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Abstract

We study the A-discriminant of toric varieties. We reduce its computation to the case of irreducible configurations and describe its behavior under specialization of some of the variables to zero. We give characterizations of dual defect toric varieties in terms of their Gale dual and classify dual defect toric varieties of codimension less than or equal to four.

Key words: Sparse discriminant, dual defect varieties.
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1. Introduction

In this paper we will study properties of the *sparse* or A-discriminant. Given a configuration $A = \{a_1, \ldots, a_n\}$ of n points in $\mathbb{Z}^d$ we may construct an ideal $I_A \subset \mathbb{C}[x_1, \ldots, x_n]$ and, if I_A is homogeneous, a projective toric variety $X_A \subset \mathbb{P}^{n-1}$. The dual variety X_A^* is, by definition, the Zariski closure of the locus of hyperplanes in $(\mathbb{P}^{n-1})^*$ which are tangent

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to X_A at a smooth point. Generically, X_A^* is a hypersurface and its defining equation $D_A(x)$, suitably normalized, is called the A-discriminant. If X_A^* has codimension greater than one then X_A is called a dual defect variety and we define $D_A = 1$.

The A-discriminant generalizes the classical notion of the discriminant of univariate polynomials. It was introduced by Gel'fand, Kapranov, and Zelevinsky (their book (Gel′fand et al., 1994) serves as the basic reference of our work) and it arises naturally in a variety of contexts including the study of hypergeometric functions (Gel′fand et al., 1989; Cattani et al., 2001; Cattani and Dickenstein, 2004) and in some recent formulations of mirror duality (Batyrev and Materov, 2002).

When studying the A-discriminant it is often convenient to consider a Gale dual of A. This is a configuration $B = \{b_1, \ldots, b_n\} \subset \mathbb{Z}^m$, where m is the codimension of X_A in $\mathbb{P}^{n-1}$. The configuration B, and by extension A, is said to be irreducible if no two vectors in B lie on the same line. Equivalently, if the matroid $\mathcal{M}_B = (B, \mathcal{I})$ defined by the family, $\mathcal{I}$, of linearly independent subsets of B is *simple*. In Theorem 11, we prove a univariate resultant formula which reduces the computation of the A-discriminant to the case of irreducible configurations. This implies, in particular, that the Newton polytope of the discriminant is unchanged, up to affine isomorphism, if we replace B by the configuration obtained by adding up all subsets of collinear vectors. This generalizes a result of Dickenstein and Sturmfels (2002) for codimension-two configurations. We point out that, in their case, this is a consequence of a complete description of the Newton polytope of the discriminant.

In the study of rational hypergeometric functions, one is interested in understanding the behavior of the A-discriminant when specializing a variable x_j to zero and its relation to the discriminant of the configuration obtained by removing the corresponding point a_j from A. Theorem 15 generalizes the known results in this direction (Cattani et al. (2001, Lemma 3.2); Cattani and Dickenstein (2004, Lemma 3.2)). This specialization result was first proved by the first author in his PhD dissertation (Curran, 2005), using the theory of coherent polyhedral subdivisions. We give a greatly simplified proof in §4, where we derive the specialization theorem as a corollary of our resultant formula.

Using tropical geometry methods, Dickenstein, Feitchner, and Sturmfels have been able to compute the dimension of the dual of a projective toric variety X_A and this, in particular, makes it possible to decide if a given toric variety is dual defect, i.e. if the dual variety has codimension greater than one. Their formula (Dickenstein et al., 2005, Corollary 4.5) involves the configuration A and the geometric lattice, $\mathcal{S}(A)$, whose elements are the supports, ordered by inclusion, of the vectors in $\ker(A)$. The information contained in $\mathcal{S}(A)$ is essentially the same as that contained in a family of flats in $\mathcal{M}_B$, for a Gale dual configuration B of A. Thus, one could say that the formula by Dickenstein, Feitchner, and Sturmfels involves both A and B information. In Theorem 18, we use Theorem 15 to show that we can decide whether a configuration is dual defect purely in terms of certain *non-splitting* flags of flats in the matroid $\mathcal{M}_B$. In Theorem 25 we obtain a decomposition of the Gale dual configuration of a toric variety and give, in terms of this decomposition, a sufficient condition for the variety to be dual defect. Although we believe this condition to also be necessary, we are not able to prove it at this point.

Dual defect varieties have been extensively studied: Beltrametti et al. (1992); Di Rocco (2004); Ein (1985, 1986); Lanteri and Struppa (1987). In particular, Dickenstein and Sturmfels have classified codimension-two dual defect varieties (Dickenstein and Sturmfels, 2002) and, by completely different methods, Di Rocco (2004) has classified dual defect projective embeddings of smooth toric varieties in terms of their associated polytopes. We give a complete classification of of dual defect toric varieties of codimension less than or equal to four in terms of the Gale duals. This implies, in particular, that in these cases the condition in Theorem 25 is necessary and sufficient. We conclude §5 by comparing Di Rocco's list, for codimension less than or equal to four, with our classification.

2. Preliminaries

We begin by setting up the notation to be used throughout. We will denote by A a $d \times n$ integer matrix or, equivalently, the configuration $A = \{a_1, \ldots, a_n\}$ of n points in $\mathbb{Z}^d$ defined by the columns of A. We will always assume that A has rank d and set $m := n - d$, the *codimension* of A. Viewing A as a map $\mathbb{Z}^n \to \mathbb{Z}^d$ we denote by $\mathcal{L}_A \subset \mathbb{Z}^n$ the kernel of A. $\mathcal{L}_A$ is a lattice of rank m. For any $u \in \mathbb{Z}^n$ we write $u = u_+ - u_-$, where $u_+, u_- \in \mathbb{N}^n$ have disjoint support. Let $I_A \subset \mathbb{C}[x_1, \ldots, x_n]$ be the lattice ideal defined by $\mathcal{L}_A$, that is the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ generated by all binomials of the form: $x^{u_+} - x^{u_-}$, where $u \in \mathcal{L}_A$. Note that for any vector $w \in \mathbb{Q}^d$ in the $\mathbb{Q}$-rowspan of A we have

$$\langle w, u_+ \rangle \ = \ \langle w, u_- \rangle$$

for all $u \in \mathcal{L}_A$ and, hence, I_A is w-weighted homogeneous.

Definition 1 *We will say that A is* homogeneous *or nonconfluent if the vector* $(1, \ldots, 1)$ *is in the $\mathbb{Q}$-rowspan of A.*

Note that in terms of the configuration in $\mathbb{Z}^d$, A is homogeneous if and only if all the points lie in a rational hyperplane not containing the origin. Throughout this paper we will be interested in properties of homogeneous configurations A which depend only on the $\mathbb{Q}$-rowspan of A. Thus, in those cases we may assume without loss of generality that the first row of A is $(1, \ldots, 1)$. We shall then say that A is in *standard form*.

Given a homogeneous configuration A, let $X_A := \mathbb{V}(I_A) \subset \mathbb{P}^{n-1}$ be the projective (though not necessarily normal) variety defined by the homogeneous ideal I_A. The map

$$t \in (\mathbb{C}^*)^d \mapsto (t^{a_1} : \cdots : t^{a_d}) \in X_A \subset \mathbb{P}^{n-1}$$

defines a torus embedding which makes X_A into a toric variety of dimension $d - 1$. Generically, its *dual variety* X_A^* is an irreducible hypersurface defined over $\mathbb{Z}$. Its normalized defining polynomial $D_A(x_1, \ldots, x_n)$ is called the sparse or A-discriminant. It is well-defined up to sign. If the dual variety X_A^* has codimension greater than one, then we define $D_A = 1$ and refer to X_A as a *dual defect variety* and to A as a *dual defect configuration.* Note that X_A, and consequently X_A^*, depend only on the rowspan of A. Indeed, it is shown in (Gel′fand et al., 1994, Proposition 1.2, Chapter 5) that X_A depends only on the affine geometry of the set $A \subset \mathbb{Z}^d$.

Alternatively, given a configuration $A = \{a_1, \ldots, a_n\}$ we consider the generic Laurent polynomial supported on A:

$$f_A(x; t) \ := \ \sum_{i=1}^{n} x_i t^{a_i} \, , \tag{1}$$

which, for a choice of coefficients $x_i \in \mathbb{C}$, we view as a regular funcion on the torus $(\mathbb{C}^*)^d$. Then, the discriminant is an irreducible polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ which vanishes whenever the specialization of f_A has a multiple root in the torus; i.e. f_A and all its derivatives $\partial f_A/\partial t_i$ vanishing simultaneously at some point in $t \in (\mathbb{C}^*)^d$. Note that when A is in standard form:

$$t_1 \frac{\partial f_A}{\partial t_1} \ = \ f_A \tag{2}$$

and, consequently, f_A and $\partial f_A/\partial t_1$ have the same zeroes on $(\mathbb{C}^*)^d$. Let $R := \mathbb{C}[x][t^{\pm 1}]$ be the ring of Laurent polynomials in t whose coefficients are polynomials in x, and denote by $J(f_A)$ the ideal in R generated by f_A and its partial derivatives with respect to the t variables. Set $\mathbb{V}_A := \mathbb{V}(J(f_A)) \subset \mathbb{C}^n_x \times (\mathbb{C}^*)^d_t$. Let ∇_A be the Zariski closure of the projection of $\mathbb{V}(J(f_A))$ in $\mathbb{C}^n_x$, then if ∇_A is a hypersurface, $\nabla_A = \{x : D_A(x) = 0\}$. If A is homogeneous and X_A is not dual defect then ∇_A is the cone over X_A^*.

We recall that if $\nu_1, \ldots, \nu_m \in \mathbb{Z}^n$ are a $\mathbb{Z}$-basis of $\mathcal{L}_A$, then the $n \times m$ matrix B, whose columns are $\nu_1, \ldots, \nu_m$ is called a *Gale dual* of A. The same name is used to denote the configuration $\{b_1, \ldots, b_n\} \subset \mathbb{Z}^m$ of row vectors of B. Gale duals are defined up to $GL(m, \mathbb{Z})$-action. We will also consider $n \times m$ integer matrices C, whose columns $\xi_1, \ldots, \xi_m \in \mathbb{Z}^n$ are a $\mathbb{Q}$-basis of $\mathcal{L}_A \otimes_{\mathbb{Z}} \mathbb{Q}$. In that case we will say that C is a $\mathbb{Q}$-dual of A. For any $n \times m$ integer matrix C of rank m we will denote by q the greatest common divisor of all maximal minors of C and call it the *index* of C. Indeed, q is the index of the lattice generated by the row vectors of C, $c_1, \ldots, c_n$, in $\mathbb{Z}^m$. An $n \times d$ integer matrix A of rank d is said to be a dual configuration of C if $A \cdot C = 0$. Note that C is a Gale dual of A if and only if it has index 1 and that, if A is dual to C, then A is homogeneous if and only if the row vectors of C add up to zero. Such a configuration C will also be called homogeneous. If $c_j = 0$ for some j, then any dual configuration A is a *pyramid*, i.e. all the vectors $a_i, i \neq j$ are contained in a hyperplane. It is easy to check that in that case X_A is dual defect.

