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Abstract

Given n polynomials in n variables with a finite number of complex roots, for any of their roots there is a local residue operator assigning a complex number to any polynomial. This is an algebraic, but generally not rational, function of the coefficients. On the other hand, the global residue, which is defined as the sum of the local residues over all roots, has invariance properties which guarantee its rational dependence on the coefficients [9],[27]. In this paper we present symbolic algorithms for evaluating that rational function.

Under the assumption that the deformation to the initial forms is flat, for some choice of weights on the variables, we express the global residue as a single residue integral with respect to the initial forms. When the input equations are a Gröbner basis with respect to a term order, this leads to an efficient series expansion algorithm for global residues, and to a vanishing theorem with respect to the corresponding cone in the Gröbner fan.

The global residue of a polynomial is shown to equal the highest coefficient of its (Gröbner basis) normal form, and, conversely, the entire normal form is expressed in terms of global residues. This yields a new method for evaluating traces over zero-dimensional complete intersections. Applications to be discussed include the counting of real roots (as in [4],[22]),

the computation of the degree of a polynomial map (cf. [12]), and the evaluation of multivariate symmetric functions (cf. [16],[21]). All results and algorithms are illustrated for an explicit system in three variables.

0. Basic Properties of Multidimensional Residues

Multidimensional residues play a fundamental role in complex analysis and geometry. Recent applications of residues in computational algebra include explicit division formulae [5],[6], the evaluation of symmetric functions [21], the membership problem for polynomial ideals [9],[10], the effective Nullstellensatz [13], and numerical algorithms for solving polynomial systems [7]. In most of these articles the emphasis lies on degree estimates and complexity results. Our goal here is to develop practical tools for computing global residues. We thus refrain from using "univariate projections" or "linear changes of coordinates"; instead we seek algorithms involving Gröbner bases and sparsity-preserving series expansions. While initially our discussion follows a path similar to [25],[27], it then proceeds to systematically develop the interplay between residues and Gröbner bases.

This paper is organized as follows. In §1 we consider n polynomials in n variables which form a Gröbner basis (or H-basis) with respect to some choice of positive weights. This hypothesis has natural geometric (1.3'), algebraic (1.5) and analytic (1.7) interpretations. In (1.17) we express the global residue as a single residue integral with respect to the initial forms. In §2 we specialize to the case of a Gröbner basis in the usual sense, with respect to a term order. In (2.3) we express the residue as a coefficient of a certain polynomial. This yields a polyhedral vanishing theorem (2.5), and a bound on the degree of the residue as a polynomial in the trailing coefficients (2.7). The constructions of §1 and §2 lead to algorithms, which will be presented in §3. In §4 we relate global residues to the coefficients in the (Gröbner basis) normal form. The global residue of a polynomial is shown to equal the highest coefficient of its normal form (4.2). This results in fast procedures for computing residues and traces (4.8). In §5 we present applications to symmetric functions, to computing the degree of a polynomial map, and to counting real roots. Finally, we study in §6 an explicit system in three variables.

In this section (§0) we review the complex-analytic definition and basic properties of multidimensional residues. Details and proofs can be found in [1],[15],[25]. For the algebraic counterpart to the analytic theory see e.g. [3],[19],[20],[23]. We shall discuss the equivalence of the algebraic and the analytic approach briefly at the end of §0.

Given n holomorphic functions $g_1, \ldots, g_n$ in an open set $U \subset \mathbb{C}^n$ with a single common zero p in U, one can associate to any holomorphic function $h \in \mathcal{O}(U)$ the local residue at p of the meromorphic n-form

$$\omega = \frac{h(\mathbf{x})\, d\mathbf{x}}{g_1(\mathbf{x}) \ldots g_n(\mathbf{x})}\ , \quad d\mathbf{x} = dx_1 \wedge \ldots \wedge dx_n\, .$$

This defines the $\mathbb{C}$-linear operator

$$(0.1) \qquad \mathcal{O}(U) \to \mathbb{C}\,, \quad h \mapsto \mathrm{Res}_p\left(\frac{h(\mathbf{x})\,d\mathbf{x}}{g_1(\mathbf{x})\dots g_n(\mathbf{x})}\right) := \frac{1}{(2\pi i)^n}\int_{\Gamma_{\mathbf{g}}(\epsilon)} \omega\,,$$

where $\Gamma_{\mathbf{g}}(\epsilon)$ is the real n-dimensional cycle

$$\Gamma_{\mathbf{g}}(\epsilon) = \{\,\mathbf{x} \in U \ :\ |g_i(\mathbf{x})| = \epsilon_i\ , i = 1, \dots, n\,\}$$

with orientation defined by the n-form $d\arg(g_1)\wedge\dots\wedge d\arg(g_n)$, and where $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ is any n-tuple of sufficiently small, positive real numbers.

The Cauchy formula in n complex variables provides the simplest example of a residue operator: If $g_i(\mathbf{x}) = (x_i - p_i)^{a_i+1}$, $a_i \in \mathbb{N}$, then

$$(0.2) \qquad \mathrm{Res}_p\left(\frac{h(\mathbf{x})\,d\mathbf{x}}{\prod_{i=1}^n (x_i - p_i)^{a_i+1}}\right) = \frac{1}{a_1!\dots a_n!}\left(\frac{\partial^{a_1+\dots+a_n} h}{\partial x_1^{a_1}\dots\partial x_n^{a_n}}\right)(p)\,.$$

Let $J_{\mathbf{g}} := \det\left(\dfrac{\partial g_i}{\partial x_j}\right)$ denote the Jacobian of $\mathbf{g} = (g_1, \dots, g_n)$. Then,

$$(0.3) \qquad \mathrm{Res}_p\left(\frac{h\,J_{\mathbf{g}}\,d\mathbf{x}}{g_1\cdots g_n}\right) = \mu_{\mathbf{g}}(p)\,h(p)\,,$$

where $\mu_{\mathbf{g}}(p)$ denotes the intersection multiplicity of $\mathbf{g} = (g_1, \dots, g_n)$ at p [15, p. 662ff.]. If p is a simple root, hence $J_{\mathbf{g}}(p) \neq 0$, then

$$(0.4) \qquad \mathrm{Res}_p\left(\frac{h\,d\mathbf{x}}{g_1\cdots g_n}\right) = \frac{h(p)}{J_{\mathbf{g}}(p)}\,.$$

Suppose now that $g_1, \dots, g_n \in \mathbb{C}[\mathbf{x}]$ are n-variate polynomials whose zero set $Z(\mathbf{g})$ is a non-empty finite subset of $\mathbb{C}^n$. We can consider the *global residue operator* ([9],[15],[25]) which assigns to a polynomial $h \in \mathbb{C}[\mathbf{x}]$ the complex number

$$(0.5) \qquad \mathrm{Res}_{\mathbf{g}}(h) = \mathrm{Res}\left(\frac{h\,d\mathbf{x}}{g_1\cdots g_n}\right) := \sum_{p\in Z(\mathbf{g})} \mathrm{Res}_p\left(\frac{h\,d\mathbf{x}}{g_1\cdots g_n}\right)\,.$$

The main functorial properties of the global residue are encapsulated in the following two results whose proof may be found, for example, in [25, II.8.3-4].

(0.6) Transformation Law. *Suppose that $f_1, \dots, f_n \in \mathbb{C}[\mathbf{x}]$ have finitely many common roots and that we can write*

$$f_i = \sum_{j=1}^n A_{ij} g_j\ ; \quad A_{ij} \in \mathbb{C}[\mathbf{x}]\ , \quad i = 1, \dots, n\,.$$

Then, for $h \in \mathbb{C}[\mathbf{x}]$,

$$\mathrm{Res}\left(\frac{h\,d\mathbf{x}}{g_1\cdots g_n}\right) = \mathrm{Res}\left(\frac{h\,\det(A_{ij})\,d\mathbf{x}}{f_1\cdots f_n}\right)\,.$$

If $g_1, \ldots, g_n \in \mathbb{C}[\mathbf{x}]$ are as above, the ideal I generated by them is zero-dimensional, and therefore $V = \mathbb{C}[\mathbf{x}]/I$ is a finite dimensional $\mathbb{C}$-vector space. Since the global residue $\mathrm{Res}_{\mathbf{g}}(h)$ vanishes for $h \in I$ (see [15, p. 650]), it defines a $\mathbb{C}$-linear map:

$$\mathrm{Res}_{\mathbf{g}}\colon V \to \mathbb{C}\ , \quad h \mapsto \mathrm{Res}_{\mathbf{g}}(h)\ .$$

(0.7) Duality. *A polynomial $h \in \mathbb{C}[\mathbf{x}]$ lies in I if and only if $\mathrm{Res}_{\mathbf{g}}(fh) = 0$ for all $f \in \mathbb{C}[\mathbf{x}]$.*

This duality law may be interpreted as follows: Let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Then the pairing

$$V \times V^* \to V^*\ , \quad (b, \phi) \to (b.\phi)\ ,$$

where $(b.\phi)(b') = \phi(b\,b')$, makes V^* into a V-module. Statement (0.7) is equivalent to the assertion that the residue operator $\mathrm{Res}_{\mathbf{g}}$ is a generator of V^* as a V-module.

We recall that $\mathrm{Res}_{\mathbf{g}}(h)$ is a rational function, with integral coefficients, in the coefficients of $g_1, \ldots, g_n$. It may be computed in simply exponential time with respect to n, the number of variables; see [9],[21],[27]. A general procedure, suggested in these articles, is to find univariate polynomials $f_1(x_1), \ldots, f_n(x_n)$ in the ideal generated by $g_1, \ldots, g_n$ and to transform the global residue applying (0.6). The global residues with respect to $f_1, \ldots, f_n$ are then computed as a sum of products of univariate global residues. This general procedure is often too slow for practical computations. We seek more efficient algorithms for "nice" situations, such as the case when $g_1, \ldots, g_n$ are a Gröbner basis for a term order. The theory for such nice situations is to be developed in the next section.

In closing let us mention the relationship between this analytic definition of global residue and the algebraic definitions: of course, they coincide. Write for each $i = 1, \ldots, n$,

$$g_i(\mathbf{y}) - g_i(\mathbf{x}) \quad = \quad \sum_{j=1}^{n} g_{ij}(\mathbf{y}, \mathbf{x})(y_j - x_j)\ .$$

and denote $\Delta := \det(g_{ij})$. Let $U \subset \mathbb{C}^n$ be the union of relatively compact open neighborhoods isolating each of the points in $Z(\mathbf{g})$, let ϵ be any n-tuple of small positive real numbers, and define $\Pi_\epsilon := \{\mathbf{x} \in U : |g_i(\mathbf{x})| < \epsilon_i\ , \forall i = 1, \ldots, n\}$. For any holomorphic function h on U, one can deduce from (0.2) and (0.6) (cf.[25, §17]) the following integral representation known as Weil's formula [26]:

$$h(\mathbf{x}) = \frac{1}{(2\pi i)^n} \int_{\Gamma_{\mathbf{g}}(\epsilon)} \frac{h(\mathbf{y})\Delta(\mathbf{y}, \mathbf{x})d\mathbf{y}}{\prod_{i=1}^{n}(g_i(\mathbf{y}) - g_i(\mathbf{x}))}\ , \quad \mathbf{x} \in \Pi_\epsilon.$$

