{U}_\infty \subset \tilde{\Sigma}$ of U_o and U_∞. By setting

$$\psi^o = (0,1) \qquad \textit{and} \qquad \psi^\infty = (1,0)$$

(in the trivialization of W used in Sections 3 and 4), these sections $\tilde{\sigma} \mapsto \psi^{\tilde{\sigma}}$ can be extended through the punctures of $\tilde{U}_o$ and $\tilde{U}_\infty$ such that

$$\psi\colon (\tilde{U}_o \cup \{o\}) \times T^2 \to \mathbb{C}^2 \qquad \textit{and} \qquad \psi\colon (\tilde{U}_\infty \cup \{\infty\}) \times T^2 \to \mathbb{C}^2$$

are C^∞ as maps depending on two variables and holomorphic in the first variable.

The main reason for carrying out the asymptotic analysis of Sections 3 and 4 within the l^1–framework (instead of the usual L^2–setting) is that l^1–convergence implies C^0–convergence. This is essential for the proof of Lemma 5.1.

Proof. It is sufficient to prove the statement for U_o since the real structure ρ exchanges U_o and U_∞ and $\psi^{\rho(\tilde{\sigma})} := -\psi^{\tilde{\sigma}} j$. In the finite genus case Lemmas 4.3 and 4.6 imply that, for small enough $\delta > 0$ and $\epsilon > 0$, we can chose $R > 0$ big enough such that

1.) the intersection of $\tilde{S}$ with $\{|a| < \epsilon\} \times (\bigcup_{c \in \Gamma'; |c| > R} (\Delta_{\epsilon/2} + 2c))$ is a graph of a function $b \mapsto a(b)$ over $\bigcup_{c \in \Gamma'; |c| > R} (\Delta_{\epsilon/2} + 2c)$,
2.) for every $c_0 \in \Gamma'$ that satisfies $|c_0| > 2R$, the intersection of $\tilde{S}$ with the polydisc $\{(a,b) \in \mathbb{C}^2 \mid |a| < \epsilon \text{ and } |b - c_0| < \epsilon\}$ consists of a pair of discs which are graphs over the coordinate planes and have a double point, and
3.) the section $\tilde{\sigma} \mapsto \psi^{\tilde{\sigma}}$ of $\tilde{\mathcal{L}}$ defined by Lemmas 4.3 and 4.6 over the preimage under the projection $\tilde{\Sigma} \to \widetilde{\mathcal{S}}$ of the part of $\widetilde{\mathcal{S}}$ which is a graph of a function $b \mapsto a(b)$ with $|a| < \epsilon$ over $\bigcup_{c \in \Gamma'; |c| > R} (B' + 2c)$ satisfies

$$\|\psi^{\tilde{\sigma}} - \psi^\infty\| < \delta, \tag{5.1}$$

where $B' = \{(a,b) \in B \mid b = \lambda_1 c_1 + \lambda_2 c_2 \text{ with } |\lambda_i| \leq 1\}$.

Denote by $\tilde{U}_o$ an open subset of $\tilde{\Sigma}$ contained in the domain of definition of $\tilde{\sigma} \mapsto \psi^{\tilde{\sigma}}$ such that the image of $\tilde{U}_o$ under the projection $\tilde{\Sigma} \to \widetilde{\mathcal{S}}$ is a graph over $\{b \in \mathbb{C} \mid |b| > 1/r\}$ for some $r > 0$. Let U_o be the image of $\tilde{U}_o$ under the projection $\tilde{\Sigma} \to \Sigma = \tilde{\Sigma}/\Gamma^*$. By construction, this set U_o is a punctured neighborhood of o with the property that the restriction of the normalization map $h\colon \Sigma \to \mathrm{Spec}(W, D)$ to U_o has a single valued logarithm whose image, in the (a,b)–coordinates (3.1), is contained in the ϵ–tube around the b–plane and is the graph of a holomorphic function $b \mapsto a(b)$ which is bounded by ϵ and defined on $\{b \in \mathbb{C} \mid |b| > 1/r\}$. Setting $x = 1/b$ we obtain a parametrization of U_o by $x \in \{x \in \mathbb{C}_* \mid |x| < r\}$. Riemann's removable singularity theorem implies that the bounded holomorphic function $x \mapsto a(x)$ extends to $x = 0$. Because $\epsilon > 0$ can be chosen arbitrarily small, this extension vanishes for $x = 0$. Using (3.1) this proves that the normalization map h is of the given form when expressed in the x–coordinate on U_o .