Given an $n \times m$ integer matrix C of rank m we will denote by $\mathcal{L}_C$ the sublattice of $\mathbb{Z}^n$ generated by the columns of C and by $J_C \subset \mathbb{C}[x_1, \ldots, x_n]$ the lattice ideal defined by $\mathcal{L}_C$. If C is a Gale dual of A, then $\mathcal{L}_C = \mathcal{L}_A$ and $I_A = J_C$ is a prime ideal. In any case, if $\xi_1, \ldots, \xi_m$ are the columns of C and we denote by J_ξ the ideal

$$J_\xi = \langle x^{\xi_1^+} - x^{\xi_1^-}, \ldots, x^{\xi_m^+} - x^{\xi_m^-} \rangle,$$

then the lattice ideal J_C is the saturation $J_C \ = \ J_\xi \colon (x_1 \cdots x_m)^\infty$.

If C is homogeneous of index q then the variety $X_C := \mathbb{V}(I_C) \subset \mathbb{P}^{n-1}$ has q irreducible components and they are all torus translates of $X_A = \mathbb{V}(I_A)$, where A is a dual of C. Similarly, the dual variety X_C^* is a union of finitely many torus translates of X_A^*. In particular if one of them is a hypersurface so is the other. In that case, we denote by $D_C \in \mathbb{C}[x_1, \ldots, x_n]$ the defining equation suitably normalized. Moreover, there exist $\theta^1, \ldots \theta^q \in (\mathbb{C}^*)^n$ such that

$$D_C(x) \ = \ \prod_{j=1}^{q} D_A(\theta^j * x)\,, \tag{3}$$

where $*$ denotes component-wise multiplication. We will say that C is dual defect if and only if A is dual defect.

The computation of the A-discriminant is well-known in the case of codimension-one homogeneous configurations. Let $B = (b_1, \ldots, b_n)^T$, $b_i \in \mathbb{Z}$, be a Gale dual of A. Reordering the columns of A, if necessary, we may assume without loss of generality that $b_i > 0$ for $i = 1, \ldots, r$ and $b_j < 0$ for $r + 1 \leq j \leq n$. Set

$$p \ = \ b_1 + \cdots + b_r \ = \ -(b_{r+1} + \cdots + b_n)\,.$$

Then, up to an integer factor

$$D_A \ = \ \prod_{j=r+1}^{n} |b_j|^{|b_j|} \ \prod_{i=1}^{r} x_i^{b_i} \ - \ (-1)^p \ \prod_{i=1}^{r} b_i^{b_i} \ \prod_{j=r+1}^{n} x_j^{|b_j|} \tag{4}$$

We recall the notion of *Horn uniformization* from (Gel′fand et al., 1994, Chapter 9). Although in Gel′fand et al. (1994) this is done only in the case of saturated lattice ideals, the generalization to arbitrary lattice ideals is straightforward. Let $C = (c_{ij})$ be an integer matrix whose rows add up to zero, the *Horn map* $h_C : \mathbb{P}^{m-1} \to (\mathbb{C}^*)^m$ is defined by the formula $h_C(\zeta_1 : \cdots : \zeta_m) = (\Psi_1(\zeta), \ldots, \Psi_m(\zeta))$, where

$$\Psi_k(\zeta_1 : \cdots : \zeta_m) = \prod_{i=1}^{n} (c_{i1}\zeta_1 + \cdots + c_{im}\zeta_m)^{c_{ik}}. \tag{5}$$

We also define $T_C : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^m$ by $T_C(x) := (x^{\xi_1}, \ldots, x^{\xi_m})$, where $\xi_1, \ldots, \xi_m$ are the column vectors of C, and set $\widetilde{\nabla}_C := h_C(\mathbb{P}^{m-1}) \subset (\mathbb{C}^*)^m$.

The following result is proved in (Gel′fand et al., 1994, Chapter 9, Theorem 3.3a) for the case of Gale duals. Its extension to $\mathbb{Q}$-duals is straightforward.

Theorem 2 *Let $A \subset \mathbb{Z}^n$ be a homogeneous configuration and $C \in \mathbb{Z}^{n \times m}$ a $\mathbb{Q}$-dual of A. Then if X_A^* is a hypersurface, so is $\widetilde{\nabla}_C$. Moreover,*

$$T_C^{-1}(\widetilde{\nabla}_C) \ = \ \nabla_C \cap (\mathbb{C}^*)^n. \tag{6}$$

3. Discriminants and Splitting Lines

In this section we will study the effect on the A-discriminant of removing from the Gale dual configuration B a set of collinear vectors which add up to zero. We will show that this operation preserves the dual defect property and the Newton polytope of the discriminant. Moreover, there is a resultant formula relating the two discriminants. We shall assume throughout this section that our configurations are homogeneous.

Theorem 3 *Let A be a configuration in $\mathbb{Z}^n$ which is not a pyramid, and $B \subset \mathbb{Z}^m$ a Gale dual. Suppose we can decompose B as*

$$B \ = \ C_1 \cup C_2\,,$$

where C_1 and C_2 are homogeneous configurations, C_1 is of rank m, and C_2 is of rank 1. Let A_1 be a dual of C_1. Then $\operatorname{codim}(\nabla_A) = \operatorname{codim}(\nabla_{A_1})$. In particular, A is dual defect if and only if A_1 is dual defect.

Proof. Let A_2 be a dual of C_2. We may assume without loss of generality that A_1 and A_2 are in standard form. We may also assume that $C_1 = \{b_1, \ldots, b_r\}$ and $C_2 = \{b_{r+1}, \ldots, b_n\}$. Since the vectors in C_1 span $\mathbb{Z}^m$ over $\mathbb{Q}$, there is a $\mathbb{Z}$-relation

$$\sum_{i=1}^{r} \gamma_i b_i \ + \sum_{j=r+1}^{n} \mu_j b_j = 0 \quad \text{with} \quad \sum_{j=r+1}^{n} \mu_j b_j \neq 0. \tag{7}$$

It is then easy to check that the matrix

$$A \ = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \\ \hline \gamma_1 \ \cdots \ \gamma_r & \mu_{r+1} \ \cdots \ \mu_n \end{array} \right) \tag{8}$$

is dual to B and, consequently, we may assume that A agrees with the matrix (8). We can write $d = d_1 + d_2 + 1$, where: $d_1 = r - m$ and $d_2 = n - r - 1$ and view A_1, A_2 as configurations in $\mathbb{Z}^{d_1}$, $\mathbb{Z}^{d_2}$, respectively. We let $t = (t_1, \ldots, t_{d_1})$, $s = (s_1, \ldots, s_{d_2})$, $x = (x_1, \ldots, x_r)$, and $y = (y_{r+1}, \ldots, y_n)$. Given $u \in \mathbb{C}^*$, we let $u^\gamma * x = (u^{\gamma_1} x_1, \ldots, u^{\gamma_r} x_r)$. We define $u^\mu * y$ in an analogous way.

If A is as in (8), $f_A(x, y; t, s, u) \ = \ f_{A_1}(u^\gamma * x; t) + f_{A_2}(u^\mu * y; s)$ and, therefore,

$$J(f_A) \ = \ \langle J(f_{A_1}(u^\gamma * x; t)), J(f_{A_2}(u^\mu * y; s)), \partial f_A / \partial u \rangle \, .$$

In particular, we get a map $\Phi : \mathbb{V}_A \to \mathbb{V}_{A_1}$ given by $\Phi(x, y, t, s, u) \ = \ (u^\gamma * x, t)$. We also define $\Psi : \mathbb{V}_A \to \mathbb{C}^* \times \nabla_A$ by $\Psi(x, y, t, s, u) \ = \ (u, x, y)$. Let $Z = Im(\Psi) \subset \mathbb{C}^* \times \nabla_A$, and let $\Pi : \mathbb{V}_{A_1} \to \nabla_{A_1}$ denote the natural projection. Finally, define $\phi : Z \to \nabla_{A_1}$ by $\phi(u, x, y) = u^\gamma * x$. Then the diagram

$$\begin{array}{ccc} \mathbb{V}_A & \xrightarrow{\Phi} & \mathbb{V}_{A_1} \\ {\scriptstyle \Psi}\big\downarrow & & \big\downarrow{\scriptstyle \Pi} \\ Z & \xrightarrow{\phi} & \nabla_{A_1} \end{array} \tag{9}$$

commutes. We note that $\dim Z = \dim \nabla_A$. Indeed, the natural projection $p\colon Z \to \nabla_A$ has finite fibers since, for any $(u, x, y) \in Z$, $u^\mu * y \in \nabla_{A_2}$. But A_2 is a codimension-one configuration and therefore its discriminant is given by (4). Hence, u must satisfy an equation of the form $u^q \ = \ c\, y^\alpha$, for some $q \in \mathbb{Z}$, $c \in \mathbb{Q}$, and $\alpha \in \mathbb{Z}^{n-r}$.

We now claim that the conclusion of Theorem 3 will follow from Lemma 5, proved below, which asserts that ϕ is generically surjective with fibers of dimension $n-r$. Indeed, we have $\dim \nabla_A = \dim Z = \dim \nabla_{A_1} + n - r$ and, consequently,

$$\operatorname{codim}(\nabla_A) = n - \dim \nabla_A = r - \dim \nabla_{A_1} = \operatorname{codim}(\nabla_{A_1})\, .$$

□

Before proving the statements on generic surjectivity and fiber dimension, we prove an auxiliary Lemma.

