1994

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DAMTP R/93/27
UMHEP-371
NSF-ITP-93-125
hep-th/9310118

Supersymmetric Self-Gravitating Solitons

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ABSTRACT

We show that the 'instantonic' soliton of five-dimensional Yang-Mills theory and the closely related BPS monopole of four-dimensional Yang-Mills/Higgs theory continue to be exact static, and stable, solutions of these field theories even after the inclusion of gravitational, electromagnetic and, in the four-dimensional case, dilatonic interactions, provided that certain non-minimal interactions are included. With the inclusion of these interactions, which would be required by supersymmetry, these exact self-gravitating solitons saturate a gravitational version of the Bogomol'nyi bound on the energy of an arbitrary field configuration.

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1. Introduction

Many non-linear field theories in flat spacetime have soliton solutions (by which we mean localised solutions of finite energy). Two examples, which will serve as a basis for our discussion are the BPS monopoles of four-dimensional ($d = 4$) Yang-Mills(YM)/Higgs theory and instantons viewed as solitons of five-dimensional ($d = 5$) YM theory*. An important feature in each of these systems is the existence of a Bogomol'nyi bound [1] on the energy in terms of the topological charge of the configuration (monopole charge or instanton number, respectively). Static soliton configurations saturate these bounds. Mathematically this implies that these solutions satisfy certain first-order equations (Bogomol'nyi or self duality, respectively). One can find analytic solutions of these first-order equations describing single solitons, as well as static multi-soliton configurations. These multi-soliton solutions are possible because the solitons satisfy a force balance with respect to each other. For example, in the case of BPS monopoles, the repulsive force due to the gauge field is exactly compensated by an attractive force due to the scalar field.

One can ask whether a similar situation holds for self-gravitating solitons. In general the answer is clearly no because coupling gravity to the systems described above disturbs the force balance; there will be a net attractive force between solitons and static multi-soliton solutions will no longer exist. We shall see below, however, that the force balance can be restored and energy bounds shown to exist, if one couples solitonic matter to the bosonic fields of ($N = 2$) *supergravity*. Such a coupling is actually quite natural. Witten and Olive [2] showed that, for the BPS monopole, the Bogomol'nyi bound may be derived by embedding the theory in one with $N = 2$ supersymmetry. The key element to this result is that the topological charge of the soliton appears as a central charge in the supersymmetry algebra. We might expect that coupling this globally supersymmetric matter theory to $N = 2$

∗ It is convenient to consider these examples together because the $d = 4$ Bogomol'nyi YM /Higgs equations are the dimensionally reduced $d = 5$ YM self-duality equations.

supergravity will lead to energy bounds for gravitating BPS monopoles.

In general relativity, the energy and momentum of an asymptotically flat four dimensional spacetime are given by the ADM 4-momentum, which is a surface integral over the 2-sphere at spatial infinity. An analogue of a Bogomol'nyi bound for a gravitating soliton should be a variant of the positive energy theorem [3,4], which bounds the ADM 4-momentum in terms of the topological charge of the solitonic matter fields. Coupling to supergravity is natural in this context as well. Witten's [4] proof of the positive energy theorem was motivated by [5] and is most clearly understood in the context of supergravity [6]. In supergravity, the global supercharges are also given by surface integrals at spatial infinity [7]. The supercharges combine with the ADM 4-momentum to form the global supersymmetry algebra. We expect that, as in the case of global supersymmetry [2], the topological charge of the matter fields will enter this algebra as a central charge.

The $N = 2$ supergravity multiplet in four dimensions contains the graviton, a pair of gravitini and a $U(1)$ gauge field, which we will refer to as the Maxwell field**. The Maxwell field introduces another force, which can restore the force balance between solitons***. If we restrict our attention to purely bosonic configurations then we are dealing with solitonic matter coupled to Einstein/Maxwell theory. Gibbons and Hull [9] have proven a version of the positive energy theorem for Einstein/Maxwell theory, which we here generalise to include a dilaton with arbitrary coupling constant b (as defined by the action (6.1) to follow). Given certain conditions on the stress-energy and charge current of the matter fields, one has that

$$M \geq \frac{1}{\sqrt{1+b^2}}\sqrt{Q^2+P^2}\ , \tag{1.1}$$

where M is the ADM mass and Q and P are the total electric and magnetic charges

** We will be considering ungauged supergravity in which the gravitini are neutral with respect to the Maxwell field. Coupling the gravitini to the Maxwell field introduces a cosmological constant.

*** We note that over the past few years many gravitating soliton solutions have been found in supergravity theories arising in the low energy limit of string theory [8]. In many of these, the force balance is restored by the antisymmetric tensor field.

of the spacetime with respect to the Maxwell field. In the context of supergravity, Q and P enter the algebra of global supersymmetry transformations as central charges. Spacetimes which saturate the bound (1.1) are those which have spinor fields which are constant with respect to a 'supercovariant' derivative operator. This property can be thought of as an analogue of the first order Bogomol'nyi equations. The puzzle remains of how the topological charge of the matter becomes a source for the Maxwell field. As we shall see below, a certain non-minimal Maxwell/matter interaction achieves precisely this. This interaction might ordinarily be disregarded because it is non-renormalisable, but it is required by $N = 2$ local supersymmetry. Recall that the Maxwell field is in the same supermultiplet as the graviton so, in the context of supersymmetry, it is not surprising that it should have non-renormalisable interactions with matter fields.

In Section 2, we look at the case of $SU(2)$ YM/Higgs matter fields in 4 dimensions, coupled to gravity and an additional Maxwell field but *without* a dilaton. In this case, we are able to show that an energy bound on solitons exists, but we are unable to find analytic solutions saturating this bound. In an appendix, we present a proof that such solutions do exist, at least for sufficiently weak values of the gravitational coupling. In Sections 3, 4 and 5, we establish results for instantonic solitons in $d = 5$ supergravity/YM theory. In this case we are able to find analytic solutions for the matter configurations. The solutions we find are non-singular, static, self-gravitating, particle-like configurations with a (multi) instanton core. For arbitrary core radius the solutions have the property that they saturate a Bogomol'nyi - type bound, which we derive following the method used for $d = 4$ Einstein/Maxwell theory in [9]. In the context of $d = 5$ supergravity the solutions preserve half the supersymmetry of the vacuum. For this reason we refer to configurations which saturate the bound as 'supersymmetric'.

We then return in Sections 5 and 6 to the $d = 4$ case, but now including a dilaton field. We first derive the bound (1.1) and find the vacuum solutions which saturate this bound. These are previously constructed charged dilaton black holes which reduce to the standard Reissner-Nordström (RN) solution in the $b \rightarrow$

0 limit. The inclusion of solitonic matter in the form of YM/Higgs monopoles is consistent with the continued existence of a super-covariantly constant spinor provided that the dilaton couples to the YM/Higgs fields in a particular way. A surprising feature is that, once the dilaton/matter coupling is fixed in this way, the $b \to 0$ limit cannot be taken because the action would be singular. The results with and without dilaton are therefore qualitatively different and must therefore be considered separately. The case without a dilaton will be considered in Section 2, as mentioned above. In Section 7 we analyse the dilatonic case. In both cases self-gravitating monopoles, should they exist, saturate a gravitational analogue of the Bogomol'nyi bound, but we are able to find exact solutions of the curved space YM/Higgs equations that saturate this bound only in the dilatonic case. In fact, in the dilatonic case one finds that these equations are solved by solutions of the *flat space* Bogomol'nyi equations. (Multi) monopole solutions of the latter are well known and the existence of self-gravitating (multi) monopoles saturating the gravitational version of the Bogomol'nyi bound are therefore guaranteed. The resulting 4-metrics are non-singular and without horizons. Without a dilaton, as shown in Section 2, the curved space YM/Higgs equations are still solved by any solution of a version of the Bogomol'nyi equations, but the latter are no longer the flat space equations and their solution requires the simultaneous solution of the Einstein equations. The fact that *exact* solutions of the curved space YM/Higgs equations can be found in the dilatonic case can be seen to be a consequence, at least for a particular value of b, of our $d = 5$ results. This is because certain results for $d = 4$ can be obtained from $d = 5$ by dimensional reduction; one simply solves the $d = 5$ equations for spaces of topology $\mathbb{R}^3 \times S^1$ instead of $\mathbb{R}^4$. The existence of exact self-gravitating (multi) monopole solutions for a particular $d = 4$ theory is therefore guaranteed from the $d = 5$ results. Our results obtained directly in $d = 4$ are consistent with this observation.

2. Four Dimensional Supergravitating Monopoles

In this section, we will search for self-gravitating monopole solutions in $3+1$ dimensions which saturate an energy bound. The strategy we follow here will be repeated in other contexts in later sections. In the present case, however, the calculations necessary to establish the new results will be shorter than those in subsequent sections because we can take advantage of a number of already established results. This section can then serve the reader as a less technical introduction to some of the methods used in the remainder of the paper.

In $3+1$ dimensions, the positive energy theorem establishes that the norm of the ADM four momentum P_μ satisfies $P_\mu P^\mu \leq 0$ (with signature $(-+++)$), with equality only for Minkowski space. A stronger result can be derived for Einstein/Maxwell theory [9,10]. If Q is the total electric charge of the Maxwell field* and $M^2 \equiv -P_\mu P^\mu$, then we have

$$M \geq \mid Q \mid \ . \tag{2.1}$$

We are looking for 'ground states', i.e. configurations which minimise the ADM mass subject to fixed electric charge. The bound (2.1) is known to be saturated if and only if the spacetime admits a complex spinor field ϵ which is constant with respect to the supercovariant derivative

$$\hat{\nabla}_\mu = \nabla_\mu + \frac{i}{4} F_{\alpha\beta}\Gamma^\alpha\Gamma^\beta\Gamma_\mu \ , \tag{2.2}$$

where $F_{\alpha\beta}$ is the Maxwell field strength. One class of such spacetimes are the extremal Reissner-Nordström solutions and their multi-black hole generalisations the, Majumdar-Papapetrou solutions. These are electro-vac solutions. In this paper, however, we are interested in spacetimes containing solitonic matter fields and for which the bound (2.1) is still saturated.