As $|g_i(\mathbf{x})| < |g_i(\mathbf{y})|$ for any $i\,, \mathbf{x} \in \Pi_\epsilon$ and $\mathbf{y} \in \Gamma_{\mathbf{g}}(\epsilon)$, the integrand may be expanded as a multiple geometric series

$$\frac{h(\mathbf{y})\Delta(\mathbf{y},\mathbf{x})}{\prod_{i=1}^n (g_i(\mathbf{y}) - g_i(\mathbf{x}))} \;=\; h(\mathbf{y})\Delta(\mathbf{y},\mathbf{x}) \sum_{\alpha_i \geq 0} \left(\prod_{i=1}^n \frac{g_i(\mathbf{x})^{\alpha_i}}{g_i(\mathbf{y})^{\alpha_i+1}} \right),$$

which converges uniformly on compact subsets of $\Gamma_{\mathbf{g}}(\epsilon) \times \Pi_\epsilon$. As a result of term-by-term integration, we deduce that (cf.[25,§5],[5],[6])

$$h(\mathbf{x}) \;=\; \mathrm{Res}_{\mathbf{g}}(h(\,\cdot\,)\;\Delta(\,\cdot\,,\mathbf{x})) + \sum_{|\alpha|\geq 1} \mathrm{Res}\left(\frac{h(\mathbf{y})\Delta(\mathbf{y},\mathbf{x})\,d\mathbf{y}}{\prod_{i=1}^n g_i(\mathbf{y})^{\alpha_i+1}} \right) \mathbf{g}^\alpha(\mathbf{x})\,, \quad \forall \mathbf{x} \in \Pi_\epsilon. \tag{0.8}$$

Note that the second summand on the right is in the ideal generated by $g_1, \ldots, g_n$ in $\mathcal{O}(\Pi_\epsilon)$ and that $\mathrm{Res}_{\mathbf{g}}(h(\,\cdot\,)\;\Delta(\,\cdot\,,\mathbf{x}))$ depends polynomially on $\mathbf{x}$. If, in addition, h is a polynomial, then the fact that $Z(\mathbf{g})$ is contained in Π_ϵ plus the fact that local analytic membership is equivalent to local algebraic membership ([24]), imply that

$$h(\mathbf{x}) = \mathrm{Res}_{\mathbf{g}}(h(\,\cdot\,)\;\Delta(\,\cdot,\mathbf{x})) \quad \text{on the quotient ring } V\,. \tag{0.9}$$

In general, (0.8) does not provide a representation of their difference as a polynomial linear combination of $g_1, \ldots, g_n$. Under the hypothesis (1.3) below, it follows from the vanishing statement in (1.18) that the series becomes a finite sum, giving an effective division formula with remainder which involves computing only finitely many global residues associated to powers of $g_1, \ldots, g_n$. In summary, the formula (0.9) is the algebraic version of the integral representation. It proves that the global residue we are considering coincides with the "trace" associated to Δ as in [19, Appendix F] and [13], and with the Kronecker symbol (i.e., the dualizing linear form associated to Δ) as in [3].

1. Gröbner Bases for a Weight Partial Order

Let $\mathbf{K}$ be any subfield of the complex numbers $\mathbb{C}$ and let $g_i(\mathbf{x})$, $i = 1, \ldots, n$, be polynomials in $S = \mathbf{K}[\mathbf{x}]$, $\mathbf{x} = (x_1, \ldots, x_n)$. Let $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$ be a positive *weight vector*. The *weighted degree* of a monomial $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n}$ is

$$\deg_{\mathbf{w}}(\mathbf{x}^{\mathbf{a}}) = \langle \mathbf{w}, \mathbf{a} \rangle = \sum_{i=1}^n w_i\, a_i\,.$$

We extend the notion of weighted degree to arbitrary polynomials in S in the usual manner. Write each polynomial $g_i(\mathbf{x})$ as

$$g_i(\mathbf{x}) = p_i(\mathbf{x}) + q_i(\mathbf{x}) \tag{1.1}$$

where p_i is $\mathbf{w}$-homogeneous, and

$$d_i = \deg_{\mathbf{w}}(p_i) = \deg_{\mathbf{w}}(g_i)\,; \quad \deg_{\mathbf{w}}(q_i) < \deg_{\mathbf{w}}(g_i)\,. \tag{1.2}$$

Throughout this section we make the following assumption:

$$p_1(\mathbf{x}) = \cdots = p_n(\mathbf{x}) = 0 \quad \text{if and only if} \quad \mathbf{x} = 0 \tag{1.3}$$

In what follows we will interpret this condition geometrically (1.3'), algebraically (1.5) and analytically (1.7). Let

$$\tilde{g}_i(t;\mathbf{x}) = t^{d_i} g_i(t^{-w_1}x_1, \ldots, t^{-w_n}x_n) \tag{1.4}$$

This is a homogeneous polynomial of degree d_i in $(t; x_1, \ldots, x_n)$ relative to the weights $(1; w_1, \ldots, w_n)$.

Let $\mathbb{P}^n_{\mathbf{w}}$ denote the weighted projective space with homogeneous coordinates $(t; x_1, \ldots, x_n)$ and weights $(1; w_1, \ldots, w_n)$. The image of the hyperplane $\{t = 1\} \subset \mathbb{C}^{n+1} \setminus \{0\}$ in $\mathbb{P}^n_{\mathbf{w}}$ is identified with $\mathbb{C}^n$. If

$$\tilde{D}_i = \{(t;\mathbf{x}) \in \mathbb{P}^n_{\mathbf{w}} \ : \ \tilde{g}_i(t;\mathbf{x}) = 0\}$$

then (1.3) is equivalent to the geometric condition

$$\tilde{D}_1 \cap \ldots \cap \tilde{D}_n \subset \mathbb{C}^n \tag{1.3$'$}$$

The algebraic meaning of (1.3) is best expressed using the following notion of a Gröbner basis: Given a polynomial $f \in S$, we denote by $\mathrm{in}_{\mathbf{w}}(f)$ its form of highest weighted degree. For any ideal $I \subset S$ we define the initial ideal $\mathrm{in}_{\mathbf{w}}(I)$ to be the ideal generated by $\mathrm{in}_{\mathbf{w}}(f)$ where f runs over I. A finite subset $\mathcal{G} \subset I$ is said to be a *Gröbner basis* for I, relative to the weight $\mathbf{w}$, provided:

$$\mathrm{in}_{\mathbf{w}}(I) \quad = \quad \langle \mathrm{in}_{\mathbf{w}}(g) \ : \ g \in \mathcal{G} \rangle\,.$$

We emphasize that $\mathrm{in}_{\mathbf{w}}(I)$ need not be a monomial ideal. Some authors prefer to call $\mathcal{G}$ an *H-basis*, a term which goes back to Macaulay in the classical case $\mathbf{w} = (1, 1, \ldots, 1)$.

(1.5) Lemma. *Suppose $\mathcal{G} = \{g_1, \ldots, g_n\} \subset S$ satisfy (1.3). Then $\mathcal{G}$ is a Gröbner basis for the ideal it generates. Conversely, suppose $\mathcal{G} = \{g_1, \ldots, g_n\} \subset S$ is a Gröbner basis , with respect to $\mathbf{w}$, for a zero-dimensional ideal I. Then $\{g_1, \ldots, g_n\}$ satisfy (1.3).*

Proof: With the same notation as above we have $p_i = \mathrm{in}_{\mathbf{w}}(g_i)$ and $q_i = g_i - \mathrm{in}_{\mathbf{w}}(g_i)$. Since $p_1, \ldots, p_n$ define a complete intersection, the Koszul complex on these forms in exact. This implies that every syzygy $\sum h_i \cdot e_i$ on $(p_1, \ldots, p_n)$ can be written as a linear combination of the basic syzygies $p_j \cdot e_k \; - \; p_k \cdot e_j$.

Suppose that $\mathcal{G}$ is not a Gröbner basis. Then, there exists a polynomial

$$f \quad = \quad \sum h_i g_i \quad = \quad \sum h_i p_i + \sum h_i q_i$$

whose initial form does not lie in $\langle p_1, \ldots, p_n \rangle$. Hence $\sum h_i p_i \; = \; 0$. By the remark above, we can write $\sum h_i \cdot e_i \; = \; \sum_{j,k} b_{jk} \cdot (p_j \cdot e_k \; - \; p_k \cdot e_j)$, and the leading term of

$$\sum h_i g_i \;\; = \; \sum h_i q_i \;\; = \; \sum_{j,k} b_{jk} \cdot (p_j q_k \; - \; p_k q_j)$$

must lie in $\langle p_1, \ldots, p_n \rangle$. This is a contradiction, completing the proof of the first statement. To prove the converse, it suffices to note $\dim(I) = \dim(\mathrm{in}_{\mathbf{w}}(I)) = 0$ (see e.g. [17]). $\diamond$

(1.6) Remarks: (i) The first part of Lemma (1.5) remains true for any set of polynomials $\mathcal{G} = \{g_1, \ldots, g_m\} \subset S$ whose initial forms define a complete intersection.

(ii) The initial ideal $\mathrm{in}_{\mathbf{w}}(I)$ is a flat deformation of the given ideal I (see e.g. [11, Ch. 6]).

(iii) The results in this section can be extended to fields other than the complex numbers using the deformation techniques in [20].

For each $t \in \mathbb{C}$, consider the map $\mathbf{g}_t \colon \mathbb{C}^n \to \mathbb{C}^n$ defined by $\mathbf{g}_t(\mathbf{x}) = (\tilde{g}_1(t;\mathbf{x}), \ldots, \tilde{g}_n(t;\mathbf{x}))$. We have the following analytic interpretation of (1.3). Recall that a map $F \colon \mathbb{C}^n \to \mathbb{C}^n$ is said to be *proper* if the inverse image of any compact set is compact.

(1.7) Lemma. *The polynomials $g_1, \ldots, g_n$ satisfy condition (1.3) if and only if the map $\mathbf{g}_t$ is proper for every $t \in \mathbb{C}$.*

Proof: At $t = 0$ we have $\mathbf{g}_0 = (p_1, \ldots, p_n)$. Since the polynomials p_i are weighted homogeneous, the inverse image $\mathbf{g}_0^{-1}(0)$ is compact if and only if $\mathbf{g}_0^{-1}(0) = \{0\}$. Thus, if $\mathbf{g}_0$ is proper, then condition (1.3) is satisfied.

For the converse it is enough to show that the map $\mathbf{g}$ is proper, since (1.3) is a condition on just the initial form of the polynomials. Let $\tilde{\mathbf{g}} \colon \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}^{n+1} \setminus \{0\}$ be defined by

$$\tilde{\mathbf{g}}(t, x_1, \ldots, x_n) = (t, \tilde{g}_1(t;\mathbf{x}), \ldots, \tilde{g}_n(t;\mathbf{x}))$$

The fact that $\mathbf{g}(\mathbf{x})$ satisfies (1.3) guarantees that $\tilde{\mathbf{g}}$ takes values in $\mathbb{C}^{n+1} \setminus \{0\}$. Since

$$\tilde{\mathbf{g}}(\lambda\, t, \lambda^{w_1}\, x_1, \ldots, \lambda^{w_n}\, x_n) = (\lambda\, t, \lambda^{d_1}\, \tilde{g}_1(t;\mathbf{x}), \ldots, \lambda^{d_n}\, \tilde{g}_n(t;\mathbf{x}))\,,$$

$\tilde{\mathbf{g}}$ defines a map from $\mathbb{P}^n_{\mathbf{w}}$ to weighted projective space $\mathbb{P}^n_{\mathbf{d}}$ with weights $(1; d_1, \ldots, d_n)$. We may now consider the embedding of $\mathbb{C}^n$ in $\mathbb{C}^{n+1} \setminus \{0\}$ as the hyperplane $\{t = 1\}$. Since t has weight one in both $\mathbb{P}^n_{\mathbf{w}}$ and $\mathbb{P}^n_{\mathbf{d}}$, the natural projection from $\mathbb{C}^{n+1} \setminus \{0\}$ to $\mathbb{P}^n_{\mathbf{w}}$ or $\mathbb{P}^n_{\mathbf{d}}$ is a homeomorphism of the hyperplane $\{t = 1\}$ to its image. Thus, $\tilde{\mathbf{g}}$ is a continuous extension of $\mathbf{g}$ to appropriate compactifications of $\mathbb{C}^n$. If $K \subset \mathbb{C}^n$ is compact then $\mathbf{g}^{-1}(K)$ is compact since it coincides with $\tilde{\mathbf{g}}^{-1}(K)$. $\diamond$

(1.8) Examples: An important special case, to be investigated in detail in §2, is that of n polynomials $g_1, \ldots, g_n \in S$ with finitely many common roots in $\mathbb{C}^n$ and such that they are a Gröbner basis with respect to some term order $\prec$. We can choose a weight vector $\mathbf{w} \in \mathbb{N}^n$ such that $\mathrm{in}_\prec(g_i) = \mathrm{in}_{\mathbf{w}}(g_i)$. Since the ideal generated by the initial monomials $\mathrm{in}_{\mathbf{w}}(g_i)$ is zero-dimensional, we may assume without loss of generality that:

$$\mathrm{in}_{\mathbf{w}}(g_i) = \alpha_i\, x_i^{r_i+1}$$

for some $\alpha_i \in \mathbf{K} \setminus \{0\}$, and therefore they satisfy (1.3). Particular examples are:

(i) The term order $\prec$ is lexicographic order: then $g_1, \ldots, g_n$ satisfy $g_i(\mathbf{x}) = g_i(x_i, \ldots, x_n)$ and g_i is monic in x_i. This is the case studied in [9]; see also [21] for the subcase when $g_i(\mathbf{x}) = g_i(x_i)$ are univariate polynomials.