Similarly, the section $\tilde{\sigma} \mapsto \psi^{\tilde{\sigma}}$ defined on $\tilde{U}_o \subset \tilde{\Sigma}$, when seen as a holomorphic map from $\tilde{U}_o$ to $C^0(T^2, \mathbb{C}^2)$, is bounded by (5.1) and Riemann's removable singularity theorem implies that it has a unique holomorphic extension to o. This extension maps o to the constant element $\psi^o = (0,1) \in C^0(T^2, \mathbb{C}^2)$, because $\delta > 0$ can be chosen arbitrarily small. Since by Lemma 2.4 the line bundle $\tilde{\mathcal{L}}$ is a holomorphic line subbundle of $C^\infty(T^2, \mathbb{C}^2)$ in the C^∞–topology, for every $m \geq 0$ the holomorphic section $\tilde{\sigma} \mapsto \psi^{\tilde{\sigma}}$ can be seen as a holomorphic map from $\tilde{U}_o$ to $C^m(T^2, \mathbb{C}^2)$ and has a Laurent series

$$\psi^{\tilde{\sigma}(x)} = \sum_{k=-\infty}^{\infty} \psi_k^m x^k$$

in $C^m(T^2, \mathbb{C}^2)$. Because in $C^0(T^2, \mathbb{C}^2)$ the section $\tilde{\sigma} \mapsto \psi^{\tilde{\sigma}}$ has a holomorphic extension through the puncture, the Laurent series for $m = 0$ is a power series and the coefficients of all negative exponents vanish. Since the embedding $C^m(T^2, \mathbb{C}^2) \to C^0(T^2, \mathbb{C}^2)$ is continuous, the uniqueness of Laurent series implies that the same is true for all m. Thus, for every $m \geq 0$ the section $\tilde{\sigma} \mapsto \psi^{\tilde{\sigma}}$ extends to a holomorphic map from $\tilde{U}_o \cup \{o\}$ to $C^m(T^2, \mathbb{C}^2)$ and hence $\psi\colon (\tilde{U}_o \cup \{o\}) \times T^2 \to \mathbb{C}^2$ is C^∞ and holomorphic in the first variable. □

Corollary 5.2. *Let Σ be the spectral curve of a quaternionic holomorphic line bundle $(W, D = \bar{\partial} + Q)$ of degree zero over a torus. Assume Σ is disconnected and hence the (disconnected) direct sum of two compact Riemann surfaces with a single puncture which are interchanged under the anti–holomorphic involution ρ. Then, except in the vacuum case when $Q \equiv 0$, both summands have genus $g \geq 1$.*

It can be shown [2] that the following classes of constrained Willmore tori in S^4 have irreducible spectral curves: Willmore tori in S^3 which are not Möbius equivalent to minimal tori in $\mathbb{R}^3$, minimal tori in the standard 4–sphere or hyperbolic 4–space that are not super–minimal, CMC tori in $\mathbb{R}^3$ and S^3.

Proof. By Theorem 4.1 we only have to show that the two components have genus $g \geq 1$. Lemma 5.1 shows that, for each of the components one of the projections which, in the (a, b)–coordinates of (3.1), are given by $(a, b) \mapsto a$ and $(a, b) \mapsto b$ extends to a non–trivial holomorphic map from the compactification of the component onto the torus $\mathbb{C}/\bar{\Gamma}'$ or $\mathbb{C}/\Gamma'$. But by the Riemann–Hurwitz formula a compact surface admitting a non–trivial holomorphic map onto a torus has genus $g \geq 1$. □

Using the identification of the Lie algebra $\mathrm{Hom}(\Gamma, \mathbb{C})$ with $\mathrm{Harm}(T^2, \mathbb{C})$ (see Section 1), the logarithmic derivative $d^\Sigma(\log(h)) \in \Omega^1_\Sigma(\mathrm{Harm}(T^2, \mathbb{C}))$ of the normalization map h can be written as

$$d^\Sigma(\log(h)) = \omega_\infty \, dz + \omega_o \, d\bar{z} \tag{5.2}$$

with z denoting the coordinate induced by the isomorphism $T^2 \cong \mathbb{C}/\Gamma$ used in the definition of the (a, b)–coordinates (3.1). The holomorphic forms ω_∞ and ω_o are derivatives $\omega_\infty = da$ and $\omega_o = db$ of the functions a and b which are, up to the Γ'–action (3.10), well defined on Σ. The following corollary is an immediate consequence of Lemma 5.1.