$\star$ We assume zero magnetic charge for simplicity

In 3 + 1 dimensions, the form of the most general metric and gauge field admitting a supercovariantly constant spinor has been given in [9,11]. If we restrict our attention to static metrics and gauge fields, then these have the form

$$ds^2 = -e^{-2\phi}dt^2 + e^{2\phi}d\mathbf{x}\cdot d\mathbf{x} \qquad A = \pm e^{-\phi}dt\ , \tag{2.3}$$

where $\phi = \phi(\mathbf{x})$ and $d\mathbf{x}\cdot d\mathbf{x}$ is the Euclidean 3-metric. We can now find what matter sources are required to produce fields of this form by simply demanding that the Einstein-Maxwell equations be solved and then determining the sources. From the Einstein and Maxwell constraint equations, we then find that the charge density ρ_q and energy density ρ_m of the matter fields (excluding the energy density of the Maxwell field) are equal (up to a sign) and given by

$$\mid \rho_q \mid = \rho_m = -\frac{1}{4\pi G}e^{-3\phi}\nabla^2\left(e^{\phi}\right)\ , \tag{2.4}$$

where ∇^2 is the Laplacian for Euclidean 3-space. Furthermore, one finds that the Einstein and Maxwell evolution equations imply that the spatial components of the matter stress energy T_{ij} and the charge current J_i all vanish

$$T_{ij} = J_i = 0\ . \tag{2.5}$$

These conditions on the matter sources characterise charge equal mass (or $q = m$) dust. Moreover, any spatial configuration of $q = m$ dust solves the Einstein-Maxwell-matter system and saturates the energy bound. One can simply choose the dust configuration by specifying a function $f(\mathbf{x})$, solve $\nabla^2\left(e^{\phi}\right) = -4\pi G f$, to obtain the metric function, and recover the energy and charge densities via $\rho_m = |\rho_q| = e^{-3\phi}f$. Note that $f(\mathbf{x}) = 0$ yields the equation $\nabla^2\left(e^{\phi}\right) = 0$, the solutions of which are the Majumdar-Papapetrou electro-vac spacetimes.

Next we shall show that solitonic matter fields saturating an appropriate Bogomol'nyi constraint can have energy and charge densities of the $q = m$ dust form.

We consider YM/Higgs matter coupled to Einstein/Maxwell theory via the action

$$
\begin{aligned}
S = \frac{1}{4\pi G} \int d^4x\, e \left\{ \frac{1}{4} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} \\
+ \frac{1}{g^2} \int d^4x \left\{ -\frac{1}{4} e\, \mathrm{tr}\left(G_{\mu\nu} G^{\mu\nu} \right) - \frac{1}{2} e\, \mathrm{tr}\left(D_\mu \Phi D^\mu \Phi \right) + \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} \mathrm{tr}\left(\Phi G_{\alpha\beta} \right) \right\} \ ,
\end{aligned}
\tag{2.6}
$$

where $e = \sqrt{-\det g_{\mu\nu}}$ with $g_{\mu\nu}$ the spacetime metric, $G_{\mu\nu}$ is the covariant field strength for a Lie-algebra-valued YM vector potential B_μ

$$
G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \ , \tag{2.7}
$$

Φ is the Higgs field in the adjoint representation and $D_\mu \Phi$ its YM covariant derivative

$$
D_\mu \Phi = \partial_\mu \Phi + [B_\mu, \Phi] \ . \tag{2.8}
$$

The YM and Higgs field are assumed to have dimensions of L^{-1}, so the $d = 4$ YM coupling constant g has dimensions of $(ML)^{-(1/2)}$. The coupling between the Maxwell and the YM/Higgs fields is motivated by supersymmetry; in a supersymmetric theory the coefficient of this term is non-zero and our results lead us to believe that the value appearing in (2.6) is that singled out by supersymmetry*. It is helpful to keep in mind that the Coulomb force law for the Maxwell field normalized as in (2.6) is $F = GQ_1Q_2/r^2$.

Observe, that to preserve parity we require either Φ or A_μ to be parity odd. Since the $N = 2$ Maxwell supermultiplet contains both a scalar and a pseudoscalar Higgs field either one can be used in a monopole solution. If we use the scalar then we should interpret A_μ as the magnetic vector potential rather than the electric vector potential. We shall suppose here that Φ is a pseudoscalar and A_μ the electric potential.

$\star$ This could be verified by a construction of the general $N = 2$ supergravity/YM/Higgs theory in the conventions of this paper.

Consider static configurations of the YM and Higgs fields, having

$$D_0\Phi = 0, \qquad G_{0i} = 0 \ , \tag{2.9}$$

and take the metric and Maxwell field to have the form (2.3). The charge density ρ_q, found by varying the action with respect to the Maxwell field is given by

$$4\pi G\rho_q = -\frac{1}{2g^2}e^{-3\phi}\varepsilon^{ijk}\mathrm{tr}\left(D_i\Phi G_{jk}\right) \ , \tag{2.10}$$

where ε^{ijk} is the 3-dimensional alternating tensor density. The matter energy density ρ_m is given by

$$\begin{aligned}4\pi G\rho_m &= \frac{1}{2}\mathrm{tr}\left(D_i\Phi D^i\Phi\right) + \frac{1}{4}\mathrm{tr}\left(G_{ij}G^{ij}\right)\\ &= \frac{1}{2g^2}e^{-2\phi}\mathrm{tr}\left[D_i\Phi D_i\Phi + \frac{1}{2}e^{-2\phi}G_{ij}G_{ij}\right] \ ,\end{aligned} \tag{2.11}$$

where the latter equality is found by making use of the fact that $g_{ij} = \delta_{ij}e^{2\phi}$. Now, if the YM and Higgs field satisfy a curved space form of the Bogomol'nyi equation, specifically

$$\sqrt{g^{(3)}}g^{il}g^{jm}G_{lm} = \pm\varepsilon^{ijk}D_k\Phi \ , \tag{2.12}$$

where $g^{(3)}$ is the determinant of the 3-metric g_{ij}, then the expression for the energy density (2.11) becomes identical to that for the charge density (2.10) and we have $\rho_m = \rho_q$. One can also check that, given (2.12), the spatial components of the matter stress energy T_{ij} and charge current J_i vanish. Note that since $g_{ij} = e^{2\phi}\delta_{ij}$, and ε^{ijk} is independent of the metric, (2.12) is equivalent to the Euclidean 3-space equation

$$G_{ij} = \pm e^{\phi}\varepsilon^{ijk}D_k\Phi \ . \tag{2.13}$$

In addition, given the form of the metric and the Maxwell field, then if (2.12) is satisfied, one can check that both the YM and Higgs equations of motion are satisfied as well. Unfortunately, the curved space Bogomol'nyi equation (2.13) depends on the metric function e^{ϕ}, and we have not been able to find analytic solutions to this coupled system. However, specialising to the case of spherical symmetry, we can show that solutions exist, at least for sufficiently weak gravitational coupling. We give a proof of this in the Appendix. We shall show later that when a dilaton field is included, the curved space Bogomol'nyi equation becomes identical to the flat space one, and so exact solutions can easily be found.

We end this section by presenting an alternative derivation of the results presented above that is closer in spirit to Bogomol'nyi's original argument [1]. We insert the result (2.3) into the matter part of the action (2.6). From the corresponding Hamiltonian the total energy is then found to be

$$\begin{aligned} E = \frac{1}{g^2} \int d^3x \Big\{ & \frac{1}{4} e^{-2\phi} \mathrm{tr}\big(G_{ij} \mp e^{\phi} \varepsilon^{ijk} D_k \Phi\big)^2 \\ & + \frac{1}{2} e^{2\phi} \big[\mathrm{tr}(G_{0i}G_{0i}) + e^{2\phi} \mathrm{tr}(D_0\Phi D_0\Phi)\big] \pm \eta \partial_i \left[e^{-\phi} B_i^{mat} \right] \Big\} , \end{aligned} \tag{2.14}$$

where

$$B_i^{mat} = \frac{1}{2\eta} \varepsilon^{ijk} \mathrm{tr}\left(\Phi G_{jk}\right) , \tag{2.15}$$

is the 'matter' magnetic field. Assuming that $\mathbf{B}^{mat} \sim 1/r$ and that $\phi \to 0$ as $r \to \infty$, we immediately derive the bound

$$E \geq \frac{\eta}{g^2} \mid P^{mat} \mid , \tag{2.16}$$

where P^{mat} is the (dimensionless) total 'matter' magnetic charge determined by the flux of $\mathbf{B}^{mat}$ through the 2-sphere at spatial infinity S_2

$$P^{mat} = \int_{S_2} B_i^{mat} dS_i . \tag{2.17}$$

By integrating (2.10) we see that

$$Q = \frac{\eta}{g^2} P^{mat} , \tag{2.18}$$

and so we have $E \geq Q$, in agreement with (2.1). The bound (2.16) is saturated when

$$G_{0i} = 0, \qquad D_0\Phi = 0, \qquad G_{ij} = \pm e^{\phi}\varepsilon^{ijk} D_k\Phi \ , \tag{2.19}$$

in agreement with (2.9) and (2.13).

3. An Energy Bound For d=5 Einstein/Maxwell Theory

The bosonic sector of $d = 5$ supergravity coupled to matter [12,13,14] has the action

$$S = \frac{1}{4\pi G_5}\int d^5x \Big\{ -\frac{1}{4}eR - \frac{1}{4}eF_{\mu\nu}F^{\mu\nu} - \frac{1}{6\sqrt{3}}\varepsilon^{\mu\alpha\beta\gamma\delta}A_\mu F_{\alpha\beta}F_{\gamma\delta}\Big\} + S_{matter} \ . \tag{3.1}$$

Here, $R = g^{\mu\nu}R^\lambda{}_{\mu\lambda\nu}$ is the scalar curvature, $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the Maxwell field-strength tensor, and $e = \sqrt{-g}$ is the determinant of the fünfbein $e_\mu{}^{\underline{\alpha}}$, where an underlined index refers to the locally inertial Lorentz frame. For $d = 5$ we shall use the 'mostly minus' metric signature. Note that A_μ is a dimensionless pseudovector. The dimensions of G_5 (the $d = 5$ Newton's constant) are L^2M^{-1}.

