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Abstract

Non-extreme black hole solutions of four dimensional, $N = 2$ supergravity theories with Calabi-Yau prepotentials are presented, which generalize certain known double-extreme and extreme solutions. The boost parameters characterizing the nonextreme solutions must satisfy certain constraints, which effectively limit the functional independence of the moduli scalars. A necessary condition for being able to take certain boost parameters independent is found to be block diagonality of the gauge coupling matrix. We present a number of examples aimed at developing an understanding of this situation and speculate about the existence of more general solutions.

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1. Introduction

Considerable effort has been devoted recently to studying black hole solutions in four-dimensional, $N = 2$ supergravity theories [1-16]. Interest has been focused, so far, on extreme black holes, which satisfy additional supersymmetry constraints and saturate a BPS bound. A key discovery [3] in this case is that the values of the scalar moduli fields of the $N = 2$ vector multiplets are actually fixed at the black hole horizon in terms of the electric and magnetic charges carried by the black hole. In particular, the horizon values of the scalar fields are independent of the values of the scalar fields at infinity. The evolution of the scalar fields moving inward from infinity towards the horizon can then be thought of as motion in a kind of attractor [3]. Of particular interest are the "double-extreme" solutions, for which the scalar fields stay fixed at their horizon values throughout the spacetime [9]. These are "doubly" extreme in the sense that, in addition to having degenerate horizons, the black hole mass, for these solutions, is minimized for the given charges. "Singly" extreme solutions with non-constant scalars are given in [11].

In this paper we will look at non-extreme black hole solutions in $N = 2$ theories in four dimensions, obtained by dimensional reduction of Type II supergravity on a Calabi-Yau threefold. Since the basic form of the extreme solutions in this case [11] is quite similar to certain supersymmetric, intersecting brane solutions of torus compactifications [17,18], a simple ansatz for the non-extreme $N = 2$ black holes arises from the known non-extreme intersecting brane solutions in torus compactifications [19]. This ansatz is also analogous to the non-extreme generalization of the extreme black branes solution of M-theory [20]. In this ansatz, given below, there is a single "non-extremality" parameter μ and a number of "boost parameters" γ_Λ related to the individual charges. We find below, however, that this ansatz does not in general solve the equations of motion. Rather, the equations of motion reduce to a condition which may be regarded as a constraint on the boost parameters. The only general (*i.e.* for all Calabi-Yau manifolds) solution to this constraint, which we have found, is when all the boost parameters are taken to be equal. For specific models, such as the STU model and others discussed below, it is possible to take separate boost parameters.

We have not yet explored these constraints fully. In the case of torus compactifications of $D = 11$ supergravity, the general non-extreme solutions of [19] may be obtained from the $D = 10$ Schwarzschild solution via various combinations of boosts, dimensional upliftings

and reductions and duality symmetries. We note that these same methods cannot be used to similarly construct the non-extreme $N = 2$ solutions. [1]

2. The Basic Setup: $N = 2$ Lagrangian

We give only a brief summary of the formalism here. A more complete treatment may be found in, *e.g.*, [9]. An $N = 2$ supergravity theory in four dimensions includes, in addition to the graviton multiplet, n_v vector multiplets and n_h hypermultiplets. In our work we consistently take the hypermultiplet fields to be constant and will ignore them below. The bosonic part of the action is then given by[2]

$$S = \int d^4x \sqrt{-G} \left[R - 2g_{A\bar{B}} \partial_\nu z^A \partial^\nu \bar{z}^B - \tfrac{1}{4} \left(F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu} \mathrm{Im} N_{\Lambda\Sigma} + F^\Lambda_{\mu\nu} * F^{\Sigma\mu\nu} \mathrm{Re} N_{\Lambda\Sigma} \right) \right], \quad (1)$$