Corollary 5.3. *The form ω_∞ is holomorphic on $\Sigma \cup \{o\}$ and has a second order pole with no residue at ∞. The form $\omega_o = \rho^* \bar{\omega}_\infty$ is holomorphic on $\Sigma \cup \{\infty\}$ and has a second order pole with no residue at o.*

The extendibility through the ends of the holomorphic sections $\tilde\sigma \mapsto \psi^{\tilde\sigma}$ of $\tilde{\mathcal{L}}$ established in Lemma 5.1 immediately implies the extendibility of $S\colon \Sigma_\nabla \to \Gamma(\mathrm{End}(W))$ through o and ∞. Recall that, on the universal cover of T^2, non–trivial elements of $\tilde{\mathcal{L}}_{\tilde\sigma}$ and $\mathcal{L}_\sigma$ coincide up to scaling by a complex function if $\tilde\sigma \in \tilde\Sigma$ is the preimage of $\sigma \in \Sigma$ under the projection $\tilde\Sigma \to \Sigma = \tilde\Sigma/\Gamma^*$. Thus S can also be defined using holomorphic sections of $\tilde{\mathcal{L}}$.

Theorem 5.4. *Let W be a quaternionic holomorphic line bundle of degree zero over a torus with spectral curve Σ of finite genus. By setting*

$$S^\infty = J \qquad \textit{and} \qquad S^o = -J,$$

the family (4.9) *of complex structures $S^\sigma \in \Gamma(\mathrm{End}(W))$ defined for $\sigma \in \Sigma_\nabla$ is extended holomorphically (in the C^∞–topology) through the points o and ∞ to a map*

$$\sigma \in \Sigma_\nabla \cup \{o,\infty\} \mapsto S^\sigma \in \Gamma(\mathrm{End}(W)).$$

In particular, the T^2–family (4.13) *of holomorphic functions $S_p\colon \Sigma \to \mathbb{CP}^1$ extends to a family of algebraic functions*

$$S_p\colon \Sigma \cup \{o,\infty\} \to \mathbb{CP}^1, \qquad p \in T^2.$$

If $\Sigma_\nabla = \Sigma$, this T^2–family of algebraic functions glues to a C^∞–map

$$S\colon (\Sigma \cup \{o,\infty\}) \times T^2 \to \mathbb{CP}^1.$$

In Section 5.3 it will be shown that the complex holomorphic line bundle belonging to S_p, the pull–back of the tautological bundle over $\mathbb{CP}^1$ by S_p, depends linearly on $p \in T^2$ as a map into the Picard group of the compactified spectral curve.

5.2. **Asymptotics of ∇^σ and the Willmore energy.** We investigate the asymptotics of the connections ∇^σ defined in (4.9) when σ approaches the ends of Σ. Theorem 5.5 shows how the Willmore energy of a quaternionic holomorphic line bundle of finite spectral genus is encoded in the asymptotics of $h\colon \Sigma \to \mathrm{Hom}(\Gamma, \mathbb{C}_*)$.

Because ρ interchanges o and ∞ but leaves ∇^σ invariant it is sufficient to investigate ∇^σ in a punctured neighborhood of ∞. By Theorem 5.4 the sections $S^\sigma \in \Gamma(\mathrm{End}(W))$ satisfy $S^\sigma(p) \neq -J(p)$ for all $p \in T^2$ provided σ is in a small enough punctured neighborhood $U_\infty \subset \Sigma_\nabla$ of ∞. Applying stereographic projection from $-J$ we write

$$S^\sigma = (1+Y^\sigma)J(1+Y^\sigma)^{-1} \tag{5.3}$$

with $Y^\sigma \in \Gamma(\mathrm{End}_-(W))$. Because $\nabla^\sigma S^\sigma = 0$ the flat connection ∇^σ can be expressed as

$$\nabla^\sigma = (1+Y^\sigma) \circ (\hat\nabla + \alpha^\sigma) \circ (1+Y^\sigma)^{-1}\,. \tag{5.4}$$