Lemma 4 *Let A be a $d \times n$ integer matrix of rank d with Gale dual $B = \{b_1, \ldots, b_n\}$. Let $x \in \mathbb{C}^n, t \in (\mathbb{C}^*)^d$. Suppose that for some $\Theta \in \mathbb{Z}^n$, $\mathbb{V}_A \subset \{f_A(\Theta * x; t) = 0\}$. Then $\Theta_1 b_1 + \cdots + \Theta_n b_n = 0$.*

Proof. Let $t_0 = (1, \ldots, 1) \in (\mathbb{C}^*)^d$. Then, the set $\{x \in \mathbb{C}^n : (x, t_0) \in \mathbb{V}_A\}$ agrees with the conormal space of X_A at the point $[1 : \cdots : 1] \in X_A \subset \mathbb{P}^{n-1}$. For each such $x = (x_1, \ldots, x_n)$ we have, by assumption

$$\Theta_1 x_1 + \cdots + \Theta_n x_n = 0.$$

Hence Θ lies in the tangent space to X_A at the point $[1 : \cdots : 1]$. Since this tangent space equals the row span of A, the result follows. $\square$

Lemma 5 *Under the hypotheses (8), the map $\phi \colon Z \to \nabla_{A_1}$ is generically surjective with fibers of dimension $n - r$.*

Proof. To prove the first statement we show that $\Phi \colon \mathbb{V}_A \to \mathbb{V}_{A_1}$ is generically surjective. Let $(\bar{x}, t) \in \mathbb{V}_{A_1}$ and choose (u, y) such that

$$D_{A_2}(u^\mu * y) \; = \; 0. \tag{10}$$

As noted above, for any choice of $y \in \mathbb{C}^{n-r}$ there are finitely many possible choices of u satisfying (10). We next choose $s \in (\mathbb{C}^*)^{d_2}$ such that $(u^\mu * y, s) \in \mathbb{V}_{A_2}$. Note that the assumption that A_2 is in standard form implies that if $(u^\mu * y, s) \in \mathbb{V}_{A_2}$ then so does $(u^\mu * y, s_\lambda)$, where $s_\lambda = (\lambda s_1, s_2, \ldots, s_{d_2})$; $\lambda \in \mathbb{C}^*$. For the given choice of u, let x be defined by $u^\gamma * x = \bar{x}$. Therefore, $(x, t) \in \mathbb{V}(J(f_{A_1}(u^\gamma * x; t))$. Thus, it suffices to show that we can choose $\lambda \in \mathbb{C}^*$ such that (x, y, t, s_λ, u) satifies

$$\frac{\partial f_A}{\partial u}(x, y; t, s_\lambda, u) \; = \; 0 \tag{11}$$

But clearly

$$u \, \frac{\partial f_A}{\partial u}(x, y; t, s_\lambda, u) \; = \; f_{A_1}(\gamma * u^\gamma * x; t) + \lambda f_{A_2}(\mu * u^\mu * y; s),$$

where $\gamma * u^\gamma * x = (\gamma_1 u^{\gamma_1} x_1, \ldots, \gamma_r u^{\gamma_r} x_r)$, and similarly for $\mu * u^\mu * y$. Lemma 4 and (7) imply that we may assume without loss of generality that (y, s, u) have been chosen so that $f_{A_2}(\mu * u^\mu * y; s) \neq 0$. Thus, if $(\bar{x}, t)$ are so that $f_{A_1}(\gamma * \bar{x}; t) \neq 0$, then we can certainly choose $\lambda \in \mathbb{C}^*$ so that (11) holds and, consequently, Φ is surjective outside the zero locus of $f_{A_1}(\gamma * \bar{x}; t)$. Appealing once again to Lemma 4 and (7), it follows that this zero locus does not contain $\mathbb{V}_{A_1}$ which completes the proof of the first assertion.

Finally, we note that the remark after (10) implies the statement about the fiber dimension of ϕ. $\square$

Suppose now that we are under the same assumptions as in Theorem 3. That is, A is a configuration in $\mathbb{Z}^n$ which is not a pyramid. $B \subset \mathbb{Z}^m$ is a Gale dual of A which may be decomposed as $B = C_1 \cup C_2$, where C_1 and C_2 are homogeneous configurations. C_1 is of rank m, and C_2 is of rank 1. Moreover, let A_1 be a dual of C_1. We then have

Theorem 6 *If C_1 has index q, then the Newton polytope $\mathcal{N}(D_A)$ is affinely isomorphic to $q \cdot \mathcal{N}(D_{A_1})$.*

Proof. By Theorem 3, $D_A = 1$ if and only if $D_{A_1} = 1$, thus we may assume $D_A \neq 1$. Let $B = \{b_1, \ldots, b_n\} \subset \mathbb{Z}^m$ and suppose that that $C_1 = \{b_1, \ldots, b_r\}$. We will then show that the projection $\pi_r : \mathbb{R}^n \to \mathbb{R}^r$ on the first r coordinates maps $\mathcal{N}(D_A)$ to $q \cdot \mathcal{N}(D_{A_1})$. Since both of these polytopes have the same dimension the result follows.

Note that since the vectors $\{b_{r+1}, \ldots, b_n\}$ are all collinear and $b_{r+1} + \cdots + b_n = 0$, we have, for all $k = 1, \ldots, m$, that the product

$$\prod_{i=r+1}^{n} (b_{i1}\zeta_1 + \cdots + b_{i,n-d}\zeta_{n-d})^{b_{ik}}$$

is a constant $\lambda_k \in \mathbb{Q}$. Hence, the defining equations $F_B(z), F_{C_1}(z)$ of $\widetilde{\nabla}_B, \widetilde{\nabla}_{C_1}$, are related through

$$F_B(z_1, \ldots, z_m) = F_{C_1}(\lambda_1 z_1, \ldots, \lambda_m z_m). \tag{12}$$

By (6), substituting z_j by x^{ν_j}, $j = 1, \ldots, m$, where ν_j is the the j-th column vector of B, into $F_B(z)$ gives the discriminant $D_A(x)$ up to a Laurent monomial factor. On the other hand, this same substitution in the right hand side of (12) yields a polynomial in $\mathbb{C}[x_1, \ldots, x_r]$ whose support equals that of D_{C_1}. Hence

$$\pi_r(\mathcal{N}(D_A)) = \mathcal{N}(D_{C_1}).$$

Since, on the other hand, (3) implies that $\mathcal{N}(D_{C_1}) = q \cdot \mathcal{N}(D_{A_1})$, the result follows. □

Definition 7 *A configuration $B = \{b_1, \ldots, b_n\} \subset \mathbb{Z}^m$ is called* irreducible *if any two vectors in B are linearly independent. If A is dual to an irreducible configuration B, we shall also call A irreducible. Given a configuration B we will denote by $\tilde{B}$ the irreducible configuration obtained by removing all vectors lying on splitting lines and replacing non-splitting subsets of collinear vectors in B by their sum.*

Remark 8 *$\mathcal{M}_B = (B, \mathcal{I})$ be the matroid defined by the family, $\mathcal{I}$, of linearly independent subsets of B. Then B is irreducible if and only if $\mathcal{M}_B$ is* simple.

Definition 9 *Let $A \subset \mathbb{Z}^d$ be a configuration and $B \subset \mathbb{Z}^m$ a Gale dual. B is said to be* degenerate *if and only if* $\mathrm{rank}(\tilde{B}) < \mathrm{rank}(B)$.

The following corollary may be viewed as a generalization of the results in (Dickenstein and Sturmfels, 2002, §4).

Corollary 10 *Let A be a $d \times n$, integer matrix of rank d defining a homogeneous configuration. Let $B = \{b_1, \ldots, b_n\}$ be a Gale dual of A. Let $\tilde{B}$ be as above. Then $\mathcal{N}(D_B)$ and $\mathcal{N}(D_{\tilde{B}})$ are affinely isomorphic.*

Proof. Let $L_1, \ldots, L_s$ denote the set of lines in $\mathbb{R}^m$ containing vectors in B. For each $j = 1, \ldots, s$, let

$$\sigma_j := \sum_{b_k \in B \cap L_j} b_k.$$

Consider the configuration

$$C := B \cup \{\sigma_1, -\sigma_1\} \cup \cdots \cup \{\sigma_s, -\sigma_s\}\,.$$

Repeated applications of Theorem 6 gives that $\mathcal{N}(D_C) \cong \mathcal{N}(D_B)$. On the other hand we may also view C as

$$C = \tilde{B} \cup C_1 \cup \cdots \cup C_s\,,$$

where $C_j = \{-\sigma_j\} \cup (B \cap L_j)$. Theorem 6 then implies that $\mathcal{N}(D_C) \cong \mathcal{N}(D_{\tilde{B}})$. $\square$

We next show that, with the notation and assumptions of Theorem 3, there is a univariate resultant formula relating the discriminants D_A and D_{A_1}.

Theorem 11 *Let A, B, A_1, C_1, and C_2 be as in Theorem 3 and let A_2 be a dual of C_2. Assume moreover that C_1 consists of the first r vectors in B. Then, there exist integers $\delta_1, \delta_2, \gamma_1, \ldots, \gamma_r, \mu_{r+1}, \ldots, \mu_n, M$ such that*

$$M\, D_A(x) \;=\; Res_u(u^{\delta_1}\, D_{A_1}(u^{\gamma} * x'), u^{\delta_2}\, D_{A_2}(u^{\mu} * x'')), \tag{13}$$

where $x = (x_1, \ldots, x_n)$, $x' = (x_1, \ldots, x_r)$, $x'' = (x_{r+1}, \ldots, x_n)$, and $*$ denotes componentwise multiplication with $u^{\gamma} = (u^{\gamma_1}, \ldots, u^{\gamma_r})$ and $u^{\mu} = (u^{\mu_{r+1}}, \ldots, u^{\mu_n})$.

Proof. If $D_A(x) = 1$, then $D_{A_1}(x') = 1$ by Theorem 3 and (13) is clearly true.