where $G_{\mu\nu}$ is the spacetime metric, z^A with $A = 1, \ldots, n_v$ are complex scalar moduli fields parametrizing a special Kähler manifold and $F^\Lambda_{\mu\nu} = 2\partial_{[\mu} A^\Lambda_{\nu]}$ with $\Lambda = 0, 1, \ldots, n_v$ are the field strengths of $n_v + 1$ $U(1)$ gauge fields A^Λ_μ. Here, the complex scalars are related to the holomorphic symplectic sections X^Λ by the inhomogeneous coordinates condition

$$z^A = \frac{X^A}{X^0} \quad (2)$$

The Kähler potential K, scalar metric $g_{A\bar{B}}$ and gauge couplings $N_{\Lambda\Sigma}$ are all determined in terms of the prepotential $F(X)$, which is a holomorphic, second-order homogeneous function. The Kähler potential K is given by

$$e^{-K} = i \left(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda \right) \quad (3)$$

where $F_\Lambda = \partial F / \partial X^\Lambda$. The Kähler metric on the scalar moduli space is then given by $g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K(z, \bar{z})$ where $\partial_{\bar{A}} = \partial / \partial \bar{z}^A$ and the gauge field couplings $N_{\Lambda\Sigma}$ by

$$N_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i(\mathrm{Im} F_{\Lambda\Delta})(\mathrm{Im} F_{\Sigma\Gamma}) X^\Gamma X^\Delta / \left(X^\Omega X^\Phi \mathrm{Im} F_{\Omega\Phi} \right) \quad (4)$$

where $F_{\Lambda\Sigma} = \partial F_\Lambda / \partial X^\Sigma$.

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[1] After this work was completed, we found that the same ansatz for the non-extreme solutions had been made in [13]. We disagree with the claim there that the ansatz generally satisfies the equations of motion.

[2] We use the normalization $\epsilon_{\hat{t}\hat{r}\hat{\vartheta}\hat{\varphi}} = 1$.

For type II supergravity compactified on a Calabi-Yau space, the prepotential takes the form

$$F(X) = \frac{d_{ABC}X^A X^B X^C}{X^0},$$ (5)

where the constants d_{ABC}, with ABC completely symmetric, are the topological intersection numbers of the manifold. We further restrict our interest here to the axion free case, in which all the moduli scalars z^A are pure imaginary. The gauge coupling matrix $N_{\Lambda\Sigma}$ is then pure imaginary, having nonzero components

$$N_{00} = -d_{ABC}z^A z^B z^C \quad , \quad N_{AB} = -6d_{ABC}z^C + 9\frac{d_{ACD}z^C z^D d_{BEF}z^E z^F}{d_{GHI}z^G z^H z^I}$$ (6)

and the Kähler metric is given by

$$g_{A\bar{B}} = \frac{N_{AB}}{4N_{00}}$$ (7)

The equations of motion following from the action (with $\mathrm{Re}N = 0$) are given by

$$\partial_\mu \left(\sqrt{-G}F^{\Lambda\mu\nu}\mathrm{Im}N_{\Lambda\Sigma} \right) = 0$$ (8)

$$16g_{A\bar{B}}\nabla^\nu\partial_\nu\bar{z}^B + 8(\partial_A g_{B\bar{C}})\partial^\mu z^B \partial_\mu \bar{z}^C - (\partial_A \mathrm{Im}N_{\Lambda\Sigma})F_{\mu\nu}^\Lambda F^{\Sigma\mu\nu} = 0$$ (9)

$$R_{\mu\nu} - 2g_{A\bar{B}}(\partial_\mu z^A)\partial_\nu \bar{z}^B - \frac{1}{2}\left(F_{\mu\sigma}^\Lambda F_\nu^{\Sigma\sigma} - \frac{g_{\mu\nu}}{4}F_{\rho\sigma}^\Lambda F^{\Sigma\rho\sigma} \right)\mathrm{Im}N_{\Lambda\Sigma} = 0.$$ (10)