Here $\hat\nabla$ denotes the unique flat quaternionic connection with $\bar\partial = \hat\nabla''$ and $\hat\nabla J = 0$ with unitary holonomy and $\alpha^\sigma \in \Omega^1(\mathrm{End}_+(W))$ is a $\hat\nabla$–closed 1–form. On the other hand, because $(\nabla^\sigma)'' = D = \bar\partial + Q$, the family ∇^σ can be written as $\nabla^\sigma = \hat\nabla + Q + \eta^\sigma$ with $\eta^\sigma \in \Gamma(K\,\mathrm{End}(W))$ and hence

$$\nabla^\sigma = \hat\nabla + Q + \eta^\sigma = \hat\nabla + \frac{1}{1+|Y^\sigma|^2}(\underbrace{\alpha^\sigma + \hat\nabla Y^\sigma Y^\sigma - Y^\sigma\alpha^\sigma Y^\sigma}_{\mathrm{End}_+}\underbrace{-\hat\nabla Y^\sigma + Y^\sigma\alpha^\sigma - \alpha^\sigma Y^\sigma}_{\mathrm{End}_-}). \tag{5.5}$$

Since $Q+\eta^\sigma$ has End_+–part of type K we obtain $(\alpha^\sigma+\hat\nabla Y^\sigma Y^\sigma-Y^\sigma\alpha^\sigma Y^\sigma)''=0$ and

$$QY^\sigma=\frac{1}{1+|Y^\sigma|^2}(-\hat\nabla Y^\sigma Y^\sigma+Y^\sigma\alpha^\sigma Y^\sigma-\alpha^\sigma Y^\sigma Y^\sigma)'' = \frac{1}{1+|Y^\sigma|^2}(\alpha^\sigma-\alpha^\sigma Y^\sigma Y^\sigma)''=(\alpha^\sigma)''. \tag{5.6}$$

The fact that the families ∇^σ and S^σ are holomorphic in σ in the sense of (4.12) implies that the families Y^σ and α^σ are also holomorphic and satisfy

$$(Y^\sigma)'=(Y^\sigma)^{\cdot}J \qquad \text{and} \qquad (\alpha^\sigma)'=(\alpha^\sigma)^{\cdot}J.$$

Without loss of generality we can assume that the punctured neighborhood $U_\infty\subset\Sigma_\nabla$ of ∞ is the parameter domain of a chart x with $x(\infty)=0$ (e.g. the chart defined in Lemma 5.1). Then Y^σ has a power series expansion (in the C^∞–topology)

$$Y^{\sigma(x)}=\sum_{k=1}^\infty Y_k x^k \tag{5.7}$$

with $Y_k\in\Gamma(\mathrm{End}_-(W))$ and $(\alpha^\sigma)''$, by (5.6), has an expansion

$$(\alpha^{\sigma(x)})''=\sum_{k=1}^\infty \alpha_k'' x^k.$$

For every $\sigma\in U_\infty$ the form $\alpha^\sigma\in\Omega^1(\mathrm{End}_+(W))\cong\Omega^1(\mathbb{C})$ is closed so it has a unique decomposition $\alpha^\sigma=\alpha^\sigma_{harm}+\alpha^\sigma_{exact}$ into a harmonic and an exact part. The multiplier h^σ is then given by $h^\sigma(\gamma)=\hat h(\gamma)e^{-\int_\gamma\alpha^\sigma_{harm}}$, where $\hat h$ denotes the holonomy of $\hat\nabla$ restricted to the i–eigenline bundle $\hat W$ of J (cf. Section 3.2). Lemma 5.1 now implies that there is a holomorphic function a on U_∞ with a first order pole at ∞ and a holomorphic function b on $U_\infty\cup\{\infty\}$ with $b(\infty)=0$ such that

$$\alpha^\sigma_{harm}=-(a(\sigma)+2\bar b_0)\,dz-b(\sigma)\,d\bar z, \tag{5.8}$$

where z is the chart on $T^2\cong\mathbb{C}/\Gamma$ and $b_0\in\mathbb{C}$ satisfies $\hat h^\gamma=e^{-\bar b_0\gamma+b_0\bar\gamma}$.