(ii) The term order is defined as total degree with ties broken by lexicographic order with $x_n > \ldots > x_1$. This is the case studied in [1, (21.3)] and [25, II.8.2]. A weighted variant, due to Aĭzenberg and Tsikh, is studied in [1, (21.5)].

In this section we are interested in studying the global residue $\mathrm{Res}_{\mathbf{g}}(h)$, for a polynomial $h \in \mathbb{C}[\mathbf{x}]$, under the hypothesis that $g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})$ satisfy (1.3). This hypothesis makes it possible to reduce the computation of $\mathrm{Res}_{\mathbf{g}}(h)$ to that of residues involving only certain powers of the initial forms $p_1(\mathbf{x}), \ldots, p_n(\mathbf{x})$. This is the content of (1.20) below. In fact, considering t as a parameter, the idea that one can recover the information from the deformation to the initial forms is the core of the geometric interpretation of Gröbner bases (see e.g. [2]).

Since the map $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ is proper, we can replace (0.5) by a single integral

$$\mathrm{Res}\left(\frac{h\, d\mathbf{x}}{g_1 \cdots g_n}\right) \;=\; \frac{1}{(2\pi i)^n} \int_{\Gamma(\mathbf{r})} \frac{h\, d\mathbf{x}}{g_1 \cdots g_n}$$

for any $\mathbf{r} = (r_1, \ldots, r_n) \in (\mathbb{R}_{>0})^n$, where $\Gamma(\mathbf{r})$ is the compact, real n-cycle

$$\Gamma(\mathbf{r}) = \{ \mathbf{x} \in \mathbb{C}^n \ : \ |g_i(\mathbf{x})| = r_i \ ; \ i = 1, \ldots n \} .$$

Similarly, for each fixed $t \in \mathbb{C}$,

$$R_h(t) := \mathrm{Res}\left(\frac{h(\mathbf{x})\, d\mathbf{x}}{\tilde{g}_1(t;\mathbf{x}) \ldots \tilde{g}_n(t;\mathbf{x})} \right) = \frac{1}{(2\pi i)^n} \int_{\tilde{\Gamma}_t(\mathbf{r})} \frac{h(\mathbf{x})\, d\mathbf{x}}{\tilde{g}_1(t;\mathbf{x}) \ldots \tilde{g}_n(t;\mathbf{x})}, \tag{1.9}$$

where the global residue is taken relative to the divisors $\{ \mathbf{x} \in \mathbb{C}^n \ : \ \tilde{g}_i(t;\mathbf{x}) = 0 \}$, $i = 1, \ldots, n$ and $\tilde{\Gamma}_t(\mathbf{r}) = \{ \mathbf{x} \in \mathbb{C}^n \ : \ |\tilde{g}_i(t;\mathbf{x})| = r_i \ ; \ i = 1, \ldots n \}$.

The family $\{\tilde{g}_1(t, \mathbf{x}), \ldots, \tilde{g}_n(t, \mathbf{x})\}$ is a Gröbner basis with respect to $\mathbf{w}' = (1, 2w_1, \ldots, 2w_n)$ for the ideal $\tilde{I}$ generated by $\{\tilde{f}, f \in I\}$ and $\mathrm{in}_{\mathbf{w}'}(\tilde{f}) = \mathrm{in}_{\mathbf{w}}(f)$, $\forall f \in S$. It then follows that the coordinates $(t, \mathbf{x})$ are in *Noether position* [8] for $\tilde{I}$ and, consequently, we may apply Theorem 3.3 in [9] to deduce that $R_h(t)$ is a polynomial in t. We will reprove this and in fact obtain the stronger result (1.18).

We set

$$\tilde{G}(t;\mathbf{x}) \ := \ \prod_{i=1}^{n} \tilde{g}_i(t;\mathbf{x}) \ = \ \sum_{j=0}^{d_{\mathbf{w}}} A_j(\mathbf{x})\, t^j \tag{1.10}$$

where $d_{\mathbf{w}} = d_1 + \ldots + d_n$. Inverting the polynomial $\tilde{G}(t;\mathbf{x})$ as a rational formal power series in t, we write

$$\tilde{G}^{-1}(t;\mathbf{x}) \ = \ \sum_{j \geq 0} B_j(\mathbf{x})\, t^j . \tag{1.11}$$

Given positive real numbers $k_1, \ldots, k_n$, let

$$T(\mathbf{k}) := \{ \mathbf{x} \in \mathbb{C}^n \ : \ |p_i(\mathbf{x})| = k_i \} .$$

(1.12) Lemma. *Given $\delta > 0$, there exist positive constants $k_1, \ldots, k_n$ so that, for $|t| \leq \delta$,*

$$R_h(t) = \frac{1}{(2\pi i)^n} \sum_{m \geq 0} \left(\int_{T(\mathbf{k})} h(\mathbf{x})\, B_m(\mathbf{x})\, d\mathbf{x} \right) t^m ,$$

and this series is uniformly convergent for $|t| \leq \delta$.

Proof: Suppose $g_1, \ldots, g_n$ satisfy (1.3) and let

$$\tilde{g}_i(t;\mathbf{x}) = p_i(\mathbf{x}) + t\,\hat{q}_i(t;\mathbf{x}).$$

Then, given $\delta > 0$, there exist positive constants $k_1, \ldots, k_n$ such that, for $\mathbf{x} \in T(\mathbf{k})$,

$$\frac{|p_i(\mathbf{x})|}{2} > |t\,\hat{q}_i(t;\mathbf{x})| \tag{1.13}$$

for all $i = 1, \ldots, n$ and $|t| \leq \delta$. Indeed, because of the weighted homogeneity property of p_i, it suffices to take $k_i = \lambda^{d_i}$ with λ sufficiently large.

The estimate (1.13) allows us to apply *Rouché's principle for residues* [25, II.8.1] and replace, for $|t| \leq \delta$, the integration cycles $\tilde{\Gamma}_t(\mathbf{r})$ in (1.9) by the fixed cycle $T(\mathbf{k})$. Thus,

$$R_h(t) = \frac{1}{(2\pi i)^n} \int_{T(\mathbf{k})} \frac{h(\mathbf{x})\,d\mathbf{x}}{\tilde{g}_1(t;\mathbf{x}) \ldots \tilde{g}_n(t;\mathbf{x})} \qquad \text{for all } |t| \leq \delta. \tag{1.14}$$

In view of (1.13), it follows that the series

$$\sum_{j \geq 0} B_j(\mathbf{x})\,t^j \quad = \quad \frac{1}{\displaystyle\prod_{i=1}^{n} p_i(\mathbf{x}) \left(1 + \frac{t\,\hat{q}_i(t;\mathbf{x})}{p_i(\mathbf{x})}\right)}$$

is uniformly convergent for $\mathbf{x} \in T(\mathbf{k})$ and $|t| \leq \delta$. Since we can now integrate (1.14) term-by-term, the result follows. $\diamond$

(1.15) Lemma. *Let $P(\mathbf{x}) = p_1(\mathbf{x}) \ldots p_n(\mathbf{x})$. Then $P^{m+1}(\mathbf{x})\,B_m(\mathbf{x})$ is a weighted homogeneous polynomial of degree $m\,(d_{\mathbf{w}} - 1)$ with respect to $\mathbf{w}$.*

Proof: Since $\tilde{G}(t;\mathbf{x})$ is weighted homogeneous of degree $d_{\mathbf{w}}$ and t has weight 1, the coefficients $A_j(\mathbf{x})$ in (1.10) are weighted homogeneous of degree $d_{\mathbf{w}} - j$. On the other hand, the series (1.11) inverts (1.10). This implies the following recursion relations:

$$\sum_{j=0}^{m} A_j\,B_{m-j} = 0 \quad , \quad m \geq 1 \tag{1.16}$$

with initial conditions $A_0\,B_0 = 1$ and $A_0(\mathbf{x}) = P(\mathbf{x})$. In particular,

$$P\,B_m \;=\; -\sum_{j=1}^{m} A_j\,B_{m-j}\,,$$

and

$$P^{m+1}\,B_m \;=\; \sum_{j=1}^{m} A_j\,P^{j-1}\,(P^{m-j+1}\,B_{m-j})\,.$$

Assuming that (1.14) holds inductively with respect to m, we obtain

$$\deg_{\mathbf{w}}(P^{m+1}\,B_m) \;=\; (d_{\mathbf{w}} - j) + (j-1)\,d_{\mathbf{w}} + (m-j)\,(d_{\mathbf{w}} - 1) \;=\; m\,(d_{\mathbf{w}} - 1)\,. \qquad \diamond$$

The following is the main result in this section.

(1.17) Theorem. *For any monomial* $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \dots x_n^{a_n}$, *set* $s(\mathbf{a}) = \langle \mathbf{w}, \mathbf{a} \rangle - d_{\mathbf{w}} + \sum_{i=1}^n w_i$. *Then*

$$R_{\mathbf{x}^{\mathbf{a}}}(t) \;=\; \mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}}) \cdot t^{s(\mathbf{a})} \,,$$

and

$$\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}}) \;=\; \frac{1}{(2\pi i)^n} \int_{T(\mathbf{k})} \mathbf{x}^{\mathbf{a}}\, B_{s(\mathbf{a})}(\mathbf{x})\, d\mathbf{x}\,.$$

Before proving (1.17), we note the following weighted version of the Euler-Jacobi Theorem [15, p. 671]. A more general toric version was given by Khovanskii in [18].

(1.18) Corollary. *$R_h(t)$ is a polynomial in t of degree at most* $\deg_{\mathbf{w}}(h) - d_{\mathbf{w}} + \sum w_i$, *and*

$$\mathrm{Res}\left(\frac{h\, d\mathbf{x}}{g_1 \dots g_n}\right) = 0 \qquad \textit{whenever} \quad \deg_{\mathbf{w}}(h) < d_{\mathbf{w}} - \sum w_i.$$

We observe also that, under the current hypothesis (1.3), this corollary implies that the terms in the series in (0.8) will vanish for $\sum_{i=1}^n \alpha_i\, d_i > \deg_{\mathbf{w}}(h)$.