3. Non-Extreme Solutions

We want to generalize certain double-extreme and extreme solutions, which were given in [9] and [11] respectively. In these solutions, the gauge field $F_{\mu\nu}^0$ carries only electric charge, while each gauge field $F_{\mu\nu}^A$ carries only magnetic charge. As discussed in [9,11], regarded as a compactification of M-theory on $S^1 \times CY$, these solutions correspond to fivebranes wrapping 4-cycles of the Calabi-Yau space, with a boost along the common string. For the special case of a torus compactification, the corresponding non-extreme solutions are given in [19]. It is straightforward to modify the solutions there to get an ansatz for the non-extreme solutions in the present case,

$$ds^2 = -e^{-2U}f dt^2 + e^{2U}\left(f^{-1}dr^2 + r^2 d\Omega^2 \right) \quad , \quad e^{2U} = \sqrt{H_0 d_{ABC}H^A H^B H^C}$$

$$f = 1 - \frac{\mu}{r} \quad , \quad z^A = iH^A H_0 e^{-2U} \quad , \quad H^A = h^A\left(1 + \frac{\mu}{r}\sinh^2\gamma_A \right)$$

$$A_t^0 = \frac{r\widetilde{H}_0'}{h_0 H_0} \quad , \quad A_\varphi^C = r^2\cos\vartheta\, \widetilde{H}^{C'} \quad , \quad \widetilde{H}^A = h^A\left(1 + \frac{\mu}{r}\cosh\gamma_A\sinh\gamma_A \right)$$ (11)

$$H_0 = h_0\left(1 + \frac{\mu}{r}\sinh^2\gamma_0 \right) \quad , \quad \widetilde{H}_0 = h_0\left(1 + \frac{\mu}{r}\cosh\gamma_0\sinh\gamma_0 \right),$$

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where prime denotes ∂_r. Nonzero components of the gauge field strengths are

$$F_{tr}^0 = \frac{\widetilde{H}_0'}{H_0^2} \quad , \quad F_{\varphi\vartheta}^A = r^2 \sin\vartheta \widetilde{H}^{A'}. \tag{12}$$

The ansatz (11) reduces to the "singly" extreme solutions given in [11] when the limit $\mu \to 0, \gamma_\Lambda \to \infty$ is taken with $\mu \sinh^2 \gamma_\Lambda \equiv k_\Lambda$ held fixed and further to the "doubly" extreme solutions, with constant moduli scalars, in [9] when all the k_Λ are the same. It can also be shown that, if the solution (11) with $H_0 = \widetilde{H}_0 = 1$ satisfies the equations of motion, then the solution with more general H_0 and $\widetilde{H}_0$, as given in (11), satisfies the equations of motion. This corresponds to a boost transformation in M-theory compactified on $S^1 \times CY$. Henceforth, in checking the equations of motion, we will set $H_0 = \widetilde{H}_0 = 1$.

It is straightforward to check that the ansatz (11) satisfies the gauge field equation of motion (8). Equation (10) for the curvature reduces to the condition

$$r^2 \mathrm{Im} N_{AB} \left(f H^{A'} H^{B'} - \widetilde{H}^{A'} \widetilde{H}^{B'} \right) = 2\mu \left(e^{2U} \right)', \tag{13}$$

and the scalar field equation (9) leads to

$$r^2 \left(\partial_A \mathrm{Im} N_{BC} \right) \left(\widetilde{H}^{C'} \widetilde{H}^{B'} - f H^{C'} H^{B'} \right) = 8\mu e^{2U} g_{A\bar{B}} \bar{z}^{B'}. \tag{14}$$

In deriving these last two equations we have made use of the fact that the extreme solutions, with $f = 1$ and $(H_0, H^A) = (\widetilde{H}_0, \widetilde{H}^A)$, satisfy the equations of motion. Note that both sides of equations (13) and (14) vanish identically in this case. We also note that $\mathrm{Im} N_{BC} = -i N_{BC}$ by virtue of (6) is a first order homogeneous function of z^A and that, in particular, $z^A \partial_A \mathrm{Im} N_{BC} = \mathrm{Im} N_{BC}$. This property can be used to "contract" equation (14) with z^A to obtain equation (13). Thus it is only necessary to show that the ansatz (11) (with $H_0 = \widetilde{H}_0 = 1$) satisfies (14).