The Einstein and Maxwell field equations are

$$\begin{aligned} G_{\mu\nu} - 2T_{\mu\nu}(F) &= 8\pi G_5 T_{\mu\nu}(mat.) \\ \nabla_\mu F^{\mu\nu} - \frac{1}{2\sqrt{3}}e^{-1}\varepsilon^{\nu\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} &= 4\pi G_5 J^\nu(mat.), \end{aligned} \tag{3.2}$$

where

$$T_{\mu\nu}(F) = -\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) , \tag{3.3}$$

and

$$T_{\mu\nu}(mat.) \equiv \frac{1}{2e}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \qquad J^\mu(mat.) \equiv \frac{1}{e}\frac{\delta S_{matter}}{\delta A_\mu} \ . \tag{3.4}$$

We are now going to establish a Bogomol'nyi-type bound on the energy of any field configuration, following the derivation in [9] of a similar bound for $d = 4$

Einstein/Maxwell theory. To this end we introduce the Nester-like tensor [15]

$$\hat{E}^{\mu\nu} = \frac{1}{2}\bar{\epsilon}\Gamma^{\mu\nu\rho}\hat{\nabla}_\rho\epsilon \ + c.c. \ , \tag{3.5}$$

where ϵ is a complex commuting $SO(1,4)$ spinor, $\bar{\epsilon}$ its Dirac conjugate $\epsilon^\dagger\Gamma^{\underline{0}}$ and

$$\hat{\nabla}_\mu\epsilon \equiv \nabla_\mu\epsilon + \frac{1}{4\sqrt{3}}\big(\Gamma^{\alpha\beta}\Gamma_\mu + 2\Gamma^\alpha\delta_\mu^\beta\big)\epsilon F_{\alpha\beta} \ . \tag{3.6}$$

Here Γ^α is a Dirac matrix and $\Gamma^{\alpha_1\cdots\alpha_n}$ is the antisymmetrised product of n Dirac matrices (with 'strength one'). The product of all five Dirac matrices equals the unit matrix up to a sign; we choose the sign such that $e\Gamma^{\alpha_1\ldots\alpha_5} = \varepsilon^{\alpha_1\ldots\alpha_5}$. The covariant derivative ∇ is defined in terms of the usual (torsion free) spin connection ω. The definition (3.6) is directly motivated by $d = 5$ supergravity since the supersymmetry transformation of the (complex) gravitino in that theory, in a background for which the gravitino vanishes, is just $\delta\psi_\mu = \hat{\nabla}_\mu\epsilon$ (although ϵ would be anticommuting in this context).

Let Σ be a spacelike hypersurface with hypersurface element dS_μ and with boundary $\partial\Sigma$ at spatial infinity. If the spacetime contains black holes, then there will also be components of $\partial\Sigma$ corresponding to the black hole horizons. As shown in reference [10], the boundary terms on the horizons can be made to vanish if ϵ satisfies $\Gamma^{\underline{n}}\Gamma^{\underline{0}}\epsilon = \epsilon$ where $\underline{n}$ indicates the direction normal to the horizon. We shall assume that this condition holds below. With this in mind we consider the integral

$$\int_\Sigma dS_\mu\, e\nabla_\nu\hat{E}^{\mu\nu} = \int_\Sigma dS_\mu\partial_\nu\big(e\hat{E}^{\mu\nu}\big) = \frac{1}{2}\int_{\partial\Sigma} dS_{\mu\nu}\, e\hat{E}^{\mu\nu} \ , \tag{3.7}$$

where $dS_{\mu\nu}$ is the element of the 3-sphere at spatial infinity. Now, $\hat{E}^{\mu\nu}$ can be written as

$$\hat{E}^{\mu\nu} = \frac{1}{2}\left\{\bar{\epsilon}\Gamma^{\mu\nu\rho}\nabla_\rho\epsilon - \frac{\sqrt{3}}{4}\bar{\epsilon}\Gamma^{\mu\nu\alpha\beta}\epsilon F_{\alpha\beta} - \frac{\sqrt{3}}{2}\bar{\epsilon}\epsilon F^{\mu\nu}\right\} \ + c.c. \ . \tag{3.8}$$

Assuming that the spacetime is asymptotically flat and that, asymptotically, F is purely electric and ϵ approaches the *constant* spinor ϵ_∞, we have that

$$\begin{aligned}\int_\Sigma dS_\mu \, e \nabla_\nu \hat{E}^{\mu\nu} &= \int_{\partial\Sigma} dS_{\mu\nu} \left[\frac{1}{4} e\bar{\epsilon}_\infty \Gamma^{\mu\nu\rho} \nabla_\rho \epsilon_\infty \; + \; c.c.\right] - \frac{\sqrt{3}}{4} (\bar{\epsilon}_\infty \epsilon_\infty) \int_{\partial\Sigma} dS_{\mu\nu} \, eF^{\mu\nu} \\ &= (\bar{\epsilon}_\infty \Gamma^\mu \epsilon_\infty) \, P_\mu^{ADM} - \frac{\sqrt{3}}{2} (\bar{\epsilon}_\infty \epsilon_\infty) \, Q \ ,\end{aligned} \tag{3.9}$$

where, in the second line, we have used the Witten-Nester expression for the ADM 5-momentum, and $Q = \frac{1}{2} \int_{\partial\Sigma} dS_{\mu\nu} \, (eF^{\mu\nu})$ is the total electric charge.

We turn now to the evaluation of $\nabla_\nu \hat{E}^{\mu\nu}$. A lengthy calculation yields the result

$$\nabla_\nu \hat{E}^{\mu\nu} = \left\{ \frac{1}{2} (\overline{\hat{\nabla}_\nu \epsilon}) \Gamma^{\mu\nu\rho} \hat{\nabla}_\rho \epsilon + c.c. \right\} + (4\pi G_5) K^\mu \ , \tag{3.10}$$

where we have used the Einstein/Maxwell field equations (3.2) and

$$K^\mu = \bar{\epsilon} \Gamma^\nu \epsilon T_\nu{}^\mu (mat.) + \frac{\sqrt{3}}{2} \bar{\epsilon} \epsilon J^\mu (mat.) \ . \tag{3.11}$$

Combining (3.10) with (3.9) we deduce that

$$\begin{aligned}(\bar{\epsilon}_\infty \Gamma^\mu \epsilon_\infty) \, P_\mu^{ADM} - \frac{\sqrt{3}}{2} (\bar{\epsilon}_\infty \epsilon_\infty) \, Q = &\int_\Sigma dS_\mu e \left[\frac{1}{2} \overline{\left(\hat{\nabla}_\nu \epsilon\right)} \Gamma^{\mu\nu\rho} \left(\hat{\nabla}_\rho \epsilon\right) \; + c.c. \right] \\ &+ (4\pi G_5) \int_\Sigma dS_\mu e K^\mu \ .\end{aligned} \tag{3.12}$$

The first term on the right hand side of (3.12) is non-negative for spinors ϵ satisfying the (modified) Witten condition [4]

$$n \cdot \hat{\nabla} \epsilon = 0 \ , \tag{3.13}$$

where n is a 5-vector normal to Σ; it vanishes if and only if

$$\hat{\nabla}_\mu \epsilon = 0 \ . \tag{3.14}$$

We shall call a spinor satisfying (3.14) a Killing spinor.

The second term on the right hand side of (3.12) is non-negative if K^μ is future-directed timelike, or zero, for all ϵ, and we shall assume this condition in what follows. This term vanishes if and only if $K^\mu = 0^\star$. Under these conditions, the right hand side of (3.12) is non-negative and hence so is the left hand side. This implies that

$$M \geq \frac{\sqrt{3}}{2} Q \ , \tag{3.15}$$

where $M = \sqrt{P^{ADM} \cdot P^{ADM}}$ is the ADM mass. This bound can be saturated only if the right hand side of (3.12) vanishes which, as we have now seen, requires that

$$\hat{\nabla}_\mu \epsilon = 0 \qquad K^\mu = 0 \ . \tag{3.16}$$

Let us define $V^\mu = \bar{\epsilon}\Gamma^\mu\epsilon$. Note that

$$\hat{\nabla}_{(\mu} V_{\nu)} = \overline{\left(\hat{\nabla}_{(\mu}\epsilon\right)} \Gamma_{\nu)}\epsilon + \bar{\epsilon}\Gamma_{(\nu}\hat{\nabla}_{\mu)}\epsilon \ , \tag{3.17}$$

so that if $\hat{\nabla}_\mu\epsilon = 0$ then V^μ is a Killing vector field. Note also the identity (for $V^\mu = \bar{\epsilon}\Gamma^\mu\epsilon$), valid in $d = 5$,

$$(\Gamma \cdot V)\,\epsilon = sgn\,(\bar{\epsilon}\epsilon)\mid V \mid \epsilon \ , \tag{3.18}$$

which implies, in particular, that

$$V \cdot V = (\bar{\epsilon}\epsilon)^2 \geq 0 \ . \tag{3.19}$$

Hence, V^μ is timelike or null. At infinity (3.18) reduces to

$$(\Gamma \cdot V_\infty)\,\epsilon_\infty = sgn\,(\bar{\epsilon}_\infty\epsilon_\infty)\mid V_\infty \mid \epsilon_\infty \ . \tag{3.20}$$

We now observe that the vanishing of the left hand side of (3.12) requires that ϵ_∞

⋆ Choose a frame for which $K^i = 0$. Then, since $K^0 \geq 0$, the integral $\int_\Sigma dS_\mu e K^\mu$ vanishes if and only if $K^0 \equiv 0$. But if the integral vanishes, K^0 must vanish in every frame and hence, given the assumed condition on K^μ, so must K^μ.

satisfy

$$\left(\Gamma \cdot P^{ADM}\right)\epsilon_\infty = sgn(Q) \mid P^{ADM} \mid \epsilon_\infty \ . \tag{3.21}$$

By comparison with (3.20) and choosing $sgn\,(\bar\epsilon_\infty \epsilon_\infty) = sgn(Q)$ we deduce that V_∞ is parallel to P^{ADM}. For static spacetimes and matter field configurations for which $V = \frac{\partial}{\partial t}$, equation (3.18) reduces to

$$\Gamma^{\underline{0}}\epsilon = \pm\epsilon \ , \tag{3.22}$$

and $K^\mu = 0$ is equivalent to

$$T_{\underline{00}}(mat.) = \frac{\sqrt{3}}{2} \mid J_{\underline{0}}(mat.) \mid \ . \tag{3.23}$$

In the interior of Σ, V may become null on an event horizon (it cannot become spacelike). In this case $(\Gamma \cdot V)\,\epsilon = 0$, but this is not consistent with the boundary condition on ϵ at event horizons that was used in [10] unless ϵ vanishes at the horizon. However, the condition $(\Gamma \cdot V)\,\epsilon = 0$ is only needed to *saturate* the bound (3.15). We shall see from our explicit solutions to follow, that *either* there are no horizons when the bound is saturated *or* (in the absence of matter) ϵ vanishes at an infinitely distant horizon.