Proof of (1.17): We begin by noting that, as in the case with unit weights [25, IV.20.1]:

(1.19) *If P and Q are weighted homogeneous polynomials in $\mathbb{C}[\mathbf{x}]$, and* $\deg_{\mathbf{w}}(P) - \deg_{\mathbf{w}}(Q) + \sum w_i \neq 0$, *then the form* $\omega = \dfrac{P(\mathbf{x})\, d\mathbf{x}}{Q(\mathbf{x})}$ *is exact.*

Indeed, we find that $\omega = (\deg_{\mathbf{w}}(P) - \deg_{\mathbf{w}}(Q) + \sum w_i)^{-1}\; d\sigma$, where

$$\sigma \;=\; \frac{P(\mathbf{x})}{Q(\mathbf{x})} \sum_{j=1}^n (-1)^{j-1} w_j\, x_j\, dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_n\,.$$

The verification of this equality is a straightforward consequence of Euler's formula for weighted homogeneous polynomials:

$$\sum_{j=1}^n w_j\, x_j\, \frac{\partial P}{\partial x_j} \;=\; \deg_{\mathbf{w}}(P)\, P\,.$$

As in Lemma (1.12), we write

$$R_{\mathbf{x}^{\mathbf{a}}}(t) = \sum_{m \geq 0} \left(\int_{T(\mathbf{k})} \mathbf{x}^{\mathbf{a}}\, B_m(\mathbf{x})\, d\mathbf{x} \right) t^m\,.$$

Since, by Lemma (1.15), $B_m(\mathbf{x})$ is a quotient of weighted homogeneous polynomials we can apply (1.19) to conclude that

$$\int_{T(\mathbf{k})} \mathbf{x}^{\mathbf{a}}\, B_m(\mathbf{x})\, d\mathbf{x} \; = 0$$

whenever

$$\deg_{\mathbf{w}}(\mathbf{x}^{\mathbf{a}}) + \deg_{\mathbf{w}}(B_m) + \sum w_i \quad \neq \quad 0\,.$$

This inequation is equivalent to $m \neq s(\mathbf{a})$. Hence all integrals in (1.12) vanish, except for the one with $m = s(\mathbf{a})$. This was precisely the claim of (1.17). $\diamond$

The second assertion of (1.17) says that we may write

$$\text{(1.20)} \qquad \operatorname{Res}\left(\frac{\mathbf{x}^{\mathbf{a}}\, d\mathbf{x}}{g_1 \dots g_n}\right) = \frac{1}{(2\pi i)^n} \int_{T(\mathbf{k})} \frac{\mathbf{x}^{\mathbf{a}}\,(P^{s(\mathbf{a})+1}(\mathbf{x})\, B_{s(\mathbf{a})}(\mathbf{x}))}{P^{s(\mathbf{a})+1}(\mathbf{x})}\, d\mathbf{x}\,.$$

The numerator is a weighted homogeneous polynomial, by Lemma (1.15). Therefore (1.20) is a residue with respect to the $(s(\mathbf{a})+1)$ power of the initial forms $p_1(\mathbf{x}), \dots, p_n(\mathbf{x})$.

We conclude this section by observing that as a direct consequence of (1.17) and the Duality Theorem (0.7) we obtain (see [25, IV.20.1] for the case of unit weights):

(1.21) Macaulay's Theorem. *Let $p_1(\mathbf{x}), \dots, p_n(\mathbf{x})$ be weighted homogeneous polynomials whose only common zero is the origin. Then, any weighted homogeneous polynomial $h(\mathbf{x})$ satisfying*

$$\deg_{\mathbf{w}}(h) > d_{\mathbf{w}} - \sum_{i=1}^{n} w_i$$

is in the ideal generated by $p_1(\mathbf{x}), \dots, p_n(\mathbf{x})$.

2. Gröbner Bases for a Term Order

In this section we specialize to the case of n polynomials $g_1, \dots, g_n \in S$ with finitely many roots in $\mathbb{C}^n$, which are a Gröbner basis with respect to a term order $\prec$. As in (1.8), we choose a positive weight $\mathbf{w} \in \mathbb{N}^n$ such that $\text{in}_{\prec}(g_i) = \text{in}_{\mathbf{w}}(g_i)$. We may assume that

$$\text{(2.1)} \qquad \text{in}_{\mathbf{w}}(g_i) = \; x_i^{r_i+1}\,, \quad i = 1, \dots, n\,.$$

Let $\mathbf{r} = (r_1, \dots, r_n)$ and $\mathbf{r}+\mathbf{1} = (r_1+1, \dots, r_n+1)$. Then, with notation as in §1, $d_{\mathbf{w}} = \langle \mathbf{w}, \mathbf{r}+\mathbf{1}\rangle$, and

$$P(\mathbf{x}) = x_1^{r_1+1} \dots x_n^{r_n+1} = \mathbf{x}^{\mathbf{r}+\mathbf{1}}\,,$$

$$s(\mathbf{a}) = \langle \mathbf{w}, \mathbf{a} \rangle - d_{\mathbf{w}} + \sum w_i = \langle \mathbf{w}, \mathbf{a} - \mathbf{r} \rangle ,$$

and (1.15) implies the following homogeneity of the coefficients of (1.11):

(2.2) *$B_m(\mathbf{x})$ is a $\mathbf{w}$-homogeneous Laurent polynomial of weighted degree $-(m + d_{\mathbf{w}})$.*

Consequently, the integrand in (1.17) is a Laurent polynomial. Since

$$\int_{T(\mathbf{k})} \mathbf{x}^{\mathbf{b}} \, d\mathbf{x} \quad = \quad 0 \qquad \text{for} \quad \mathbf{b} \neq (-1, \ldots, -1),$$

we obtain the following result.

(2.3) Theorem. *The residue* $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$ *equals the* $x_1^{-1} \ldots x_n^{-1}$*-coefficient of* $\mathbf{x}^{\mathbf{a}} \, B_{s(\mathbf{a})}(\mathbf{x})$ *.*

Theorem (2.3) is essentially a restatement of a formula for computing global residues due to Aĭzenberg and Tsikh [1, (21.5)], which, in a simpler version pointed out to us by J. Petean, says that $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$ equals the $x_1^{-1} \ldots x_n^{-1}$-coefficient in the expression

$$\sum_{|\alpha| \leq \langle \mathbf{w}, \mathbf{a} - \mathbf{r} \rangle} \left((-1)^{|\alpha|} \, \mathbf{x}^{\mathbf{a} - (\mathbf{r} + \mathbf{1})} \prod_{i=1}^{n} (q_i(\mathbf{x}) / x_i^{r_i + 1})^{\alpha_i} \right) .$$

The introduction of the homogenizing parameter t organizes the computation of this Laurent series and the search for the desired coefficient, as evidenced in Algorithm (3.1) below. Keeping track of the homogeneity properties of the coefficients $B_j(\mathbf{x})$, also allows us to get more precise information about the global residues, such as Theorems (2.5) and (2.7).

(2.4) Remark. Note that (2.2) implies that $\mathbf{x}^{\mathbf{a}} \, B_m(\mathbf{x})$ may contain a term of the form $\alpha \, x_1^{-1} \ldots x_n^{-1}$ only if

$$\langle \mathbf{w}, \mathbf{a} \rangle - (m + d_{\mathbf{w}}) = - \sum w_i$$

that is, only if

$$m = \langle \mathbf{w}, \mathbf{a} - \mathbf{r} \rangle.$$

Combined with (1.12), this gives a simpler proof of the first statement in (1.17) in the case when $g_1, \ldots, g_n$ are a Gröbner basis with respect to some term order.

A similar argument combined with the recursive relations (1.16) makes it possible to improve on the vanishing statement in Corollary (1.18). Let $\mathcal{W}^*$ denote the polyhedral cone in $\mathbb{R}^n$ which is positively spanned by all lattice points of the form $\mathbf{r} + \mathbf{1} - \mathbf{b}$, where $\mathbf{x}^{\mathbf{b}}$ runs over all monomials appearing in the expansion of $g_1(\mathbf{x}) g_2(\mathbf{x}) \ldots g_n(\mathbf{x})$.

(2.5) Theorem. *The residue* $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$ *vanishes if* $\mathbf{a} - \mathbf{r}$ *lies outside the cone* $\mathcal{W}^*$*.*

Proof: The condition on $\mathbf{b}$ in the definition of $\mathcal{W}^*$ is equivalent to saying that $\mathbf{x}^{\mathbf{b}}$ appears in one of the coefficients $A_j(\mathbf{x})$, $j = 1, \ldots, d_{\mathbf{w}}$, in the expansion (1.10) of $\tilde{G}(t;\mathbf{x})$ as a polynomial in t. The recursion relations (1.16) imply that $B_m(\mathbf{x})$ consists of terms $k_\alpha \mathbf{x}^\alpha$ where the n-tuples $\alpha \in \mathbb{Z}^n$ are in the translated cone $-((\mathbf{r}+\mathbf{1}) + \mathcal{W}^*)$. Indeed, if $k_\alpha \mathbf{x}^\alpha$ is a term in $B_m(\mathbf{x})$, then (1.16) implies that for some $j = 1, \ldots, m$, the Laurent polynomial $A_j(\mathbf{x})\, B_{m-j}(\mathbf{x})$ contains a term of the form $c_\alpha \mathbf{x}^{\alpha+\mathbf{r}+\mathbf{1}}$ and therefore,

$$\alpha + \mathbf{r} + \mathbf{1} \quad = \quad \mathbf{b} + \beta,$$

where $\mathbf{x}^{\mathbf{b}}$ is a monomial in $A_j(\mathbf{x})$ and $\mathbf{x}^\beta$ is a monomial in $B_{m-j}(\mathbf{x})$. Then

$$\alpha + \mathbf{r} + \mathbf{1} \quad = \quad (\mathbf{b} - (\mathbf{r}+\mathbf{1})) + (\beta + \mathbf{r} + \mathbf{1})$$

and the assertion follows by induction on m.

Now, Theorem (2.3) implies that $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}}) = 0$ unless the coefficient $B_{s(\mathbf{a})}(\mathbf{x})$ contains a term which is a non-zero multiple of $x_1^{-(a_1+1)} \ldots x_n^{-(a_n+1)}$. But this is possible only if $-(\mathbf{a}+\mathbf{1}) \in -((\mathbf{r}+\mathbf{1}) + \mathcal{W}^*)$, or equivalently, if $\mathbf{a} - \mathbf{r} \in \mathcal{W}^*$. $\diamond$

Theorem (2.3) may also be used to study the dependence of $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$ on the coefficients of the polynomials $g_1, \ldots, g_n$. We write

$$g_i(\mathbf{x}) = x_i^{r_i+1} - \sum_{j=1}^{\nu_i} c_{ij}\, \mathbf{x}^{\mathbf{a}_{ij}}, \tag{2.6}$$

and let $\mathcal{W} \subset \mathbf{R}^n$ denote the closed convex cone of all vectors $\mathbf{w} \in \mathbf{R}^n$ such that

$$\langle \mathbf{w}, \mathbf{a}_{ij} \rangle \le \langle \mathbf{w}, (r_i+1)\, \mathbf{e}_i \rangle \; ; \quad \text{for all } i = 1 \ldots, n\,;\ j = 1, \ldots, \nu_i.$$

This cone is the polar dual of the cone $\mathcal{W}^*$ defined above. By assumption, $\mathcal{W}$ has non-empty interior. Note that $\mathcal{W}$ is the cone in the *Gröbner fan* of I corresponding to the given term order (see e.g. [14, §3.1]). We have the following result.