It is not difficult to see that, for an arbitrary choice of the constants d_{ABC} in the prepotential, the condition (14) is not satisfied unless the parameters γ_A are taken to be equal. This differs from the case of intersecting branes on a torus [19], for parameters γ_A may be specified independently for each set of branes. We do not at present fully understand the significance of the restrictions placed by (14) on the parameters γ_A. Note that, if all the boost parameters, including γ_0, are set equal to some common value γ in (11), then the scalars z^A will be constant, having values

$$z^A = i h^A h_0, \tag{15}$$

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where the asymptotic flatness condition, $h_0 d_{ABC} h^A h^B h^C = 1$, has been used. This case is then a non-extreme version of the "doubly" extreme black holes in [9]. Taking γ_0 to be different, as may always be done, makes the scalars z^A non-constant, but keeps their ratios constants. Clearly, if some, or all, of the γ_A's may also be taken unequal, then there will be additional functional independence between the scalars. In the next section, we will explore some simple examples of prepotentials for which some, or all, of the γ_A's may be specified independently.

4. Examples

We list below some choices for the d_{ABC} which allow some of the γ_A's also to be different from each other. It follows from (13), that a *necessary* condition for (at least) some of the γ_A's to be independent is that the gauge coupling matrix $\text{Im} N_{AB}$ be block diagonal. In this case there turns out to be one independent parameter per block. From this point of view, it seems consistent that γ_0 may always be specified independently of γ_A, since N_{0A} vanishes as evident by (6), and hence N_{00} forms a 1×1 block.

Our first example is the *STU* model [9] for which the only nonzero d_{ABC} is d_{123}. In this case the coupling matrix $\text{Im} N_{BC}$ is diagonal and all three parameters $\gamma_1, \gamma_2, \gamma_3$ may all be specified independently. However, when quantum corrections are added to the *STU* model [9,11] d_{333} becomes nonzero. This makes the coupling matrix $\text{Im} N_{BC}$ completely nondiagonal, which in turn implies that the γ_A's must be taken equal.

As a second example, we can take only the constants d_{1AB} to be nonzero, where $A, B \neq 1$ (a similar model is considered in [5]). The coupling matrix $\text{Im} N_{BC}$ in this case is block diagonal, having a 1×1 block and an $(n_v - 1) \times (n_v - 1)$ block. It follows that γ_1 can be chosen independently of the γ_A for $A \neq 1$, which must all be the same.

A specialization of the previous example is to take only d_{12B} nonzero with $B = 3 \dots n_v$. This makes $\text{Im} N_{BC}$ block diagonal with two 1×1 blocks and one $(n_v - 2) \times (n_v - 2)$ block and one can have three different γ's: γ_1, γ_2 and one more γ_B for $B = 3 \dots n_v$. [3]

As a final example we consider a simple toy model where only d_{112} and d_{111} are nonzero. In this case $\text{Im} N_{BC}$ is diagonal if and only if $d_{111} = 0$, *i.e.* $\gamma_1 = \gamma_2$ is required unless $d_{111} = 0$. In each of these cases block diagonality of the gauge coupling matrix $\text{Im} N_{BC}$ appears to be both a necessary and a sufficient condition to be able to take independent γ's, though we have not been able to show this generally.

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[3] Notice that if one specializes this last example one step further one ends up with the *STU* model (without the quantum correction).

5. Physical Parameters and Discussion

We examine the physical properties of the non-extreme solutions. In particular, we want to check, given the restrictions on the γ_A's, that the charges may still be specified arbitrarily, as they can in the extreme limit [9,11]. We will first display all formulae as if the γ_A's can be specified independently and then discuss the actual solutions, in which the γ_A's are restricted. After imposing the asymptotic flatness condition, the set of independent parameters for the solutions can be taken to be $\{\mu, \gamma_0, h^A, \gamma_A\}$. These can be exchanged for the more physical set $\{E, q_0, p^A, \gamma_A\}$, where E is the ADM mass, q_0 the electric charge for $F_{\mu\nu}^0$ and p^A the magnetic charges for $F_{\mu\nu}^A$. The ADM energy is given by[4]