Suppose $D_{A_1} \neq 1$. Let q be the index of C_1 and let w be a $\mathbb{Z}$-generator of the one-dimensional lattice $\mathbb{Z}\langle b_{r+1}, \ldots, b_n\rangle$. Since B has index 1, q is the smallest positive integer such that $q\,w \in \mathbb{Z}\langle b_1, \ldots, b_r\rangle$. We can find integers $\gamma_1, \ldots, \gamma_r, \mu_{r+1}, \ldots, \mu_n$ such that

$$\gamma_1 b_1 + \cdots + \gamma_r b_r \;=\; q\,w \;=\; -\mu_{r+1} b_{r+1} - \cdots - \mu_n b_n\,. \tag{14}$$

We may then assume that A is as in (8) and therefore, since both A_1 and A_2 are in standard form, it follows from (2) that if $D_A(x) = 0$ then the discriminants $D_{A_1}(u^{\gamma} * x')$ and $D_{A_2}(u^{\mu} * x'')$ vanish simultaneously for some $u \in \mathbb{C}^*$. Let $\delta_1, \delta_2 \in \mathbb{Z}$ be such that $u^{\delta_1}\, D_{A_1}(u^{\gamma} * x')$ and $u^{\delta_2} D_{A_2}(u^{\mu} * x'')$ are polynomials in u with non-zero constant term. Then there exists a polynomial $F(x)$ such that

$$Res_u(u^{\delta_1}\, D_{A_1}(u^{\gamma} * x'), u^{\delta_2}\, D_{A_2}(u^{\mu} * x'')) \;=\; F(x)\, D_A(x)\,. \tag{15}$$

The proof of Theorem 6 implies that the degree of $D_A(x)$ in the variables x' equals $q\,\deg(D_{A_1}(x'))$. On the other hand, the degree of the left-hand side of (15) is the u-degree of $u^{\delta_2}\, D_{A_2}(u^{\mu} * x'')$ times $\deg(D_{A_1}(x'))$. By definition of w, we can write $b_j = \beta_j\, w$, $\beta_j \in \mathbb{Z}$, $j = r+1, \ldots, n$, and therefore

$$q = -\mu_{r+1}\, \beta_{r+1} - \cdots - \mu_n\, \beta_n$$

but then it follows from the expression (4) for the discriminant of a codimension-one configuration that

$$\deg_u(u^{\delta_2}\, D_{A_2}(u^{\mu} * x'')) \;=\; q\,.$$

Hence both sides of (13) have the same degree in the variables x' and, consequently, $F(x)$ depends only on $x'' = (x_{r+1}, \ldots, x_n)$.

Suppose $F(x'')$ is not constant. We can write

$$u^{\delta_1} \cdot D_{A_1}(u^\gamma * x') = g_l(x')u^\ell + \cdots + g_1(x')u + g_0(x'). \tag{16}$$

$$u^{\delta_2} \cdot D_{A_2}(u^\mu * x'')) = u^q \prod_{\beta_j>0} \beta_j^{\beta_j} \prod_{\beta_j<0} x_j^{-\beta_j} - \prod_{\beta_j<0} \beta_j{}^{-\beta_j} \prod_{\beta_j>0} x_j^{\beta_j}. \tag{17}$$

Choose $a'' = (a_{r+1}, \ldots, a_n)$ with $F(a'') = 0$. Then

$$Res_u(u^{\delta_1} \cdot D_{A_1}(u^\gamma * x'), u^{\delta_2} \cdot D_{A_2}(u^\mu * a'')) = 0$$

for all $x' = (x_1, \ldots, x_r)$. This means Equations (16) and (17) are solvable in u for all (x', a''). There are at most q possible values for u which solve (17), which means that (16) must be the zero polynomial which is a contradiction since the monomials appearing in $g_i(x')$ are distinct monomials of D_{A_1}. Thus $F(x)$ is a constant $M \in \mathbb{Z}$. $\square$

Remark 12 *We note that there are many possible choices for $\delta_1, \delta_2, \gamma, \mu$ in Theorem 11. Indeed, it suffices that γ and μ satisfy (14) and that δ_1 and δ_2 be chosen so that the products $u^{\delta_1} D_{A_1}(u^\gamma * x')$ and $u^{\delta_2} D_{A_2}(u^\mu * x'')$ be polynomials in u with non-zero constant term. In fact, if we replace (14) by*

$$\gamma_1' b_1 + \cdots + \gamma_r' b_r \;=\; q'\, w \;=\; -\mu_{r+1}' b_{r+1} - \cdots - \mu_n' b_n\,, \tag{18}$$

where $q' = kq$, with k a positive integer, then its effect is to make a change of variable $u \mapsto u^k$ in the resultant and therefore we would have:

$$M\, D_A(x)^k \;=\; Res_u(u^{\delta_1'}\, D_{A_1}(u^{\gamma'} * x'), u^{\delta_2'}\, D_{A_2}(u^{\mu'} * x'')) \tag{19}$$

for suitable integers δ_1', δ_2'.

The following corollary which will be needed in the next section describes the effect on the discriminant of adding to the B configuration a vector and its negative.

Corollary 13 *Let $A \in \mathbb{Z}^{d\times n}$ be a homogeneous configuration and let $B = \{b_1, \ldots, b_n\} \subset \mathbb{Z}^m$, $m = n - d$, be a Gale dual. Let $v \in \mathbb{Z}^m$ be a non-zero vector and let*

$$B^\sharp \;:=\; B \cup \{v, -v\}\,.$$

Let $A^\sharp$ be dual to $B^\sharp$. Let $x = (x_1, \ldots, x_n)$ and $D_A \in \mathbb{C}[x]$, $D_{A^\sharp} \in \mathbb{C}[x; y_+, y_-]$, the discriminants associated with A and $A^\sharp$, respectively. Then

$$D_A(x) \;=\; D_{A^\sharp}(x, y_+, y_-)|_{y_+=1, y_-=-1}.$$

Proof. Since B has index 1, we can write

$$v \;=\; \sum_{j=1}^n \gamma_j\, b_j \;;\; \gamma_j \in \mathbb{Z},$$

and setting $\mu_{n+1} = 0$, $\mu_{n+2} = 1$, we can apply (13) and obtain

$$D_{A^\sharp}(x; y_+, y_-) \;=\; Res_u(u^{\delta_1} \cdot D_A(u^\gamma * x), y_+ + u\, y_-)$$

for a suitable integer δ_1. We may specialize this resultant to $y_+ = 1$, $y_- = -1$ since that does not change the u-degrees of the polynomials involved and obtain:

$$D_{A^\sharp}(x, y_+, y_-)|_{y_+=1, y_-=-1} \;=\; Res_u(u^{\delta_1} \cdot D_A(u^\gamma * x), 1-u) = D_A(x).$$

□

We end this section with a simple example to illustrate how we can use Theorem 11 and Corollary 13 to reduce the computation of discriminants to that of irreducible configurations and univariate resultants.

Example. We work directly on the B side and consider a configuration B consisting of seven vectors $\{b_1, \ldots, b_7\}$, where

$$b_1 = (0,1), \quad b_2 = (-3,1), \quad b_3 = (2,-3), \quad b_4 = (-1,1),$$

$$b_5 = (1,0), \quad b_6 = (3,0), \quad b_7 = (-2,0).$$

The last 3 vectors lie on a line L and $\sigma(L) = (2,0)$. As before, we set

$$B^\sharp = B \cup \{\sigma(L), -\sigma(L)\} = C_1 \cup C_2,$$

where $C_1 = \{b_1, b_2, b_3, b_4, \sigma(L)\}$ and $C_2 = \{b_5, b_6, b_7, -\sigma(L)\}$. We let $\{x_1, \ldots, x_7\}$ denote variables ssociated with $\{b_1, \ldots, b_7\}$, respectively, and let y_+, y_- be associated with $\sigma(L)$ and $-\sigma(L)$.

We note that C_1 and C_2 are homogeneous configurations satisfying the assumptions in Theorem 11 and $\mathrm{index(C_1)} = 1$. Following the notation of Theorem 11 we have $w = (1,0)$ and therefore

$$b_1 - b_4 \;=\; w \;=\; -(-1)b_5. \tag{20}$$

On the other hand, using Singular (Greuel et al., 2001) we compute

$$D_{C_1}(x_1, x_2, x_3, x_4, y_+) \;=\; 256x_2^5x_3^6x_4 + 13824x_1x_2^6x_3^3x_4^2 + 186624x_1^2x_2^7x_4^3 - 432x_2^2x_3^8y_+^2 - 24224x_1x_2^3x_3^5x_4y_+^2 - 359856x_1^2x_2^4x_3^2x_4^2y_+^2 - 432x_1x_3^7y_+^4 - 24696x_1^2x_2x_3^4x_4y_+^4 - 1210104x_1^3x_2^2x_3x_4^2y_+^4 - 823543x_1^4x_4^2y_+^6.$$

While clearly

$$D_{C_2}(x_5, x_6, x_7, y_-) \;=\; 8x_5x_6^3 - 27x_7^2y_-^2.$$

Thus, given (20), we may apply Theorem 11 with $\delta_1 = 0$, $\delta_2 = 1$ and obtain

$$D_{B^\sharp}(x, y_+, y_-) \;=\; \mathrm{Res}_u(D_{C_1}(ux_1, x_2, x_3, u^{-1}x_4, y_+), uD_{C_2}(u^{-1}x_5, x_6, x_7, y_-)) \;=\; 5038848x_2^5x_3^6x_4x_7^6y_-^6 - 746496x_1x_3^7y_+^4x_5^2x_6^6x_7y_-^2 - 421654016x_1^4x_4^2y_+^6x_5^3x_6^9 - 2098680192x_1^2x_2^4x_3^2x_4^2y_+^2x_5x_6^3x_7^4y_-^4 - 42674688x_1^2x_2x_3^4x_4y_+^4x_5^2x_6^6x_7^2y_-^2 - 2519424x_2^2x_3^8y_+^2x_5x_6^3x_7^4y_-^4 - 141274368x_1x_2^3x_3^5x_4y_+^2x_5^1x_6^3x_7^4y_-^4 + 272097792x_1x_2^6x_3^3x_4^2x_7^6y_-^6 + 3673320192x_1^2x_2^7x_4^3x_7^6y_-^6.$$

According to Corollary 13 setting $y_+ = 1, y_- = -1$ yields $D_B(x)$. Since y_- appears only raised to even powers, the expression for $D_B(x)$ is obtained from that for $D_{B^\sharp}(x, y_+, y_-)$ erasing y_+ and y_-. Finally, note that if, instead of (20), we use the relation:

$$\sigma(L) \,=\, 2w \,=\, -(-\sigma(L)),$$

then, as noted in Remark 12

$$D^2_{B^\sharp}(x, y_+, y_-) \,=\, \mathrm{Res}_u(D_{C_1}(x, uy_+), D_{C_2}(x, uy_-)).$$

4. Specialization of the A-discriminant

The main result of this section is a specialization theorem for the A-discriminant generalizing Lemma 3.2 in (Cattani et al., 2001) and Lemma 3.2 in (Cattani and Dickenstein, 2004). In these references, the lemmata in question play an important role in the study of rational hypergeometric functions.