(2.7) Theorem. *The residue* $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$ *is a polynomial function in the coefficients* c_{ij}. *Its degree in the variable* c_{ij} *is bounded above by*

$$\min_{\mathbf{w} \in \mathcal{W}} \frac{\langle \mathbf{w}, \mathbf{a} - \mathbf{r} \rangle}{\langle \mathbf{w}, (r_i+1)\, \mathbf{e}_i - \mathbf{a}_{ij} \rangle} \tag{2.8}$$

and the total degree in the variables $\mathbf{c} = (c_{ij})$ *is bounded by*

$$\min_{\mathbf{w} \in int(\mathcal{W}) \cap \mathbb{Z}^n} \langle \mathbf{w}, \mathbf{a} - \mathbf{r} \rangle. \tag{2.9}$$

Proof: Given $g_i(\mathbf{x})$ as in (2.6), its weighted homogenization with respect to a weight $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n \cap \mathcal{W}$, is given by

$$\tilde{g}_i(t;\mathbf{x}) = x_i^{r_i+1} - \sum_{j=1}^{\nu_i} c_{ij}\, t^{\langle \mathbf{w}, (r_i+1)\,\mathbf{e}_i - \mathbf{a}_{ij}\rangle}\, \mathbf{x}^{\mathbf{a}_{ij}}$$

where $\mathbf{e}_i$ denotes the i-th unit vector. Set $\rho_{ij} = (r_i+1)\,\mathbf{e}_i - \mathbf{a}_{ij}$.

According to (2.3), $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$ equals the $x_1^{-1} \ldots x_n^{-1}$-coefficient of $\mathbf{x}^{\mathbf{a}}\, B_{s(\mathbf{a})}(\mathbf{x})$, which is equal to the coefficient of $t^{\langle \mathbf{w}, \mathbf{a}-\mathbf{r}\rangle}\, x_1^{-1} \ldots x_n^{-1}$ in the expansion of

$$\frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{x}^{\mathbf{r+1}}} \sum_{i_1,\ldots,i_n \geq 0} \left(\sum_{j=1}^{\nu_1} c_{1j}\, t^{\langle \mathbf{w}, \rho_{1j}\rangle}\, \mathbf{x}^{-\rho_{1j}} \right)^{i_1} \cdots \left(\sum_{j=1}^{\nu_n} c_{nj}\, t^{\langle \mathbf{w}, \rho_{nj}\rangle}\, \mathbf{x}^{-\rho_{nj}} \right)^{i_n} . \tag{2.10}$$

It is now clear that the residue depends polynomially on the coefficients c_{ij} and that, for a given choice of $\mathbf{w} \in \mathcal{W}$, its degree in c_{ij} is bounded above by $\langle \mathbf{w}, \mathbf{a}-\mathbf{r}\rangle / \langle \mathbf{w}, (r_i+1)\,\mathbf{e}_i - \mathbf{a}_{ij}\rangle$.

To prove the bound in (2.9) it suffices to observe that if a monomial $\mathbf{c}^{\mathbf{k}}$ appears in $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$, then by (2.10)

$$\sum_{i,j} k_{ij} \langle \mathbf{w}, \rho_{ij}\rangle \;=\; \langle \mathbf{w}, \mathbf{a} - \mathbf{r}\rangle.$$

As $\langle \mathbf{w}, \rho_{ij}\rangle \geq 1$ for $\mathbf{w} \in int(\mathcal{W})$ integral and all i, j, the claim follows. $\diamond$

(2.11) Remarks : i) By the Cauchy-Schwarz inequality, the bound (2.8) is minimized by vectors $\mathbf{w} \in \mathcal{W}$ which are as far as possible from $\mathbf{a} - \mathbf{r}$ and as close as possible to $(r_i+1)\,\mathbf{e}_i - \mathbf{a}_{ij}$.

ii) If we are given a system of polynomials $f_1, \ldots, f_n$ which satisfy (2.1), and supposing that only the constant coefficients are perturbed, say

$$g_i(\mathbf{x}) \;=\; f_i(\mathbf{x}) - c_i \qquad i = 1, \ldots, n\,,$$

then the bound in (2.9) implies the following: If a monomial $c_1^{k_1} \ldots c_n^{k_n}$ appears in $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$, then $\langle \mathbf{w}, \mathbf{k}\rangle \leq \langle \mathbf{w}, \mathbf{a} - \mathbf{r}\rangle$, that is, the weighted degree with respect to $\mathbf{w}$ of $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$ in $(c_1, \ldots, c_n)$ is bounded by $\langle \mathbf{w}, \mathbf{a} - \mathbf{r}\rangle$ for all integral vectors $\mathbf{w} \in int(\mathcal{W})$.

3. Deformation Algorithms for Global Residues

Let $\{g_1, \ldots, g_n\} \subset S$ be a Gröbner basis as in §2, and let $\mathbf{w} \in \mathbb{N}^n$ such that (2.1) holds. We have shown that for any polynomial

$$h \;=\; \sum_{\langle \mathbf{w}, \mathbf{a}\rangle \leq d} c_{\mathbf{a}}\, \mathbf{x}^{\mathbf{a}} \quad \in \quad S\,,$$

the computation of the global residue

$$\mathrm{Res}\left(\frac{h\,d\mathbf{x}}{g_1\cdots g_n}\right) = \sum_{\langle \mathbf{w},\mathbf{a}\rangle \le d} c_{\mathbf{a}}\,\mathrm{Res}\left(\frac{\mathbf{x}^{\mathbf{a}}\,d\mathbf{x}}{g_1\cdots g_n}\right)$$

can be reduced to a sum of residues (at the origin) with respect to the family of monomials $\{\,x_1^{r_1+1},\ldots,x_n^{r_n+1}\,\}$ and some of their powers. Theorem (2.3) gives the following algorithm for computing all global residues up to a given weighted degree. Note that Algorithm (3.1) respects possible sparsity of the input polynomials.

(3.1) Algorithm.

Input: $\mathbf{w} \in \mathbb{N}^n$, $g_1,\ldots,g_n \in S$ *satisfying (2.1), and* $d \in \mathbb{N}$.
Output: *The global residues* $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$, *for all* $\mathbf{a} \in \mathbb{N}^n$ *such that* $\langle \mathbf{w},\mathbf{a}\rangle \le d$.

Step 1: Define the weighted homogenizations $\tilde{g}_i(t;\mathbf{x}) = t^{w_i(r_i+1)}\, g_i(t^{-w_1}x_1,\ldots,t^{-w_n}x_n)$.
Step 2: If $\langle \mathbf{w},\mathbf{a}-\mathbf{r}\rangle \le 0$, then $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}}) = 0$.
Step 3: Set $d' = d - \langle \mathbf{w},\mathbf{r}\rangle$. Compute the Taylor polynomial $\sum_{j=0}^{d'} B_j(\mathbf{x})\,t^j$ of degree d' for $(\prod \tilde{g}_i(t;\mathbf{x}))^{-1}$.
Step 4: For each $\mathbf{a}$ such that $\langle \mathbf{w},\mathbf{r}\rangle \le \langle \mathbf{w},\mathbf{a}\rangle \le d$, find the coefficient of $\mathbf{x}^{-(\mathbf{a}+\mathbf{1})}$ in the Laurent polynomial $B_{\langle \mathbf{w},\mathbf{a}-\mathbf{r}\rangle}(\mathbf{x})$. It equals $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$.

The verification that $\mathcal{G} = \{g_1,\ldots,g_n\}$ is a Gröbner basis and, if so, the choice of a compatible weight $\mathbf{w} \in \mathbb{N}^n$ may be accomplished in at most $O(n^{n+2}\,m^{2n-1}\,d^{(2n+1)n})$ arithmetic operations, where m is a bound for the number of monomials in each g_i and d is a bound for their degrees. This was shown in [14, §3.2].

Naturally, Algorithm (3.1) will be most efficient when it is known *a priori* that $\mathcal{G}$ is a Gröbner basis and a "small" compatible weight is given. However, even if the given equations $\mathcal{G}$ are not a Gröbner basis for any $\mathbf{w}$, then Algorithm (3.1) still serves as a useful subroutine. To illustrate this, we describe a general procedure for computing the global residue associated to *any* complete intersection zero dimensional ideal:

(3.2) Algorithm.

Input: *Polynomials* $g_1,\ldots,g_n$ *in* S *whose ideal* I *is zero-dimensional.*
Output: *The global residue* $\mathrm{Res}_{\mathbf{g}}(h)$, *for any specified polynomial* $h \in S$.

Step 1: Choose a "good" term order "$\prec$".
Step 2: Starting with $\{g_1,\ldots,g_n\}$, run the Buchberger algorithm towards a Gröbner basis, until the current basis of I contains polynomials $f_1,\ldots,f_n$ with $\mathrm{in}_{\prec}(f_i) = x_i^{r_i+1}$.
Step 3: By keeping track of coefficients during Step 2, we obtain an $n\times n$-matrix $A = (A_{ij})$

of polynomials such that $f_i = \sum_{i=1}^n A_{ij}\, g_j$, for $i = 1, \ldots, n$.
Step 4: Compute the desired residue via the following formula:

$$\text{(3.3)} \qquad \operatorname{Res}\left(\frac{h\, d\mathbf{x}}{g_1 \cdots g_n}\right) \quad = \quad \operatorname{Res}\left(\frac{h\, \det(A)\, d\mathbf{x}}{f_1 \cdots f_n}\right).$$

(3.4) Remarks. (i) In Step 1 we may take "$\prec$" to be *optimal* in the precise sense of [14, §3.3]. Such a choice is possible at almost no extra computational cost if the *Gröbner basis detection* procedure of [14, §3.2] had been run beforehand to test applicability of (3.1).

(ii) The termination and correctness of Step 2 follows from $\dim(\text{in}_\prec(I)) = \dim(I) = 0$.

(iii) The correctness of (3.3) is just the Tranformation Law (0.6). In order the evaluate the right hand side of (3.3), we may use either Algorithm (3.1), in case many residues are desired, or the formula to be presented in (4.2) below, in case only one residue is desired.

(iv) As shown in [9, Theorem 3.3], it is possible to find polynomials $f_1, \ldots, f_n$ and A_{ij}, $1 \le i, j \le n$ with degrees bounded by $nd^{2n} + d^n + d$, where $d = \max(3, \max\{\deg(g_i)\})$.

(v) If the polynomials $g_1, \ldots, g_n$ satisfy (1.3) relative to some weight $\mathbf{w}$, but not necessarily (2.1), then the weighted version of the following argument due to Tsikh [25, II.8.3] describes how to find polynomials $f_1, \ldots, f_n$ in the ideal $I = \langle g_1, \ldots, g_n \rangle$ such that

$$\text{(3.5)} \qquad \text{in}_{\mathbf{w}}(f_i) \quad = \quad x_i^\rho \quad \text{for some } \rho.$$

Indeed, Macaulay's Theorem (1.21) implies that, if $\rho > d_{\mathbf{w}} - (w_1 + \ldots + w_n)$, then

$$x_i^\rho \;\in\; \text{in}_{\mathbf{w}}(I) \;=\; \langle p_1, \ldots, p_n \rangle.$$

This degree bound allows the use of linear algebra (over $\mathbf{K}$) to determine polynomials A_{ij}, $1 \le i, j \le n$, such that $x_i^\rho = \sum_{i=1}^n A_{ij}\, p_j$. Moreover, $\deg_{\mathbf{w}}(A_{ij}) = \rho\, w_i - \deg_{\mathbf{w}}(g_j)$. Let

$$f_i \;:=\; \sum_{i=1}^n A_{ij}\, g_j \;=\; x_i^\rho + \sum_{i=1}^n A_{ij}\, q_j$$

Since $\deg_{\mathbf{w}}\left(\sum A_{ij}\, q_j\right) < \rho\, w_i$, (3.5) holds, and (3.1) or (4.2) are applicable. Naturally, we may use the Buchberger Algorithm to organize the computation of the A_{ij}, which leads to a version of Algorithm (3.2) which is applied to the initial forms $p_1, \ldots, p_n$ rather than the entire equations $g_1, \ldots, g_n$. In summary, there is plenty of room for experimentation !