Because both $(\alpha^\sigma)''$ and $(\alpha^\sigma_{harm})''$ extended holomorphically through the point $\sigma=\infty$ with $(\alpha^\infty)''=0$ and $(\alpha^\infty_{harm})''=0$, the same is true for $(\alpha^\sigma_{exact})''=(\alpha^\sigma-\alpha^\sigma_{harm})''$. Moreover, the Fourier expansions of the exact forms α^σ_{exact} have no constant terms and the Fourier coefficients of $(\alpha^\sigma_{exact})'$ and $(\alpha^\sigma_{exact})''$ coincide up to multiplicative constants independent of σ such that $(\alpha^\sigma_{exact})'$, like $(\alpha^\sigma_{exact})''$, extends holomorphically through ∞ with $(\alpha^\infty_{exact})'=0$. Since the two latter components in the decomposition $\alpha^\sigma=(\alpha^\sigma_{harm})'+(\alpha^\sigma_{exact})'+(\alpha^\sigma)''$ extend holomorphically through $\sigma=\infty$ and $(\alpha^\sigma_{harm})'=-(a(\sigma)+2\bar b_0)\,dz$ has a first order pole, we obtain that α^σ has a Laurent series of the form

$$\alpha^{\sigma(x)}=\sum_{k=-1}^\infty \alpha_k x^k \tag{5.9}$$

with closed $\alpha_k\in\Omega^1(T^2,\mathbb{C})$. The coefficient α_{-1} is a non–trivial holomorphic 1–form on the torus. Plugging (5.7) and (5.9) into (5.5) and taking the $\bar K\,\mathrm{End}_-(W)$–part yields $Q=Y_1\alpha_{-1}$. By plugging this into (5.6) we obtain $\alpha_1''=Y_1\alpha_{-1}Y_1$. Hence

$$Res_{x=0}\left(\int_{T^2}\alpha\wedge\frac{\partial\alpha}{\partial x}\right)dx=\int_{T^2}\alpha_{-1}\wedge\alpha_1-\alpha_1\wedge\alpha_{-1}= =-2\int_{T^2}\alpha_1''\wedge\alpha_{-1}=-2\int_{T^2}Y_1\alpha_{-1}Y_1\wedge\alpha_{-1}=i\,\mathcal{W}, \tag{5.10}$$

where $\mathcal{W}$ is the Willmore energy of the bundle which is given by

$$\mathcal{W} = 2\int_{T^2} Q \wedge *Q = 2\int_{T^2} Y_1\alpha_{-1} \wedge Y_1 * \alpha_{-1}.$$

Using again the identification of Section 1 between the Lie algebra of $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$ and $\mathrm{Harm}(T^2, \mathbb{C})$, the formula $h^\sigma(\gamma) = \hat{h}^\sigma(\gamma)e^{-\int_\gamma \alpha^\sigma_{harm}}$ implies $\alpha^\sigma_{harm} = -\log(h^\sigma) + \beta$ for some $\beta \in \mathrm{Harm}(T^2, \mathbb{C})$ which, like $\log(h)$, is only well determined up to adding an element of Γ^*, that is, a $2\pi i\mathbb{Z}$–periodic harmonic form. Because α in (5.10) can be replaced by its harmonic part α_{harm}, we have proven the following theorem due to Grinevich and Schmidt, see (47), (52) in [9] or (44) in [23].

Theorem 5.5. *Let (W, D) be a quaternionic holomorphic line bundle of degree zero over a torus. In case (W, D) has finite spectral genus its Willmore energy is given by*

$$\mathcal{W} = iRes_o\left(\Omega\left(\log(h), d^\Sigma \log(h)\right)\right) = -iRes_\infty\left(\Omega\left(\log(h), d^\Sigma \log(h)\right)\right) .$$

Here Ω denotes the canonical symplectic form

$$\Omega(\beta_1, \beta_2) := \int_{T^2} \beta_1 \wedge \beta_2 , \qquad \beta_1, \beta_2 \in \mathrm{Harm}(T^2, \mathbb{C}) ,$$

on the Lie algebra $\mathrm{Hom}(\Gamma, \mathbb{C}) \cong \mathrm{Harm}(T^2, \mathbb{C})$ of $\mathrm{Hom}(\Gamma, \mathbb{C}_*)$ and $\log(h)$ denotes the logarithm of $h\colon \Sigma \to \mathrm{Spec}(W, D) \subset \mathrm{Hom}(\Gamma, \mathbb{C}_*)$ which is single valued in punctured neighborhoods of o and ∞.