We begin with a general result on the variable grouping in the A-discriminant.

Proposition 14 *Let A be a $d \times n$, integer matrix of rank d and $B = \{b_1, \ldots, b_n\} \subset \mathbb{Z}^m$ a Gale dual of A. Let $D_A(x)$, $x = (x_1, \ldots, x_n)$, be the sparse discriminant. Then, if b_k and b_ℓ, $1 \leq k, \ell \leq n$, are positive multiples of each other,*

$$D_A|_{x_k=0} = D_A|_{x_\ell=0}.$$

Proof. Define $\omega_k \in \mathbb{R}^n$ by $(\omega_k)_j = -\delta_{kj}$, $j = 1, \ldots, n$,. It is clear that the initial form $in_{\omega_k}(D_A)$ of D_A relative to the weight ω_k agrees with the restriction $D_A|_{x_k=0}$. Thus, it suffices to show that

$$in_{\omega_k}(D_A) = in_{\omega_l}(D_A). \tag{21}$$

We recall (Gel′fand et al., 1994, Chapter 10, Theorem 1.4 a) that the secondary fan $\Sigma(A)$ is the normal fan to the Newton polytope $\mathcal{N}(E_A)$ of the principal $A-$determinant (we refer to (Gel′fand et al., 1994, Chapter 10) for the definition and main properties of the principal $A-$determinant). Then

$$in_{\omega_k}(E_A) = in_{\omega_\ell}(E_A) \tag{22}$$

if and only if ω_k and ω_ℓ are in the same relatively open cone of $\Sigma(A)$.

On the other hand, it follows from (Billera et al., 1990, Lemma 4.2), that the linear map $-B^T : \mathbb{R}^n \to \mathbb{R}^m$ defines an isomorphism of fans between the secondary fan, $\Sigma(A)$, and its image, a polytopal fan $\mathcal{F}$ defined on $\mathbb{R}^{n-d}$. Hence, (22) holds if and only if $-B^T \cdot \omega_k$ and $-B^T \cdot \omega_\ell$ are in the same relatively open cone of $\mathcal{F}$. But $-B^T \cdot \omega_k = b_k$ is a positive multiple of $-B^T \cdot \omega_l = b_l$ by assumption, so they must be in the same relatively open cone of $\mathcal{F}$.

Since D_A is a factor of E_A by (Gel′fand et al., 1994, Chapter 10, Theorem 1.2), the normal fan of E_A refines that of D_A. Then, any two vectors giving the same initial form on E_A give the same initial form on D_A. This proves equation (21) and concludes the proof of the Proposition. $\square$

As before, let $A = \{a_1, \ldots, a_n\}$ be a homogeneous configuration in $\mathbb{Z}^d$ which is not a pyramid. For any index set $I \subset \{1, \ldots, n\}$, we denote by $A(I)$ the subconfiguration of A consisting of $\{a_i, i \in I\}$. Let $B \subset \mathbb{R}^m$ be a Gale dual of A. Given a line $\Lambda \subset \mathbb{R}^m$, let

$$I_\Lambda = \{j : b_j \notin \Lambda\} \subset \{1, \ldots, n\}\,;\ J_\Lambda = \{1, \ldots, n\} \setminus I_\Lambda.$$

and

$$\sigma(\Lambda) \ := \ \sum_{j \in J_\Lambda} b_j\,.$$

If Λ is a non-splitting line, let w be the $\mathbb{Z}$-generator of $\mathbb{Z}\langle b_j ; j \in J_\Lambda \rangle$ in the same direction as $\sigma(\Lambda)$ and, for $j \in J_\Lambda$ write $b_j = \beta_j w$. We set $J_\Lambda^+ = \{j \in J_\Lambda, \beta_j > 0\}$ and define J_Λ^- accordingly.

We may now prove the main result of this section

Theorem 15 *Let A be a homogeneous, $d \times n$ integer matrix of rank d, and let Λ be a non-splitting line. Then, for any $j \in J_\Lambda^+$,*

$$D_{A(I_\Lambda)} \ \text{divides} \ D_A|_{x_j=0}.$$

Proof. We may assume that $I_\Lambda = \{1, \ldots, r\}$, and let us denote by $x' = (x_1, \ldots, x_r)$, $x'' = (x_{r+1}, \ldots, x_n)$. Let $B^\sharp = B \cup \{\sigma(\Lambda), -\sigma(\Lambda)\}$ and $A^\sharp$ a dual of $B^\sharp$. As we have done before, let us denote by y_+, respectively y_-, the variable associated with $\sigma(\Lambda)$, respectively $-\sigma(\Lambda)$. By Corollary 13

$$D_A(x) \ = \ D_{A^\sharp}(x, y_+, y_-)|_{y_+=1, y_-=-1}.$$

On the other hand, we can write $B^\sharp = C_1 \cup C_2$, where

$$C_1 = \{b_1, \ldots, b_r, \sigma(\Lambda)\}\ ; \quad C_2 = \{b_{r+1}, \ldots, b_n, -\sigma(\Lambda)\}.$$

Let w be a generator of $\mathbb{Z}\langle b_{r+1}, \ldots, b_n \rangle$ so that $\sigma(\Lambda) = cw$ with c a positive integer. Let q be the index of C_1. Then we may write:

$$q \cdot \sigma(\Lambda) \ = \ c \cdot q \cdot w \ = \ -q \cdot (-\sigma(\Lambda))\,.$$

Thus, it follows from (19) that, up to constant,

$$(D_{A^\sharp}(x))^c \ = \ Res_u(D_{A_1}(x', u^q \cdot y_+), D_{A_2}(x'', u^q \cdot y_-))\,,$$

since we can choose $\delta'_1 = \delta'_2 = 0$. Consequently

$$(D_A(x))^c = Res_u(D_{A_1}(x', u^q \cdot y_+), D_{A_2}(x'', u^q \cdot y_-))|_{y_+=1, y_-=-1}.$$

On the other hand, let $b_j = \beta_j \cdot w$, $\beta_j \in \mathbb{Z}$, $j = r+1, \ldots, n$. Then, since $-\sigma(\Lambda) = -c \cdot w$,

$$D_{A_2}(x'', u^q \cdot y_-) = K_1 \prod_{j \in J_\Lambda^+} x_j^{\beta_j} - K_2\, u^{cq}\, y_-^c \prod_{j \in J_\Lambda^-} x_j^{-\beta_j},$$

where K_1 and K_2 are integers. It then follows that we may specialize $x_j = 0$, $j \in J_\Lambda^+$, in the resultant since that does not change the leading term of $D_{A_2}(x'', u^q \cdot y_-)$. Hence, up to constants and monomials:

$$(D_A(x))^c|_{x_j=0} \ = \ D_{A_1}(x', u^q \cdot y_+)^{cq}|_{u=0, y_+=1} \ = \ D_{A_1}(x', y_+)^{cq}|_{y_+=0}.$$

But, since $\sigma(\Lambda)$ is the unique vector in the line Λ in the configuration C_1, it follows that $A(I_\Lambda)$ is a non-facial circuit in A_1 and therefore by (Cattani et al., 2001, Lemma 3.2), $D_{A(I_\Lambda)}$ divides $D_{A_1}(x', y_+)|_{y_+=0}$ and the result follows. $\square$

5. Dual Defect Varieties

In this section we apply the specialization Theorem 15 and recent results in (Dickenstein et al., 2005) to prove, in Theorem 18, a Gale dual characterization of dual defect toric varieties. This leads to a classification of dual defect toric varieties of codimension less than or equal to four. Motivated by this classification, we prove that the Gale dual of a configuration may be decomposed as a disjoint union of non dual-defect configurations which are maximal in an appropriate sense. Using this decomposition we give a sufficient condition for a configuration to be dual defect. We believe that this condition is necessary as well. Indeed, it follows from Theorems 20 and 21, that this is the case for codimension less than or equal to four.

Throughout this section, we let A be a homogeneous configuration of n points in $\mathbb{Z}^d$ which is not a pyramid. We assume moreover that the elements of A span the lattice $\mathbb{Z}^d$. As always, if convenient, we will view A as a $d \times n$ integer matrix of rank d. Let X_A denote the associated projective toric variety and $X_A^* \subset \mathbb{P}^{n-1}$ its dual variety. Let $\mathcal{S}(A)$ denote the geometric lattice whose elements are the supports, ordered by inclusion, of the vectors in $\ker(A)$. The following result is proved in (Dickenstein et al., 2005) using tropical geometry methods.

Theorem 16 (Dickenstein et al. (2005, Corollary 4.5)) *Let A be as above. The dimension of X_A^* is one less than the largest rank of any matrix $(A^t, \sigma_1, \ldots, \sigma_{n-d-1})$, where $\sigma_1, \ldots, \sigma_{n-d-1}$ is a proper maximal chain in $\mathcal{S}(A)$.*

Let $B \subset \mathbb{Z}^m$, $m = n - d$, be a Gale dual of A and let $\mathcal{M}_B = (B, \mathcal{I})$ be the matroid defined by the family, $\mathcal{I}$, of linearly independent subsets of B. Given a subset $B' \subset B$, the *rank* of B' is defined as the cardinality of the maximal element of $\mathcal{I}$ completely contained in B'. A subset $F \subset B$ is called a *k-flat* if it is a maximal, rank-k subset of B. Clearly every subset $B' \subset B$ spans a subspace $\langle B' \rangle \subset \mathbb{R}^m$ whose dimension equals the rank of B'. A subspace $W \subset \mathbb{R}^m$ is said to be B-spanned if $\dim(W) = \mathrm{rank}(B \cap W)$. Given a flat $F \subset B$ we denote

$$\sigma(F) \ = \ \sigma(\langle F \rangle) \ = \ \sum_{b \in F} b \,.$$

A subset $C \subset B$ such that $\sigma(C) = 0$ will be called a *homogeneous subconfiguration* (or a homogeneous flat if C is a flat in B).