The fact that the coefficients $B_j(\mathbf{x})$ are weighted-homogeneous implies that we can fix the value of one of the variables, say $x_n = 1$, and obtain a non-homogeneous version of Algorithm (3.1): Set $x_n = 1$ everywhere, and, for each $\mathbf{a}$ with $\langle \mathbf{w}, \mathbf{a} \rangle \le d$, find the coefficient of $x_1^{-(a_1+1)} \ldots x_{n-1}^{-(a_{n-1}+1)}$ in $B_{\langle \mathbf{w}, \mathbf{a}-\mathbf{r}\rangle}(x_1, \ldots, x_{n-1}, 1)$. We note that other terms of the form $k\, x_1^{-(a_1+1)} \ldots x_{n-1}^{-(a_{n-1}+1)}$ may appear in different coefficients $B_j(x_1, \ldots, x_{n-1}, 1)$, but the homogenizing parameter t keeps track of the only one contributing to the residue.

In the classical case $\mathbf{w} = (1, \ldots, 1)$, we can apply an argument of Yuzhakov [27] to give the following geometric interpretation of (3.1). As in §1, we imbed $\mathbb{C}^n$ in $\mathbb{P}^n$; the n-form $\dfrac{\mathbf{x}^{\mathbf{a}}\, d\mathbf{x}}{g_1(\mathbf{x}) \ldots g_n(\mathbf{x})}$ may be extended to a meromorphic form Φ in $\mathbb{P}^n$, which has $\{t = 0\}$ as a polar divisor of order $\langle \mathbf{w}, \mathbf{a} - \mathbf{r}\rangle + 1$ if and only if $\langle \mathbf{w}, \mathbf{a} - \mathbf{r}\rangle \geq 0$. The global residue in $\mathbb{C}^n$ may now be expressed as a single residue at a point at ∞: if $P = (0, \ldots, 0, 1)$, then

$$\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}}) \quad = \quad (-1)^n \, \mathrm{Res}_P\, \Phi \quad =$$

$$\frac{1}{\langle \mathbf{w}, \mathbf{a}-\mathbf{r}\rangle !} \mathrm{Res}_0\!\left(x_1^{a_1} \ldots x_{n-1}^{a_{n-1}} \cdot \frac{\partial^{\langle \mathbf{w}, \mathbf{a}-\mathbf{r}\rangle}}{\partial t^{\langle \mathbf{w}, \mathbf{a}-\mathbf{r}\rangle}} \left(\frac{1}{\prod_{i=1}^n \tilde{g}_i(x_1, \ldots, x_{n-1}, 1; t)} \right)(0)\; dx_1 \wedge \ldots \wedge dx_{n-1} \right)$$

which, in turn, may be seen to equal the coefficient of $x_1^{-(a_1+1)} \ldots x_{n-1}^{-(a_{n-1}+1)}$ in $B_{\langle \mathbf{w}, \mathbf{a}-\mathbf{r}\rangle}(x_1, \ldots, x_{n-1}, 1)$. With suitable modifications, the same interpretation holds for arbitrary weights.

4. Normal Forms

In this section we give a formula expressing the coefficients of the (Gröbner basis) normal form in terms of global residues. In particular, the global residue of a polynomial equals the highest coefficient of its normal form (4.2). This leads to a fast algorithm for computing residues as well as traces over a zero-dimensional complete intersection.

Suppose $g_1, \ldots, g_n \in \mathbf{K}[\mathbf{x}]$ satisfy (2.1) with $\mathrm{in}_{\mathbf{w}}(g_i) = x_i^{r_i+1}$. Then $V = \mathbf{K}[\mathbf{x}]/I$ is an Artinian ring of $\mathbf{K}$-dimension $(r_1+1)(r_2+1)\cdots(r_n+1)$. Abbreviating

$$\mathcal{I} \quad := \quad \{\, \mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n \;:\; 0 \leq i_j \leq r_j\,,\; j = 1, \ldots, n \,\},$$

the set of monomials $\{\, \mathbf{x}^{\mathbf{i}} \;:\; \mathbf{i} \in \mathcal{I} \,\}$ is a $\mathbf{K}$-vectorspace basis of V. Every polynomial $h \in \mathbf{K}[\mathbf{x}]$ has a unique *normal form*

$$\mathcal{NF}(h) \quad = \quad \sum_{\mathbf{i} \in \mathcal{I}} c_{\mathbf{i}}(h)\, \mathbf{x}^{\mathbf{i}}. \tag{4.1}$$

The scalars $c_{\mathbf{i}}(h) \in \mathbf{K}$ are uniquely defined by the property that $h \equiv \mathcal{NF}(h) \pmod{I}$. They are computed using the division algorithm modulo the Gröbner basis $\{g_1, \ldots, g_n\}$.

(4.2) Lemma. *With the notation as above, every polynomial $h \in \mathbf{K}[\mathbf{x}]$ satisfies*

$$\mathrm{Res}\left(\frac{h\, d\mathbf{x}}{g_1 \cdots g_n} \right) \quad = \quad c_{r_1, \ldots, r_n}(h)$$

Proof: By linearity of the residue operator, $\mathrm{Res}_{\mathbf{g}}(h) = \sum_{\mathbf{i}\in\mathcal{I}} c_{\mathbf{i}} \,\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{i}})$. However, for $\mathbf{i} \in \mathcal{I}$, $\mathbf{i} \neq \mathbf{r}$, we have $\langle \mathbf{w}, \mathbf{i} - \mathbf{r}\rangle < 0$ and, consequently, $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{i}}) = 0$ by Corollary (1.18). On the other hand, it follows from Theorem (1.17) that

$$\mathrm{Res}\left(\frac{\mathbf{x}^{\mathbf{r}}\, d\mathbf{x}}{g_1 \cdots g_n}\right) \quad = \quad \mathrm{Res}\left(\frac{\mathbf{x}^{\mathbf{r}}\, d\mathbf{x}}{x_1^{r_1+1} \cdots x_n^{r_n+1}}\right) \quad = \quad 1.$$

This proves Lemma (4.2). $\diamond$

We now show that all coefficients of the normal form may be computed using residues:

(4.3) Theorem. *Fix an order on the index set $\mathcal{I}$, and let M be the symmetric $|\mathcal{I}| \times |\mathcal{I}|$-matrix defined by*

$$M_{\mathbf{ij}} \quad := \quad \mathrm{Res}\left(\frac{\mathbf{x}^{\mathbf{i}}\,\mathbf{x}^{\mathbf{j}}\, d\mathbf{x}}{g_1 \cdots g_n}\right), \quad \mathbf{i}, \mathbf{j} \in \mathcal{I}\,.$$

Then M is invertible and for any $h \in \mathbf{K}[\mathbf{x}]$,

$$\big(c_{\mathbf{i}}(h)\big)_{\mathbf{i}\in\mathcal{I}} \quad = \quad M^{-1} \cdot \Big(\mathrm{Res}\big(\frac{h\,\mathbf{x}^{\mathbf{j}}\, d\mathbf{x}}{g_1 \cdots g_n}\big)\Big)_{\mathbf{j}\in\mathcal{I}}$$

Proof: The Duality Theorem (0.7) implies that the symmetric bilinear form

$$V \times V \to \mathbf{K}\,, \qquad (h_1, h_2) \mapsto \mathrm{Res}\left(\frac{h_1\, h_2\, d\mathbf{x}}{g_1 \cdots g_n}\right) \tag{4.4}$$

is non-degenerate. The symmetric matrix M represents (4.4) relative to the basis $\{\, \mathbf{x}^{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\,\}$. Therefore M is non-singular. The second claim follows from the fact that

$$\mathrm{Res}\left(\frac{\mathbf{x}^{\mathbf{i}}\, h\, d\mathbf{x}}{g_1 \cdots g_n}\right) \;=\; \sum_{\mathbf{j}\in\mathcal{I}} c_{\mathbf{j}}(h)\, \mathrm{Res}\left(\frac{\mathbf{x}^{\mathbf{i}}\,\mathbf{x}^{\mathbf{j}}\, d\mathbf{x}}{g_1 \cdots g_n}\right) \;=\; \sum_{\mathbf{j}\in\mathcal{I}} M_{\mathbf{ij}}\, c_{\mathbf{j}}(h)\,. \;\diamond$$

(4.5) Remark. As observed in [3], if we choose the lexicographical order in $\mathcal{I}$, then the matrix M has the triangular form

$$M \quad = \quad \begin{pmatrix} & & & 1 \\ & 0 & \cdot & \\ & & \cdot & \\ & \cdot & & * \\ 1 & & & \end{pmatrix}$$

Consequently, $\det M = \pm 1$ and it is easy to compute the inverse of M.

We have seen in §0 that the Duality Law may also be interpreted as saying that the global residue operator $\mathrm{Res}_{\mathbf{g}}$ is a generator of V^* as a V-module. There is another element in this V-module which is of special interest in computational algebra, namely, the morphism $\mathrm{tr} \in V^*$ which assigns to a polynomial h the trace of the endomorphism of V given by multiplication by h. The *trace* can be computed by normal form reduction as follows:

$$\mathrm{tr}(h) \quad = \quad \sum_{\mathbf{i} \in \mathcal{I}} c_{\mathbf{i}}\big(h \cdot \mathbf{x}^{\mathbf{i}}\big). \tag{4.6}$$

On the other hand, it is known (see e.g. [23, Satz (4.2)]) that the trace may be expressed in terms of the global residue:

$$\mathrm{tr}(h) \quad = \quad \sum_{p \in Z(\mathbf{g})} \mu_{\mathbf{g}}(p) h(p) \quad = \quad \mathrm{Res}\left(\frac{h\, J_{\mathbf{g}}\, d\mathbf{x}}{g_1 \cdots g_n}\right). \tag{4.7}$$

This expression, together with (4.2), gives the following formula for computing the trace:

(4.8) Algorithm. *Compute the trace of an element $h \in V$ by* $\mathrm{tr}(h) = c_{r_1 \ldots r_n}(h \cdot J_{\mathbf{g}})$.

Thus, to find the trace of an element of V over $\mathbf{K}$, it suffices to run a single normal form reduction. We found Algorithm (4.8) to be quite efficient in practice. Additional speed can be gained by simple tricks, such as replacing $J_{\mathbf{g}}$ by $\mathcal{NF}(J_{\mathbf{g}})$ in Algorithm (4.8), and by storing previously computed normal forms of monomials.

From Theorem 2.7 we can derive bounds on the degree of the trace for $g_1, \ldots, g_n$ as in (2.6) which satisfy (2.1). Note that the corresponding cone $\mathcal{W}$ in the Gröbner fan has nonempty interior. For each value of the parameters c_{ij}, let $Z_{\mathbf{c}}$ denote the zero set of $g_1, \ldots, g_n$.

(4.9) Theorem. *For any $h \in K[\mathbf{x}]$, the parametric trace*

$$\mathrm{tr}(h)(\mathbf{c}) \quad = \quad \sum_{p \in Z_{\mathbf{c}}} \mu_{\mathbf{g}}(p)\, h(p)$$

is a polynomial function of $\mathbf{c} = (c_{ij})$ with degree bounded above by $\min_{\mathbf{w} \in int(\mathcal{W}) \cap \mathbb{Z}^n} \deg_{\mathbf{w}}(h)$.