It would be of interest to determine the general configuration compatible with the constraint $\hat\nabla_\mu \epsilon = 0$, as has been done for $d = 4$ Einstein/Maxwell [11], but we shall be content here to establish the existence of a simple class of solutions found via the ansatz

$$\begin{aligned} &e_0{}^{\underline{i}} = e_i{}^{\underline{0}} = 0 \\ &e_i{}^{\underline{j}} = e^\phi \delta_i^j \qquad \dot\phi = 0 \\ &\dot e_0{}^{\underline{0}} = 0 \ . \end{aligned} \tag{3.24}$$

It follows that

$$\omega_{0\underline{i}\,\underline{0}} = -e^{-\phi}\partial_i e_0{}^{\underline{0}} \qquad \omega_{i\underline{j}\underline{k}} = -2\delta_{i[j}\partial_{k]}\phi \ , \tag{3.25}$$

all other components of the spin connection vanishing. Note that we need not distinguish between indices i and $\underline{i}$ on the right-hand sides of these equations since all tensors and derivative operators become those of Euclidean 4-space. Using the condition (3.22) equation (3.14) can now be shown to imply

$$e_0{}^{\underline{0}} = e^{-2\phi} \qquad F_{0i} = \mp\frac{\sqrt{3}}{2}\partial_i\big(e^{-2\phi}\big) \ , \tag{3.26}$$

and

$$\epsilon = e^{-\phi}\epsilon_\infty \ , \tag{3.27}$$

where ϵ_∞ is a constant spinor which we may identify, without loss of generality, as the normalised constant spinor introduced previously. These results imply that the metric has the form

$$ds^2 = e^{-4\phi}dt^2 \ - \ e^{2\phi}d\mathbf{x}\cdot d\mathbf{x} \ , \tag{3.28}$$

where $d\mathbf{x}\cdot d\mathbf{x}$ is the Euclidean 4-metric, and that the Maxwell potential one-form is

$$A = \pm\frac{\sqrt{3}}{2}e^{-2\phi}dt \ . \tag{3.29}$$

In the following section we shall determine and solve the conditions on ϕ that are required for this supersymmetric Einstein/Maxwell field configuration to solve the field equations (3.2), and study the particular case of no matter.

4. d=5 Supersymmetric Black Holes

Given (3.28) and (3.29), the left hand sides of the field equations (3.2) may be computed. The result is

$$\begin{aligned} G_{\underline{00}} - 2T_{\underline{00}}(F) &= -\frac{3}{2}e^{-4\phi}\Box e^{2\phi} \\ \partial_i\big(eF^{i0}\big) &= \mp\frac{\sqrt{3}}{2}\Box e^{2\phi} \ , \end{aligned} \tag{4.1}$$

all other components vanishing *identically*. Here, $\Box$ is the Laplacian for *Euclidean*

4-space. In the absence of matter we deduce that

$$\Box e^{2\phi} = 0 \ . \tag{4.2}$$

Let us impose the boundary condition $\phi = 0$ at spatial infinity (so $\epsilon \to \epsilon_\infty$ as required). The general solution with four-dimensional spherical symmetry is then

$$e^{2\phi} = 1 + \frac{a^2}{r^2} \ , \tag{4.3}$$

where r is the radial distance in the Euclidean 4-metric and a is a constant. The 5-metric (3.28) is then

$$ds^2 = \left(1 + \frac{a^2}{r^2}\right)^{-2} dt^2 - \left(1 + \frac{a^2}{r^2}\right) d\mathbf{x} \cdot d\mathbf{x} \ , \tag{4.4}$$

Rewriting this metric in terms of a new radial coordinate $\tilde{r}$ defined by $r^2 = \tilde{r}^2 - a^2$, it becomes

$$ds^2 = \left(1 - \frac{a^2}{\tilde{r}^2}\right)^{2} dt^2 - \left(1 - \frac{a^2}{\tilde{r}^2}\right)^{-2} d\tilde{r}^2 - \tilde{r}^2 d\Omega_3^2 \ , \tag{4.5}$$

where $d\Omega_3^2$ is the round metric on the unit 3-sphere. Substitution of this metric into the expression given earlier for the total mass M yields

$$a^2 = \frac{4G_5 M}{3\pi} \ . \tag{4.6}$$

This metric was found previously in [16]. Here we have shown that it is supersymmetric in the sense that it admits a Killing spinor.

To analyse the behaviour as $\tilde{r} \to a$ we define the new dimensionless variable λ by

$$\tilde{r} = a\left(1 + \frac{\lambda}{2}\right) \ . \tag{4.7}$$

Then the metric becomes

$$ds^2 = \left[\lambda^2 dt^2 - \frac{a^2}{4\lambda^2} d\lambda^2 - a^2 d\Omega_3^2\right] \{1 + O(\lambda)\} \ . \tag{4.8}$$

The asymptotic form of the metric near $\lambda = 0$ is found by neglecting the $O(\lambda)$

terms in this expression. The resulting metric is simply that of ad$S_2 \times S^3$. This fact implies, incidentally, that ad$S_2 \times S^3$ is an allowed compactification of pure $d = 5$ supergravity. Thus, as for the $d = 4$ RN black hole [17], the $d = 5$ black hole metric (4.4) interpolates between two vacuum solutions of $d = 5$ Einstein/Maxwell theory.

5. Supersymmetric Self-Gravitating Solitons For $d = 5$

It is clear from the result summarised in (4.1) that any 'matter' source for the Einstein/Maxwell equations that is consistent with the existence of a Killing spinor must be such that *the only non-vanishing components of the matter stress-tensor and electric current are* $T_{00}(mat.)$ *and* $J_0(mat.)$ and, furthermore, that

$$T_{\underline{00}}(mat.) = \frac{\sqrt{3}}{2} \mid J_{\underline{0}}(mat.) \mid \ . \tag{5.1}$$

Note that such matter currents satisfy the condition $K^\mu = 0$ of Section 2 which we found there to be necessary for saturation of the energy bound $M = \frac{\sqrt{3}}{2} \mid Q \mid$. This condition is satisfied by 'extremal charged dust' but here we are interested in matter in the form of field theory solitons. We now specify the 'matter' to be a Lie-algebra-valued YM potential B_μ with field-strength $G_{\mu\nu}$ ⋆

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \ . \tag{5.2}$$

We shall take the action to have the form

$$S_{matter} = \frac{1}{g_5^2} \int d^5x \left\{ -\frac{1}{4} e \, \mathrm{tr}\big(G_{\mu\nu} G^{\mu\nu}\big) + \frac{c}{4\sqrt{3}} \varepsilon^{\mu\alpha\beta\gamma\delta} A_\mu \mathrm{tr}\big(G_{\alpha\beta} G_{\gamma\delta}\big) \right\} \ , \tag{5.3}$$

for some dimensionless constant c †. The YM field will be assumed to have dimensions of L^{-1}, so the $d = 5$ YM coupling constant g_5 has dimensions of $M^{-(1/2)}$.

⋆ It should be obvious from the context when $G_{\mu\nu}$ refers to the field-strength of B_μ and when it refers to the Einstein tensor.

† Note that c *is not* the speed of light, which has been set to 1 throughout.

From this action, it follows that

$$
\begin{aligned}
T_{\mu\nu}(mat.) = T_{\mu\nu}(G) &\equiv -\frac{1}{g_5^2}\mathrm{tr}\left(G_{\mu\lambda}G_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}G^{\alpha\beta}G_{\alpha\beta}\right) \\
J^\mu(mat.) = J^\mu(G) &\equiv \frac{c}{4\sqrt{3}g_5^2}e^{-1}\varepsilon^{\mu\alpha\beta\gamma\delta}\mathrm{tr}\big(G_{\alpha\beta}G_{\gamma\delta}\big)\ .
\end{aligned}
\tag{5.4}
$$

Consider any YM field configuration for which $G_{i0} = 0$ and

$$
\sqrt{g^{(4)}}G^{ij} = \pm\frac{1}{2}\varepsilon^{ijkl}G_{kl}\ , \tag{5.5}
$$

where $g^{(4)} = \det g_{ij}$. It is not difficult to show that for such configurations all components of $T_{\mu\nu}(G)$ and $J_\mu(G)$ vanish except $T_{00}(G)$ and $J_0(G)$ and, furthermore, that

$$
T_{\underline{00}}(G) = \frac{c\sqrt{3}}{2}\mid J_{\underline{0}}(G)\mid\ . \tag{5.6}
$$

Hence, to satisfy (5.1) we must set

$$
c = 1\ . \tag{5.7}
$$

This is precisely the value given by coupling super YM theory to $d = 5$ supergravity [13].