$$E = \tfrac{1}{2}\left[\mu + \tfrac{1}{2}\left(k_0 + 3h_0 d_{ABC} h^A h^B \mathcal{K}^C\right)\right] \tag{16}$$

where $\mathcal{K}^C \equiv h^C k_C$ and $k_\Lambda = \mu \sinh^2 \gamma_\Lambda$ as above. The electric charge q_0 and magnetic charges p^A are defined by

$$q_0 = \frac{1}{4\pi}\int *F_{\vartheta\varphi}^0 \mathrm{Im} N_{00}\ d\vartheta d\varphi\ , \quad p^A = \frac{1}{4\pi}\int F_{\vartheta\varphi}^A\ d\vartheta d\varphi. \tag{17}$$

We find

$$q_0 = \frac{\mu h_0 \sinh 2\gamma_0}{2}\ , \quad p^A = \frac{\mu h^A \sinh 2\gamma_A}{2} \tag{18}$$

The Hawking temperature is

$$T = \frac{1}{4\pi\mu\sqrt{\lambda_0 d_{ABC}\lambda^A\lambda^B\lambda^C}} \tag{19}$$

where $\lambda_0 = h_0 \cosh^2 \gamma_0$ and $\lambda^A = h^A \cosh^2 \gamma_A$ and the Bekenstein entropy is

$$S = \pi\mu^2\sqrt{\lambda_0 d_{ABC}\lambda^A\lambda^B\lambda^C}. \tag{20}$$

First, note that equation (18) implies that, even in the case that all boost parameters are set equal, the charges q_0, p^A may still be chosen arbitrarily by virtue of the constants h^A and the single boost parameter γ. As we observed above, the restrictions on the γ_A should be regarded as restrictions on the functional independence of the scalars z^A, with respect to one another. Next, we note that, for all the examples discussed in the last section, the formulae for the temperature (19) and the entropy (20) simplify considerably.

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[4] In order to simplify the formulae we explicitly display h_0 bearing in mind that it can be regarded as a function of h^A.

The square roots in (19) and (20) can be "gotten rid of", in these cases, because the λ factors appearing in the each term of the sums are identical. For example, in the d_{1AB} model, the entropy (20) reduces to

$$S = \pi \mu^2 \cosh \gamma_0 \cosh \gamma_1 \cosh^2 \gamma, \tag{21}$$

where $\gamma = \gamma_A$ for $A = 2 \ldots n_v$.

It remains an open question, whether, or not, more general non-extreme solutions (static, axion-free and carrying only the charges q_0 and p^A) exist. These might, for example, have independent boost parameters for each of the Calabi-Yau 4-cycles. In the case of orthogonally intersecting branes on a torus [19], there are at most four independent parameters corresponding to a boost and three sets of branes. However, the most general black hole solutions in type II theory compactified to 4-dimensions on a torus are described by 28 electric and 28 magnetic charges (see *e.g.* [21]). The extreme solutions in this case arise via collections of branes intersecting *non-orthogonally* [22]. It may be necessary to look at a non-extreme solution based on branes intersecting at angles to get the most general solution in the Calabi-Yau case as well. It would also be interesting to try to construct the solutions, which we have found here, using the available symmetry transformations, which in the present case include boosts in the time direction and symplectic transformations.

Finally, it should also be possible to find nonextreme solutions in $N = 2$ theories with prepotentials not of the Calabi-Yau form. We note that since (13) and (14) are derived using the extreme solution and since they are displayed not in terms of the particular prepotential we have used in this paper, they are generally applicable to finding non-extreme black hole solutions for other prepotentials. In particular the block diagonality of $\mathrm{Im}N_{AB}$ is a *necessary* condition for the existence of more than one γ_A. We emphasize that the derivation of (13) and (14) does not depend on any specific expression for e^{2U} and depends only on the fact that $\mathrm{Re}N = 0$, $F^0_{\mu\nu} = 0$, and $F^A_{\mu\nu}$ carries only magnetic charge.

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