Proof : Let $J_{\mathbf{g}}(\mathbf{x})$ be the Jacobian of $g_1, \ldots, g_n$ with respect to the variables $x_1, \ldots, x_n$. Given any polynomial h and $\mathbf{w} \in int(\mathcal{W}) \cap \mathbb{Z}^n$, we know by the second bound in Theorem 2.7 that the degree of $\mathrm{tr}(h)(\mathbf{c}) = \mathrm{Res}_{\mathbf{g}}(h \cdot J_{\mathbf{g}})$ in the $\mathbf{c}$ variables is bounded by $\deg_{\mathbf{w}}(h \cdot J_{\mathbf{g}}) - \langle \mathbf{w}, \mathbf{r} \rangle$. As $\deg_{\mathbf{w}}(J_{\mathbf{g}}) = \langle \mathbf{w}, \mathbf{r} \rangle$, the claim follows. $\diamond$

5. Real Roots, Degree, and Symmetric Polynomials

We present three applications of the computation of global residues: counting real roots using the trace form (following Pedersen-Roy-Szpirglas [22], Becker-Wörmann [4]), computing the degree of a polynomial map (following Eisenbud-Levine [12]), and evaluating elementary symmetric polynomials in a multivariate setting (following classical work of Junker [16]).

We assume as above that $g_1, \ldots, g_n \in \mathbf{K}[\mathbf{x}]$ satisfy (2.1) with $\mathrm{in}_{\mathbf{w}}(g_i) = x_i^{r_i+1}$, and again let $\mathcal{I} := \{ \mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n : 0 \le i_j \le r_j\,,\ j = 1, \ldots, n \}$. Suppose that $\mathbf{K}$ is a subfield of the real numbers $\mathbb{R}$ and let T be the symmetric $|\mathcal{I}| \times |\mathcal{I}|$-matrix T defined over $\mathbf{K}$ by

$$T_{\mathbf{ij}} \quad := \quad \mathrm{tr}\left(\mathbf{x}^{\mathbf{i}}\, \mathbf{x}^{\mathbf{j}}\right)\,, \quad \mathbf{i}, \mathbf{j} \in \mathcal{I}\,. \tag{5.1}$$

The following result is due to Becker-Wörmann and Pedersen-Roy-Szpirglas.

(5.2) Theorem. ([4],[22]) *The rank of T equals the number of distinct complex roots in $Z(\mathbf{g})$. The signature of T equals the number of distinct real roots in $Z(\mathbf{g}) \cap \mathbb{R}^n$.*

Recall that the *signature* of T equals the number of positive eigenvalues minus the number of negative eigenvalues (all eigenvalues of a real symmetric matrix are real). As is pointed out in [22, Prop. 2.8], the signature can be read off directly from (the number of sign variations in) the characteristic polynomial of T. A straightforward generalization of (5.2) states that, for any $h \in V$, the signature of the matrix $T^h = \left(\mathrm{tr}(\mathbf{x}^{\mathbf{i}}\, \mathbf{x}^{\mathbf{j}} \cdot h)\right)_{\mathbf{i},\mathbf{j} \in \mathcal{I}}$ equals the number of distinct real roots with $h > 0$ minus the number of distinct real roots with $h < 0$. Algorithms (4.8) and (3.1) provide subroutines for computing T and hence for counting real zeros of zero-dimensional complete intersections.

Viewing now $(g_1, \ldots, g_n)$ as a proper map $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$, its *degree* is defined as

$$\deg(\mathbf{g}) := \sum_{p \in \mathbf{g}^{-1}(q)} \deg_p(\mathbf{g})$$

where q is a regular value of $\mathbf{g}$ and $\deg_p(\mathbf{g})$ is ± 1 depending on whether $J_{\mathbf{g}}(p)$ is positive or negative. The degree is a topological invariant of $\mathbf{g}$.

Let M be the non-singular, symmetric matrix M defined, as in (4.3), by

$$M_{\mathbf{ij}} \quad := \quad \mathrm{Res}\left(\frac{\mathbf{x}^{\mathbf{i}}\, \mathbf{x}^{\mathbf{j}}\, d\mathbf{x}}{g_1 \cdots g_n}\right)\,, \quad \mathbf{i}, \mathbf{j} \in \mathcal{I}\,.$$

The following result is essentially contained in [12]; although the results there are local, the passage to the global situation may be done as in [22].

(5.3) Theorem. *The degree of $\mathbf{g}$ equals the signature of M.*

We may apply Algorithm (3.1) or Lemma (4.2) to compute the matrix M and, consequently, the degree of $\mathbf{g}$.

For our third application we need to review the concept of symmetric polynomials in a multivariate setting. This theory is classical (see Junker [16], who refers to even earlier work of MacMahon and Schläfli). It reappeared in the recent computer algebra literature in [21]. Let $A = (\alpha_{ij})$ be an $N \times n$-matrix of indeterminates over $\mathbf{K}$. The symmetric group S_N acts on the polynomial ring $\mathbf{K}[\alpha_{ij}]$ by permuting rows of A. We are interested in the invariant subring $\mathbf{K}[\alpha_{ij}]^{S_N}$, whose elements are called *symmetric polynomials*. It is known that $\mathbf{K}[\alpha_{ij}]^{S_N}$ is generated by symmetric polynomials of total degree at most N, but, in contrast to the familiar $n = 1$ case, this $\mathbf{K}$-algebra $\mathbf{K}[\alpha_{ij}]^{S_N}$ is not free for $n \geq 2$. An important set of generators are the *elementary symmetric polynomials* $e_{\mathbf{j}}(A)$, which are defined as the coefficients of the following auxiliary polynomial in $u_1, u_2, \ldots u_n$:

$$(5.4) \qquad \prod_{i=1}^{N} (1 + \alpha_{i1}u_1 + \alpha_{i2}u_2 + \cdots + \alpha_{in}u_n) \quad = \sum_{j_1+\ldots+j_n \leq N} e_{j_1,\ldots,j_n}(A) \cdot u_1^{j_1} u_2^{j_2} \cdots u_n^{j_n}$$

Another set of generators is given by the *power sums :*

$$(5.5) \quad h_{\mathbf{j}} \quad := \quad \sum_{i=1}^{N} \alpha_{i1}^{j_1} \alpha_{i2}^{j_2} \cdots \alpha_{in}^{j_n}, \qquad \text{for } \mathbf{j} = (j_1, j_2, \ldots, j_n) \in \mathbb{N}^n, \; j_1 + \ldots + j_n \leq N$$

Algorithms and formulas for writing the $e_{\mathbf{j}}$ in terms of the $h_{\mathbf{j}}$ and conversely are studied in detail by Junker [16]. One of his methods will be presented in (5.8)-(5.9) below.

Returning to our zero-dimensional complete intersection, let $N = \dim_{\mathbf{K}}(V)$ be the cardinality of the multiset $Z(\mathbf{g}) \subset \mathbb{C}^n$ (counting multiplicities). We fix any bijection between the rows of $A = (\alpha_{ij})$ and $Z(\mathbf{g})$. This defines a natural $\mathbf{K}$-algebra homomorphism

$$(5.6) \qquad \phi \; : \; \mathbf{K}[\alpha_{ij}]^{S_N} \quad \rightarrow \quad \mathbf{K},$$

where the indeterminate α_{ij} gets mapped to the j-th coordinate of the i-th point in $Z(\mathbf{g})$. Our objective is to evaluate the map ϕ using only operations in $\mathbf{K}$. In particular, we are interested in the problem of evaluating the elementary symmetric polynomials $e_{\mathbf{j}}$ under ϕ.

The punch line of our discussion is that it is easy to evaluate the power sums via the trace:

$$(5.7) \qquad \phi(h_{\mathbf{j}}) \quad = \quad \mathrm{tr}(\mathbf{x}^{\mathbf{j}}).$$

Thus to compute (5.7) we use Algorithm (4.8). We then proceed using the following method due to Junker and MacMahon. Consider the image of (5.4) under ϕ,

$$(5.8) \qquad R(\mathbf{u}) \quad = \quad \prod_{p \in Z(\mathbf{g})} (1 + p_1u_1 + \cdots + p_nu_n)^{\mu_{\mathbf{g}}(p)} \quad = \quad \sum_{\mathbf{j}} \phi(e_{\mathbf{j}}) \cdot \mathbf{u}^{\mathbf{j}}$$

The polynomial $R(\mathbf{u})$ is the *Chow form* of the zero-dimensional scheme defined by I. In computer algebra it is also known as the *U-resultant*. Following[16, pp. 233, Eq. (4)], the formal logarithm of (5.8) equals

$$\log\bigl(R(\mathbf{u})\bigr) \quad = \quad \sum_{d=1}^{\infty} \frac{(-1)^{d-1}}{d} \cdot \sum_{|\mathbf{j}|=d} \binom{d}{\mathbf{j}} \phi(h_{\mathbf{j}})\, \mathbf{u}^{\mathbf{j}}. \tag{5.9}$$

Here $|\mathbf{j}| = j_1 + \cdots + j_n$ and $\binom{d}{\mathbf{j}} = \frac{d!}{j_1! j_2! \cdots j_n!}$. Using (5.7) and (4.8), we can compute the formal power series (5.9) up to any desired degree d'. We then formally exponentiate this truncated series (using operations only in $\mathbf{K}$) to get the Chow form (5.8) up to the same degree d'. In order to determine (5.8) completely, which means to evaluate all elementary symmetric polynomials, it suffices to expand (5.9) up to degree $d' = N = \dim_{\mathbf{K}}(V)$.

6. An Example

In this section we apply our results and algorithms to the specific trivariate system:

$$g_1 \;=\; \underline{x_1^5} + x_2^3 + x_3^2 - 1, \quad g_2 \;=\; x_1^2 + \underline{x_2^2} + x_3 - 1, \quad g_3 \;=\; x_1^6 + x_2^5 + \underline{x_3^3} - 1. \tag{6.1}$$

This example is taken from [14, Example 3.1.2], where it served to illustrate the problem of *Gröbner Basis Detection*. Indeed, the polynomials g_1, g_2, g_3 are a Gröbner basis, namely, for the weight vector $\mathbf{w} = (3, 4, 7)$. With respect to these weights, the initial monomials are the pure powers underlined above. We see that, counting possible multiplicities, the set $Z(\mathbf{g})$ consists of 30 points in $\mathbb{C}^3$. Our basic problem is to evaluate the global residue

$$\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}}) \quad = \quad \mathrm{Res}\left(\frac{x_1^{a_1} x_2^{a_2} x_3^{a_3}\, d\mathbf{x}}{g_1(\mathbf{x}) g_2(\mathbf{x}) g_3(\mathbf{x})}\right), \tag{6.2}$$

for any non-negative integer vector $\mathbf{a} = (a_1, a_2, a_3)$.

It is interesting to compare the relative efficiency of Algorithm (3.1) and the Gröbner basis reduction method deduced from Lemma (4.2). In step 1 of Algorithm (3.1) we compute the weighted homogenizations

$$\tilde{g}_1 \;=\; x_1^5 + t^3 x_2^3 + t^7 x_3^2 - t^{15}, \quad \tilde{g}_2 \;=\; x_2^2 + t x_3 + t^2 x_1^2 - t^8, \quad \tilde{g}_3 \;=\; x_3^3 + t x_2^5 + t^3 x_1^6 - t^{21}.$$

We then consider the expression

$$\frac{1}{\tilde{g}_1(t; \mathbf{x}) \cdot \tilde{g}_2(t; \mathbf{x}) \cdot \tilde{g}_3(t; \mathbf{x})} \tag{6.3}$$

as a rational function in t, and we compute its Taylor expansion $\sum_{j=0}^{d} B_j(\mathbf{x})\, t^j$ up to some degree d which exceeds $3(a_1-4)+4(a_2-1)+7(a_3-2)$. Here the coefficients $B_j(\mathbf{x})$ are $\mathbf{w}$-homogeneous Laurent polynomials in x_1, x_2, x_3; for instance,

$$B_2(\mathbf{x}) \quad = \quad \frac{1}{x_1^{10}x_2^4} - \frac{1}{x_1^3x_2^4x_3^3} + \frac{x_2}{x_1^5x_3^5} + \frac{x_2^3}{x_1^{10}x_3^4} + \frac{x_3}{x_1^{15}x_2^2} + \frac{x_2^8}{x_1^5x_3^9} + \frac{1}{x_1^5x_2^6x_3}.$$

Now set $j = 3(a_1-4)+4(a_2-1)+7(a_3-2)$. The desired residue (6.2) equals the coefficient of $x_1^{-a_1-1}x_2^{-a_2-1}x_3^{-a_3-1}$ in the Laurent polynomial $B_j \in \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]$.