Theorem 13.17 in [10] is the analogue to Theorem 5.5 for the energy of harmonic tori $T^2 \to S^3$ (instead of the Willmore energy of conformal tori $f\colon T^2 \to S^4$ with finite spectral genus). To make the analogy more explicit we give a slight reformulation of the theorem. For this we define the a skew symmetric product $(,)_p$ on the space of meromorphic 1–forms with single pole and no residue at p by

$$(\omega_1, \omega_2)_p = Res_p(\omega_1 F_2),$$

where F_2 denotes a local primitive of ω_2, i.e., a holomorphic function with $dF_2 = \omega_2$. Plugging (5.2) into the formula for the Willmore energy we obtain

$$\mathcal{W} = 4(\omega_\infty, \omega_o)_\infty Vol(\mathbb{C}/\Gamma) = -4(\omega_\infty, \omega_o)_o Vol(\mathbb{C}/\Gamma) \tag{5.11}$$

(recall that the forms ω_∞ and ω_o defined by (5.2) depend on the choice of a chart z on T^2 which defines an isomorphism $T^2 \cong \mathbb{C}/\Gamma$). For a positive basis γ_1 and γ_2 of the lattice Γ, define $\theta = \omega_\infty\gamma_1 + \omega_o\bar{\gamma}_1$ and $\tilde{\theta} = \omega_\infty\gamma_2 + \omega_o\bar{\gamma}_2$. Because $\gamma_1\bar{\gamma}_2 - \bar{\gamma}_1\gamma_2 = -2i\, Vol(\mathbb{C}/\Gamma)$ we obtain

$$\mathcal{W} = 2i(\theta, \tilde{\theta})_\infty = -2i(\theta, \tilde{\theta})_o, \tag{5.12}$$

the direct analogue to the Energy formula given in [10].

As a direct application of (5.11) we show now that the Willmore energy determines the "speed" at which the spectrum $\mathrm{Spec}(W, D)$ converges to the vacuum $\mathrm{Spec}(W, \bar{\partial})$ when h goes to ∞: by Lemma 5.1, a punctured neighborhood of ∞ in Σ can be parametrized by a parameter $|x| > r$ for which

$$\log(h^x) = (\bar{b}_0 + 1/x)\, dz + (b_0 + \lambda x + O(x^2))\, d\bar{z}.$$

In this coordinate we thus have $\omega_\infty = da = -1/x^2 dx$ and $\omega_o = db = (\lambda + O(x))dx$ such that formula (5.11) implies

$$\mathcal{W} = -4\lambda\, Vol(\mathbb{C}/\Gamma). \tag{5.13}$$

5.3. **The linar flow.** Let W be a quaternionic holomorphic line bundle of degree 0 over a 2–torus T^2 with spectral curve Σ of finite genus. The kernel bundle $\mathcal{L} \to \Sigma$ does not extend to the compactified spectral curve $\bar{\Sigma} = \Sigma \cup \{o, \infty\}$ since the monodromy h^σ of elements of $\mathcal{L}_\sigma$ has essential singularities at $\sigma = o$ and ∞. However, evaluating sections in $\mathcal{L}_\sigma$ at a point $p \in T^2$ gives rise to a complex holomorphic line bundle $E_p \to \Sigma$, a subbundle of the trivial bundle $\Sigma \times W_p$, which extends to the compactification $\bar{\Sigma}$. Its extension $E_p \to \bar{\Sigma}$ is the pull back of the tautological bundle over $\mathbb{CP}^1$ under the algebraic function $S_p\colon \bar{\Sigma} \to \mathbb{CP}^1$ defined in Theorem 5.4. We now prove that the resulting T^2–family of complex holomorphic line bundles $E_p \to \bar{\Sigma}$ moves linearly in the Jacobian of $\bar{\Sigma}$ when the point $p \in T^2$ moves linearly on the torus.

Theorem 5.6. *Let W be a quaternionic holomorphic line bundle of degree 0 over a 2–torus T^2 with spectral curve Σ of finite genus and let $p_0 \in T^2$ be fixed. Then the map*

$$T^2 \to Jac(\bar{\Sigma}): \quad p \mapsto E_p E_{p_0}^{-1}$$

is a group homomorphism.