It remains to check whether (5.5) satisfies the YM field equation. This is

$$
D_\mu\big(eG^{\mu\nu}\big)\ +\ \frac{c}{2\sqrt{3}}\varepsilon^{\nu\alpha\beta\rho\sigma}F_{\alpha\beta}G_{\rho\sigma} = 0\ , \tag{5.8}
$$

where D_μ is the YM covariant derivative. Since F is purely electric and G is purely 'magnetic' the time component of this equation is trivially satisfied. The space component reads

$$D_i\big(eG^{ij}\big) - \frac{c}{\sqrt{3}}\varepsilon^{jklm}F_{0k}G_{lm} = 0 \ . \tag{5.9}$$

Into this equation we substitute

$$e = e_0{}^{\underline{0}}\sqrt{g^{(4)}} = e^{-2\phi}\sqrt{g^{(4)}} \qquad\qquad F_{0i} = \mp\frac{\sqrt{3}}{2}\partial_i\big(e^{-2\phi}\big) \ , \tag{5.10}$$

to reduce it to

$$e^{-2\phi}\partial_i\left(\sqrt{g^{(4)}}G^{ij}\right) - \partial_j\big(e^{-2\phi}\big)\left(\sqrt{g^{(4)}}G^{ij} \mp \frac{c}{2}\varepsilon^{ijkl}G_{kl}\right) = 0 \ , \tag{5.11}$$

which is solved by any solution of (5.5), provided again that $c = 1$.

Observe that for any conformally flat 4-metric, (5.5) is equivalent to the *Euclidean* 4-*space* YM self-duality equation.

$$G_{ij} = \pm\frac{1}{2}\varepsilon^{ijkl}G_{kl} \ . \tag{5.12}$$

We have therefore shown that any *flat space* YM instantonic-soliton configuration, together with a metric and Maxwell one-form of the form (3.28) and (3.29), is a supersymmetric solution of the Euler-Lagrange equations of the action

$$\begin{aligned} S =& \frac{1}{4\pi G_5}\int d^5x\left\{-\frac{1}{4}eR - \frac{1}{4}eF_{\mu\nu}F^{\mu\nu} + \frac{1}{6\sqrt{3}}\varepsilon^{\mu\alpha\beta\gamma\delta}A_\mu F_{\alpha\beta}F_{\gamma\delta}\right\} \\ &+ \frac{1}{g_5^2}\int d^5x\left\{-\frac{1}{4}e\,\mathrm{tr}\big(G_{\mu\nu}G^{\mu\nu}\big) + \frac{1}{4\sqrt{3}}\varepsilon^{\mu\alpha\beta\gamma\delta}A_\mu\mathrm{tr}\big(G_{\alpha\beta}G_{\gamma\delta}\big)\right\} \ . \end{aligned} \tag{5.13}$$

Note that this includes multi-soliton solutions.

To fully specify the solution we must determine ϕ for a given soliton source. To do this we return to (3.2) and use the results (4.1) and (5.4) to find that

$$\Box e^{2\phi} = -\frac{1}{6}\left(\frac{4\pi G_5}{g_5^2}\right) \Big| \, \varepsilon^{ijkl}\mathrm{tr}\big(G_{ij}G_{kl}\big) \, \Big| \ , \tag{5.14}$$

which now replaces (4.2). For any choice of instanton or multi-instanton localised within a finite 'core' region of 4-space, the solution of this equation has the asymptotic form

$$e^{2\phi} \sim 1 + \left(\frac{16\pi G_5}{3g_5^2}\right)\frac{|\,\nu\,|}{r^2} \ , \tag{5.15}$$

where ν is the instanton number

$$\nu = \frac{1}{16\pi^2}\int d^4x \, \varepsilon^{ijkl}\mathrm{tr}\big(G_{ij}G_{kl}\big) \ . \tag{5.16}$$

Comparison with the extreme $d = 5$ black hole solution of the previous section yields the result

$$M = \left(\frac{2\pi}{g_5}\right)^2 |\,\nu\,| \ . \tag{5.17}$$

For non-zero core radius (or radii in the multi-soliton case) the resulting spacetime is non-singular and without event horizons. In the limit of vanishing soliton core radius the asymptotic solution (5.15) becomes exact and we recover the extreme $d = 5$ RN solution with mass M. Note that not all values of the mass are obtained in this way because (5.17) is a quantisation condition. Note also that since $|\,Q\,| = (2/\sqrt{3})M$, we also have the relation

$$|\,Q\,| = \frac{8\pi^2}{\sqrt{3}g_5^2} \ |\,\nu\,| \tag{5.18}$$

between the electric charge Q and the soliton's topological charge ν. Finally we remark that if the 4-space is asymptotically $\mathbb{R}^3 \times S^1$ instead of $\mathbb{R}^4$, solutions of (5.12) yield (multi) monopole solutions of the dimensionally reduced $d = 4$ theory, but we shall deal directly with the $d = 4$ case in the following sections.

6. An Energy Bound For $d = 4$ Einstein/Maxwell Theory With Arbitrary Dilaton Coupling

Motivated by consideration of the dimensional reduction of $d = 5$ supergravity to $d = 4$ we now turn our attention to a $d = 4$ action of the form

$$S = \frac{1}{4\pi G}\int d^4x\, e\left[\frac{1}{4}R - \frac{1}{4}e^{2b\sigma}F^2 - \frac{1}{2}(\partial\sigma)^2\right] \;+\; S_{matter}\ , \tag{6.1}$$

where σ is the dilaton field and b the dilaton coupling constant. We might also have included an axion field arising from the fifth component of the $d = 5$ Maxwell potential but since only A_0 was non-zero for the $d = 5$ solutions found above, we omit it. The $d = 4$ Newton's constant G has dimensions of LM^{-1}. *Note also that we now use a 'mostly plus' signature.*

The dimensional reduction and subsequent truncation of $d = 5$ supergravity to an action of the form (6.1) yields a value of $\sqrt{3}$ for the dilaton coupling constant b, and if $b = -1$ the action (6.1) is a truncation of $N = 4$ supergravity. At least for these values of b, there is an underlying $N = 2$ supergravity model. We believe that (6.1) is the truncation of some model of $N = 2$ supergravity coupled to a scalar multiplet for any value of b. All the results to follow are consistent with this hypothesis, but we have not checked it explicitly.

The field equations of (6.1) are

$$\begin{aligned} G_{\mu\nu} - 2T_{\mu\nu}(F) - 2T_{\mu\nu}(\sigma) &= (8\pi G)T_{\mu\nu}(mat.)\\ \nabla_\mu\left(e^{2b\sigma}F^{\mu\nu}\right) &= (4\pi G)J^\nu(mat.)\\ \partial_\mu\left(eg^{\mu\nu}\partial_\nu\sigma\right) - \frac{g}{2}ee^{2b\sigma}F^{\mu\nu}F_{\mu\nu} &= -(4\pi G)\frac{\delta S_{matter}}{\delta\sigma}\ , \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} T_{\mu\nu}(F) &= e^{2b\sigma}\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F^2\right)\\ T_{\mu\nu}(\sigma) &= \left(\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}g_{\mu\nu}(\partial\sigma)^2\right)\ , \end{aligned} \tag{6.3}$$

and

$$T_{\mu\nu}(mat.) \equiv \frac{1}{e}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \qquad J^{\mu}(mat.) \equiv -\frac{1}{e}\frac{\delta S_{matter}}{\delta A_{\mu}} \ . \tag{6.4}$$

As we did for $d = 5$, we now introduce the Nester-like tensor $\hat{E}^{\mu\nu}$ defined as before in terms of a modified covariant derivative $\hat{\nabla}_{\mu}$ acting on a complex spinor ϵ;

$$\hat{E}^{\mu\nu} = \frac{1}{2}\bar{\epsilon}\Gamma^{\mu\nu\rho}\hat{\nabla}_{\rho}\epsilon \ + c.c. \ . \tag{6.5}$$

For $d = 4$, and including the dilaton field, this modified covariant derivative is

$$\hat{\nabla}_{\mu}\epsilon = \nabla_{\mu}\epsilon + \frac{i}{4\sqrt{1+b^2}}e^{b\sigma}\Gamma^{\alpha\beta}\Gamma_{\mu}\epsilon\, F_{\alpha\beta} \ . \tag{6.6}$$

It will also be necessary to define the quantity

$$\delta\lambda \equiv \frac{1}{\sqrt{2}}\left[\Gamma^{\mu}\epsilon\, \partial_{\mu}\sigma + \frac{ib}{4\sqrt{1+b^2}}e^{b\sigma}\Gamma^{\alpha\beta}\epsilon\, F_{\alpha\beta}\right] \ . \tag{6.7}$$

The specific factors that appear in these definitions are motivated *a posteriori* as those that are required to derive the energy bound to follow, but they also have an *a priori* motivation as the supersymmetry transformation laws of the gravitino and dilatino fields in the associated supergravity model.