This Taylor expansion is a fairly space consuming process since the polynomials $B_j(\mathbf{x})$ grow quite large. This is witnessed by the following table, which shows the number of terms of $B_j(\mathbf{x})$ for some values of j between 2 and 40:

j :	2	5	10	15	20	25	30	35	40
# :	7	41	216	569	1102	1803	2682	3744	4964

On the other hand, the normal form method of Lemma (4.2) is quite efficient for evaluating individual residues. Let I denote the ideal in $\mathbb{Q}[x_1, x_2, x_3]$ generated by (6.1). The quotient ring $V = \mathbb{Q}[x_1, x_2, x_3]/I$ is a 30-dimensional $\mathbb{Q}$-vector space. Every element $h \in V$ is uniquely represented by its normal form $\mathcal{NF}(h)$ modulo the reduction relations:

$$(6.4) \qquad x_1^5 \longrightarrow -x_2^3 - x_3^2 + 1, \qquad x_2^2 \longrightarrow -x_1^2 - x_3 + 1, \qquad x_3^3 \longrightarrow -x_1^6 - x_2^5 + 1.$$

By Lemma (4.2), the residue (6.2) is equal to the coefficient of $x_1^4x_2x_3^2$ in $\mathcal{NF}(h)$.

For instance, for the Jacobian

$$J(\mathbf{x}) \;=\; \det\left(\frac{\partial f_i}{\partial x_j}\right) \;=\; 18x_1^5x_2^2 - 24x_1^5x_2x_3 - 25x_1^4x_2^4 + 30x_1^4x_2x_3^2 + 20x_1x_2^4x_3 - 18x_1x_2^2x_3^2$$

it takes 10 reductions modulo (6.4) to reach the normal form

$$\begin{aligned} \mathcal{NF}(J) \;=\; & \underline{30x_1^4x_2x_3^2} - 25x_1^4x_3^2 - 152x_1^4x_2 + 146x_1^4x_3 - 251x_1^3x_2x_3 + 83x_1^3x_3^2 + 16x_1^4 \\ & + 229x_1^3x_2 + 8x_1^3x_3 - 196x_1^2x_2x_3 + 226x_1^2x_3^2 - 114x_1x_2x_3^2 - 73x_1^3 + 240x_1^2x_2 \\ & + 34x_1^2x_3 + 254x_1x_2x_3 - 62x_1x_3^2 + 69x_2x_3^2 - 260x_1^2 - 140x_1x_2 - 78x_1x_3 \\ & + 108x_2x_3 - 49x_3^2 + 140x_1 - 177x_2 - 128x_3 + 177. \end{aligned}$$

Indeed, we see that the coefficient of $x_1^4x_2x_3^2$ equals $\mathrm{Res}_{\mathbf{g}}(J) = \mathrm{tr}(1) = \dim(V) = 30$. Here is a slightly more serious example: it takes 62 reductions modulo (6.4), running less than two minutes in MAPLE on a Sparc 2, in order to find the global residue

$$\mathrm{Res}_{\mathbf{g}}(x_1^{15}x_2^{15}x_3^{15}) \quad = \quad -258,756,707,658,424,020,014,953,731,203.$$

We made the observation that the efficiency of the two methods is comparable when computing all residues of the form $\mathrm{Res}_{\mathbf{g}}(\mathbf{x}^{\mathbf{a}})$ with $\langle \mathbf{w}, \mathbf{a} \rangle \leq d$ for some fixed d. This is the case, for example, in the computation of the matrix M defined in §4. This is a symmetric, 30×30-matrix whose computation using Algorithm (3.1) requires the knowledge of $B_j(\mathbf{x})$ for $j \leq 30$. Using MAPLE on a Sparc 2 these may be obtained in 321 seconds. It takes an additional 247 seconds to read off the desired 465 coefficients. On the other hand, it takes 324 seconds to build up the matrix M using Lemma (4.2). The signature of M is zero, and hence so is the degree of the map $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^3$ by Theorem (5.3).

For further combinatorial analysis we may wish to compute the two polyhedral cones in Section 2. We first obtain the 3-dimensional quadrangular cone

$$\begin{aligned} \mathcal{W} &= \big\{ (w_1, w_2, w_3) \in \mathbb{R}^3 \ : \ 5w_1 \geq 2w_3,\ 2w_2 \geq w_3,\ w_3 \geq 2w_1,\ 3w_3 \geq 5w_2 \big\} \\ &= \mathrm{pos}\big\{(4,5,10),\ (1,1,2),\ (5,6,10),\ (2,3,5)\big\}. \end{aligned}$$

The interior of $\mathcal{W}$ consists of all weight vectors which select the underlined monomials in (6.1) to be initial. The cone polar to $\mathcal{W}$ equals

$$\begin{aligned} \mathcal{W}^* &= \mathrm{pos}\big\{(5,0,-2),\ (0,2,-1),\ (-2,0,1),\ (0,-5,3)\big\} \\ &= \big\{ (a_1, a_2, a_3) \in R^3 \ : \ 4a_1 + 5a_2 + 10a_3 \geq 0,\ a_1 + a_2 + 2a_3 \geq 0, \\ &\qquad\qquad 5a_1 + 6a_2 + 10a_3 \geq 0,\ 2a_1 + 3a_2 + 5a_3 \geq 0 \big\}. \end{aligned}$$

By Theorem (2.5), the residue (6.2) vanishes whenever

$$(a_1, a_2, a_3) \quad \notin \quad (4,1,2) + \mathcal{W}^*, \qquad \text{or equivalently,}$$

$$4a_1{+}5a_2{+}10a_3 < 41 \ \text{ or } \ a_1{+}a_2{+}2a_3 < 9 \ \text{ or } \ 5a_1{+}6a_2{+}10a_3 < 46 \ \text{ or } \ 2a_1{+}3a_2{+}5a_3 < 21.$$

For instance, $(6,1,1)$ satisfies the first inequality and therefore $\mathrm{Res}_{\mathbf{g}}(x_1^6 x_2 x_3) = 0$.

In Section 4 we have shown that the trace $\mathrm{tr}(h)$ of an element h in $V = \mathbb{Q}[\mathbf{x}]/I$ can be computed easily as the coefficient of $x_1^4 x_2 x_3^2$ in $\mathcal{NF}(h \cdot J)$. Using this technique, let us now analyze the zero set $Z(\mathbf{g})$ with respect to multiple roots, real roots, etc... We compute the symmetric, integer 30×30-matrix representing the trace form T as in (4.8). The largest entry in T appears in the lower right corner:

$$\mathrm{tr}(x_1^4 x_2 x_3^2 \cdot x_1^4 x_2 x_3^2) \quad = \quad \mathrm{Res}_{\mathbf{g}}\big(x_1^8 x_2^2 x_3^4 \cdot J(\mathbf{x})\big) \quad = \quad 16,049,138,278.$$

The rank of the matrix T equals 20. By Theorem (5.2), this is the number of *distinct* roots of $\mathbf{g}$. The characteristic polynomial of T has 13 positive real roots and 7 negative real roots. Therefore the signature of T equals 6, and this is the number of distinct *real* roots of $\mathbf{g}$. It turns out that there are four rational roots, and they account for all multiplicities:

the root $(1,0,0)$ has multiplicity 3, the root $(0,1,0)$ has multiplicity 4, the root $(0,0,1)$ has multiplicity 6, while the root $(-1,1,-1)$ is simple. The remaining 16 roots, two real and 14 imaginary, are all simple and they are conjugates over $\mathbb{Q}$.

We finally come to the problem of computing the Chow form

$$R(u_1,u_2,u_3) \quad = \prod_{(\alpha_1,\alpha_2,\alpha_3)\in Z(\mathbf{g})} (1+\alpha_1 u_1+\alpha_2 u_2+\alpha_3 u_3)^{\mu_{\mathbf{g}}(\alpha)}.$$

Note that each of the three non-simple roots appears with its multiplicity in this product. The $\binom{33}{3} = 5,456$ rational coefficients of $R(u_1,u_2,u_3)$ are the values of the elementary symmetric polynomials at the roots of $\mathbf{g} = (g_1,g_2,g_3)$. Following (5.9), (5.7) and using Algorithm (4.8), we compute the following formal power series up to a chosen degree:

$$\begin{aligned}
\log\bigl(R(u_1,u_2,u_3)\bigr) \quad &= \quad \mathrm{tr}(x_1)u_1+\mathrm{tr}(x_2)u_2+\mathrm{tr}(x_3)u_3-\frac{1}{2}\cdot\bigl(\mathrm{tr}(x_1^2)u_1^2+2\mathrm{tr}(x_1x_2)u_1u_2\\
&+2\mathrm{tr}(x_1x_3)u_1u_3+\mathrm{tr}(x_2^2)u_2^2+2\mathrm{tr}(x_2x_3)u_2u_3+\mathrm{tr}(x_3^2)u_3^2\bigr)\ +\frac{1}{3}\cdot\bigl(\mathrm{tr}(x_1^3)u_1^3+\ldots\\
&= \quad 5u_2-5u_3+37u_1u_2-121u_1u_3-\frac{35}{2}u_2^2+106u_2u_3-\frac{485}{2}u_3^2+17u_1^3-74u_1^2u_2\\
&+177u_1^2u_3-172u_1u_2^2+536u_1u_2u_3-686u_1u_3^2+\frac{185}{3}u_2^3-667u_2^2u_3+1084u_2u_3^2\ +\ \ldots
\end{aligned}$$

By formally exponentiating this series, we obtain the Chow form

$$\begin{aligned}
R(u_1,u_2,u_3) \ &= \ 1\ +\ 5u_2-5u_3\ +\ 37u_1u_2-121u_1u_3-5u_2^2+81u_2u_3-230u_3^2\ +\ 17u_1^3\\
&-74u_1^2u_2+177u_1^2u_3+13u_1u_2^2-254u_1u_2u_3-81u_1u_3^2-5u_2^3-112u_2^2u_3-596u_2u_3^2\ +\ \ldots
\end{aligned}$$

and hence all elementary symmetric polynomials. For instance, we see that $\sum \alpha_1\beta_2\gamma_3 = -254$, where the sum is taken over all triples of roots $(\alpha_1,\alpha_2,\alpha_3)$, $(\beta_1,\beta_2,\beta_3)$, $(\gamma_1,\gamma_2,\gamma_3) \in Z(\mathbf{g})$.

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Note added in proof: During the MEGA 94 meeting we became aware of the paper: [M. Kreuzer and E. Kunz: Traces in strict Frobenius algebras and strict complete intersections. *J. reine angew. Math.* **381** (1987), 181-204]. Our assumption (1.3) is equivalent, by their Proposition (4.2), to the statement that the K-algebra $V = K[\mathbf{x}]/I$ is a strict complete intersection. Consequently, Theorem (1.17), in the case $s(\mathbf{a}) \leq 0$ (in particular the Euler-Jacobi Theorem (Corollary (1.18)) is contained in their Corollary (4.6) and Theorem (4.8) .

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