Definition 17 *A k-flag of flats $\mathcal{F}$ is a flag $F_0 \subset F_1 \subset \cdots \subset F_k$, where $F_j \subset B$ is a j-flat. The flag is said to be* non-splitting *if and only if $\sigma(F_j) \notin \langle F_{j-1} \rangle$, for all $j = 1, \ldots, k$.*

Note that $F_0 = \emptyset$ and $\langle F_0 \rangle = \{0\}$, so we will usually drop it from the notation. If $\mathcal{F}$ is a non-splitting flag then, for all $j = 1, \ldots, k$, $\langle F_j \rangle$ is a B-spanned subspace and $\sigma(F_j) \neq 0$. Moreover, $\langle F_j \rangle$ projects to a non-splitting line in $\mathbb{R}^m / \langle F_{j-1} \rangle$. Clearly, the projection of a non-splitting k-flag $\mathcal{F}$ to $\mathbb{R}^m / \langle F_1 \rangle$ is a non-splitting $(k-1)$-flag in the configuration defined by the projection of B.

The following is a characterization of dual defect toric varieties which parallels that contained in Theorem 16 although it only involves the Gale dual B.

Theorem 18 *Let $A \subset \mathbb{Z}^d$ be as above and $B \subset \mathbb{Z}^m$ a Gale dual of A. Then X_A is dual defect if and only if B does not have any non-splitting $(m-1)$-flags.*

Proof. We prove the if direction by induction on the codimension m. The result is obviously true for $m = 1$. Assuming it to be true for configurations of codimension $m-1$, let B be a codimension m configuration with a non-splitting $(m-1)$-flag $F_1 \subset \cdots \subset F_{m-1}$. Let $\pi_1 : \mathbb{Z}^m \to \mathbb{Z}^{m-1}$ denote the projection onto a rank $m-1$ lattice complementary to $\langle F_1 \rangle \cap \mathbb{Z}^m$ and let $G_j = \pi_1(F_{j+1})$. Clearly, $G_1 \subset \cdots \subset G_{m-2}$ is a non-splitting $(m-2)$-flag for $\pi_1(B_1)$, where $B_1 := \{b \in B : b \notin F_1\}$. We recall that $\pi_1(B_1)$ is a Gale dual for the configuration $A_1 := \{a_i \in A : b_i \in B_1\}$. By induction hypothesis, A_1 is not dual defect and, by Theorem 15, the discriminant D_{A_1} must divide an appropriate specialization of D_A. Hence A is not dual defect.

We also prove the converse by induction on the codimension m. Once again, the case $m = 1$ is clear. We begin by considering the special case of a configuration A with an irreducible Gale dual B. If A is not dual defect, by Theorem 16, there exists a proper maximal chain in $\mathcal{S}(A)$, $\sigma_1, \ldots, \sigma_{n-d-1}$, such that the matrix $M := (A^t, \sigma_1, \ldots, \sigma_{n-d-1})$ has rank $n-1$. After reordering the columns of A, and consequently the entries of σ_j, we may assume that $\mathrm{supp}(\sigma_j) = \{1, \ldots, k_j\}$ with $k_1 < \cdots < k_{n-d-1}$.

We claim that there exists an index i, $k_{n-d-2} < i \leq k_{n-d-1}$ such that the matrix M_i, obtained by removing the i-th row and the last column of M, has rank $n-2$. Indeed, if the columns of M_i are linearly dependent then, since the corresponding columns of M are independent, it follows that the basis vector e_i may be written as a linear combination of the first $n-2$ columns of M. If this were true for every i, $k_{n-d-2} < i \leq k_{n-d-1}$, we could write the vector

$$\sigma_{n-d-1} - \sigma_{n-d-2} \;=\; \sum_{k_{n-d-2} < j \leq k_{n-d-1}} e_j$$

as a linear combination of the first $n-2$ columns of M, a contradiction.

We fix now an index i, as above, such that $\mathrm{rank}(M_i) = n-2$. Let A' be configuration obtained by removing the i-th column of A. Notice that the vectors $\sigma'_1, \ldots, \sigma'_{n-d-2}$ obtained, also, by removing the zero in the i-th entry from the corresponding σ_j, define a proper maximal chain in $\mathcal{S}(A')$. We then have, by Theorem 16, that A' is not dual defect and, therefore any Gale dual B' of A' must contain a non-splitting $(m-2)$-flag $G_1 \subset \cdots \subset G_{m-2}$. Now, since B is irreducible, B' agrees —up to $\mathbb{Q}$-linear isomorphism— with the projection of B onto $\mathbb{R}^m/\langle b_i \rangle$. Then, denoting by V_j the lifting of $\langle G_{j-1} \rangle$ to $\mathbb{R}^m$, $j = 2, \ldots, m-1$, and setting

$$F_j \;:=\; V_j \cap B \;; \quad j = 2, \ldots, m-1,$$

$F_1 = \{b_i\}$, we have that $F_1 \subset F_2 \subset \cdots F_{m-1}$ is a non-splitting flag of flats in B.

Finally, consider the general case. That is, let A be a non dual-defect configuration whose Gale dual B is not necessarily irreducible. As before, let $\tilde{B}$ be the irreducible configuration obtained from B by replacing all subsets of collinear vectors in B by their sum. Note that $\tilde{B}$ need not have index one, but we may still consider a dual A_1 of $\tilde{B}$. It follows from Corollary 10 that $D_{A_1} \neq 1$. Moreover, a Gale dual B_1 of A_1, being $\mathbb{Q}$-linearly isomorphic to $\tilde{B}$, is irreducible. Therefore B_1 has a non-splitting $(m-1)$-flag. But then so do $\tilde{B}$ and B. □

Corollary 19 *Let $A \subset \mathbb{Z}^d$ be a homogeneous configuration and let B be a Gale dual. Then if B is degenerate, A is dual defect.*

Proof. If $\text{codim}(A) = m$ but B is degenerate, then $\text{rank}(\tilde{B}) < m$ and $\tilde{B}$ may not contain any non-splitting $(m-1)$-flags and, therefore, neither does B. $\square$

Note that, by Theorem 18, if A is not a pyramid and $\text{codim}(A) = 2$, then $D_A = 1$ if and only if a Gale dual B has no non-splitting one-flags, i.e. if and only if every line is splitting or, equivalently, if $\tilde{B} = \emptyset$. This classification of codimension-two dual defect toric varieties is contained in Corollary 4.5 of (Dickenstein and Sturmfels, 2002). This observation may be generalized to the codimension-three case:

Theorem 20 *Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^d$ be a homogeneous configuration of codimension three, which is not a pyramid. Let $B \subset \mathbb{Z}^3$ be a Gale dual of A. Then $D_A = 1$ if and only if B is degenerate.*

Proof. By the above Corollary and Theorem 18 it suffices to show that if B is an irreducible configuration of rank three, then B has a non-splitting two-flag. Let b and b' be distinct elements in B and set F_2 be the two-flat containing $\{b, b'\}$. If, $\sigma(F_2) \neq 0$ then we may assume $\sigma(F_2) \notin \langle b \rangle$ and $\{b\} \subset F_2$ is a non-splitting two-flag. On the other hand, suppose every B-spanned plane $P \subset \langle B \rangle$ satisfies $\sigma(P) = 0$. Then, fixing an element $b \in B$, and denoting by $P_1, ..., P_r$ the distinct B-spanned planes containing b we would have that $0 = \sigma(B) = \sigma(P_1) + \cdots + \sigma(P_r) - (r-1) \cdot b$. But, we have assumed $\sigma(P_i) = 0$ for all $i = 1, \ldots, r$. Hence $r = 1$, and this implies that $\text{rank}(B) = 2$, a contradiction. $\square$

We consider now the case of codimension-four configurations:

Theorem 21 *Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^d$ be a homogeneous configuration of codimension four, which is not a pyramid. Let $B \subset \mathbb{Z}^4$ be a Gale dual of A. Then $D_A = 1$ if and only if either B is degenerate, or there exist planes $P, Q \subset \mathbb{R}^4$, such that $P \cap Q = \{0\}$, and every non-splitting line lies either in P or in Q.*

Proof. Let A be such that $D_A = 1$ and suppose B is non-degenerate. Let $\tilde{B}$ be the irreducible configuration as in Definition 7. Since B is non-degenerate the vectors in $\tilde{B}$ span $\mathbb{R}^4$ and, by Corollary 10, $D_A = 1$ if and only if $D_{\tilde{B}} = 1$. Thus, we may assume without loss of generality that B is irreducible. We note that if $B = C_1 \cup C_2$, where C_1 and C_2 are homogeneous configurations contained in complementary planes P and Q, respectively, then B may not contain any non-splitting three-flags and, therefore, A is dual defect.

In order to prove the only-if direction of Theorem 21 we begin with two lemmas which hold for arbitrary rank.

Lemma 22 *Let B be a homogeneous configuration of rank m and let $\Lambda \subset \langle B \rangle$ be a line. Suppose B has a non-splitting flag of rank k. Then, B has a non-splitting flag $\mathcal{G}$ of rank k such that $\langle G_k \rangle \cap \Lambda = \{0\}$.*

Proof. We prove the result by induction on k, $1 \leq k \leq m-1$. The result is obvious for $k = 1$ since $m \geq 2$ and B is homogeneous which means that the number of non-splitting one-flats in B is either zero or at least three. Assume it to be true for non-splitting flags of rank less than k, and let $\mathcal{F}$ be a non-splitting flag $\mathcal{F}$ in B of rank $k \geq 2$. We can assume that $\langle F_1 \rangle \neq \Lambda$. Consider the projection $\pi(B)$ to $\langle B \rangle / \langle F_1 \rangle$. Λ projects to a line $\bar{\Lambda}$

in $\langle B\rangle/\langle F_1\rangle$. Moreover, the projection of $\mathcal{F}$ defines a non-splitting flag of rank $k-1$ in $\pi(B)$. By inductive hypothesis there exists a non-splitting flag $\bar{\mathcal{G}}$ in $\pi(B)$ of rank $k-1$ such that $\langle \bar{G}_{k-1}\rangle \cap \bar{\Lambda} = \{0\}$. Let $W_{j+1} \subset \langle B\rangle$ be the unique subspace of dimension $j+1$ containing $\langle F_1\rangle$ and projecting onto $\langle \bar{G}_j\rangle$, $j = 1, \ldots, k-1$. Notice that by construction $W_k \cap \Lambda \subset \langle F_1\rangle$ but, since $\Lambda \cap \langle F_1\rangle = \{0\}$ we have $W_k \cap \Lambda = \{0\}$. Setting $G_1 = F_1$, $G_j = W_j \cap B$ for $j = 2, \ldots, k$, we get the desired non-splitting k-flag in B. $\square$

Lemma 23 *Let $A \subset \mathbb{Z}^d$ be a homogeneous configuration of codimension m and B a Gale dual. If B is non-degenerate, then there exists a flat $F \subset B$ of rank $m-1$ such that $\sigma(F) \neq 0$. Moreover, if we denote by B_F the homogeneous configuration in $\langle F\rangle$ defined by $B_F := F \cup \{-\sigma(H)\}$, then, if B_F is non dual-defect, B is not dual defect.*

Proof. If every flat of rank $m-1$ is homogeneous, let $s < m-1$ be the maximal rank of a non-homogeneous flat F in B. We have $s > 0$ since B is non-degenerate. Choose a flat G of rank s with $\sigma(G) \neq 0$ and let $\Theta_1, ..., \Theta_r$ be the rank $s+1$ flats which contain G. By assumption, $\sigma(\Theta_i) = 0$ for all $i = 1, \ldots, r$. Then,

$$0 = \sigma(B) = \sum_{i=1}^{r} \sigma(\Theta_i) - (r-1) \cdot \sigma(G) = -(r-1) \cdot \sigma(G).$$

Hence $r = 1$ and therefore B has rank $s+1$. Since $s+1 < m$ this implies that B is degenerate, a contradiction.