Remark 5.7. In the special case of a quaternionic holomorphic line bundle that corresponds to a harmonic map $f\colon T^2 \to S^2$ from the 2–torus to the 2–sphere the above theorem is shown in Chapter 7 of [10]. The holomorphic line bundles E_p in that case coincide with the holonomy eigenline bundles of the holomorphic family of flat $\mathrm{SL}_2(\mathbb{C})$–connections defined by the harmonic map f, cf. Section 6.3 of [8] and Section 6.4 of [2]. To prove the result for general quaternionic holomorphic line bundles of degree 0 of finite spectral genus (rather than bundles corresponding to harmonic maps), we apply similar arguments as in Chapter 7 of [10]. In our situation the analog of the harmonic map family of flat $\mathrm{SL}_2(\mathbb{C})$–connections is the family ∇^σ of flat quaternionic connections introduced in Section 4.3.

Proof. Let $V_\infty = U_\infty \cup \{\infty\}$ be a neighborhood of ∞ in $\bar{\Sigma}$ with U_∞ as in Section 5.2 and denote by $V_o = \rho(V_\infty)$ the corresponding neighborhood of o. We compute the change of E_p in $p \in T^2$ by representing bundles in terms of Čech–cohomology classes with respect to the open cover Σ, V_∞ and V_o of $\bar{\Sigma}$. Denote by ψ_∞ a $\hat{\nabla}$–parallel section with monodromy of the quaternionic line bundle $W \to T^2$ with complex structure J that satisfies $J\psi_\infty = \psi_\infty i$, where as before $\hat{\nabla}$ denotes the unique flat connection with $\hat{\nabla}'' = \bar{\partial}$ and $\hat{\nabla} J = 0$ that has unitary holonomy. The restriction of E_p to V_∞ can then be holomorphically trivialized by taking the evaluation at $p \in T^2$ of the section

$$\psi_\infty^\sigma := (1 + Y^\sigma)\psi_\infty$$

with Y^σ as defined in (5.3). Similarly, the restriction of E_p to V_o can be trivialized by taking the evaluation of $\psi_o^\sigma := \psi_\infty^{\rho(\sigma)} j$.

In order to trivialize the restriction of E_p to Σ, we fix $p_0 \in T^2$ and a nowhere vanishing holomorphic section of the restriction of E_{p_0} to Σ. Taking the parallel transport with respect to ∇^σ, we obtain a family ψ_Σ^σ of holomorphic sections of the pullback $\tilde{W}$ of $W \to T^2 = \mathbb{C}/\Gamma$ to the universal cover $\mathbb{C}$ of T^2 whose restriction $\psi_\Sigma^\sigma(z)$ to $z \in \mathbb{C}$ is a holomorphic section of E_p for $p \in T^2 = \mathbb{C}/\Gamma$, the point represented by z.

The bundle E_{p_0} is represented by the Čech–cocycle $f_\infty\colon U_\infty \to \mathbb{C}_*$ and $f_o\colon U_o \to \mathbb{C}_*$ given by $\psi_\Sigma^\sigma(z_0) = \psi_\infty^\sigma(p_0) f_\infty^\sigma$ for every $\sigma \in U_\infty = V_\infty \cap \Sigma$ and $\psi_\Sigma^\sigma(z_0) = \psi_o^\sigma(p_0) f_o^\sigma$ for every $\sigma \in U_o = V_o \cap \Sigma$, where $z_0 \in \mathbb{C}$ denotes a point representing $p_0 \in T^2 = \mathbb{C}/\Gamma$. Equation (5.4) implies that the bundle E_p is then represented by the Čech–cocycle $\sigma \mapsto f_\infty^\sigma \exp(-\int_{z_0}^z \alpha^\sigma)$ and $\sigma \mapsto f_o^\sigma \exp(-\int_{z_0}^z \overline{\alpha^{\rho(\sigma)}})$ defined on U_∞ and U_o respectively for $z \in \mathbb{C}$

a point representing $p \in T^2 = \mathbb{C}/\Gamma$. In Section 5.2 we have seen that the exact part of α^σ extends holomorphically through ∞. Hence E_p is represented by the equivalent cocycle $\sigma \mapsto f_\infty^\sigma \exp(a(\sigma)(z - z_0))$ and $\sigma \mapsto f_o^\sigma \exp(\overline{a(\rho(\sigma))(z - z_0)})$ on U_∞ and U_o respectively, where $a(\sigma)$ is the holomorphic function on U_∞ defined in (5.8). Changing the point $p \in T^2$ thus amounts to a linear change of the Čech–cohomology class representing E_p which proves the theorem. $\square$

References

[1] A. Bobenko, All constant mean curvature tori in R^3, S^3, H^3 in terms of theta functions. *Math. Ann.* **290** (1991), 209–245.