As before, let us choose Σ to be a spacelike hypersurface with hypersurface element dS_{μ} and boundary $\partial\Sigma$ at spatial infinity. We can write $\hat{E}^{\mu\nu}$ as

$$\hat{E}^{\mu\nu} = \frac{1}{2}\left\{\bar{\epsilon}\Gamma^{\mu\nu\rho}\nabla_{\rho}\epsilon - \frac{i}{\sqrt{1+b^2}}\bar{\epsilon}\left(F^{\mu\nu} + \frac{1}{2}\Gamma^{\mu\nu\alpha\beta}F_{\alpha\beta}\right)\epsilon e^{b\sigma}\right\} + c.c. \ . \tag{6.8}$$

Assuming that ϵ is asymptotic to a *constant spinor* ϵ_{∞} and that $\sigma \sim 0$ at spatial infinity, we deduce (using the fact that $e\Gamma^{\mu\nu\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta}\gamma_5$ where $\gamma_5 = \Gamma^{\underline{0}}\cdots\Gamma^{\underline{3}}$)

that

$$\int\limits_{\Sigma} dS_\mu \, e\nabla_\nu \hat{E}^{\mu\nu} = \int\limits_{\partial\Sigma} dS_{\mu\nu} \left[\frac{1}{4} e\bar{\epsilon}_\infty \Gamma^{\mu\nu\rho}\nabla_\rho \epsilon_\infty + c.c.\right]$$
$$- \frac{i}{2\sqrt{1+b^2}} \int\limits_{\partial\Sigma} dS_{\mu\nu} \left[e\bar{\epsilon}_\infty \epsilon_\infty F^{\mu\nu} + \bar{\epsilon}_\infty \gamma_5 \epsilon_\infty \tilde{F}^{\mu\nu}\right] \tag{6.9}$$
$$= \left(\bar{\epsilon}_\infty \Gamma^\mu \epsilon_\infty\right) P^{ADM}_\mu - \frac{i}{\sqrt{1+b^2}} \bar{\epsilon}_\infty \left(Q + \gamma_5 P\right) \epsilon_\infty \ ,$$

where P^{ADM}_μ is the ADM 4-momentum, $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ and

$$Q = \frac{1}{2} \int\limits_{\partial\Sigma} dS_{\mu\nu} \left(eF^{\mu\nu}\right) \qquad P = \frac{1}{2} \int\limits_{\partial\Sigma} dS_{\mu\nu} \tilde{F}^{\mu\nu} \ , \tag{6.10}$$

are the total electric and magnetic charges respectively.

A lengthy calculation yields the result

$$\nabla_\nu \hat{E}^{\mu\nu} = \left\{ -\frac{1}{2} \overline{\left(\hat{\nabla}_\nu \epsilon\right)} \Gamma^{\mu\nu\rho} \hat{\nabla}_\rho \epsilon - \frac{1}{2} \overline{\delta\lambda} \Gamma^\mu \delta\lambda + c.c. \right\} - \left(4\pi G\right) K^\mu \ , \tag{6.11}$$

where

$$K^\mu = \bar{\epsilon} \Gamma^\nu \epsilon T_\nu{}^\mu (mat.) + \frac{ie^{-b\sigma}}{\sqrt{1+b^2}} \bar{\epsilon}\epsilon J^\mu (mat.) \ . \tag{6.12}$$

Now upon using equations (6.9) and (6.11) we deduce that

$$\left(\bar{\epsilon}_\infty \Gamma^\mu \epsilon_\infty\right) P^{ADM}_\mu - \frac{i}{\sqrt{1+b^2}} \bar{\epsilon}_\infty \left(Q + \gamma_5 P\right) \epsilon_\infty = \int\limits_{\Sigma} dS_\mu e \left[-\frac{1}{2} \overline{\left(\hat{\nabla}_\nu \epsilon\right)} \Gamma^{\mu\nu\rho} \hat{\nabla}_\rho \epsilon \right.$$
$$\left. - \frac{1}{2} \overline{\delta\lambda} \Gamma^\mu \delta\lambda + c.c. \right] - \left(4\pi G\right) \int\limits_{\Sigma} dS_\mu e K^\mu \ , \tag{6.13}$$

The first term on the right hand side of (6.13) is non-negative for spinors ϵ satisfying the (modified) Witten condition

$$n \cdot \hat{\nabla} \epsilon = 0 \ , \tag{6.14}$$

where n is a 4-vector normal to Σ. The last term on the right hand side of (6.13) is non-negative if K^μ is future-directed timelike for all ϵ, and henceforth we shall assume this to be the case.

Under these conditions the right hand side of (6.13) is non-negative, and vanishes if and only if

$$\hat{\nabla}_\mu \epsilon = 0 \qquad \delta\lambda = 0 \qquad K^\mu = 0 \; . \tag{6.15}$$

Hence the left hand side of (6.13) is non-negative. This implies that

$$M \geq \frac{1}{\sqrt{1+b^2}} \sqrt{Q^2 + P^2} \; , \tag{6.16}$$

and the bound is saturated when the conditions (6.15) hold. Saturation of the bound implies that the left hand side of (6.13) vanishes, so that ϵ_∞ satisfies

$$\left(\Gamma^\mu P^{ADM}_\mu \right) \epsilon_\infty = \frac{i}{\sqrt{1+b^2}} \left(Q + \gamma_5 P \right) \epsilon_\infty \; . \tag{6.17}$$

As in the $d = 5$ case, we define

$$V^\mu = \bar{\epsilon} \Gamma^\mu \epsilon \; , \tag{6.18}$$

which is a Killing vector field when $\hat{\nabla}_\mu \epsilon = 0$. Note the identity, valid in $d = 4$,

$$\left(\Gamma \cdot V \right) \epsilon = \left[\bar{\epsilon}\epsilon + \left(\bar{\epsilon}\gamma_5\epsilon \right) \gamma_5 \right] \epsilon \; , \tag{6.19}$$

which implies that

$$V \cdot V = \left(\bar{\epsilon}\epsilon \right)^2 + \left(\bar{\epsilon}\gamma_5\epsilon \right)^2 \geq 0 \; . \tag{6.20}$$

Hence V is timelike or null. At infinity (6.19) reduces to (6.17) with $V \propto P^{ADM}$ and

$$\frac{P}{Q} = \frac{\bar{\epsilon}_\infty \gamma_5 \epsilon_\infty}{\bar{\epsilon}_\infty \epsilon_\infty} \; . \tag{6.21}$$

We shall be interested in static spacetimes admitting a Killing spinor and with

$P = 0$. In this case we can choose $\bar\epsilon\gamma_5\epsilon = 0$. Then equation (6.19) reduces to[⋆]

$$i\Gamma^{\underline{0}}\epsilon = \pm\epsilon \ , \tag{6.22}$$

where the timelike Killing vector field V is given by $V = \frac{\partial}{\partial t}$. In this case the condition $K^\mu = 0$ reduces to

$$T_{\underline{00}}(mat.) = \frac{e^{-b\sigma}}{\sqrt{1+b^2}} \mid J_{\underline{0}}(mat.) \mid \ . \tag{6.23}$$

Note that (6.22) is not compatible with the boundary condition on ϵ at event horizons, but again (6.22) is only required for *saturation* of the bound (6.16). As for $d = 5$, spacetimes that saturate the bound either have no horizons or horizons at an infinite distance at which ϵ vanishes.

We shall again seek solutions that can be found via the ansatz (3.24). We should note that the change of signature from the five dimensional case means that the components of the spin connection now become

$$\omega_{0\underline{i0}} = e^{-\phi}\partial_i e_0{}^{\underline{0}} \qquad \omega_{i\underline{jk}} = +2\delta_{i[j}\partial_{k]}\phi \ . \tag{6.24}$$

This now leads to a metric of the form

$$ds^2 = -e^{-2\phi}dt^2 + e^{2\phi}d\mathbf{x}\cdot d\mathbf{x} \ , \tag{6.25}$$

where $d\mathbf{x}\cdot d\mathbf{x}$ is the Euclidean 3-metric. It leads also to the Maxwell one-form

$$A = \pm\frac{1}{\sqrt{1+b^2}}e^{-(1+b^2)\phi}dt \ , \tag{6.26}$$

and to the dilaton

$$\sigma = b\phi \ . \tag{6.27}$$

The solution of $\hat\nabla_\mu\epsilon = 0$, $\delta\lambda = 0$ is then found to be

$$\epsilon = e^{-\frac{1}{2}\phi}\epsilon_\infty \ , \tag{6.28}$$

where ϵ_∞ is the constant spinor introduced previously, since we shall impose the

⋆ Observe that $\left(i\Gamma^{\underline{0}}\right)^2 = 1$ for the 'mostly plus' metric signature that we use for $d = 4$.

boundary condition that $\phi \sim 0$ as spatial infinity is approached.

7. Supersymmetric Dilaton Black Holes For d=4

Given the results just quoted, the Einstein, Maxwell and dilaton field equations of the action (6.1) become

$$\begin{aligned}(4\pi G)T_{\underline{00}}(mat.) &= -\frac{1}{1+b^2}e^{-(3+b^2)\phi}\nabla^2\left[e^{(1+b^2)\phi}\right]\\ (4\pi G)e^{-b\sigma}J_{\underline{0}}(mat.) &= \mp\frac{1}{\sqrt{1+b^2}}e^{-(3+b^2)\phi}\nabla^2\left[e^{(1+b^2)\phi}\right]\\ (4\pi G)\frac{\delta S_{matter}}{\delta\sigma} &= -\frac{b}{1+b^2}e^{-(1+b^2)\phi}\nabla^2\left[e^{(1+b^2)\phi}\right] \ ,\end{aligned} \tag{7.1}$$

where ∇^2 is the Laplacian of three-dimensional Euclidean space. In the absence of matter all three equations reduce to

$$\nabla^2\left[e^{(1+b^2)\phi}\right] = 0 \ . \tag{7.2}$$

The spherically symmetric solution corresponding to the boundary condition that ϕ vanish at spatial infinity is

$$e^{(1+b^2)\phi} = 1 + \frac{\mu}{r} \ , \tag{7.3}$$

for constant μ. Rewriting the metric in terms of the new radial coordinate $\tilde{r}$, now defined by $r = \tilde{r} - \mu$, we find that

$$ds^2 = -\left(1-\frac{\mu}{\tilde{r}}\right)^{\frac{2}{1+b^2}} dt^2 + \left(1-\frac{\mu}{\tilde{r}}\right)^{-\frac{2}{1+b^2}} d\tilde{r}^2 + \left(1-\frac{\mu}{\tilde{r}}\right)^{\frac{2b^2}{1+b^2}} \tilde{r}^2 d\Omega_2^2 \ , \tag{7.4}$$

where $d\Omega_2^2$ is the 'round' metric on S^2 and the dilaton is given by

$$e^{\sigma} = \left(1-\frac{\mu}{\tilde{r}}\right)^{-\frac{b}{1+b^2}} \ . \tag{7.5}$$

Substitution of (7.4) into the Witten-Nester expression for the total mass M yields the relation

$$\mu = (1 + b^2) GM \ , \tag{7.6}$$

between M and the constant μ. This extreme 'dilatonic' black hole solution has been found previously [18]. Here we have shown that it is supersymmetric.