Suppose now that B_H is not dual defect. By Theorem 18, B_H has a non-splitting flag $\mathcal{G}$ of rank $m-2$ and, by Lemma 22, we may assume that $G_j \cap \langle \sigma(F)\rangle = \{0\}$. But then,

$$G_1 \subset \cdots \subset G_{m-2} \subset F$$

is a non-splitting flag of rank $m-1$ in B. Applying Theorem 18 again we deduce that B is not dual defect. $\square$

Corollary 24 *Let $A \subset \mathbb{Z}^d$ be a homogeneous configuration of codimension four and suppose a Gale dual $B \subset \mathbb{R}^4$ of A is irreducible. Suppose B does not have any non-splitting three-flags and let F be a rank-three flat with $\sigma(F) \neq 0$. Then $\sigma(F) \in F$ and the elements $\{b \in F : b \neq \sigma(F)\}$ span a plane $P \subset \langle F\rangle$, with $\sigma(P) = 0$.*

Proof. Let B_F be as in Lemma 23. Since B is dual defect so is B_H and hence, by Theorem 20, B_F must be degenerate. Since F has rank three and B is irreducible, this can only happen if $\sigma(F) \in F$, so that $\{\sigma(F), -\sigma(F)\}$ define a splitting line. The second assertion is then clear by Theorem 20. $\square$

We now return to the proof of Theorem 21. Because of Corollary 10 and Theorem 18, it suffices to prove that if $B \subset \mathbb{R}^4$ is an irreducible, non-degenerate configuration which does not have any non-splitting three-flags, then $B = C_1 \cup C_2$, where C_1 and C_2 are homogeneous, rank-two configurations.

Let $F \subset B$ be a rank three flat with $\sigma(F) \neq 0$. By Corollary 24, $F \cap B = C_1 \cup \sigma(F)$ and C_1 is a rank-two flat with $\sigma(C_1) = 0$. Let $C_2 := B \backslash C_1$. We claim that C_2 does not have any non-splitting two-flags. Indeed, suppose $G_1 \subset G_2$ is a non-splitting two-flag. Let $b \in C_1 \backslash G_2$. Such b exists since $C_1 \neq G_2$. Then, letting G_3 be the smallest three-flat containing $G_2 \cup \{b\}$, we would have that $G_1 \subset G_2 \subset G_3$ would be a non-splitting three-flag in B, contradicting our assumption. But, it is easy to see that the argument used in the proof of Theorem 20 implies that since C_2 is irreducible and has no non-splitting two-flags, it must have rank two and $\sigma(C_2) = 0$. Since B has rank four, the planes $\langle C_1 \rangle$ and $\langle C_2 \rangle$ must be complementary. $\square$

Theorem 21 motivates the following decomposition theorem which gives a sufficient condition for a Gale configuration to be dual defect.

Theorem 25 *Let B be a homogeneous, irreducible configuration of rank m. Then, we can write*

$$B = C_1 \cup \cdots \cup C_s, \tag{23}$$

where the C_i's are homogeneous, disjoint, non dual-defect subconfigurations of B. Moreover, C_i is a flat in $C_i \cup C_{i+1} \cup \cdots \cup C_s$ and the C_i's are maximal with these properties. Moreover, the rank of a non-splitting flag in B is bounded by

$$\rho = \rho(B) := \sum_{i=1}^{s} \operatorname{rank}(C_i) - s. \tag{24}$$

Hence if $\rho \leq m-2$, B is dual defect.

Remark 26 *It follows from Theorems 20 and 21 that the condition $\rho \geq m-1$ is a necessary and sufficient condition for a configuration B, of rank at most four, to be dual defect. We expect this to be the case in general. This would give a complete classification of dual defect toric varieties in terms of their Gale configuration.*

Proof. The following two lemmas, necessary for the proof of Theorem 25, may be of independent interest as well.

Lemma 27 *Let B be a homogeneous non dual-defect configuration of rank m. Suppose $V \subset \langle B \rangle$ is a k-dimensional subspace, $0 \leq k < m$. Then, B has a non-splitting flag $\mathcal{F}$ of rank $m-1$ such that $\langle F_1 \rangle \cap V = \{0\}$.*

Proof. We proceed by induction on m. The result is clear for $m = 2$. Assume our statement holds for configurations of rank $m-1$. Let $\mathcal{G}$ be a non-splitting flag of rank $m-1$ in B. If $\langle G_1 \rangle \cap V = \{0\}$ we are done. Assume then that $G_1 \subset V$ and consider the projection $\pi(B)$ to $\langle B \rangle / \langle F_1 \rangle$. Then, $\pi(B)$ is not dual defect and, by inductive hypothesis, there exists a non-splitting $(m-2)$-flag $\bar{\mathcal{F}}$ in $\pi(B)$ such that $\langle \bar{F}_1 \rangle \cap \pi(V) = \{0\}$. Let W_{j+1} be the unique subspace of $\langle B \rangle$ containing $\langle G_1 \rangle$ and projecting to $\langle \bar{F}_j \rangle$ and set $F_{j+1} = W_{j+1} \cap B$. Note that $\sigma(F_{j+1}) \notin \langle F_j \rangle$ since $\bar{\mathcal{F}}$ is non-splitting. Now $\langle \bar{F}_1 \rangle \cap \pi(V) = \{0\}$ implies that $\langle F_2 \rangle \cap V = \langle G_1 \rangle$. Now, since F_2 is spanned by non-splitting one-flats, there exists a one-flat $F_1 \subset F_2$, with $\langle F_1 \rangle \neq \langle G_1 \rangle$, and such that $\sigma(F_2) \notin F_1$. The flag $F_1 \subset F_2 \subset \cdots \subset F_{m-1}$ is a non-splitting flag in B with $\langle F_1 \rangle \cap V = \{0\}$. $\square$

Lemma 28 *Let B be an irreducible, homogeneous, dual defect configuration and let Λ a line in $\langle B \rangle$. Then there exists a homogeneous, non dual-defect flat $C \subset B$ of rank k, $2 \leq k < m$, such that $\langle C \rangle \cap \Lambda = \{0\}$.*

Proof. We proceed by induction on $m = \mathrm{rank}(B)$. If $m \leq 3$ then, by Theorem 20, there are no irreducible, non dual-defect configurations. So assume that $m \geq 4$ and that the result holds for configurations of rank less than m. Let $k < m-1$ be the largest rank of a non-splitting flag in B. We may assume that $k \geq 2$. Otherwise, given any one-flat F_1 in B, every two-flat containing it must be homogeneous, but this is impossible since B is irreducible. Moreover, by Lemma 22, we may assume that B has a non-splitting k-flag $\mathcal{F}$ such that $\langle F_k \rangle \cap \Lambda = \{0\}$.

Let $\Theta_0, \ldots, \Theta_q$ be the distinct $(k+1)$-flats in B containing F_k. Since $m > k+1$, $q \geq 1$, and at most one $(k+1)$-flat may contain both $\langle F_k \rangle$ and Λ. Hence may assume $\Lambda \cap \langle \Theta_j \rangle = \{0\}$ for $j \geq 1$. If $\sigma(\Theta_j) = 0$ for some $j \geq 1$, then we can take $C = \Theta_j$ and we are done. If not, let Let $W = \langle \Theta_1 \rangle$ and $B_W = \Theta_1 \cup \{-\sigma(\Theta_1)\}$. Then B_W is a homogeneous configuration of rank $k+1$, which may or may not be irreducible. Let $\tilde{B}_W$ be as in Definition 7.

Suppose $\mathrm{rank}(\tilde{B}_W) = k$. Then, since B is irreducible, $C := \tilde{B}_W$ is a homogeneous B-flat of rank k which, we claim, is not dual defect. Indeed, let j be such that $\sigma(\Theta_1) \in F_j \backslash F_{j-1}$, we can define a non-splitting flag $F'_1 \subset \cdots \subset F'_{k-1}$, of rank $k-1$ in C, by $F'_i = F_i$ for $i < j$ and $F'_i = F_{i+1} \cap C$ for $i = j, \ldots, k-1$.

If, on the other hand, $\mathrm{rank}(\tilde{B}_W) = k+1$, then note that $\tilde{B}_W$ is dual defect. Indeed, suppose $\tilde{B}_W$ has a non-splitting k-flag $G_1 \subset \cdots \subset G_k$. Then, by Lemma 22, we may assume without loss of generality that $\langle G_k \rangle \cap \langle \sigma(\Theta_1) \rangle = \{0\}$. But then $G_1 \subset \cdots \subset G_k \subset \Theta_1$ would be a non-splitting flag of rank $k+1$ in B, a contradiction. Hence, by inductive hypothesis, $\tilde{B}_W$ has a homogeneous, non dual-defect flat C of rank at least two and such that $\langle C \rangle \cap \langle \sigma(\Theta_1) \rangle = \{0\}$. Therefore, C is a flat in B as well and the proof is complete. □

We return now to the proof of Theorem 25. We prove the existence of (23) by induction on the rank m. If $m = 2$ then, being irreducible, B is not dual defect and we may take $B = C_1$.