[2] C. Bohle, Constrained Willmore tori in the 4–sphere. `http://arxiv.org/abs/0803.0633`.

[3] C. Bohle, K. Leschke, F. Pedit, and U. Pinkall, Conformal maps from a 2–torus to the 4–sphere. `http://arxiv.org/abs/0712.2311`.

[4] F. Burstall, D. Ferus, K. Leschke, F. Pedit, and U. Pinkall, *Conformal Geometry of Surfaces in S^4 and Quaternions*. Lecture Notes in Mathematics, 1772, Springer-Verlag, Berlin, 2002.

[5] F. Burstall, F. Pedit, and U. Pinkall, Schwarzian derivatives and flows of surfaces, pp. 39–61 in *Differential Geometry and Integrable Systems*, edited by M.A. Guest, R. Miyaoka, Y. Ohnita, *Contemp. Math.* **308** (2002).

[6] B.A. Dubrovin, I.M. Krichever, and S.P. Novikov, The Schrödinger equation in a periodic field and Riemann surfaces. *Soviet Math. Dokl.* **17** (1976), 947–952.

[7] J. Feldmann, H. Knörrer, and E. Trubowitz, *Infinite Genus Riemann Surfaces.* AMS, 2003.

[8] D. Ferus, K. Leschke, F. Pedit, and U. Pinkall, Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2–tori. *Invent. Math.* **146** (2001), no. 3, 507–59.

[9] P. G. Grinevich and M. U. Schmidt, Conformal invariant functionals of immersions of tori into R^3. *J. Geom. Phys.* **26** (1998), 51–78.

[10] N. Hitchin, Harmonic maps from a 2–torus to the 3–sphere. *J. Differential Geom.* **31** (1990), no. 3, 627–710.

[11] B. G. Konopelchenko, Induced surfaces and their integrable dynamics. *Stud. Appl. Math.* **96** (1996), 9–51.

[12] B. G. Konopelchenko, Weierstrass representations for surfaces in 4D spaces and their integrable deformations via DS hierarchy. *Ann. Glob. Anal. Geom.* **18** (2000), 61–74.

[13] I. Krichever, Spectral theory of two–dimensional periodic operators and its applications. *Russian Math. Surveys* **44** (1989), 145–225.

[14] P. Kuchment, *Floquet theory for partial differential equations.* Birkhäuser, Basel, 1993.

[15] I. McIntosh, The quaternionic KP hierarchy and conformally immersed 2–tori in the 4–sphere. `http://arxiv.org/abs/0902.3598`

[16] S.P. Novikov, Two–dimensional Schrödinger operators in periodic fields. *J. of Soviet Math.* **28** (1985), 1–20.

[17] F. Pedit and U. Pinkall, Quaternionic analysis on Riemann surfaces and differential geometry. *Doc. Math. J. DMV*, Extra Volume ICM Berlin 1998, Vol. II, 389–400.

[18] U. Pinkall and I. Sterling, On the classification of constant mean curvature tori. *Ann. of Math.* **130** (1989), 407–451.

[19] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press, New York, 1978.

[20] M. U. Schmidt, A proof of the Willmore conjecture. `http://arxiv.org/abs/math/0203224`.

[21] I. A. Taimanov, Modified Novikov–Veselov equation and differential geometry of surfaces. *Amer. Math. Soc. Transl.* **179** (1997), 133–151.

[22] I. A. Taimanov, The Weierstrass representation of closed surfaces in $\mathbb{R}^3$. *Funct. Anal. Appl.* **32** (1998), 49–62.

[23] I. A. Taimanov, Two–dimensional Dirac operator and surface theory. *Russian Math. Surveys* **61** (2006), 79–159.

[24] F. Warner, *Foundations of differentiable manifolds and Lie groups.* Scott, Foresman and Co., Glenview, Ill.-London, 1971.

Christoph Bohle, Institut für Mathematik, Technische Universität Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany

Franz Pedit, Mathematisches Institut der Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, and, Department of Mathematics, University of Massachusetts, Amherst, MA 01003, USA

Ulrich Pinkall, Institut für Mathematik, Technische Universität Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany

E-mail address: `bohle@math.tu-berlin.de, pedit@mathematik.uni-tuebingen.de, pinkall@math.tu-berlin.de`