The metric (7.4) is singular at $\tilde{r} = \mu$, but the rescaled metric $e^{2b\sigma} ds^2$ is non-singular. Using equations (7.4) and (7.5) we find the rescaled metric to be

$$e^{2b\sigma} ds^2 = -\left(1 - \frac{\mu}{\tilde{r}}\right)^{\frac{2(1-b^2)}{(1+b^2)}} dt^2 + \left(1 - \frac{\mu}{\tilde{r}}\right)^{-2} d\tilde{r}^2 + \tilde{r}^2 d\Omega_2^2 \ . \tag{7.7}$$

To analyse the behaviour as $\tilde{r} \to \mu$ we define the new (dimensionless) variable λ by

$$\tilde{r} = \mu \left(1 + \lambda\right) \ . \tag{7.8}$$

Then the rescaled metric becomes

$$e^{2b\sigma} ds^2 = \left[-\lambda^{\frac{2(1-b^2)}{(1+b^2)}} dt^2 + \left(\frac{\mu}{\lambda} d\lambda\right)^2 + \mu^2 d\Omega_2^2 \right] \left\{1 + O\left(\lambda\right)\right\} \ . \tag{7.9}$$

The asymptotic form of the metric near $\lambda = 0$ is found by neglecting the $O(\lambda)$ terms in this expression. Thus, introducing the new variable $\rho = \mu \ln \lambda$, we see that the rescaled metric has the asymptotic form

$$e^{2b\sigma} ds^2 \sim -e^{-\frac{2(1-b^2)}{(1+b^2)} \frac{\rho}{\mu}} dt^2 + d\rho^2 + \mu^2 d\Omega_2^2 \ , \tag{7.10}$$

and the dilaton has the asymptotic form

$$\sigma \sim \frac{-b}{(1 + b^2)} \frac{\rho}{\mu} \ . \tag{7.11}$$

If $b \neq \pm 1$ the resulting metric is simply that of $adS_2 \times S^2$, whilst if $b = \pm 1$ it is a metric for $\mathcal{M}^2 \times S^2$. Hence dilatonic extreme black holes interpolate between four-dimensional Minkowski space and a compactified vacuum spacetimes with a linear dilaton, at least for one choice of 'conformal gauge'. This behaviour is similar to that of the extreme fivebrane solution of $d = 10$ supergravity in string-theory conformal gauge [19].

8. Supersymmetric Self-Gravitating Monopoles With A Dilaton

We now return to the dilatonic action of (6.1) and take the matter action to be

$$\begin{aligned} S_{matter} =& \frac{1}{g^2}\int d^4x\Big\{ -\frac{1}{4}e^{\frac{(1-b^2)}{b}\sigma} e\,\mathrm{tr}(G_{\mu\nu}G^{\mu\nu}) - \frac{1}{2}e^{-\frac{(1+b^2)}{b}\sigma} e\,\mathrm{tr}(D_\mu\Phi D^\mu\Phi) \\ &+ \frac{c\sqrt{1+b^2}}{4}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}\mathrm{tr}(\Phi G_{\alpha\beta})\Big\}\ , \end{aligned} \tag{8.1}$$

where c is a constant (again, not to be confused with the speed of light which is set equal to unity) soon to be determined. The particular coupling of the dilaton to the YM/Higgs fields in (8.1) are those required for consistency of the YM/Higgs equations with the Einstein/Maxwell /dilaton equations and the assumption that the metric, Maxwell one-form and dilaton have the form found in Section 6. From (8.1) and the definitions (6.4) we have that

$$\begin{aligned} T_{\mu\nu}(mat.) =& \frac{1}{g^2}e^{\frac{(1-b^2)}{b}\sigma}\mathrm{tr}\left(G_{\mu\lambda}G_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}G_{\alpha\beta}G^{\alpha\beta}\right) \\ &+ \frac{1}{g^2}e^{-\frac{(1+b^2)}{b}\sigma}\mathrm{tr}\left(D_\mu\Phi D_\nu\Phi - \frac{1}{2}g_{\mu\nu}D_\alpha\Phi D^\alpha\Phi\right) \\ J^\nu(mat.) =& \frac{ce^{-1}}{2g^2}\sqrt{1+b^2}\varepsilon^{\nu\mu\rho\sigma}\mathrm{tr}\left(D_\mu\Phi G_{\rho\sigma}\right)\ . \end{aligned} \tag{8.2}$$

Observe that when $b=\sqrt{3}$, which is what would be found by reduction (and truncation) to $d=4$ of $d=5$ supergravity, then the dilaton coupling to the YM field is $1/\sqrt{3}$, which is also what is found on dimensional reduction from $d=5$. The choice of Lagrangian (8.1) is therefore consistent with what we would find on reduction of supergravity/YM from $d=5$ but we are now allowing for arbitrary *non-zero* b. Once the coupling of the dilaton to the YM field is chosen its coupling to the Higgs field can be fixed, and has been so fixed in (8.1), by requiring invariance of the action under the rigid scaling $\sigma\to\sigma+const.$, which is achieved by assigning scaling weights to all fields and then choosing powers of σ to compensate any lack of scale invariance. What makes this non-trivial is the presence of the non-minimal terms in the action involving epsilon tensors for which the scale weight must add to zero. This symmetry is not necessary for consistency of the theory, of course, but it would be required by supersymmetry.

Using the relation $\sigma = b\phi$, from (6.27), the equations of motion for B and Φ can be written as

$$D_\mu \left[e e^{(1-b^2)\phi} G^{\mu\nu} \right] - e^{-(1+b^2)\phi} e \left[\Phi, D^\nu \Phi\right] - \frac{c}{2}\sqrt{1+b^2}\, \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} D_\mu \Phi = 0 \ , \quad (8.3)$$

and

$$D_\mu \left[e e^{-(1+b^2)\phi} D^\mu \Phi \right] + \frac{c\sqrt{1+b^2}}{4} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} G_{\mu\nu} = 0 \ . \quad (8.4)$$

Using (6.25) and (6.26), and assuming that G_{0i} and $D_0\Phi$ vanish, these equations reduce to the *Euclidean* 3*-space* equations

$$D_j G_{ji} - [\Phi, D_i \Phi] - (1+b^2)\partial_j \phi \left[G_{ji} \mp c\, \varepsilon^{ijk} D_k \Phi \right] = 0 \ , \quad (8.5)$$

and

$$D_i \left(D_i \Phi \right) - (1+b^2)\partial_i \phi \left[D_i \Phi \pm \frac{c}{2} \varepsilon^{ijk} G_{jk} \right] = 0 \ . \quad (8.6)$$

Provided that $c^2 = 1$, these equations are solved by any YM-Higgs configuration that solves the *flat space* Bogomol'nyi equations

$$G_{ij} = \mp c \varepsilon^{ijk} D_k \Phi \ . \quad (8.7)$$

Here we see the principal difference with the results of the section 2; unlike the corresponding equation of that section, equation (8.7) is *independent* of ϕ. Its solutions are well known and include multi-monopole configurations. For example, the one-monopole solution for YM group $SO(3)$ is

$$\Phi^a = n^a \left[\frac{1}{r} - \frac{\cosh r}{\sinh r} \right] \qquad\qquad B_i^a = \varepsilon^{iab} n^b \left[\frac{1}{\sinh r} - \frac{1}{r} \right] \ , \quad (8.8)$$

where a, b are $SO(3)$-vector indices and $n^a = \frac{r^a}{r}$.

As a further illustration of the difference between the dilatonic and non-dilatonic cases, we may choose $c = -1$ and use the specific form of the metric (6.25) and dilaton (6.27) to rewrite (8.7) as

$$e^{\frac{\sigma}{b}} \sqrt{g^{(3)}} g^{il} g^{jm} G_{lm} = \pm \varepsilon^{ijk} D_k \Phi \ , \tag{8.9}$$

which may be compared with (2.12). Observe that (8.9) has no $b \to 0$ limit, so that (2.12) cannot be obtained as a limit of (8.9).

We now turn to the Einstein, Maxwell and dilaton equations. Firstly, we note that, under the same conditions used to derive (8.5), the only non-zero components of the matter stress-tensor and electric current are

$$\begin{aligned} T_{\underline{00}}(mat.) &= \frac{1}{g^2} e^{(3+b^2)\phi} \mathrm{tr}(D\Phi \cdot D\Phi) \\ e^{-b\sigma} J_{\underline{0}}(mat.) &= \mp \frac{c}{g^2} \sqrt{1+b^2} e^{-(3+b^2)\phi}\, \mathrm{tr}(D\Phi \cdot D\Phi) \ , \end{aligned} \tag{8.10}$$

where $D\Phi \cdot D\Phi$ indicates the *Euclidean* 3-vector scalar product. Recall that consistency requires that

$$T_{\underline{00}}(mat.) = \frac{e^{-b\sigma}}{\sqrt{1+b^2}} \mid J_{\underline{0}}(mat.) \mid \ , \tag{8.11}$$

which we now see is true *provided again that* $c^2 = 1$. The Einstein and Maxwell equations may now be seen to be equivalent to the *Euclidean* 3-*space* Poisson equation

$$-\nabla^2 \left[e^{\left(1+b^2\right)\phi} \right] = \left(\frac{4\pi G}{g^2} \right) \left(1+b^2\right) \mathrm{tr}\left(D\Phi \cdot D\Phi\right) \ . \tag{8.12}$$

This is also the dilaton equation; the choice of dilaton coupling to the YM field in (8.1) was chosen to make this happen.