Suppose the theorem holds for configurations of rank less than m and let B be an irreducible, dual defect configuration of rank m. By Lemma 28, there exists a homogeneous, non dual-defect, B-flat $C_1 \subset B$. We may assume that C_1 is not contained in any larger, homogeneous, non dual-defect B-flat and $\mathrm{rank}(C_1) < m$. Let $B_1 = B \backslash C_1$. Clearly, B_1 is homogeneous and irreducible. If B_1 is not dual defect then taking $C_2 = B_1$ we are done. On the other hand, if B_1 is dual defect and of rank less than m, then we may apply the inductive hypothesis to write $B_1 = C_2 \cup \cdots \cup C_s$ where the C_j are maximal, homogeneous, disjoint, non dual-defect subconfigurations of B_1 and, for $i \geq 2$, C_i is a flat in $C_i \cup C_{i+1} \cup \cdots C_s$. Finally, if $\mathrm{rank}(B_1) = m$, we repeat the argument and write B_1 as a disjoint union $B_1 = C_2 \cup B_2$, where C_2 is a homogeneous non dual-defect B_1 flat. Since at each step the cardinality of the remaining homogeneous configuration B_j strictly decreases, it is clear that this process terminates.

In order to prove the second assertion, consider a non-splitting flag $\mathcal{F}$ of rank k in B. We claim that, for each $p \leq k$, there exist C_i-flats $F_{i,p} \subset C_i \cap F_p$ such that

(1) $\langle F_p \rangle = \langle F_{1,p} \rangle \oplus \cdots \oplus \langle F_{s,p} \rangle$ and

(2) if $\sigma(F_{i,p}) \in \langle F_{i,p-1} \rangle$, then $F_{i,p} = F_{i,p-1}$.

Clearly, this would imply the result since the distinct flats among the $F_{i,p}$, $p = 1, \ldots, k$ would define a non-splitting flag in C_i whose rank would, therefore, be bounded by $\mathrm{rank}(C_i) - 1$. To prove the claim we proceed by induction on p. If $p = 1$, then we may assume $F_1 \subset C_1$ and it suffices to choose $F_{1,1} = F_1$ and $F_{i,1} = \emptyset$ for $i > 1$. Suppose now that we have constructed $F_{i,p-1}$, $i = 1, \dots, s$ and set $G_{i,p} := C_i \cap F_p$. Then F_p is the disjoint union of the C_i-flats $G_{i,p}$, for $i = 1, \dots, s$. Let i_0 be the first index such that $\sigma(G_{i_0,p}) \notin \langle F_{p-1} \rangle$. Such an index exists since $\mathcal{F}$ is a non-splitting flag. Since $\sigma(G_{i_0,p}) \notin \langle F_{p-1} \rangle$, there exists a C_{i_0}-flat $F_{i_0,p}$ such that $F_{i_0,p-1} \subset F_{i_0,p} \subset G_{i_0,p}$ and $\operatorname{rank}(F_{i_0,p}) = 1 + \operatorname{rank}(F_{i_0,p-1})$. Set $F_{i,p} = F_{i,p-1}$ for $i \neq i_0$. Note that since $F_{i_0,p} \not\subset F_{p-1}$ $\operatorname{rank}(F_{1,p} \cup \dots \cup F_{s,p})$ must be strictly larger than $p-1$. Hence $\langle F_p \rangle = \langle F_{1,p} \rangle + \dots + \langle F_{s,p} \rangle$ and, for dimensional reasons, this must be a direct sum. $\square$

We have shown in Theorem 21 that if $A \subset \mathbb{Z}^d$ is a dual defect homogeneous configuration of codimension four, which is not a pyramid, and B is a Gale dual then either B is degenerate or $\tilde{B} = C_1 \cup C_2$, and $\langle C_1 \rangle$ and $\langle C_2 \rangle$ are complementary planes. In this case, if $\tilde{A}$ is a dual of $\tilde{B}$ then $\tilde{A}$ is a union of homogeneous, codimension-two configurations lying in complementary subspaces of $\mathbb{Z}^d$. Similarly, if B is a degenerate configuration consisting of vectors in a splitting line and in a complementary three-dimensional space, then A is a union of two homogeneous configurations, of codimension one and three respectively, lying in complementary subspaces of $\mathbb{Z}^d$. In either case, the projective toric variety X_A is obtained from a join of two varieties by attaching codimension-one configurations according to (8).

More generally, if B is decomposed as in (23) and A is a dual of B, then A will be a Cayley configuration of s configurations $A_0, \dots, A_{s-1}$ in $\mathbb{Z}^q$, where $q = |B| - \operatorname{rank}(B) - s$, in the following sense:

Definition 29 *Let $A_0, \dots, A_k \subset \mathbb{Z}^r$ be configurations. The configuration*

$$\operatorname{Cay}(A_0, \dots, A_k) := (\{e_0\} \times A_0) \cup \dots \cup (\{e_k\} \times A_k) \subset \mathbb{Z}^{k+1} \times \mathbb{Z}^r,$$

where $e_0, \dots, e_k$ is the standard basis of $\mathbb{Z}^{k+1}$, is called the Cayley configuration of $A_0, \dots, A_k$.

In the special case when $B = C_1 \cup \dots \cup C_s$, as in Theorem 25, is an irreducible configuration such that

$$\langle B \rangle = \langle C_1 \rangle \oplus \dots \oplus \langle C_s \rangle,$$

then, if $A \subset \mathbb{Z}^d$ is dual to B, the toric variety X_A is a join of varieties $X_{A_1}, \dots, X_{A_s}$ lying in disjoint linear subspaces and the dual variety X_A^* has codimension s. However, as the following example shows, for codimension greater than four, it is no longer true that every dual defect toric variety is obtained from a join by attaching codimension-one configurations according to (8).

Example. Let A be the Cayley configuration in $\mathbb{Z}^4$,

$$A := \operatorname{Cay}(\{0,1,2\}, \{0,1,2\}, \{0,1,2\}).$$

The variety X_A is a smooth three-fold in $\mathbb{P}^8$. It is easy to show that a Gale dual $B \subset \mathbb{Z}^5$ may be decomposed as $B = C_1 \cup C_2 \cup C_3$, where C_i is an irreducible, homogeneous, codimension-two configuration and, therefore, non dual-defect. Let $\rho(B)$ be as in (24). Then $\rho(B) = 3 = \operatorname{rank}(B) - 2$ and, by Theorem 25, B is dual defect. In fact using Theorem 16 one can show that X_A^* is a six-dimensional subvariety of $\mathbb{P}^8$.

Di Rocco has obtained a classification of dual defect projective embeddings of smooth toric varieties in terms of their associated polytopes (Di Rocco, 2004). Recall that a homogeneous configuration A is said to be *saturated* if $A = \{a_1, \dots, a_n\}$ consists of all the integer points of a $d-1$ dimensional polytope with integer vertices, P, lying on a hyperplane off the origin. Moreover, the projective toric variety X_A is smooth, if and only if the polytope P is *Delzant*, that is, for each vertex v of P, there exist $w_1, \ldots, w_d \in \mathbb{Z}^d$, such that $\{w_1, \ldots, w_d\}$ is a lattice basis of $\mathbb{Z}^d$, and $P = v + \sum_{j=1}^{d} \mathbb{R}_+ \cdot w_j$ near v. It is well known that projective embeddings of smooth toric varieties are in one-to-one correspondence with Delzant polytopes.

Di Rocco's classification theorem (Di Rocco, 2004, Theorem 5.12), which is proved by techniques completely different to the ones in this paper, may now be stated as follows:
Theorem 30 *Let A be a saturated, homogeneous, configuration in $\mathbb{Z}^d$ which is not a pyramid and such that $P = \mathrm{conv}(A)$ is Delzant. Then A is dual defect if and only if*

$$A = \mathrm{Cay}(A_0, \ldots, A_k),$$

where k is such that $\max(2, \frac{d}{2}) \leq k \leq d-1$, $A_0, \ldots, A_k$ are saturated and the polytopes $P_i := \mathrm{conv}(A_i) \subset \mathbb{R}^{d-k-1}$ are all Delzant polytopes of the same combinatorial type.

Thus, we see that the smoothness condition puts very strong conditions on the type of Cayley configuration we may consider. To illustrate this, we will list all smooth dual defect projective toric varieties of codimension at most four.

We note first of all that in these cases, the configurations A_i in Theorem 30 must be one-dimensional. In fact, let A be a dual defect, saturated, homogeneous, configuration in $\mathbb{Z}^d$ which is not a pyramid and such that $P = \mathrm{conv}(A)$ is Delzant, and write $A = \mathrm{Cay}(A_0, \ldots, A_k)$, as in Theorem 30. Then, if $\mathrm{codim}(X_A) \leq 5$, each polytope P_i must be one-dimensional. Indeed, let us consider the simplest case when the polytopes P_i are two-dimensional. Then $d = k+3$ and since by assumption $k \geq (k+3)/2$, we must have $k \geq 3$. The fewest number of integral points in a Delzant polytope in $\mathbb{R}^2$ is three. Hence $n = |A| \geq 12$ and $m = n - 6 \geq 6$.

Let $[p]$ denote the configuration $\{0, 1, \ldots, p\} \subset \mathbb{Z}$. An easy counting argument now shows that the smooth dual defect toric varieties of codimension less than or equal to four are the ones associated with the Cayley configurations listed below:

Codimension 2: $\mathrm{Cay}([1],[1],[1])$.

Codimension 3: $\mathrm{Cay}([1],[1],[2])$; $\mathrm{Cay}([1],[1],[1],[1])$.

Codimension 4: $\mathrm{Cay}([1],[2],[2])$; $\mathrm{Cay}([1],[1],[3])$; $\mathrm{Cay}([1],[1],[1],[2])$;
$\mathrm{Cay}([1],[1],[1],[1],[1])$.

The Gale duals of the configurations in the above list are easily computed. Indeed, it is easy to see that each Cayley factor $A_i = [1]$ contributes a splitting line containing two vectors from B, and this vectors are primitive relative to the lattice $\mathbb{Z}^m$. Similarly, each factor $A_j = [k]$ contributes a homogeneous subconfiguration C_j of rank k and containing exactly $k+1$ primitive vectors in B. Thus, for example, in the codimension four case, the configuration $\mathrm{Cay}([1],[2],[2])$ has a Gale dual B whose reduced configuration $\tilde{B}$ decomposes as $C_1 \cup C_2$, where C_i are homogeneous configurations of rank two, lying in complementary planes, and consisting of three primitive vectors each.

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