Using the Bogomol'nyi equation, (8.12) can be rewritten as

$$\nabla^2 \left[e^{(1+b^2)\phi} \right] = -\eta \left(\frac{4\pi G}{g^2} \right) (1+b^2) \mid \partial_i B_i^{mat} \mid , \tag{8.13}$$

where the 'matter' magnetic field B_i^{mat} is again given by

$$B_i^{mat} = \frac{1}{2\eta} \varepsilon^{ijk} \mathrm{tr}(\Phi G_{jk}) . \tag{8.14}$$

The existence of a unique non-singular solution for ϕ to equation (8.13) that vanishes at infinity is guaranteed. We have not been able to find this solution analytically, but its asymptotic form is

$$e^{(1+b^2)\phi} \sim 1 + (1+b^2)\alpha \frac{\mid P^{mat} \mid}{\eta r} , \tag{8.15}$$

where P^{mat} is the again the (dimensionless) total 'matter' magnetic charge determined by the flux of $\mathbf{B}^{mat}$ through the sphere at spatial infinity and α is the dimensionless constant

$$\alpha = \frac{G\eta^2}{g^2} . \tag{8.16}$$

Note that ϕ is non-singular, and hence there are no event horizons, for *any* value of α. By comparison with the corresponding formula (7.3) for a black hole, and using (7.6), we see that

$$M = \left(\frac{\eta}{g^2} \right) \mid P^{mat} \mid . \tag{8.17}$$

Since the self-gravitating monopole solution we have found is supersymmetric and saturates the bound (6.16) with $P = 0$, we find the following relation between the electric charge and the 'matter' magnetic charge:

$$\mid Q \mid = \sqrt{(1+b^2)} \left(\frac{\eta}{g^2} \right) \mid P^{mat} \mid . \tag{8.18}$$

The bound on the total energy implied by supersymmetry may therefore be expressed either in terms of Q or in terms of P^{mat}.

As in section 2, we conclude with the alternative derivation of the energy bound that is closer in spirit to Bogomol'nyi's original argument. We insert the ansatz (6.25) and (6.26) into the action (8.1). From the corresponding Hamiltonian the total energy is then found to be

$$E = \frac{1}{g^2}\int d^3x \Big\{ \frac{1}{4} e^{-(1+b^2)\phi} \mathrm{tr}\big(G_{ij} \mp \varepsilon_{ijk} D_k \Phi\big)^2 \\ + \frac{1}{2} e^{(3-b^2)\phi}\big[\mathrm{tr}(G_{0i}G_{0i}) + \mathrm{tr}(D_0\Phi D_0\Phi)\big] \pm \eta \partial_i \left[e^{-(1+b^2)\phi} B_i^{mat}\right] \Big\}. \tag{8.19}$$

Assuming that $\mathbf{B}^{mat} \sim 1/r$ and that $\phi \to 0$ as $r \to \infty$, we immediately derive the bound

$$E \geq \frac{\eta}{g^2} \mid P^{mat} \mid , \tag{8.20}$$

which is saturated when

$$G_{0i} = 0 \qquad D_0\Phi = 0 \qquad G_{ij} = \pm\varepsilon_{ijk} D_k \Phi \ , \tag{8.21}$$

in agreement with (8.7). Thus, a solution of (8.21) has mass M given by (8.17).

9. Conclusions

Minkowski space BPS monopoles saturate a Bogomol'nyi bound on their mass in terms of their magnetic charge and are therefore stable. Reissner-Nordström black holes are stable for a similar reason; they saturate a gravitational analogue of the Bogomol'nyi bound on their ADM mass in terms of a combination of electric and magnetic charges. These two examples of stable 'solitons' can be viewed as extreme cases of the more general situation of a self-gravitating BPS monopole. The strength of the gravitational field relative to the YM/Higgs fields is measured by the dimensionless constant $\alpha = \frac{G\eta^2}{g^2}$. If $\alpha \ll 1$, gravitational effects can be ignored and we have, effectively, a flat space BPS monopole. If $\alpha \gg 1$ one might expect the monopole core to be hidden behind a horizon, with only the long range

$U(1)$ subgroup of the YM group in evidence, in which case we have, effectively, a magnetic Reissner-Nordström black hole. In the latter case one expects an *approximate* bound on the energy in terms of the magnetic charge, but this must fail when $\alpha \sim 1$ because the replacement of the YM field by its long range $U(1)$ component is then no longer justifiable. Since the mass of the black hole is no longer bounded by the magnetic charge one might expect it to exhibit an instability. Such an instability has been demonstrated by Lee, Nair and Weinberg [20].

We have shown that the inclusion of certain non-minimal couplings of the YM/Higgs fields to an additional $U(1)$ 'Maxwell' field implies that the mass of any field configuration *is* bounded by the YM monopole charge. Configurations that saturate this bound are necessarily stable. This is true whether or not there is a dilaton field present, but if there is one its coupling to the YM/Higgs fields cannot be arbitrarily chosen. Accumulated experience suggests that these non-minimal couplings and the required dilaton couplings (when applicable) are those which would be required for a coupling of $N = 2$ supergravity to $N = 2$ super YM theory. We have not proven this here but if this proposition is accepted we see that, as expected, instabilities of the type found in [20] cannot occur in supersymmetric theories.

Our results for supersymmetric self-gravitating monopoles in the presence of a dilaton show that there is no event horizon whatever the monopole core radius, so one cannot 'hide' a monopole inside a black hole. In the absence of a dilaton, the situation is less clear. It may be that in this case one can 'hide' a monopole inside a black hole. In any event, any black hole/monopole configuration saturating the gravitational version of the Bogomol'nyi bound is necessarily stable.

A curious fact that emerges from the analysis of this paper is the qualitative difference in supersymmetric field configurations occasioned by the presence of a dilaton. With a dilaton the full gravitationally-corrected YM/Higgs equations are solved by any solution of the *flat-space* Bogomol'nyi equations. We mentioned in the introduction that this result can be understood, via dimensional reduction, as being a consequence of the similar result for $d = 5$, at least for a special value of the dilaton coupling constant b. It seems likely that this observation could be extended to all non-zero values of b by inclusion of a dilaton in $d = 5$.

One aim of this work was to understand in the gravitational context the much studied Bogomol'nyi bounds saturated by Minkowski space soliton solutions. We believe that we have achieved this fully in $d = 5$ but there remain questions in $d = 4$. In particular, we have not yet found solutions representing self-gravitating dyons. It will probably be important to fill this gap before attempting to investigate the nature of the metric on the moduli space of multi self-gravitating BPS monopoles because Minkowski space results suggest that dyons could be produced by scattering monopoles. However this much seems to be rather plausible: as a manifold the moduli space must be identical to the flat space moduli space, but the metric may differ. For arbitrary monopole number, $N = 2$ supersymmetry strongly suggests that it is Kähler. For two monopoles the rotational symmetries of flat $\mathbb{R}^3$ should give rise to a diagonal Bianchi IX metric.

Acknowledgements

One of the authors, L.A.J.L., would like to thank E.A.F. Plessiet for useful and informative discussions. D.K., P.K.T. and J.T. thank the Institute for Theoretical Physics for its hospitality and support under grant NSF-PHY89-04035. JT is supported in part by grant NSF-THY-8714-684-A01.

APPENDIX

In order to show that spherically symmetric monopoles exist for the system described in Section 2, we have to solve three coupled equations. We will show that these equations have solutions, at least in the limit of small gravitational coupling⋆. Make the ansatze for the fields (appropriate to isotropic coordinates)

$$\Phi^a = \hat{r}^a s(r), \qquad B_i^a = \epsilon_{iab}\frac{\hat{r}^b}{r}e^{-\phi}\left(-1+gv(r)\right) \ . \tag{A.1}$$

The Bogomol'nyi equation (2.12) is then given by

$$\psi \equiv v'(r) - gv(r)s(r)e^{\phi} = 0, \qquad \eta \equiv s'(r) - \frac{1}{r^2}\left(-\frac{1}{g}+gv^2\right)e^{-\phi} = 0 \ . \tag{A.2}$$

This can be used to put the constraint equation in the form

$$\partial_r\left(r^2\partial_r e^{\phi}\right) = -\frac{4\pi G}{g^2}\partial_r\left(s(-\frac{1}{g}+gv^2)\right) \ . \tag{A.3}$$

Which can, after using equations (A.2), be integrated to give

$$e^{\phi} = e^{\phi_0}e^{-2\pi G s^2(r)/g^2} \ . \tag{A.4}$$

Since $s\to\eta$ as $r\to\infty$, we find that $e^{\phi}\to\left(1+\frac{M}{r}\right)$, where $M=\frac{4\pi\eta}{g^2}$ is the ADM mass of the spacetime.

We now want to argue for the existence of solutions to equations (A.2) for $\alpha=\frac{G\eta^2}{g^2}$ sufficiently small. Substitute the expression (A.4) for e^{ϕ} into (A.2). Then the left hand sides are functionals ψ, η of $s(r)$, $v(r)$ and the parameter α. Let $\mathcal{F}(s,v;\alpha)=(\psi,\eta)$. We seek solutions $s(\alpha)$, $v(\alpha)$ such that $\mathcal{F}(s,v;\alpha)=0$. We know that $\mathcal{F}(\bar{s},\bar{v};0)=0$, where $\bar{s}$ and $\bar{v}$ are the known flat space solutions. Hence,

⋆ Similar arguments are made in [21,22] for the existence of gravitating monopoles solutions without the extra interaction with the Maxwell field considered here.

by the implicit function theorem, it suffices to show that (a) $\frac{\partial\mathcal{F}}{\partial\alpha}\Big|_{\alpha=0}$ is continuous and that (b) there are no zero modes of

$$\mathcal{DF} \equiv \left[\frac{\partial\mathcal{F}}{\partial s}\delta s + \frac{\partial\mathcal{F}}{\partial v}\delta v\right]\Bigg|_{\alpha=0}, \qquad (A.5)$$

or of $\mathcal{DF}^*$. Condition (a) is easily checked directly, using the behavior of $\bar{s}$ and $\bar{v}$. Condition (b) follows from the stability of the flat space solution to small perturbations.

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